UNIVERSIDAD DE INVESTIGACIÓN DE TECNOLOGÍA EXPERIMENTAL YACHAY

Escuela de Ciencias Físicas y Nanotecnología

TÍTULO: SCATTERING OF SCALAR RELATIVISTIC PARTICLE BY THE LAMBERT-*W* POTENTIAL

Trabajo de integración curricular presentado como requisito para la obtención del título de Físico

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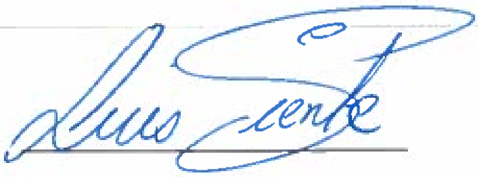

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Dedicatoria

Este trabajo se lo dedico con todo cariño a mis padres por su amor, sacrificio y esfuerzo, por apoyarme en mis estudios y brindarme una profesión y las herramientas necesarias para afrontar los desafíos que se presentan en la vida. Siempre han confiado en mi capacidad y han estado brindándome constantemente su comprensión, cariño y afecto, estoy seguro que nos deparará un futuro prometedor. Me han dado buenos valores, me han guiado en el camino correcto y continuamente me han orientado con sabios principios poniendo a Dios en primer lugar.

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Resumen

En este trabajo derivamos la ecuación de Klein-Gordon que es una ecuación de onda relativista. Esta ecuación describe correctamente a todas las partículas con espín cero. Como una teoría de campo cuantificada, la ecuación de Klein-Gordon describe bosones. Particularmente, vamos a estudiar las soluciones de dispersión de la ecuación de Klein-Gordon de una dimensión con la barrera de potencial Lambert-W. También, estudiaremos las soluciones de dispersión del potencial de tangente hiperbólica y la barrera de potencial de paso. Estos potenciales idealizados son estudiados en esta investigación porque son relativamente fáciles de entender y representan excelentes aproximaciones de lo que ocurre en el mundo real. Las soluciones de dispersión son derivadas en términos de funciones híper geométricas y discutidos en términos de un potencial de barrera con un alto valor. Dividimos nuestra investigación en tres regiones, observando superradiancia en uno de ellos. Por último, analizamos el fenómeno conocido como la Paradoja de Klein cuando una mayor cantidad de partículas son reflejadas por un potencial que partículas que inciden en estas barreras de potencial.

Palabras clave:

Ecuación de Klein-Gordon, potencial Lambert-W, estados de dispersión, funciones híper geométricas, funciones Heun confluentes.

Abstract

In this work we derive the Klein-Gordon equation that is a relativistic wave equation. This equation describes all spinless particles with positive, negative as well as zero charge. As a quantized field theory, the Klein-Gordon equation describes bosons. Particularly, we are going to study the scattering solutions of the one-dimensional Klein-Gordon equation with the Lambert-W potential barrier. We also study the scattering solutions of the hyperbolic tangent potential and the step potential. These idealized potentials are studied in this research because they are relatively easy to understand and they are exemplary approximations to real ones. The scattering solutions are derived in terms of hypergeometric functions, and discussed in terms of the height of the potential barrier. We divide our research into three regions, observing superradiance in one of them. At last, we discuss the phenomenon known as Klein Paradox when more particles are reflected by a potential than are incident on it.

Keywords:

Klein-Gordon equation, Lambert-W potential, scattering states, hypergeometric functions, confluent Heun functions.

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Chapter 1

Introduction

In the context of quantum mechanics the physical state of a free particle of mass m is described by a wave function $\psi(\vec{x},t)$ encompassing all space-time information which is itself a classical field having a probabilistic interpretation. This work analyzes the relativistic equation for wave functions, called the Klein-Gordon equation and we will restrict the discussion to motion in one dimension. For a single free particle a wave function, in the relativistic case, it is solution to the Klein-Gordon equation*. A system of N interacting particles will be described by a wave function $\psi(\vec{x}_1,\vec{x}_2,...,\vec{x}_N;t)$ whose squared modulus $\left|\psi(\vec{x}_1,\vec{x}_2,...,\vec{x}_N;t)\right|^2$ represents the probability density $\rho(x)$ of finding the particles at the points $\vec{x}_1,\vec{x}_2,...,\vec{x}_N$ at the time t. The normalization of $\psi(\vec{x},t)$ is fixed by requiring that the probability of finding the particle anywhere in space at any time t be one

$$\int \left|\psi(\vec{x},t)\right|^2 d^3x = 1. \tag{1.1}$$

*The famous equation suggested by Erwin Schrödinger was studied with great attention by Oskar Benjamin Klein and Walter Gordon. We know that from elementary quantum mechanics non-relativistic Schrödinger equation is $i\hbar\frac{\partial}{\partial t}\psi(\vec{x},t) = \left[-\frac{\hbar^2}{2m_0}\vec{\nabla}^2 + \vec{V}(\vec{x})\right]\psi(\vec{x},t)$.Remember that the Schrödinger equation for the quantum wave function is based on the non-relativistic expression for the energy of a particle.

In quantum mechanics, the one-dimensional potential is an idealized system employed to model incident, reflected and transmitted matter waves. In order to determine the quantum mechanics about this physical situation, we have to make a model and then solve the Klein Gordon equation to find the convenient wave functions. In quantum mechanics, it is useful to define the concept of particle flux, which in one dimension it is defined as the average number of particles passing a point per unit time[1].

1.1 Problem Statement

The physical situation we are modeling is really peculiar. A particle is moving with constant velocity in the positive x-direction and encounters at some point with a potential energy function $V(x)$. There exists some potentials for which the stationary Klein-Gordon equation is exactly determined in terms of special functions. For non-relativistic quantum physics the basic equation to be solved is the Schrödinger equation, in a certain way it is more common to consider the Schrödinger equation in order to discuss the properties of the potentials, as well as reflection and transmission of a particle when interacts with different barriers or obstacles. In response to this problem, our study proposes to investigate the scattering solutions for Lambert-W potential and analyze their resemble the characteristics of both step and hyperbolic tangent potentials starting from the Klein-Gordon equation.

1.2 General and Specific Objectives

We apply a innovative solution to determine reflection and transmission equations for Lambert-W potential, these solutions of this problem are written in terms of the confluent Heun equations. We will explain properly the superradiance phenomenon, when the reflection coefficient (R), is greater than one.

This undergraduate work is organized of the following way. Chapter 2 shows the scattering solutions of the Klein-Gordon equation for the step potential barrier and hyperbolic tangent potential in terms of hypergeometric functions. In both cases the behaviour of the reflection (R) and transmission (T), coefficients are studied in different regions of energy. In chapter 3 the results of scattering solutions and the behaviour of R and T coefficients for the Lambert-W potential are shown. This research ends with the chapter 4, where conclusions are discussed.

Chapter 2

Methodology

2.1 The Klein-Gordon Equation

The description of the phenomena at high energies requires the investigation of relativistic wave equation[2]. The principal features of the Klein-Gordon theory for the relativistic description of spin-0 particles * are explained in this section. Here we deal with negative energy states, which can be related to antiparticles. Furthermore, we discuss the range of validity of the Klein-Gordon one-particle and show two examples of interpretational potentials. The transition from nonrelativistic to a relativistic description implies the next concepts:

1. A relativistic particle cannot be localized more accurately than $\approx \hbar/m_0 c$, where m_0 denotes the rest mass of the particle.

2. If the position of the particle is uncertain, so that if $\Delta x > \frac{\hbar}{m_0 c}$, the time is also uncertain, because $\Delta t \sim \frac{\Delta x}{c} > \frac{\hbar}{m_0 c}$. This means that relativistic particle cannot be localized more precisely than an area whose linear extend is large related to the particle's *Compton wave*

*Those particles with integer spins, such as 0,1,2, are konow as bosons.

length[3] $\lambda_c = \hbar/(m_0 c)$.

In nonrelativistic quantum mechanics the starting point is the energy-momentum of a free particle

$$E = \frac{\vec{p}^2}{2m}, \tag{2.1}$$

when classical quantities are replaced by the operators energy and momentum, we have respectively

$$E \to i\hbar\frac{\partial}{\partial t}, \qquad \vec{p} \to \frac{\hbar}{i}\vec{\nabla}, \qquad p^\mu \to i\hbar\,\partial^\mu.$$

leads to the free time-dependent Schrödinger equation

$$i\hbar\frac{\partial}{\partial t}\psi(\vec{x},t) = -\frac{\hbar^2}{2m}\vec{\nabla}^2\psi(\vec{x},t). \tag{2.2}$$

This research analyzes at the simplest relativistic equation for wave functions, called the Klein-Gordon equation. The discussion will be based to a spinless particle in empty space, where there is no potential energy. Solutions with a definite value E for the energy take the form $\Psi = c\mathrm{e}^{-iEt/\hbar}\psi$. The Klein-Gordon equation is given by

$$-\frac{1}{c^2}\frac{\partial^2}{\partial t^2}\psi(\vec{x},t) + \nabla^2\psi(\vec{x},t) = \left(\frac{mc^2}{\hbar c}\right)^2\psi(\vec{x},t) \tag{2.3}$$

Recall first how the Klein-Gordon equation arose. For the Minkowski metric the contravariant and covariant metric tensor are identical ($g^{\mu\nu} = g_{\mu\nu}$). For the description of the space-time coordinates the contravariant four-vector for x^μ and p^μ are represented respectively by

$$x^\mu = g^{\mu\nu}x_\nu = \left(x^0, x^1, x^2, x^3\right) = (ct, x, y, z), \tag{2.4}$$

$$p^\mu = g^{\mu\nu}p_\nu = \left(p^0, p^1, p^2, p^3\right) = \left(\frac{E}{c}, p_x, p_y, p_z\right). \tag{2.5}$$

