UNIVERSIDAD DE INVESTIGACIÓN DE TECNOLOGÍA EXPERIMENTAL YACHAY

Escuela de Ciencias Matemáticas y Computacionales

TÍTULO: Frequencies of oscillations of free boundaries in a perfect fluid

Trabajo de integración curricular presentado como requisito para la obtención del título de Matemático

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Urcuquí, julio del 2020.

UNIVERSIDAD YACHAY TECH

Urcuquí, 26 de marzo de 2020

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Dedicatoria

Quiero dedicar este trabajo a mi familia, de manera especial a mis padres, Mario y Lorena. Papitos, ustedes son el pilar fundamental de mi vida y les agradezco por creer en mí siempre. Nada de esto hubiera sido posible sin su amor, apoyo incondicional y esfuerzo. ¡Este logro es tan suyo como mío!

María Lorena Correa Abendaño

Agradecimiento

Me gustaría agradecer a todas la personas sin las cuales la realización de este trabajo no hubiera sido posible. No solo a las que me ayudaron a nivel académico, sino también a las que me ayudaron a crecer como persona.

Primeramente quiero agradecer a mi familia, a mis papás que me dieron su amor y me apoyaron incondicionalmente para la realización de este sueño, incluso cuando eso significó estar lejos de ellos. No existen las palabras suficientes para agradecerle todo lo que han hecho por mí. A mis hermanos, Andre y Mayes, y mis sobrinos, Paulo, Emma, Giulia y Allegra, por estar cuando los necesito y ser una motivación para mí. A mis abuelitas, Hilda y María Eugenia, cuyo amor ha estado siempre presente a lo largo de mi vida. A mi tía Mónica, a quien considero una segunda mamá. A Paúl, Estefy, Belén, Cami y Sebas, a quienes considero mis hermanos. Gracias familia por su amor y su apoyo.

Quiero agradecer de manera especial a Jonnathan, por estar siempre y ayudarme muchísimo en la realización de este trabajo. Ayudándome desde entender conceptos físicos que se me escapaban, hasta darme palabras de aliento que necesitaba para seguir. Gracias por brindarme tu amor incondicional y creer en mí, incluso cuando yo no podría creer. Y finalmente, gracias por hacer de mi paso por Yachay Tech una mejor experiencia.

A mis amigos, quienes hicieron que Yachay Tech se sienta como un hogar. Se convirtieron en una segunda familia para mí.

A mi tutor, Juan Carlos, gracias por sus enseñanzas, por guiarme y por su paciencia (ya que en ocasiones necesitó muchísima paciencia). Gracias también porque además de ser un gran profesor, se convirtió en un gran amigo.

A Yachay Tech, sus profesores y ex-profesores, de manera especial a los profes Cédric M. Campos y Juan Mayorga, quienes siempre se preocuparon por transmitir sus conocimientos de la mejor manera posible y además se preocuparon por el bienestar de sus estudiantes. Al profe Francesc Antón, por permitirme usar las computadoras de los laboratorios para correr mis códigos. Y finalmente, a toda la Escuela de Ciencias Matemáticas y Computacionales, por formarme y permitirme conocer grandes profesionales y amigos.

María Lorena Correa Abendaño

Resumen

Se calcula las frecuencias naturales para las oscilaciones de la frontera libre de las ondas estacionarias capilares en contacto con un contenedor sólido. Primero, estudiamos el caso del semiplano. Deducimos una ecuación evolutiva integrodiferencial para la frontera libre linealizada e imponemos condiciones de borde fijo y borde libre. Para ambos casos, se proporcionan las frecuencias de oscilaciones naturales para las superficies libres y se comparan con las frecuencias en ausencia de paredes. Luego, mediante mapeos conformes, se puede hacer el mismo análisis en contenedores arbitrarios 2D, con toda la información sobre su geometría contenida en una matriz, que aparece como un factor en un sistema lineal para el cálculo de frecuencias propias. En particular, hacemos el análisis a una tira vertical infinita y un contenedor redondo.

Palabras Clave: frecuencias, oscilaciones, extremo fijo, extremo libre, ondas capilares, superficie libre.

Abstract

We compute the natural frequencies for the oscillations of the free boundary of capillary standing waves in contact with a solid container. First, we study the case of the half-plane. We deduce an integrodifferential evolutionary equation for the linearized free boundary and impose pinned-end and free-end boundary conditions. For both cases, the natural oscillations frequencies for the free surfaces are provided and compared with the frequencies in the absence of solid walls. Then, by conformal mappings, the same analysis can be done to arbitrary 2D containers, with all the information on their geometry contained into a matrix that appears as a factor in a linear system for the computation of eigenfrequencies. In particular, we make the analysis to a vertical-infinite strip and a rounded container

Keywords: frequencies, oscillations, pinned-end, free-end, capillary waves, free surface.

Contents

1 Introduction

In the present work, we describe the motion of a perfect and incompressible fluid delimited above by a free surface under capillary action. We are going to compute the frequencies for the oscillations of the free boundary of these capillary waves in contact with solid walls. During this analysis, we neglect gravity since we are working on a small scale. We want to emphasize that the manner that waves arise depends on the presence of walls and the contact with the free surface. Besides, we study different two-dimensional geometrical cases, namely: the half-plane, an infinite vertical strip, and a semicircular shaped container. We also compare our results with those in the literature: the natural frequencies for the gravity-capillary case in different geometries are shown in [1] and [2]. The analysis will allow us to determine the manner the container geometry affects the frequencies of oscillations.

The methodology for the half-plane is as follows: we first deduce an integrodifferential evolutionary equation for the linearized Euler equations. We also impose two different boundary conditions: Dirichlet homogeneous or pinned-end condition, and Neumann homogeneous or free-end condition; in this last case, we consider a contact angle of $\pi/2$ between the wave and the solid wall, which has been used before in the literature (see [3]). For the purpose of this work, we are going to use the separation of variables technique; and, in this way, transform our problem to an ordinary differential equation in the time variable and an eigenvalue problem in the space-variable. The eigenvalue problem can be approximated numerically by finite matrices. Once we get the frequencies for the free surfaces, we can compare them with the frequencies in the absence of solid walls (sine and cosine frequencies). For this comparison, MATLAB is a valuable tool since it allows us to numerically approximate systems in each case and, for instance, to determine the free surface shape in both cases: in the presence or in the absence of walls. Besides, MATLAB is also useful in checking the convergence of the eigenvalues.

In order to study the remaining geometries, the main tool to be used is the conformal mapping technique to transform any two-dimensional domain into a simpler one. We first consider the technique for a general geometry and then apply it to the geometries in mention: we will conformally map both the infinite vertical strip and the semicircular geometry into the half-plane.

The motivation to work on this topic is that the technique is useful in engineering. The technique is related to the ink-jet print technology. Besides the conventional printing on paper technique, the ink-jet print technology goes further. For example, we can find some of its application in the displays industry to create the transparent electrodes that criss-cross the front and rear surfaces of computer displays [4], in biology and medicine to build pregnancy and diabetes tests [4], in chemistry for nanoelectrospray ionization [5], in physics for droplet generators, which has a lot of applications by itself (mass spectroscopy, fuel processing, multilayer parts and circuits manufacture, etc.) [6, 7] and in fluid mechanics when one tries to control the surface by injecting fluid through the boundary, as is done in the treatment of mining disposals [8].

The structure of the work is organized in the following way: first, we will introduce some important results on Fourier and Hilbert transforms and Tchebyshev's first and second-kind polynomials which will be useful later when decomposing the Hilbert transform into an orthogonal L^2 basis. Then, we present the deduction of the conservation equations in order to get the Euler incompressible equations and also the Navier-Stokes equations. Once this framework is settled down, we formulate the problem and proceed to linearize the related equations. As mentioned before, we are going to deduce an integrodifferential evolutionary equation to find the natural oscillation frequencies. Next, by employing conformal mappings, we are going to extend our results to other geometries and compare them. Finally, we will discuss the results, conclude, and give an outlook for future research.

2 Objectives

2.1 General Objective

To compute the natural frequencies for the oscillations of the free boundary of capillary waves in contact with an specific solid container: half-plane, vertical infinite-strip and rounded container.

2.2 Specific Objectives

- To deduce conservation equations in order to get the incompressible Euler equations.
- To linearize the Euler equations in order to solve it explicitly, after a domain transformation.
- To solve the eigenvalue problem arising from the linear conservation laws, for each boundary condition and each container geometry, in order to get the free surface profile.
- To approximate numerically the eigenvalue problem by using MATLAB.
- To explore the application of this technique by considering different geometries.

3 Preliminaries

In the present chapter we will deduce some important results. These results will be used further in the following sections. We will first introduce the so-called Fourier transform that owes its name to Jean-Baptiste Joseph Fourier; in his publications *Mémoire sur la propagation de la chaleur dans les corps solides* (1807) and *Théorie analytique de la chaleur* (1822), Fourier showed that there are functions that can be expressed as trigonometric series. In a simple definition, the Fourier transform is a linear transform that decomposes a signal into its contributing frequencies[9]. The applications of Fourier transform can be found in many fields: in signal analysis, for example when processing seismic waves[10] or when obtaining the first image of a black hole[11]; in communication theory to understand how a signal passes through communication channels[12]; in physics, for spectral estimation[13]; among others.

Furthermore, Tchebyshev polynomials of first and second-kind will be introduced. The main reason to choose this basis instead of the Fourier basis is that the weight in the linearized conservation equations makes Tchebyshev polynomials the most reasonable option to work with. Some of the properties of these polynomials are also presented. Finally, the explicit way of obtaining the curvature of the free surface that we will be later working with; and the deduction of the conservation equations that later poses our problem are showed during this section.

3.1 The Fourier transform

It is important to first mention the Fourier series. For introducing the series we will rely on [14]. Let $\Omega = [-\pi, \pi] \subset \mathbb{R}$ and let's consider the Fourier basis

$$\mathcal{F} = \{1_f\} \cup \{\mathcal{C}_n / n \in \mathbb{N}\} \cup \{\mathcal{S}_n / n \in \mathbb{N}\} \subseteq C(\Omega),$$

where 1_f is defined as follows

$$\begin{aligned} 1_f : \Omega &\longrightarrow \mathbb{R} \\ t &\longmapsto 1_f(t) = 1, \end{aligned}$$

and, for $n \in \mathbb{N}$,

$$\mathcal{C}_n(t) = \cos(nt), \quad \mathcal{S}_n(t) = \sin(nt).$$

Let f be a $2\pi-$periodic function on Ω. The Fourier series allow us to represent the function f in terms of simpler functions in the following way:

$$f = \frac{a_0}{2} 1_f + \sum_{n=1}^{\infty} a_n \mathcal{C}_n + \sum_{n=1}^{\infty} b_n \mathcal{S}_n, \tag{3.1}$$

implying

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(nt) + \sum_{n=1}^{\infty} b_n \sin(nt), \quad \text{a.e } \forall t \in \Omega, \tag{3.2}$$

with

$$a_0 = \frac{1}{\pi} \int_\Omega f(t)\, dt, \quad a_n = \frac{1}{\pi} \int_\Omega f(t) \cos(nt)\, dt, \quad \text{and} \quad b_n = \frac{1}{\pi} \int_\Omega f(t) \sin(nt)\, dt.$$

The Fourier transform is an extension of the Fourier series; in this case the period of the function can approach infinity. In the case of the Fourier transform, the sine and coefficients are written as complex exponential coefficients by using the Euler's formula. We will now introduce the Fourier transform as presented in [15]. The Fourier transform of a function $f \in L^2[-\infty, \infty]$ is given by:

$$F[f](k) = \hat{f}(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{ikx} f(x) dx, \tag{3.3}$$

and its inversion formula, also given in [15], is:

$$F^{-1}[f](k) = \check{f}(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ikx} f(x) dx, \tag{3.4}$$

Theorem 3.1 (Derivative of the transform)**.** *Let $u \in L^1(\mathbb{R})$ such that $|x|^n u \in L^1(\mathbb{R})$. Then $\hat{u} \in C^n(\mathbb{R})$ and the following holds:*

$$\frac{d^\alpha}{dk^\alpha} \hat{u}(k) = F\left[(ix)^\alpha u\right](k), \quad \textit{for } \alpha \textit{ such that } |\alpha| \leq n. \tag{3.5}$$

Proof. We know that

$$\frac{\hat{u}(k) - \hat{u}(k_0)}{k - k_0} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} u(x) e^{ik_0 x} \frac{e^{ik(k-k_0)} - 1}{k - k_0} dx.$$

Let $\varphi(x, v) = \frac{e^{ixv} - 1}{v}$. Then,

$$|\varphi(x, v)| \leq |x|,$$

and

$$\lim_{v \to 0} \varphi(x, v) = ix.$$

Applying the dominated convergence theorem, we finally obtain that

$$\begin{aligned} \lim_{k \to k_0} \frac{\hat{u}(k) - \hat{u}(k_0)}{k - k_0} &= \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} ixu(x) e^{ik_0 x} dx \\ &= F[ixu](k_0) \end{aligned}$$

□

Theorem 3.2 (Transform of the derivative)**.** *Let $u \in C_c^n(\mathbb{R})$ such that $\frac{d^\alpha u}{dx^\alpha} \in L^1(\mathbb{R})$ for all $|\alpha| \leq n$. Then,*

$$F\left(\frac{d^\alpha u}{dx^\alpha}\right)(k) = (-ik)^\alpha \hat{u}(k). \tag{3.6}$$

Proof. Let's prove it by mathematical induction.

Base case: for $\alpha = 1$. We need to prove that:

$$F\left[u'(x)\right](k) = (-ik) \hat{u}(k).$$

By using integration by parts and given the fact that u has compact support we get that:

$$\begin{aligned} F\left[u'(x)\right](k) &= \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} u(x) e^{ikx} dx \\ &= -\frac{ik}{\sqrt{2\pi}} \int_{\mathbb{R}} u(x) e^{ikx} dx \\ &= (-ik) \hat{u}(k). \end{aligned}$$

Inductive hypothesis: let $\alpha \in \mathbb{N}$ such that $1 \leq \alpha \leq n-1$. Then,

$$F\left[\frac{d^{\alpha}u}{dx^{\alpha}}\right](k) = (-ik)^{\alpha}\,\hat{u}(k).$$

Inductive step: for $\alpha = n$, let's prove that:

$$F\left[\frac{d^{n}u}{dx^{n}}\right](k) = (-ik)^{n}\,\hat{u}(k).$$

We know that

$$\begin{aligned} F\left[\frac{d^{n}u}{dx^{n}}\right](k) &= \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}}\frac{d^{n}u}{dx^{n}}e^{ikx}dx \\ &= \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}}\frac{du}{dx}\frac{d^{n-1}u}{dx^{n-1}}e^{ikx}dx. \end{aligned}$$

By using integration by parts, and the inductive hypothesis, we obtain

$$\begin{aligned} F\left[\frac{d^{n}u}{dx^{n}}\right](k) &= \frac{-ik}{\sqrt{2\pi}}\int_{\mathbb{R}}\frac{d^{n-1}u}{dx^{n-1}}e^{ikx}dx \\ &= -ikF\left[\frac{d^{n-1}u}{dx^{n-1}}\right](k) \\ &= (-ik)(-ik)^{n-1}\,\hat{u}(k) \\ &= (-ik)^{n}\,\hat{u}(k). \end{aligned}$$

□

Theorem 3.3 (Convolution theorem)**.** *Let $u, v \in L^2[-\infty,\infty]$. Recall that the convolution of u and v is defined as*

$$(u * v)(x) = \int_{-\infty}^{\infty} u(z)\,v(x-z)\,dz.$$

The Fourier transform of the convolution of u and v is given by

$$F[u * v](k) = \sqrt{2\pi}\left(F[u](k)\cdot F[v](k)\right). \tag{3.7}$$

Proof. Let $u, v \in L^2[-\infty,\infty]$, generic. Then,

$$\begin{aligned} F[u * v] &= \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} e^{ikx}(u * v)(x)\,dx \\ &= \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} e^{ikx}\left(\int_{\mathbb{R}} u(z)\,v(x-z)\,dz\right)dx. \end{aligned}$$

By appliyng Fubini and then the change of variable $y = x - z$, we get

$$\begin{aligned} F[u * v] &= \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} u(z)\left(\int_{\mathbb{R}} e^{ikx}v(x-z)\,dx\right)dz \\ &= \int_{\mathbb{R}} e^{ikz}u(z)\left(\frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} e^{iky}v(y)\,dx\right)dz \\ &= \int_{\mathbb{R}} e^{ikz}u(z)\,dz\cdot F[v](k) \\ &= \sqrt{2\pi}\left(F[u](k)\cdot F[v](k)\right) \end{aligned}$$

We conclude by the arbitrariness of u and v. □

3.1.1 The Hilbert transform

In this section we will focus on establishing a relation between the Fourier transform and the Hilbert transform. We will rely on the results shown in [15]. The Hilbert transform of $\phi \in L^2(\mathbb{R})$ is the operator defined as:

$$f(x) = \frac{1}{\pi} \int_{-\infty}^{\infty^*} \frac{\phi(y)}{x-y} dy, \quad f(x) \in L^2(\mathbb{R}). \tag{3.8}$$

The * above the integral indicates that it is a Cauchy principal value integral. This means that it cannot be calculated as an improper integral because of the point $y = x$. The Cauchy principal value integral is defined as:

$$\int_{-\infty}^{\infty^*} \frac{\phi(y)}{x-y} dy = \lim_{\epsilon \to 0} \left(\int_{-\infty}^{x-\epsilon} \frac{\phi(y)}{x-y} dy + \int_{x+\epsilon}^{\infty} \frac{\phi(y)}{x-y} dy \right).$$

Theorem 3.4. *For f as in (3.8), we have that:*

$$H\big(H(\phi)\big) = -\phi. \tag{3.9}$$

In order to prove the theorem, we first need to prove the following two lemmas:

Lemma 3.5.

$$\int_{-\infty}^{\infty} e^{isx} \int_{-\infty}^{\infty^*} \frac{\phi(y)}{x-y} dy dx = \int_{-\infty}^{\infty} e^{isx} \phi(x)\, dx \int_{-\infty}^{\infty^*} \frac{e^{isy}}{y} dy. \tag{3.10}$$

Proof. First, the left-hand side of (3.10) can be rewritten as:

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty^*} e^{isx} \frac{\phi(y)}{x-y} dy dx. \tag{3.11}$$

Now, apply Fubini's theorem to (3.11) and then start solving:

$$\begin{aligned}
\int_{-\infty}^{\infty} \int_{-\infty}^{\infty^*} e^{isx} \frac{\phi(y)}{x-y} dx dy &= \int_{-\infty}^{\infty} \phi(y) \int_{-\infty}^{\infty^*} \frac{e^{is(y+x-y)}}{x-y} dx dy \\
&= \int_{-\infty}^{\infty} \phi(y)\, e^{isy} \int_{-\infty}^{\infty^*} \frac{e^{is(x-y)}}{x-y} dx dy \\
&= \int_{-\infty}^{\infty} \phi(y)\, e^{isy} \int_{-\infty}^{\infty^*} \frac{e^{isw}}{w} dw dy \\
&= \int_{-\infty}^{\infty} \phi(y)\, e^{isy} dy \int_{-\infty}^{\infty^*} \frac{e^{isw}}{w} dw, \quad \text{(by Fubini)}.
\end{aligned}$$

Then, the lemma has been proved. □

Lemma 3.6.

$$\int_0^{\infty} \frac{\sin(sx)}{x} dx = \frac{\pi}{2} sgn(s). \tag{3.12}$$

Proof. By using Fubini's theorem we get:

$$\int_0^{\infty} \left(\int_0^{\infty} e^{-xy} \sin(sx) dy \right) dx = \int_0^{\infty} \left(\int_0^{\infty} e^{-xy} \sin(sx) dx \right) dy, \tag{3.13}$$

where

$$\begin{aligned}
\int_0^{\infty} e^{-xy} \sin(sx) dy &= -e^{-xy} \frac{\sin(sx)}{x} \bigg|_0^{\infty} \\
&= \lim_{y \to \infty} \left(e^{-xy} \frac{\sin(sx)}{x} \right) + \frac{\sin(sx)}{x} \\
&= \frac{\sin(sx)}{x},
\end{aligned} \tag{3.14}$$

and

$$\begin{aligned}\int_0^\infty e^{-xy}\sin(sx)dx &= -\int_0^\infty ye^{-xy}\frac{\cos(sx)}{s}dx - e^{-xy}\frac{\cos(sx)}{s}\bigg|_0^\infty \\ &= -\frac{y}{s}\int_0^\infty e^{-xy}\cos(sx)dx + \frac{1}{s} \\ &= -\frac{y}{s}\left(\int_0^\infty ye^{-xy}\frac{\sin(x)}{s}dx + e^{-xy}\frac{\sin(x)}{s}\bigg|_0^\infty\right) + \frac{1}{s} \\ &= -\frac{y^2}{s^2}\int_0^\infty e^{-xy}\sin(sx)dx + \frac{1}{s}\end{aligned}$$

Therefore,

$$\left(1+\frac{y^2}{s^2}\right)\int_0^\infty e^{-xy}\sin(sx)dx = \frac{1}{s},$$

which implies,

$$\int_0^\infty e^{-xy}\sin(sx)dx = \frac{s}{s^2+y^2}. \tag{3.15}$$

By replacing (3.14) and (3.15) in (3.13) we get:

$$\begin{aligned}\int_0^\infty \frac{\sin(sx)}{x}dx &= \int_0^\infty \frac{s}{s^2+y^2}dy \\ &= \frac{1}{s}\int_0^\infty \frac{1}{1+\left(\frac{y}{s}\right)^2}dy \\ &= \frac{1}{s}\cdot s\cdot \arctan\left(\frac{y}{s}\right)\bigg|_0^\infty \\ &= \arctan\left(\frac{y}{s}\right)\bigg|_0^\infty \\ &= \lim_{y\to\infty}\arctan\left(\frac{y}{s}\right) - 0 \\ &= \begin{cases}\frac{\pi}{2}, & s\geq 0\\ -\frac{\pi}{2}, & s<0\end{cases} \\ &= \frac{\pi}{2}\operatorname{sgn}(s).\end{aligned}$$

Thus, the lemma has been proved. □

Now that both lemmas are proved, we can use these results to prove theorem 3.4 as follows:

Proof. We begin by computing the following integral by using (3.10),

$$\begin{aligned}\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{isx}\int_{-\infty}^{\infty^*}\frac{\phi(y)}{x-y}dydx &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{isx}\phi(x)\,dx\int_{-\infty}^{\infty^*}\frac{e^{isy}}{y}dy \\ &= F(\phi)\int_{-\infty}^{\infty^*}\frac{e^{isy}}{y}dy,\end{aligned}$$

where F is the Fourier transform, and by using (3.12), we have

$$\begin{aligned}\int_{-\infty}^{\infty^*}\frac{e^{isy}}{y}dy &= \int_{-\infty}^{\infty^*}\frac{\cos(sy)}{y}dy + i\int_{-\infty}^{\infty^*}\frac{\sin(sy)}{y}dy \\ &= 2i\int_0^{\infty^*}\frac{\sin(sy)}{y}dy \\ &= \pi i\operatorname{sgn}(s).\end{aligned}$$

Therefore, by applying Fourier transform to (3.8) we get:

$$i\ \mathrm{sgn}\left(s\right)F\left(\phi\right)=F\left(f\right), \tag{3.16}$$

from where we obtain

$$\phi\left(x\right)=-\frac{1}{\pi}\int_{-\infty}^{\infty^*}\frac{f\left(y\right)}{x-y}dy. \tag{3.17}$$

We will denote the operator defined in (3.8) as H. Thus, (3.8) and (3.17) can be rewritten as:

$$\begin{aligned}H\left(\phi\right)&=f\\ \phi&=-H\left(f\right),\end{aligned}$$

so the statement of the theorem has been proved. Besides, from these two equations it follows that:

$$\begin{aligned}H\left(H\left(\phi\right)\right)&=H\left(f\right)\\ &=-\phi.\end{aligned}$$

By (3.16) we also get:

$$\begin{aligned}F\left(H\left(\phi\right)\right)&=F\left(f\right)\\ &=i\ \mathrm{sgn}\left(s\right)F\left(\phi\right),\end{aligned}$$

which indicates that $H\left(\phi\right)\in L^2\left(-\infty,\infty\right)$. □

An special case of the transform is the finite Hilbert transform that comes from the airfoil problem in aerodynamics (see [15]). This modified Hilbert transform is of the form

$$\frac{1}{\pi}\int_{-1}^{1^*}\frac{\phi\left(z\right)}{x-z}dz=H\left[\phi\left(z\right)\right]\left(x\right)=f\left(x\right),\quad f\left(x\right)\in L^2\left[-1,1\right], \tag{3.18}$$

and its solution leads to the following inversion formula (see Appendix A for details)

$$\phi\left(x\right)=-\frac{1}{\pi}\int_{-1}^{1^*}\frac{\sqrt{1-x^2}}{\sqrt{1-z^2}}\frac{f\left(z\right)}{x-z}dz. \tag{3.19}$$

3.2 Tchebyshev polynomials

In the present section we will introduce the first and second-kind Tchebyshev polynomials as well as its main properties. It is important to mention these polynomials since they form the L^2 we are looking for our problem. The following definitions and the properties statements are based on [16], and [17].

3.2.1 The first-kind polynomial T_n

The Tchebyshev polynomial of the first kind, denoted $T_n\left(x\right)$, is a polynomial in x of degree n, given by:

$$T_n\left(x\right)=\cos\left(n\theta\right),\quad \text{when } x=\cos\left(\theta\right), \tag{3.20}$$

and $-1\leq x\leq 1$.

The recurrence relation

$$T_n\left(x\right)=2xT_{n-1}\left(x\right)-T_{n-2}\left(x\right),\quad n=2,3,... \tag{3.21}$$

together with the initial conditions

$$T_0\left(x\right)=1,\quad T_1\left(x\right)=x, \tag{3.22}$$

generates all the polynomials recursively. The first five first-kind polynomials are shown in Figure 1.

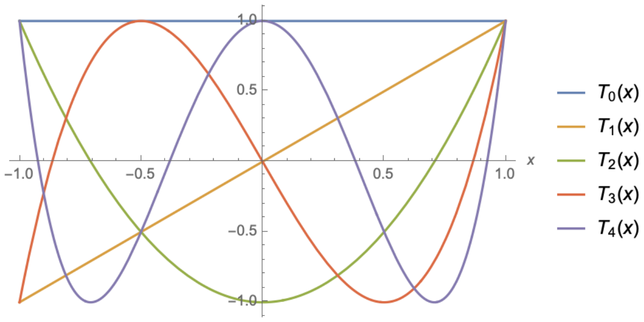


Figure 1: first-kind Tchebyshev polynomials up to degree 4.

3.2.2 The second-kind polynomial U_n

The Tchebyshev polynomial of the second kind, denoted as $U_n(x)$, is a polynomial in x of degree n, given by:

$$U_n(x) = \frac{\sin\left[(n+1)\theta\right]}{\sin(\theta)}, \quad \text{when } x = \cos(\theta), \tag{3.23}$$

and $-1 \leq x \leq 1$.

The recurrence relation

$$U_n(x) = 2xU_{n-1}(x) - U_{n-2}(x), \quad n = 2, 3, ... \tag{3.24}$$

along with the initial conditions

$$U_0(x) = 1, \quad U_1(x) = 2x, \tag{3.25}$$

generates all the polynomials recursively. The first five second-kind polynomials are shown in Figure 2.

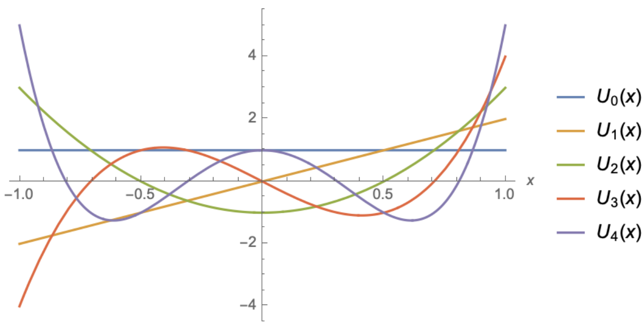


Figure 2: second-kind Tchebyshev polynomials up to degree 4.

