UNIVERSIDAD DE INVESTIGACIÓN DE TECNOLOGÍA EXPERIMENTAL YACHAY

Escuela de Ciencias Físicas y Nanotecnología

TÍTULO: Anisotropic interior solutions and Buchdahl's limit in the context of gravitational decoupling

Trabajo de integración curricular presentado como requisito para la obtención del título de Física

Autora:

Arias Pruna Cynthia Belén

Tutor:

Ph. D. Medina Dagger Ernesto Antonio

Co-Tutor:

Ph. D. Contreras Herrada Ernesto José

Urcuquí, julio 2020

Urcuquí, 6 de marzo de 2020

SECRETARÍA GENERAL
(Vicerrectorado Académico/Cancillería)
ESCUELA DE CIENCIAS FÍSICAS Y NANOTECNOLOGÍA
CARRERA DE FÍSICA
ACTA DE DEFENSA No. UITEY-PHY-2020-00003-AD

En la ciudad de San Miguel de Urcuquí, Provincia de Imbabura, a los 6 días del mes de marzo de 2020, a las 16:30 horas, en el Aula 1 de la Universidad de Investigación de Tecnología Experimental Yachay y ante el Tribunal Calificador, integrado por los docentes:

Presidente Tribunal de Defensa	Dr. RAMIREZ VELASQUEZ JOSE MANUEL , Ph.D.
Miembro No Tutor	Dra. ROJAS CELY CLARA INES , Ph.D.
Tutor	Dr. MEDINA DAGGER, ERNESTO ANTONIO, Ph.D.

Se presenta el(la) señor(ita) estudiante **ARIAS PRUNA, CYNTHIA BELEN**, con cédula de identidad No. **0550118798**, de la **ESCUELA DE CIENCIAS FÍSICAS Y NANOTECNOLOGÍA**, de la Carrera de **FÍSICA**, aprobada por el Consejo de Educación Superior (CES), mediante Resolución **RPC-SO-39-No.456-2014**, con el objeto de rendir la sustentación de su trabajo de titulación denominado: **Anisotropic interior solutions and Buchdahl ´s limit in the context of gravitational decoupling**, previa a la obtención del título de **FÍSICO/A.**

El citado trabajo de titulación, fue debidamente aprobado por el(los) docente(s):

Tutor	Dr. MEDINA DAGGER, ERNESTO ANTONIO, Ph.D.

Y recibió las observaciones de los otros miembros del Tribunal Calificador, las mismas que han sido incorporadas por el(la) estudiante.

Previamente cumplidos los requisitos legales y reglamentarios, el trabajo de titulación fue sustentado por el(la) estudiante y examinado por los miembros del Tribunal Calificador. Escuchada la sustentación del trabajo de titulación, que integró la exposición de el(la) estudiante sobre el contenido de la misma y las preguntas formuladas por los miembros del Tribunal, se califica la sustentación del trabajo de titulación con las siguientes calificaciones:

Tipo	Docente	Calificación
Tutor	Dr. MEDINA DAGGER, ERNESTO ANTONIO, Ph.D.	9,9
Presidente Tribunal De Defensa	Dr. RAMIREZ VELASQUEZ JOSE MANUEL , Ph.D.	7,7
Miembro Tribunal De Defensa	Dra. ROJAS CELY CLARA INES , Ph.D.	9,9

Lo que da un promedio de: **9.2 (Nueve punto Dos)**, sobre 10 (diez), equivalente a: **APROBADO**

Para constancia de lo actuado, firman los miembros del Tribunal Calificador, el/la estudiante y el/la secretario ad-hoc.

ARIAS PRUNA, CYNTHIA BELEN
Estudiante

Dr. RAMIREZ VELASQUEZ JOSE MANUEL , Ph.D.
Presidente Tribunal de Defensa

Firmado electrónicamente por:
JOSE MANUEL
RAMIREZ
VELASQUEZ

Dr. MEDINA DAGGER, ERNESTO ANTONIO, Ph.D.
Tutor

Firmado electrónicamente por:
ERNESTO ANTONIO
MEDINA DAGGER

Dra. ROJAS CELY CLARA INES , Ph.D.
Miembro No Tutor

Firmado electrónicamente por:
CLARA INES ROJAS CELY

CIFUENTES TAFUR, EVELYN CAROLINA
Secretario Ad-hoc

Firmado electrónicamente por:
EVELYN CAROLINA CIFUENTES TAFUR

AUTORÍA

Yo, **CYNTHIA BELÉN ARIAS PRUNA**, con cédula de identidad 0550118798, declaro que las ideas, juicios, valoraciones, interpretaciones, consultas bibliográficas, definiciones y conceptualizaciones expuestas en el presente trabajo; así como, los procedimientos y herramientas utilizadas en la investigación, son de absoluta responsabilidad de la autora del trabajo de integración curricular. Así mismo, me acojo a los reglamentos internos de la Universidad de Investigación de Tecnología Experimental Yachay.

Urcuquí, julio del 2020.

Cynthia Belén Arias Pruna
CI: 0550118798

AUTORIZACIÓN DE PUBLICACIÓN

Yo, **CYNTHIA BELÉN ARIAS PRUNA**, con cédula de identidad 0550118798, cedo a la Universidad de Tecnología Experimental Yachay, los derechos de publicación de la presente obra, sin que deba haber un reconocimiento económico por este concepto. Declaro además que el texto del presente trabajo de titulación no podrá ser cedido a ninguna empresa editorial para su publicación u otros fines, sin contar previamente con la autorización escrita de la Universidad.

Asimismo, autorizo a la Universidad para que realice la digitalización y publicación de este trabajo de integración curricular en el repositorio virtual, de conformidad a lo dispuesto en el Art. 144 de la Ley Orgánica de Educación Superior.

Urcuquí, julio del 2020.

Cynthia Belén Arias Pruna
CI: 0550118798

Acknowledgements

I would like to express my sincere gratitude to Ernesto Contreras, Ph.D., for his continuous support, patience, motivation and immense knowledge during the realization of this project.

My sincere thanks to all my professors for inspire me by sharing the best of their knowledge and experience although the difficult times. I thank to my classmates, specially to my closest friends for making every day lighter and special.

Last but not the least, I would like to thank my family, specially to my parents, for their warm support and encouragement along this five years. Ending this stage of my life successfully is dedicated to them.

Cynthia Belén Arias Pruna

Abstract

The stellar evolution is a process through which a star changes along the time depending mainly of its mass. In this context, there exists a condition known as the Buchdahl's limit which describes the amount of mass that a spherically compact object can have before undergoing into gravitational collapse. In other words, the Buchdahl's limit give us an idea about the minimum radius that delimits a star under certain conditions. This kind of systems have been studied in terms of an extended version of the well known isotropic interior solutions, which corresponds to the study of anisotropic solutions. Anisotropic sources can be included by means of the gravitational decoupling using the Minimal Geometric Deformation method (MGD) to understand better the properties of this more realistic stellar systems under different field conditions. With the gravitational decoupling by MGD we can start from a simple spherically symmetric gravitational source and add to it more and more complex gravitational sources, as long as the spherical symmetry is preserved. The aim of the present project is to know whether the Buchdahl's limit is modified by applying the gravitational decoupling by MGD method to compact objects. To accomplish the purpose of this research, the Tolman IV interior solution was studied where bounds and conditions over its parameters were applied to let a plausible solution. It was found that MGD allows to map unstable isotropic solutions to anisotropic stable configurations with extra packing of mass.

Keywords: MGD, Buchdahl's-limit, Tolman IV, anisotropy, extra-packing.

Resumen

La evolución estelar es un proceso a través del cual una estrella cambia a lo largo del tiempo, dependiendo principalmente de su masa. En este contexto, existe una condición conocida como el límite de Buchdahl que describe la cantidad de masa que puede tener un objeto esférico compacto antes de sufrir un colapso gravitacional. En otras palabras, el límite de Buchdahl nos da una idea sobre el radio mínimo que delimita una estrella bajo ciertas condiciones. Este tipo de sistemas se han estudiado en términos de una versión extendida de las conocidas soluciones interiores isotrópicas, que corresponde al estudio de soluciones anisotrópicas. Las fuentes anisotrópicas pueden incluirse mediante desacoplamiento gravitacional utilizando el método de deformación geométrica mínima (MGD) para comprender mejor las propiedades de estos sistemas estelares mucho mÃąs realistas en diferentes condiciones de campo. Con el desacoplamiento gravitacional de MGD podemos comenzar desde una fuente gravitacional esférica simétrica simple y agregarle fuentes gravitacionales más complejas, siempre que se conserve la simetría esférica. El objetivo del presente proyecto de investigación es saber si el límite de Buchdahl se modifica aplicando el desacoplamiento gravitacional por el método MGD a objetos compactos. Para lograr el propósito de esta investigación, se estudió la solución interior Tolman IV donde se aplicaron límites y condiciones sobre sus parámetros para permitir una solución plausible. Se descubrió que MGD permite mapear soluciones isotrópicas inestables a configuraciones estables anisotrópicas con masa adicional en sus configuraciones.

Palaras Clave: MGD, límite de Buchdahl, Tolman IV, anisotropía.

Contents

List of Figures

List of Tables

List of Papers

[1] Arias, C., Tello-Ortiz, F. & Contreras, E. Extra packing of mass of anisotropic interiors induced by MGD. Eur. Phys. J. C 80, 463 (2020). https://doi.org/10.1140/epjc/s10052-020-8042-3

Chapter 1

Introduction

The theory of General Relativity (GR) was proposed by Albert Einstein (1879-1955) in the early part of the 20^{th} century and is one of the most significant scientific advances of our time. GR postulates that gravitational effects may be explained by the curvature of spacetime which is modeled by a four-dimensional pseudo-Riemannian manifold, and that gravity should not be regarded as a force in the conventional sense [1]. Within the theory Einstein's equations were developed to provide a precise formulation of the relationship between spacetime geometry and the properties of matter.

The study of relativistic stellar structure began with the discovery of the Schwarzschild's exterior and interior solutions [2,3]. For a long time the star interior was considered to be made of perfect fluid, ı.e, equal radial and tangential pressure ($p_r = p_t = p$). Regardingly, the work developed by Tolman [4] about spherically symmetric and static fluid spheres driven by an isotropic matter distribution, began a new stage to seek analytic solutions to Einstein's field equations describing exciting compact structures such as neutron stars, white dwarfs, etc, that provide information to understand its behavior in the strong gravitational field regime.

Buchdahl's studies [5] have determined that for an isotropic matter distribution the ratio M/R can not exceed the upper bound $M/R \leq 4/9$. However, it is well known that celestial bodies are not necessarily made of isotropic matter. This is supported by the theory developed by Ruderman [6] in 1972, who described more realistic stellar models which have anisotropic behavior ($p_r \neq p_t$) at least in certain very high density ranges ($\rho > 10^{15}\mathrm{g/cm^3}$), where the nuclear interactions must be treated in a relativistic framework.

Following this direction, Bowers and Liang [7] reported the importance of locally anisotropic equations of state for relativistic spheres. They found that the contributions coming from local anisotropies into the Tolman-Oppenheimer-Volkoff (TOV) equation [6,8] have Newtonian origin. Besides, they studied the impact of these anisotropies on the surface gravitational red-shift $z(r)$. To address this point they considered an incompressible fluid (energy-density ρ =constant) and determined under this condition that if the fluid has an isotropic behavior the maximum $z(r)$ is 4.77

and if the material contains anisotropies, this value can be exceeded and even infinite. Although this is not a real situation, this work was the departure point to understand how local anisotropies influence on the main properties of astrophysical bodies. Nowadays, the study of high density objects has become a wide and active research branch from the theoretical and observational point of view. The effects of the anisotropic matter distribution on the main properties of compact objects in the arena of GR have been extensively studied.

1.1 Problem Statement

Assuming local isotropy when studying massive objects is really common. Nevertheless, it has been raised the need of studying situations beyond the trivial cases, such as the interior of stellar structures with gravitational sources more complex than the ideal perfect fluid[9–21]. Some models have been proposed in order to describe more realistic stellar systems. They suggest that stellar matter may be anisotropic at least in certain density ranges[6,22].

An anisotropy could be introduced by the existence of a solid core, by the presence of type P superfluid, or by other physical phenomena[23]. Several works has been already introduced anisotropic matter and it has been shown that some properties may differ drastically from the isotropic spheres[7,24]. The introduction of anisotropies in Einstein's equation leads to technical difficulties associated to the finding of analytical solutions. However, as we shall explain further, a method to introduce anisotropies is the so called Minimal Geometric Deformation (MGD) method[25–27], which is a simple, systematic and direct approach to extend isotropic and physically accepted solutions to anisotropic domains.

Moreover, Buchdahl's studies[5] have determined that for an isotropic matter distribution the ratio M/R can not exceed the upper bound $M/R \leq 4/9$. However, celestial bodies are not necessarily made of isotropic matter. A study made by Böhmer and Harko[28] about the derivation of the corresponding upper bound for the mass-radius relation for an anisotropic matter distribution, suggests that it is true that the Buchdahl' s limit is modified when we introduce an anisotropy to the system. This was shown in presence of a cosmological constant Λ.

The main concern of this project is to discover whether the introduction of anisotropies to the system through MGD still leads to modification of the Buchdahl's limit.

1.2 General and Specific Objectives

The general objective of this project is to study the modifications on the Buchdahl's limit which can be induced by the introduction of anisotropies through the MGD method. To reach this, first it is needed the study of the plausibility conditions of anisotropic solutions given by the MGD method. Once that conditions are achieved, a specific solution is considered which, in our case, corresponds to the extended case of the Tolman IV isotropic solution to anisotropic domains by MGD. At the end, finding a condition of extra packing of mass will establish the modification on the Buchdahl's limit.