The metric tensor $g_{\mu\nu}$ yields the covariant components

$$x_\mu = g_{\mu\nu}x^\nu = (x_0, x_1, x_2, x_3) = (ct, -x, -y, -z), \tag{2.6}$$

$$p_\mu = g_{\mu\nu}p^\nu = (p_0, p_1, p_2, p_3) = \left(\frac{E}{c}, -\vec{p}\right). \tag{2.7}$$

Using the equation (2.4) and (2.5) we can write the four-momentum operator as

$$\begin{aligned} \hat{p}^{\mu} = i\hbar\frac{\partial}{\partial x_{\mu}} &= \left[i\hbar\frac{\partial}{\partial(ct)}, i\hbar\frac{\partial}{\partial x_1}, i\hbar\frac{\partial}{\partial x_2}, i\hbar\frac{\partial}{\partial x_3}\right], \\ &= \left[i\hbar\frac{\partial}{\partial(ct)}, -i\hbar\frac{\partial}{\partial x}, -i\hbar\frac{\partial}{\partial y}, i\hbar\frac{\partial}{\partial z}\right], \\ &= i\hbar\left[\frac{\partial}{\partial(ct)}, -\vec{\nabla}\right]. \end{aligned} \tag{2.8}$$

The invariant scalar product of the four-momentum is given by

$$p^{\mu}p_{\mu} = \frac{E^2}{c^2} - \vec{p}^2 = p_0^2 - \vec{p}^2 = m_0^2c^2 \quad , \tag{2.9}$$

arriving to the corresponding relativistic energy-momentum relation for free particles,

$$E = \sqrt{\vec{p}^2c^2 + m_0^2\, c^4}. \tag{2.10}$$

where m_0 denotes the rest mass of the particle and c indicates the velocity of light in the vacuum. So, the **Klein-Gordon equation** in Lorentz-covariant form for free particles is given by the next expression

$$\hat{p}^{\mu}\hat{p}_{\mu}\psi(x) = m_0^2c^2\psi(x) \quad . \tag{2.11}$$

Additionally, we can write (2.11) in the form

$$\begin{aligned} \hat{p}^{\mu}\hat{p}_{\mu} &= i\hbar\frac{\partial}{\partial x_{\mu}} i\hbar\frac{\partial}{\partial x^{\mu}}, \\ &= -\hbar^2\frac{\partial}{\partial x_{\mu}}\frac{\partial}{\partial x^{\mu}}, \\ &= -\hbar^2\left(\frac{\partial}{\partial x_0}\frac{\partial}{\partial x^0} + \frac{\partial}{\partial x_1}\frac{\partial}{\partial x^1} + \frac{\partial}{\partial x_2}\frac{\partial}{\partial x^2} + \frac{\partial}{\partial x_3}\frac{\partial}{\partial x^3}\right), \\ &= -\hbar^2\left(\frac{\partial^2}{c^2\partial t^2} - \frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2} - \frac{\partial^2}{\partial z^2}\right), \\ &= -\hbar^2\left(\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \vec{\nabla}^2\right). \end{aligned} \tag{2.12}$$

As is known from electrodynamics, the **d'Alembert operator** $\Box \equiv \partial_\mu \partial^\mu = \frac{\partial}{\partial x_\mu}\frac{\partial}{\partial x^\mu} = \left(\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \vec{\nabla}^2\right)$ is invariant under Lorentz transformations.Then

$$\hat{p}^\mu \hat{p}_\mu = -\hbar^2 \Box, \tag{2.13}$$

so, (2.13) can be written as

$$-\hbar^2 \Box\, \psi = m_0^2 c^2 \psi,$$

$$\left(\Box + \frac{m_0^2 c^2}{\hbar^2}\right)\psi = 0, \tag{2.14}$$

which is widely known as the Klein-Gordon equation. It is a quantum relativistic wave equation, used in the description of particles with spin 0[4]. We recognize (2.14) as the classical wave equation[†] including the mass term $m_0^2 c^2/\hbar^2$. A possible solution for the wave equation is a plane monochromatic wave

$$\psi(\vec{x},t) = \psi_0 \mathrm{e}^{i(\vec{\mathrm{k}}.\vec{x} - \omega t)}, \tag{2.15}$$

propagating in the direction of the vector $\vec{\mathrm{k}}$. Where ω and $\vec{\mathrm{k}}$ are constants related to the frequency f and wavelength λ of the wave by[5]

$$w = 2\pi f, \qquad \lambda = \frac{2\pi}{|\vec{\mathrm{k}}|}.$$

Applying into account Einstein's relation $E = \hbar\omega$, and De Broglie's one, $\vec{p} = \hbar\,\vec{\mathrm{k}}$, we can rewrite the equation for the plane waves (2.15) in terms of the energy and momentum, as

$$\psi(\vec{x},t) = \psi_0 \mathrm{e}^{i(\vec{p}.\vec{x} - Et)/\hbar}. \tag{2.16}$$

2.1.1 Lorentz Invariance of the Klein-Gordon Equation

In this section we will demonstrate the Lorentz invariance of the Klein-Gordon equation that is a direct consequence of the invariance of energy-momentum relation (2.9). Now it is convenient

[†]The classical wave equation: $\frac{\partial^2 \psi}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 \psi}{\partial t^2} = 0$

to express the Klein-Gordon equation (2.14) in the following form

$$\left[\frac{\partial}{\partial x_\mu}\frac{\partial}{\partial x^\mu}+\left(\frac{m_0 c}{\hbar}\right)^2\right]\psi(x_\mu)=0.$$

Therefore, in the transformed system the Klein-Gordon equation it follows that

$$\left[\frac{\partial}{\partial x'_\mu}\frac{\partial}{\partial x'^\mu}+(\kappa)^2\right]\psi'(x'_\mu)=0, \tag{2.17}$$

where $\kappa = m_0 c/\hbar$, therefore, we must show that the operator $(\partial/\partial x'_\mu)(\partial/\partial x'^\mu)$ is invariant under Lorentz transformations. This is achieved by

$$\hat{p}_\mu = +i\hbar\frac{\partial}{\partial x^\mu},$$

and, consequently,

$$\frac{\partial}{\partial x_\mu}\frac{\partial}{\partial x^\mu}\approx \hat{p}_\mu \hat{p}^\mu. \tag{2.18}$$

Lorentz transformation $x \to x'$ implies $\psi \to \psi'$ where

$$\psi(x)=\psi'(x'), \tag{2.19}$$

refers to the same space-time point.

2.1.2 Free Solutions of the Klein-Gordon Equation

The equation (2.14) is known as the **free Klein-Gordon equation**, in order to distinguish it from generalizations that additionally contain external potentials or electromagnetic. There are two free solutions in the form of plane waves

$$\psi^{(1)}_{\vec{p}}(\vec{x},t)=\mathrm{e}^{-i(cp_0 t-\vec{p}.\vec{x})/\hbar}, \qquad \psi^{(2)}_{\vec{p}}(\vec{x},t)=\mathrm{e}^{+i(cp_0 t-\vec{p}.\vec{x})/\hbar},$$

with

$$p_0=+\sqrt{\vec{p}^2+m_0^2\,c^2}>0. \tag{2.20}$$

Note that the Klein-Gordon equation leads to solutions with positive energy eigenvalues $E = +cp_0$ and negative energy eigenvalues $E = -cp_0$. While the positive solutions can be interpreted as particle wave functions, the physical interpretation of the negative solutions $(\psi_{\vec{p}}^{(2)})$ is not so easy. Both positive and negative energies occur here and the energy is not bounded from below.

2.1.3 Interpretation of the negative solutions

Starting from the equation (2.10), we have $E = \pm c\sqrt{\vec{p}^2 + m_0^2\, c^2}$. Thus, as we have noticed there exist solutions both for positive $E = +c\ (\vec{p}^2 + m_0^2\ c^2)^{1/2}$ as well as for negative $E = -c\ (\vec{p}^2 + m_0^2\ c^2)^{1/2}$ energies respectively (see Figure 2.1). Negative solutions can be related to *antiparticles*[3]. At this point we should consider a new degree of freedom, the *electric charge*[‡]. That is, positive solutions describe particles that would carry the charge $-e$, while negative solutions describe antiparticles that would carry the charge $+e$. In experimental results we see generally the bound states directly below the positive energy continuum with $E < m_0 c^2$.

[‡]In the nonrelativistic theory describes states with only one charge sign.

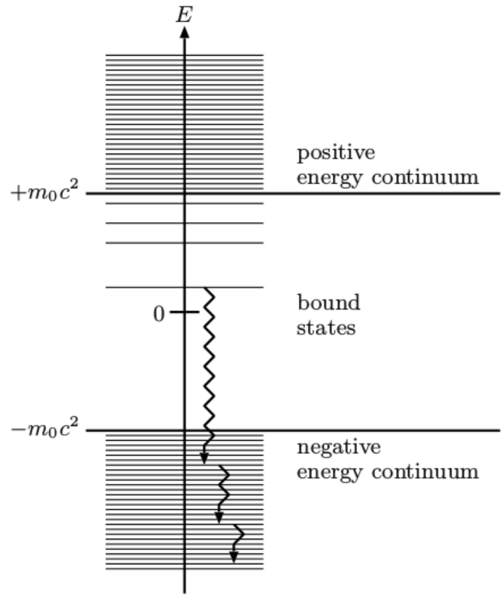


Figure 2.1: Energy spectrum of a antiparticle (specifically a pion atom)

