UNIVERSIDAD DE INVESTIGACIÓN DE TECNOLOGÍA EXPERIMENTAL YACHAY

Escuela de Ciencias Matemáticas y Computacionales

TÍTULO: Existence of a solution for a discontinuous problem involving the p-Laplacian operator

Trabajo de integración curricular presentado como requisito para la obtención del título de Matemático

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Dedicatoria

Para mi madre Nancy, y mis dos amadas abuelitas Marcela y Dolores.

Darwin Xavier Tallana Chimarro

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Darwin Xavier Tallana Chimarro

Resumen

En este trabajo se estudia el siguiente problema de valor de frontera asociado con el p-Laplaciano:

$$\begin{cases} -\Delta_p u(x) = h(x)f(u(x)) + q(x), & x \in \Omega, \\ u(x) = 0, & x \in \partial\Omega, \end{cases} \tag{PP}$$

donde $p > 1$, $\Omega \subseteq \mathbb{R}^N$ es un dominio suave y acotado, $q \in L^{p'}(\Omega)$, $h \in L^\infty(\Omega)$, y $f : \mathbb{R} \to \mathbb{R}$ es una función discontinua, la cual satisface:

($\mathbf{F_1}$) Existe $a \in \mathbb{R}$ tal que

a) $f \in C(\mathbb{R}\backslash\{a\}, \mathbb{R})$,
b) $f(a^-) < f(a^+)$,
c) $f(a) \in [f(a^-), f(a^+)]$,

($\mathbf{F_2}$) existen $\alpha, C_1, C_2 > 0$, con $1 + \alpha \in [p, p^\star]$, tales que

$$\forall s \in \mathbb{R} : \quad |f(s)| \leq C_1 + C_2|s|^\alpha,$$

donde

$$p^\star = \begin{cases} \dfrac{Np}{N-p}, & p < N, \\ +\infty, & p \geq N. \end{cases}$$

Al considerar el gradiente generalizado (Clarke) de $I : W_0^{1,p}(\Omega) \to \mathbb{R}$ en un elemento u, denotado $\partial I(u)$, donde

$$I(u) = \frac{1}{p}\int_\Omega |\nabla u(x)|^p dx - \int_\Omega q(x)u(x)dx - \int_\Omega \int_0^{u(x)} f(s)h(x)dsdx,$$

se demuestra que la condición $0 \in \partial I(u)$ es equivalente a

$$-\Delta_p u(x) - q(x) \in \hat{\phi}(x, u(x)), \text{ a.e. } \Omega,$$

donde

$$\hat{\phi}(x, s) = \begin{cases} \{h(x)f(s)\}, & s \neq a, \\ [h(x)f(a^-), h(x)f(a^+)], & s = a, x \in \Omega_+, \\ [h(x)f(a^+), h(x)f(a^-)], & s = a, x \in \Omega_-, \end{cases}$$

y

$$\Omega_+ := \left\{x \in \Omega \,/\, h(x) \geq 0\right\} \quad \text{y} \quad \Omega_- := \left\{x \in \Omega \,/\, h(x) < 0\right\}.$$

También, asumiendo ciertas condiciones sobre la imagen de q y que $0 \in \partial I(u)$, se demuestra que u es una solución débil casi en todas partes del problema (PP).

Además, al asumir que u es un punto de mínimo local de I y la medida de Ω_- es cero, o u es un punto de máximo local de I y la medida de Ω_+ es cero, se demuestra que u es también una solución débil casi en todas partes.

Finalmente, como una aplicación, suponiendo que $|\Omega_-| = 0$, $\alpha = p-1$, $M \neq 0$, y $C_1 < \lambda_1/M$ (donde λ_1 es el primer autovalor del p-Laplaciano y $M = \|h\|_{L^\infty(\Omega)}$), se demuestra que (PP) tiene una solución débil casi en todas partes.

Palabras Clave— p-Laplaciano, problema de valor de frontera, discontinuidad, gradiente generalizado.