Chapter 2

Methodology

2.1 Einstein's field equations

Einstein's field equations, relate the geometry of spacetime with the distribution of matter within it. This equation was first published by Einstein in 1915 in the form of a tensor equation[29] relating the local spacetime curvature (expressed by the Einstein tensor) with the local energy and momentum within that spacetime (given by the stress-energy tensor)[30]. The Einstein's field equations for General Relativity tells us how the curvature of spacetime reacts to the presence of energy-momentum and are given by

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = kT_{\mu\nu}, \tag{2.1}$$

where $k = \frac{-8\pi G}{c^4}$.
At the left hand side we have the following components:

- $R_{\mu\nu}$ is the *Ricci tensor* is defined as the contraction of the *Riemann tensor* $R_{\lambda\mu\rho\nu}$ as:

$$R_{\mu\nu} = g^{\lambda\rho}R_{\lambda\mu\rho\nu} = R^{\rho}_{\mu\rho\nu}. \tag{2.2}$$

The *Riemann tensor*, also called the curvature tensor, is defined in terms of the Christoffel symbols, namely $\Gamma^{\sigma}_{\lambda\mu}$, and their derivatives as:

$$R^{\lambda}_{\mu\rho\nu} = \partial_{\rho}\Gamma^{\lambda}_{\mu\nu} - \partial_{\nu}\Gamma^{\lambda}_{\mu\rho} + \Gamma^{\lambda}_{\sigma\rho}\Gamma^{\sigma}_{\mu\nu} - \Gamma^{\lambda}_{\sigma\nu}\Gamma^{\sigma}_{\mu\rho}. \tag{2.3}$$

The Christoffel symbols are related to the partial derivatives of the *metric tensor* $g_{\mu\nu}$ as follows:

$$\Gamma^{\nu}_{\lambda\mu} = \frac{1}{2}g^{\nu\sigma}(\partial_{\mu}g_{\sigma\lambda} + \partial_{\lambda}g_{\sigma\mu} - \partial_{\sigma}g_{\mu\lambda}). \tag{2.4}$$

- R is the *curvature scalar* or *Ricci scalar*, which is defined as the contraction of the *Ricci tensor* as:

$$R = g^{\mu\nu} R_{\mu\nu} = R^{\mu}_{\nu}. \tag{2.5}$$

At the right hand side we have the following elements:

- G is the Newton's constant of gravitation.
- c is the speed of light.
- $T_{\mu\nu}$ is the energy-momentum tensor, which is a generalization of the mass density and it is the source of the gravitational field. For a perfect fluid it reads:

$$T_{\mu\nu} = (\rho + p) U_{\mu} U_{\nu} - p g_{\mu\nu}, \tag{2.6}$$

  where ρ is the energy density*

$$\rho = T^{0}_{0},$$

  p is the isotropic pressure which reads

$$p = -T^{1}_{1} = -T^{2}_{2} = -T^{3}_{3},$$

  and U is the fluid's four velocity.

Einstein's equations are second-order differential equations for the metric tensor field $g_{\mu\nu}$. Since both sides of Einstein's equation (2.1) are symmetric two-index tensors, there are ten independent equations for ten unknown functions of the metric components to be determined: six components of $g_{\mu\nu}$, three components of the four-vector velocity U^{μ}, and the density of the matter ρ or its pressure p. As differential equations, the system is extremely complicated to solve since the Ricci scalar and tensor are contractions of the Riemann tensor composed by derivatives and products of the Christoffel symbols, which in turn involve the inverse metric and derivatives of the metric. Furthermore, the energy-momentum tensor $T_{\mu\nu}$ will generally involve the metric also and the equations are nonlinear, so that two known solutions cannot be superposed to find a third one. It is therefore very difficult to solve Einstein's equations, so it is usually necessary to make some simplifying assumptions.

These equations were originally derived by assuming the following requirements:

- Field equations should be tensor equations in order to exhibit covariance because the choice of the reference system is arbitrary and laws of nature must be formally the same for any coordinate system (x^0, x^1, x^2, x^3).

*We are assuming signature (+ - - -) for the metric tensor

- As any field equation of physics, they must be partial differential equations of, at most, second order in time for the components of the metric tensor $g_{\mu\nu}$, which are the functions to be determined.
- The spatial derivatives are of, at most, second order and the equations must be also linear in the highest derivatives.
- They should reduce to the Poisson equation in the appropriate limit of the Newtonian theory

$$\nabla^2\phi = 4\pi G\rho, \tag{2.7}$$

in which ϕ is the potential of the gravitational field and ρ is the mass density of the source. The so called "Newtonian limit" is obtained by assuming a weak and static field and that the velocities of the sources of the latter are very small compared to the velocity of light c.

- The energy-momentum tensor $T_{\mu\nu}$ should be the source of the gravitational field.

2.2 Classical solutions of Einstein's equations

2.2.1 Schwarzschild's exterior solution

As we saw, solving the field equations for $g_{\mu\nu}$ is not easy due to the high degree of nonlinearity of the equations. The problem becomes easier if one looks for special solutions, for example those which involve symmetries and extra constraints. In this sense, the first exact solution was obtained by K. Schwarzschild in 1916[2] by considering the following assumptions:

- Static field.
- Spherically symmetric field.
- Empty spacetime.
- Asymptotically flat spacetime.

It means that Schwarzschild sought the metric tensor field representing the static spherically symmetric gravitational field in the empty spacetime surrounding some massive spherical object like a star. This solution is valid only outside the gravitating body. In other words, for a spherical body of radius R, the solution is valid for $r > R$. To describe the gravitational field both inside and outside the gravitating body, the Schwarzschild's solution must be matched with some suitable interior solution at $r = R$, such as the interior Schwarzschild's metric developed by himself afterwards[3]. Schwarzschild also assumed that spacetime could be expressed by coordinates (t, r, θ, ϕ), where t was a timelike coordinate, θ and ϕ were polar angles, and r was some radial coordinate[1]. He then postulated the line element

$$ds^2 = e^{2\alpha(r)}dt^2 - e^{2\beta(r)}dr^2 - r^2 d\Omega^2, \tag{2.8}$$

where $d\Omega = d\theta^2 + \sin^2\theta d\phi^2$, also $\alpha(r)$ and $\beta(r)$ were unknown functions of r obtained by solving the field equations. Let us now use $g_{\mu\nu}$, obtained from the line element (2.8) as a trial solution for the empty spacetime field equations. Let us begin by evaluating the corresponding non-zero Christoffel symbols, which are given by:

- $\Gamma^t_{tr} = \partial_r\alpha$,
- $\Gamma^r_{tt} = e^{2(\alpha-\beta)}\partial_r\alpha$,
- $\Gamma^r_{rr} = \partial_r\beta$,
- $\Gamma^r_{\theta\theta} = -re^{-2\beta}$,
- $\Gamma^r_{\phi\phi} = -re^{-2\beta}\sin^2\theta$,
- $\Gamma^\theta_{r\theta} = \frac{1}{r}$,
- $\Gamma^\theta_{\phi\phi} = -\sin\theta\cos\theta$,
- $\Gamma^\phi_{\theta\phi} = \cot\theta$.

The coordinates were labeled according to $x^0 \equiv t$, $x^1 \equiv r$, $x^2 \equiv \theta$, $x^3 \equiv \phi$. After substitution of this coefficients on the Riemann tensor it reads:

$$R^t_{rtr} = \partial_r\alpha\partial_r\beta - \partial_r^2\alpha - (\partial_r\alpha)^2, \tag{2.9}$$
$$R^t_{\theta t\theta} = -re^{-2\beta}\partial_r\alpha, \tag{2.10}$$
$$R^t_{\phi t\phi} = -re^{-2\beta}\sin^2\theta\partial_r\alpha, \tag{2.11}$$
$$R^r_{\theta r\theta} = re^{-2\beta}\partial_r\beta, \tag{2.12}$$
$$R^r_{\phi r\phi} = re^{-2\beta}\sin^2\theta\partial_r\beta, \tag{2.13}$$
$$R^\theta_{\phi\theta\phi} = (1 - e^{-2\beta})\sin^2\theta. \tag{2.14}$$
$$\tag{2.15}$$

Then, taking the contraction yields to the Ricci tensor:

$$R_{tt} = e^{2(\alpha-\beta)}\left[\partial_r^2\alpha + (\partial_r\alpha)^2 - \partial_r\alpha\partial_r\beta + \frac{2}{r}\partial_r\alpha\right], \tag{2.16}$$
$$R_{rr} = -\partial_r^2\alpha - (\partial_r\alpha)^2 + \partial_r\alpha\partial_r\beta + \frac{2}{r}\partial_r\beta, \tag{2.17}$$
$$R_{\theta\theta} = e^{-2\beta}\left[r(\partial_r\beta - \partial_r\alpha) - 1\right] + 1, \tag{2.18}$$
$$R_{\phi\phi} = \sin^2\theta R_{\theta\theta}. \tag{2.19}$$
$$\tag{2.20}$$

Now, we would like to set this Ricci tensor equal to zero. Since R_{tt} and R_{rr} vanish independently, we can write:

$$0 = e^{2(\alpha-\beta)}R_{tt} + R_{rr} = \frac{2}{r}(\partial_r\alpha + \partial_r\beta),$$

which implies that $\alpha = -\beta$.
Then, from $R_{\theta\theta}$=0

$$e^{2\alpha}(2r\partial_r\alpha + 1) = 1 \implies \partial_r(re^{2\alpha}) = 1.$$

After solving it we obtain

$$e^{2\alpha} = 1 - \frac{\kappa}{r}, \tag{2.21}$$

where the constant κ is given by:

$$\kappa = 2GM.$$

Finally, the Schwarzschild's exterior solution for the empty spacetime outside a spherical body of mass M is:

$$ds^2 = \left(1 - \frac{2GM}{r}\right)dt^2 - \left(1 - \frac{2GM}{r}\right)^{-1} dr^2 - r^2 d\Omega. \tag{2.22}$$

It is worth to mention that the present work is about the study of the stellar interior (where the radius of the object is greater than the event horizon) of a central massive object surrounded by an empty spacetime, so the Schwarzschild's exterior solution will be used to match the interior solution with its exterior and to ensure a radial pressure that vanishes at the surface of the object. In other words, it will be used to fulfill the continuity of the first and second fundamental form, conditions that will be discussed later.

2.2.2 Schwarzschild's interior solution

In 1916, Schwarzschild found this solution[3] months after he obtained the exterior solution in (2.22), to determine the spherically gravitational field inside the matter. The approximation for a perfect fluid is used to describe the energy-momentum tensor associated to matter

$$T_{\mu\nu} = \begin{pmatrix} \rho & 0 & 0 & 0 \\ 0 & -p & 0 & 0 \\ 0 & 0 & -p & 0 \\ 0 & 0 & 0 & -p \end{pmatrix} \tag{2.23}$$

So, we can construct a vector such that

$$U_\mu U^\mu = -1 \iff g^{\mu\nu} U_\mu U\nu = -1.,$$

where U_μ=$(U_0, 0, 0, 0)$ (zero for the spatial part), are the comoving coordinates.
Then,

$$g^{\mu\nu} U_\mu U_\nu = g^{00} U_0 U_0 = -e^{2\alpha} U_0^2 = -1.$$

Therefore,

$$U_0 = e^\alpha \implies U_\mu = (e^\alpha, 0, 0, 0) \tag{2.24}$$

Now, the energy-momentum tensor reads:

$$\begin{aligned} T_{tt} &= (\rho + p)U_0U_0 + pg_{00} = \rho e^{2\alpha}, && (2.25)\\ T_{rr} &= pe^{2\beta}, && (2.26)\\ T_{\theta\theta} &= r^2 p, && (2.27)\\ T_{\phi\phi} &= r^2 \sin^2\theta p. && (2.28) \end{aligned}$$

Given that these objects transform as a tensor, they are invariant under a general coordinate transformation. So, we need to solve:

$$\begin{aligned} \frac{1}{r^2}e^{-2\beta}(2r\partial_r\beta - 1 + e^{2\beta}) &= 8\pi G\rho, && (2.29)\\ \frac{1}{r^2}e^{-2\beta}(2r\partial_r\alpha + 1 - e^{2\beta}) &= 8\pi G p, && (2.30)\\ e^{-2\beta}\left(\partial_r^2\alpha + (\partial_r\alpha)^2 - \partial_r\alpha\partial_r\beta\right) + \frac{1}{r}(\partial_r\alpha - \partial_r\beta) &= 8\pi G p. && (2.31) \end{aligned}$$

We have four unknowns and three partial differential equations. So, we need extra information from an equation of state (EOS). One of these EOS is the barotropic equation of state

$$p = p(\rho). \tag{2.32}$$

Moreover, in order to match with Schwarzschild's exterior solution, the structural form of the interior solution is given by

$$e^{2\beta} \equiv \frac{1}{1 - \frac{2Gm(r)}{r}}. \tag{2.33}$$

It is important to mention that the total mass of the star and the mass function $m(r)$ are related in the following way:

$$M = \int_0^R m(r)dr. \tag{2.34}$$

Replacing (2.33) in the first Einstein's field equation, we obtain that

$$\frac{dm}{dr} = 4\pi r^2\rho \implies m(r) = 4\pi\int_0^r \rho(r')r'^2 dr', \tag{2.35}$$

which corresponds to the mass function in terms of the energy density ρ. Now, the total mass is given by

$$M = 4\pi\int_0^R \rho(r')r'^2 dr'. \tag{2.36}$$

In terms of the mass function, we obtain that

$$\frac{d\alpha}{dr} = \frac{Gm(r) + 4\pi G r^3 p}{r\left[r - 2Gm(r)\right]}. \tag{2.37}$$

Then, we can use the conservation equation $\nabla_\mu T^{\mu\nu}$=0, which holds

$$(\rho + p)\frac{d\alpha}{dr} = -\frac{dp}{dr}, \tag{2.38}$$

from where the Tolman-Oppenheimer-Volkoff (TOV) equation reads

$$\frac{dp}{dr} = \frac{-(\rho + p)\left[Gm(r) + 4\pi G r^3 p\right]}{r\,[r - 2Gm(r)]}. \tag{2.39}$$

Now, let us obtain an interior solution by considering the following simplest model for ρ=ρ_0:

$$m(r) = \begin{cases} \frac{4}{3}\pi r^3 \rho_0 & r < R, \\ \frac{4}{3}\pi R^3 \rho_0 & r > R. \end{cases}$$

Replacing $m(r)$ in TOV equation (2.39), it becomes

$$p = \rho_0 \left(\frac{(R\sqrt{R - 2GM} - \sqrt{R^3 - 2GMr^2}}{\sqrt{R^2 - 2GMr^2} - 3R\sqrt{R^2 - 2GM}} \right). \tag{2.40}$$

Now, from equation (2.37) we obtain:

$$e^{\alpha} = \frac{3}{2}\left(1 - \frac{2GM}{R}\right)^{1/2} - \frac{1}{2}\left(1 - \frac{2GMr^2}{R^3}\right)^{1/2}. \tag{2.41}$$