3.2.3 Properties

Proposition 3.7 (Symmetry)**.** *If the Tchebyshev polynomials are of an even order, then they have even symmetry, and, besides they only have even powers of x. If the Tchebyshev polynomials are of an odd order, then they have odd symmetry, and, besides they only have odd powers of x, i.e.,*

$$T_n(-x) = (-1)^n T_n(x) \tag{3.26}$$

$$= \begin{cases} T_n(x), & n \text{ even}, \\ -T_n(x), & n \text{ odd}. \end{cases} \tag{3.27}$$

And

$$U_n(-x) = (-1)^n U_n(x) \tag{3.28}$$

$$= \begin{cases} U_n(x), & n \text{ even}, \\ -U_n(x), & n \text{ odd}. \end{cases} \tag{3.29}$$

Proof. We will work with the first-kind polynomial since the proof for the second kind is analogous. We are going to prove the result inductively.

- **Base case:** it holds for n=0:
$$T_0(-x) = 1 = (-1)^0 T_0(x).$$
- **Inductive hypothesis:** we assume the statement is true for $0 \leq n \leq k$, i.e., it holds that
$$T_n(-x) = (-1)^n T_n(x).$$
- **Inductive step:** we are now going to prove it for $n = k+1$. By definition of T_n and the inductive hypothesis, we have that
$$\begin{aligned} T_{k+1}(-x) &= -2xT_k(-x) - T_{k-1}(-x) \\ &= -2x(-1)^k T_k(x) - (-1)^{k-1} T_{k-1}(x) \\ &= 2x(-1)^{k+1} T_k(x) - (-1)^{k+1} T_{k-1}(x) \\ &= (-1)^{k+1}\left[2xT_k(x) - T_{k-1}(x)\right] \\ &= (-1)^{k+1} T_{k+1}(x). \end{aligned}$$

□

Proposition 3.8 (Roots and extrema)**.** *A Tchebyshev polynomial of degree n has n different simple roots, called Tchebyshev roots, in the interval* $[-1,1]$.

The roots of T_n are given by
$$x_k = \cos\left(\frac{\pi\left(k+1/2\right)}{n}\right), \quad k = 0, ..., n-1.$$

In a similar way, the roots of U_n are given by
$$x_k = \cos\left(\frac{k}{n+1}\pi\right), \quad k = 1, ..., n.$$

The extrema of T_n on $[-1,1]$ are located at:
$$x_k = cos\left(\frac{k}{n}\pi\right), \quad k = 0, ..., 1.$$

Both the first and second kinds of Tchebyshev polynomial have extrema at the endpoints, given by:
$$\begin{aligned} T_n(1) &= 1, \\ T_n(-1) &= (-1)^n, \\ U_n(1) &= n+1, \\ U_n(-1) &= (n+1)(-1)^n. \end{aligned}$$

Proof. Let's first find the roots of the first-kind polynomials T_n. By definition $T_n(x) = \cos(n\theta)$, then $T_n(x) = 0$ implies that
$$\cos(n\theta) = 0,$$
from were we know that
$$n\theta = \left(k + 1/2\right)\pi,$$
and finally we can solve for θ:
$$\theta = \frac{\pi\left(k+1/2\right)}{n}.$$

Since $x = \cos(\theta)$, then

$$x_k = \cos\left(\frac{\pi\left(k + 1/2\right)}{n}\right), \quad k = 0, ..., n-1.$$

Besides,

$$\frac{dx}{d\theta} = \frac{d}{dx}\cos(\theta) = -\sin(\theta),$$

implying,

$$\frac{d\theta}{dx} = -\frac{1}{\sin(\theta)}. \tag{3.30}$$

Now let's find where its extrema are located. For this we need to use (3.30),

$$\begin{aligned} 0 &= \frac{d}{dx}T_n(x) \\ &= \frac{d}{dx}\cos(n\theta) \\ &= \frac{d\theta}{dx}\frac{d}{d\theta}\cos(n\theta) \\ &= \frac{n\sin(n\theta)}{\sin(\theta)}. \end{aligned}$$

Therefore the above result states that

$$\sin(n\theta) = 0,$$

thus

$$\theta = \frac{k}{n}\pi,$$

and since we know that $x = \cos(\theta)$ we have that

$$x_k = \cos\left(\frac{k}{n}\pi\right), \quad k = 0, ..., n.$$

If we evaluate T_n in x_0 and x_n, we find that polynomial endpoints give also extrema:

$$\begin{aligned} T_n(x_0) = T_n(1) = 1, \\ T_n(x_n) = T_n(-1) = (-1)^n. \end{aligned}$$

Now let's find the roots of the second-kind polynomials U_n. By definition $U_n(x) = \dfrac{\sin\left[(n+1)\theta\right]}{\sin(\theta)}$, then by letting

$$U_n(x) = 0,$$

we get that, in particular,

$$\sin\left[(n+1)\theta\right] = 0,$$

then,

$$(n+1)\theta = k\pi,$$

and

$$\theta = \frac{k}{n+1}\pi.$$

Since $x = \cos(\theta)$, we have that

$$x_k = \cos\left(\frac{k}{n+1}\pi\right), \quad k = 1, ..., n.$$

□

Proposition 3.9 (Differentiation and integration)**.** *The derivatives of the Tchebyshev polynomials are given by:*

$$\frac{dT_n}{dx} = nU_{n-1}, \tag{3.31}$$

$$\frac{dU_n}{dx} = \frac{(n+1)\,T_{n+1} - xU_n}{x^2-1}, \tag{3.32}$$

$$\frac{d^2T_n}{dx^2} = n\frac{nT_n - xU_{n-1}}{x^2-1} = n\frac{(n+1)\,T_n - U_n}{x^2-1}. \tag{3.33}$$

Concerning integration,

$$\int_{-1}^{1} U_n dx = \frac{T_{n+1}}{n+1}. \tag{3.34}$$

Proof. Let's begin by proving (3.31),

$$\begin{aligned}\frac{dT_n}{dx} &= \frac{d\theta}{dx}\frac{d}{d\theta}\cos(n\theta)\\ &= n\frac{\sin(n\theta)}{\sin(\theta)}\\ &= nU_{n-1}.\end{aligned}$$

Now we prove (3.32):

$$\begin{aligned}\frac{dU_n}{dx} &= \frac{d\theta}{dx}\frac{d}{d\theta}\frac{\sin\left[(n+1)\,\theta\right]}{\sin(\theta)}\\ &= -\frac{1}{\sin(\theta)}\frac{(n+1)\cos\left[(n+1)\,\theta\right]\sin(\theta) - \sin\left[(n+1)\,\theta\right]\cos(\theta)}{\sin^2(\theta)}\\ &= \frac{1}{\sin(\theta)}\frac{(n+1)\,T_{n+1}\sin(\theta) - U_n\sin(\theta)\,x}{x^2-1}\\ &= \frac{(n+1)\,T_{n+1} - xU_n}{x^2-1}.\end{aligned}$$

Next, we are going to prove (3.33):

$$\begin{aligned}\frac{d^2T_n}{dx^2} &= \frac{d}{dx}\left(\frac{dT_n}{dx}\right)\\ &= \frac{d\theta}{dx}\frac{d}{d\theta}\left(\frac{dT_n}{dx}\right)\\ &= \frac{d\theta}{dx}\frac{d}{d\theta}\left(n\frac{\sin(n\theta)}{\sin(\theta)}\right)\\ &= -\frac{n}{\sin(\theta)}\frac{n\cos(n\theta)\sin(\theta) - \sin(n\theta)\cos(\theta)}{\sin^2(\theta)}\\ &= \frac{n}{\sin(\theta)}\frac{nT_n\sin(\theta) - U_{n-1}\sin(\theta)\,x}{x^2-1}\\ &= n\frac{nT_n - xU_{n-1}}{x^2-1}.\end{aligned}$$

Finally, let us prove (3.34). Using (3.31), we get:

$$\frac{d\,(T_{n+1})}{dx} = (n+1)\,U_n,$$

which implies that

$$\frac{T_{n+1}}{n+1} = \int_{-1}^{1} U_n dx.$$

$\square$

Proposition 3.10 (Orthogonality)**.** *The first-kind polynomials T_n are orthogonal with respect to the weight*

$$\frac{1}{\sqrt{1-x^2}},$$

on the interval $[-1,1]$, i.e.,

$$\int_{-1}^{1} T_n(x)\,T_m(x)\,\frac{dx}{\sqrt{1-x^2}} = \begin{cases} 0, & n \neq m, \\ \pi, & n = m = 0, \\ \frac{\pi}{2}, & n = m \neq 0. \end{cases} \tag{3.35}$$

Similarly, the second-kind polynomials U_n are orthogonal with respect to the weight

$$\sqrt{1-x^2},$$

on the interval $[-1,1]$, i.e.,

$$\int_{-1}^{1} U_n(x)\,U_m(x)\,\sqrt{1-x^2}dx = \begin{cases} 0, & n \neq m, \\ \frac{\pi}{2}, & n = m. \end{cases} \tag{3.36}$$

Proof. Let's begin studying the first-kind polynomials.

- $n \neq m$. Let $x = \cos(\theta)$, thus $dx = -\sin(\theta)\,d\theta$ and

$$\begin{aligned} \int_{-1}^{1} T_n(x)\,T_m(x)\,\frac{dx}{\sqrt{1-x^2}} &= \int_{0}^{\pi} T_n\big[\cos(\theta)\big]\,T_m\big[\cos(\theta)\big]\,d\theta \\ &= \int_{0}^{\pi} \cos(n\theta)\cos(m\theta)\,d\theta \\ &= \frac{1}{2}\int_{0}^{\pi} \Big[\cos\big[(n+m)\,\theta\big] + \cos\big[(n-m)\,\theta\big]\Big]\,d\theta \\ &= \frac{1}{2}\left[\frac{\sin\big[(n+m)\,\theta\big]}{n+m} + \frac{\sin\big[(n-m)\,\theta\big]}{n-m}\right]\Bigg|_{0}^{\pi} \\ &= 0. \end{aligned}$$

- $n = m = 0$. Let $x = \cos(\theta)$, thus $dx = -\sin(\theta)\,d\theta$ and

$$\begin{aligned} \int_{-1}^{1} T_n(x)\,T_m(x)\,\frac{dx}{\sqrt{1-x^2}} &= \int_{0}^{\pi} T_n\big[\cos(\theta)\big]\,T_m\big[\cos(\theta)\big]\,d\theta \\ &= \int_{0}^{\pi} \cos(n\theta)\cos(m\theta)\,d\theta \\ &= \int_{0}^{\pi} \cos^2(0)\,d\theta \\ &= \int_{0}^{\pi} d\theta \\ &= \theta\Big|_{0}^{\pi} \\ &= \pi. \end{aligned}$$

- $n = m \neq 0$. Let $x = \cos(\theta)$, thus $dx = -\sin(\theta)\, d\theta$ and

$$\begin{aligned}
\int_{-1}^{1} T_n(x)\, T_m(x)\, \frac{dx}{\sqrt{1-x^2}} &= \int_{0}^{\pi} T_n\left[\cos(\theta)\right] T_m\left[\cos(\theta)\right] d\theta \\
&= \int_{0}^{\pi} \cos(n\theta)\cos(m\theta)\, d\theta \\
&= \frac{1}{2}\int_{0}^{\pi} \Big[\cos\left[(n+m)\,\theta\right] + \cos\left[(n-m)\,\theta\right]\Big] d\theta \\
&= \frac{1}{2}\int_{0}^{\pi} \left[\cos(2n\theta) + \cos(0)\right] d\theta \\
&= \frac{1}{2}\left[\frac{\sin(2n\theta)}{2n} + \theta\right]\Bigg|_{0}^{\pi} \\
&= \frac{\pi}{2}.
\end{aligned}$$

Now let's work with the second-kind polynomials.

- $n \neq m$. Let $x = \cos(\theta)$, thus $dx = -\sin(\theta)\, d\theta$ and

$$\begin{aligned}
\int_{-1}^{1} U_n(x)\, U_m(x)\, \sqrt{1-x^2} dx &= \int_{0}^{\pi} U_n\left[\cos(\theta)\right] U_m\left[\cos(\theta)\right] \sin^2(\theta)\, d\theta \\
&= \int_{0}^{\pi} \sin\left[(n+1)\,\theta\right] \sin\left[(m+1)\,\theta\right] d\theta \\
&= \frac{1}{2}\int_{0}^{\pi} \Big[-\cos\left[(n+m+2)\,\theta\right] + \cos\left[(n-m)\,\theta\right)\Big] d\theta \\
&= \frac{1}{2}\left[-\frac{\sin\left[(n+m+2)\,\theta\right]}{n+m+2} + \frac{\sin\left[(n-m)\,\theta\right]}{n-m}\right]\Bigg|_{0}^{\pi} \\
&= 0.
\end{aligned}$$

- $n = m$. Let $x = \cos(\theta)$, thus $dx = -\sin(\theta)\, d\theta$ and

$$\begin{aligned}
\int_{-1}^{1} U_n(x)\, U_m(x)\, \sqrt{1-x^2} dx &= \int_{0}^{\pi} U_n\left[\cos(\theta)\right] U_m\left[\cos(\theta)\right] \sin^2(\theta)\, d\theta \\
&= \int_{0}^{\pi} \sin\left[(n+1)\,\theta\right] \sin\left[(m+1)\,\theta\right] d\theta \\
&= \frac{1}{2}\int_{0}^{\pi} \Big[-\cos\left[(n+m+2)\,\theta\right] + \cos\left[(n-m)\,\theta\right)\Big] d\theta \\
&= \frac{1}{2}\int_{0}^{\pi} \Big[-\cos\left[(n+1)\,2\theta\right] + \cos(0)\Big] d\theta \\
&= \frac{1}{2}\left[-\frac{\sin\left[(n+1)\,2\theta\right]}{2\,(n+1)} + \theta\right]\Bigg|_{0}^{\pi} \\
&= \frac{\pi}{2}.
\end{aligned}$$

□

3.3 Curvature

The objective of this section is to get an analytic expression for the curvature of a graph of a function. The curvature, denoted as κ, is a measurement of how much a curve bends [18]. While moving along a curve, one can notice that the direction of the tangent changes as the curve bends. Since we are interested in direction of the tangent instead of its magnitude, we shall consider the unit tangent vector.

In order to define the curvature, we need some previous definitions. In a curve, its arc length s is defined as the distance between two points along it. For a curve parametrized by $r(t)$, it is given by

$$s(t) = \int_{t_0}^{t} |v(\tau)| d\tau, \tag{3.37}$$

where $v = dr/dt$.

Let C be a smooth curve parametrized by $r(t)$. Now, we also know that $s = s(t)$ as it can be seen in (3.37). Therefore, from this last equation, it is possible to get t such that $t = t(s)$. As a consequence, any curve parametrized in terms of t can be also parametrized in terms of s: we are able to rewrite $r = r\big(t(s)\big) = r(s)$.

For a smooth curve C parametrized by $r(t)$, let $v = dr/dt$ be its velocity vector. The velocity vector is tangent to $r(t)$. We define the unit tangent vector T as follows

$$T = \frac{v(s)}{\|v(s)\|},$$

and T is a differentiable function of s as long as v is also a differentiable function of s.

We are now able to define what the curvature is. Let C be a smooth curve with position vector $r(s)$, where s is the arc length parameter. Then, the curvature κ of C is

$$\kappa = \left\|\frac{dT}{ds}\right\|, \tag{3.38}$$

where T is the unit tangent vector.

The following theorem give us an alternative way to compute the curvature.

Theorem 3.11. *Let C be a smooth curve with position vector $r(t)$. Then, the following formula can be used to compute the curvature, κ,*

$$\kappa = \frac{\|r'(t) \times r''(t)\|}{\|r'(t)\|^3}. \tag{3.39}$$

Proof. Let's first compute r', r'' and its cross product.

1. Let's compute $r'(t)$.

   By definition,

   $$T = \frac{r'(t)}{\|r'(t)\|},$$

   and, by applying fundamental theorem of calculus in (3.37), we get

   $$\frac{ds}{dt} = \|r'\|. \tag{3.40}$$

   Therefore, we get that

   $$r' = \frac{ds}{dt} T. \tag{3.41}$$

2. Let's compute $r''(t)$. By the last part,

   $$r'' = \frac{d}{dt}\left(\frac{ds}{dt}T\right) = \frac{d^2s}{dt^2}T + \frac{ds}{dt}T'. \tag{3.42}$$

3. Now, let's compute the cross product $r'(t) \times r''(t)$ and its norm. By (3.41) and (3.42),

   $$r' \times r'' = \frac{ds}{dt}\frac{d^2s}{dt^2}(T \times T) + \left(\frac{ds}{dt}\right)^2 (T \times T').$$

Since T' is tangent to T, $\theta = \pi/2$ is the angle between them, and

$$\begin{aligned}\|r' \times r''\| &= \left(\frac{ds}{dt}\right)^2 \|T \times T'\| \\ &= \left(\frac{ds}{dt}\right)^2 \|T\|\|T'\| \sin\theta \\ &= \left(\frac{ds}{dt}\right)^2 \|T'\| \sin\left(\frac{\pi}{2}\right),\end{aligned}$$

and by using (3.40), we get

$$\|r'(t) \times r''(t)\| = \|r'(t)\|^2 \|T'\|.$$

Therefore,

$$\|T'\| = \frac{\|r'(t) \times r''(t)\|}{\|r'(t)\|^2}.$$

Now, by applying (3.40), we have

$$\begin{aligned}\kappa &= \left\|\frac{dT}{ds}\right\| \\ &= \left\|\frac{dt}{ds}\left(\frac{d}{dt}T\right)\right\| \\ &= \frac{\|T'(t)\|}{\|r'(t)\|} \\ &= \frac{\|r'(t) \times r''(t)\|}{\|r'(t)\|^3}.\end{aligned}$$

□

Finally, we introduce a theorem that allow us to compute the curvature of the graph of a function.

Theorem 3.12. *If C is a curve given by $y = f(x)$, with f twice differentiable, then*

$$\kappa = \frac{|f''(x)|}{\left(1 + \left(f'(x)\right)^2\right)^{3/2}}. \tag{3.43}$$

Proof. To parametrize the curve given by $y = f(x)$ as a 3D parametric curve, we use

$$\begin{cases} x = x \\ y = f(x) \\ z = 0 \end{cases}$$

Then, the position vector of C is $r(x) = \left(x, f(x), 0\right)$ and it follows that

$$\begin{aligned} r'(x) &= \left(1, f'(x), 0\right), \\ r''(x) &= \left(0, f''(x), 0\right). \end{aligned}$$

Hence,

$$r' \times r'' = \left(0, 0, f''(x)\right),$$

therefore,

$$\|r' \times r''\| = |f''(x)|.$$

We also know that

$$\|r'\| = \sqrt{1 + \left(f'(x)\right)^2}$$

Thus,

$$\begin{aligned}\kappa &= \frac{\|r' \times r''\|}{\|r'(x)\|^3} \\ &= \frac{|f''(x)|}{\left(1 + \left(f'(x)\right)^2\right)^{3/2}}.\end{aligned}$$

□

3.4 Conservation equations

During the development of this section, we are going to deduce the mass and momemtum conservation laws by making use of the Reynold's transport theorem. The aim will be to later deduce the Euler equations as well as Navier-Stokes equations. We will base on the results presented in [19] and [20].

3.4.1 Eulerian and Lagrangian approach

In order to derive mass and momemtum conservation equations, first it is essential to choose a coordinate system: the eulerian or the lagrangian system. Before choosing the coordinate system, it is of key importance to define a control volume. According to [21], a control volume is a volume in space (independent of mass) through which fluid may flow. Furthermore, as any volume in space can be considered as a control volume, we have to establish which control volume will be working with for the deduction of the equations. For the purpose of this section, we might assume an arbitrary control volume.

On one hand, the eulerian coordinate system can be used when a control volume is fixed and we focus on the fluid passing through it. In this case, at different times, the portion of the fluid we are seeing is also different but the control volume remains the same. Besides, the independent variables are the spatial coordinates x, y, z and also the time variable t.

On the other hand, in the lagrangian system we choose a portion of the fluid and follow it during a time interval. Therefore, at different times we still have the same portion of fluid but at different spatial coordinates. In this case, the independent variables are x_0, y_0, z_0 and t, where x_0, y_0 and z_0 denote the spatial coordinates of the portion of the fluid at time t_0. For convenience, we choose this last approach in order to derive the conservation equations.

3.4.2 Material derivative

Let α be a given property of the fluid and $\mathbf{u}$ denote the velocity vector of the fluid. Then, the material derivative is defined as follows:

$$\frac{D\alpha}{Dt} = \frac{\partial \alpha}{\partial t} + (\mathbf{u} \cdot \nabla)\, \alpha. \tag{3.44}$$

As mentioned before, we will deduce the mass and momemtum conservation laws in the lagrangian approach. First, the left-hand side of (3.44) shows the total change in α in the lagrangian system, that is, how α changes as we follow a particular portion of the fluid as it flows. In the right-hand side of (3.44), the total change in α in the eulerian approach is represented: the first term represents the eulerian time derivative, that is how α changes as t does it (recall that the portion of fluid is different for different times); while the second-term shows how α changes in a system that does not depend on time.

3.4.3 Reynold's transport theorem

The Reynold's transport theorem will allow us to relate the eulerian and the lagrangian approaches. Let V be an arbitrarily shaped control volume, let α denote an arbitrary property of the fluid and let $\mathbf{u}$ denote the velocity vector of the fluid. Then,

$$\frac{D}{Dt} \int_V \alpha dV = \int_V \left[\frac{\partial \alpha}{\partial t} + \nabla \cdot (\alpha \mathbf{u}) \right] dV, \tag{3.45}$$

where $\nabla \cdot (\alpha \mathbf{u})$ denotes the divergence of $\alpha \mathbf{u}$. Equivalently,

$$\frac{D}{Dt}\int_V \alpha dV = \int_V \left[\frac{\partial \alpha}{\partial t} + \sum_k \frac{\partial}{\partial x_k}(\alpha \mathbf{u}_k)\right] dV. \tag{3.46}$$

3.4.4 Mass conservation law

Let us consider an specific mass of fluid with an arbitrary volume V. If this fluid mass is followed as it flows, then it can be seen that despite the fluid shape changes, its mass remains constant. This is called the conservation of mass principle. If fluid density is denoted as ρ, the principle states that

$$\frac{D}{Dt}\int_V \rho dV = 0. \tag{3.47}$$

By using Reynold's transport theorem and setting $\alpha = \rho$, we get

$$\int_V \left[\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{u})\right] dV = 0,$$

and since V was arbitrarily chosen, the only way that the equality can be satisfied is by setting the integrand equal to zero, i.e.,

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{u}) = 0, \tag{3.48}$$

or, by (3.44),

$$\frac{\partial \rho}{\partial t} + \mathbf{u} \cdot \nabla \rho + \rho \nabla \cdot \mathbf{u} = 0 \iff \frac{D\rho}{Dt} + \rho \nabla \cdot \mathbf{u} = 0. \tag{3.49}$$

3.4.5 Momentum conservation law

The momentum conservation law is a direct consequence of the Newton's second law of motion. The law states that the rate of change of momentum variation is equal to the sum of the resultant forces, that is, for an arbitrary volume V,

$$\frac{D}{Dt}\int_V \rho \mathbf{u} dV = \int_V \mathbf{f}_c + \mathbf{f}_e dV, \tag{3.50}$$

where $\mathbf{f}_c$ denotes the contact density forces and $\mathbf{f}_e$ denotes the external density forces. Let $\mathbf{u} = (u_1, u_2, ..., u_n)$.

To obtain an explicit formula from (3.50), we can work by components as follows:

$$\begin{aligned}
\int_V (\mathbf{f}_c + \mathbf{f}_e)_i \, dx &= \frac{D}{Dt}\int_V \rho u_i dx \\
&= \int_V \left[\frac{\partial}{\partial t}(\rho u_i) + \nabla \cdot (\rho u_i \mathbf{u})\right] dx \\
&= \int_V \left[\frac{\partial \rho}{\partial t} u_i + \rho \frac{\partial u_i}{\partial t} + \nabla(\rho u_i) \cdot \mathbf{u} + \rho u_i \nabla \cdot \mathbf{u}\right] dx \\
&= \int_V \left[\frac{\partial \rho}{\partial t} u_i + \rho \frac{\partial u_i}{\partial t} + (\rho \nabla u_i + u_i \nabla \rho) \cdot \mathbf{u} + u_i \rho \nabla \cdot \mathbf{u}\right] dx \\
&= \int_V \left[u_i\left(\frac{\partial \rho}{\partial t} + \nabla \rho \cdot \mathbf{u} + \rho \nabla \cdot \mathbf{u}\right) + \rho\left(\frac{\partial u_i}{\partial t} + \nabla u_i \cdot \mathbf{u}\right)\right] dx,
\end{aligned}$$

which, by (3.49), becomes

$$\int_V (\mathbf{f}_c + \mathbf{f}_e)_i \, dx = \int_V \rho\left(\frac{\partial u_i}{\partial t} + \nabla u_i \cdot \mathbf{u}\right) dx.$$

Since this equality is true for any domain V, we conclude

$$(\mathbf{f}_c + \mathbf{f}_e)_i = \rho \left(\frac{\partial u_i}{\partial t} + \nabla u_i \cdot \mathbf{u} \right).$$

Writing equation above in vector form, give us

$$\mathbf{f}_c + \mathbf{f}_e = \rho \left(\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla)\, \mathbf{u} \right).$$

This last result holds since, for a fixed component i, $\nabla u_i \cdot \mathbf{u} = \sum_{k=1}^{n} u_k \partial_k u_i$ is the i-th component of $(\mathbf{u} \cdot \nabla)\, \mathbf{u}$.

Furthermore, the contact forces $\mathbf{f}_c$ are the sum of a pressure and a viscosity component, that is, $\mathbf{f}_c = \mathbf{f}_p + \mathbf{f}_v$. Johann and Daniel Bernoulli as well as Leonhard Euler worked in describing the contact forces components. They managed to describe the pressure component as $\mathbf{f}_p = -\nabla p$ (see [20]). Finally, the momentum conservation law has the form

$$\rho \left(\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla)\, \mathbf{u} \right) = -\nabla p + \mathbf{f}_v\,(x,t) + \mathbf{f}_e\,(x,t)\,, \tag{3.51}$$

where $\mathbf{f}_v$ represents the viscous effects on the fluid.

4 Theoretical framework of the project

Once the mass and momemtum conservation equations has been deduced, we can proceed studying the Euler equations equation as well as the Navier-Stokes equations. As we will see later, the Euler equations are of great importance in the present work: we are going to linearize these equations and solve the system in particular domains of $\mathbb{R}^2$. The results are based on results in [19], [20], and [22].

4.1 Euler equations

Let $\mathbf{u} = (u_1, u_2, ..., u_n)$ be the velocity vector of the fluid. The inconvenience with the system described in (3.49) − (3.51) is that the system give us $n+1$ equations while it has $n+2$ unknowns ($\mathbf{u}, \rho$ and p). Thus, to overcome the difficulties of this system, Bernoulli and Euler (XVIII century) proposed to find reasonable conditions to reduce the problem to one that actually can be mathematically analyzed. This reduction consists of two considerations, the first is to consider fluids that cannot be compressed, known as incompressible fluids; and the second is to consider fluids that do not suffer viscous effects, known as perfect fluids.

4.1.1 Incompressibility

This incompressible fluid condition states the following:

$$\frac{D\rho}{Dt} = 0.$$

Besides, in order to subject the fluid to the mass conservation law in (3.49), it also happens that

$$\nabla \cdot \mathbf{u} = 0.$$

If we add spatial homogeneity $\rho = \rho\,(t)$, then, from $D\rho/Dt = 0$ we can get $\partial\rho/\partial t = 0$; that is, ρ should be constant in time and space, therefore, ρ is not a variable of the system anymore.

4.1.2 Perfect fluids

For fluids that are sensible to pressure but not to shear stress, the only contact force that will play a role will be the pressure component, where $\mathbf{f}_p = -\nabla p$ and the dynamic equation (3.51) becomes

$$\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} = -\frac{1}{\rho} \left[(\nabla p) + \mathbf{f}_e\,(x,t) \right]. \tag{4.1}$$

By adding the incompressibility hypothesis, we have

$$\nabla \cdot \mathbf{u} = 0. \tag{4.2}$$

It is common to assume homogeneity, thus ρ is a constant. The Euler system for perfect fluids consists of equations (4.1) and (4.2). Thus, this new system has $n + 1$ unknown variables as well as $n + 1$ equations.