Now we can determinate the four-current density j_μ connected with Klein-Gordon equation. We take the complex conjugate of the equation (2.11)

$$\left(\hat{p}_\mu \hat{p}^\mu - m_0^2 c^2\right) \psi^* = 0. \tag{2.21}$$

Multiplying the equation (2.11) by ψ^* and equation (2.21) by ψ and calculating the difference, we obtain

$$\begin{aligned}
\psi^* \left(\hat{p}_\mu \hat{p}^\mu - m_0^2 c^2\right) \psi - \psi \left(\hat{p}_\mu \hat{p}^\mu - m_0^2 c^2\right) \psi^* &= 0, \\
-\psi^* \left(\hbar^2 \vec{\nabla}_\mu \vec{\nabla}^\mu + m_0^2 c^2\right) \psi + \psi \left(\hbar^2 \vec{\nabla}_\mu \vec{\nabla}^\mu + m_0^2 c^2\right) \psi^* &= 0, \\
-\psi^* \hbar^2 \vec{\nabla}_\mu \vec{\nabla}^\mu \psi + \psi\, \hbar^2 \vec{\nabla}_\mu \vec{\nabla}^\mu \psi^* &= 0, \\
\vec{\nabla}_\mu \left(\psi^* \vec{\nabla}^\mu \psi - \psi \vec{\nabla}^\mu \psi^*\right) = \vec{\nabla}_\mu j^\mu &= 0.
\end{aligned} \tag{2.22}$$

Motion in quantum mechanics is probabilistic, hence, this motion about is how the probability for finding the particle moves around with time. Then, the principal idea we need is to find a *probability current* that relates to how the probability for locating the particle might be changing with time. Multiplying by $i\hbar/2m_0$, so that zero component j_0 has the dimension of a probability density $(1/cm^3)$. Then, the four-current density is

$$j_\mu = \frac{i\hbar}{2m_0}\left(\psi^*\vec{\nabla}_\mu\psi - \psi\vec{\nabla}_\mu\psi^*\right). \tag{2.23}$$

This scalar theory does not contain spin and could only describe particles with zero spin. One of the problems of the Klein-Gordon equation occurs when interpreting the function $\psi(x)$ as probability amplitude[6]. Interpretation of $\psi(x)$ as probability amplitude is only possible if there exists a probability density $\rho(x)$ and a current $\vec{j}(x)$ that satisfy a continuity equation that ensures conservation of charge

$$\frac{\partial}{\partial t}\rho(x) + \vec{\nabla}.\vec{j}(x) = 0. \tag{2.24}$$

In this work, it has been successfully derived the Klein-Gordon equation.

2.2 The interaction of Particles with an Electromagnetic Field

In this section we are interested in the case of relativistic particles with spin 0 particle. The electromagnetic field denoted by the four-vector is defined as

$$A^\mu = \left\{A_0, \vec{A}\right\} = \{A_0, A_x, A_y, Az\} = g^{\mu\nu}A_\nu, \tag{2.25}$$

likewise

$$A_\mu = \left\{A_0, -\vec{A}\right\} = \{A_0, -A_x, -A_y, -Az\} = g_{\mu\nu}A^\nu, \tag{2.26}$$

$$\hat{E} \Rightarrow i\hbar\frac{\partial}{\partial t} - eA_0, \qquad \vec{\hat{p}} \Rightarrow -i\hbar\vec{\nabla} - \frac{e}{c}\vec{A}, \tag{2.27}$$

which can be rewritten to the four-dimensional and covariant form in the following way

$$\hat{p}^\mu \Rightarrow \hat{p}^\mu - \frac{e}{c}A^\mu \quad \text{or} \quad \hat{p}_\mu \Rightarrow \hat{p}_\mu - \frac{e}{c}A_\mu, \tag{2.28}$$

$$\left(\hat{p}^{\mu}-\frac{e}{c}A^{\mu}\right)\left(\hat{p}_{\mu}-\frac{e}{c}A_{\mu}\right)\psi=m_0^2c^2\psi, \tag{2.29}$$

or

$$\left[g^{\mu\nu}\left(i\hbar\frac{\partial}{\partial x^{\nu}}-\frac{e}{c}A_{\nu}\right)\left(i\hbar\frac{\partial}{\partial x^{\mu}}-\frac{e}{c}A_{\mu}\right)\right]\psi=m_0^2c^2\psi, \tag{2.30}$$

$$\left(g^{00}\ \frac{i\hbar}{c}\frac{\partial\psi}{\partial t}-g^{00}\ \frac{e}{c}A_0\psi\right)^2+g^{ii}\ \hbar^2\ \frac{\partial^2\psi}{\partial x^{i2}}=m_0^2c^2\psi,$$

$$\left\{\left[\frac{E}{c}-\frac{V(x)}{c}\right]^2-m_0^2c^2+\hbar^2\vec{\nabla}^2\right\}\psi(x)=0,$$

$$\left\{\left[E-V(x)\right]^2-m_0^2c^4+\hbar^2c^2\vec{\nabla}^2\right\}\psi(x)=0,$$

$$\hbar^2c^2\frac{d^2\psi(x)}{dx^2}+\left\{\left[E-V(x)\right]^2-m_0^2c^4\right\}\psi(x)=0,$$

then, the one-dimensional Klein-Gordon equation to solve in natural units $\hbar=c=1$, is

$$\frac{d^2\psi(x)}{dx^2}+\left\{\left[E-V(x)\right]^2-m_0^2\right\}\psi(x)=0. \tag{2.31}$$

For simplicity, we also set the mass of the particle equal to one ($m_0=1$), we get

$$\frac{d^2\psi(x)}{dx^2}+\left\{\left[E-V(x)\right]^2-1\right\}\psi(x)=0, \tag{2.32}$$

where E is the total energy of the particle, $V(x)$ the potential energy function and $\psi(x)$ the spatial part of wave function.

2.3 Finding Solutions to Klein-Gordon Equation

2.3.1 Step Potential Barrier

In order to start our investigation of simple one-particle systems, we shall explore the behavior of the solutions to the Klein-Gordon equation for a particle whose potential energy $V(x)$ can be represented by

$$V(x) = \begin{cases} 0, & \text{for} \quad x < 0, \\ V_0, & \text{for} \quad x \geq 0. \end{cases} \tag{2.33}$$

This potential is know as *step potential barrier*, illustrate in Figure 2.2. Where V_0 is a positive constant energy and the barrier is positioned at $x = 0$. Note that the potential energy of the particle is zero when it is to left of the step. The result we obtain for this potential will allow us to explain several characteristic quantum mechanical phenomena. Assume that a particle is moving toward the point $x = 0$ at which the $V(x)$ suddenly changes its value.

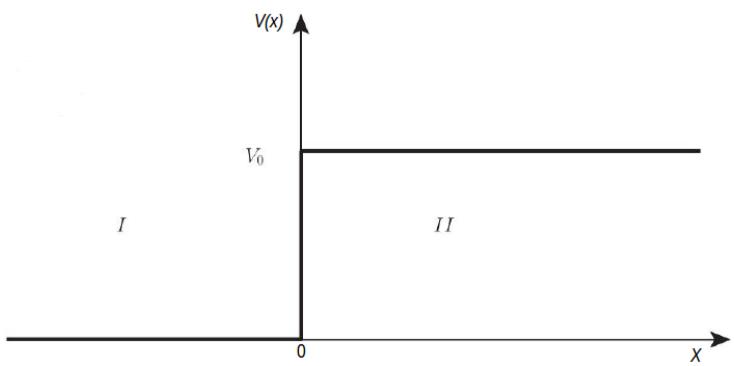


Figure 2.2: A schematic representation of a step potential of height V_0

With the purpose of determine the motion of the particle for this case, the x axis breaks up into two region and then we use the stationary Klein-Gordon equation (2.32). The region I where $x < 0$ (left of the step) we have $V(x) = 0$, $\psi(x) = \psi_I(x)$, and the region II where $x \geq 0$ (right of the step), we have $V(x) = V_0$ and $\psi(x) = \psi_{II}(x)$. So the behavior of the particle is a solution to the simple K-G equation (2.32). For both regions, we obtain

$$\frac{d^2\psi_I(x)}{dx^2} + \left(E^2 - 1\right)\psi_I(x) = 0 \qquad (x < 0), \tag{2.34}$$

$$\frac{d^2\psi_{II}(x)}{dx^2} + \left[(E - V_0)^2 - 1\right]\psi_{II}(x) = 0 \qquad (x \geq 0). \tag{2.35}$$

The two equations are solved separately. The differential equation (2.34) can be expressed as

$$\frac{d^2\psi_I(x)}{dx^2} + k^2\psi_I(x) = 0. \tag{2.36}$$

Note that k^2 is a positive constant

$$k = \sqrt{E^2 - 1} \tag{2.37}$$

The general solution of traveling wave in this region is easy enough to write down. We obtain

$$\psi_I(x) = A\ \mathrm{e}^{ikx} + B\ \mathrm{e}^{-ikx}, \tag{2.38}$$

where A and B are arbitrary constants, and we can determine these from the conditions assumed to apply at certain position. Applying two boundary conditions, at $x = 0$ we get

$$\psi_I(x)\ \big|_{x=0} = \psi_{II}(x)\ \big|_{x=0} \qquad (2.39a) \qquad \frac{\psi_I(x)}{dx}\bigg|_{x=0} = \frac{d\psi_{II}(x)}{dx}\bigg|_{x=0} \qquad (2.40a)$$

Now, evaluate the probability current (2.23) for the wave function (2.38), for which

$$j_L = k\ (AA^* - BB^*) \qquad x < 0. \tag{2.41}$$

So that we can identify the current when $x \to -\infty$ as

$$j_L = j_{inc} - j_{ref}, \tag{2.42}$$

where j_{inc} and j_{ref} are the incident and reflected current, respectively. We can note

$$j_{inc} = k\ AA^*, \tag{2.43}$$

as the probability current incident on the barrier from the left and

$$j_{ref} = k\ BB^*, \tag{2.44}$$

as the probability current reflected from the barrier. Resulting that the reflection coefficient is expressed by

$$R = \frac{j_{ref}}{j_{inc}} = \frac{BB^*}{AA^*}, \tag{2.45}$$

and the transmission coefficient for this scattering example is given by

$$T = \frac{j_{trans}}{j_{inc}}. \tag{2.46}$$

Next we consider the differential equation for the region in which $V(x) = V_0$.