And from equation (2.33) we get:

$$e^{2\beta} = \frac{1}{1 - \frac{2GMr^2}{R^3}}. \tag{2.42}$$

Finally, replacing the obtained expressions (2.41) and (2.42) in the line element (2.8), we get the interior Schwarzschild's solution for an incompressible fluid, which reads:

$$ds^2 = \frac{1}{4}\left(3\sqrt{1 - \frac{2GM}{R}} - \sqrt{1 - \frac{2GMr^2}{R^3}}\right)^2 dt^2 - \left(1 - \frac{2GMr^2}{R^3}\right)^{-1} dr^2 - r^2 d\Omega^2. \tag{2.43}$$

2.2.3 Tolman IV solution

Besides the well known Schwarzschild's solutions, there are other explicit solutions of physical interest. In reference [4], Tolman found a method to obtain analytic interior solutions. The method consists in to consider geometrical constraints to close the system. Solution IV (or Tolman IV solution) for perfect fluids, is the first original obtained solution. For this study, let us recall the Schwarzschild-like coordinates for a static spherically symmetric metric in the following way:

$$ds^2 = e^{\nu}(r)dt^2 - \frac{dr^2}{\mu(r)} - r^2 d\Omega. \tag{2.44}$$

In this case the geometric constraint reads

$$\frac{e^{\nu}\nu'}{2r} = const.$$

Then, the resulting solution is

$$e^{\nu} = B^2\left(1 + \frac{r^2}{A^2}\right), \tag{2.45}$$

$$\mu = e^{-\lambda} = \frac{\left(1 - \frac{r^2}{C^2}\right)\left(1 + \frac{r^2}{A^2}\right)}{1 + \frac{2r^2}{A^2}}, \tag{2.46}$$

$$\rho = \frac{3A^4 + A^2(3C^2 + 7r^2) + 2r^2(C^2 + 3r^2)}{k^2C^2(A^2 + 2r^2)^2}, \tag{2.47}$$

and

$$p = \frac{C^2 - A^2 - 3r^2}{k^2C^2(A^2 + 2r^2)}. \tag{2.48}$$

Here, A, B and C are constants of integration which are determined from the matching conditions between the above interior solution and the exterior metric, in this case the Schwarzschild's exterior solution. The matching conditions are given by:

- Continuity of the first fundamental form at the surface of the star $\sum$ defined by $r = R$

  $$[ds^2]_{\Sigma} = 0, \tag{2.49}$$

  where $[F]_{\Sigma} \equiv F(r \to R^+) - F(r \to R^-) \equiv F^+_R - F^-_R$, for any function $F = F(r)$.

  That states that an interior solution must be matched continuously to the exterior Schwarzschild solution (2.22), namely

  $$e^{\nu(R)} = \mu(R) = 1 - \frac{2M}{R}. \tag{2.50}$$

- Continuity of the second fundamental form which says that the radial pressure must vanish at the surface

  $$[G_{\mu\nu}r^{\nu}]_{\Sigma} = 0, \tag{2.51}$$

  where r_{μ} is an unit radial vector.

From the above conditions we obtain that:

$$\frac{A^2}{R^2} = \frac{R - 3M_0}{M_0}, \tag{2.52}$$

$$B^2 = 1 - \frac{3M_0}{R}, \tag{2.53}$$

$$\frac{C^2}{R^2} = \frac{R}{M_0}, \tag{2.54}$$

where M_0 is the total mass of the star.

2.3 Physical plausibility conditions of interior solutions

In this section, we describe the conditions that the interior solutions for Einstein's field equations might fulfill for a suitable description of a physical system. Some of the conditions that are useful for the present work are described below.

Matching conditions

This condition was previously discussed in last section (2.2.3) and it corresponds to the continuity of the first and the second fundamental form.

Geometric and matter sector

On one hand, the metric potentials of a physically acceptable interior solution must be positive, finite and free of singularities in the interior and centre by satisfying $e^{-\lambda}(0) = 1$ and $e^{\nu(0)} = constant$. On the other hand, conditions on the matter sector state that the density and the pressures should be positive inside the star. Besides, they should reach a maximum at the centre, so $\rho'(0) = p_r'(0) = p_t'(0) = 0$ and should decrease monotonously outwards, $\rho' \leqslant 0$, $p_r' \leqslant 0$, $p_t' \leqslant 0$.

Redshift condition

The interior redshift z is defined as $z(r) = e^{-\nu/2} - 1$. Note that, given that ν is an increasing and positive function, the redshift z should decrease as long as r increases. Additionally, the surface redshift is related with the compactness parameter, $u = M/R$ through

$$z(R) = (1 - u)^{-1/2} - 1. \tag{2.55}$$

Energy conditions

- An interior solution must satisfy the dominant energy condition (DEC). This condition states that the speed of energy flow of matter must be less than the speed of light for any observer, so in order to fulfill this requirement, the matter content must satisfy

$$\rho \geqslant |p_r|, \tag{2.56}$$

$$\rho \geqslant |p_t|. \tag{2.57}$$

When the pressures are positive, DEC is equivalent to the weak energy condition (WEC). It is desirable that even the strong energy condition (SEC) $\rho \geqslant p_r + 2p_t$ is satisfied.

- *Causality condition*, from which the radial and tangential speeds of sound should not overcome the speed of light. The speeds of sound are defined as $v_r^2 = dp_r/d\rho$ and $v_t^2 = dp_t/d\rho$. Therefore the condition reads:

$$0 < \frac{dp_r}{d\rho} \leqslant 1 \tag{2.58}$$

$$0 < \frac{dp_t}{d\rho} \leqslant 1. \tag{2.59}$$

Adiabatic index

The adiabatic index Γ allows us to connect the relativistic structure of a spherical static object and the equation of state of the interior fluid serving as a criterion of stability. It index is the ratio of two specific heats and should be bigger than $4/3$ for stability[31–33]. It is defined as

$$\Gamma = \frac{\rho + p_r}{p_r}\frac{dp_r}{d\rho} \geqslant \frac{4}{3}. \tag{2.60}$$

2.4 Minimal Geometric Deformation method

The Minimal Geometric Deformation method is a simple, systematic and direct approach to decouple gravitational sources in general relativity[25–27]. It is useful when looking for new spherically symmetric solutions of Einstein's field equations. This method allows us to extend isotropic solutions to anisotropic domains as follows.
Let us consider the Einstein's field equations (2.1)

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = kT_{\mu\nu}^{(tot)}. \tag{2.61}$$

Given a complex energy-momentum tensor $T_{\mu\nu}^{(tot)}$, it can be decomposed into

$$T_{\mu\nu}^{(tot)} = T_{\mu\nu}^{(isotropic)} + \alpha\theta_{\mu\nu}, \tag{2.62}$$

where $T_{\mu\nu}^{(isotropic)}$ is the 4-dimensional energy-momentum tensor of a perfect fluid, and $\theta_{\mu\nu}$ is the source that in general produces anisotropies in self-gravitating systems. Besides, α is a coupling constant. Note that, since the Einstein tensor is divergence free, the total energy-momentum tensor (2.62) must satisfy the conservation equation

$$\nabla_\nu T^{(tot)\mu\nu} = 0. \tag{2.63}$$

Now, from Schwarzschild-like coordinates, the spherically symmetric metric reads

$$ds^2 = e^\nu dt^2 - e^\lambda dr^2 - r^2(d\theta^2 + \sin^2\theta d\phi^2), \tag{2.64}$$

where the metric functions $\nu = \nu(r)$ and $\lambda = \lambda(r)$ are functions of the radial coordinate only, ranging from $r = 0$ (the center of the star) to some $r = R$ (the star surface). Then, from equation (2.64) as solution of Einstein's field equations, we get

$$k^2\tilde{\rho} = \frac{1}{r^2} + e^{-\lambda}\left(\frac{\lambda'}{r} - \frac{1}{r^2}\right), \tag{2.65}$$

$$k^2\tilde{p}_r = -\frac{1}{r^2} + e^{-\lambda}\left(\frac{\nu'}{r} + \frac{1}{r^2} +\right), \tag{2.66}$$

$$k^2\tilde{p}_t = \frac{1}{4}e^{-\lambda}\left[2\nu'' + \nu'^2 - \nu'\lambda' + 2\frac{\nu' - \lambda'}{r}\right], \tag{2.67}$$

where primed quantities indicates derivation respect to the radial component. Moreover we have defined:

$$\tilde{\rho} = \rho + \alpha\theta_0^0, \tag{2.68}$$

$$\tilde{p}_r = p - \alpha\theta_1^1, \tag{2.69}$$

$$\tilde{p}_t = p - \alpha\theta_2^2, \tag{2.70}$$

as effective quantities for density $\tilde{\rho}$, radial $\tilde{p}_r$ and tangential pressure $\tilde{p}_t$. From here, we can notice that the decomposition of the total energy-momentum tensor in equation (2.62) is just the separation of its components in the matter sector.

Through the Minimal Geometric method (MGD), it is possible to obtain two set of equations: the perfect fluid equations and the ones corresponding to the decoupler sector. This does not happen in other contexts due to the non-linearity of the Einstein's equations[34]. The method consists in to introduce a geometric deformation in the metric functions, namely

$$\nu = \xi + \alpha g, \tag{2.71}$$

$$e^{-\lambda} = \mu + \alpha f, \tag{2.72}$$

where g and f are called decoupling functions and α is a the same free parameter introduced before which controls the deformation. Although it is possible to deform both components of the metric, for this particular case we will consider $g = 0$ and $f \neq 0$. Now, we obtain the two sets of differential equations:

a) Perfect fluid equations

These are the Einstein's field equations that describe an isotropic system with perfect fluid energy-momentum tensor as source. Explicitly:

$$k^2\rho = \frac{1 - r\mu' - \mu}{r^2}, \tag{2.73}$$

$$k^2 p = \frac{r\mu\nu' + \mu - 1}{r^2}, \tag{2.74}$$

$$k^2 p = \frac{\mu'(r\nu' + 2) + \mu(2r\nu'' + r\nu'^2 + 2\nu')}{4r}, \tag{2.75}$$

with

$$\nabla_\mu T^{(isotropic)\mu\nu} = p' - \frac{\nu'}{2}(\rho + p) = 0. \tag{2.76}$$

b) Decoupler matter equations

The second set of equations are the quasi-Einstein's field equations sourced by $\theta_{\mu\nu}$ is given by:

$$k^2\theta_0^0 = -\frac{rf' + f}{r^2}, \tag{2.77}$$

$$k^2\theta_1^1 = -\frac{rf\nu' + f}{r^2}, \tag{2.78}$$

$$k^2\theta_2^2 = -\frac{f'(r\nu' + 2) + f(2r\nu'' + r\nu'^2 + 2\nu')}{4r}. \tag{2.79}$$

Note that, whenever $\theta_1^1 \neq \theta_2^2$, it is induced a local anisotropy. Moreover, the conservation equation $\nabla_\nu \theta^{\mu\nu} = 0$ explicitly reads

$$(\theta_1^1)' - \frac{\nu'}{2}(\theta_0^0 - \theta_1^1) - \frac{2}{r}(\theta_2^2 - \theta_1^1) = 0, \tag{2.80}$$

which is a linear combination of decoupler sector equations (2.77)-(2.79). Under these conditions, there is no exchange of energy-momentum between the perfect fluid and the anisotropic source $\theta_{\mu\nu}$, so their interaction is purely gravitational[35].

As it was mentioned before, MGD method can be used to extend isotropic solutions to anisotropic domains. Given the metric functions ν, μ that solve the Einstein field equations (2.73)-(2.75) for a perfect fluid with ρ, p, we can find the deformation function $f(r)$ from equations (2.77)-(2.79) after choosing suitable conditions on the anisotropic source $\theta_{\mu\nu}$. Suppose that instead of solving the Einstein's equations for perfect fluid, we simply choose a known solution with physical relevance. Parameters such as constants in the metric functions, can be determined from the matching conditions between the interior and exterior metric.

However, it is needed additional information to close the system through the decoupler matter equations (2.77)-(2.79). For this, we can impose an equation of state for the source $\theta_{\mu\nu}$ or some physical restriction on $f(r)$. In any case they must lead to solutions that are physically acceptable. In this sense, we can establish mimic constraints for pressure and density that lead the exterior solution to be compatible with the regular interior matter. A particular case will be developed later using the modified Tolman IV solution as isotropic sector.

2.5 Buchdahl's limit

Systems such as static and spherically symmetric compact stars that immersed in Schwarzschild vacuum are bounded by the Buchdahl limit[36–40]. This bound states that the ratio between the mass and radius of any regular and thermodynamically stable perfect fluid star must be:

$$\frac{2M}{r} < \frac{8}{9}. \tag{2.81}$$

This bound can be proved by using some conditions such as:

- Regular and smooth metric functions in the stellar interior.
- Matching conditions at the boundary of the star where the spacetime is soothly matched to Schwarzschild's exterior.
- Thermodynamic stability, it means that the average density must be a decreasing function of the radial coordinate.
- For an isotropic matter distribution the ratio M/R can not exceed the upper bound $M/R \leq 4/9$ which corresponds to a maximum gravitational surface red-shift of

$$z(r)_{max} = 2. \tag{2.82}$$

For a detailed derivation of the Buchdahl's limit, please review APPENDIX A.