4.2 Navier-Stokes equations

The Navier-Stokes system (NSS) is a set of non-linear PDEs describing the motion of an incompressible fluid. In contrast to Euler system, NSS takes viscous effects into consideration. The NSS owes its name to Claude-Louis Navier and sir George Gabriel Stokes since they determined the viscosity component in (3.51).

The force due to viscosity is defined as follows

$$\mathbf{f}_v = \nabla \tau \delta V = \nabla \tau \delta x \delta y \delta z, \tag{4.3}$$

where τ is a shear stress. The shear stress is a tensor, and therefore, has three forces in each direction as represented in Figure 3.

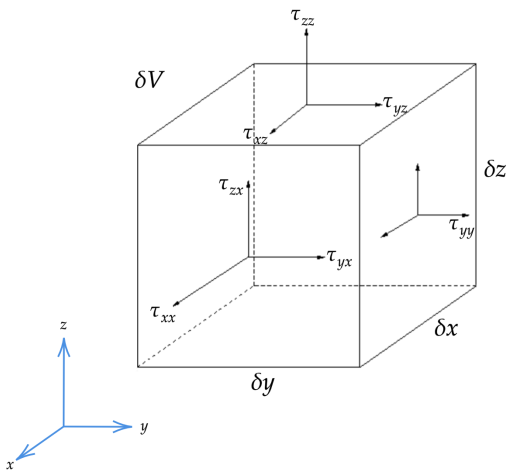


Figure 3: representation of the components of the forcing viscosity terms in a small region of volume δV.

For each direction x, y and z, we can sum the forces due to viscosity and get

$$\frac{\partial \tau_{xx}}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z} = \mathbf{f}_{v_x}, \tag{4.4}$$

$$\frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \tau_{yy}}{\partial y} + \frac{\partial \tau_{zy}}{\partial z} = \mathbf{f}_{v_y}, \tag{4.5}$$

$$\frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + \frac{\partial \tau_{zz}}{\partial z} = \mathbf{f}_{v_z}. \tag{4.6}$$

For a fluid with constant viscosity, commonly referred as a newtonian fluid, the stress is proportional to the rate of deformation. This means that

$$\tau_{ij} = \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right) \mu, \quad \text{for } i, j \in \{1, 2, 3\},$$

where $\mathbf{u} = (u_1, u_2, u_3)$ and $(x_1, x_2, x_3) = (x, y, z)$. In particular,

$$\tau_{xy} = \tau_{yx} = \left(\frac{\partial u_1}{\partial y} + \frac{\partial u_2}{\partial x}\right)\mu, \tag{4.7}$$

$$\tau_{xz} = \tau_{zx} = \left(\frac{\partial u_1}{\partial z} + \frac{\partial u_3}{\partial x}\right)\mu, \tag{4.8}$$

$$\tau_{yz} = \tau_{zy} = \left(\frac{\partial u_2}{\partial z} + \frac{\partial u_3}{\partial y}\right)\mu, \tag{4.9}$$

$$\tau_{xx} = 2\mu\frac{\partial u_1}{\partial x}, \tag{4.10}$$

$$\tau_{yy} = 2\mu\frac{\partial u_2}{\partial y} \tag{4.11}$$

$$\tau_{zz} = 2\mu\frac{\partial u_3}{\partial z}. \tag{4.12}$$

Replacing (4.7) – (4.12) in (4.4), we now get

$$\begin{aligned}
\mathbf{f}_{v_x} &= \frac{\partial}{\partial x}\left(\mu\frac{\partial u_1}{\partial x} + \mu\frac{\partial u_1}{\partial x}\right) + \frac{\partial}{\partial y}\left(\frac{\partial u_1}{\partial y} + \frac{\partial u_2}{\partial x}\right)\mu + \frac{\partial}{\partial z}\left(\frac{\partial u_1}{\partial z} + \frac{\partial u_3}{\partial x}\right)\mu \\
&= \mu\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)u_1 + \mu\frac{\partial}{\partial x}\left(\frac{\partial}{\partial x}u_1 + \frac{\partial}{\partial y}u_2 + \frac{\partial}{\partial z}u_3\right) \\
&= \mu\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)u_1 + \mu\frac{\partial}{\partial x}\nabla\cdot\mathbf{u} \\
&= \mu\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)u_1,
\end{aligned}$$

since we are assuming that $\nabla\cdot\mathbf{u} = 0$. Analogously, we can obtain explicit expressions for $\mathbf{f}_{v_y}$ and $\mathbf{f}_{v_z}$,

$$\begin{aligned}
\mathbf{f}_{v_y} &= \mu\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)u_2, \\
\mathbf{f}_{v_z} &= \mu\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)u_3.
\end{aligned}$$

Then, with these results and (4.3) , we have that

$$\mathbf{f}_v = \mu\Delta\mathbf{u}\delta V.$$

Now, we are able to replace the viscosity term in (3.51) to get

$$\frac{\partial\mathbf{u}}{\partial t} + (\mathbf{u}\cdot\nabla)\,\mathbf{u} = -\frac{1}{\rho}\left(\nabla p + \mu\Delta\mathbf{u}\delta V + \mathbf{f}_e\,(x,t)\right)$$

Let $\nu = \mu/\rho$ be the parameter characterizing the viscosity property which depends on each fluid, we finally get the NSS with constant density and viscosity:

$$\frac{\partial\mathbf{u}}{\partial t} + (\mathbf{u}\cdot\nabla)\,\mathbf{u} + \frac{1}{\rho}\nabla p - \nu\Delta\mathbf{u} = -\frac{1}{\rho}\mathbf{f}_e\,(x,t)\,, \tag{4.13}$$

$$\nabla\cdot\mathbf{u} = 0. \tag{4.14}$$

4.3 Problem statement

The problem considered in the present work consists of analyzing the waves in a container with a small hole, around the origin, of size 2 at the center of the top wall. That is, we want to study incompressible, perfect fluids, on general domains in $\mathbb{R}^2$. We start by considering the lower half-plane case, $\Omega = \mathbb{R}^2_- := \{(x, y)\, / x \in \mathbb{R}$ and $y \leq 0\}$. The container geometry is represented in Figure 4, where the top walls are assumed to extend to infinity.

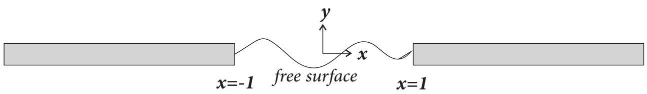


Figure 4: half-plane container geometry.

Some assumptions on the liquid inside the container are that it has a constant density ρ and a surface tension σ, which appears on the system as an external force, related to the pressure. Therefore, our fluid will obey the system of equations (4.1) – (4.2). The external force component to be considered is the gravity acceleration denoted as $-ge_2$, with $g > 0$ and $e_2 = (0, 1)$, the unit normal vector in the vertical direction. Thus, the fluid is ruled by the following system:

$$\nabla \cdot \mathbf{u} = 0, \tag{4.15}$$

$$\rho \left[\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla)\, \mathbf{u} \right] = -\nabla p - g e_2. \tag{4.16}$$

In our case, gravity effects on the fluid can be neglected since we are working on a microscale and there is almost no contribution in comparison to surface tension. Let the velocity vector $\mathbf{u}\,(x, y) = (u, v)$, where u and v denote the velocity components in x and y, respectively. We will consider the following conditions for the asymptotic behaviour at infinity:

$$u, v \to 0, \quad \text{as } y \to -\infty \quad \text{or } |x| \to \infty. \tag{4.17}$$

We also impose an impermeability condition on the walls, i.e.,

$$v = 0, \quad \text{at } y = 0 \text{ and } |x| \geq 1. \tag{4.18}$$

Let the equation of the free surface be of the form

$$y = h\,(x, t)\,, \quad \text{for } |x| \leq 1. \tag{4.19}$$

At the free surface, consider that the pressure is given by the following expression

$$p = \sigma \kappa, \tag{4.20}$$

where σ is the surface tension and κ is the curvature of the surface. In this case, by (3.43),

$$\kappa = -h_{xx} / \left(1 + h_x^2\right)^{3/2}. \tag{4.21}$$

We can define $\Gamma\,(x, y, t) = y - h\,(x, t)$. Thus, by the mass conservation and (4.19) we have that

$$\begin{aligned} \frac{d\Gamma}{dt} &= \frac{\partial \Gamma}{\partial x}\frac{dx}{dt} + \frac{\partial \Gamma}{\partial y}\frac{dy}{dt} + \frac{\partial \Gamma}{\partial t} \\ &= -u h_x + v - h_t \\ &= 0. \end{aligned}$$

Therefore,

$$h_t = -u h_x + v. \tag{4.22}$$

4.4 Linearized equations

In this section we are going to linearize Euler equations. Then, by using the Fourier transform method, we will write the system on the free surface only, as a scalar equation involving an integral operator.

Before, let us reformulate the problem in terms of the velocity potential φ. Assume an incompressible, inviscid and irrotational flow fluid satisfying the conservation equations. Then, we can consider a potential function $\varphi(x, y)$ of $\mathbf{u}$ such that $\mathbf{u} = \nabla\varphi$.

Then, from (4.15) we get that for $(x, y) \in \Omega$:

$$\begin{aligned} 0 &= \nabla \cdot \mathbf{u} \\ &= \nabla \cdot \nabla\varphi \\ &= \Delta\varphi. \end{aligned}$$

Now, from (4.16) and considering that

$$(\mathbf{u} \cdot \nabla) \mathbf{u} = \nabla \left(\frac{1}{2} \mathbf{u} \cdot \mathbf{u} \right) - \mathbf{u} \times (\nabla \times \mathbf{u}),$$

we get

$$\rho \left[\frac{\partial \nabla\varphi}{\partial t} + \nabla \frac{1}{2} |\nabla\varphi|^2 - \nabla\varphi \times (\nabla \times \nabla\varphi) \right] = -\nabla p,$$

but for any scalar φ, it holds that

$$\nabla \times \nabla\varphi = 0.$$

Therefore, we get:

$$\begin{aligned} \rho \left[\frac{\partial \nabla\varphi}{\partial t} + \nabla \frac{1}{2} |\nabla\varphi|^2 \right] &= -\nabla p \\ \rho \nabla \left[\frac{\partial\varphi}{\partial t} + \frac{1}{2} |\nabla\varphi|^2 \right] &= -\nabla p, \end{aligned}$$

implying,

$$\rho \left[\frac{\partial\varphi}{\partial t} + \frac{1}{2} |\nabla\varphi|^2 \right] = -p + c,$$

where we can assume the constant c to be zero.

Hence, for $(x, y) \in \Omega$,

$$\frac{\partial\varphi}{\partial t} + \frac{1}{2} |\nabla\varphi|^2 + \frac{1}{\rho} p = 0.$$

Finally, the boundary conditions in (4.17) for $\mathbf{u}$, can be rewritten in terms of φ, as

$$\frac{\partial\varphi}{\partial x}, \frac{\partial\varphi}{\partial y} \to 0, \quad \text{as } y \to -\infty \quad \text{or } |x| \to \infty.$$

Summarizing, we have

$$\Delta\varphi = 0, \quad \text{for } (x, y) \in \mathbb{R}^2_-, \tag{4.23}$$

$$\frac{\partial\varphi}{\partial t} + \frac{1}{2} |\nabla\varphi|^2 + \frac{1}{\rho} p = 0, \quad \text{for } (x, y) \in \mathbb{R}^2_-, \tag{4.24}$$

$$\frac{\partial\varphi}{\partial x}, \frac{\partial\varphi}{\partial y} \to 0, \quad \text{as } y \to -\infty \quad \text{or } |x| \to \infty. \tag{4.25}$$

System (4.23) – (4.25), is known as the Bernoully formulation of Euler equations. As we mentioned above, we are going to linearize this system, which will allow us to find an explicit solution.

Let

$$h(x, t) = \varepsilon f(x, t), \quad |x| < 1. \tag{4.26}$$

Then, it is expected that the velocity potential be a small perturbation of the trivial solution for the Laplacian. Therefore, let

$$\varphi = c + \varepsilon\phi. \tag{4.27}$$

Next, we are going to write system (4.23) – (4.25) in terms of ϕ and f. From (4.23), and the above equation we obtain:

$$\Delta\varphi = \varepsilon\Delta\phi = 0, \quad \text{for } y < 0.$$

Since ε is arbitrary, it follows that

$$\Delta\phi = 0, \quad \text{for } y < 0.$$

By replacing (4.20), (4.21), and (4.27) into (4.24), we get

$$0 = \varepsilon\frac{\partial\phi}{\partial t} + \frac{\varepsilon^2}{2}|\nabla\varphi|^2 + \frac{1}{\rho}\sigma\left(-\frac{h_{xx}}{(1+h_x^2)^{3/2}}\right) = \varepsilon\frac{\partial\phi}{\partial t} + \frac{\varepsilon^2}{2}|\nabla\varphi|^2 - \frac{\varepsilon}{\rho}\sigma\left[\frac{f_{xx}}{(1+\varepsilon^2 f_x^2)^{3/2}}\right].$$

Dividing by ε, we obtain

$$\frac{\partial\phi}{\partial t} + \frac{\varepsilon}{2}|\nabla\varphi|^2 = \frac{\sigma}{\rho}\left[\frac{f_{xx}}{(1+\varepsilon^2 f_x^2)^{3/2}}\right],$$

$$\frac{\partial\phi}{\partial t} = \frac{\sigma}{\rho}f_{xx} + \mathcal{O}(\varepsilon).$$

Since the $\mathcal{O}(\varepsilon)$ terms are negligible, we get:

$$\frac{\partial\phi}{\partial t} = \frac{\sigma}{\rho}f_{xx}.$$

By (4.26) and (4.22) we get the following equation:

$$\varepsilon f_t = -\varepsilon u f_x + v, \quad \text{for } |x| \le 1 \text{ and } y = 0.$$

We also know that $(u, v) = \big(\varphi_x, \varphi_y\big) = \big(\varepsilon\phi_x, \varepsilon\phi_y\big)$, then

$$\begin{aligned} \varepsilon f_t &= -\varepsilon^2\phi_x f_x + \varepsilon\phi_y, \\ f_t &= \phi_y - \varepsilon\phi_x f_x \\ &= \phi_y + \mathcal{O}(\varepsilon), \quad \text{for } |x| \le 1 \text{ and } y = 0. \end{aligned}$$

Thus, by taking a first order approximation, we finally get:

$$f_t = \frac{\partial\phi}{\partial y}, \quad \text{for } |x| \le 1 \text{ and } y = 0.$$

By (4.27) and condition in (4.18), we obtain

$$\frac{\partial\phi}{\partial y} = \frac{1}{\varepsilon}\frac{\partial\varphi}{\partial y} = 0, \quad \text{for } |x| > 1 \text{ and } y = 0.$$

The following conditions follow directly from (4.25):

$$\frac{\partial\phi}{\partial x}, \frac{\partial\phi}{\partial y} \to 0, \quad \text{as } y \to -\infty \text{ or } |x| \to \infty.$$

To summarize, the Bernoully system, in terms of ϕ and f, becomes

$$\Delta\phi = 0, \quad \text{for } y < 0, \tag{4.28}$$

$$\frac{\partial\phi}{\partial t} = \frac{\sigma}{\rho}f_{xx}, \quad \text{for } y = 0 \text{ and } |x| \le 1, \tag{4.29}$$

$$f_t = \frac{\partial\phi}{\partial y}, \quad \text{for } |x| \le 1 \text{ and } y = 0, \tag{4.30}$$

$$\frac{\partial\phi}{\partial y} = 0, \quad \text{for } |x| > 1 \text{ and } y = 0, \tag{4.31}$$

$$\frac{\partial\phi}{\partial x}, \frac{\partial\phi}{\partial y} \to 0, \quad \text{as } y \to -\infty \text{ or } |x| \to \infty. \tag{4.32}$$

Remark 4.1. *Our system needs boundary conditions (4.29) and (4.30) because part of the boundary (free surface f) is unknown and it is precisely what we want to determine. This is what is known in the literature as a free boundary problem.*

4.5 Integrodifferential equation

From (4.28), we know that ϕ satisfies Laplace equation on the lower half plane. Then, we can find a solution for ϕ under conditions (4.31) and (4.30) by applying the Fourier transform in x as in (3.6). That is:

$$\begin{aligned} F\left(\frac{\partial^2}{\partial y^2}\phi(x,y)+\frac{\partial^2}{\partial x^2}\phi(x,y)\right) &= \frac{\partial^2}{\partial y^2}F(\phi)-k^2F(\phi) \\ &= 0. \end{aligned}$$

Besides, we know that the general solution of a second order ordinary differential equation with constant coefficients is of the form:

$$F[\phi](k,y,t)=c_1e^{|k|y}+c_2e^{-|k|y}.$$

By (4.32), we also get that:

$$\frac{\partial}{\partial y}F[\phi]=c_1|k|e^{|k|y}-c_2|k|e^{-|k|y}\xrightarrow[y\to-\infty]{}0,$$

which implies that $c_2=0$, i.e.,

$$F[\phi](k,y,t)=c_1e^{|k|y}.$$

At y=0, we have:

$$F[\phi](k,0,t)=c_1,$$

thus,

$$F[\phi](k,y,t)=F[\phi](k,0,t)\,e^{|k|y}. \tag{4.33}$$

Since the inverse transform of $e^{|k|y}$ corresponds to the Poisson kernel $P_y(x)=-\frac{\sqrt{2}y}{\sqrt{\pi}(x^2+y^2)}$, and by using the convolution theorem in (3.7), we have that:

$$\begin{aligned} F[\phi](k,y,t) &= F[\phi](k,0,t)\,e^{|k|y} \\ &= F[\phi](k,0,t)\cdot F\left(P_y\right) \\ &= \frac{1}{\sqrt{2\pi}}F\left[\phi(k,0,t)*\frac{-\sqrt{2}y}{\sqrt{\pi}\left(x^2+y^2\right)}\right]. \end{aligned}$$

By using the inverse tranform once again we get:

$$\phi(x,y,t)=-\frac{y}{\pi}\int_{-\infty}^{\infty}\frac{\phi(z,0,t)}{(x-z)^2+y^2}dz.$$

By taking partial derivative with respect to y in (4.33) and evaluating at $y=0$ we obtain:

$$\begin{aligned} \left.\frac{\partial F[\phi]}{\partial y}\right|_{y=0} &= |k|F[\phi](k,0,t) \\ &= (i)\operatorname{sgn}(k)(-ik)\,F[\phi](k,0,t)\,, \end{aligned}$$

by using (3.6) and (3.16), the equation becomes:

$$\begin{aligned} \left.\frac{\partial F[\phi]}{\partial y}\right|_{y=0} &= (i)\operatorname{sgn}(k)\,F\left[\frac{\partial\phi}{\partial x}(k,0,t)\right] \\ &= F\left[H\left[\frac{\partial\phi}{\partial x}(k,0,t)\right]\right], \end{aligned}$$

where H denotes the Hilbert transform.

Taking the inverse Fourier transform, we get

$$\left.\frac{\partial \phi}{\partial y}\right|_{y=0} = H\left[\left.\frac{\partial \phi}{\partial x}\right|_{y=0}\right].$$

By using (3.9) and (4.31), we obtain

$$\begin{aligned} \left.\frac{\partial \phi}{\partial x}\right|_{y=0} &= -H\left[\left.\frac{\partial \phi}{\partial y}\right|_{y=0}\right] \\ &= -\frac{1}{\pi}\int_{-\infty}^{\infty^*} \frac{(\partial\phi/\partial y)(z,0,t)}{x-z}dz \\ &= -\frac{1}{\pi}\int_{-1}^{1^*} \frac{(\partial\phi/\partial y)(z,0,t)}{x-z}dz, \end{aligned} \tag{4.34}$$

by (4.30), we get

$$\left.\frac{\partial \phi}{\partial x}\right|_{y=0} = -\frac{1}{\pi}\int_{-1}^{1^*} \frac{f_t(z,t)}{x-z}dz.$$

By derivating with respect to t in both sides, and using (4.29), we get

$$\frac{\sigma}{\rho} f_{xxx}(x,t) = -\frac{1}{\pi}\int_{-1}^{1^*} \frac{f_{tt}(z,t)}{x-z}dz.$$

We can assume the term $\sigma/\rho = 1$ to obtain

$$f_{xxx}(x,t) = -\frac{1}{\pi}\int_{-1}^{1^*} \frac{f_{tt}(z,t)}{x-z}dz. \tag{4.35}$$

4.6 Solution of the integrodifferential equation

In this subsection we focus on solving (4.35). First, apply the separation of variables method to rewrite the equation as an eigenvalue problem, for which we will use the linearized system in (4.28) − (4.32). Besides, in this subsection we use Tchebyshev polynomials as a basis to decompose the spatial frequencies.

Let's assume that

$$f(x,t) = A(t)\,S(x).$$

By replacing this on (4.35), we have

$$S'''(x)\,A(t) = -\frac{1}{\pi}A''(t)\int_{-1}^{1^*} \frac{S(z)}{x-z}dz, \quad \forall t>0, \forall x \in (-1,1).$$

Thus,

$$\frac{S'''(x)}{-\frac{1}{\pi}\int_{-1}^{1^*}\frac{S(z)}{x-z}dz} = \frac{A''(t)}{A(t)} = -\lambda, \quad \forall t>0, \forall x\in(-1,1).$$

Therefore, we have two problems: (1) an ODE in the time variable and (2) an eigenvalue problem in space.

1. $A''(t) + \lambda A(t) = 0.$

   We can assume $\lambda \neq 0$, which is the case we are interested in. We know that the characteristic equation associated to the above ode is:

   $$m^2 + \lambda = 0 \implies m = \pm\sqrt{-\lambda}.$$

   - **Case 1.** $\lambda < 0$, thus we have real roots and the solution is given by:

     $$A(t) = c_1 e^{\sqrt{-\lambda}t} + c_2 e^{-\sqrt{-\lambda}t}.$$

- **Case 2.** $\lambda > 0$, thus we have complex roots and the solution is given by:

$$A(t) = c_1 \sin\left(\sqrt{\lambda} t\right) + c_2 \cos\left(\sqrt{\lambda} t\right).$$

2. $S'''(x) = \dfrac{\lambda}{\pi} \displaystyle\int_{-1}^{1^*} \frac{S(z)}{x-z} dz.$

In this case,

$$S'''(x) = H\left[\lambda S(z)\right](x),$$

where H denotes the finite Hilbert transform in 3.18. Thus, by using the corresponding inversion in (3.19), we get

$$\begin{aligned} \lambda S(x) &= -\frac{1}{\pi} \int_{-1}^{1^*} \frac{\sqrt{1-x^2}}{\sqrt{1-z^2}} \frac{S'''(z)}{x-z} dz \\ &= -\frac{1}{\pi} \sqrt{1-x^2} \int_{-1}^{1^*} \frac{S'''(z)}{\sqrt{1-z^2}(x-z)} dz. \end{aligned} \tag{4.36}$$

We have to complement last equation with proper boundary conditions. We have two possible cases: (1) homogeneous Dirichlet boundary condition, i.e, $S(\pm 1) = 0$; or, (2) homogeneous Neumann boundary condition, i.e., $\frac{\partial S}{\partial n}(x)\Big|_{\pm 1} = 0$.

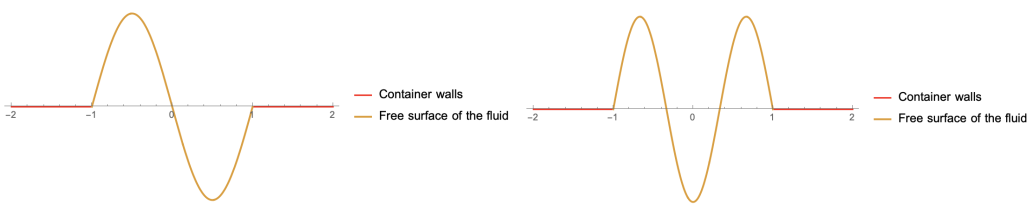
- **Case 1.** Dirichlet homogeneous, $S(\pm 1) = 0$.

It is also known as *the pinned-end case.* As it is a linear problem we can expect a Fourier expansion of $S(x)$, satisfying $S(\pm 1) = 0$. The first kind of solution is the anti-symmetric one, as shown in Figure 5a; this solution is of the form:

$$S(x) = \sum_{n=1}^{\infty} c_n \sin(n\pi x).$$

The second kind of solution is the symmetric solution, which is shown in Figure 5b; this solution is of the form:

$$S(x) = \sum_{n=1}^{\infty} c_n \cos\left(\left(n - \frac{1}{2}\right)\pi x\right).$$



(a) Anti-symmetric (b) Symmetric

Figure 5: pinned-end edge case

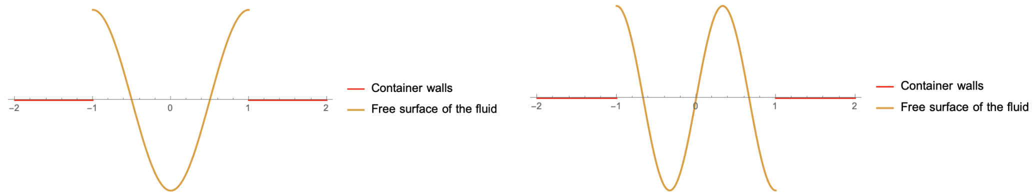
- **Case 2.** Neumann homogeneous, $\dfrac{\partial S}{\partial n}\Big|_{\pm 1} = 0.$

It is also known as the *free-end edge condition.* Physically, this case means that waves form an angle of $\frac{\pi}{2}$ with the side walls. In this case the symmetric solution, shown in Figure 6a, is given by:

$$S(x) = \sum_{n=1}^{\infty} c_n \cos(n\pi x).$$

The anti-symmetric solution, as in Figure 6b is given by:

$$S\left(x\right)=\sum_{n=1}^{\infty}c_n\sin\left(\left(n-\frac{1}{2}\right)\pi x\right).$$



(a) Symmetric (b) Anti-symmetric

Figure 6: free-end edge case

Let's proceed to study each case individually.

4.6.1 Anti-symmetric pinned-end boundary condition

$$S\left(x\right)=\sum_{n=1}^{\infty}a_n\sin\left(n\pi x\right).$$

Replacing the condition in (4.36), we obtain:

$$\begin{aligned}\lambda\sum_{n=1}^{\infty}a_n\sin\left(n\pi x\right)&=-\frac{1}{\pi}\sqrt{1-x^2}\int_{-1}^{1^*}\frac{1}{\sqrt{1-z^2}\left(x-z\right)}\left[-\sum_{n=1}^{\infty}a_n\left(n\pi\right)^3\cos\left(n\pi z\right)\right]dz\\&=\sqrt{1-x^2}\sum_{n=1}^{\infty}\left(n\pi\right)^3a_n\frac{1}{\pi}\int_{-1}^{1^*}\frac{\cos\left(n\pi z\right)}{\sqrt{1-z^2}\left(x-z\right)}dz.\end{aligned}$$

Because of the weight in the integral, Tchebyshev polynomials are a suitable basis to work with. Let

$$\cos\left(n\pi x\right)=\sum_{k=0}^{\infty}c_{kn}T_k\left(x\right),\tag{4.37}$$

with

$$c_{kn}=\begin{cases}\frac{1}{\pi}\int_{-1}^{1}\cos\left(n\pi x\right)\frac{1}{\sqrt{1-x^2}}dx, & k=0,\\ \frac{2}{\pi}\int_{-1}^{1}\cos\left(n\pi x\right)\frac{T_k(x)}{\sqrt{1-x^2}}dx, & k\geq 1.\end{cases}$$

Recall that $T_k\left(x\right)$ is the first-kind Tchebyshev polynomial, defined as in (3.21). Thus,

$$\begin{aligned}\lambda\sum_{n=1}^{\infty}a_n\sin\left(n\pi x\right)&=\sqrt{1-x^2}\sum_{n=1}^{\infty}a_n\left(\pi n\right)^3\frac{1}{\pi}\int_{-1}^{1^*}\frac{\sum_{k=0}^{\infty}c_{kn}T_k\left(z\right)}{\sqrt{1-z^2}\left(x-z\right)}dz\\&=\sqrt{1-x^2}\sum_{n\geq 1}\sum_{k\geq 0}c_{kn}a_n\left(\pi n\right)^3\frac{1}{\pi}\int_{-1}^{1^*}\frac{T_k\left(z\right)}{\sqrt{1-z^2}\left(x-z\right)}dz\\&=-\sqrt{1-x^2}\sum_{n,k\geq 1}c_{kn}a_n\left(\pi n\right)^3U_{k-1}\left(x\right),\end{aligned}\tag{4.38}$$

where $U_k\left(x\right)$ is the second-kind Tchebyshev polynomial defined as in (3.25).