Abstract

In this work we study the following boundary value problem involving the p-Laplacian operator:

$$\begin{cases} -\Delta_p u(x) = h(x)f(u(x)) + q(x), & x \in \Omega, \\ u(x) = 0, & x \in \partial\Omega, \end{cases} \tag{PP}$$

where $p > 1$, $\Omega \subseteq \mathbb{R}^N$ is a smooth bounded domain, $q \in L^{p'}(\Omega)$, $h \in L^\infty(\Omega)$, and $f : \mathbb{R} \to \mathbb{R}$ is a discontinuous function satisfying:

($\mathbf{F_1}$) There exists $a \in \mathbb{R}$ such that

a) $f \in C(\mathbb{R}\backslash\{a\}, \mathbb{R})$,
b) $f(a^-) < f(a^+)$,
c) $f(a) \in [f(a^-), f(a^+)]$,

($\mathbf{F_2}$) there exist $\alpha, C_1, C_2 > 0$, with $1 + \alpha \in [p, p^\star]$, such that

$$\forall s \in \mathbb{R} : \quad |f(s)| \leq C_1 + C_2|s|^\alpha,$$

where

$$p^\star = \begin{cases} \dfrac{Np}{N-p}, & p < N, \\ +\infty, & \text{otherwise.} \end{cases}$$

By considering the Clarke's generalized gradient of $I : W_0^{1,p}(\Omega) \to \mathbb{R}$ at u, denoted $\partial I(u)$, where

$$I(u) = \frac{1}{p}\int_\Omega |\nabla u(x)|^p dx - \int_\Omega q(x)u(x)dx - \int_\Omega \int_0^{u(x)} f(s)h(x)dsdx,$$

we show that condition $0 \in \partial I(u)$ is equivalent to

$$-\Delta_p u(x) - q(x) \in \hat{\phi}(x, u(x)), \text{ a.e. } \Omega,$$

where

$$\hat{\phi}(x,s) = \begin{cases} \{h(x)f(s)\}, & s \neq a, \\ [h(x)f(a^-), h(x)f(a^+)], & s = a, x \in \Omega_+, \\ [h(x)f(a^+), h(x)f(a^-)], & s = a, x \in \Omega_-, \end{cases}$$

with

$$\Omega_+ := \left\{x \in \Omega \,/\, h(x) \geq 0\right\} \quad \text{and} \quad \Omega_- := \left\{x \in \Omega \,/\, h(x) < 0\right\}.$$

We also show that, under certain conditions on the image of q, u is an almost everywhere weak solution to the problem (PP) if $0 \in \partial I(u)$.

Besides, by assuming that u is a point of local minimum of I and the measure of Ω_- is zero, or u is a point of local maximum of I and the measure of Ω_+ is zero, we show that u is also an almost everywhere solution.

Finally, as an application, by assuming $|\Omega_-| = 0$, $\alpha = p - 1$, $M \neq 0$, and $C_1 < \lambda_1/M$ (λ_1 being the first eigenvalue of the p-Laplacian and $M = \|h\|_{L^\infty(\Omega)}$), we show that problem (PP) has an almost everywhere weak solution.

Keywords— p-Laplacian, boundary value problem, discontinuity, generalized gradient.

Contents

1 Introduction

Partial Differential Equations have a long history of applications to many fields, starting with the link between Calculus and Physics. Nowadays, it is widely used to make predictions about the behavior of different phenomena. Special examples are Hamiltonian equations in mechanics, the heat equation in thermodynamics, Lotka-Volterra system in population dynamics, or SIR models to study the evolution of diseases.

Although this area is known commonly by its applications in mathematical modeling, it constitutes a huge branch of mathematics on its own. The extensive development of this theory goes beyond the most basic situation of two independent variables (x,t), allowing the study of mathematical problems on infinite-dimensional spaces, linked other abstract fields, and being useful in the rigorous set up of physical theories.

The main concern is to find solutions to partial differential equations (PDE) and study their qualitative behaviour. To achieve this, different methods have been used by combining varied fields of mathematics like functional analysis, topology, and calculus of variations. Among these methods, we have the variational ones, which basically consist of associating a functional to the PDE and finding critical points of this functional, i.e., points where the differential of the functional nulls. These critical points give us, in a weak sense, solutions to our problem. However, to apply these results we have to assume conditions of "smoothness" over the functional, which is not always possible. This fact motivated a more general theory to search for critical points of nonsmooth functionals. In this work we study a PDE involving a functional of this kind.