Chapter 3

Results & Discussion

3.1 Anisotropic interior solutions and Buchdahl's limit in MGD context

As it is well known, the Buchdahl's limit is modified when we introduce an anisotropy to the system. This was shown by Böhmer and Harko[28] in their derivation of the corresponding upper bound for the mass-radius relation for an anisotropic matter distribution in presence of a cosmological constant Λ (APPENDIX B). They obtained the general expression

$$\frac{2M}{R} \leq \left(1 - \frac{8\pi}{3}\lambda R^2\right)\left[1 - \frac{1}{9}\frac{(1 - 2\lambda/\langle\rho\rangle)^2}{\left(1 - \frac{8\pi}{3}\lambda R^2\right)(1+f)^2}\right], \tag{3.1}$$

being $\langle\rho\rangle$ the mean energy density and f is a function proportional to the anisotropy $\Delta = p_t - p_r$. This function f is given by

$$f = 2\frac{\Delta(R)}{\langle\rho\rangle}\left[\frac{\arcsin\left(\sqrt{\frac{2M\chi(R)}{R}}\right)}{\sqrt{\frac{2M\chi(R)}{R}}} - 1\right], \tag{3.2}$$

where χ stands for

$$\chi(r) \equiv 1 + \frac{4\pi}{3}\Lambda\frac{r^3}{m(r)}.$$

As we are interested in studying space-time without cosmological constant from now on we will set off Λ. Then the upper bound (3.1) becomes

$$2u \leq 1 - \frac{1}{9(1+f)^2}. \tag{3.3}$$

Since we have eliminated Λ then $\chi(r) = 1$ and f turns

$$f = \frac{R^2\Delta(u)}{\frac{3u}{8\pi}}\left[\frac{\sin^{-1}\left(\sqrt{2u}\right)}{\sqrt{2u}} - 1\right]. \tag{3.4}$$

Note that f is a positive quantity. Otherwise, the condition $\Delta \geq 0$ could be violated. As a consequence, non–vanishing values of f, equation (3.3) allows extra packing of mass in compact stellar structures. Second, from equation (3.4), it is straightforward to show that the upper bound of the compactness parameter u, is defined by

$$u \geq \frac{4}{9}. \tag{3.5}$$

So, our next task is to analyze for which values of compactness parameters in (3.5), our interior solution fulfills the conditions of physical acceptability described in section 2.3. The Tolman IV anisotropic interior solution is studied in the next section to discuss in detail how the local anisotropies introduced in the stellar interior by gravitational decoupling through MGD approach contributes on the maximum mass-radius ratio value allowable for relativistic anisotropic fluid spheres in GR.

3.1.1 Tolman IV solution: Anisotropic case

Let us choose a known solution with physical relevance, like the well-known Tolman IV solution ν, μ, ρ, p for perfect fluids[4], namely:

$$e^{\nu(r)} = B^2\left(1 + \frac{r^2}{A^2}\right), \tag{3.6}$$

$$\mu(r) = \frac{\left(1 - \frac{r^2}{C^2}\right)\left(1 + \frac{r^2}{A^2}\right)}{1 + \frac{2r^2}{A^2}}, \tag{3.7}$$

$$\rho(r) = \frac{3A^4 + A^2(3C^2 + 7r^2) + 2r^2(C^2 + 3r^2)}{k^2C^2(A^2 + 2r^2)^2}, \tag{3.8}$$

$$p(r) = \frac{C^2 - A^2 - 3r^2}{k^2C^2(A^2 + 2r^2)}. \tag{3.9}$$

Now, thorough MGD and assuming the mimic constraint for the density, namely $\rho = \theta_0^0$, the above configuration becomes an anisotropic solution. Then, the radial metric component reads

$$e^{-\lambda(r)} = \mu(r) + \alpha f(r), \tag{3.10}$$

where

$$f(r) = -\frac{r^2\left(A^2 + C^2 + r^2\right)}{C^2\left(A^2 + 2r^2\right)}. \tag{3.11}$$

Through this equation and the matching conditions, the compactness parameter $u = M/R$ of the anisotropy and the compactness parameter of the isotropic solution $u_0 = M_0/R$ are related by

$$u = u_0 + \frac{\alpha}{2}\frac{R^2}{C^2}\left(\frac{A^2 + C^2 + R^2}{A^2 + 2R^2}\right), \tag{3.12}$$

and

$$B = \sqrt{\left(\frac{1}{1+\frac{R^2}{A^2}}\right)\left[1 - 2u_0 - \alpha\frac{R^2}{C^2}\left(\frac{A^2+C^2+R^2}{A^2+2R^2}\right)\right]}. \tag{3.13}$$

Moreover, the continuity of the second fundamental form yields

$$C = \sqrt{\frac{(1+\alpha)\left(A^2+R^2\right)\left(A^2+3R^2\right)}{\left(A^2+R^2\right)-\alpha\left(A^2+3R^2\right)}}. \tag{3.14}$$

On the other hand, by using these metric functions, the effective isotropic radial pressure becomes

$$\tilde{p}_r(r,\alpha) = p(r) - \alpha\frac{(A^2+C^2+r^2)(A^2+3r^2)}{k^2C^2\left(A^2+r^2\right)\left(A^2+2r^2\right)}. \tag{3.15}$$

Meanwhile, using the mimic constraint for density yields

$$\tilde{\rho}(r,\alpha) = (1+\alpha)\rho(r). \tag{3.16}$$

Therefore, the effective tangential pressure reads

$$\tilde{p}_t(r,\alpha) = \tilde{p}_r(r,\alpha) + \frac{\alpha r^2}{k^2\left(A^2+r^2\right)^2}. \tag{3.17}$$

With the anisotropy thus given by

$$\Delta(r,\alpha) = \frac{\alpha r^2}{k^2(A^2+r^2)^2}, \tag{3.18}$$

from where $\alpha > 0$ to ensure $\Delta \geq 0$.

Considering equation (3.12) and solving it for A leads to

$$A = \frac{\sqrt{\alpha - 3u(1+\alpha) + 3u_0(1+\alpha)}}{\sqrt{(u-u_0)(1+\alpha)}}. \tag{3.19}$$

Since A appears naturally as a positive quantity in the equations (3.13) and (3.14), the expressions inside the square root in both, numerator and denominator, must be positive. Starting from the numerator we impose that

$$\alpha - 3u(1+\alpha) + 3u_0(1+\alpha) > 0, \tag{3.20}$$

in order to determine from (3.20) the following conditions that let α, u and u_0 fulfill $A > 0$

$$u > \frac{1}{3}(1+3u_0), \tag{3.21}$$

and

$$\alpha < \frac{-3u+3u_0}{-1+3u-3u_0}. \tag{3.22}$$

Moreover, from the denominator of equation (3.19), it is positive if

$$u > u_0. \tag{3.23}$$

Then, analyzing the expression for the compactness parameter

$$u = u_0 + \frac{\alpha}{2}\frac{R^2}{C^2}\left(\frac{A^2 + C^2 + R^2}{A^2 + 2R^2}\right), \tag{3.24}$$

it is clear that for $u_0 = 4/9$, the anisotropic solution acquire an extra packing in the sense that

$$u \geq \frac{4}{9}, \tag{3.25}$$

due to the fact that the factor containing α is a positive quantity.

Our task now is to explore the space of parameters, and it become easier starting from the fact that the radial pressure must decrease monotonously as long as the radius increases. Let us propose the following initial values which fulfill this behavior: $\alpha = 0.19$ and A = 1. Also we give an initial value for u_0=0.39. This value was chosen from the following interval (3.26) given by the condition in (3.22) which leads

$$0.112 < u_0 < 0.392 \tag{3.26}$$

Next, from the initial values of α, A and u_0, we will be able to find the anisotropic compactness parameter $u \geq 4/9$ which indicates the extra packing condition and fulfill the conditions of physical acceptability as well.

Replacing in equation (3.12) the initial values: $\alpha = 0.19$, $A = 1$, $u_0 = 0.39$ and the normalized radius $R = 1$, lead to the value for the anisotropic compactness parameter $u = 0.43$. In this case, the obtained value does not correspond to the condition for extra packing that we have established in (3.5).

To overcome the Buchdahl's limit, we need to explore the space of parameters until reaching the maximum extra packing of mass as possible. So we just need to increase the values for α keeping in mind that the physical acceptability conditions and the limit for extra packing must be fulfilled. In other words, as long as the parameter α is increased, we must check whether the values B, C still are real and positive, and the we must check the behavior of the density, pressure, metric functions and adiabatic index.

In the first two columns of table (3.1), there are shown the values for α and its corresponding compactness parameter u. We have considered $\alpha = 0.19$ (our initial setting) and we kept increasing it until 0.69. We see that at higher α values, the upper bound $u < 1/2$ was overgone.

α	u	B	C
0.19	0.43	0.26	2.77
0.29	0.45	0.23	3.5
0.39	0.46	0.20	5.03
0.49	0.47	0.17	17.26
0.50	Indeterminate	-	-
0.59	0.48	0.13	5.94 i
0.69	0.49	0.09	4.22 i
0.79	0.50	0.02 i	3.51 i

Table 3.1: Bounded parameters according plausibility conditions.

Until now, the parameters are bounded enough based on the minimum and maximum bounds that the compactness parameter u can have (quantities in red). At this point, the behavior of the physical quantities were checked and they fulfill the required conditions graphically.

From this range of values we wanted to find the maximum compactness parameter in such a way that B and C are positive quantities. As we can see in the third and fourth columns from table (3.1), real and positive values are given from a maximum $\alpha = 0.49$, just before the indetermination.

Now, we have discovered the values for a considerable extra packing, and the most important, that lead to physical acceptability conditions which we can appreciate in the plots that we show in the next section.

3.1.2 Graphic Representation

Remember that the set of parameters obtained which bound our interior solution, in this case the anisotropic like–Tolman IV solution, must fulfill the plausibility conditions in order to describe a system which is physically acceptable. The appropriate way of discovering the behavior of our physical quantities is by representing them graphically in terms of the parameters that were found.

Metric Potentials

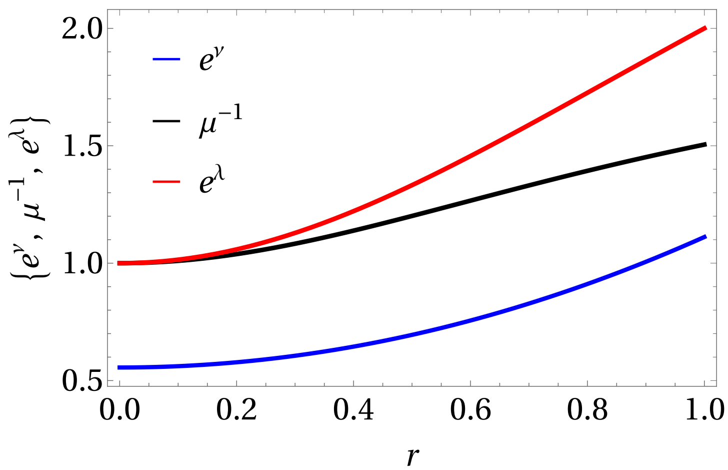

Figure 3.1: Increasing behavior of metric functions $\{e^\nu, e^{-\lambda}, \mu\}$.

In figure 3.1 we show the behavior of the metric functions $\{e^\nu, e^{-\lambda}, \mu\}$ for $u = 0.47$, $u_0 = 0.39$ and $\alpha = 0.49$. Note that, as expected, the functions are monotonously growing. Besides, $e^{-\lambda(0)} = \mu(0) = 1$. In this sense both the isotropic and the anisotropic solutions satisfy the requirements demanded on interior solutions for the chosen set of parameters.

Matter sector quantities

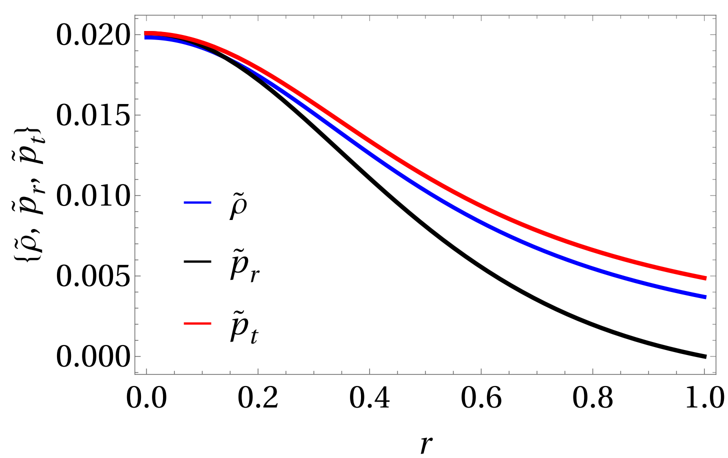

Figure 3.2: Decreasing behavior of density and pressures of the anisotropic solution.

In figure 3.2, for the same values, $u = 0.47$, $u_0 = 0.39$ and $\alpha = 0.49$ it is shown the behavior of the density, and the effective pressures of the anisotropic solution. Note that all the quantities shown in the profiles are monotonously decreasing and reach their maximum value at the center of the star as expected. Besides, the radial pressure vanishes a the surface of the star.

Redshift

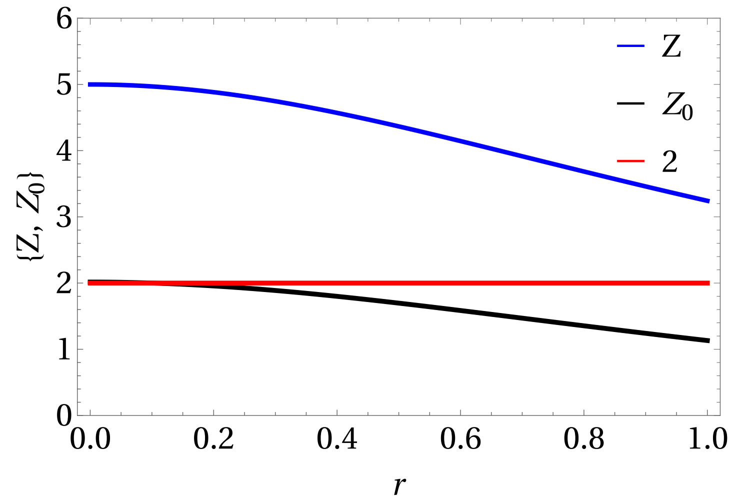

Figure 3.3: Redshift parameter $z(r)$ for the isotropic and anisotropic Tolman IV interior solution.

In figure (3.3), we see that the redshift z_0 corresponding to the isotropic solution (blue line), is less than 2, which agrees with the condition defined in equation (2.82), section 2.5. However, in the extended anisotropic case, we see that the redshift z (red line) is greater than 2. It makes sense, from the fact that the redshift parameter is related with the compactness parameter as it is stated in equation (2.55). In other words, modification of the Buchdahl's limit, of course, lead the modification in the redshift condition, in our case that limit was overgone.