$$\frac{d^2\psi_{II}(x)}{dx^2} + q^2\psi_{II}(x) = 0, \tag{2.47}$$

where q^2 is a positive constant

$$q = \sqrt{(E - V_0)^2 - 1}. \tag{2.48}$$

We calculate its solution

$$\psi_{II}(x) = C\,\mathrm{e}^{iqx} + D\,\mathrm{e}^{-iqx}. \tag{2.49}$$

Since the energy is greater than the potential energy for $x > 0$, the solutions are given by (2.49), where the D term generates a probability current flowing to the left for $x > 0$[7]. The D term causes a probability current flowing to the left for $x > 0$. So, we are interested in which particles are incident on the potential step only from the left, then we can establish $D = 0$ in (2.49), we find

$$\psi_{II}(x) = C\mathrm{e}^{iqx} \qquad x > 0, \tag{2.50}$$

Summarizing the general solutions

$$\psi(x) = \begin{cases} A\,\mathrm{e}^{ikx} + B\,\mathrm{e}^{-ikx} & x < 0, \\ C\,\mathrm{e}^{iqx} & x > 0, \end{cases}$$

and substituting the wave function (2.50) into (2.23), we have

$$j_R = j_{trans} = CC^*, \tag{2.51}$$

substituting (2.51) and (2.43) into (2.46), we obtain the transmission coefficient for the step potential

$$T = \frac{q}{k}\frac{CC^*}{AA^*} \tag{2.52}$$

Let's now examine the reflection and transmission coefficient in terms of k and q. The arbitrary constants A, B, and C of equation (2.38) and (2.50) must be so chosen that the total eingenfunction satisfies the requirements concerning finiteness and continuity, of $\psi(x)$ and $\psi'(x)$. Continuity of $\psi(x)$ is obtained by satisfying the relation

$$A\,\mathrm{e}^{ikx}\Big|_{x=0} + B\,\mathrm{e}^{-ikx}\Big|_{x=0} = C\,\mathrm{e}^{iqx}\Big|_{x=0}, \tag{2.53}$$

and continuity of the derivative

$$ikA\,\mathrm{e}^{ikx}\Big|_{x=0} - ikB\,\mathrm{e}^{-ikx}\Big|_{x=0} = iq\,C\,\mathrm{e}^{iqx}\Big|_{x=0}, \tag{2.54}$$

which yield

$$A + B = C \tag{2.55}$$

$$ik\,(A - B) = iq\,C \tag{2.56}$$

Solving this system of equations,we have

$$C = \frac{2k}{k+q}A \quad \text{and} \quad B = \frac{k-q}{k+q}A \tag{2.57}$$

Note that we have satisfied the boundary conditions for any value of the energy[7]. Using (2.45) and (2.52), we have

$$R = \frac{(k-q)^2}{(k+q)^2} = \left|\frac{(q-k)}{(q+k)}\right|^2, \tag{2.58}$$

the reflection coefficient. On the other hand, we find

$$T = \frac{4kq}{(k+q)^2} = \frac{q}{k}\left|\frac{2k}{q+k}\right|^2, \tag{2.59}$$

the transmission coefficient. Note that $R + T = 1$.

2.3.2 Hyperbolic Tangent Potential

At this moment, we solve the scattering solutions of the Klein Gordon equation in the presence of the hyperbolic tangent potential. For this potential we calculate the reflection (R_{HT}) and transmission (T_{HT}) coefficients. The hyperbolic tangent potential is given by

$$V(x) = \frac{V_0}{2}\left[1 + \tanh(bx)\right], \tag{2.60}$$

where V_0 represents the height of the potential and b the smoothness of the curve. When $b \to \infty$ this potential goes into the step potential, we obtain the curve of $V(x)$ as a function of x shown in Figure 2.3. First, to consider the scattering solutions, we replace (2.60) in (2.32) and give us the

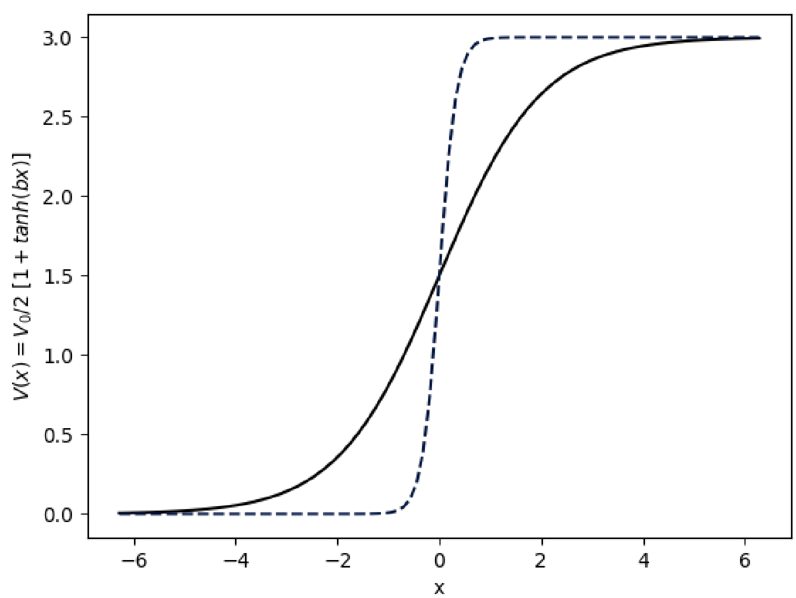


Figure 2.3: The hyperbolic tangent potential with $b = 0.5$ (solid line) and $b = 3$ (dashed line), for $V_0 = 3$ in both cases.

following differential equation

$$\frac{d^2\psi(x)}{dx^2} + \left\{ \left[E - \frac{V_0}{2}[1 + \tanh(bx)]\right]^2 - 1 \right\} \psi(x) = 0. \tag{2.61}$$

where $\tanh(bx)$ in terms of exponential functions is given by

$$\tanh(bx) = \frac{e^{bx} - e^{-bx}}{e^{bx} + e^{-bx}}. \tag{2.62}$$

By making the substitution $y = -e^{2bx}$,

$$\begin{aligned}\frac{d^2\psi}{dx^2} &= \frac{d}{dy}\left(2by\frac{d\psi}{dy}\right)\frac{dy}{dx} \\ &= \frac{d}{dy}\left(2by\frac{d\psi}{dy}\right)\left(-2be^{2bx}\right) \\ &= 4b^2y\frac{d}{dy}\left(y\frac{d\psi}{dy}\right)\end{aligned} \tag{2.63}$$

and equation (2.62) becomes

$$\tanh(bx) = \frac{-iy^{1/2} - iy^{-1/2}}{-iy^{1/2} + iy^{-1/2}} = -\frac{1+y}{1-y}. \tag{2.64}$$

Substituting from (2.63) and (2.64) into (2.61) we get

$$4b^2y\frac{d}{dy}\left(y\frac{d\psi}{dy}\right) + \left[\left(E + V_0\,\frac{y}{1-y}\right)^2 - 1\right]\psi(y) = 0$$

$$4b^2y(1-y)^2\frac{d}{dy}\left(y\frac{d\psi}{dy}\right) + \left[\left(E(1-y) + V_0\,y\right)^2 - (1-y)^2\right]\psi = 0 \tag{2.65}$$

Plugging $\psi = y^{\alpha}(1-y)^{\beta}f(y)$ into (2.65). This leads to

$$\begin{aligned}&y^{\alpha+1}(1-y)^{\beta+1}\Big[\alpha^2y^{-1}(1-y)f - \alpha\beta f + \alpha(1-y)f' - \beta(\alpha+1)f + \beta(\beta-1)\,y\,(1-y)^{-1}f - \beta yf' + \\ &\quad(\alpha+1)(1-y)f' - \beta yf' + y(1-y)f''\Big] + \frac{1}{4b^2}\left\{\left[E(1-y) + V_0\,y\right]^2 - (1-y)^2\right\}y^{\alpha}(1-y)^{\beta}f = 0\end{aligned} \tag{2.66}$$

After some calculations, we arrive

$$\begin{aligned}&y^{\alpha+1}(1-y)^{\beta+1}\left\{y(1-y)f'' + [(2\alpha+1) - (2\alpha+2\beta+1)\,y\,]f'\right\} + y^{\alpha}(1-y)^{\beta} \\ &\underbrace{\left\{\alpha^2(1-y)^2 - \alpha\,\beta\,y(1-y) - \beta(\alpha+1)\,y(1-y) + \beta(\beta-1)y^2 + \frac{1}{4b^2}\left[\left(E(1-y) + V_0\,y\right)^2 - (1-y)^2\right]\right\}}_{Ay(1-y)} f =\end{aligned} \tag{2.67}$$

$$y^{\alpha+1}(1-y)^{\beta+1}\left\{y(1-y)f''+[(2\alpha+1)-(2\alpha+2\beta+1)\,y\,]f'\right\}+y^{\alpha}(1-y)^{\beta}Ay(1-y)f=0 \tag{2.68}$$

$$y(1-y)f''+[(2\alpha+1)-(2\alpha+2\beta+1)\,y\,]f'+Af=0 \tag{2.69}$$

The equation (2.69) has the form of hypergeometric differential equation[8]

$$(1-z)\frac{d^2w}{az^2}+\left[c'-(a'+b'+1)\,z\,\right]\frac{dw}{dz}-a'b'w=0$$

Thus, (2.69) becomes

$$y(1-y)f''+\left[(1+2\alpha)-(2\alpha+2\beta+1)\,y\,\right]f'-(\alpha+\beta-\gamma)(\alpha+\beta+\gamma)f=0, \tag{2.70}$$

note that the prime notation indicates derivatives with respect to y, and also this equation has the general solution in terms of Gaussian hypergeometric function[§] ${}_2F_1(k_{HT},q_{HT},\lambda;y)$

$$\begin{aligned} f &= C_1\,{}_2F_1(\alpha+\beta-\gamma,\alpha+\beta+\gamma,1+2\alpha;y)\\ &\quad +C_2\,y^{-2\alpha}\,{}_2F_1(-\alpha+\beta+\gamma,-\alpha+\beta-\gamma,1-2\alpha;y), \end{aligned} \tag{2.71}$$

where the involved parameters are given as

$$\alpha=ik_{HT}\quad \text{with}\quad k_{HT}=\frac{\sqrt{E^2-1}}{2b}, \tag{2.72}$$

$$\beta=\lambda\quad \text{with}\quad \lambda=\frac{b+\sqrt{b^2-V_0^2}}{2b}, \tag{2.73}$$

$$\gamma=iq_{HT}\quad \text{with}\quad q_{HT}=\frac{\sqrt{(E-V_0)^2-1}}{2b}. \tag{2.74}$$