In concrete, we are interested in solutions for

$$\begin{cases} -\Delta_p u(x) = h(x)f(u(x)) + q(x), & x \in \Omega, \\ u(x) = 0, & x \in \partial\Omega, \end{cases} \tag{PP}$$

where

$$\Delta_p u = \nabla \cdot (|\nabla u|^{p-2}\nabla u),$$

denotes the p-Laplacian operator and $\Omega \subseteq \mathbb{R}^N$ is a smooth bounded domain. The following conditions are assumed

($\mathbf{F_1}$) For some $a \in \mathbb{R}$,

a) $f \in C(\mathbb{R}\backslash\{a\}, \mathbb{R})$,

b) $f(a^-) < f(a^+)$,

c) $f(a) \in [f(a^-), f(a^+)]$.

($\mathbf{F_2}$) There exist $\alpha, C_1, C_2 > 0$, with $1 + \alpha \in [p, p^\star]$, such that

$$\forall s \in \mathbb{R} : \quad |f(s)| \le C_1 + C_2|s|^\alpha.$$

Here

$$p^\star = \begin{cases} \dfrac{Np}{N-p}, & p < N, \\ +\infty, & \text{otherwise.} \end{cases}$$

The function f is, in general, nonlinear and is called the nonlinear term of the problem. This situation is a generalization of a problem studied by Ambrosseti and Badiale, [1],

$$\begin{cases} -\Delta u(x) = f(u(x)) + q(x), & x \in \Omega, \\ u(x) = 0, & x \in \partial\Omega, \end{cases} \tag{P1}$$

which is associated with a physical model set, [3], which represents the steady-state distribution of temperature in an electric arc under forced convection. In the model, the function f has a discontinuity. Ambrosseti and Badiale used a variational approach called Clarke's Dual Action Principle, introduced in [17], which was fully expanded and applied in [19] and [6]. This approach basically allowed to find a smooth functional associated with (P1), called the dual functional, and then to apply the classical theory to obtain its critical points. However, to employ this method, they had to consider an additional condition on the growth of f, namely

($\mathbf{F_2'}$) There exists $\nu : \mathbb{R} \to \mathbb{R}$ such that $\nu(s) = ms + f(s)$, strictly increasing, for some $m \geq 0$.

Arcoya and Calahorrano [4], in an attempt to generalize the problem for the p-Laplacian, changed the variational approach of Ambrosseti and Badiale by one developed by Chang in [13]. This change was done because the nonlinearity of the p-Laplacian complicates finding the desired dual functional. Thus, by applying Chang's machinery, they work with the generalized gradient of the functional associated with

$$\begin{cases} -\Delta_p u(x) = f(u(x)) + q(x), & x \in \Omega, \\ \quad u(x) = 0, & x \in \partial\Omega. \end{cases} \tag{P2}$$

In this way, condition ($\mathbf{F_2'}$) was no more necessary.

Another generalization was obtained by Mayorga-Zambrano and Calahorrano, [8]. They considered the following generalization of (P2),

$$\begin{cases} -\Delta_p u(x) = h(x) f(u(x)) + q(x), & x \in \Omega, \\ \quad u(x) = 0, & x \in \partial\Omega, \end{cases}$$

assuming

($\mathbf{H_1}$) $h \in L^\infty(\Omega)$ and $h > 0$.

In this work we extend this problem once more by considering any bounded function h, i.e., by replacing ($\mathbf{H_1}$) by ($\mathbf{H_2}$),

($\mathbf{H_2}$) $h \in L^\infty(\Omega)$.

To produce our results, we use the machinery developed in [13]. We shall work with a non-Frechet differentiable but locally Lipschitz continuous functional (associated to (PP)),

$$I : W_0^{1,p}(\Omega) \to \mathbb{R},$$

given by

$$I(u) = \frac{1}{p} \int_\Omega |\nabla u(x)|^p dx - \int_\Omega q(x) u(x) dx - \int_\Omega F(u(x)) h(x) dx,$$

where

$$F(t) = \int_0^t f(s) ds.$$

We consider two kinds of solution. First, we say that $u \in W_0^{1,p}(\Omega)$ is a *multivalued solution* of (PP) if

$$-\Delta_p u(x) - q(x) \in \hat{\phi}(x, u(x)), \text{ a.e. } \Omega.$$

with

$$\hat{\phi}(x,s) = \begin{cases} \{h(x) f(s)\}, & s \neq a, \\ [h(x) f(a^-), h(x) f(a^+)], & s = a, x \in \Omega_+, \\ [h(x) f(a^+), h(x) f(a^-)], & s = a, x \in \Omega_-, \end{cases}$$

where

$$\Omega_+ := \left\{ x \in \Omega \,/\, h(x) \geq 0 \right\} \quad \text{ and } \quad \Omega_- := \left\{ x \in \Omega \,/\, h(x) < 0 \right\}.$$