Dominant Energy Condition (DEC)

From the DEC described in equation (2.56) and (2.57), we obtain:

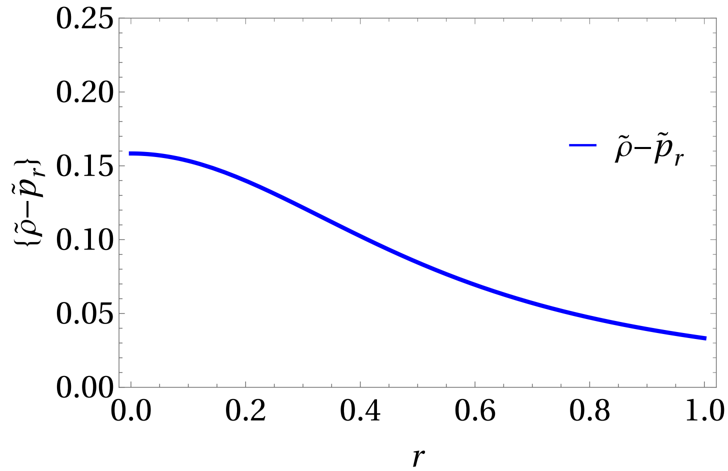

Figure 3.4: DEC regarding the radial pressure. As expected, it lies between 0 and 1.

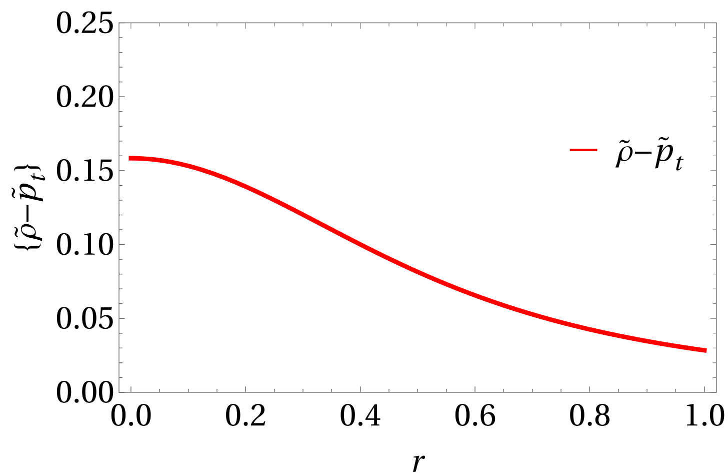

Figure 3.5: DEC regarding the tangential pressure. As expected, it lies between 0 and 1.

In figures (3.4) and (3.5), we see that DEC is satisfied for the chosen parameters. The speed of energy flow of matter is less than the speed of light, as expected.

Strong Energy Condition (SEC)

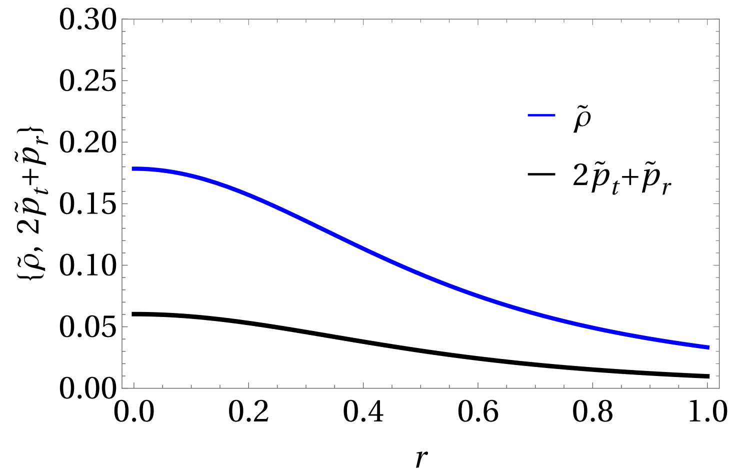

Figure 3.6: SEC condition $\tilde{\rho} \geqslant 2p_t + p_r$ is fulfilled.

Similarly, the requirement for SEC is satisfied since ρ is greater than $2p_t + p_r$ as expected.

Causality

Remember that this condition ensures that either radial and tangential sound velocities are less than the speed of light, in our work we have:

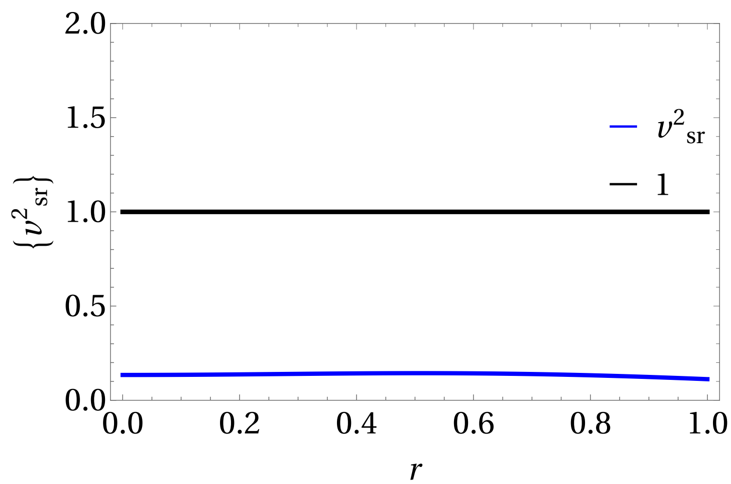

Figure 3.7: Sound velocity regarding to the radial pressure bounded between 0 and 1, as expected.

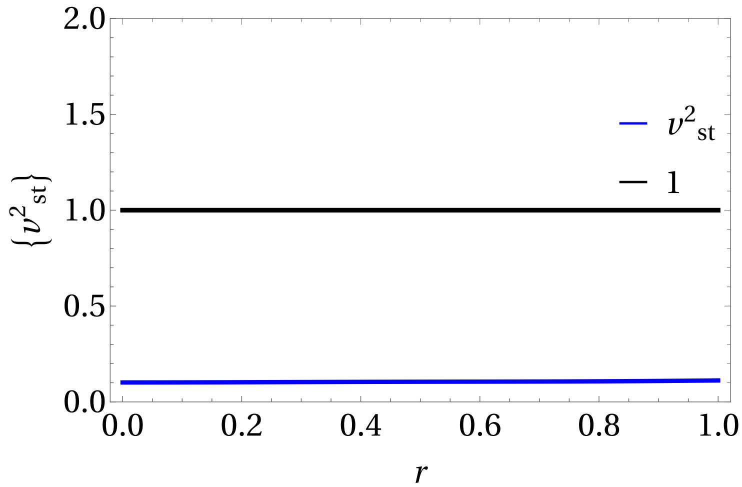

Figure 3.8: Sound velocity regarding to the tangential pressure bounded between 0 an 1, as expected.

In fact, in both cases the sound velocity is positive and less than 1, so the requirement is fulfilled.

Adiabatic index

The last step consist in to check the stability of the solution bounded by the parameters that we have found by calculating the adiabatic index as

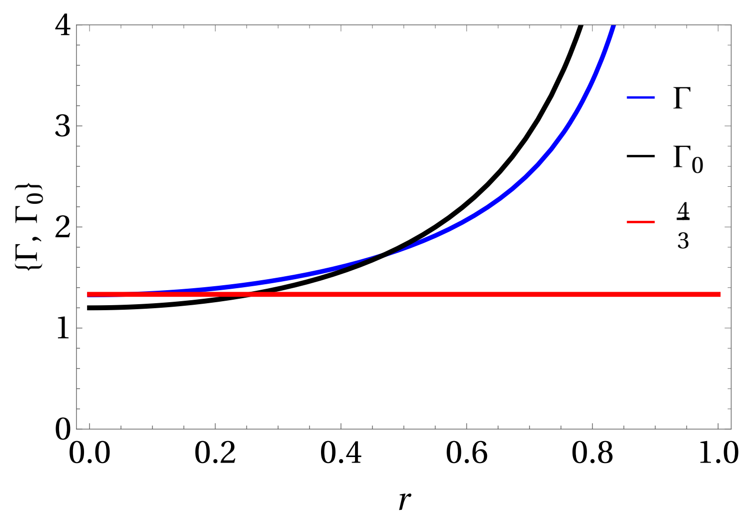

Figure 3.9: Adiabatic index showing stability for the anisotropic interior solution case (greater than 4/3).

In figure 3.9, the adiabatic index reveals that for $u = 0.47$, we obtain a stable interior configuration from the anisotropic solution. Remarkably, by turning-off α (namely, isotropic case) we lie in unstable configurations since the index is below the bound $4/3$.

As we have seen, the set of parameters chosen satisfy all the considered requirements for this study. It means that the anisotropic Tolman IV interior solution by MGD, with parameters: $u = 0.47$, $u_0 = 0.39$ and $\alpha = 0.49$ describes a physically accepted system.

Before concluding this section, we would like to emphasize that through the solution obtained here, we have demonstrated that the MGD method does not only allow to extend isotropic solutions to anisotropic domains, but it can be used to map a non acceptable solution as the Tolman IV for $u_0 = 0.39$ to an anisotropic interior solution which satisfies the required acceptability physical conditions and with extra packing of mass, in this case, an anisotropic solution with $u = 0.47$.

Chapter 4

Conclusions & Outlook

The central objective of the present study was to found whether the Buchdahl's limit is modified under the introduction of local anisotropies by applying the Minimal Geometric Deformation method. This was done by using the well known Tolman IV solution for spherically symmetric systems where through a deep analysis of the space of parameters were performed. The parameters were given by the set of equations corresponding to the anisotropic Tolman IV solution where constraints in the density ρ were considered.

The determination of the parameters were based on the conditions that an interior solution must fulfill in order to be physically accepted. In fact, the plausibility requirements were held by showing graphically that the behavior of the metric potentials, matter sector quantities, and energy conditions fit with theoretical expectations. Moreover, graphically we can appreciate that the values that let us bound the parameters indicated extra packing of mass, which is the required condition to conclude that indeed the Buchdahl's limit is modified in the presence of anisotropies induced by MGD.

Finally, the behavior of the adiabatic index demonstrates that MGD is a tool that does not only allow to extend isotropic solutions to anisotropic domains, but let us to map unstable and non acceptable isotropic solutions to anisotropic stable configurations with extra packing of mass. In addition, it could be interesting to match this results with some model of neutron, boson or black stars, so it is open for further research.

Appendix A

Buchdahl's limit proof

The Buchdahl's limit can be proved after establishing the metric and field equations.

Metric and equations

Consider a static spherical system with metric functions $A(r)$ and $B(r)$:

$$ds^2 = A^2 dt^2 - B^2 dr^2 - r^2 d\Omega^2, \tag{A.1}$$

or alternatively, $A = e^{\phi}$ and $B = (1 - 2m/r)^{-1/2}$ where $\phi(r)$ and $m(r)$.
Then, the equation becomes:

$$ds^2 = e^{2\phi} dt^2 - \left(1 - \frac{2m}{r}\right)^{-1} dr^2 - r^2 d\Omega^2. \tag{A.2}$$

With this metric the Newtonian limit is ensured, then $\phi(r)$ is the Newtonian potential.
The Einstein tensor for this metric is:

$$G^t_t = -\frac{2m'}{r^2}, \tag{A.3}$$

$$G^r_r = -\frac{2m}{r^3} + \frac{2}{r}\left(1 - \frac{2m}{r}\right)\phi' = \frac{1}{r^2}\left(\frac{1}{B^2} - 1\right) + \frac{1}{B^2}\left(\frac{2}{r}\frac{A'}{A}\right), \tag{A.4}$$

$$G^\theta_\theta = G^\phi_\phi = \frac{1}{B^2}\left[\frac{A''}{A} - \frac{A'}{A}\frac{B'}{B} + \frac{1}{r}\left(\frac{A'}{A} - \frac{B'}{B}\right)\right]. \tag{A.5}$$

The four-velocity U^μ and acceleration a_μ follows from Killing vector $e^{\phi}U^\mu$ as follows:

$$U^\mu \partial_\mu = e^{-\phi}\partial_t, \quad U_\mu dx^\mu = e^{\phi} dt, \tag{A.6}$$

$$a_\mu = U^\nu \nabla_\nu U_\mu = -\partial_\mu \phi, \tag{A.7}$$

allows the energy-momentum tensor to be written as:

$$T^{\mu}_{\nu} = \rho U^{\mu} U_{\nu} - p\gamma^{\mu}_{\nu}, \quad \gamma^{\mu}_{\nu} = \delta^{\mu}_{\nu} - U^{\mu} U_{\nu}, \tag{A.8}$$

$$T^{t}_{t} = \rho, \;\; T^{r}_{r} = T^{\theta}_{\theta} = T^{\phi}_{\phi} = -p. \tag{A.9}$$

Pressure isotropy implies:

$$\frac{A''}{A} - \frac{A'}{A}\frac{B'}{B} - \frac{1}{r}\frac{A'}{A} - \frac{1}{r}\left(\frac{1}{r} + \frac{B'}{B}\right) + \frac{B^2}{r^2} = 0. \tag{A.10}$$

Field equation $G^{t}_{t} = -8\pi T^{t}_{t}$ yields the mass-density relation:

$$\frac{dm}{dr} = 4\pi r^2 \rho. \tag{A.11}$$

In terms of $B(r)$, this relation is written as:

$$\frac{1}{B^2}\left(\frac{1}{r^2} - \frac{2}{r}\frac{B'}{B}\right) - \frac{1}{r^2} = -8\pi\rho. \tag{A.12}$$

The equations of motion $\nabla_{\nu} T^{\mu\nu} = 0$ can be written as:

$$U^{\mu}[\dot{\rho} + (p+\rho)U^{\alpha}_{;\alpha}] + (p+\rho)a^{\mu} - \gamma^{\mu\nu}\nabla_{\nu} p = 0. \tag{A.13}$$

Since the fluid is static, the U^{μ} component vanishes, and the pressure gradient is:

$$p' = -(p+\rho)\phi'. \tag{A.14}$$

The TOV equilibrium equation is:

$$p' = -\left(\frac{p+\rho}{r^2}\right)\left(\frac{m + 4\pi r^3 p}{1 - \frac{2m}{r}}\right). \tag{A.15}$$

Substituting for p' in equation (A.14) provides:

$$\phi' = \frac{m + 4\pi r^3 p}{r^2\left(1 - \frac{2m}{r}\right)}. \tag{A.16}$$

Given ρ, one can integrate equation (A.11) for metric component $m(r)$. Then given p, equation (A.16) yields the second metric component. We will use these equations with graphs and inequalities to prove Buchdahl' s theorem.