[§]Gaussian or ordinary hypergeometric function is a solution of a second-order linear ordinary differential equation (ODE)

Then, (2.71) in terms of x becomes

$$\psi(x) = \underbrace{C_1\left(-\mathrm{e}^{2bx}\right)^{ik_{HT}}\left(1+\mathrm{e}^{2bx}\right)^{\lambda}\ {}_2F_1\left(ik_{HT}+\lambda-iq_{HT}, ik_{HT}+\lambda+iq_{HT}, 1+2ik_{HT}; -e^{2bx}\right)}_{\text{incident wave}}$$
$$+\underbrace{C_2\left(-\mathrm{e}^{2bx}\right)^{-ik_{HT}}\left(1+\mathrm{e}^{2bx}\right)^{\lambda}\ {}_2F_1\left(-ik_{HT}+\lambda+iq_{HT}, -ik_{HT}+\lambda-iq_{HT}, 1-2ik_{HT}; -e^{2bx}\right)}_{\text{reflected wave}}. \tag{2.75}$$

This equation provides the incident and reflected waves.Using the following expression

$$\begin{aligned} {}_2F_1(a,b,c;z) =& \frac{\Gamma(c)\Gamma(b-a)}{\Gamma(b)\Gamma(c-a)}(-z)^{-a}\ {}_2F_1\left(a, 1-c+a, 1-b+a; z^{-1}\right) \\ &+\frac{\Gamma(c)\Gamma(a-b)}{\Gamma(a)\Gamma(c-b)}(-z)^{-b}\ {}_2F_1\left(b, 1-c+b, 1-a+b; z^{-1}\right), \end{aligned} \tag{2.76}$$

we can find the transmitted wave

$$\psi_{trans} = C_3\left(-\mathrm{e}^{2bx}\right)^{iq_{HT}}\left(1+\mathrm{e}^{2bx}\right)^{\lambda}\left(\mathrm{e}^{2bx}\right)^{-\lambda}\ {}_2F_1\left(ik_{HT}+\lambda-iq_{HT}, -ik_{HT}+\lambda-iq_{HT}, 1+2iq_{HT}, -e^{2bx}\right) \tag{2.77}$$

We define the next expression

$$\psi_{trans} = A\psi_{inc} + B\ \psi_{ref}, \tag{2.78}$$

where

$$A = \frac{\Gamma(1-2iq_{HT})\ \Gamma(-2ik_{HT})}{\Gamma(-ik_{HT}+\lambda-iq_{HT})\ \Gamma(1-ik_{HT}-\lambda-iq_{HT})}, \tag{2.79}$$

$$B = \frac{\Gamma(1-2iq_{HT})\ \Gamma(2ik_{HT})}{\Gamma(ik_{HT}+\lambda-iq_{HT})\ \Gamma(1+ik_{HT}-\lambda-iq_{HT})}. \tag{2.80}$$

Summarizing, we have

$$\psi_{inc}(x) = A\ \mathrm{e}^{2ib\ k_{HT}x}\left(1+\mathrm{e}^{2bx}\right)^{\lambda}\ {}_2F_1\left(ik_{HT}+\lambda-iq_{HT}, ik_{HT}+\lambda+iq_{HT}, 1+2ik_{HT}; -e^{2bx}\right) \tag{2.81}$$

$$\psi_{ref}(x) = B\ \mathrm{e}^{-2ib\ k_{HT}}\left(1+\mathrm{e}^{2bx}\right)^{\lambda}\ {}_2F_1\left(-ik_{HT}+\lambda+iq_{HT}, -ik_{HT}+\lambda-iq_{HT}, 1-2ik_{HT}; -e^{2bx}\right). \tag{2.82}$$

In (2.60) when $x \to \infty$ the $V \to V_0$ and the other hand when $x \to -\infty$ the $V \to 0$ the asymptotic behaviour of (2.77), (2.81) and (2.82) result flat waves given by

$$\psi_{inc}(x) \;\to\; A\ \mathrm{e}^{2ib\ k_{HT}x} \tag{2.83}$$

$$\psi_{ref}(x) \;\to\; B\ \mathrm{e}^{-2ib\ k_{HT}x} \tag{2.84}$$

$$\psi_{trans}(x) \to\; \mathrm{e}^{2ib\ q_{HT}x}. \tag{2.85}$$

Using the four-current density (2.23) we can write the reflection and transmission coefficients as

$$R_{HT} = \frac{j_{ref}}{j_{inc}} = \left|\frac{B}{A}\right|^2, \tag{2.86}$$

$$T_{HT} = \frac{j_{trans}}{j_{inc}} = \frac{q_{HT}}{k_{HT}}\left|\frac{1}{A}\right|^2. \tag{2.87}$$

2.4 The Klein's Paradox ¶ and Superradiance

Oskar Klein was a pioneer in the studies of Dirac's equation in the presence of a step potential. In nonrelativistic quantum mechanics, electron tunneling into a barrier is observed, with exponential damping. However, Klein demonstrated that an electron beam propagating in a region with a large enough potential barrier V can appear without the exponential damping expected‖ [9]. In standard scattering processes, incident waves lose energy due to interaction with the potential they traverse. Specifically, their incoming amplitude is larger than the amplitude of the reflected waves [10]. However, **superradiance** occurs if a single mode on the potential barrier has a reflected current greater than its incoming current (that is, the amplitude of the reflected wave is larger than the amplitude of the incoming one**). Instead, pair creation occur if the vacuum expectation value

¶The topic of Klein's paradox is commonly treated as a component of an introductory discussion of relativistic quantum mechanics.

‖This is the first treatment of what came to be known as the Klein paradox.

**This means energy is extracted from the system.

of the current operator is nonzero [11]. The reflection and transmission coefficients (R and T) depend on the detailed shape of the barrier and the general solutions between the coefficients (known as Wronskian relations) can be get from current conservation. Superradiance is a phenomenon where the energy is extracted from the barrier, also appears in relativistic quantum mechanics when the Klein-Gordon equation is applied for an abrupt or smooth potential barrier. Then, in order to analyze the occurrence of superradiance, we need a conservation relation, which can be obtained by the spatial derivative of the Wronskian,

$$\mathscr{W} = f_1 \left(\frac{d}{dx} f_2 \right) - f_2 \left(\frac{d}{dx} f_1 \right). \tag{2.88}$$

Let us consider an incident superradiant massless bosonic wave with charge e and energy $w < eV$ and evaluating the Wronskian (2.88), we see that

$$|R|^2 = |I|^2 - \frac{w - eV}{w} |T|^2 \qquad (0 < w < eV), \tag{2.89}$$

it is possible to have superradiance of the reflected current [9] particularly $|R| > |I|$. We know that all particles incident on the potential barrier are reflected, although the incident beam stimulates pair creation at the barrier, which emits particles and antiparticles. When $w > eV$ the superradiance does not occur, and $|R|^2 \leq |I|^2$. We note that $|R|^2 + |T|^2 = |I|^2$. After all, one should remember that superradiant scattering strongly depends on the spin of the field that is being scattered [12].

Chapter 3

Results & Discussion

3.1 The Lambert-W step-potential

The concept new in this research is the use of the Lambert-W function. We will find this solution of a practical problem, the step potential barrier, provides a introduction to the Lambert W function as one of the family of special functions, which are useful not only for quantum problems but for a diversity of problems in distinct fields. We introduce a new exactly solvable potential for the stationary K-G equation. The Lambert-W step-potential affords the exact solution to the one-dimensional stationary Klein-Gordon equation in terms of the confluent Heun functions. The potential is given in terms of the Lambert W-function*, which is an implicitly elementary function $w = W(z)$ also known as the product logarithm [13]that is the analytic multi-branch solution of $W(z)\mathrm{e}^{W(z)} = w\mathrm{e}^{w} = z$, where z is the complex argument of $W(z)$. Thus W is the inverse of the function $g(w) = w\mathrm{e}^{w}$. It has infinite number of branches distinguished by a subscript $k = 0, \pm1, \pm2, \pm3, \pm\infty$, particularly W_0 is called the principal branch that has domain $z \geq -1/\mathrm{e}$ and range $W(z) \geq -1$. Likewise, the branch satisfying $W(z) \leq -1$ is denoted by $W_{-1}(z)$ and it is

*The Lambert-W function is named after Johann Heinrich Lambert.

defined on $-1/\mathrm{e} \leq z < 0$. Comparing the Lambert function with the natural logarithm function $log(z)$, we note that they are closely related; $W = log(z)$ is the multi-branch analytic function[14] that solves the equation $\mathrm{e}^{log(z)} = \mathrm{e}^{w} = z$. The Lambert-potential is an asymmetric step potential of height V_0 whose steepness and asymmetry are controlled by parameter σ (see Figure 3.1). The Lambert-W step-potential is given by

$$V_L(x) = \frac{V_0}{1 + W(\mathrm{e}^{-x/\sigma})}, \tag{3.1}$$

where W is the Lambert function.