We need to expand the left hand side in order to have both sides in function of the second-kind polynomial. Let

$$\frac{\sin(n\pi x)}{\sqrt{1-x^2}} = \sum_{r=1}^{\infty} e_{rn} U_{r-1}(x), \tag{4.39}$$

where

$$e_{rn} = \frac{2}{\pi}\int_{-1}^{1} \sin(n\pi x)\, U_{r-1}(x)\, dx.$$

Replacing it into (4.38), it becomes

$$\lambda \sum_{n,r\geq 1} e_{rn} a_n U_{r-1}(x) = -\sum_{n,k\geq 1} c_{kn} a_n (\pi n)^3 U_{k-1}(x),$$
$$\lambda \sum_{n\geq 1} e_{rn} a_n = -\sum_{n\geq 1} c_{kn} a_n (\pi n)^3, \quad \forall k, r \geq 1.$$

In matrix notation:

$$\lambda E\vec{a} = -C\left(\operatorname{diag}(n\pi)^3\right)\vec{a}, \tag{4.40}$$

From (4.37), (4.39) and the orthogonality of the Tchebyshev polynomials, we can explicitly get the coefficients of matrices C and E.

Proposition 4.2. *If C is defined as above, and*

$$D = d_{kn} = \int_{-1}^{1} \cos(n\pi x)\, T_k(x)\, dx, \quad \forall k \geq 0, \forall n \geq 1, \tag{4.41}$$

then $D^T = C^{-1}$,

Proof. We know that

$$\cos(n\pi x) = \sum_{k=0}^{\infty} c_{kn} T_k(x),$$

thus,

$$\begin{aligned}\int_{-1}^{1} \cos(n\pi x)\frac{T_m(x)}{\sqrt{1-x^2}}dx &= \int_{-1}^{1}\sum_{k=0}^{\infty} c_{kn}\frac{T_k(x)\,T_m(x)}{\sqrt{1-x^2}}dx\\ &= \sum_{k=0}^{\infty} c_{kn}\int_{-1}^{1}\frac{T_k(x)\,T_m(x)}{\sqrt{1-x^2}}dx\\ &= \begin{cases}\pi c_{mn}, & m = n = 0,\\ \frac{\pi}{2} c_{mn}, & m = n \neq 0.\end{cases}\end{aligned}$$

From where, we obtain that

$$c_{kn} = \begin{cases}\frac{1}{\pi}\int_{-1}^{1}\cos(n\pi x)\frac{1}{\sqrt{1-x^2}}dx, & k = 0,\\ \frac{2}{\pi}\int_{-1}^{1}\cos(n\pi x)\frac{T_k(x)}{\sqrt{1-x^2}}dx, & k \geq 1.\end{cases}$$

Analogous to (4.37), we can assume that

$$\cos(n\pi x) = \sum_{k=0}^{\infty}\alpha_{kn}\frac{T_k(x)}{\sqrt{1-x^2}}.$$

By orthogonality,

$$\alpha_{kn} = \begin{cases}\frac{1}{\pi}\int_{-1}^{1}\cos(n\pi x)\, T_k(x)\, dx = \frac{1}{\pi}d_{kn}, & k = 0,\\ \frac{2}{\pi}\int_{-1}^{1}\cos(n\pi x)\, T_k(x)\, dx = \frac{2}{\pi}d_{kn}, & k \geq 1.\end{cases}$$

then,

$$\cos(n\pi x) = \frac{1}{\pi} d_{0n} \frac{1}{\sqrt{1-x^2}} + \frac{2}{\pi} \sum_{k\geq 1} d_{kn} \frac{T_k(x)}{\sqrt{1-x^2}}.$$

We also know that

$$\begin{aligned}
\delta_{nm} &= \int_{-1}^{1} \cos(n\pi x)\cos(m\pi x)\, dx \\
&= \int_{-1}^{1} \left[\sum_{k=1}^{\infty} c_{kn} T_k(x)\right] \left[\frac{1}{\pi} d_{0n} \frac{1}{\sqrt{1-x^2}} + \frac{2}{\pi} \sum_{\hat{k}\geq 1} d_{\hat{k}n} \frac{T_{\hat{k}}(x)}{\sqrt{1-x^2}}\right] dx \\
&= \frac{1}{\pi} c_{0n} d_{0n} \int_{-1}^{1} \frac{1}{\sqrt{1-x^2}} dx + \frac{2}{\pi} \sum_{k,\hat{k}\geq 1} c_{kn} d_{\hat{k}m} \int_{-1}^{1} \frac{T_k(x) T_{\hat{k}}(x)}{\sqrt{1-x^2}} dx \\
&= c_{0n} d_{0n} + \sum_{k\geq 1} c_{kn} d_{kn} \\
&= \sum_{k\geq 0} c_{kn} d_{kn},
\end{aligned}$$

which, by (3.35), becomes

$$\delta_{nm} = \sum_{k=0}^{\infty} c_{kn} d_{km},$$

or, in matrix notation

$$C^T D = I \implies D^T C = I,$$

therefore, the proposition has been proved. □

We already know that (4.40) holds for $k, n \geq 1$, while (4.41) holds for $k \geq 0$ and $n \geq 1$. Therefore, before multiplying (4.40) by D^T, we need to add a zero first-row to E and a first-row of coefficients c_{0n} to C, for $n \geq 1$. These completed matrices are going to be denoted as $\overline{E}$ and $\overline{C}$, respectively.

The system that is going to be multiplied by D^T is the following:

$$\lambda \overline{E} \vec{a} = -\overline{C} \operatorname{diag}(n\pi)^3, \tag{4.42}$$

and this is valid as long as the following condition holds

$$\begin{aligned}
\sum_{n\geq 1} c_{0n} (n\pi)^3 a_n &= \frac{1}{\pi} \int_{-1}^{1} \frac{\sum_{n\geq 1} a_n (n\pi)^3 \cos(n\pi x)}{\sqrt{1-x^2}} dx \\
&= -\frac{1}{\pi} \int_{-1}^{1} \frac{S'''(x)}{\sqrt{1-x^2}} dx \\
&= 0.
\end{aligned}$$

Finally, multiplying (4.42) by D^T, we have:

$$\lambda D^T \overline{E} \vec{a} = -\operatorname{diag}(n\pi)^3 \vec{a}.$$

This eigenvalue problem can be solved by using MATLAB. In order to solve the problem, we first need to truncate it. Let us consider the problem for dimension $N = 50$, then the obtained eigenvalues λ_i for $i = 1:50$, and their associated eigenvectors a^i are the ones given in Table 1. Furthermore, notice that for a given i, $a^i = [a_1, a_2, ..., a_{50}]^T$.

	$\lambda_1 = 34.31338847$	$\lambda_2 = 262.2696816$	$\lambda_3 = 870.0457763$	$\lambda_4 = 2043.686572$	$\lambda_5 = 3969.230952$
a_1	-1	0.114282556	-0.100089552	-0.092637944	0.08692467
a_2	-0.007266829	-1	0.078165716	0.062956624	-0.05758339
a_3	0.001155386	-0.015954596	-1	-0.065430934	0.048908545
a_4	-0.000320826	0.003777839	-0.021632909	1	-0.059016131
a_5	0.000118319	-0.001355745	0.006190956	0.025444976	1
a_6	-5.21E-05	0.000597542	-0.002555324	-0.008135654	0.028148725
a_7	2.60E-05	-0.000300007	0.001253401	0.003666506	-0.009679945
a_8	-1.41E-05	0.000165155	-0.000684589	-0.00193096	0.004633667
a_9	8.26E-06	-9.74E-05	0.000403365	0.00111826	-0.002566991
a_{10}	-5.10E-06	6.07E-05	-0.000251681	-0.000691965	0.001551898
a_{11}	3.29E-06	-3.95E-05	0.000164307	0.000450068	-0.000996409
a_{12}	-2.21E-06	2.67E-05	-0.000111293	-0.000304485	0.000669167

Table 1: first eigenvalues and its associated eigenvectors for the anti-symmetric pinned-end case. This data was obtained by using MATLAB and by letting $N = 50$.

From Figure 7, it seems that the frist eigenvalue is convergent as $N \to \infty$. In fact, λ_1 converges to 34.3134 as N grows.

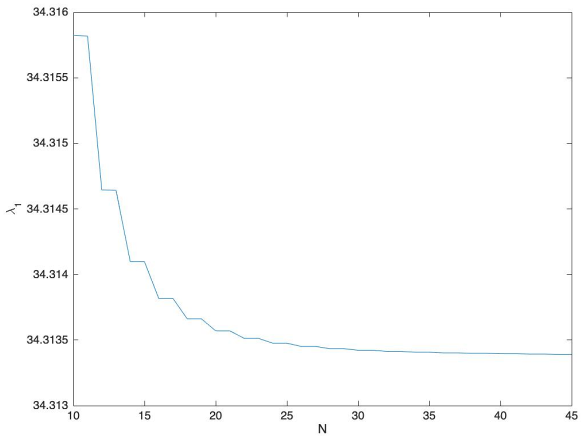
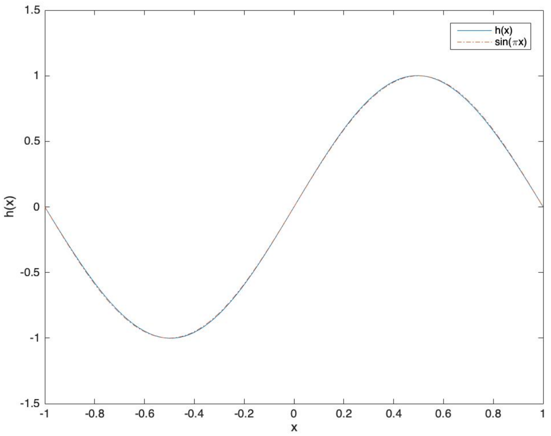
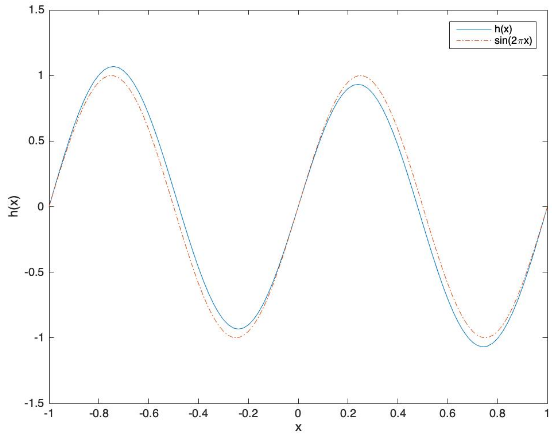


Figure 7: different values for λ_1 as dimension N grows.

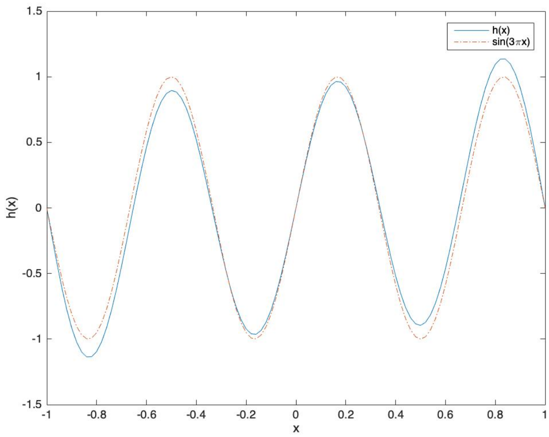
Since we have solved the eigenvalue problem, we are now able to plot the free surface for a fixed value of t. Let us consider $t = 0$, then $h(x,t) = h(x) = S(x)$. The free surface of the fluid in presence of walls differs from the case where no walls are considered, this can be seen in Figure 8.



(a) Eigenfunction associated to λ_1 in comparison with $\sin(\pi x)$.



(b) Eigenfunction associated to λ_2 in comparison with $\sin(2\pi x)$.



(c) Eigenfunction associated to λ_3 in comparison with $\sin(3\pi x)$.

Figure 8: the free surface $h(x)$ in presence of walls compared to the free surface in absence of walls (sine frequencies).

4.6.2 Symmetric pinned-end boundary

$$S(x) = \sum_{n=1}^{\infty} b_n \cos\left(\left(n - \frac{1}{2}\right)\pi x\right).$$

In this case, it is possible that the free-surface is as illustrated in Figure 9. Clearly, the area under the curve is not zero and as a consequence, there is no mass conservation.

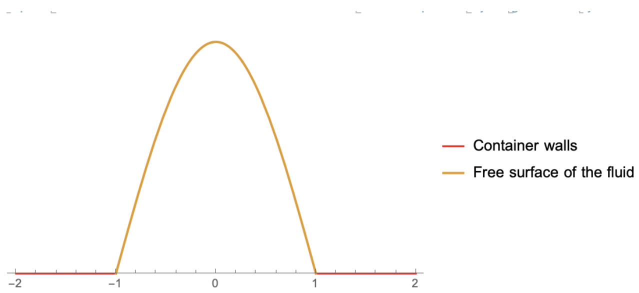


Figure 9: symmetric pinned-end case without mass conservation.

But we want our surface to be subjected to the mass conservation law. Thus, we need to impose an extra condition on $f(x,t) = A(t)\,S(x)$ as follows:

$$\begin{aligned} 0 &= \int_{-1}^{1} f(x,t)\,dx \\ &= A(t)\int_{-1}^{1} S(x)\,dx \\ &= A(t)\sum_{n=1}^{\infty} b_n \int_{-1}^{1} \cos\left(\left(n-\frac{1}{2}\right)\pi x\right)dx. \end{aligned} \tag{4.43}$$

Recall that

$$\begin{aligned} \int_{-1}^{1} \cos\left(\left(n-\frac{1}{2}\right)\pi x\right)dx &= \frac{1}{\left(n-\frac{1}{2}\right)\pi}\sin\left(\left(n-\frac{1}{2}\right)\pi x\right)\Bigg|_{-1}^{1} \\ &= (-1)^{n+1}\frac{2}{\left(n-\frac{1}{2}\right)\pi}, \end{aligned}$$

and, thus for

$$\omega_n = (-1)^{n+1}\frac{2}{\left(n-\frac{1}{2}\right)\pi}, \tag{4.44}$$

(4.43) becomes

$$\sum_{n=1}^{\infty} \omega_n b_n = 0. \tag{4.45}$$

We also know that replacing the symmetric pinned end condition in (4.36), we obtain:

$$\begin{aligned} \lambda\sum_{n=1}^{\infty} b_n \cos\left(\left(n-\frac{1}{2}\right)\pi x\right) &= -\frac{1}{\pi}\sqrt{1-x^2}\int_{-1}^{1^*} \frac{\sum_{n=1}^{\infty}\left(\left(n-\frac{1}{2}\right)\pi\right)^3 b_n \sin\left(\left(n-\frac{1}{2}\right)\pi z\right)}{\sqrt{1-z^2}\,(x-z)}dz \\ &= -\sqrt{1-x^2}\sum_{n=1}^{\infty}\left[\left(n-\frac{1}{2}\right)\pi\right]^3 b_n \frac{1}{\pi}\int_{-1}^{1^*} \frac{\sin\left(\left(n-\frac{1}{2}\right)\pi z\right)}{\sqrt{1-z^2}\,(x-z)}dz. \end{aligned}$$

Let

$$\sin\left(\left(n-\frac{1}{2}\right)\pi x\right) = \sum_{k=1}^{\infty} c_{kn} T_k(x), \tag{4.46}$$

where,

$$c_{kn} = \frac{2}{\pi}\int_{-1}^{1} \sin\left(\left(n-\frac{1}{2}\right)\pi x\right)\frac{T_k(x)}{\sqrt{1-x^2}}dx.$$

Thus,

$$\begin{aligned}\lambda\sum_{n=1}^{\infty}b_n\cos\left(\left(n-\frac{1}{2}\right)\pi x\right) &= -\sqrt{1-x^2}\sum_{n=1}^{\infty}\left(\left(n-\frac{1}{2}\right)\pi\right)^3 b_n\frac{1}{\pi}\int_{-1}^{1^*}\frac{\sum_{k=0}^{\infty}c_{kn}T_k\left(z\right)}{\sqrt{1-z^2}\left(x-z\right)}dz\\ &= -\sqrt{1-x^2}\sum_{n\geq 1}\sum_{k\geq 0}\left(\left(n-\frac{1}{2}\right)\pi\right)^3 c_{kn}b_n\frac{1}{\pi}\int_{-1}^{1^*}\frac{T_k\left(z\right)}{\sqrt{1-z^2}\left(x-z\right)}dz\\ &= \sqrt{1-x^2}\sum_{n,k\geq 1}\left(\left(n-\frac{1}{2}\right)\pi\right)^3 c_{kn}a_nU_{k-1}\left(x\right).\end{aligned}\tag{4.47}$$

Let

$$\frac{\cos\left(\left(n-\frac{1}{2}\right)\pi x\right)}{\sqrt{1-x^2}}=\sum_{r=1}^{\infty}e_{rn}U_{r-1}\left(x\right),$$

where,

$$e_{rn}=\frac{2}{\pi}\int_{-1}^{1}\cos\left(\left(n-\frac{1}{2}\right)\pi x\right)U_{r-1}\left(x\right)dx.$$

Using this identity in (4.47), we get:

$$\lambda\sum_{n,r\geq 1}b_ne_{rn}U_{r-1}\left(x\right)=\sum_{n,k\geq 1}\left(\left(n-\frac{1}{2}\right)\pi\right)^3 c_{kn}b_nU_{k-1}\left(x\right).$$

Therefore,

$$\lambda\sum_{n\geq 1}b_ne_{rn}=\sum_{n\geq 1}\left(\left(n-\frac{1}{2}\right)\pi\right)^3 c_{kn}b_n,\quad \forall k,r\geq 1$$

or, in matrix notation,

$$\lambda E\vec{b}=C\left(\operatorname{diag}\left[\left(n-\frac{1}{2}\right)\pi\right]^3\right)\vec{b}.\tag{4.48}$$

Again, we can explicitly get the coefficients of matrices C and E.

Proposition 4.3. *If C is defined as above, and*

$$D=d_{kn}=\int_{-1}^{1}\sin\left(\left(n-\frac{1}{2}\right)\pi x\right)T_k\left(x\right)dx,\quad \forall k,n\geq 1,$$

then

$$D^T=C^{-1}$$

Proof. We know that

$$\sin\left(\left(n-\frac{1}{2}\right)\pi x\right)=\sum_{k=1}^{\infty}c_{kn}T_k\left(x\right),\tag{4.49}$$

thus,

$$\begin{aligned}\int_{-1}^{1}\sin\left(\left(n-\frac{1}{2}\right)\pi x\right)\frac{T_m\left(x\right)}{\sqrt{1-x^2}}dx &= \int_{-1}^{1}\sum_{k=1}^{\infty}c_{kn}\frac{T_k\left(x\right)T_m\left(x\right)}{\sqrt{1-x^2}}dx\\ &= \sum_{k=1}^{\infty}c_{kn}\int_{-1}^{1}\frac{T_k\left(x\right)T_m\left(x\right)}{\sqrt{1-x^2}}dx\\ &= \frac{\pi}{2}c_{mn}.\end{aligned}$$

From this, we obtain that

$$c_{kn} = \frac{2}{\pi} \int_{-1}^{1} \sin\left(\left(n - \frac{1}{2}\right)\pi x\right) \frac{T_k(x)}{\sqrt{1-x^2}} dx.$$

Analogous to (4.46), let us assume

$$\sin\left(\left(n - \frac{1}{2}\right)\pi x\right) = \sum_{k=1}^{\infty} \alpha_{kn} \frac{T_k(x)}{\sqrt{1-x^2}}.$$

By orthogonality, it holds that

$$\alpha_{kn} = \frac{2}{\pi} \int_{-1}^{1} \sin\left(\left(n - \frac{1}{2}\right)\pi x\right) T_k(x)\, dx = \frac{2}{\pi} d_{kn}.$$

Then,

$$\sin\left(\left(n - \frac{1}{2}\right)\pi x\right) = \frac{2}{\pi} \sum_{k=1}^{\infty} d_{kn} \frac{T_k(x)}{\sqrt{1-x^2}}.$$

Besides, we know that

$$\begin{aligned}
\delta_{nm} &= \int_{-1}^{1} \sin\left(\left(n - \frac{1}{2}\right)\pi x\right) \sin\left(\left(m - \frac{1}{2}\right)\pi x\right) dx \\
&= \frac{2}{\pi} \int_{-1}^{1} \left(\sum_{k=1}^{\infty} c_{kn} T_k(x)\right) \left(\sum_{\hat{k}=1}^{\infty} d_{\hat{k}m} \frac{T_{\hat{k}}(x)}{\sqrt{1-x^2}}\right) dx \\
&= \frac{2}{\pi} \sum_{k,\hat{k} \geq 1} c_{kn} d_{\hat{k}m} \int_{-1}^{1} \frac{T_k(x)\, T_{\hat{k}}(x)}{\sqrt{1-x^2}} dx \\
&= \sum_{k=1}^{\infty} c_{kn} d_{km},
\end{aligned}$$

or, in matrix notation,

$$C^T D = I \implies D^T C = I.$$

Therefore, the proposition has been proven. □

By multiplying (4.48) by D^T, we get:

$$\lambda D^T E \vec{b} = \operatorname{diag}\left[\left(n - \frac{1}{2}\right)\pi\right]^3 \vec{b}, \tag{4.50}$$

but this system does not consider mass conservation law, so we need to impose condition (4.45).
Therefore, consider the system

$$\lambda \overline{(D^T E)} \vec{b} = \overline{\operatorname{diag}}\left[\left(n - \frac{1}{2}\right)\pi\right]^3 \vec{b},$$

with $\overline{(D^T E)}$ being the $D^T E$ matrix adding a zero first row and $\overline{\text{diag}}$ being the same diagonal matrix added coefficients $w_n, n \geq 1$ in the first row. The sytem in mention satisfies both (4.45) and (4.50), hence it is the system we are working with in order to get the eigenvalues.

Again, we solve this eigenvalue problem with MATLAB and we truncate it. Let us consider the problem for dimension $N = 50$, then the obtained eigenvalues λ_i for $i = 1:50$, and their associated eigenvectors a^i are the ones given in Table 2. Furthermore, notice that for a given i, $= b^i = [b_1, b_2, ..., b_{50}]^T$

	$\lambda_1 = 89.07560592$	$\lambda_2 = 447.185654330556$	$\lambda_3 = 1263.38586834285$	$\lambda_4 = 2725.0041811121$
b_1	-0.010916623	-0.007633327	-0.005956717	-0.004884362
b_2	-0.06120303	0.011168489	0.007733839	0.006275705
b_3	0.002288109	0.102746236	-0.011931031	-0.007800237
b_4	-0.000684191	-0.004210158	-0.139787439	0.012657817
b_5	0.00029081	0.00149753	0.005762789	0.173926497
b_6	-0.000147563	-0.000722119	-0.002236615	-0.007021789
b_7	8.37E-05	0.000402494	0.001158436	0.002870289
b_8	-5.14E-05	-0.000245425	-0.000684334	-0.001555826
b_9	3.34E-05	0.000159369	0.000437674	0.000955686
b_{10}	-2.27E-05	-0.000108494	-0.000295706	-0.000632051
b_{11}	1.60E-05	7.66E-05	0.00020814	0.000439552
b_{12}	-1.17E-05	-5.58E-05	-0.000151331	-0.000317342

Table 2: first eigenvalues and its associated eigenvectors for the symmetric pinned-end case. This data was obtained by using MATLAB and by letting $N = 50$.

4.6.3 Symmetric free-end boundary condition

$$S(x) = \sum_{n=1}^{\infty} \widetilde{b_n} \cos(n\pi x).$$

Replacing the condition in (4.36), we obtain:

$$\begin{aligned}
\lambda \sum_{n=1}^{\infty} \widetilde{b_n} \cos(n\pi x) &= -\frac{1}{\pi}\sqrt{1-x^2} \int_{-1}^{1^*} \frac{\sum_{n=1}^{\infty} (n\pi)^3 \widetilde{b_n} \sin(n\pi z)}{\sqrt{1-z^2}\,(x-z)} dz \\
&= -\sqrt{1-x^2} \sum_{n=1}^{\infty} (n\pi)^3 \widetilde{b_n} \frac{1}{\pi} \int_{-1}^{1^*} \frac{\sin(n\pi z)}{\sqrt{1-z^2}\,(x-z)} dz.
\end{aligned}$$

Let

$$\sin(n\pi x) = \sum_{k=1}^{\infty} c_{kn} T_k(x), \tag{4.51}$$

with

$$c_{kn} = \frac{2}{\pi} \int_{-1}^{1} \sin(n\pi x) \frac{T_k(x)}{\sqrt{1-x^2}} dx.$$

Thus,

$$\begin{aligned}
\lambda \sum_{n=1}^{\infty} \widetilde{b_n} \cos(n\pi x) &= -\sqrt{1-x^2} \sum_{n=1}^{\infty} (n\pi)^3 \widetilde{b_n} \frac{1}{\pi} \int_{-1}^{1^*} \frac{\sum_{k=1}^{\infty} c_{kn} T_k(z)}{\sqrt{1-z^2}\,(x-z)} dz \\
&= -\sqrt{1-x^2} \sum_{n,k\geq 1} (n\pi)^3 c_{kn} \widetilde{b_n} \frac{1}{\pi} \int_{-1}^{1^*} \frac{T_k(z)}{\sqrt{1-z^2}\,(x-z)} dz \\
&= \sqrt{1-x^2} \sum_{n,k\geq 1} (n\pi)^3 c_{kn} \widetilde{b_n} U_{k-1}(x).
\end{aligned} \tag{4.52}$$

Let

$$\frac{\cos(n\pi x)}{\sqrt{1-x^2}} = \sum_{r=1}^{\infty} e_{rn} U_{r-1}(x),$$

with

$$e_{rn} = \frac{2}{\pi} \int_{-1}^{1} \cos(n\pi x) U_{r-1}(x)\, dx.$$

Using this identity in (4.52), we get:

$$\lambda \sum_{n,r\geq 1} \widetilde{b_n} e_{rn} U_{r-1}(x) = \sum_{n,k\geq 1} (n\pi)^3 c_{kn} \widetilde{b_n} U_{k-1}(x).$$

Therefore,

$$\lambda \sum_{n\geq 1} \widetilde{b_n} e_{rn} = \sum_{n\geq 1} (n\pi)^3 c_{kn} \widetilde{b_n}, \quad \forall k, r \geq 1,$$

or, in matrix notation,

$$\lambda E \vec{\tilde{b}} = C \left(\operatorname{diag} (n\pi)^3 \right) \vec{\tilde{b}}. \tag{4.53}$$

Again, we can explicitly get the coefficients of matrices C and E.