Proof

1. In $m(r)$, at $m(0) = 0$, the slope of the curve is positive since $m' = 4\pi r^2\rho$ and $\rho \geq 0$ everywhere, therefore the curve is concave upwards from the origin, shaped as $m \sim r^{2+}$.
2. Consider $F(r) \equiv 8\pi\rho - 6m/r^3$. Near $r = 0$, $F \leq 0$:

$$
\begin{aligned}
F' &= 8\pi\rho' + \frac{18m}{r^4} - \frac{6m'}{r^3} \\
&= 8\pi\rho' + \frac{18m}{r^4} - \frac{6(4\pi r^2\rho)}{r^3} \\
&= 8\pi\rho' + \frac{3}{r}\left(\frac{6m}{r^3} - 8\pi\rho\right) \\
&= 8\pi\rho' - \frac{3F}{r}.
\end{aligned}
$$

Since $\rho' \leq 0$ (ρ is non-increasing outwards):

$$
\begin{aligned}
F' &\leq -\frac{3F}{r} \text{ which implies} \\
F &\leq 0 \text{ everywhere.}
\end{aligned}
$$

3. We use equation (A.12) to substitute for B'/B in the pressure isotropy equation, and obtain:

$$
\begin{aligned}
\frac{A''}{A} - \frac{A'}{A}\frac{B'}{B} - \frac{1}{r}\frac{A'}{A} - \frac{1}{r^2} - \frac{1}{2r^2} + \frac{B^2}{2r^2} - 4\pi\rho B^2 + \frac{B^2}{r^2} &= 0 \\
\frac{Br}{A}\left(\frac{A'}{Br}\right)' - \frac{3}{2}\frac{B^2}{r^2}\left(\frac{1}{B^2} - 1\right) - 4\pi\rho B^2 &= 0
\end{aligned}
$$

Using

$$
\frac{1}{B^2} - 1 = -\frac{2m}{r}, \quad \frac{B^2}{2} = 4\pi\rho B^2 - 3m\frac{B^2}{r^3}, \tag{A.17}
$$

the pressure isotropy equation becomes:

$$
\begin{aligned}
\frac{Br}{A}\left(\frac{A'}{Br}\right)' - \frac{B^2}{2}F &= 0 \\
\left(\frac{A'}{Br}\right)' &= \left(\frac{AB}{2r}\right)F.
\end{aligned}
$$

Since $F \leq 0$ everywhere, and $A > 0$, $B > 0$, it follows that $(A'/Br)' \leq 0$. With $B = (1 - 2m/r)^{-1/2}$ the inequality can be written:

$$
\left[\frac{A'}{r}(1 - 2m/r)^{1/2}\right]' \leq 0 \tag{A.18}
$$

Trapped surfaces are excluded, $r_b > 2M$, and $B_b \equiv B(r_b) = (1 - 2M/r_b)^{-1/2} > 0$.
4. Recall $F = 8\pi\rho - 6m/r^3$, rewritten as $2m/r^3 = 8\pi\rho/3 - F/3$. With $\rho > 0$, $F < 0$, and ρ and F decreasing outwards, it follows that $(8\pi\rho - F)/3$ decreases toward the boundary:

$$\frac{2m}{r^3} \geq \frac{2M}{r_b^3}.$$

Thus:

$$\begin{aligned}
\frac{2m}{r} &\geq \left(\frac{r}{r_b}\right)^2 \left(\frac{2M}{r_b}\right) \\
\left(1 - \frac{2m}{r}\right)^{1/2} &\leq \left[1 - \left(\frac{r}{r_b}\right)^2 \left(\frac{2M}{r_b}\right)\right]^{1/2} \\
\left(1 - \frac{2m}{r}\right)^{1/2} &\geq \left[1 - \left(\frac{r}{r_b}\right)^2 \left(\frac{2M}{r_b}\right)\right]^{1/2}
\end{aligned} \tag{A.19}$$

5. From inequality (A.18) $(A'/r)(1 - 2m/r)^{1/2} \geq (A'_b/r_b)(1 - 2M/r_b)^{1/2}$

$$\begin{aligned}
A_b &\geq \left(\frac{A'_b}{r_b B_b}\right)\left(-\frac{r_b^3}{2M}\right)\left(1 - \frac{2M}{r_b}\right)^{1/2} \\
&\geq \left(\frac{A'_b r_b}{B_b}\right)\left(-\frac{r_b}{2M}\right)\left(\frac{1}{B_b} - 1\right) \\
\text{Note} \qquad -\frac{r_b}{2M} &= \frac{1}{B_b^{-2} - 1} \\
A_b &\geq (A'_b r_b)\left(\frac{1 - B_b}{1 - B_b^2}\right) \geq (A'_b r_b)\frac{1}{1 + B_b}.
\end{aligned} \tag{A.20}$$

6. To establish the value of $A'_b r_b$, we start with equation (A.16) at the boundary

$$\phi'_b = \frac{M}{r_b^2(1 - 2M/r_b)} = \frac{MB_b^2}{r_b^2}.$$

We also know that $A'/A = \phi'$ so that $A'_b = A_b \phi'_b$. Thus

$$A'_b = \frac{A_b M B_b^2}{r_b^2}.$$

At the vacuum Schwarzschild boundary $A_b = 1/B_b$. We have

$$A'_b r_b = \frac{2M}{2A_b r_b} = \frac{1 - A_b^2}{2A_b}.$$

7. Inequality (A.20) becomes

$$A_b \geq \frac{1-A_b^2}{2A_b}\left(\frac{1}{1+1/A_b}\right) \geq \frac{1-A_b^2}{2(1+A_b)} \geq \frac{1-A_b}{2}$$

or

$$\begin{aligned} 3A_b &\geq 1 \;\Rightarrow\; 9A_b^2 \geq 1 \;\Rightarrow\; 9\left(1-\frac{2M}{r_b}\right) \geq 1 \\ \frac{2M}{r_b} &\leq \frac{8}{9}. \end{aligned}$$

Appendix B

Buchdahl's limit for anisotropic stars derivation

The interior line element for for a static general relativistic spherically symmetric matter configuration is given by

$$ds^2 = e^{\nu} dt^2 - e^{\lambda} dr^2 - r^2 d\Omega^2. \tag{B.1}$$

The Einstein's equations for this metric read:

$$k^2 \tilde{\rho} = \frac{1}{r^2} - e^{-\lambda}\left(\frac{1}{r^2} - \frac{\lambda'}{r}\right), \tag{B.2}$$

$$k^2 \tilde{p}_r = -\frac{1}{r^2} + e^{-\lambda}\left(\frac{1}{r^2} + \frac{\nu'}{r}\right), \tag{B.3}$$

$$k^2 \tilde{p}_t = \frac{e^{-\lambda}}{4}\left(2\nu'' + \nu'2 - \lambda'\nu' + 2\frac{\nu' - \lambda'}{r}\right), \tag{B.4}$$

where $k = \frac{8\pi G}{c^4}$. Moreover, $\tilde{\rho}$ is the effective energy density, $\tilde{p}_r$ and $\tilde{p}_t$ are the radial and tangential effective pressures respectively, and we have defined them as:

$$\tilde{\rho} = \rho + \alpha\theta_0^0, \tag{B.5}$$

$$\tilde{p}_r = p - \alpha\theta_1^1, \tag{B.6}$$

$$\tilde{p}_t = p - \alpha\theta_2^2. \tag{B.7}$$

As it is shown in section 2.4, after applying the MGD we obtain two set of equations: the perfect fluid equations and the ones corresponding to the decoupler sector.

a) Perfect fluid equations

$$k^2\rho = \frac{1 - r\mu' - \mu}{r^2}, \quad \text{(B.8)}$$

$$k^2 p = \frac{r\mu\nu' + \mu - 1}{r^2}, \quad \text{(B.9)}$$

$$k^2 p = \frac{\mu'(r\nu' + 2) + \mu(2r\nu'' + r\nu'^2 + 2\nu')}{4r}. \quad \text{(B.10)}$$

b) Decoupler matter equations

$$k^2\theta_0^0 = -\frac{rf' + f}{r^2}, \quad \text{(B.11)}$$

$$k^2\theta_1^1 = -\frac{rf\nu' + f}{r^2}, \quad \text{(B.12)}$$

$$k^2\theta_2^2 = -\frac{f'(r\nu' + 2) + f(2r\nu'' + r\nu'^2 + 2\nu')}{4r}. \quad \text{(B.13)}$$

Since we are considering that the system has an anisotropic fluid distribution, the corresponding components of the energy-momentum tensor are:

$$T_0^0 = \tilde{\rho},$$
$$T_1^1 = -\tilde{p}_r,$$
$$T_2^2 = T_3^3 = -\tilde{p}_t.$$

Let us suppose that inside the system $\tilde{p}_r \neq \tilde{p}_t$, $\forall r \neq 0$, so we can define the anisotropy parameter as

$$\Delta = \tilde{p}_t - \tilde{p}_r = \alpha(\theta_1^1 - \theta_2^2). \quad \text{(B.14)}$$

The properties of an anisotropic compact object can be described by the following gravitational structure equations:

$$\frac{dm}{dr} = 4\pi\tilde{\rho}r^2, \quad \text{(B.15)}$$

$$\frac{d\tilde{p}_r}{dr} = -\frac{(\rho + \tilde{p}_r)[m + 4\pi r^3 \tilde{p}_r]}{r^2(1 - \frac{2m}{r})} + \frac{2\Delta}{r}, \quad \text{(B.16)}$$

$$\frac{d\nu}{dr} = -\frac{2}{\rho + \tilde{p}_r}\frac{d\tilde{p}_r}{dr} + \frac{4\Delta}{r(\tilde{\rho} + \tilde{p}_r)}, \quad \text{(B.17)}$$

where $m = m(r)$ is the mass inside the radius r.

Derivation of the gravitational structure equations.

- To get equation (B.15), evaluating the radial metric function $\lambda = -\ln(1 - \frac{2m}{r})$ into equation B.2 leads

$$8\pi\tilde{\rho} = \frac{1}{r^2} - \left(1 - \frac{2m}{r}\right)\left[\frac{1}{r^2} + \frac{\frac{2m}{r^2} - \frac{2m'}{r}}{r\left(1 - \frac{2m}{r}\right)}\right],$$

  and then solving for the first derivative of the mass with respect to the radius r, we get

$$m' = 4\pi\tilde{\rho}r^2 \iff \frac{dm}{dr} = 4\pi\tilde{\rho}r^2. \quad \text{(B.18)}$$

- To get equation (B.16), solving equation (B.3) for ν' leads

$$\nu' = \frac{2m + 8\pi r^3 \tilde{p}_r}{r^2 - 2rm}, \tag{B.19}$$

and taking its second derivative with respect to the radius r, we obtain

$$\nu'' = \frac{1}{r^2(r-2m)^2}\left\{4m^2 - 4m\left[r + 4\pi r^3(2\tilde{p}_r + r\tilde{p}_r')\right] + 2r^2\left[m' + 4\pi r^2(\tilde{p}_r + 2\tilde{p}_r m' + r\tilde{p}_r')\right]\right\}. \tag{B.20}$$

Now, let us replace equations (B.19) and (B.20) into equation (B.4), then it reads

$$8\pi\tilde{p}_t = \frac{1}{r^2(r-2m)}\left\{16\pi^2 r^5 \tilde{p}_r^2 + mm' + 4\pi r^2 \tilde{p}_r\left[-3m + r(2+m')\right] + 4\pi r^3(r-2m)\tilde{p}_r'\right\}. \tag{B.21}$$

Solving for $\tilde{p}_r'$, and using the expression for the anisotropy in (B.14), we get

$$\tilde{p}_r' = -\frac{1}{4\pi r^3(r-2m)}\left\{m\left[4\pi r^2(\tilde{p}_r + 4\Delta) + m'\right] + 4\pi r^3\left[-2\Delta + \tilde{p}_r(4\pi r^2\tilde{p}_r + m')\right]\right\}. \tag{B.22}$$

Replacing the expression for m' given by equation (B.15) it reads

$$\tilde{p}_r' = -\frac{1}{r(r-2m)}\left[4\pi r^3\tilde{p}_r^2 - 2r\Delta + 4\pi r^3\tilde{p}_r\tilde{\rho} + m(\tilde{p}_r + 4\Delta + \tilde{\rho})\right]. \tag{B.23}$$

Finally, collecting terms and after some simplification steps we obtain

$$\tilde{p}_r' = -\frac{(m + 4\pi r^3\tilde{p}_r)(\tilde{p}_r + \tilde{\rho})}{r(r-2m)} + \frac{2\Delta}{r} \iff \frac{d\tilde{p}_r}{dr} = -\frac{(\rho + \tilde{p}_r)(m + 4\pi r^3\tilde{p}_r)}{r^2(1-\frac{2m}{r})} + \frac{2\Delta}{r}. \tag{B.24}$$

- To get the last structure equation (B.17), first let us define the anisotropy Δ. For this, solve the equation (B.3) for $\tilde{p}_r$, which gives us

$$\tilde{p}_r = \frac{-2m + r(r-2m)\nu'}{8\pi r^3}, \tag{B.25}$$

and solve equation (B.4) for $\tilde{p}_t$ to lead

$$\tilde{p}_t = \frac{1}{32\pi r^3}\left[(2 + r\nu')\left(m(2 - 2r\nu') + r(-2m' + r\nu')\right) + 2r^2(r-2m)\nu''\right]. \tag{B.26}$$

Now, replacing in equation (B.14) the anisotropy becomes

$$4m' + r\left[32\pi r\Delta + \nu'(2 + 2m' - r\nu') - 2r\nu''\right] + m\left[-\frac{12}{r} + 2\nu'(-3 + r\nu') + 4r\nu''\right] = 0. \tag{B.27}$$

Replacing the expression for ν'', last expression becomes

$$\frac{4}{r(r-2m)}\left\{-4m^2 + 4\pi r^4\left(\tilde{p}_r - 2\Delta + 2\tilde{p}_r m' + r\tilde{p}_r'\right) + m\left[r + 2rm' - 8\pi r^3\left(2\tilde{p}_r - 2\Delta + r\tilde{p}_r'\right)\right]\right\} + r(r-2m)\nu'2 = 2\left(r - 3 + rm'\right)\nu'. \tag{B.28}$$

Then, solving equation (B.3) for m which reads

$$m = \frac{r^2\left(-8\pi r\tilde{p}_r + \nu'\right)}{2 + 2r\nu'}. \tag{B.29}$$

Replacing it in the previous expression (B.28), we get

$$r\left[8\pi r\left(-2\Delta + r\tilde{p}_r'\right) + \left(4\pi r^2\tilde{p}_r + m'\right)\nu'\right] = 0. \tag{B.30}$$