Second, we say that $u \in W_0^{1,p}(\Omega)$ is a *solution* or an *almost everywhere weak solution* of (PP) if

$$-\Delta_p u(x) - q(x) = h(x) f(u(x)), \text{ a.e. } \Omega.$$

Also, we say that $u \in W_0^{1,p}(\Omega)$ is a *generalized critical point* of I if and only if $0 \in \partial I(u)$, where $\partial I(u)$ is the generalized gradient of I at u.

Next, we will shortly state our results. In one hand, $u \in W_0^{1,p}(\Omega)$ is a generalized critical point of I if and only if it is a multivalued solution of (PP). In other hand, by assuming that u is a generalized critical point and

$$-q(x) \notin \left[\alpha^-, \alpha^+\right], \quad \text{ a.e. } \Omega,$$

where

$$\alpha^- := \min\{\alpha^-_{<0}, \alpha^-_{\geq 0}\},$$
$$\alpha^+ := \max\{\alpha^+_{<0}, \alpha^+_{\geq 0}\},$$

with

$$\alpha^-_{\geq 0} := \min\left\{m_+ f\left(a^-\right), M_+ f\left(a^-\right)\right\},$$
$$\alpha^+_{\geq 0} := \max\left\{m_+ f\left(a^+\right), M_+ f\left(a^+\right)\right\},$$
$$\alpha^-_{<0} := \min\left\{m_- f\left(a^+\right), M_- f\left(a^+\right)\right\},$$
$$\alpha^+_{<0} := \max\left\{m_- f\left(a^-\right), M_- f\left(a^-\right)\right\},$$

and

$$m_+ := \operatorname*{ess\,inf}_{x\in\Omega_+}(h(x)) \quad M_+ := \operatorname*{ess\,sup}_{x\in\Omega_+}(h(x))$$
$$m_- := \operatorname*{ess\,inf}_{x\in\Omega_-}(h(x)) \quad M_- := \operatorname*{ess\,sup}_{x\in\Omega_-}(h(x)),$$

we will show that u is an almost everywhere weak solution of (PP). Finally, by considering one of the following conditions

(i) $|\Omega_-| = 0$ and I has a point of local minimum at $u \in W_0^{1,p}(\Omega)$,

(ii) $|\Omega_+| = 0$ and I has a point of local maximum at $u \in W_0^{1,p}(\Omega)$,

we will prove that u is also an almost everywhere weak solution. After that we will show an application of these results.

Our work is organized as follows:

- In Section 2 we will present concepts and results that will be used during this work. We start by quickly introducing Baire, Borel, and Lebesgue measurable functions. Next, we briefly discuss L^p and Sobolev spaces. We state some important theorems like the Riesz representation for L^p spaces and embedding theorems for Sobolev spaces. After that, we examine, in short, the variational approach of partial differential equations, in particular, we discuss the relation between critical points and the Euler equation. In the following subsection we state essential definitions and some results of nonlinear analysis. In concrete, we introduce the Palais-Smale condition, a Deformation Lemma, and a version of the Mountain Pass theorem. This subsection is presented as additional material that can be useful for the reader to compare it with variational methods applied to non-Fréchet differentiable functionals. Next, we introduce the concept of generalized gradient, which is the main tool to work with the functional I. We construct the generalized gradient of a locally Lipschitz continuous functional by using the concept of generalized directional derivative. We present some properties and results about this gradient, and state results used to find the generalized gradient of I. We present analogous objects as those for the case of Fréchet differentiable functionals. At the end, we quickly introduce the p-Laplacian operator and discuss some properties of its first eigenvalue, which will be used to state the application.

- In Section 3 we present our results. We open this section by commenting some previous works associated with our setting. Also, we list our assumptions and explain why they are considered. Specifically, we redo the proof to find the functional I from (PP) in a formal way. After this, we state and prove our main result, which gives conditions to get multivalued solutions and almost everywhere weak solutions of (PP). Finally, we apply our main result to find solutions of one specific PDE equation.