Proposition 4.4. *If C is defined as above, and*

$$D = d_{kn} = \int_{-1}^{1} \sin(n\pi x) T_k(x)\, dx,$$

then

$$D^T = C^{-1}$$

Proof. We know that

$$\sin(n\pi x) = \sum_{k=1}^{\infty} c_{kn} T_k(x), \tag{4.54}$$

thus,

$$\begin{aligned}
\int_{-1}^{1} \sin(n\pi x) \frac{T_m(x)}{\sqrt{1-x^2}} dx &= \int_{-1}^{1} \sum_{k=1}^{\infty} c_{kn} \frac{T_k(x) T_m(x)}{\sqrt{1-x^2}} dx \\
&= \sum_{k=1}^{\infty} c_{kn} \int_{-1}^{1} \frac{T_k(x) T_m(x)}{\sqrt{1-x^2}} dx \\
&= \frac{\pi}{2} c_{mn}.
\end{aligned}$$

From this, we obtain that

$$c_{kn} = \frac{2}{\pi} \int_{-1}^{1} \sin(n\pi x) \frac{T_k(x)}{\sqrt{1-x^2}} dx.$$

Analogous to (4.51), assume

$$\sin(n\pi x) = \sum_{k=1}^{\infty} \alpha_{kn} \frac{T_k(x)}{\sqrt{1-x^2}}.$$

By orthogonality, it holds that

$$\alpha_{kn} = \frac{2}{\pi} \int_{-1}^{1} \sin(n\pi x) T_k(x)\, dx = \frac{2}{\pi} d_{kn}.$$

Then,

$$\sin(n\pi x) = \frac{2}{\pi} \sum_{k=1}^{\infty} d_{kn} \frac{T_k(x)}{\sqrt{1-x^2}}.$$

Besides, we know that

$$\begin{aligned}
\delta_{nm} &= \int_{-1}^{1} \sin(n\pi x)\sin(m\pi x)\,dx \\
&= \frac{2}{\pi}\int_{-1}^{1}\left[\sum_{k=1}^{\infty} c_{kn}T_k(x)\right]\left[\sum_{\hat{k}=1}^{\infty} d_{\hat{k}m}\frac{T_{\hat{k}}(x)}{\sqrt{1-x^2}}\right]dx \\
&= \frac{2}{\pi}\sum_{k,\hat{k}\geq 1} c_{kn}d_{\hat{k}m}\int_{-1}^{1}\frac{T_k(x)\,T_{\hat{k}}(x)}{\sqrt{1-x^2}}dx \\
&= \sum_{k=1}^{\infty} c_{kn}d_{km},
\end{aligned}$$

or, in matrix notation,

$$C^T D = I \implies D^T C = I.$$

Therefore, the proposition has been proven. □

By multiplying (4.53) by D^T, we finally get:

$$\lambda D^T E\vec{\tilde{b}} = \operatorname{diag}(n\pi)^3\,\vec{\tilde{b}}.$$

Again, we solve this eigenvalue problem with MATLAB and we truncate it. Let us consider the problem for dimension $N = 50$, then the obtained eigenvalues λ_i for $i = 1 : 50$, and their associated eigenvectors a^i are the ones given in Table 3. Furthermore, notice that for a given i, $\widetilde{b}^i = \left[\widetilde{b}_1, \widetilde{b}_2, ..., \widetilde{b}_{50}\right]^T$.

	$\lambda_1 = 34.31336564$	$\lambda_2 = 262.2693226$	$\lambda_3 = 870.0439734$	$\lambda_4 = 2043.680906$	$\lambda_5 = 3969.217186$
$\widetilde{b_1}$	1	-0.057140528	0.033362423	-0.023158677	0.017384073
$\widetilde{b_2}$	0.014533452	1	-0.05210861	0.031476691	-0.023031761
$\widetilde{b_3}$	-0.003466077	0.023930969	1	-0.049069802	0.029342457
$\widetilde{b_4}$	0.001283261	-0.007555277	0.028841721	1	-0.047207557
$\widetilde{b_5}$	-0.000591566	0.003389125	-0.010317288	0.031802324	1
$\widetilde{b_6}$	0.000312668	-0.001792466	0.005110054	-0.012201692	0.033772334
$\widetilde{b_7}$	-0.000181656	0.00104991	-0.002924192	0.006415269	-0.013549076
$\widetilde{b_8}$	0.000113157	-0.000660534	0.001825267	-0.003861134	0.007412068
$\widetilde{b_9}$	-7.44E-05	0.00043843	-0.001209859	0.002515492	-0.004619301
$\widetilde{b_{10}}$	5.10E-05	-0.00030347	0.000838748	-0.001729447	0.003102818
$\widetilde{b_{11}}$	-3.62E-05	0.000217289	-0.00060231	0.001237319	-0.002191345
$\widetilde{b_{12}}$	2.65E-05	-0.000159975	0.000444977	-0.000912981	0.001605036

Table 3: first eigenvalues and its associated eigenvectors for the symmetric free-end case. This data was obtained by using MATLAB and by letting $N = 50$.

From Figure 10, it seems the first eigenvalue is convergent as $N \to \infty$. In this figure, we see that λ_1 converges to 34.3133 as N grows.

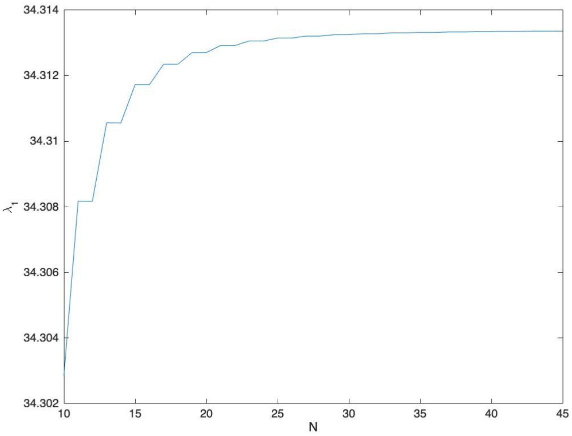
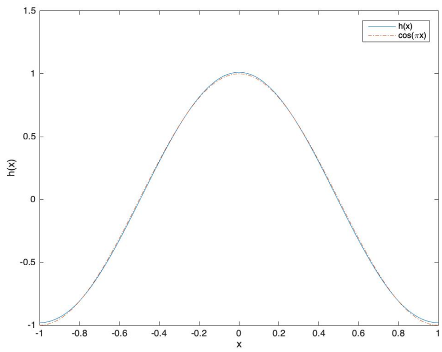
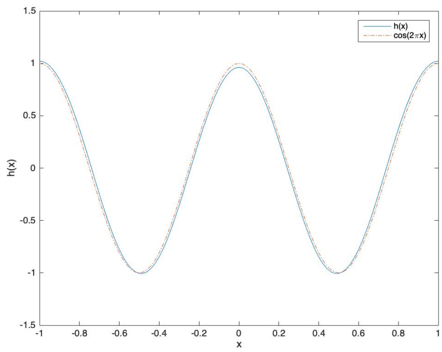


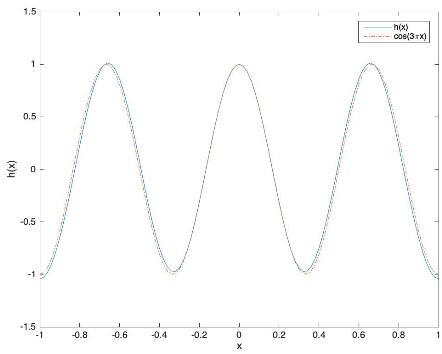
Figure 10: different values for λ_1 as dimension N grows.



(a) Eigenfunction associated to λ_1 in comparison with $\cos(\pi x)$.



(b) Eigenfunction associated to λ_2 in comparison with $\cos(2\pi x)$.



(c) Eigenfunction associated to λ_3 in comparison with $\cos(3\pi x)$.

Figure 11: the free surface $h(x)$ in presence of walls compared to the free surface in absence of walls (cosine frequencies).

We are able to plot the free surface for a fixed value of t. Let us consider $t = 0$, then $h(x, t) = h(x) = S(x)$. The free surface of the fluid in presence of walls differs from the case where no walls are considered, this can be seen in Figure 11.

4.6.4 Anti-symmetric free-end boundary condition

$$S(x) = \sum_{n=1}^{\infty} \widetilde{a_n} \sin\left(\left(n - \frac{1}{2}\right)\pi x\right).$$

Replacing the condition in (4.36), we obtain:

$$\begin{aligned}\lambda \sum_{n=1}^{\infty} \widetilde{a_n} \sin\left(\left(n - \frac{1}{2}\right)\pi x\right) &= \frac{1}{\pi}\sqrt{1-x^2} \int_{-1}^{1^*} \frac{\sum_{n=1}^{\infty}\left(\left(n-\frac{1}{2}\right)\pi\right)^3 \widetilde{a_n} \cos\left(\left(n-\frac{1}{2}\right)\pi z\right)}{\sqrt{1-z^2}\,(x-z)} dz \\ &= \sqrt{1-x^2} \sum_{n=1}^{\infty} \left(\left(n-\frac{1}{2}\right)\pi\right)^3 \widetilde{a_n} \frac{1}{\pi} \int_{-1}^{1^*} \frac{\cos\left(\left(n-\frac{1}{2}\right)\pi z\right)}{\sqrt{1-z^2}\,(x-z)} dz.\end{aligned}$$

Let

$$\cos\left(\left(n - \frac{1}{2}\right)\pi x\right) = \sum_{k=0}^{\infty} c_{kn} T_k(x), \tag{4.55}$$

with

$$c_{kn} = \begin{cases} \frac{1}{\pi}\int_{-1}^{1} \cos\left(\left(n-\frac{1}{2}\right)\pi x\right) \frac{1}{\sqrt{1-x^2}} dx, & k = 0, \\ \frac{2}{\pi}\int_{-1}^{1} \cos\left(\left(n-\frac{1}{2}\right)\pi x\right) \frac{T_k(x)}{\sqrt{1-x^2}} dx, & k \geq 1. \end{cases}$$

Thus,

$$\begin{aligned}\lambda \sum_{n=1}^{\infty} \widetilde{a_n} \sin\left(\left(n - \frac{1}{2}\right)\pi x\right) &= \sqrt{1-x^2} \sum_{n,k\geq 1} \left(\left(n-\frac{1}{2}\right)\pi\right)^3 c_{kn}\widetilde{a_n} \frac{1}{\pi} \int_{-1}^{1^*} \frac{T_k(z)}{\sqrt{1-z^2}\,(x-z)} dz \\ &= -\sqrt{1-x^2} \sum_{n,k\geq 1} \left(\left(n-\frac{1}{2}\right)\pi\right)^3 c_{rn}\widetilde{a_n} U_{k-1}(x).\end{aligned} \tag{4.56}$$

Besides, let

$$\frac{\sin\left(\left(n-\frac{1}{2}\right)\pi x\right)}{\sqrt{1-x^2}} = \sum_{r=1}^{\infty} e_{rn} U_{r-1}(x), \tag{4.57}$$

$$e_{rn} = \frac{2}{\pi}\int_{-1}^{1} \sin\left(\left(n-\frac{1}{2}\right)\pi x\right) U_{r-1}(x)\, dx.$$

Replacing it into (4.38), it becomes

$$\lambda \sum_{n,r\geq 1} e_{rn}\widetilde{a_n} U_{r-1}(x) = -\sum_{n,k\geq 1} \left(\left(n-\frac{1}{2}\right)\pi\right)^3 c_{kn}\widetilde{a_n} U_{k-1}(x),$$

$$\lambda \sum_{n\geq 1} e_{rn}\widetilde{a_n} = -\sum_{n\geq 1} \left(\left(n-\frac{1}{2}\right)\pi\right)^3 c_{kn}\widetilde{a_n}, \quad \forall k, r \geq 1.$$

In matrix notation:

$$\lambda E\vec{a} = -C\left(\operatorname{diag}\left[\left(n-\frac{1}{2}\right)\pi\right]^3\right)\vec{a}. \tag{4.58}$$

Once again, we can explicitly get the coefficients of matrices C and E.

Proposition 4.5. *If C is defined as above, and*

$$D = d_{kn} = \int_{-1}^{1} \cos\left(\left(n-\frac{1}{2}\right)\pi x\right) T_k(x)\, dx, \quad \forall k \geq 0, \forall n \geq 1, \tag{4.59}$$

then $D^T = C^{-1}$.

Proof. We know that

$$\cos\left(\left(n-\frac{1}{2}\right)\pi x\right)=\sum_{k=0}^{\infty}c_{kn}T_k\left(x\right),$$

thus,

$$\begin{aligned}\int_{-1}^{1}\cos\left(\left(n-\frac{1}{2}\right)\pi x\right)\frac{T_m\left(x\right)}{\sqrt{1-x^2}}dx &= \int_{-1}^{1}\sum_{k=0}^{\infty}c_{kn}\frac{T_k\left(x\right)T_m\left(x\right)}{\sqrt{1-x^2}}dx\\ &= \sum_{k=0}^{\infty}c_{kn}\int_{-1}^{1}\frac{T_k\left(x\right)T_m\left(x\right)}{\sqrt{1-x^2}}dx\\ &= \begin{cases}\pi c_{mn}, & m=n=0,\\ \frac{\pi}{2}c_{mn}, & m=n\neq 0.\end{cases}\end{aligned}$$

From where, we obtain that

$$c_{kn}=\begin{cases}\frac{1}{\pi}\int_{-1}^{1}\cos\left(\left(n-\frac{1}{2}\right)\pi x\right)\frac{1}{\sqrt{1-x^2}}dx, & k=0,\\ \frac{2}{\pi}\int_{-1}^{1}\cos\left(\left(n-\frac{1}{2}\right)\pi x\right)\frac{T_k(x)}{\sqrt{1-x^2}}dx, & k\geq 1.\end{cases}$$

Analogous to (4.55), we can assume that

$$\cos\left(\left(n-\frac{1}{2}\right)\pi x\right)=\sum_{k=0}^{\infty}\alpha_{kn}\frac{T_k\left(x\right)}{\sqrt{1-x^2}}.$$

By orthogonality,

$$\alpha_{kn}=\begin{cases}\frac{1}{\pi}\int_{-1}^{1}\cos\left(\left(n-\frac{1}{2}\right)\pi x\right)dx=\frac{1}{\pi}d_{kn}, & k=0,\\ \frac{2}{\pi}\int_{-1}^{1}\cos\left(\left(n-\frac{1}{2}\right)\pi x\right)T_k\left(x\right)dx=\frac{2}{\pi}d_{kn}, & k\geq 1,\end{cases}$$

then,

$$\cos\left(\left(n-\frac{1}{2}\right)\pi x\right)=\frac{1}{\pi}\frac{d_{0n}}{\sqrt{1-x^2}}+\frac{2}{\pi}\sum_{k=1}^{\infty}d_{kn}\frac{T_k\left(x\right)}{\sqrt{1-x^2}}.$$

We also know that

$$\begin{aligned}\delta_{nm} &= \int_{-1}^{1}\cos\left(\left(n-\frac{1}{2}\right)\pi x\right)\cos\left(\left(m-\frac{1}{2}\right)\pi x\right)dx\\ &= \int_{-1}^{1}\left(\sum_{k=0}^{\infty}c_{kn}T_k\left(x\right)\right)\left(\frac{1}{\pi}\frac{d_{0n}}{\sqrt{1-x^2}}+\frac{2}{\pi}\sum_{\hat{k}=1}^{\infty}d_{\hat{k}n}\frac{T_{\hat{k}}\left(x\right)}{\sqrt{1-x^2}}\right)dx\\ &= \frac{1}{\pi}c_{0n}d_{0n}\int_{-1}^{1}\frac{1}{\sqrt{1-x^2}}dx+\frac{2}{\pi}\sum_{k,\hat{k}\geq 1}c_{kn}d_{\hat{k}m}\int_{-1}^{1}\frac{T_k\left(x\right)T_{\hat{k}}\left(x\right)}{\sqrt{1-x^2}}dx\\ &= c_{0n}d_{0n}+\sum_{k\geq 1}c_{kn}d_{kn}\\ &= \sum_{k\geq 0}c_{kn}d_{kn},\end{aligned}$$

or, in matrix notation

$$C^TD=I \implies D^TC=I,$$

therefore, the proposition has been proved. □

We already know that (4.58) holds for $k, n \geq 1$, while (4.59) holds for $k \geq 0$ and $n \geq 1$. Therefore, before multiplying (4.40) by D^T, we need to add a zero first-row to E and a first-row of coefficients c_{0n} to C, for $n \geq 1$. These completed matrices are going to be denoted as $\overline{E}$ and $\overline{C}$, respectively.

The system that is going to be multiplied by D^T is the following:

$$\lambda \overline{E}\vec{a} = -\overline{C}\ \mathrm{diag}\,(n\pi)^3\,, \tag{4.60}$$

and this is valid as long as the following condition holds

$$\begin{aligned}\sum_{n\geq 1} c_{0n}\left(\left(n-\frac{1}{2}\right)\pi\right)^3 a_n &= \frac{1}{\pi}\int_{-1}^{1}\frac{\sum_{n\geq 1} a_n\left(\left(n-\frac{1}{2}\right)\pi\right)^3\cos\left(\left(n-\frac{1}{2}\right)\pi x\right)}{\sqrt{1-x^2}}dx\\ &= -\frac{1}{\pi}\int_{-1}^{1}\frac{S'''(x)}{\sqrt{1-x^2}}dx\\ &= 0.\end{aligned}$$

Finally, multiplying (4.60) by D^T, we have:

$$\lambda D^T\overline{E}\vec{a} = -\ \mathrm{diag}\left[\left(n-\frac{1}{2}\right)\pi\right]^3\vec{a}.$$

As before, we solve this eigenvalue problem with MATLAB and we truncate it. Let us consider the problem for dimension $N = 50$, then the obtained eigenvalues λ_i for $i = 1 : 50$, and their associated eigenvectors a^i are the ones given in Table 4. Furthermore, notice that for a given i, $\widetilde{a}^i = [\widetilde{a}_1, \widetilde{a}_2, ..., \widetilde{a}_{50}]^T$.

	$\lambda_1 = 5.005739892$	$\lambda_2 = 113.9431184$	$\lambda_3 = 509.724646$	$\lambda_4 = 1378.354237$	$\lambda_5 = 2905.865588$
$\widetilde{a_1}$	-1	-0.078454478	0.04092459	-0.026253864	0.018674219
$\widetilde{a_2}$	-0.008835174	1	-0.062644329	0.037292738	-0.026643463
$\widetilde{a_3}$	0.001473296	0.023195705	1	-0.055374691	0.033155007
$\widetilde{a_4}$	-0.000448371	-0.00645288	0.029497567	1	-0.051572518
$\widetilde{a_5}$	0.000182643	0.002649954	-0.009897411	0.032859789	1
$\widetilde{a_6}$	-8.86E-05	-0.001313684	0.004665272	-0.012129809	0.034930139
$\widetilde{a_7}$	4.84E-05	0.000732449	-0.002566623	0.006179825	-0.013655862
$\widetilde{a_8}$	-2.87E-05	-0.000443288	0.001551749	-0.003622984	0.007309918
$\widetilde{a_9}$	1.82E-05	0.000285177	-0.001001827	0.002308539	-0.004471194
$\widetilde{a_{10}}$	-1.21E-05	-0.000192374	0.00067937	-0.001557381	0.002954909
$\widetilde{a_{11}}$	8.36E-06	0.0001348	-0.000478796	0.001096159	-0.002057393
$\widetilde{a_{12}}$	-5.97E-06	-9.75E-05	0.000348134	-0.000797576	0.001488515

Table 4: first eigenvalues and its associated eigenvectors for the anti-symmetric free-end case. This data was obtained by using MATLAB and by letting $N = 50$.

From Figure 12, it seems the first eigenvalue is convergent as $N \to \infty$. In this figure, we see that λ_1 converges to 5.0057 as N grows.

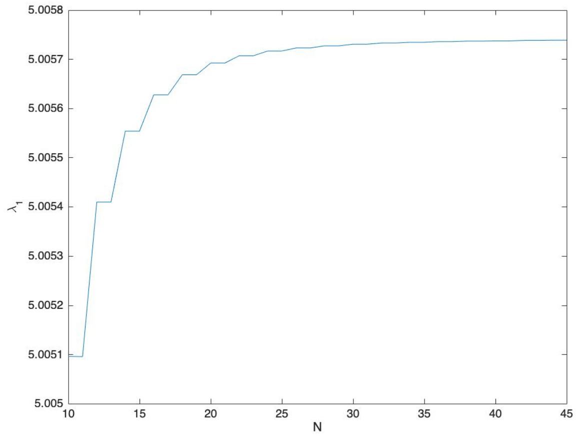
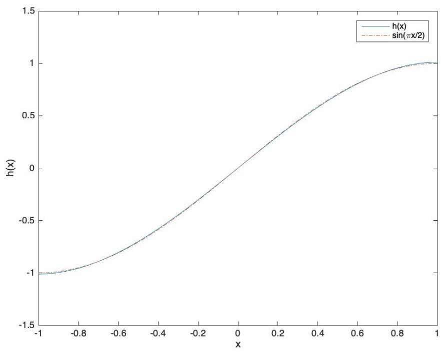
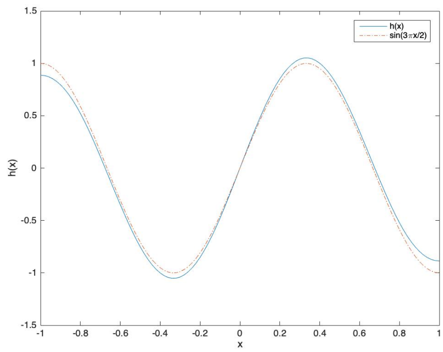


Figure 12: different values for λ_1 as dimension N grows.

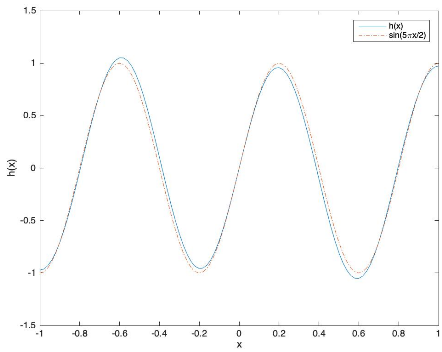
Once again, we are able to plot the free surface for a fixed value of t. Let us consider $t = 0$, then $h\left(x,t\right) = h\left(x\right) = S\left(x\right)$. The free surface of the fluid in presence of walls differs from the case where no walls are considered, this can be seen in Figure 13.



(a) Eigenfunction associated to λ_1 in comparison with $\sin\left(\frac{\pi x}{2}\right)$.



(b) Eigenfunction associated to λ_2 in comparison with $\sin\left(\frac{3\pi x}{2}\right)$.



(c) Eigenfunction associated to λ_3 in comparison with $\sin\left(\frac{5\pi x}{2}\right)$.

Figure 13: the free surface $h\left(x\right)$ in presence of walls compared to the free surface in absence of walls (sine frequencies).

4.7 Conformal mapping

In this section we will try to solve system (4.28) − (4.32) for any arbitrary geometry of the container. The technique consists on making a change of variable such that the geometry in mention is transformed into the lower half-plane, where the problem has already been solved. The main idea is to get an expression for the normal derivative in terms of the conformal map. This technique is widely described in [23].

Let $\psi : D \subseteq \mathbb{C} \to \mathbb{R}$ be defined as follows:

$$\psi(x, y) = \psi(z).$$

Then, for $w = x' + iy' \in \widetilde{D}$, $\widetilde{\psi} : \widetilde{D} \subseteq \mathbb{C} \to \mathbb{R}$ is defined as:

$$\widetilde{\psi}(w) := \psi\left(f^{-1}(w)\right) = \psi\left(x(x', y'), y(x', y')\right),$$

where $z = x + iy \in D$, f is injective on $D \cup \partial D$ and it is holomorphic.

By definition:

$$\psi(x, y) = \widetilde{\psi}\left(x'(x, y), y'(x, y)\right),$$

where

$$w = f(z), \tag{4.61}$$

is our conformal mapping.

Let C be a curve in the z-plane written as $z = z(t)$, we have that the normal derivative of ψ is defined in [24] as follows: since

$$\begin{aligned}
\operatorname{Im}\left[\left(\frac{\partial\psi}{\partial x} - i\frac{\partial\psi}{\partial y}\right)\frac{dz}{dt}\right] &= \operatorname{Im}\left[\frac{\partial\psi}{\partial x}\left(\frac{dx}{dt} + i\frac{dy}{dt}\right) - i\frac{\partial\psi}{\partial y}\left(\frac{dx}{dt} + i\frac{dy}{dt}\right)\right] \\
&= \operatorname{Im}\left(\frac{\partial\psi}{\partial t} + i\frac{\partial\psi}{\partial x}\frac{dy}{dt} + \frac{\partial\psi}{\partial t} - i\frac{\partial\psi}{\partial y}\frac{dx}{dt}\right) \\
&= \frac{\partial\psi}{\partial t}\frac{dy}{dt} - \frac{\partial\psi}{\partial y}\frac{dx}{dt} \\
&= \left|\frac{dz}{dt}\right|\frac{\partial\psi}{\partial n},
\end{aligned}$$

then,

$$\frac{\partial\psi}{\partial n} = \frac{1}{\left|\frac{dz}{dt}\right|}\operatorname{Im}\left[\left(\frac{\partial\psi}{\partial x} - i\frac{\partial\psi}{\partial y}\right)\frac{dz}{dt}\right].$$

Under (4.61), C is mapped into $C^* : w(t) = f\left(z(t)\right)$. By applying the chain rule, if $\widetilde{\psi}(x', y') = \psi\left(x(x', y'), y(x', y')\right)$,

$$\begin{aligned}
\left|\frac{dw}{dt}\right|\frac{\partial\widetilde{\psi}}{\partial\widetilde{n}} &= \operatorname{Im}\left[\left(\frac{\partial\widetilde{\psi}}{\partial x'} - i\frac{\partial\widetilde{\psi}}{\partial y'}\right)\frac{dw}{dt}\right] \\
&= \operatorname{Im}\left[\left\{\frac{\partial\psi}{\partial x}\left(\frac{\partial x}{\partial x'} - i\frac{\partial x}{\partial y'}\right) + \frac{\partial\psi}{\partial y}\left(\frac{\partial y}{\partial y'} - i\frac{\partial y}{\partial x'}\right)\right\}\frac{dw}{dt}\right] \\
&= \operatorname{Im}\left[\left(\frac{\partial\psi}{\partial x} - i\frac{\partial\psi}{\partial y}\right)\frac{dz}{dw}\frac{dw}{dt}\right] \\
&= \operatorname{Im}\left[\left(\frac{\partial\psi}{\partial x} - i\frac{\partial\psi}{\partial y}\right)\frac{dz}{dt}\right] \\
&= \left|\frac{dz}{dt}\right|\frac{\partial\psi}{\partial n}.
\end{aligned}$$

Then,

$$\frac{\partial \widetilde{\psi}}{\partial \widetilde{n}} = \left| \frac{dz}{dw} \right| \frac{\partial \psi}{\partial n}$$
$$= \frac{1}{|f'(z)|} \frac{\partial \psi}{\partial n}. \tag{4.62}$$

4.8 Solution of the integro-differential equation in a container with vertical walls

As before, let us consider the case of a container with a small hole, around the origin, of size 2 at the center of the top wall, but this time with vertical walls at $x = \pm(b+1)$, for some positive b. The idea is to solve the problem by using a conformal map. In fact, the conformal map that transforms the geometry in mention into the lower half-plane is

$$f(z) = \sin\left[\frac{\pi z}{2(b+1)}\right]. \tag{4.63}$$

The domain in mention can be denoted as

$$D = \left\{ (x,y) : |x| < b+1, y < \left\{ \begin{array}{ll} 0, & -(1+b) < x < -1; \\ h(x,t), & |x| < 1; \\ 0, & 1 < x < b+1, \end{array} \right. \right\},$$

and its boundary is defined by the free surface and the walls. The new container geometry is shown in Figure 14.

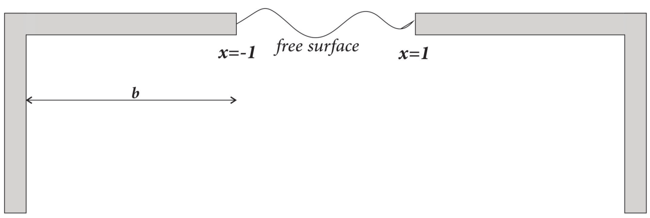


Figure 14: container with vertical walls.

Again, ϕ is the velocity potential satisfying:

$$\Delta\phi = 0 \text{ in } D, \tag{4.64}$$
$$\frac{\partial \phi}{\partial n} = 0 \text{ on the walls.} \tag{4.65}$$

The mapping defined in (4.63) transforms D into $\widetilde{D}$, defined as:

$$\widetilde{D} = \left\{ (x', y') : |x'| < \infty, y' < \left\{ \begin{array}{ll} 0, & x' < -\sin\left(\frac{\pi}{2(1+b)}\right); \\ f\left(h(x,t)\right) = H(x',t), & |x'| < \sin\left(\frac{\pi}{2(1+b)}\right); \\ 0, & x' > \sin\left(\frac{\pi}{2(1+b)}\right), \end{array} \right. \right\}$$

where x' and y' are coordinates in the $w-$plane. This is a variation of our first domain (lower half-plane case).

By means of the conformal mapping, our new problem is given by

$$\Delta\widetilde{\phi}(x', y') = 0 \text{ in } \widetilde{D}, \tag{4.66}$$
$$\frac{\partial \widetilde{\phi}}{\partial \widetilde{n}}(x', y') = 0 \text{ on the corresponding walls,} \tag{4.67}$$

where $\widetilde{n}$ is the normal vector in the $w-$plane.