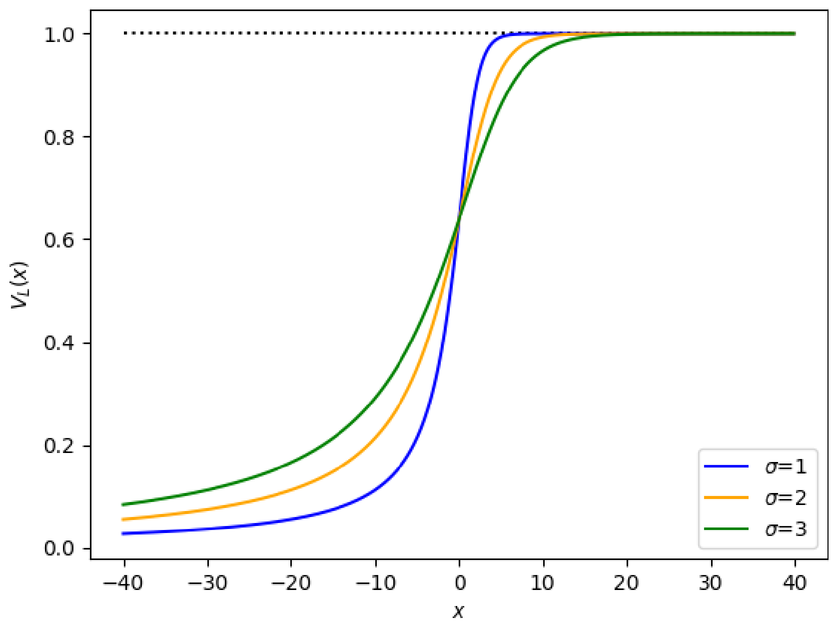


Figure 3.1: The Lambert potential (3.1), with $V_0 = 1$ and $\sigma = 1, 2, 3$.

3.1.1 Comparison between potentials barriers

We present the only concept new which is the use of the Lambert W function. The hyperbolic-tangent potential, provides a comprehensible introduction to the Lambert W function as one of the family of special functions, which are useful not only for quantum problems but for a diversity of problems in different fields [14]. Another expression for the hyperbolic-tangent potential barriers is given by

$$V_{HT}(x) = \frac{V_0}{1+(\mathrm{e}^{-x/d})}, \tag{3.2}$$

Figure 3.2 shows the comparison of (3.1) with (3.2). Figure 3.3 shows the difference of the three

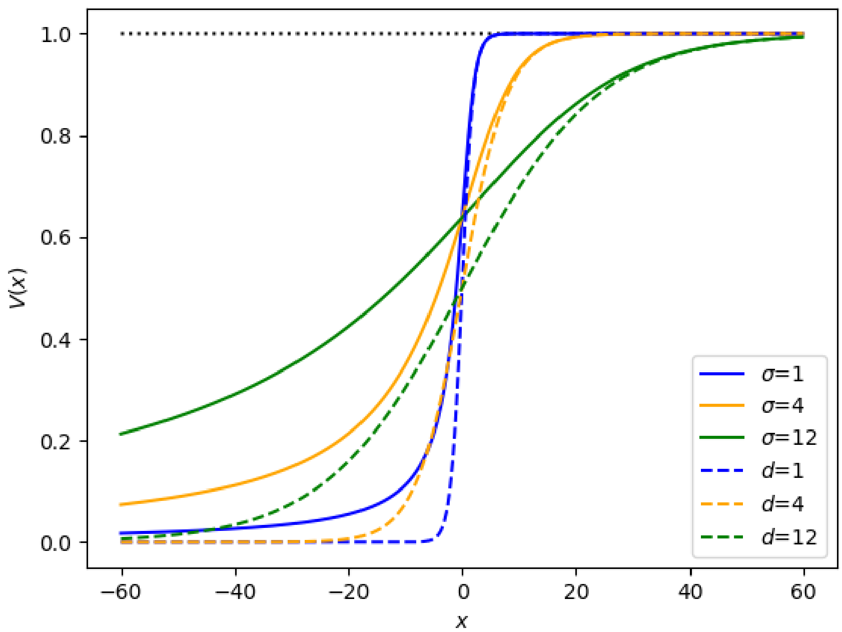


Figure 3.2: The Lambert (solid lines) and hyperbolic tangent (3.2) (dashed lines) potentials, $V_0 = 1$ and $\sigma = d = 1, 4, 12$.

potentials studied.

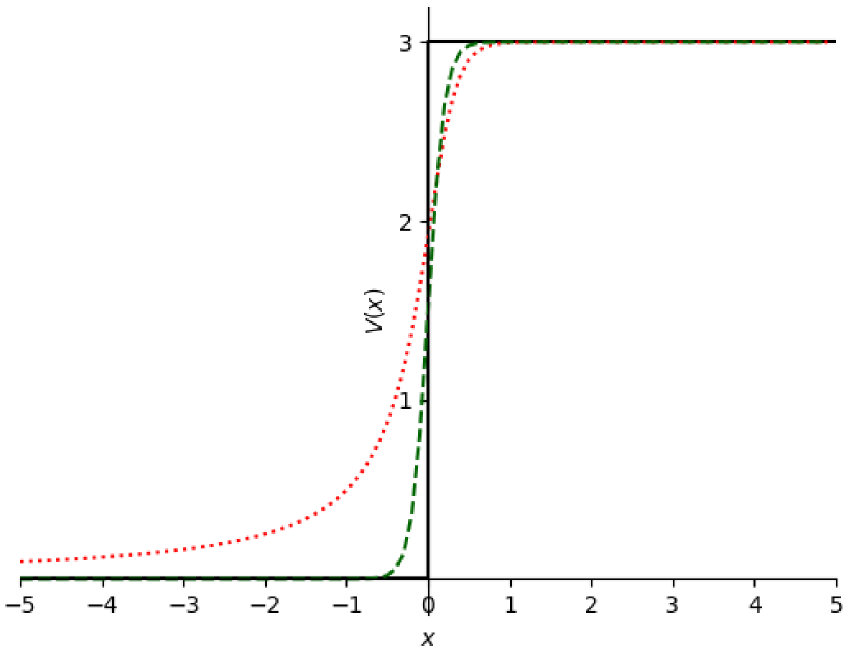


Figure 3.3: Comparison of the potential barriers, in all cases $V_0 = 3$. Lambert-W potential with $\sigma = 0.15$ (dotted line), hyperbolic tangent potential with $b = 0.5$ (dashed line), and step potential (solid line).

Using the K-G equation (2.32) with $V(x) = V_L(x)$ and $m_0 = m$

$$\frac{d^2\psi(x)}{dx^2} + \left\{ \left[E - V_L(x)\right]^2 - m^2 \right\} \psi(x) = 0. \tag{3.3}$$

Replacing (3.1) into (3.3) we have

$$\frac{d^2\psi(x)}{dx^2} + \left\{ \left[E - \frac{V_0}{1 + W\left(\mathrm{e}^{-x/\sigma}\right)}\right]^2 - m^2 \right\} \psi(x) = 0. \tag{3.4}$$

The solutions of the confluent Heun's differential equation has three singular points[15]: two regular ones $-z = 0$ and $z = 1$, and one irregular one $z = \infty$. Solving the standart **confluent Heun**

function (HeunC $(\alpha, \beta, \gamma, \delta, \eta, z)$) is a unique particular solution

$$\begin{aligned}\frac{d^2 f(z)}{dz^2} - \frac{[-\alpha' z^2 + (-\beta' + \alpha' - \delta' - 2)z + \beta' + 1]}{z(z-1)} \frac{df(z)}{dz} \\ - \frac{\{[(-\beta' - \delta' - 2)\alpha' - 2\delta']z + (\beta' + 1)\alpha' + (-\delta' - 1)\beta' - 2\eta' - \delta'\}}{2z(z-1)} f(z) = 0\end{aligned} \tag{3.5}$$

which is regular around the regular singular point $z = 0$. Then, $f(0) = 1$

$$\left.\frac{df}{dz}\right|_{z=0} = \frac{(-\alpha' + 1 + \delta')\,\beta' + \delta' - \alpha' + 2\eta'}{2(\beta' + 1)} \tag{3.6}$$

Obtaining the following solution

$$f(z) = C_1 \text{HeunC}\,(\alpha', \beta', \gamma', \delta', \eta', z) + C_2\, z^{-\beta} \text{HeunC}\,(\alpha', -\beta', \gamma', \delta', \eta', z) \tag{3.7}$$

In order to write (3.4) similar to (3.5)

$$y = -W\left(\mathrm{e}^{-x/\sigma}\right) \tag{3.8}$$

The derivative† of W is

$$\frac{dW(x)}{dx} = \frac{W(x)}{x[1 + W(x)]} \tag{3.9}$$

$$\frac{dy}{dx} = \frac{W(\mathrm{e}^{-x\sigma})}{\sigma[1 + W(\mathrm{e}^{-x/\sigma})]} \tag{3.10}$$

$$\frac{d\psi}{dx} = \frac{d\psi}{dy}\frac{dy}{dx} \tag{3.11}$$

$$= \frac{d\psi}{dy}\frac{W(\mathrm{e}^{-x/\sigma})}{\sigma[1 + W(\mathrm{e}^{-x/\sigma})]}$$

$$= -\frac{y}{\sigma(1-y)}\frac{d\psi}{dy} \tag{3.12}$$

†This derivative was calculated in *Maple*

$$
\begin{aligned}
\frac{d}{dx}\left(\frac{d\psi}{dx}\right) &= \frac{d}{dy}\left(\frac{d\psi}{dx}\right)\frac{dy}{dx} = \frac{d^2\psi}{dx^2} \qquad (3.13)\\
&= \frac{d}{dy}\left[-\frac{y}{\sigma(1-y)}\frac{d\psi}{dy}\right]\frac{dy}{dx}\\
&= \left\{-\frac{d}{dy}\left[\frac{y}{\sigma(1-y)}\right]\frac{d\psi}{dy} - \frac{y}{\sigma(1-y)}\frac{d^2\psi}{dy^2}\right\}\left[-\frac{y}{\sigma(1-y)}\right]\\
&= \left\{-\left[\frac{\sigma(1-y)+\sigma y}{\sigma^2(1-y)^2}\right]\frac{d\psi}{dy} - \frac{y}{\sigma(1-y)}\frac{d^2\psi}{dy^2}\right\}\left[-\frac{y}{\sigma(1-y)}\right]\\
&= \left[-\frac{1}{\sigma(1-y)^2}\frac{d\psi}{dy} - \frac{y}{\sigma(1-y)}\frac{d^2\psi}{dy^2}\right]\left[-\frac{y}{\sigma(1-y)}\right]\\
&= \frac{y^2}{\sigma(1-y)^2}\frac{d^2\psi}{dy^2} + \frac{y}{\sigma^2(1-y)^3}\frac{d\psi}{dy} \qquad (3.14)
\end{aligned}
$$