2 Theoretical framework

2.1 Topics from Mathematical Analysis

In this section, we present some concepts and notations used in this manuscript. Our main references are [33], [27] and [32]. Consider $x = (x_1, x_2, \dots, x_N) \in \mathbb{R}^N$. Its euclidean norm, denoted $|x|$, is defined with the formula,

$$|x| := [x_1^2 + x_2^2 + \cdots + x_N^2]^{1/2}.$$

Let X be a linear space and $x, y \in X$. A *segment* $[x, y]$ from x to y is defined as the set of points such that for any $t \in [0, 1]$

$$tx + (1 - t)y \in A.$$

A subset A of X is *convex* iff

$$\forall x, y \in A : \quad [x, y] \subseteq A,$$

and $f : X \to (-\infty, +\infty]$ on a convex set X is said to be a *convex function* if it satisfies the following condition

$$\forall t \in [0, 1], \forall x, y \in X : \quad f(tx + (1 - t)y) \leq tf(x) + (1 - t)f(y).$$

We say that any function f, whose codomain is $\mathbb{R}$, is a *functional*. The functional $f : X \to \mathbb{R}$ is called *positively homogeneous* if

$$\forall \alpha > 0, \forall x \in X : \quad f(\alpha x) = \alpha f(x),$$

and *subadditive* if

$$\forall x, y \in X : \quad f(x + y) \leq f(x) + f(y).$$

When $X = \mathbb{R}$ we denote

$$f(x^+) := \lim_{x \downarrow x_0} f(x),$$

$$f(x^-) := \lim_{x \uparrow x_0} f(x).$$

Let us consider a topological space $(X, \mathcal{T})$ and $A \subseteq X$. We say that A is *compact* if and only if for any open covering of A,

$$(B_\lambda)_{\lambda \in \Lambda} \subseteq \mathcal{T},$$

we can find a finite set $I \in \Lambda$ such that $(B_\lambda)_{\lambda \in I}$ is still a covering of A. In other words, A is compact iff from any open covering of A we can extract a finite open subcovering. We also say that A is *sequentially compact* iff for any sequence in A there exists a subsequence converging in its closure, $\overline{A}$. This last property is also known as *Bolzano-Weierstrass property*. A is *relatively compact* or *precompact* if $\overline{A}$ is compact. When (X, d) is a metric space, A is compact iff it is sequentially compact. A point $x \in X$ is called an *accumulation point* of A iff

$$\forall U \in \mathcal{N}(x) : \quad (U \backslash \{x\}) \cap A \neq \emptyset,$$

where $\mathcal{N}(x)$ is the set of *neighborhoods* of x. The set of all accumulation points of A is denoted by acc(A).

Remark 2.1.1. In the case that $\dim(X) < +\infty$, $A \subseteq X$ is compact if and only if A is closed and bounded.

Consider $(Y, \mathcal{G})$, another topological space. We say that $\phi : X \to Y$ is a *homeomorphism* iff it is a continuous bijection and its inverse is also continuous.

Let us define a norm on a linear space X. We say that $\| \cdot \| : X \to \mathbb{R}$ is a *norm* if,

(i) $\forall u \in X : \quad \|u\| = 0 \iff u = 0;$

(ii) $\forall \lambda \in \mathbb{R}, \forall u \in X : \quad \|\lambda \cdot u\| = |\lambda| \cdot \|u\|;$

(iii) (**triangle inequality**) $\forall u, v \in X : \quad \|u + v\| \leq \|u\| + \|v\|.$

Also, we say that $||| \cdot ||| : X \to \mathbb{R}$ is a *seminorm* on X if conditions (ii) and (iii) hold, and additionally

(i') $\forall v \in X : \quad |||v||| \geq 0.$

Let $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ be two normed spaces. We denote by $\mathcal{T}_{\|\cdot\|_X}$, $\mathcal{T}_{\|\cdot\|_Y}$ the topologies induced by the corresponding norms. Thus, for $\mathcal{T}_{\|\cdot\|_X}$, $\mathcal{T}_{\|\cdot\|_Y}$ we consider the set of *continuous functions* from X to Y, represented by $C(X,Y)$, which consists of functions f such that

$$\forall u_0 \in X, \forall \varepsilon > 0, \exists \delta > 0 : \quad \|u - u_0\|_X < \delta \implies \|f(u) - f(u_0)\|_Y < \varepsilon.$$