By (4.62), we know that

$$\frac{\partial\widetilde{\phi}}{\partial\widetilde{n}} = \frac{2\left(b+1\right)}{\pi\cos\left[\frac{\pi x'}{2(b+1)}\right]}\frac{\partial\phi}{\partial n}.$$

Analogous to (4.34), we get

$$\left.\frac{\partial\widetilde{\phi}}{\partial x'}\right|_{y'=0} = -\frac{1}{\pi}\int_{-\sin\left[\frac{\pi}{2(b+1)}\right]}^{\sin\left[\frac{\pi}{2(b+1)}\right]^*}\frac{\left(\partial\widetilde{\phi}/\partial\widetilde{n}\right)\left(z,0,t\right)}{x'-z}dz. \tag{4.68}$$

By (4.62), we obtain

$$\frac{1}{\left|f'\left(z\right)\right|}\left.\frac{\partial\phi}{\partial x}\right|_{y=0} = -\frac{1}{\pi\left|f'\left(z\right)\right|}\int_{-\sin\left[\frac{\pi}{2(b+1)}\right]}^{\sin\left[\frac{\pi}{2(b+1)}\right]^*}\frac{\left(\partial\phi/\partial n\right)\left(z,0,t\right)}{x'-z}dz \tag{4.69}$$

$$\left.\frac{\partial\phi}{\partial x}\right|_{y=0} = -\frac{1}{\pi}\int_{-\sin\left[\frac{\pi}{2(b+1)}\right]}^{\sin\left[\frac{\pi}{2(b+1)}\right]^*}\frac{\left(\partial\phi/\partial n\right)\left(z,0,t\right)}{x'-z}dz. \tag{4.70}$$

Let $x' = \sin\left[\frac{\pi x}{2(b+1)}\right]$, then

$$\left.\frac{\partial\phi}{\partial x}\right|_{y=0} = -\frac{1}{2\left(b+1\right)}\cos\left[\frac{\pi x}{2\left(b+1\right)}\right]\int_{-1}^{1^*}\frac{\left(\partial\phi/\partial n\right)\left(z,0,t\right)}{\sin\left[\frac{\pi x}{2(b+1)}\right]-\sin\left[\frac{\pi z}{2(b+1)}\right]}dz. \tag{4.71}$$

By derivating both sides with respect to t, we finally get

$$\frac{\sigma}{\rho}h_{xxx} = -\frac{1}{2\left(b+1\right)}\cos\left[\frac{\pi x}{2\left(b+1\right)}\right]\int_{-1}^{1^*}\frac{h_{tt}\left(z,t\right)}{\sin\left[\frac{\pi x}{2(b+1)}\right]-\sin\left[\frac{\pi z}{2(b+1)}\right]}dz. \tag{4.72}$$

If once again we consider $h\left(x,t\right) = A\left(t\right)S\left(x\right)$ and we apply the separation of variables method, we obtain

$$A''\left(t\right)+\lambda A\left(t\right) = 0,$$

already solved in last section, and

$$\frac{\sigma}{\rho}S'''\left(x\right) = \frac{\lambda}{2\left(b+1\right)}\cos\left[\frac{\pi x}{2\left(b+1\right)}\right]\int_{-1}^{1^*}\frac{S\left(z\right)}{\sin\left[\frac{\pi x}{2(b+1)}\right]-\sin\left[\frac{\pi z}{2(b+1)}\right]}dz.$$

We cannot proceed as before since the weight inside the integral does not allow us to work with Tchebyshev polynomials anymore. Then, let's study the problem by cases. We will consider the following two cases: $(i)\,b\to\infty$ and $(ii)\,b=0$.

4.8.1 First case: $b\to\infty$

In this case we know that

$$\lim_{b\to\infty}\cos\left(\frac{\pi}{2\left(b+1\right)}\right) = \cos\left(0\right) = 1.$$

Let us consider the change of variable $u = \frac{\pi x}{2(b+1)}$, then

$$\begin{aligned}\lim_{b\to\infty}\frac{\sin\left(\frac{\pi}{2(b+1)}\right)}{\frac{\pi}{2(b+1)}} &= \lim_{u\to0}\frac{\sin\left(u\right)}{u}\\ &= 1,\end{aligned}$$

which means that

$$\lim_{b\to\infty} \sin\left(\frac{\pi}{2(b+1)}\right) = \frac{\pi}{2(b+1)}.$$

Plugging these results into (4.72) and assuming that $\sigma/\rho = 1$, we get

$$h_{xxx} = -\frac{1}{\pi}\int_{-1}^{1^*} \frac{h_{tt}(z,t)}{x-z}dz,$$

which is the same as (4.35), which was already studied.

4.8.2 Second case: $b = 0$

In this case we consider a container with vertical walls at $x = \pm 1$. Given the fact that we already know the solution for the temporary part of $h(x,t)$, we can assume that $h(x,t) = e^{i\omega t}S(x)$. Thus, (4.72) becomes

$$\begin{aligned} \frac{\sigma}{\rho}e^{i\omega t}S'''(x) &= \frac{\omega^2}{2}e^{i\omega t}\cos\left(\frac{\pi x}{2}\right)\int_{-1}^{1^*}\frac{S(z)}{\sin\left(\frac{\pi x}{2}\right) - sin\left(\frac{\pi z}{2}\right)}dz \\ \frac{\sigma}{\rho}S'''(x) &= \frac{\omega^2}{2}\cos\left(\frac{\pi x}{2}\right)\int_{-1}^{1^*}\frac{S(z)}{\sin\left(\frac{\pi x}{2}\right) - sin\left(\frac{\pi z}{2}\right)}dz. \end{aligned} \tag{4.73}$$

Let's first consider the anti-symmetric pinned-end case. In this case,

$$S(x) = \sum_n a_n \sin(n\pi x).$$

Let

$$\sin(n\pi x) = \sum_r q_{rn} T_{2r-1}\left(\sin\left(\frac{\pi x}{2}\right)\right),$$

where, by orthogonality,

$$q_{rn} = \int_{-1}^{1} \sin(n\pi x)\, T_{2r-1}\left(\sin\left(\frac{\pi x}{2}\right)\right)dx.$$

By using the identity

$$\frac{1}{\pi}\int_{-1}^{1^*}\frac{T_z(z)}{\sqrt{1-z^2}(x-z)}dz = -U_{r-1}(x), \tag{4.74}$$

we get

$$\begin{aligned} \int_{-1}^{1^*}\frac{T_{2r-1}\left(\sin\left(\frac{\pi z}{2}\right)\right)}{\sin\left(\frac{\pi x}{2}\right) - \sin\left(\frac{\pi z}{2}\right)}dz &= -2U_{2r-2}\left(\sin\left(\frac{\pi x}{2}\right)\right) \\ &= 2(-1)^r\frac{\cos\left((r-1/2)\pi x\right)}{\cos(\pi x/2)}. \end{aligned} \tag{4.75}$$

By using the anti-symmetric pinned-end conditon and (4.75), the right-hand side of (4.73) becomes

$$\begin{aligned} \frac{\omega^2}{2}\cos\left(\frac{\pi x}{2}\right)\int_{-1}^{1^*}\frac{S(z)}{\sin\left(\frac{\pi x}{2}\right) - sin\left(\frac{\pi z}{2}\right)}dz &= \frac{\omega^2}{2}\sum_{n,r} a_n q_{rn}\cos\left(\frac{\pi x}{2}\right)\int_{-1}^{1^*}\frac{T_{2r-1}\left(\sin\left(\frac{\pi x}{2}\right)\right)}{\sin\left(\frac{\pi x}{2}\right) - sin\left(\frac{\pi z}{2}\right)}dz \\ &= \omega^2\sum_{n,r} a_n q_{rn}(-1)^r\cos\left(\left(r-\frac{1}{2}\right)\pi x\right). \end{aligned}$$

In the same way, the left-hand side of (4.73) becomes

$$\frac{\sigma}{\rho}S'''(x) = \frac{\sigma}{\rho}\sum_{n,r}(n\pi)^3 a_n (-1)^r p_{rn}\cos\left(\left(r-\frac{1}{2}\right)\pi x\right),$$

where,

$$p_{rn} = (-1)^{r+1} \int_{-1}^{1} \cos(n\pi x) \cos\left(\left(r - \frac{1}{2}\right)\pi x\right) dx.$$

Finally, if we assume $\sigma/\rho = 1$, then (4.73) is now of the form

$$\sum_{n,r} (n\pi)^3 a_n (-1)^r p_{rn} \cos\left(\left(r - \frac{1}{2}\right)\pi x\right) = \omega^2 \sum_{n,r} a_n q_{rn} (-1)^r \cos\left(\left(r - \frac{1}{2}\right)\right),$$

or, given $r \geq 1$,

$$\sum_{n} (n\pi)^3 (-1)^r p_{rn} a_n = \omega^2 \sum_{n} (-1)^r q_{rn} a_n. \tag{4.76}$$

Now, let us consider the symmetric pinned-end case, where

$$S(x) = \sum_{n} b_n \cos\left(\left(n - \frac{1}{2}\right)\pi x\right).$$

We need to impose the mass conservation condition, i.e., $\int_{-1}^{1} h(x,t) = 0$, which leads to (4.43). The mass conservation condition is, again,

$$\sum_{n=1}^{\infty} \omega_n b_n = 0 \tag{4.77}$$

where,

$$\omega_n = (-1)^{n+1} \frac{2}{\left(n - \frac{1}{2}\right)\pi}. \tag{4.78}$$

Let

$$\cos\left(\left((n - \frac{1}{2}\right)\pi x\right) = \sum_{r} t_{rn} \cos(r\pi x),$$

and

$$\sin\left(\left((n - \frac{1}{2}\right)\pi x\right) = \sum_{r} s_{rn} \sin(r\pi x),$$

where

$$t_{rn} = \int_{-1}^{1} \cos\left(\left((n - \frac{1}{2}\right)\pi x\right) \cos(r\pi x)\, dx,$$

and

$$s_{rn} = \int_{-1}^{1} \sin\left(\left((n - \frac{1}{2}\right)\pi x\right) \sin(r\pi x)\, dx.$$

Replacing these expansions into (4.73) and by letting $\sigma/\rho = 1$, we get

$$\sum_{n,r} \left(\left(n - \frac{1}{2}\right)\pi\right)^3 s_{rn} b_n \sin(r\pi x) = \frac{\omega^2}{2} \sum_{n,r} t_{rn} b_n \cos\left(\frac{\pi x}{2}\right) \int_{-1}^{1^*} \frac{\cos(r\pi z)}{\sin\left(\frac{\pi x}{2}\right) - \sin\left(\frac{\pi z}{2}\right)} dz. \tag{4.79}$$

At this moment, identity (4.74) and the following identities are useful

$$T_n(\cos\theta) = \cos(n\theta), \tag{4.80}$$

$$U_n(\cos\theta) = \frac{\sin((n+1)\theta)}{\sin\theta}. \tag{4.81}$$

Therefore, (4.79) becomes

$$\sum_{n,r}\left(\left(n-\frac{1}{2}\right)\pi\right)^3 s_{rn}b_n\sin(r\pi x)=\frac{\omega^2}{2}\sum_{n,r}t_{rn}b_n\cos\left(\frac{\pi x}{2}\right)\int_{-1}^{1*}\frac{T_{2r}\left(\cos(\pi z/2)\right)}{\sin\left(\frac{\pi x}{2}\right)-\sin\left(\frac{\pi z}{2}\right)}dz$$
$$=\omega^2\sum_{n,r}t_{rn}b_n\cos\left(\frac{\pi x}{2}\right)(-1)^{r+1}U_{2r-1}\left(\sin\left(\frac{\pi x}{2}\right)\right).$$

For the free cases with a contact angle of $\pi/2$, the problem has a classical solution (see [25]). The eigenfunctions for this solution are $\cos(n\pi x)$ and $\sin\left(\left(n-1/2\right)\pi x\right)$ and the frequencies are given by $\omega_n^2=\left(\sigma/\rho\right)(n\pi)^3$.

Let's first consider the symmetric pinned-end case, i.e.,

$$h(x,t)=e^{i\omega t}\cos(n\pi x),$$

Thus, the right-hand side of (4.73) becomes

$$\frac{\omega^2}{2}\cos\left(\frac{\pi x}{2}\right)\int_{-1}^{1^*}\frac{S(z)}{\sin\left(\frac{\pi x}{2}\right)-\sin\left(\frac{\pi z}{2}\right)}dz=\frac{\omega^2}{2}\cos\left(\frac{\pi x}{2}\right)\int_{-1}^{1^*}\frac{\cos(n\pi z)}{\sin\left(\frac{\pi x}{2}\right)-\sin\left(\frac{\pi z}{2}\right)}dz$$
$$=\frac{\omega^2}{2}\cos\left(\frac{\pi x}{2}\right)\int_{-1}^{1^*}\frac{T_{2n}\left(\cos(\pi z/2)\right)}{\sin\left(\frac{\pi x}{2}\right)-\sin\left(\frac{\pi z}{2}\right)}dz$$
$$=\frac{\omega^2}{2}\cos\left(\frac{\pi x}{2}\right)(-1)^{r+1}U_{2n-1}\left(\sin\left(\frac{\pi x}{2}\right)\right).$$

While, the left-hand side of (4.73) becomes

$$\frac{\sigma}{\rho}S'''(x)=\frac{\sigma}{\rho}(n\pi)^3\sin(n\pi x).$$

A similar analysis can be done with the anti-symmetric free case, where $h(x,t)=e^{i\omega t}\sin\left(\left(n-1/2\right)\pi x\right)$.

4.9 Solution of the integro-differential equation in a rounded container

In this section, we want to solve our problem for the following domain:

$$D=\{(x,y):|x|<1,-\sqrt{1-x^2}<y<h(x,t)\},\tag{4.82}$$

which is represented in Figure (15),

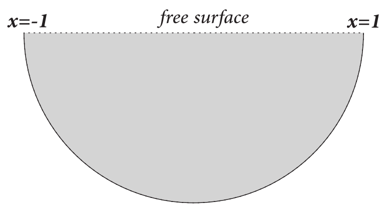


Figure 15: rounded container geometry.

The map to transform the half unit disk into the half-plane is given by

$$f(z)=\frac{1}{J(z)},\tag{4.83}$$

where $J(z)$ is known as the Joukowski map and it is given by

$$J(z)=\frac{1}{2}\left(z+\frac{1}{z}\right).\tag{4.84}$$

Let's see that $f(z)$ maps D into the lower half-plane. For $z = e^{i\theta}$, such that $\pi < \theta < 2\pi$, we have that

$$\begin{aligned}J(z) &= \frac{1}{2}\left(e^{i\theta} + \frac{1}{e^{i\theta}}\right)\\ &= \frac{1}{2}\left(e^{i\theta} + e^{-i\theta}\right)\\ &= \frac{1}{2}\left(\cos(\theta) + i\sin(\theta) + \cos(\theta) - i\sin(\theta)\right)\\ &= \cos(\theta),\end{aligned}$$

thus,

$$w = f(z) = \sec(\theta), \text{ for } z = e^{i\theta} \text{ such that } \pi < \theta < 2\pi. \tag{4.85}$$

As θ varies from π to 2π, then $w \in (-\infty, -1) \cup (1, \infty)$. Besides, $z = x + iy$ such that $x \in [-1, 1]$ and $y = 0$ is mapped into $w = x' + iy'$ such that $x' = [-1, 1]$ and $y' = 0$.

Since f is conformal and injective, it follows that $f(D)$ is either the upper or the lower half-plane (see [26]). In order to determine which half is the image of D, let's check the image of a point of the domain. If we take $z = -0.5i \in D$, then

$$\begin{aligned}f\left(-\frac{i}{2}\right) &= \frac{1}{\frac{1}{2}\left(-\frac{i}{2} + 2i\right)}\\ &= \frac{1}{\frac{1}{2}\left(\frac{3}{2}i\right)}\\ &= \frac{4}{3i}\\ &= -\frac{4}{3}i.\end{aligned}$$

Since $f(-0.5i)$ is on the lower half-plane, therefore $D' = f(D)$ coincides with the lower half-plane.

Analogous to (4.34) and using (4.62), we get

$$\begin{aligned}\left.\frac{\partial\widetilde{\phi}}{\partial\widetilde{x}'}\right|_{y'=0} &= -\frac{1}{\pi}\int_{-1}^{1*} \frac{\partial\widetilde{\phi}/\partial\widetilde{n}(z, 0, t)}{x' - z}dz\\ &= -\frac{1}{\pi}\int_{-1}^{1*} \frac{\partial\phi/\partial n(z, 0, t)}{|f'(z)|(x' - z)}dz.\end{aligned}$$

Now, proceed as before and take the derivative with respect to t in both sides. Let $f(x', t) = A(t)S(x')$. Thus,

$$A(t)S'''(x') = -\frac{1}{\pi}\int_{-1}^{1*} \frac{A''(t)S(z)}{|f'(z)|(x' - z)}dz. \tag{4.86}$$

By using separation of variables, we obtain

$$A''(t) + \lambda A(t) = 0, \tag{4.87}$$

$$S'''(x') = \frac{\lambda}{\pi}\int_{-1}^{1*} \frac{S(z)}{|f'(z)|(x' - z)}dz. \tag{4.88}$$

The first ordinary differential equation was already solved in last section, thus let us focus on (4.88). By inverting, the equation becomes

$$\frac{\lambda S(x')}{|f'(x')|} = -\frac{1}{\pi}\sqrt{1 - x'^2}\int_{-1}^{1^*} \frac{S'''(z)}{\sqrt{1 - z^2}(x' - z)}dz.$$

Namely,

$$\lambda S(x') = -\frac{1}{\pi}\sqrt{1 - x'^2}|f'(x)|\int_{-1}^{1^*} \frac{S'''(z)}{\sqrt{1 - z^2}(x' - z)}dz. \tag{4.89}$$

We want an explicit expression for $|f'(z)|$. If $x \in [-1,1]$ and $y = 0$, then $x' \in [-1,1]$ and $y' = 0$, which means that

$$x' = f(x) = \frac{2}{x + \frac{1}{x}}, \tag{4.90}$$

and

$$\begin{aligned} x'\left(x^2+1\right) &= 2x \\ x'^2\left(x^2+1\right) &= 2xx' \\ x^2x'^2 + x'^2 - 2xx' &= 0 \\ \left(xx'\right)^2 - 2xx' + 1 &= 1 - x'^2 \\ \left(xx'-1\right)^2 &= 1 - x'^2 \\ xx' - 1 &= \pm\sqrt{1-x'^2}. \end{aligned}$$

Since $f(0) = 0$, the only possible solution is

$$xx' = 1 - \sqrt{1-x'^2}. \tag{4.91}$$

Besides, we know that

$$\begin{aligned} |f'(x)| = \frac{dx'}{dx} &= 2\frac{1-x^2}{\left(x^2+1\right)^2} \\ &= \frac{2}{\left(x^2+1\right)^2} - \frac{2x^2}{\left(x^2+1\right)^2} \\ &= \frac{2x^2}{\left(x^2+1\right)^2}\left(\frac{1}{x^2} - 1\right) \\ &= \frac{1}{2}\left(\frac{2x}{x^2+1}\right)^2\left(\frac{1}{x^2} - 1\right), \end{aligned}$$

where we can apply (4.90) to get

$$\begin{aligned} |f'(x)| &= \frac{1}{2}x'^2\left(\frac{1}{x^2} - 1\right) \\ &= \frac{1}{2}x'^2\left(\frac{2}{xx'} - 2\right) \\ &= x'^2\left(\frac{1}{xx'} - 1\right). \end{aligned}$$

We can now use (4.91) and obtain

$$\begin{aligned} |f'(x)| &= x'^2\left(\frac{1}{1-\sqrt{1-x'^2}} - 1\right) \\ &= \frac{x'^2 - x'^2\left(1-\sqrt{1-x'^2}\right)}{1-\sqrt{1-x'^2}} \\ &= \frac{x'^2\sqrt{1-x'^2}}{1-\sqrt{1-x'^2}} \\ &= \left(\frac{\sqrt{1-x'^2}+1-x'^2}{1-\sqrt{1-x'^2}}\right)\left(1-\sqrt{1-x'^2}\right) \\ &= \sqrt{1-x'^2} + 1 - x'^2. \end{aligned}$$

Finally, we get an explicit expression for $|f'(x)|$, given by

$$|f'(x)| = \sqrt{1-x'^2}\left(1+\sqrt{1-x'^2}\right). \tag{4.92}$$

By plugging last equality into (4.89), equation now becomes

$$\lambda S(x') = -\frac{1}{\pi}\left(1-x'^2\right)\left(1+\sqrt{1-x'^2}\right)\int_{-1}^{1^*}\frac{S'''(z)}{\sqrt{1-z^2}(x'-z)}dz. \tag{4.93}$$

It is already known that $S(x')$ can be expressed as the sum of a symmetric and an anti-symmetric function. We can assume either pinned-end or free-end boundary condition as it was done before. Let's proceed to study each case in the following subsections.

4.9.1 Anti-symmetric pinned-end boundary condition

$$S(x') = \sum_{n=1}^{\infty} a_n \sin(n\pi x').$$

Let

$$\cos(n\pi x') = \sum_{r\geq 0} c_{rn} T_r(x'),$$

and

$$\sin(n\pi x') = \sum_{r\geq 1} e_{rn}\sqrt{1-x'^2}U_{r-1}(x'),$$

with

$$c_{kn} = \begin{cases} \frac{2}{\pi}\int_{-1}^{1}\cos(n\pi x')\frac{T_k(x')}{\sqrt{1-x'^2}}dx', & k\geq 1,\\ \frac{1}{\pi}\int_{-1}^{1}\cos(n\pi x')\frac{1}{\sqrt{1-x'^2}}dx', & k=0,\end{cases}$$

and

$$e_{rn} = \frac{2}{\pi}\int_{-1}^{1}\sin(n\pi x')U_{r-1}(x')dx'.$$

Thus, by replacing into (4.93) we get

$$\begin{aligned}\lambda\sum_{n\geq 1}\sum_{r\geq 1} e_{rn}a_n\sqrt{1-x'^2}U_{r-1}(x') &= \left(1-x'^2\right)\left(1+\sqrt{1-x'^2}\right)\sum_{n\geq 1}\sum_{k\geq 0} c_{kn}a_n(n\pi)^3\frac{1}{\pi}\int_{-1}^{1^*}\frac{T_k(z')}{\sqrt{1-z^2}(x'-z)}\\ &= -\left(1-x'^2\right)\left(1+\sqrt{1-x'^2}\right)\sum_{n,k\geq 1} c_{kn}a_n(n\pi)^3U_{k-1}(x').\end{aligned}$$

Namely,

$$\lambda\sum_{n,r\geq 1} e_{rn}a_n\frac{U_{r-1}(x')}{\sqrt{1-x'^2}\left(1+\sqrt{1-x'^2}\right)} = -\sum_{n,k\geq 1} c_{kn}a_n(n\pi)^3U_{k-1}(x'). \tag{4.94}$$

Let

$$\frac{U_{r-1}(x')}{\sqrt{1-x'^2}\left(1+\sqrt{1-x'^2}\right)} = \sum_{k\geq 1} d_{kr}U_{k-1}(x'),$$

where, by orthogonality,

$$d_{kr} = \frac{2}{\pi}\int_{-1}^{1}\frac{U_{r-1}(x')U_{k-1}(x')}{1+\sqrt{1-x'^2}}.$$

Therefore, (4.94) becomes

$$-\lambda\sum_{n,r,k\geq 1} d_{kr}e_{rn}a_nU_{k-1}(x') = \sum_{n,k\geq 1} c_{kn}a_n(n\pi)^3U_{k-1}(x'),$$

and we get the following eigenvalue problem:

$$\lambda \sum_{n,r\geq 1} d_{kr} e_{rn} a_n = -\sum_{n\geq 1} (n\pi)^3 c_{kn} a_n, \quad \forall k \geq 1,$$

or in matrix notation,

$$\lambda DE\vec{a} = -C \text{ diag}\,(n\pi)^3\,\vec{a}. \tag{4.95}$$

Proposition 4.6. *If*

$$F = f_{kn} = \int_{-1}^{1} \cos\left(n\pi x'\right) T_k\left(x'\right) dx', \quad \forall k \geq 0, \forall n \geq 1,$$

then $F^T = C^{-1}$.

Proof. We know that

$$\cos\left(n\pi x'\right) = \sum_{r\geq 0} c_{rn} T_r\left(x'\right),$$

thus

$$\int_{-1}^{1} \frac{\cos\left(n\pi x'\right) T_m\left(x'\right)}{\sqrt{1-x'^2}} dx' = \sum_{k\geq 0} c_{kn} \int_{-1}^{1} \frac{T_k\left(x'\right) T_m\left(x'\right)}{\sqrt{1-x'^2}} dx',$$

from where we get

$$c_{kn} = \begin{cases} \frac{2}{\pi} \int_{-1}^{1} \cos\left(n\pi x'\right) \frac{T_k\left(x'\right)}{\sqrt{1-x'^2}} dx', & k \geq 1, \\ \frac{1}{\pi} \int_{-1}^{1} \cos\left(n\pi x'\right) \frac{1}{\sqrt{1-x'^2}} dx', & k = 0. \end{cases}$$

Furthermore, we can assume that

$$\cos\left(n\pi x'\right) = \sum_{k\geq 0} \alpha_{kn} \frac{T_k\left(x'\right)}{\sqrt{1-x'^2}}.$$

By orthogonality,

$$\alpha_{kn} = \begin{cases} \frac{2}{\pi} f_{kn}, & k \geq 1, \\ \\ \frac{1}{\pi} f_{kn}, & k = 0, \end{cases}$$

then,

$$\cos\left(n\pi x'\right) = \begin{cases} \frac{2}{\pi} \sum_k f_{kn} \frac{T_k\left(x'\right)}{\sqrt{1-x'^2}}, & k \geq 1, \\ \frac{1}{\pi} \sum_k f_{kn} \frac{1}{\sqrt{1-x'^2}}, & k = 0. \end{cases}$$

We also know that

$$\begin{aligned} \delta_{nm} &= \int_{-1}^{1} \cos\left(n\pi x'\right) \cos\left(m\pi x'\right) dx' \\ &= \begin{cases} \frac{2}{\pi} \int_{-1}^{1} \left[\sum_k c_{kn} T_k\left(x'\right)\right] \left[\sum_{\hat{k}} f_{\hat{k}m} \frac{T_{\hat{k}}\left(x'\right)}{\sqrt{1-x'^2}}\right] dx', & k.\hat{k} \geq 1, \\ \\ \frac{1}{\pi} \int_{-1}^{1} \left[\sum_k c_{kn} T_k\left(x'\right)\right] \left[\sum_{\hat{k}} f_{\hat{k}m} \frac{1}{\sqrt{1-x'^2}}\right] dx', & k, \hat{k} = 0. \end{cases} \\ &= \sum_k c_{kn} f_{km}, \end{aligned}$$

or,

$$C^T F = I \implies F^T C = I.$$

Thus, we have proven the proposition. □

Since (4.95) is true for $k \geq 1$, we need to add a zero-row in its left-hand side and c_{0n}, for $n \geq 1$, in the right-hand side. This is true as long as

$$\begin{aligned} 0 &= \int_{-1}^{1} \frac{S'''(x)}{\sqrt{1-x^2}} dx \\ &= \sum_{n\geq 1} c_{0n} (n\pi)^3 a_n \\ &= \sum_{n\geq 1} n^2 \pi^3 a_n \int_{-1}^{1} \frac{\cos(n\pi x')}{\sqrt{1-x'^2}} dx' \end{aligned}$$

Thus, (4.106) can be rewritten as follows

$$\lambda F^T \overline{DE} \vec{a} = -\operatorname{diag}(n\pi)^3 \vec{a}., \tag{4.96}$$

where $\overline{DE}$ is the DE matrix with zeros in the first-row.

This eigenvalue problem can be solved by using MATLAB. In order to solve the problem, we first need to truncate it. Let us consider the problem for dimension $N = 50$, then the obtained eigenvalues λ_i for $i = 1:50$, and their associated eigenvectors a^i are the ones given in Table 5. Even more, notice that for a given i, $a^i = [a_1, a_2, ..., a_{50}]^T$.