The equation (3.4) becomes

$$
\begin{aligned}
\frac{y^2}{\sigma^2(1-y)^2}\frac{d^2\psi}{dy^2} + \frac{y}{\sigma^2(1-y)^3}\frac{d\psi}{dy} + \left[\left(E-\frac{V_0}{1-y}\right)^2 - m^2\right]\psi &= 0\\
\frac{d^2\psi}{dy^2} + \frac{1}{y(1-y)}\frac{d\psi}{dy} + \frac{\sigma^2(1-y)^2}{y^2}\left[\left(E-\frac{V_0}{1-y}\right)^2 - m^2\right]\psi &= 0\\
\frac{d^2\psi}{dy^2} + \frac{1}{y(1-y)}\frac{d\psi}{dy} + \frac{\sigma^2(1-y)^2}{y^2}\left\{\frac{[E(1-y)-V_0]^2 - m^2(1-y)^2}{(1-y)^2}\right\}\psi &= 0\\
\frac{d^2\psi}{dy^2} + \frac{1}{y(1-y)}\frac{d\psi}{dy} + \sigma^2\left\{\frac{[E(1-y)-V_0]^2 - m^2(1-y)^2}{y^2}\right\}\psi &= 0 \qquad (3.15)
\end{aligned}
$$

$$
\psi(y) = C_1 \mathrm{e}^{\frac{\alpha}{2}y} y^{\frac{1}{2}\beta}\,\mathrm{HeunC}\left(\alpha,\beta,\gamma,\delta,\eta,y\right) + C_2 \mathrm{e}^{\frac{\alpha}{2}y} y^{-\frac{1}{2}\beta}\; z^{-\beta}\,\mathrm{HeunC}\left(\alpha,-\beta,\gamma,\delta,\eta,y\right) \qquad (3.16)
$$

Where

$$\alpha = 2\sigma\sqrt{m^2 - E^2},$$
$$\beta = 2\sigma\sqrt{m^2 - E^2 + 2EV_0 - V_0^2},$$
$$\gamma = -2,$$
$$\delta = 2\sigma^2\left(m^2 - E^2 + EV_0\right),$$
$$\eta = 1 - 2\sigma^2\left(m^2 - E^2 + EV_0\right).$$

Equation (3.15) must be written as (3.5), then

$$\frac{1}{y(1-y)} = -\frac{[-\alpha' y^2 + (-\beta' + \alpha' - \delta' - 2)y + \beta' + 1]}{y(y-1)} = -\frac{1}{y(y-1)} \tag{3.17}$$

$$\sigma^2\left\{\frac{[E(1-y) - V_0]^2 - m^2(1-y)^2}{y^2}\right\} = -\frac{\{[(-\beta' - \delta' - 2)\alpha' - 2\gamma]y + (\beta' + 1)\alpha' + (-\gamma' - 1)\beta' - 2\eta' - \delta'\}}{2y(y-1)}. \tag{3.18}$$

The derivation itself represents tedious mathematical calculation, after some steps we get the following expression

$$\psi(x) = C_1 \mathrm{e}^{-\frac{1}{2}\alpha W(\mathrm{e}^{-x/\sigma})}\mathrm{HeunC}\left[\alpha, \beta, \gamma, \delta, \eta, -W(\mathrm{e}^{-x/\sigma})\right] W(\mathrm{e}^{-x/\sigma})^{+\frac{1}{2}\beta} + C_2 \mathrm{e}^{-\frac{1}{2}\alpha W(\mathrm{e}^{-x/\sigma})}\mathrm{HeunC}\left[\alpha, -\beta, \gamma, \delta, \eta, -W(\mathrm{e}^{-x/\sigma})\right] W(\mathrm{e}^{-x/\sigma})^{-\frac{1}{2}\beta} \tag{3.19}$$

The raw variables in (3.17) and (3.18) are unknown variables $(\alpha', \beta', \gamma', \delta', \eta')$. Following the procedure and making the equations (3.17) and (3.18) to equal zero we get

$$\alpha' y^2 + (-\alpha' + \beta' + \delta' + 2)y - \beta' = 0 \tag{3.20}$$

$$2\sigma^2\left(E^2 - m^2\right)y^3 + \left[(-6E^2 + 4EV_0 + 6m^2)\sigma^2 + (-\beta' - \delta' - 2)\alpha' - 2\delta'\right]y^2 + \left[(6E^2 - 8EV_0 + 2V_0^2 - 6m^2)\sigma^2 + (\beta' + 1)\alpha' + (-\delta' - 1)\beta' - \gamma' - 2\eta'\right]y - 2\sigma^2(E^2 - 2EV_0 + V_0^2 - m^2) = 0 \tag{3.21}$$

First System: y^3

$$\begin{cases} 2\sigma^2(E^2-m^2)-\alpha'=0 & \text{(A.3a)} \\ \sigma^2(-6E^2+4EV_0+6m^2)+(-\beta'-\gamma'-2)\alpha'-2\gamma'-(-\alpha'+\beta'+\gamma'+2)\ =0 & \text{(A.3b)} \\ \sigma^2(6E^2-8EV_0+2V_0^2-6m^2)+(\beta'+1)\alpha'+(-\gamma'-1)\beta'-\gamma'-2\eta'+\beta'=0 & \text{(A.3c)} \end{cases} \tag{3.22}$$

Here: $-2\sigma^2(E^2-2EV_0+V_0^2)=0$

Second system: y^2

$$\begin{cases} \sigma^2(-6E^2+4EV_0+6m^2)+(-\beta'-\gamma'-2)\alpha'-2\delta'-\alpha'=0 & \text{(A.4a)} \\ \sigma^2(6E^2-8EV_0+2V_0^2+6m^2)+(\beta'+1)\alpha'+(-\gamma'-1)\beta'-\gamma'-2\eta'-(-\alpha'+\beta'+\gamma'+2)=0 & \text{(A.4b)} \\ -2\sigma^2(E^2-2EV_0+V_0^2-m^2)+b=0 & \text{(A.4c)} \end{cases} \tag{3.23}$$

3.2 Reflection and Transmission Coefficients

The reflection coefficient R, and the transmission coefficient T satisfy the unitary relation $T+R=1$. In summary, R and T for all cases are expressed as follow,

A. Case 1:

For the hyperbolic Lambert-W potential barrier we have[13 16]

$$R_{LW}=\mathrm{e}^{-2\pi\sigma q}\frac{\sinh\left[\frac{\pi\sigma}{2k}(k-q)^2\right]}{\sinh\left[\frac{\pi\sigma}{2k}(k+q)^2\right]} \tag{3.24}$$

$$T_{LW}=1-R_{LW}, \tag{3.25}$$

where for a relativistic particle $k=\sqrt{E^2-1}$, and $q=\sqrt{(E-V_0)^2-1}$. The behavior of the reflection and transmission coefficients for the Lambert-W potential barrier is represented in Figure 3.5.

B. Case 2:

For the hyperbolic tangent potential barrier the equivalent of (2.86) and (2.87), respectively are

$$R_{HT} = \left|\frac{B}{A}\right|^2, \tag{3.26}$$

$$T_{HT} = \frac{q_{HT}}{k_{HT}} \left|\frac{1}{A}\right|^2, \tag{3.27}$$

where the coefficients A and B in (3.26) and (3.27) can be expressed in terms of the Gamma functions is as follows

$$A = \frac{\Gamma(1-2iq_{HT})\ \Gamma(-2ik_{HT})}{\Gamma(-ik_{HT}+\lambda-iq_{HT})\ \Gamma(1-ik_{HT}-\lambda-iq_{HT})}, \tag{3.28}$$

$$B = \frac{\Gamma(1-2iq_{HT})\ \Gamma(2ik_{HT})}{\Gamma(ik_{HT}+\lambda-iq_{HT})\ \Gamma(1+ik_{HT}-\lambda-iq_{HT})}, \tag{3.29}$$

where $k_{HT} = \frac{\sqrt{E^2-1}}{2b}$, $q_{HT} = \frac{\sqrt{(E-V_0)^2-1}}{2b}$, and $\lambda = \frac{b+\sqrt{b^2-V_0^2}}{2b}$.

C. Case 3:

The equivalent of (2.58) and (2.59) for the step potential barrier are therefore given by

$$R_{SP} = \left|\frac{(q-k)}{(q+k)}\right|^2, \tag{3.30}$$

$$T_{SP} = \frac{q}{k}\left|\frac{2k}{q+k}\right|^2. \tag{3.31}$$

The dispersion relation k and q must be positive because they correspond to an incident particle moved from left to right and, their signs depend on the group velocity, which is calculated by taking the derivative of each dispersion relation with respect to the energy E

$$\frac{dE}{dk'} = \frac{k'}{E} \geq 0, \tag{3.32}$$

$$\frac{dE}{dq'} = \frac{q'}{E-V_0} \geq 0. \tag{3.33}$$

For these potentials we have three regions, which are showed in Table 3.1.

Region I	$E > V_0 + m$	$k' > 0$	$k \in \mathbb{R}$	$q' > 0$	$q \in \mathbb{R}$
Region II	$V_0 + m > E > V_0 - m$	$k' > 0$	$k \in \mathbb{R}$		$q \in \mathbb{I}$
Region III	$V_0 - m > E > m$	$k' > 0$	$k \in \mathbb{R}$	$q' < 0$	$q \in \mathbb{R}$

Table 3.1: Different regions in which we divided our study. We pay special attention to region III.