Solving for ν' and replacing $m' = 4\pi r^2\tilde{\rho}$

$$\nu' = \frac{4\Delta - 2r\tilde{p}_r'}{r\tilde{p}_r + r\tilde{\rho}} \iff \frac{d\nu}{dr} = -\frac{2}{\rho + \tilde{p}_r}\frac{d\tilde{p}_r}{dr} + \frac{4\Delta}{r(\tilde{\rho} + \tilde{p}_r)}. \tag{B.31}$$

Using the equations of structure (B.15-B.17), let us show that $\xi = e^{\nu/2}>0,\ \forall r \in [0, R]$ obeys the equation

$$\frac{y}{r}\frac{d}{dr}\left(\frac{y}{r}\frac{\xi}{dr}\right) = \frac{\xi}{r}\left(\frac{d}{dr}\frac{m}{r^3} + \frac{8\pi\Delta}{r}\right). \tag{B.32}$$

First, let us start building the RHS of equation (B.32) by multiplying each side of the third equation of structure (B.17) by $\xi = e^{\nu/2}$. Redefining it as

$$2\frac{d\xi}{dr} = \left(-\frac{2}{\rho + \tilde{p}_r}\frac{d\tilde{p}_r}{dr} + \frac{4\Delta}{r(\tilde{\rho} + \tilde{p}_r)}\right)\xi. \tag{B.33}$$

Then, work this expression to reach formally the RHS as

$$\frac{y}{r}\frac{d}{dr}\left[\frac{y}{r}\frac{d\xi}{dr}\right] = \frac{y}{r}\frac{d}{dr}\left[\frac{y}{r}\left(-\frac{1}{\rho + \tilde{p}_r}\frac{d\tilde{p}_r}{dr} + \frac{2\Delta}{r(\tilde{\rho} + \tilde{p}_r)}\right)\xi\right] \tag{B.34}$$

Now, replacing the expression for $d\tilde{p}_r/dr$, which is the second equation of structure (B.16), we obtain

$$\begin{aligned}
\frac{y}{r}\frac{d}{dr}\left[\frac{y}{r}\frac{d\xi}{dr}\right] &= \frac{y}{r}\frac{d}{dr}\left[\frac{y}{r}\left(\frac{m + 4\pi r^3\tilde{p}_r}{r^2 - 2rm}\right)\xi\right] \\
&= \frac{y\xi}{r^4(r-2m)^2}\left\{y\left[4m^2 + (r^2 + 8\pi r^4\tilde{p}_r)m' + 4\pi r^5\tilde{p}_r' - rm\left(3 + 8\pi r^2(\tilde{p}_r + r\tilde{p}_r')\right)\right] + r(r-2m)(m + 4\pi r^3\tilde{p}_r)y'\right\} \\
&\quad + \frac{y^2\xi' r(r-2m)(m + 4\pi r^3\tilde{p}_r)}{r^4(r-2m)^2}.
\end{aligned} \tag{B.35}$$

Replacing the expression for $\tilde{p}_r'$:

$$\begin{aligned}
\frac{y}{r}\frac{d}{dr}\left[\frac{y}{r}\frac{d\xi}{dr}\right] &= \frac{y^2\xi}{r^4(r-2m)^2}\left\{4m^2 + 8\pi r^4\left[\Delta - 2\pi r^2\tilde{p}_r(\tilde{p}_r + \tilde{\rho})\right] - rm[3 + 4\pi r^2(3\tilde{p}_r + 4\Delta + \tilde{\rho})] + r^2(1 + 8\pi r^2\tilde{p}_r)m'\right\} \\
&\quad \times \frac{y\xi}{r^4(r-2m)^2}\left[r(r-2m)(m + 4\pi r^3\tilde{p}_r)y'\right] + \frac{y^2\xi' r(r-2m)(m + 4\pi r^3\tilde{p}_r)}{r^4(r-2m)^2}.
\end{aligned}$$

Then let us solve equations (B.3) and (B.2) to get $\tilde{p}_r$ and $\tilde{\rho}$ respectively:

$$\tilde{p}_r = \frac{-2m + r^2\nu' - 2rm\nu'}{8\pi r^3}, \tag{B.36}$$

$$\tilde{\rho} = \frac{m'}{4\pi r^2}. \tag{B.37}$$

Replacing it in the last expression, we get

$$\frac{y}{r}\frac{d}{dr}\left[\frac{y}{r}\frac{d\xi}{dr}\right] = \frac{y}{r^4(r-2m)^2}\frac{\xi(r-2m)}{2}\left\{2y\left(-3m+8\pi r^3\Delta + rm'\right) + r\left[y(-m+rm') + r(r-2m)y'\right]\nu'\right\}. \tag{B.38}$$

Using the definition of $y(r) \equiv (1-2m/r)^{1/2}$, the last expression becomes

$$\frac{y}{r}\frac{d}{dr}\left[\frac{y}{r}\frac{d\xi}{dr}\right] = \frac{y}{r^4(r-2m)^2}\left[\xi(r-2m)y\left(-3m+8\pi r^3\Delta + rm'\right)\right].$$

Taking r^4 as common factor

$$\frac{y}{r}\frac{d}{dr}\left[\frac{y}{r}\frac{d\xi}{dr}\right] = \frac{y}{r^4(r-2m)^2}\left[\xi(r-2m)yr^4\left(-\frac{3m}{r^4} + \frac{8\pi\Delta}{r} + \frac{m'}{r^3}\right)\right].$$

Let us identify $\frac{d}{dr}\frac{m}{r^3} = -\frac{-3m}{r^4} + \frac{m'}{r^3}$, then

$$\frac{y}{r}\frac{d}{dr}\left[\frac{y}{r}\frac{d\xi}{dr}\right] = \frac{y}{r^4(r-2m)^2}\xi r\left(1-\frac{2m}{r}\right)yr^4\left[\frac{d}{dr}\frac{m}{r^3} + \frac{8\pi\Delta}{r}\right].$$

Finally, after some simplification steps using the definition for $y(r)$

$$\frac{y}{r}\frac{d}{dr}\left[\frac{y}{r}\frac{d\xi}{dr}\right] = \frac{\xi}{r}\left[\frac{d}{dr}\frac{m}{r^3} + \frac{8\pi\Delta}{r}\right]. \tag{B.39}$$

Let us define a new function:

$$\eta(r) \equiv 8\pi\int_0^r \frac{r'}{y(r')}\left[\int_0^{r'}\frac{\Delta(r'')}{y(r'')}\frac{\zeta(r'')}{r''}dr''\right]dr', \tag{B.40}$$

where $\zeta(r)$ is a new independent variable:

$$\zeta(r) \equiv \int_0^r \frac{r'}{y(r')}dr'.$$

Moreover, let us denote

$$\psi(r) \equiv \xi(r) - \eta(r).$$

Taking in account that $\psi(0)>0$, we find that

$$\frac{d\psi}{d\xi} \le \frac{\psi}{\eta}, \tag{B.41}$$

which after solving becomes

$$\frac{y(r)}{r}\left[-\frac{8\pi r}{y(r)}\int_0^r \frac{e^{\frac{\nu(r')}{2}}\Delta(r')}{y(r')\,r'}dr' + \frac{1}{2}e^{\frac{\nu(r)}{2}}\nu'(r)\right] \le \frac{e^{\frac{\nu(r)}{2}} - 8\pi\int_0^r \frac{r'}{y(r')}\left[\int_0^{r'}\frac{e^{\frac{\nu(r'')}{2}}\Delta(r'')}{y(r'')r''}dr''\right]dr'}{\int_0^r \frac{r'}{y(r')}dr'}. \tag{B.42}$$

For stable stellar-type compact objects m/r^3 does not increase outwards. Then, the following condition holds for all the points of the star

$$\frac{m(r')}{r'} \ge \frac{m(r)}{r}\left(\frac{r'}{r}^2\right). \tag{B.43}$$

Then the anisotropy function (which decreases monotonically) holds the following condition

$$\frac{\Delta\left(r''\right)e^{\frac{\nu(r'')}{2}}}{r''} \geq \frac{\Delta\left(r'\right)e^{\frac{\nu(r')}{2}}}{r'} \geq \frac{\Delta(r)e^{\frac{\nu(r)}{2}}}{r}. \tag{B.44}$$

The denominator of the RHS of equation (B.42) holds

$$\int_0^r \frac{r'}{y\left(r'\right)}dr' \geq \int_0^r r'\left[1-\frac{2m(r)}{r^3}(r')^2\right]^{-1/2} dr'. \tag{B.45}$$

Integrating and evaluating the limits of integration

$$\int_0^r r'\left[1-\frac{2m(r)}{r^3}(r')^2\right]^{-1/2} dr' = \frac{r^3}{2m(r)}\left[1-y(r)\right].$$

Now, let us do the same for the second term in bracket on the LHS of equation (B.42)

$$\int_0^r \frac{e^{\frac{\nu(r')}{2}}\Delta\left(r'\right)}{y\left(r'\right)r'}dr' \geq \frac{e^{\frac{\nu(r)}{2}}\Delta(r)}{r}\int_0^r \left[1-\frac{2m(r)}{r^3}(r')^2\right]^{-1/2} dr'. \tag{B.46}$$

Integrating:

$$\int_0^r \left[1-\frac{2m(r)}{r^3}(r')^2\right]^{-1/2} dr' = r\sqrt{\frac{r}{2m(r)}}\arcsin\sqrt{\frac{2m(r)}{r}}. \tag{B.47}$$

Then, the LHS becomes

$$\int_0^r \frac{e^{\frac{\nu(r')}{2}}\Delta\left(r'\right)}{y\left(r'\right)r'}dr' \geq \frac{e^{\frac{\nu(r)}{2}}\Delta(r)}{r}\int_0^r \left[1-\frac{2m(r)}{r^3}(r')^2\right]^{-1/2} dr' = e^{\frac{\nu(r)}{2}}\Delta(r)\sqrt{\frac{r}{2m(r)}}\arcsin\left(\sqrt{\frac{2m(r)}{r}}\right). \tag{B.48}$$

Now, let us repeat the process for the numerator of the RHS of the equation (B.42)

$$\begin{aligned}
\int_0^r \frac{r'}{y\left(r'\right)}\left[\int_0^{r'} \frac{e^{\frac{\nu(r'')}{2}}\Delta\left(r''\right)}{y\left(r''\right)r''}dr''\right]dr' &\geq \int_0^r r\left[\frac{2m\left(r'\right)}{r'}\right]^{-1/2}\left(\frac{r'2}{r^2}\right)\left[\int_0^{r'} \frac{e^{\frac{\nu(r'')}{2}}\Delta\left(r''\right)}{y\left(r''\right)r''}dr''\right]dr' \\
&= \int_0^r \frac{r'2}{r}\left[\frac{2m\left(r'\right)}{r'}\right]^{-1/2}\frac{e^{\frac{\nu(r')}{2}}\Delta\left(r'\right)}{r'}\left[\int_0^{r'} \frac{1}{y\left(r''\right)}dr''\right]dr' \\
&\geq \int_0^r r'\left[\frac{2m\left(r'\right)}{r'}\right]^{-1/2}\left[\frac{e^{\frac{\nu(r')}{2}}\Delta\left(r'\right)}{r'}\int_0^{r'} \frac{1}{y\left(r''\right)}dr''\right]dr' \\
&\geq \int_0^r r'\left[\frac{2m\left(r'\right)}{r'}\right]^{-1/2}\left\{\frac{e^{\frac{\nu(r')}{2}}\Delta\left(r'\right)}{r'}\int_0^{r'} \left[1-\frac{2m\left(r'\right)}{r'3}r''^2\right]^{-1/2} dr''\right\}dr'
\end{aligned}$$

$$\begin{aligned}
\int_0^r \frac{r'}{y(r')}\left(\int_0^{r'} \frac{e^{\frac{\nu(r'')}{2}}\Delta(r'')}{y(r'')\,r''}dr''\right)dr' &\geq \int_0^r r'\left(\frac{2m(r')}{r'}\right)^{-1/2}\left(\frac{e^{\frac{\nu(r')}{2}}\Delta(r')}{r'}\right)r'\sqrt{\frac{r'}{2m(r')}}\arcsin\left(\sqrt{\frac{2m(r')}{r'}}\right)dr' \\
&= \int_0^r \frac{r'2}{y(r')}\left[\frac{e^{\frac{\nu(r')}{2}}\Delta(r')}{r'}\right]\sqrt{\frac{r'}{2m(r')}}\arcsin\left(\sqrt{\frac{2m(r')}{r'}}\right)dr' \\
&\geq \frac{e^{\frac{\nu(r)}{2}}\Delta(r)}{r}\int_0^r r'2\left[1-\frac{2m(r)}{r^3}r'2\right]^{-1/2}\left(\frac{2m(r)}{r^3}r'2\right)^{1/2}\arcsin\left(\sqrt{\frac{2m(r)}{r^3}}r'\right)dr' \\
&= \frac{e^{\frac{\nu(r)}{2}}\Delta(r)}{r}\int_0^r r'2\left[\frac{\frac{2m(r)}{r^3}r'2}{1-\frac{2m(r)}{r^3}r'2}\right]^{1/2}\arcsin\left(\sqrt{\frac{2m(r)}{r^3}}r'\right)dr' \\
&= e^{\frac{\nu(r)}{2}}\Delta(r)r^2\left(\frac{2m(r)}{r}\right)^{-3/2}\left[\sqrt{\frac{2m(r)}{r}}-y(r)\arcsin\left(\sqrt{\frac{2m(r)}{r}}\right)\right]. \qquad (B.49)
\end{aligned}$$

Replacing the obtained results for each expression in the inequality (B.42), we get:

$$\frac{y(r)}{r}\left\{-\frac{8\pi r}{y(r)}e^{\frac{\nu(r)}{2}}\Delta(r)\sqrt{\frac{r}{2m(r)}}\arcsin\left(\sqrt{\frac{2m(r)}{r}}\right)+\frac{1}{2}e^{\frac{\nu(r)}{2}}\nu'(r)\right\}\leq\frac{e^{\frac{\nu(r)}{2}}-8\pi\left\{e^{\frac{\nu(r)}{2}}\Delta(r)r^2\left(\frac{2m(r)}{r}\right)^{-3/2}\left[\sqrt{\frac{2m(r)}{r}}-y(r)\arcsin\left(\sqrt{\frac{2m(r)}{r}}\right)\right]\right\}}{\frac{r^3}{2m(r)}(1-y(r))}$$