	$\lambda_1 = 34.5371909$	$\lambda_2 = 237.683905$	$\lambda_3 = 762.8201969$	$\lambda_4 = 1763.282708$	$\lambda_5 = 3392.374965$
a_1	1	-0.011749505	0.158864149	-0.193297445	0.216205884
a_2	-0.018096007	1	0.196038358	0.040254537	-0.070465041
a_3	0.002200392	-0.073582931	1	0.339260795	0.014808558
a_4	-0.000514854	0.01425807	-0.143722189	1	0.484246245
a_5	0.000169097	-0.004272845	0.036615337	-0.22033077	1
a_6	-6.85E-05	0.001641838	-0.012819593	0.068191237	-0.302027205
a_7	3.19E-05	-0.000741438	0.00546377	-0.026739837	0.109062192
a_8	-1.65E-05	0.000375176	-0.002660431	0.012300352	-0.046850253
a_9	9.25E-06	-0.000206628	0.001426307	-0.006334939	0.022913636
a_{10}	-5.51E-06	0.000121522	-0.000822517	0.003547709	-0.012343203
a_{11}	3.45E-06	-7.53E-05	0.000502273	-0.00211922	0.007158007
a_{12}	-2.25E-06	4.87E-05	-0.000321226	0.001332503	-0.004398419

Table 5: first eigenvalues and its associated eigenvectors for the anti-symmetric pinned-end case. This data was obtained by using MATLAB and by letting $N = 50$.

From Figure 16, it seems that the first eigenvalues is convergent as $N \to \infty$. In this figure, it can be seen that λ_1 converges to 34.5371 as N grows.

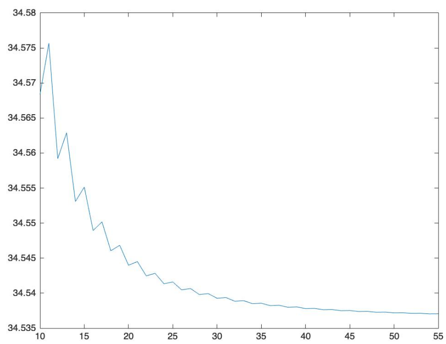
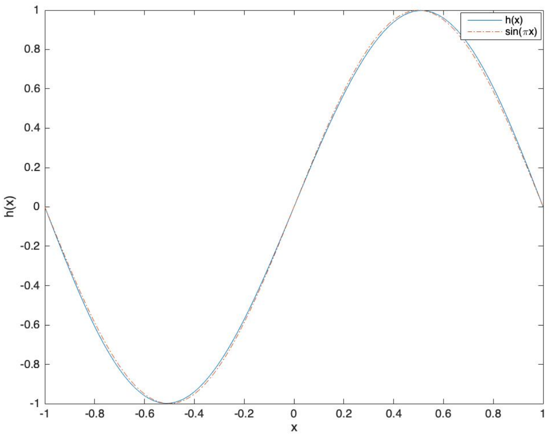
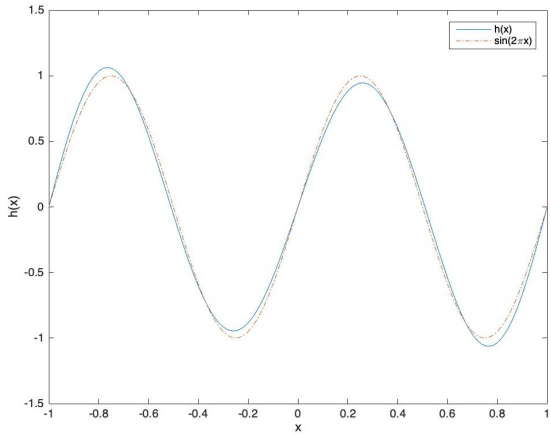


Figure 16: values of λ_1 as dimension N grows.

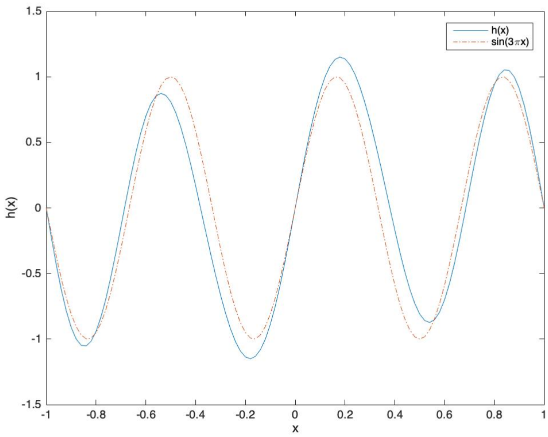
Since we have solved the eigenvalue problem, we are now able to plot the free surface for a fixed value of t. Let us consider $t = 0$, then $h(x,t) = h(x) = S(x)$, the free surface can be seen in Figure 17.



(a) Eigenfunction associated to λ_1 in comparison with $\sin(\pi x)$.



(b) Eigenfunction associated to λ_2 in comparison with $\sin(2\pi x)$.



(c) Eigenfunction associated to λ_3 in comparison with $\sin(3\pi x)$.

Figure 17: the free surface $h(x)$ in presence of walls compared to the free surface in absence of walls (sine frequencies).

4.9.2 Symmetric pinned-end boundary condition

$$S(x') = \sum_{n=1}^{\infty} b_n \cos\left(\left(n - \frac{1}{2}\right)\pi x'\right).$$

As in last section, we need to impose mass conservation extra condition. In the same way we did before, we get the following conditon:

$$\sum_n \omega_n b_n = 0, \tag{4.97}$$

where

$$\omega_n = \int_{-1}^{1} \frac{\cos\left(\left(n - \frac{1}{2}\right)\pi x'\right)}{\sqrt{1 - x'^2}\left(1 + \sqrt{1 - x'^2}\right)} dx'. \tag{4.98}$$

Besides, let

$$\cos\left(\left(n-\frac{1}{2}\right)\pi x'\right)=\sum_{r\geq 1}e_{rn}\sqrt{1-x'^2}U_{r-1}\left(x'\right),$$

and

$$\sin\left(\left(n-\frac{1}{2}\right)\pi x'\right)=\sum_{k\geq 1}c_{kn}T_k\left(x'\right),$$

where

$$e_{rn}=\frac{2}{\pi}\int_{-1}^{1}\cos\left(\left(n-\frac{1}{2}\right)\pi x'\right)U_{r-1}\left(x'\right),$$

and

$$c_{kn}=\frac{2}{\pi}\int_{-1}^{1}\sin\left(\left(n-\frac{1}{2}\right)\pi x'\right)\frac{T_k\left(x'\right)}{\sqrt{1-x'^2}}dx'.$$

Thus, by replacing into (4.93) we get

$$\begin{aligned}\lambda\sum_{n,r\geq 1}e_{rn}b_n\sqrt{1-x'^2}U_{r-1}\left(x'\right)&=-\left(1-x'^2\right)\left(1+\sqrt{1-x'^2}\right)\sum_{n,k\geq 1}c_{kn}b_n\left(\left(n-\frac{1}{2}\right)\pi\right)^3\frac{1}{\pi}\int_{-1}^{1^*}\frac{T_k\left(z\right)}{\sqrt{1-z^2}\left(x'-z\right)}\\&=\left(1-x'^2\right)\left(1+\sqrt{1-x'^2}\right)\sum_{n,k\geq 1}c_{kn}b_n\left(\left(n-\frac{1}{2}\right)\pi\right)^3U_{k-1}\left(x'\right).\end{aligned}$$

Namely,

$$\lambda\sum_{n,r\geq 1}e_{rn}b_n\frac{U_{r-1}\left(x'\right)}{\sqrt{1-x'^2}\left(1+\sqrt{1-x'^2}\right)}=\sum_{n,k\geq 1}c_{kn}b_n\left(\left(n-\frac{1}{2}\right)\pi\right)^3U_{k-1}\left(x'\right).\tag{4.99}$$

Let

$$\frac{U_{r-1}\left(x'\right)}{\sqrt{1-x'^2}\left(1+\sqrt{1-x'^2}\right)}=\sum_{k\geq 1}d_{kr}U_{k-1}\left(x'\right),$$

where, by orthogonality,

$$d_{kr}=\frac{2}{\pi}\int_{-1}^{1}\frac{U_{r-1}\left(x'\right)U_{k-1}\left(x'\right)}{1+\sqrt{1-x'^2}}.$$

Therefore, (4.99) becomes

$$\lambda\sum_{n,r,k\geq 1}d_{kr}e_{rn}b_nU_{k-1}\left(x'\right)=\sum_{n,k\geq 1}c_{kn}b_n\left[\left(n-\frac{1}{2}\right)\pi\right]^3U_{k-1}\left(x'\right),$$

and we get the following eigenvalue problem:

$$\lambda\sum_{n,r\geq 1}d_{kr}e_{rn}b_n=\sum_{n\geq 1}\left(\left(n-\frac{1}{2}\right)\pi\right)^3c_{kn}b_n,\quad\forall k\geq 1,$$

or in matrix notation,

$$\lambda DE\vec{b}=C\ \text{diag}\left[=(\left(n-\frac{1}{2}\right)\pi\right)^3\vec{b},$$

which is equivalent to

$$\lambda C^{-1}DE\vec{b}=\ \text{diag}\left[\left(n-\frac{1}{2}\right)\pi\right]^3\vec{b}.\tag{4.100}$$

Proposition 4.7. *If*

$$F = f_{kn} = \int_{-1}^{1} \sin\left(\left(n-\frac{1}{2}\right)\pi x'\right) T_k(x')\, dx', \quad \forall k, n \geq 1,$$

then $F^T = C^{-1}$.

Proof. We know that

$$\sin\left(\left(n-\frac{1}{2}\right)\pi x'\right) = \sum_{k\geq 1} c_{kn} T_k(x'),$$

thus

$$\int_{-1}^{1} \frac{\sin\left(\left(n-\frac{1}{2}\right)\pi x'\right) T_m(x')}{\sqrt{1-x'^2}} dx' = \sum_{k\geq 1} c_{kn} \int_{-1}^{1} \frac{T_k(x') T_m(x')}{\sqrt{1-x'^2}} dx',$$

from where we get

$$c_{kn} = \frac{2}{\pi} \int_{-1}^{1} \sin\left(\left(n-\frac{1}{2}\right)\pi x'\right) \frac{T_k(x')}{\sqrt{1-x'^2}} dx'.$$

Furthermore, we can assume that

$$\sin\left(\left(n-\frac{1}{2}\right)\pi x'\right) = \sum_{k\geq 1} \alpha_{kn} \frac{T_k(x')}{\sqrt{1-x'^2}}.$$

By orthogonality,

$$\alpha_{kn} = \frac{2}{\pi} f_{kn},$$

then,

$$\sin\left(\left(n-\frac{1}{2}\right)\pi x'\right) = \frac{2}{\pi} \sum_{k\geq 1} f_{kn} \frac{T_k(x')}{\sqrt{1-x'^2}}.$$

We also know that

$$\begin{aligned} \delta_{nm} &= \int_{-1}^{1} \sin\left(\left(n-\frac{1}{2}\right)\pi x'\right) \sin\left(\left(m-\frac{1}{2}\right)\pi x'\right) dx' \\ &= \frac{2}{\pi} \int_{-1}^{1} \left[\sum_{k\geq 1} c_{kn} T_k(x')\right] \left[\sum_{\hat{k}\geq 1} f_{\hat{k}m} \frac{T_{\hat{k}}(x')}{\sqrt{1-x'^2}}\right] dx' \\ &= \sum_{k\geq 1} c_{kn} f_{km}, \end{aligned}$$

or,

$$C^T F = I \implies F^T C = I.$$

Thus, we proved the proposition. □

By replacing this last result in (4.100), the system we get is

$$\lambda F^T DE\vec{b} = \operatorname{diag}(n\pi)^3 \vec{b}.$$

As we already did, we need to impose the mass conservation condition to the system, from where we get

$$\lambda F^T \overline{DE}\vec{b} = \overline{\operatorname{diag}}(n\pi)^3 \vec{b}, \tag{4.101}$$

where $\overline{DE}$ is the DE matrix with a zero first-row and $\overline{\text{diag}}$ is the diagonal matrix adding a ω_n first-row, for $n \geq 1$.

This eigenvalue problem can be solved by using MATLAB. Let us consider the problem for dimension $N = 50$, then the first obtained eigenvalues λ_i for $i = 1 : 50$, and their associated eigenvectors b^i are the ones given in Table 6. Even more, notice that for a given i, $b^i = [b_1, b_2, ..., b_{50}]^T$.

	$\lambda_1 = 230.462753328928$	$\lambda_2 = 1033.25945399075$	$\lambda_3 = 2794.34728973065$	$\lambda_4 = 5896.23667957228$
b_1	0.0117503830609352	-0.0116682850493653	-0.0103058546089242	0.00908256966009043
b_2	-0.179476253951130	0.0243460861408048	0.0200673240656862	-0.0171754462362053
b_3	0.0315127140495187	-0.144729669147775	-0.0209068627961600	0.0163466151413946
b_4	-0.0101306596924217	0.0536990914015066	0.111720959303023	-0.0185954975041048
b_5	0.00440911559175099	-0.0239654485006214	-0.0640062906193418	0.0849398802474573
b_6	-0.00229665443646346	0.0124747602697866	0.0360864166706974	-0.0652492246984256
b_7	0.00134652421502298	-0.00725612153295742	-0.0216511398617178	0.0440674945344670
b_8	-0.000858214561045399	0.00457927436158034	0.0138315802912714	-0.0297679881154231
b_9	0.000581942365752926	-0.00307412695303720	-0.00932101837699070	0.0206493874222004
b_{10}	-0.000413887855585407	0.00216555377923844	0.00656488520437511	-0.0147726539208492
b_{11}	0.000305713368224388	-0.00158539144747019	-0.00479552524676257	0.0108825445253124
b_{12}	-0.000232861833943009	0.00119773545128140	0.00361130525339064	-0.00823051641759035

Table 6: first eigenvalues and its associated eigenvectors for the anti-symmetric free-end case. This data was obtained by using MATLAB and by letting $N = 50$.

4.9.3 Symmetric free-end boundary condition

$$S(x') = \sum_{n=1}^{\infty} \widetilde{b_n} \cos(n\pi x').$$

Let

$$\cos(n\pi x') = \sum_{r\geq 1} e_{rn} \sqrt{1-x'^2}\, U_{r-1}(x'),$$

and

$$\sin(n\pi x') = \sum_{k\geq 1} c_{kn} T_k(x'),$$

with

$$e_{rn} = \frac{2}{\pi} \int_{-1}^{1} \cos(n\pi x')\, U_{r-1}(x')\, dx'$$

and

$$c_{kn} = \frac{2}{\pi} \int_{-1}^{1} \sin(n\pi x') \frac{T_k(x')}{\sqrt{1-x'^2}} dx'.$$

Thus, by replacing into (4.93) we get

$$\begin{aligned} \lambda \sum_{n,r\geq 1} e_{rn} \widetilde{b_n} \sqrt{1-x'^2}\, U_{r-1}(x') &= -\left(1-x'^2\right)\left(1+\sqrt{1-x'^2}\right) \sum_{n,k\geq 1} c_{kn} \widetilde{b_n} (n\pi)^3 \frac{1}{\pi} \int_{-1}^{1^*} \frac{T_k(z)}{\sqrt{1-z^2}\,(x'-z)} \\ &= \left(1-x'^2\right)\left(1+\sqrt{1-x'^2}\right) \sum_{n,k\geq 1} c_{kn} \widetilde{b_n} (n\pi)^3 U_{k-1}(x'). \end{aligned}$$

Namely,

$$\lambda \sum_{n,r\geq 1} e_{rn} \widetilde{b_n} \frac{U_{r-1}(x')}{\sqrt{1-x'^2}\left(1+\sqrt{1-x'^2}\right)} = \sum_{n,k\geq 1} c_{kn} \widetilde{b_n} (n\pi)^3 U_{k-1}(x'). \tag{4.102}$$

Let

$$\frac{U_{r-1}(x')}{\sqrt{1-x'^2}\left(1+\sqrt{1-x'^2}\right)} = \sum_{k\geq 1} d_{kr} U_{k-1}(x'),$$

where, by orthogonality,

$$d_{kr} = \frac{2}{\pi}\int_{-1}^{1}\frac{U_{r-1}\left(x'\right)U_{k-1}\left(x'\right)}{1+\sqrt{1-x'^2}}.$$

Therefore, (4.102) becomes

$$\lambda\sum_{n,r,k\geq 1} d_{kr}e_{rn}\widetilde{b_n}U_{k-1}\left(x'\right) = \sum_{n,k\geq 1} c_{kn}\widetilde{b_n}\left(n\pi\right)^3 U_{k-1}\left(x'\right),$$

and we get the following eigenvalue problem:

$$\lambda\sum_{n,r\geq 1} d_{kr}e_{rn}\widetilde{b_n} = \sum_{n\geq 1}\left(n\pi\right)^3 c_{kn}\widetilde{b_n}, \quad \forall k\geq 1,$$

or in matrix notation,

$$\lambda DE\vec{\widetilde{b}} = C\ \mathrm{diag}\left(n\pi\right)^3\vec{\widetilde{b}},$$

which is equivalent to

$$\lambda C^{-1}DE\vec{\widetilde{b}} = \ \mathrm{diag}\left(n\pi\right)^3\vec{\widetilde{b}}. \tag{4.103}$$

Proposition 4.8. *If*

$$F = f_{kn} = \int_{-1}^{1}\sin\left(n\pi x'\right)T_k\left(x'\right)dx', \quad \forall k,n\geq 1,$$

then $F^T = C^{-1}$.

Proof. We know that

$$\sin\left(n\pi x'\right) = \sum_{k\geq 1} c_{kn}T_k\left(x'\right),$$

thus

$$\int_{-1}^{1}\frac{\sin\left(n\pi x'\right)T_m\left(x'\right)}{\sqrt{1-x'^2}}dx' = \sum_{k\geq 1} c_{kn}\int_{-1}^{1}\frac{T_k\left(x'\right)T_m\left(x'\right)}{\sqrt{1-x'^2}}dx',$$

from where we get

$$c_{kn} = \frac{2}{\pi}\int_{-1}^{1}\sin\left(n\pi x'\right)\frac{T_k\left(x'\right)}{\sqrt{1-x'^2}}dx'.$$

Furthermore, we can assume that

$$\sin\left(n\pi x'\right) = \sum_{k\geq 1}\alpha_{kn}\frac{T_k\left(x'\right)}{\sqrt{1-x'^2}}.$$

By orthogonality,

$$\alpha_{kn} = \frac{2}{\pi}f_{kn},$$

then,

$$\sin\left(n\pi x'\right) = \frac{2}{\pi}\sum_{k\geq 1} f_{kn}\frac{T_k\left(x'\right)}{\sqrt{1-x'^2}}.$$

We also know that

$$\begin{aligned}
\delta_{nm} &= \int_{-1}^{1}\sin\left(n\pi x'\right)\sin\left(m\pi x'\right)dx' \\
&= \frac{2}{\pi}\int_{-1}^{1}\left(\sum_{k\geq 1} c_{kn}T_k\left(x'\right)\right)\left(\sum_{\hat{k}\geq 1} f_{\hat{k}m}\frac{T_{\hat{k}}\left(x'\right)}{\sqrt{1-x'^2}}\right)dx' \\
&= \sum_{k\geq 1} c_{kn}f_{km},
\end{aligned}$$

or,

$$C^T F = I \implies F^T C = I.$$

Thus, we proved the proposition. □

By replacing this last result in (4.103), the system we get is

$$\lambda F^T D E \vec{\widetilde{b}} = \operatorname{diag}(n\pi)^3 \vec{\widetilde{b}}. \tag{4.104}$$

This eigenvalue problem can be solved by using MATLAB. Let us consider the problem for dimension $N = 50$, then the first obtained eigenvalues λ_i for $i = 1 : 50$, and their associated eigenvectors $\widetilde{b}^i$ are the ones given in Table 7. Even more, notice that for a given i, $\widetilde{b}^i = \left[\widetilde{b}_1, \widetilde{b}_2, ..., \widetilde{b}_{50}\right]^T$.

	$\lambda_1 = 34.2175586272960$	$\lambda_2 = 300.557895998749$	$\lambda_3 = 1025.35445855159$	$\lambda_4 = 2437.22062288403$
$\widetilde{b}_1$	1	-0.502920749237398	0.145159683202213	-0.102054838998293
$\widetilde{b}_2$	-0.0515531686145348	-1	-0.603362498736936	0.0890614719726122
$\widetilde{b}_3$	0.0101796570090243	0.139459671406230	-1	-0.791194266006952
$\widetilde{b}_4$	-0.00328802075595610	-0.0381132638500681	0.237934316980673	-1
$\widetilde{b}_5$	0.00137561477588171	0.0145463800669990	-0.0805838360851182	0.345252329326353
$\widetilde{b}_6$	-0.000675645063586782	-0.00675443336895904	0.0345008428641132	-0.138312614217566
$\widetilde{b}_7$	0.000370236892861313	0.00356672127107907	-0.0172333678149811	0.0649412580344996
$\widetilde{b}_8$	-0.000219686543035991	-0.00206253560718778	0.00958163012496778	-0.0343820877602204
$\widetilde{b}_9$	0.000138487874635901	0.00127617192316913	-0.00576102980667357	0.0199065071452755
$\widetilde{b}_{10}$	-9.15577648315637e-05	-0.000832066840944857	0.00367635894969585	-0.0123370079997454
$\widetilde{b}_{11}$	6.28962749340519e-05	0.000565607675542838	-0.00245825241621866	0.00806242470784695
$\widetilde{b}_{12}$	-4.47123978114624e-05	-0.000398482184078008	0.00170901090731494	-0.00550238080160748

Table 7: first eigenvalues and its associated eigenvectors for the anti-symmetric free-end case. This data was obtained by using MATLAB and by letting $N = 50$.

From Figure 18, it can seems that the first eigenvalue converges as $N \to \infty$. For the case of the first eigenvalue, it can be seen that it converges to 34.0801 as N grows.

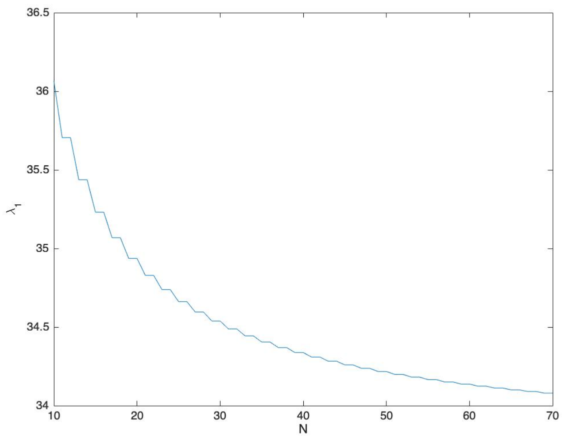
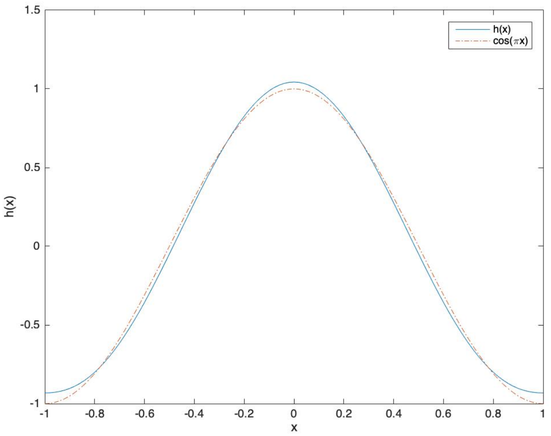
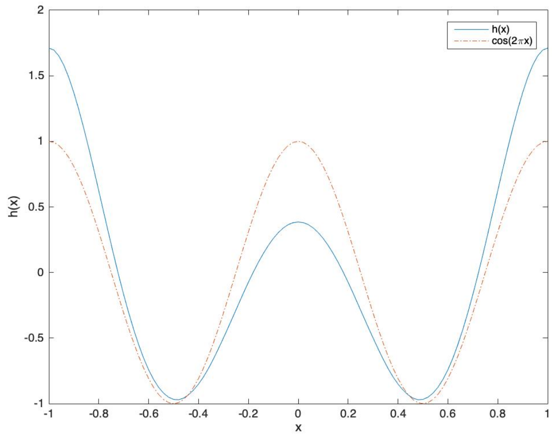
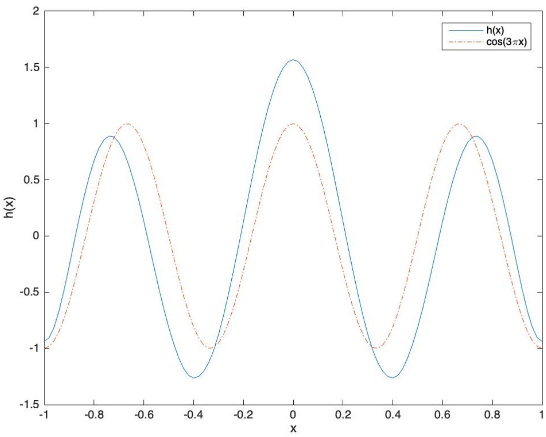


Figure 18: values of λ_1 as dimension N grows.





(a) Eigenfunction associated to λ_1 in comparison with $\cos(\pi x)$. (b) Eigenfunction associated to λ_2 in comparison with $\cos(2\pi x)$.



(c) Eigenfunction associated to λ_3 in comparison with $\cos(3\pi x)$.

Figure 19: the free surface $h(x)$ in presence of walls compared to the free surface in absence of walls (cosine frequencies).

4.9.4 Anti-symmetric free-end boundary condition

$$S(x') = \widetilde{a_n} \sum_{n=1}^{\infty} \sin\left(\left(n - \frac{1}{2}\right)\pi x'\right).$$

Let

$$\cos\left(\left(n - \frac{1}{2}\right)\pi x'\right) = \sum_{k\geq 0} c_{kn} T_k(x'),$$

and

$$\sin\left(\left(n - \frac{1}{2}\right)\pi x'\right) = \sum_{r\geq 1} e_{rn}\sqrt{1 - x'^2}\, U_{r-1}(x'),$$

where

$$c_{kn} = \begin{cases} \frac{2}{\pi}\int_{-1}^{1}\cos\left(\left(n - \frac{1}{2}\right)\pi x'\right)\frac{T_k(x')}{\sqrt{1-x'^2}}dx', & k \geq 1, \\ \frac{1}{\pi}\int_{-1}^{1}\cos\left(\left(n - \frac{1}{2}\right)\pi x'\right)\frac{1}{\sqrt{1-x'^2}}dx', & k = 0, \end{cases}$$

and

$$e_{rn} = \frac{2}{\pi} \int_{-1}^{1} \sin\left(\left(n - \frac{1}{2}\right)\pi x'\right) U_{r-1}\left(x'\right).$$

Thus, by replacing into (4.93) we get

$$\lambda \sum_{n,r\geq 1} e_{rn}\widetilde{a_n}\sqrt{1-x'^2}U_{r-1}\left(x'\right) = -\left(1-x'^2\right)\left(1+\sqrt{1-x'^2}\right)\sum_{n\geq 1}\sum_{k\geq 0} c_{kn}\widetilde{a_n}\left(\left(n-\frac{1}{2}\right)\pi\right)^3 \frac{1}{\pi}\int_{-1}^{1^*}\frac{T_k\left(z\right)}{\sqrt{1-z^2}\left(x'-z\right)}$$
$$= \left(1-x'^2\right)\left(1+\sqrt{1-x'^2}\right)\sum_{n,k\geq 1} c_{kn}\widetilde{a_n}\left(\left(n-\frac{1}{2}\right)\pi\right)^3 U_{k-1}\left(x'\right).$$

Namely,

$$\lambda \sum_{n,r\geq 1} e_{rn}\widetilde{a_n}\frac{U_{r-1}\left(x'\right)}{\sqrt{1-x'^2}\left(1+\sqrt{1-x'^2}\right)} = \sum_{n,k\geq 1} c_{kn}\widetilde{a_n}\left(\left(n-\frac{1}{2}\right)\pi\right)^3 U_{k-1}\left(x'\right). \tag{4.105}$$

Let

$$\frac{U_{r-1}\left(x'\right)}{\sqrt{1-x'^2}\left(1+\sqrt{1-x'^2}\right)} = \sum_{k\geq 1} d_{kr}U_{k-1}\left(x'\right),$$

where, by orthogonality,

$$d_{kr} = \frac{2}{\pi}\int_{-1}^{1}\frac{U_{r-1}\left(x'\right)U_{k-1}\left(x'\right)}{1+\sqrt{1-x'^2}}.$$

Therefore, (4.105) becomes

$$\lambda \sum_{n,r,k\geq 1} d_{kr}e_{rn}\widetilde{a_n}U_{k-1}\left(x'\right) = \sum_{n,k\geq 1} c_{kn}\widetilde{a_n}\left[\left(n-\frac{1}{2}\right)\pi\right]^3 U_{k-1}\left(x'\right),$$

and we get the following eigenvalue problem:

$$\lambda \sum_{n,r} d_{kr}e_{rn}\widetilde{a_n} = \sum_{n}\left(\left(n-\frac{1}{2}\right)\pi\right)^3 c_{kn}\widetilde{a_n}, \quad \forall k\geq 1,$$

or in matrix notation,

$$\lambda DE\vec{\widetilde{a}} = C \text{ diag}\left[\left(n-\frac{1}{2}\right)\pi\right]^3 \vec{\widetilde{a}}. \tag{4.106}$$

Proposition 4.9. *If*

$$F = f_{kn} = \int_{-1}^{1}\cos\left(\left(n-\frac{1}{2}\right)\pi x'\right)T_k\left(x'\right)dx', \quad \forall k\geq 0, \forall n\geq 1,$$

then $F^T = C^{-1}$.