It is important to note that in the region II ($V_0 + m > E > V_0 - m$) and the dispersion relations k and q are pure imaginary number and the transmitted wave is attenuates, that is to say $R = 1$. In the region III ($V_0 - m > E > m$) $k' > 0$ and $q' < 0$ we have that $R > 1$, so superradiance occurs[8].

The Figure 3.4 shows the graphs for the three reflection coefficients for region I ($E > V_0 + m$). We note that in this region the range of reflection is distributed over the range $[0, 1]$. Whereas, figures 3.5, 3.6, and 3.7 show the reflection R and transmission T coefficients for different parameters of each potential. We observed in the figures that in the region III the reflection coefficient R is larger than 1, whereas the transmission coefficient T becomes negative, so we observed superradiance, likewise the R and T coefficients satisfy the unitary condition $T + R = 1$.

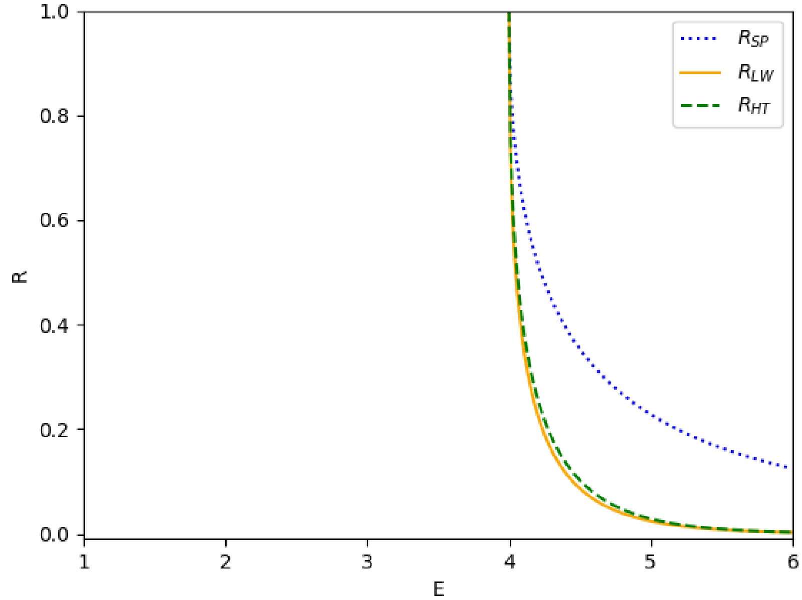
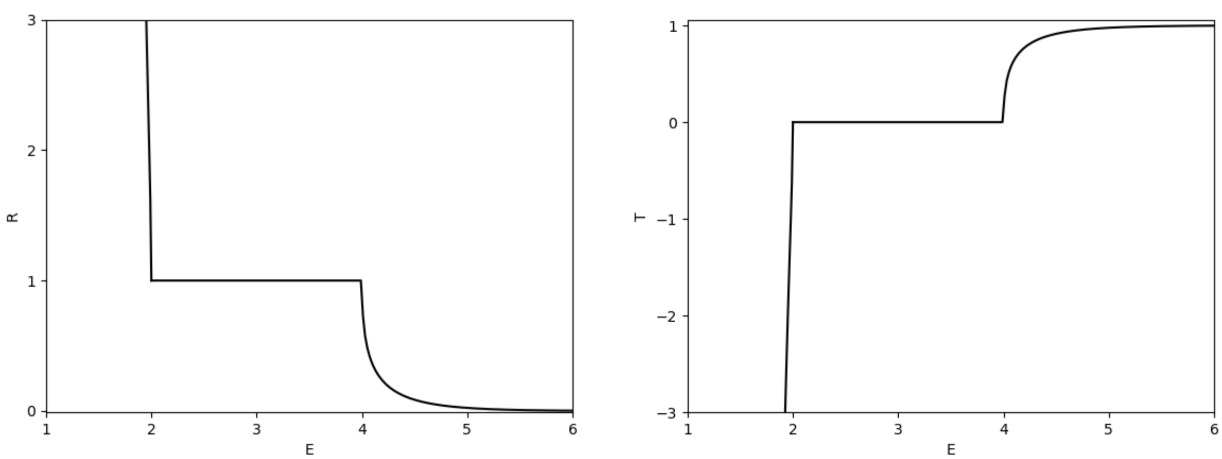
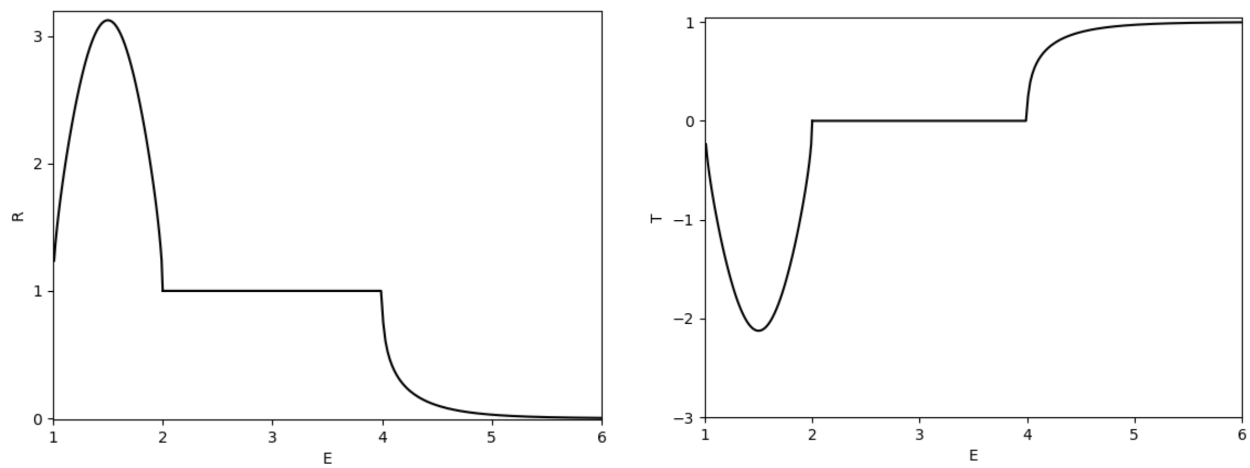


Figure 3.4: Comparison between the reflection R_{LW} (solid line), R_{SP} (dotted line), and R_{HT} (dashed line), scaled to the range $E > V_0 + m$ with $V_0 = 3$, $\sigma = 0.15$, $b = 3$, and $m = 1$.



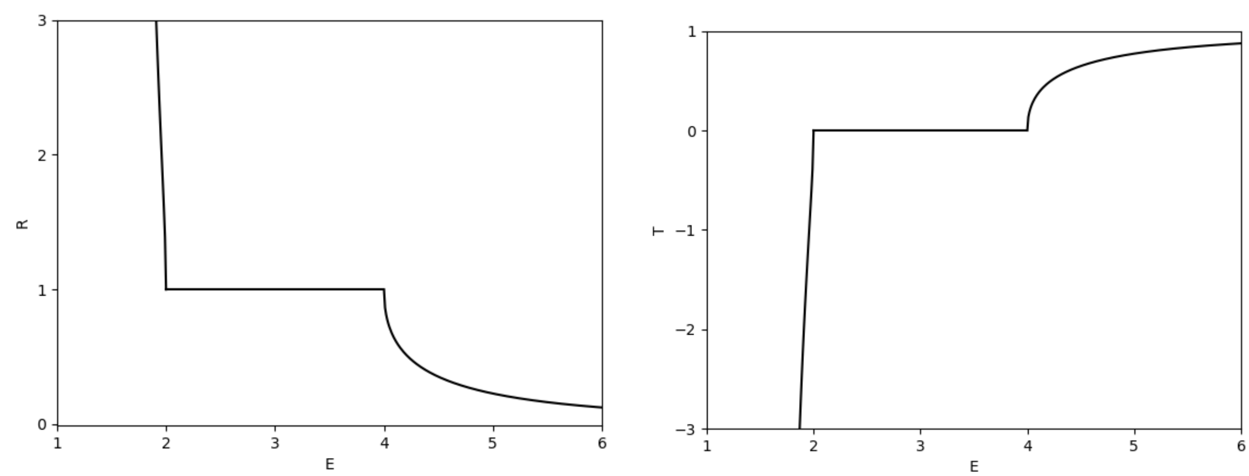
(a) (b)

Figure 3.5: The reflection R and transmission T coefficients varying energy E for the relativistic Lambert-W potential barrier with $V_0 = 3$, $\sigma = 0.15$, and $m = 1$.



(a) (b)

Figure 3.6: The reflection R and transmission T coefficients varying energy E for the relativistic hyperbolic tangent potential barrier with $V_0 = 3$, $b = 3$, and $m = 1$.



(a) (b)

Figure 3.7: The reflection R and transmission T coefficients varying energy E for the relativistic step potential barrier with $V_0 = 3$ and $m = 1$.

Chapter 4

Conclusions & Outlook

In this work we have studied the solution of the klein-Gordon equation for three different barrier potentials. The analysis was presented in mathematical and physical detail to stimulate further research in this and related avenues. The study of fields in strong electromagnetic backgrounds, known as the Klein paradox include two distinct phenomena superradiance and pair creation. Wave scattering processes are characterized by the interaction between an incident wave and a physical barrier. Superradiance occurs when the reflected current is greater than the incoming current, that is to say the reflection coefficient R is larger than one so the transmission coefficient T becomes negative. The scattering solutions are discussed in terms of the height of the potential barrier. For the region $V_0 - m > E > m$ we observe superradiance for the Lambert-W potential barrier. Finally, the problems and cases addressed in this research of the Klein-Gordon equation are approximations to scenarios related with problems that one finds in the real world and real. The solutions of these simplified potentials can give us insight into the behavior, both qualitative and quantitative, of actual physical systems.

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