Taking $e^{\nu(r)/2}$ as common factor in both sides, and then eliminating it:

$$\frac{y(r)}{r}\left\{-\frac{8\pi r}{y(r)}\Delta(r)\sqrt{\frac{r}{2m(r)}}\arcsin\left(\sqrt{\frac{2m(r)}{r}}\right)+\frac{1}{2}\nu'(r)\right\}\leq\frac{1-8\pi\Delta(r)r^2\left(\frac{2m(r)}{r}\right)^{-3/2}\left[\sqrt{\frac{2m(r)}{r}}-y(r)\arcsin\left(\sqrt{\frac{2m(r)}{r}}\right)\right]}{\frac{r^3}{2m(r)}[1-y(r)]} \qquad (B.50)$$

Let us start working in the RHS of previous expression (B.50)

$$\frac{y}{r}\left\{-\frac{8\pi r}{y}\Delta\sqrt{\frac{r}{2m}}\arcsin\left(\sqrt{\frac{2m}{r}}\right)+\frac{1}{2}\nu'\right\}\leq\frac{1}{(1-y)}\left[\frac{2m}{r^3}-8\pi\Delta\frac{r^2\left(\frac{2m}{r}\right)^{-3/2}\sqrt{\frac{2m}{r}}}{\frac{r^3}{2m}}+8\pi\Delta\frac{r^2\left(\frac{2m}{r}\right)^{-3/2}y}{\frac{r^3}{2m}}\arcsin\left(\sqrt{\frac{2m}{r}}\right)\right]$$

$$-8\pi\Delta\sqrt{\frac{r}{2m}}\arcsin\left(\sqrt{\frac{2m}{r}}\right)(1-y)+\frac{1}{2}\frac{y}{r}\nu'(1-y)\leq\frac{2m}{r^3}-8\pi\Delta+8\pi\Delta\left(\frac{r}{2m}\right)^{1/2}y\arcsin\left(\sqrt{\frac{2m}{r}}\right)$$

$$-8\pi\Delta\sqrt{\frac{r}{2m}}\arcsin\left(\sqrt{\frac{2m}{r}}\right)+\frac{1}{2}\frac{y}{r}\nu'(1-y)\leq\frac{2m}{r^3}-8\pi\Delta$$

$$\frac{1}{2}\frac{y}{r}\nu'(1-y)\leq\frac{2m}{r^3}-8\pi\Delta+8\pi\Delta\sqrt{\frac{r}{2m}}\arcsin\left(\sqrt{\frac{2m}{r}}\right)$$

$$\frac{1}{2}\frac{y}{r}\nu'(1-y)\leq\frac{2m}{r^3}-8\pi\Delta\left[\frac{\arcsin\left(\sqrt{\frac{2m}{r}}\right)}{\sqrt{\frac{2m}{r}}}-1\right]$$

Now, let us work on the LHS of expression (B.50)

$$\left[1-\left(1-\frac{2m}{r}\right)^{1/2}\right]\frac{1}{2}\frac{y}{r}\left(-\frac{2\tilde{p}_r'}{\tilde{p}_r+\tilde{\rho}}+\frac{4\Delta}{r\tilde{p}_r+r\tilde{\rho}}\right)\leq\frac{2m}{r^3}+8\pi\Delta\left[\frac{\arcsin\left(\sqrt{\frac{2m}{r}}\right)}{\sqrt{\frac{2m}{r}}}-1\right], \tag{B.51}$$

where

$$-\frac{2\tilde{p}_r'}{\tilde{p}_r+\tilde{\rho}}+\frac{4\Delta}{r\tilde{p}_r+r\tilde{\rho}}=\frac{2m+8\pi r^3\tilde{p}_r}{r^2-2rm}$$

$$\left[1-\left(1-\frac{2m}{r}\right)^{1/2}\right]\left(1-\frac{2m}{r}\right)^{1/2}\frac{m+4\pi r^3\tilde{p}_r}{r^3\left(1-\frac{2m}{r}\right)}\leq\frac{2m}{r^3}+8\pi\Delta\left[\frac{\arcsin\left(\sqrt{\frac{2m}{r}}\right)}{\sqrt{\frac{2m}{r}}}-1\right].$$

Finally,

$$\left[1-\left(1-\frac{2m}{r}\right)^{1/2}\right]\frac{m+4\pi r^3\tilde{p}_r}{r^3\left(1-\frac{2m}{r}\right)^{1/2}}\leq\frac{2m}{r^3}+8\pi\Delta\left[\frac{\arcsin\left(\sqrt{\frac{2m}{r}}\right)}{\sqrt{\frac{2m}{r}}}-1\right]. \tag{B.52}$$

Equation (B.52) is valid for all r inside the star. It is independent of the anisotropy Δ sign. Now, let us consider the isotropic case where Δ=0 and the evaluated radius is $r = R$.

$$\begin{aligned}
\left[1-\left(1-\frac{2M}{R}\right)^{1/2}\right]\frac{M+4\pi R^3\tilde{p}_r}{R^3\left(1-\frac{2M}{R}\right)^{1/2}} &\leq \frac{2M}{R^3}\\
\frac{1}{\left(1-\frac{2M}{R}\right)^{1/2}} &\leq 2\left[1-\left(1-\frac{2M}{R}\right)^{1/2}\right]^{-1}\\
\frac{1-\left(1-\frac{2M}{R}\right)^{1/2}}{\left(1-\frac{2M}{R}\right)^{1/2}} &\leq 2\\
\frac{1}{\left(1-\frac{2M}{R}\right)^{1/2}}-1 &\leq 2\\
\frac{1}{\left(1-\frac{2M}{R}\right)^{1/2}} &\leq 3\\
\frac{1}{3} &\leq \left(1-\frac{2M}{R}\right)^{1/2}\\
\frac{1}{9} &\leq 1-\frac{2M}{R}\\
\frac{2M}{R} &\leq 1-\frac{1}{9}\\
\frac{2M}{R} &\leq \frac{8}{9}.
\end{aligned}$$

Which corresponds to the well know Buchdahl's limit.
Now, let us consider the case when $\Delta \neq 0$. So, let us start defining

$$f(M,R,\Delta) \equiv \frac{8\pi R^3\Delta}{3M}\left[\frac{\arcsin\left(\sqrt{\frac{2M}{R}}\right)}{\sqrt{\frac{2M}{R}}} - 1\right] \implies \Delta = \frac{3M}{8\pi R^3}\frac{f}{\left[\frac{\arcsin\left(\sqrt{\frac{2M}{R}}\right)}{\sqrt{\frac{2M}{R}}} - 1\right]}. \tag{B.53}$$

After evaluating Δ in the expression (B.52), we obtain

$$\left[\frac{\left(1 - \sqrt{1 - \frac{2M}{R}}\right)\left(M + 4\pi R^3\tilde{p}_r(R)\right)}{\sqrt{1 - \frac{2M}{R}}R^3}\right] \leq \frac{2M}{R^3} + \frac{3fM}{R^3}.$$

For simplicity, let us use $u = 2M/R$, which corresponds to the compactness parameter. Moreover, remember that $\tilde{p}_r(R)$= 0.

$$\frac{\left(1 - \sqrt{1-u}\right)}{\sqrt{1-u}} \leq 2 + 3f.$$

Then, solving for u last expression becomes

$$u = 1 - \frac{1}{9(1+f)^2} \implies \frac{2M}{R} = 1 - \frac{1}{9(1+f)^2}, \tag{B.54}$$

which corresponds to the restriction on the mass-radius ratio for compact anisotropic objects.

Bibliography

[1] Foster, J.; Nightingale, J. *A Short Course in General Relativity*; Longman Mathematical Texts; Longman, 1979.

[2] Schwarzschild, K. Über das Gravitationsfeld eines Massenpunktes nach der Einsteinschen Theorie. *Sitzungsberichte der Königlich Preußischen Akademie der Wissenschaften (Berlin)* **1916**, 189–196.

[3] Schwarzschild, K. Über das Gravitationsfeld einer Kugel aus inkompressibler Flüssigkeit nach der Einsteinschen Theorie. **1916**, 424–434.

[4] Tolman, R. C. Static Solutions of Einstein's Field Equations for Spheres of Fluid. *Phys. Rev.* **1939**, *55*, 364–373.

[5] Buchdahl, H. A. General Relativistic Fluid Spheres. *Phys. Rev.* **1959**, *116*, 1027–1034.

[6] Ruderman, M. Pulsars: Structure and Dynamics. *Annual Review of Astronomy and Astrophysics* **1972**, *10*, 427–476.

[7] Bowers, R. L.; Liang, E. P. T. Anisotropic Spheres in General Relativity. *apj* **1974**, *188*, 657.

[8] Oppenheimer, J. R.; Volkoff, G. M. On Massive Neutron Cores. *Phys. Rev.* **1939**, *55*, 374–381.

[9] Maurya, S. K.; Govender, M. A family of charged compact objects with anisotropic pressure. *The European Physical Journal C* **2017**, *77*, 420.

[10] Dzhunushaliev, V.; Folomeev, V.; Myrzakulov, R.; Singleton, D. Non-singular solutions to Einstein-Klein-Gordon equations with a phantom scalar field. *Journal of High Energy Physics* **2008**, *2008*, 094.

[11] Chakraborty, S.; SenGupta, S. Solving higher curvature gravity theories. *European Physical Journal C* **2016**, *76*, 552.

[12] Kokkotas, K. D.; Konoplya, R. A.; Zhidenko, A. Non-Schwarzschild black-hole metric in four dimensional higher derivative gravity: Analytical approximation. *prd* **2017**, *96*, 064007.

[13] Jaime, L. G.; Patino, L.; Salgado, M. Robust approach tof(R)gravity. *Physical Review D* **2011**, *83*.

[14] Alvarez-Gaume, L.; Kehagias, A.; Kounnas, C.; Lüst, D.; Riotto, A. Aspects of quadratic gravity. *Fortschritte der Physik* **2016**, *64*, 176–189.

[15] Vernieri, D.; Carloni, S. On the anisotropic interior solutions in Horava gravity and Einstein theory. *EPL (Europhysics Letters)* **2018**, *121*, 30002.

[16] Eling, C.; Jacobson, T. Spherical solutions in Einstein-aether theory: static aether and stars. *Classical and Quantum Gravity* **2006**, *23*, 5625–5642.

[17] Stuchlik, Z.; Hledik, S.; Novotny, J. General relativistic polytropes with a repulsive cosmological constant. *Physical Review D* **2016**, *94*.

[18] Novotný, J.; Hladík, J.; Stuchlík, Z. c. v. Polytropic spheres containing regions of trapped null geodesics. *Phys. Rev. D* **2017**, *95*, 043009.

[19] Stuchlik, Z.; Schee, J.; Toshmatov, B.; Hladik, J.; Novotny, J. Gravitational instability of polytropic spheres containing region of trapped null geodesics: a possible explanation of central supermassive black holes in galactic halos. *jcap* **2017**, *2017*, 056.

[20] Ilyas, M.; Yousaf, Z.; Bhatti, M. Z.; Masud, B. Existence of relativistic structures in $f(R,T)$ gravity. *Astrophys. Space Sci.* **2017**, *362*, 237.

[21] Carloni, S.; Vernieri, D. Covariant Tolman-Oppenheimer-Volkoff equations. II. The anisotropic case. *prd* **2018**, *97*, 124057.

[22] Canuto, V. Neutron Stars: General Review. 1973.

[23] Cosenza, M.; Herrera, L.; Esculpi, M.; Witten, L. Some Models of Anisotropic Spheres in General Relativity. *Journal of Mathematical Physics* **1981**, *22*, 118.

[24] Herrera, L.; Ruggeri, G. J.; Witten, L. Adiabatic contraction of anisotropic spheres in general relativity. *apj* **1979**, *234*, 1094–1099.

[25] Ovalle, J. Searching Exact Solutions for Compact Stars in Braneworld:. a Conjecture. *Modern Physics Letters A* **2008**, *23*, 3247–3263.

[26] Ovalle, J. Braneworld Stars:. Anisotropy Minimally Projected onto the Brane. 2010.

[27] Ovalle, J. Decoupling gravitational sources in general relativity: From perfect to anisotropic fluids. *Physical Review D* **2017**, *95*.

[28] Böhmer, C. G.; Harko, T. Bounds on the basic physical parameters for anisotropic compact general relativistic objects. *Classical and Quantum Gravity* **2006**, *23*, 6479–6491.

[29] Einstein, A. Die Feldgleichungen der Gravitation. *Sitzungsberichte der Königlich Preußischen Akademie der Wissenschaften (Berlin), Seite 844-847.* **1915**,

[30] Misner, C. W.; Thorne, K. S.; Wheeler, J. A. In *San Francisco: W.H. Freeman and Co., 1973*; Misner, C. W., Thorne, K. S., & Wheeler, J. A.,, Ed.; 1973.

[31] Herrera, L.; Santos, N. Local anisotropy in self-gravitating systems. *Physics Reports* **1997**, *286*, 53 – 130.

[32] Heintzmann, H.; Hillebrandt, W. Neutron stars with an anisotropic equation of state: mass, redshift and stability. *aap* **1975**, *38*, 51–55.

[33] Chan, R. Radiating gravitational collapse with shear viscosity. *mnras* **2000**, *316*, 588–604.

[34] Torres-Sánchez, V. A.,; Contreras, E., Anisotropic neutron stars by gravitational decoupling. *Eur. Phys. J. C* **2019**, *79*, 829.

[35] Ovalle, J.; Sotomayor, A. A simple method to generate exact physically acceptable anisotropic solutions in general relativity. *The European Physical Journal Plus* **2018**, *133*.

[36] Buchdahl, H. A. General Relativistic Fluid Spheres. *Phys. Rev.* **1959**, *116*, 1027–1034.

[37] Buchdahl, H. A. General Relativistic Fluid Spheres. II. General Inequalities for Regular Spheres. *apj* **1966**, *146*, 275.

[38] Bondi, H. The gravitational redshift from static spherical bodies. *mnras* **1999**, *302*, 337–340.

[39] Islam, J. N. Some general relativistic inequalities for a star in hydrostatic equilibrium. *mnras* **1969**, *145*, 21.

[40] Wald, R. M. *General relativity*; 1984.