Proof. We know that

$$\cos\left(\left(n-\frac{1}{2}\right)\pi x'\right) = \sum_{k\geq 0} c_{kn}T_k\left(x'\right),$$

thus

$$\int_{-1}^{1}\frac{\cos\left(\left(n-\frac{1}{2}\right)\pi x'\right)T_m\left(x'\right)}{\sqrt{1-x'^2}}dx' = \sum_{k\geq 0} c_{kn}\int_{-1}^{1}\frac{T_k\left(x'\right)T_m\left(x'\right)}{\sqrt{1-x'^2}}dx',$$

from where we get

$$c_{kn} = \begin{cases} \frac{2}{\pi}\int_{-1}^{1}\cos\left(\left(n-\frac{1}{2}\right)\pi x'\right)\frac{T_k(x')}{\sqrt{1-x'^2}}dx', & k \geq 1, \\ \\ \frac{1}{\pi}\int_{-1}^{1}\cos\left(\left(n-\frac{1}{2}\right)\pi x'\right)\frac{1}{\sqrt{1-x'^2}}dx', & k = 0. \end{cases}$$

Furthermore, we can assume that

$$\cos\left(\left(n-\frac{1}{2}\right)\pi x'\right) = \sum_{k\geq 0}\alpha_{kn}\frac{T_k(x')}{\sqrt{1-x'^2}}.$$

By orthogonality,

$$\alpha_{kn} = \begin{cases} \frac{2}{\pi}f_{kn}, & k \geq 1, \\ \\ \frac{1}{\pi}f_{kn}, & k = 0, \end{cases}$$

then,

$$\cos(n\pi x') = \begin{cases} \frac{2}{\pi}\sum_k f_{kn}\frac{T_k(x')}{\sqrt{1-x'^2}}, & k \geq 1, \\ \frac{1}{\pi}\sum_k f_{kn}\frac{T_k(x')}{\sqrt{1-x'^2}}, & k = 0. \end{cases}$$

We also know that

$$\begin{aligned}
\delta_{nm} &= \int_{-1}^{1}\cos\left(\left(n-\frac{1}{2}\right)\pi x'\right)\cos\left(\left(m-\frac{1}{2}\right)\pi x'\right)dx' \\
&= \frac{2}{\pi}\int_{-1}^{1}\left(\sum_k c_{kn}T_k(x')\right)\left(\sum_{\hat{k}} f_{\hat{k}m}\frac{T_{\hat{k}}(x')}{\sqrt{1-x'^2}}\right)dx' + \frac{1}{\pi}c_{0n}f_{0m}\int_{-1}^{1}\left(\sum_k c_{kn}T_k(x')\right)\left(\sum_{\hat{k}}\frac{1}{\sqrt{1-x'^2}}\right)dx' \\
&= \sum_{k\geq 1}c_{kn}f_{km} + c_{0n}f_{om} \\
&= \sum_{k\geq 0}c_{kn}f_{km},
\end{aligned}$$

or,

$$C^T F = I \implies F^T C = I.$$

Thus, we have proven the proposition. □

Since (4.106) is true for $k \geq 1$, we need to add a zero-row in its left-hand side and c_{0n}, for $n \geq 1$, in the right-hand side. This is true since

$$\begin{aligned}
\sum_{n\geq 1}c_{0n}\pi\left(n-\frac{1}{2}\right)\widetilde{a_n} &= \frac{1}{\pi}\int_{-1}^{1}\frac{\sum_n \widetilde{a_n}\left(\left(n-\frac{1}{2}\right)\pi\right)^3\cos\left(\left(n-\frac{1}{2}\right)\pi x'\right)}{\sqrt{1-x'^2}}dx' \\
&= \frac{1}{\pi}\int_{-1}^{1}\frac{S'''(x')}{\sqrt{1-x'^2}}dx' \\
&= 0.
\end{aligned}$$

Thus, (4.106) can be rewritten as follows

$$\lambda F^T\overline{DE}\vec{\widetilde{a}} = \text{ diag}\left[\left(n-\frac{1}{2}\right)\pi\right]^3\vec{\widetilde{a}}, \tag{4.107}$$

where $\overline{DE}$ is the DE matrix with zeros in the first-row.

This eigenvalue problem can be solved by using MATLAB. Let us consider the problem for dimension $N = 50$, then the first obtained eigenvalues λ_i for $i = 1 : 50$, and their associated eigenvectors $\widehat{a^i}$ are the ones given in Table 8. Even more, notice that for a given i, $a^i = [a_1, a_2, ..., a_{50}]^T$.

	$\lambda_1 = 3.86095364427419$	$\lambda_2 = 127.056248760284$	$\lambda_3 = 594.338943011707$	$\lambda_4 = 1634.25532825826$
$\widetilde{a_1}$	1	-0.495080904664665	0.184192436151974	-0.124196571593678
$\widetilde{a_2}$	-0.0135369424178429	-1	-0.521278403012665	0.107548862539826
$\widetilde{a_3}$	0.00192722754746661	0.0910908001306199	-1	-0.680885339819841
$\widetilde{a_4}$	-0.000534078338660874	-0.0213308109713834	0.186320360255083	-1
$\widetilde{a_5}$	0.000204254688261910	0.00750108505138570	-0.0568882881153110	0.289298281773372
$\widetilde{a_6}$	-9.45500439668107e-05	-0.00330644323041500	0.0230174187370253	-0.106742686663164
$\widetilde{a_7}$	4.96744705988181e-05	0.00168332034941237	-0.0110899869518590	0.0478711426585793
$\widetilde{a_8}$	-2.85609898764373e-05	-0.000947171930926002	0.00601091532861416	-0.0246192219481397
$\widetilde{a_9}$	1.75691442511158e-05	0.000573688234802021	-0.00354532814705515	0.0139693090697572
$\widetilde{a_{10}}$	-1.13901960561351e-05	-0.000367683087906929	0.00222835309251833	-0.00852844942194157
$\widetilde{a_{11}}$	7.70109393216882e-06	0.000246448031264094	-0.00147177105887005	0.00550861706810430
$\widetilde{a_{12}}$	-5.38752955327135e-06	-0.000171281355613452	0.00101136349086949	-0.00372013373317179

Table 8: first eigenvalues and its associated eigenvectors for the anti-symmetric free-end case. This data was obtained by using MATLAB and by letting $N = 50$.

From Figure 20, it seems that the first eigenvalue converges as $N \to \infty$. In this case, λ_1 converges to 34.8447 as N grows.

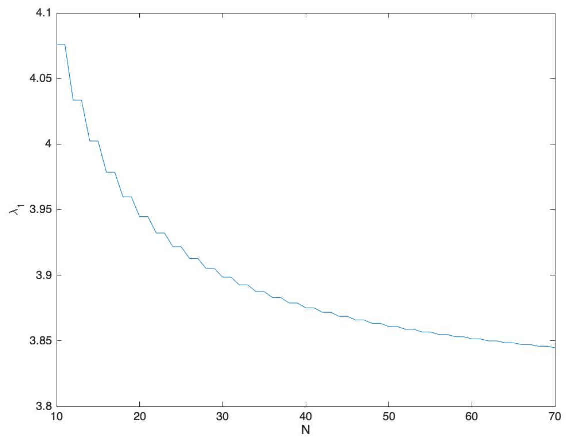
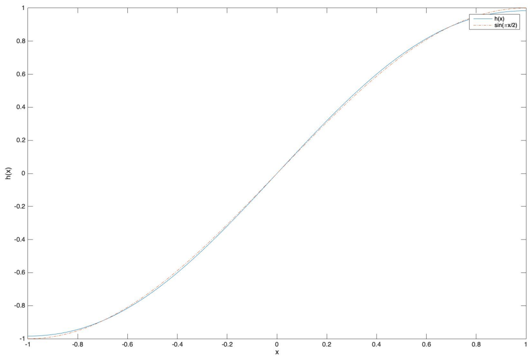
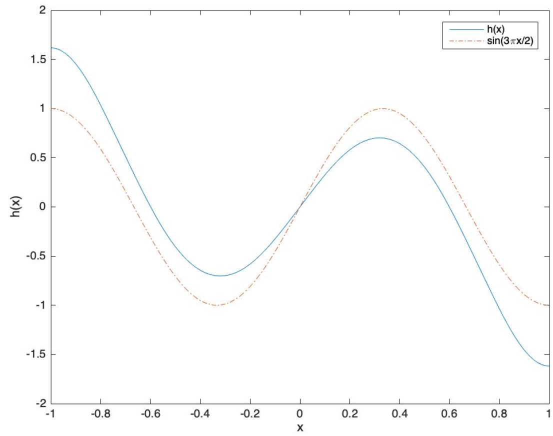
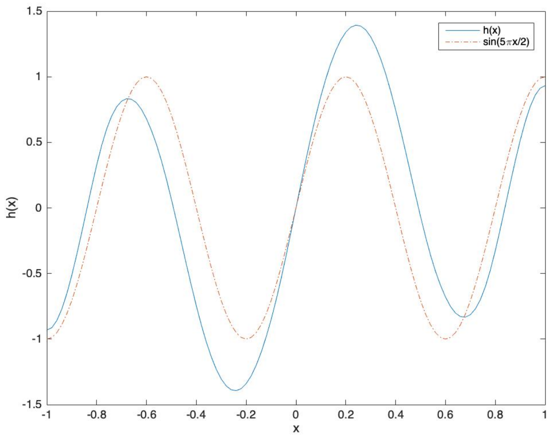


Figure 20: values of λ_1 as dimension N grows.





(a) Eigenfunction associated to λ_1 in comparison with $\sin(\pi x/2)$. (b) Eigenfunction associated to λ_2 compared to $\sin(3\pi x/2)$.



(c) Eigenfunction associated to λ_3 in comparison with $\sin(5\pi x/2)$.

Figure 21: the free surface $h(x)$ in presence of walls compared to free surface in absence of walls (sine frequencies).

5 Conclusions and future work

We have developed a method to compute natural frequencies of a liquid surface, in three different geometries: the half-plane, an infinite vertical-strip and a rounded container. We considered two different boundary conditions for each geometry: first, when the waves of the free surface are pinned to the container and second, when these waves form a contact angle of $\pi/2$ with the walls of the container. We first introduce a linearized integrodifferential equation involving the Hilbert transform, which can be solved in natural basis formed by Tchebyshev polynomials. As a result, we get an eigenvalue problem which can be approximated by truncating the associated infinite matrices. Nonetheless, as $N \to \infty$ the solution quickly converges to the solution of the original system. With this method, we found the eigenvalues of the linear Euler equations.

To study the remaining cases, we applied the conformal mapping technique. The tecnhique consists of collecting the information of the new geometry in a matrix D, that appears as a factor. As a consequence, a given geometry can be conformally mapped into the half-plane, where we already now how to obtain the eigenvalues.

In all the studied cases it is clear that the presence of walls directly affects the free surface behavior. In general, it seems that for the free-end edge condition, the frequencies for the anti-symmetric case were larger than the ones for the symmetric case; this is regardless of the container geometry. On the other hand, the eigenvalues depends on container geometry, but the variation from one geometry to another is not very substantial.

For future work, it is of interest to study a container with two free surfaces separated by a given distance d. For this problem, the Duhamel's principle may be of use. It is also of interest to see how much it changes in comparison to our problem and how much one surface affects the other.

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Appendices

A Inversion formula for the finite Hilbert transform

Let us consider the following equation:

$$\frac{1}{\pi}\int_0^{\pi^*}\frac{\sin(z)}{\cos(x)-\cos(z)}\phi(z)\,dz = f(x), \quad f\in L^2[0,\pi]. \tag{A.1}$$

Let $\phi=\sin(nx)$ $(n\in\mathbb{N})$ in the equation above. Then, we obtain

$$\begin{aligned} f(x) &= \frac{1}{\pi}\int_0^{\pi^*}\frac{\sin(z)\sin(nz)}{\cos(x)-\cos(z)}dz \\ &= \frac{1}{2\pi}\int_0^{\pi^*}\frac{\cos\big[(n-1)z\big]-\cos\big[(n+1)z\big]}{\cos(x)-\cos(z)}dz, \end{aligned}$$

which by the parity of the cosine function becomes

$$\begin{aligned} f(x) &= \frac{1}{4\pi}\int_{-\pi}^{\pi^*}\frac{\cos\big[(n-1)z\big]-\cos\big[(n+1)z\big]}{\cos(x)-\cos(z)}dz \\ &= \frac{1}{8\pi}\int_{-\pi}^{\pi^*}\left[\frac{e^{i(n-1)z}-e^{i(n+1)z}}{\cos(x)-\cos(z)}dz+\frac{e^{-i(n-1)z}-e^{-i(n+1)z}}{\cos(x)-\cos(z)}\right]dz \\ &= \frac{1}{4\pi}\int_{-\pi}^{\pi^*}\frac{e^{i(n-1)z}-e^{i(n+1)z}}{\cos(x)-\cos(z)}dz. \end{aligned}$$

Let us consider the change of variable $e^{iz}=y$. Then,

$$\begin{aligned} f(x) &= \frac{1}{4\pi i}\int_{|y|=1}\frac{y^{n-1}-y^{n+1}}{\cos(x)-\frac{y^2+1}{2y}}\frac{dy}{y} \\ &= \frac{1}{2\pi i}\int_{|y|=1}\frac{y^{n-1}-y^{n+1}}{2y\cos(x)-y^2-1}dy \\ &= -\frac{1}{2\pi i}\int_{|y|=1}\frac{y^{n-1}-y^{n+1}}{y^2-2y\cos(x)+1}dy. \end{aligned}$$

Notice that

$$\begin{aligned} \left(y-e^{ix}\right)\left(y-e^{-ix}\right) &= y^2-y\left(e^{ix}+e^{-ix}\right)+1 \\ &= y^2-2y\cos(x)+1. \end{aligned}$$

Therefore, replacing into the equation above,

$$f(x) = -\frac{1}{2\pi i}\int_{|y|=1}\frac{y^{n-1}-y^{n+1}}{\left(y-e^{ix}\right)\left(y-e^{-ix}\right)}dy.$$

Let $g(y)=\dfrac{y^{n-1}-y^{n+1}}{\left(y-e^{ix}\right)\left(y-e^{-ix}\right)}$, thus g has two simple poles at: $y=e^{ix}$ and $y=e^{-ix}$. Let us compute the residues at both poles.

- Residue at pole $y=e^{ix}$:

$$\begin{aligned} \operatorname{Res}\left(g(y),e^{ix}\right) &= \lim_{y\to e^{ix}}\frac{y^{n-1}-y^{n+1}}{y-e^{-ix}} \\ &= \frac{e^{ix(n-1)}-e^{ix(n+1)}}{e^{ix}-e^{-ix}} \\ &= -e^{ixn}. \end{aligned}$$

- Residue at pole $y = e^{-ix}$:

$$\begin{aligned}\operatorname{Res}\left(g\left(y\right), e^{-ix}\right) &= \lim_{y\to e^{-ix}} \frac{y^{n-1}-y^{n+1}}{y-e^{ix}}\\ &= \frac{e^{-ix(n-1)}-e^{-ix(n+1)}}{e^{-ix}-e^{ix}}\\ &= -e^{-ixn}.\end{aligned}$$

Therefore, by the residue theorem, we have that

$$\begin{aligned}f\left(x\right) &= -\frac{1}{2\pi i}\left(-\pi i\right)\left(e^{ixn}+e^{-ixn}\right)\\ &= \cos(nx).\end{aligned} \tag{A.2}$$

In this last step, we only consider half the residues since $x \in (0, \pi)$.
Back to (A.1), we can take the sine expansion of ϕ as follows:

$$\phi\left(x\right) = \sum_{n\geq 1} a_n \sqrt{\frac{2}{\pi}} \sin\left(nx\right),$$

thus, by (A.2),

$$f\left(x\right) = \sum_{n\geq 1} a_n \sqrt{\frac{2}{\pi}} \cos\left(nx\right).$$

The a_n coefficients are determined by using the cosine orthonogality,

$$\begin{aligned}\int_0^{\pi} f\left(x\right)\cos\left(nx\right)dx &= \sum_{m\geq 1} a_m \sqrt{\frac{2}{\pi}} \int_0^{\pi} \cos\left(mx\right)\cos\left(nx\right)dx\\ &= a_n\sqrt{\frac{\pi}{2}},\end{aligned}$$

implying,

$$a_n = \sqrt{\frac{2}{\pi}} \int_0^{\pi} f\left(z\right)\cos\left(nz\right)dz.$$

Therefore,

$$\begin{aligned}\phi\left(x\right) &= \sum_{n\geq 1} \sqrt{\frac{2}{\pi}} \int_0^{\pi} f\left(z\right)\cos\left(nz\right)dz \sqrt{\frac{2}{\pi}}\cos\left(nx\right)\\ &= -\frac{1}{\pi}\int_0^{\pi^*} \frac{\sin\left(x\right)}{\cos\left(x\right)-\cos\left(z\right)} f\left(z\right)dz.\end{aligned} \tag{A.3}$$

Recall that the Hilbert transformation of $\phi \in L^2\left[-1, 1\right]$ is given by:

$$\frac{1}{\pi}\int_{-1}^{1^*} \frac{\phi\left(z\right)}{x-z}dz = f\left(x\right).$$

Let us consider the change of variables given by:

$$x = \cos\left(\mu\right), \quad z = \cos\left(\eta\right),$$

$$g\left(\mu\right) = f\left(\cos\left(\mu\right)\right), \quad \psi\left(\eta\right) = \phi\left(\cos\left(\eta\right)\right).$$

Thus, we have that

$$\frac{1}{\pi}\int_0^{\pi^*} \frac{\sin\left(\eta\right)}{\cos\left(\mu\right)-\cos\left(\eta\right)}\psi\left(\eta\right)dz = g\left(\mu\right),$$

where, by using (A.3), we get

$$\psi(\mu) = -\frac{1}{\pi}\int_0^{\pi^*} \frac{\sin(\mu)}{\cos(\mu)-\cos(\eta)} g(\eta)\, d\eta$$
$$= -\frac{1}{\pi}\int_0^{\pi^*} \frac{\sqrt{1-\cos^2(\mu)}}{\cos(\mu)-\cos(\eta)} g(\eta)\, d\eta.$$

Now, we go back to the original variables to obtain

$$\phi(x) = -\frac{1}{\pi}\int_{-1}^{1^*} \frac{\sqrt{1-x^2}}{\sqrt{1-z^2}} \frac{f(z)}{x-z} dz,$$

giving us the inversion formula we wanted to prove.

B Anti-symmetric pinned-end boundary condition codes

In this section, the codes for the half-plane geometry are given. The codes for the other geometries are alike with the difference of the matrix F that carries information about the geometry.

B.1 Function to compute the E matrix

```
%Lorena Correa
%Yachay Tech University
function E=matrixE_antipinned(N)
E=zeros(N,N);
for r=1:floor(N/2)
    for n=1:N
        ee=@(x) sin(n*pi*x).*chebyshevU(2*r-1,x);
        E(2*r,n)=(2/pi)*integral(ee,-1,1);
    end
end
end
```

B.2 Function to compute the D matrix

```
%Lorena Correa
%Yachay Tech University
function D=matrixD_antipinned(N)
D=zeros(N,N);
for r=1:floor(N/2)
    for n=1:N
        d=@(x) cos(n*pi*x).*chebyshevT(2*r,x);
        D(2*r,n)=integral(d,-1,1);
    end
end
end
```

B.3 Function to compute the diagonal matrix

```
%Lorena Correa
%Yachay Tech University
function diag=diagonal_antipinned(N)
diag=zeros(N,N);
for n=1:N
    diag(n,n)=(n*pi)^3;
end
end
```

B.4 Function to compute eigenvalues and eigenvectors

```
%Lorena Correa
%Yachay Tech University
function [V,vp]=main_antipinned(N)
D=matrixD_antipinned(N); %compute D matrix of dimension NxN
Ep=matrixE_antipinned(N); %compute E matrix of dimension NxN
diag=diagonal_antipinned(N); %compute diagonal matrix of dimension NxN
E=zeros(N,N); %complete E matrix
for i=2:N
    for j=1:N
       E(i,j)=Ep(i-1,j);
    end
end
P=D'*E;
Q=-diag;
[V,L]=eig(P,Q); %return eigenvalues and eigenvectors for P*x=lambda*Q*x
vp=zeros(N,1);
for i=1:N
    vp(i,1)=1/L(i,i); %eingevalues for our problem
end
end
```

B.5 Plotting convergence of an specific eigenvalue

```
%Lorena Correa
%Yachay Tech University
%plot convergence for the first eigenvalue
%we consider matrix dimension NxN, from N=10 to N=45
lambda=zeros(36,1);
Nvalue=[10:1:45];
for N=10:45
    [V,vp]=main_antipinned(N);
    lambda(N-9,1)=vp(1,1); %select the first eigenvalue for the given N
end
plot(Nvalue,lambda)
```

B.6 Plotting the free surface function

We know that the free surface function depends on t and x. For plotting the surface, we consider a fixed t. In this case, we plot for $t = 1$.

```
%Lorena Correa
%Yachay Tech
%first load eigenvectors for N=50, use eigenvectors with positive diagonal
for i=1:50
    if V(i,i)<0
        for j=1:50
            V(i,j)=-1*V(i,j);
        end
    end
end
%
% %Para lambda_1
% valor=@(j,x) (V(j,1)*sin(j*pi*x));
% x=-1:0.02:1;
% suma=zeros(length(x),1);
```

```
% for i=1:length(x)
%     for j=1:50
%         suma(i,1)=suma(i,1)+valor(j,x(1,i));
%     end
% end
% plot(x,suma)
% hold on
% plot(x, sin(pi*x),'-.')
% hold off
% %
%Para lambda_2
valor=@(j,x) (V(j,2)*sin(j*pi*x));
x=-1:0.02:1;
suma=zeros(length(x),1);
for i=1:length(x)
    for j=1:50
        suma(i,1)=suma(i,1)+valor(j,x(1,i));
    end
end
plot(x,suma)
hold on
plot(x, sin(2*pi*x),'-.')
hold off
%
% %Para lambda_3
% valor=@(j,x) (V(j,3)*sin(j*pi*x));
% x=-1:0.02:1;
% suma=zeros(length(x),1);
% for i=1:length(x)
%     for j=1:50
%         suma(i,1)=suma(i,1)+valor(j,x(1,i));
%     end
% end
% plot(x,suma)
% hold on
% plot(x, sin(3*pi*x),'.-')
% hold off
```

C Symmetric pinned-end condition codes

Plotting the free surface and the convergence of the eigenvalues is analogous to the anti-symmetric pinned-end case. The rest of the codes are listed below.

C.1 Function to compute the E matrix

```
%Lorena Correa
%Yachay Tech
function E=matrixE_symmpinned(N)
E=zeros(N,N);
for r=1:ceil(N/2)
    for n=1:N
        ee=@(x) cos((n-0.5)*pi*x).*chebyshevU(2*r-2,x);
        E(2*r-1,n)=(2/pi)*integral(ee,-1,1);
    end
```

```
end
end
```

C.2 Function to compute the D matrix

```
%Lorena Correa
%Yachay Tech
function D=matrixD_symmpinned(N)
D=zeros(N,N);
for r=1:ceil(N/2)
    for n=1:N
        d=@(x) sin((n-0.5)*pi*x).*chebyshevT(2*r-1,x);
        D(2*r-1,n)=integral(d,-1,1);
    end
end
end
```

C.3 Function to compute the diagonal matrix

```
%Lorena Correa
%Yachay Tech
function diag=diagonal_symmpinned(N)
diag=zeros(N,N);
for n=1:N
    diag(n,n)=((n-0.5)*pi)^3;
end
end
```

C.4 Function to compute mass conservation condition

```
%Lorena Correa
%Yachay Tech
%compute mass conservation condition for an NxN matrix
function omega=mass_conservation(N)
omega=zeros(1,N);
for n=1:N
    omega(1,n)=((-1)^(n+1))*2/((n-0.5)*pi);
end
end
```

C.5 Function to compute eigenvalues and eigenvectors

```
%Lorena Correa
%Yachay Tech
function [V,vp]=main_symmpinned(N)
vp=zeros(N,1);
omega=mass_conservation(N);
D=matrixD_symmpinned(N); %compute D matrix of dimension NxN
E=matrixE_symmpinned(N); %compute E matrix of dimension NxN
diag=diagonal_symmpinned(N); %compute diagonal matrix of dimension NxN
Pi=D'*E;
P=zeros(N,N);
for n=1:N
    P(1,n)=omega(n); %impose mass conservation condition
end
for r=2:N
    for n=1:N
```

```
            P(r,n)=Pi(r-1,n);
        end
end
Q=zeros(N,N);
for i=2:N %first row remains as zeros (mass conservation)
        for j=1:N
            Q(i,j)=diag(i-1,j);
        end
end
[V,L]=eig(P,Q);
for i=1:N
      vp(i,1)=1/L(i,i); %select  the  first  eigenvalue  for  the  given N
end
end
```

D Symmetric free-end condition codes

Computing eigenvalues and eigenvectors, and plotting the free surface and convergence of eigenvalues are analogous to the anti-symmetric pinned-end case. Functions to compute E, D and diagonal matrices are below.

D.1 Function to compute the E matrix

```
%Lorena Correa
%Yachay Tech
function E=matrixE_symmfree(N)
E=zeros(N,N);
for r=2:ceil(N/2)
    for n=1:N
        ee=@(x) cos(n*pi*x).*chebyshevU(2*r-2,x);
        E(2*r-1,n)=(2/pi)*integral(ee,-1,1);
    end
end
end
```

D.2 Function to compute the D matrix

```
%Lorena Correa
%Yachay Tech
function D=matrixD_symmfree(N)
D=zeros(N,N);
 for r=1:ceil(N/2)
    for n=1:N
        d=@(x) sin(n*pi*x).*chebyshevT(2*r-1,x);
        D(2*r-1,n)=integral(d,-1,1);
    end
end
end
```

D.3 Function to compute the diagonal matrix

```
%Lorena Correa
%Yachay Tech
function diag=diagonal_symmfree(N)
diag=zeros(N,N);
for n=1:N
    diag(n,n)=(n*pi)^3;
```

```
end
end
```

E Anti-symmetric free-end condition codes

Computing eigenvalues and eigenvectors, and plotting the free surface and convergence of eigenvalues are analogous to the anti-symmetric pinned-end case. Functions to compute E, D and diagonal matrices are below.

E.1 Function to compute the E matrix

```
%Lorena Correa
%Yachay Tech
function E=matrixE_antifree(N)
E=zeros(N,N);
for r=1:floor(N/2)
    for n=1:N
        ee=@(x) sin((n-0.5)*pi*x).*chebyshevU(2*r-1,x);
        E(2*r,n)=(2/pi)*integral(ee,-1,1);
    end
end
end
```

E.2 Function to compute the D matrix

```
%Lorena Correa
%Yachay Tech
function D=matrixD_antifree(N)
D=zeros(N,N);
for r=1:floor(N/2)
    for n=1:N
        d=@(x) cos((n-0.5)*pi*x).*chebyshevT(2*r,x);
        D(2*r,n)=integral(d,-1,1);
    end
end
end
```

E.3 Function to compute the diagonal matrix

```
%Lorena Correa
%Yachay Tech
function diag=diagonal_antifree(N)
diag=zeros(N,N);
for n=1:N
    diag(n,n)=((n-0.5)*pi)^3;
end
end
```