For normed spaces (in general for Hausdorff spaces), we have that a function f is continuous iff,

$$\forall (u_n)_{n\in\mathbb{N}} \subseteq X : \quad \lim_{n\to\infty} u_n = u \implies \lim_{n\to\infty} f(u_n) = f(u). \tag{1}$$

We define

$$\mathcal{L}(X,Y) := \{S : X \to Y \,/\, S \text{ is linear}\}.$$

An operator $T : X \to Y$ is *bounded* iff

$$\exists c > 0, \forall u \in X : \quad \|T(u)\|_Y \leq c\|u\|_X.$$

We define

$$B(X,Y) := \{T \in \mathcal{L}(X,Y) \,/\, T \text{ is bounded}\}.$$

This set is a normed space with norm $\|\cdot\| : B(X,Y) \to \mathbb{R}$ given by

$$\|T\| = \inf\left(\mathcal{O}_{\mathcal{T}}\right)$$

where

$$\mathcal{O}_{\mathcal{T}} = \left\{c > 0 \,/\, \forall u \in X : \|T(u)\|_Y \leq c\|u\|_X\right\}.$$

We also define

$$X^* := B(X, \mathbb{R}),$$

the dual space of X. The set X^* is a *Banach space*, i.e. a complete normed space. Its norm is called the *dual norm* and can be computed by

$$\|T\| = \inf\left(\mathcal{O}_{\mathcal{T}}\right) = \sup_{u\neq 0} \frac{|T(u)|}{\|u\|_X} = \sup_{\|u\|=1} |T(u)|,$$

where

$$\mathcal{O}_{\mathcal{T}} = \left\{c > 0 \,/\, \forall u \in X : \quad |T(u)| \leq c\|u\|_X\right\}.$$

For functions on $\mathcal{L}(X,Y)$ we have the following result

Theorem 2.1.2. *Let $T \in \mathcal{L}(X,Y)$. Then T is bounded iff T is continuous.*

A proof can be found in [27].

We define the *limit superior* and *limit inferior* of a set $B \subseteq \mathbb{R}$ as follows

$$\limsup(B) := \sup(\operatorname{acc}(B)),$$
$$\liminf(B) := \inf(\operatorname{acc}(B)).$$

For the case of a functional f on a normed space X, we define the limit superior and limit inferior by

$$\begin{aligned}
\overline{\lim_{x\to x_0}}\, f(x) &= \limsup_{x\to x_0} f(x) \\
&:= \lim_{\varepsilon\to 0} \sup\{f(x) \,/\, x \in B(x_0,\varepsilon)\backslash\{x_0\}\} \\
&= \inf_{\varepsilon>0}(\sup\{f(x) \,/\, x \in B(x_0,\varepsilon)\backslash\{x_0\}\}), \\
\underline{\lim_{x\to x_0}}\, f(x) &= \liminf_{x\to x_0} f(x) \\
&:= \lim_{\varepsilon\to 0} \inf\{f(x) \,/\, x \in B(x_0,\varepsilon))\backslash\{x_0\}\} \\
&= \sup_{\varepsilon>0}(\inf\{f(x) \,/\, x \in B(x_0,\varepsilon)\backslash\{x_0\}\}),
\end{aligned}$$

where $B(x_0, \varepsilon)$ is the ball of radius ε and centered at x_0.

When $(X, \|\cdot\|)$ is a Banach space, we consider two additional types of topologies. The first one is the $\sigma(X, X^*)$ or *weak topology*, which is defined as the *initial topology* (see [7]) given by the family

$$(\eta)_{\eta \in X^*},$$

that is, it is the weakest topology that keeps continuous all elements of the dual space. The second one is a topology on X^*, called $\sigma(X^*, X)$ or *weak * topology*, defined as follows. Consider the canonical mapping $J : X \to X^{**}$ given by

$$J(u) = \psi_u,$$

where $\psi_u : X^* \to \mathbb{R}$ is defined by

$$\forall \eta \in X^* : \quad \psi_u(\eta) = \langle \eta, u \rangle := \eta(u).$$

Thus, the weak* topology is the initial topology correlated to $(\psi_u)_{u \in X}$, that is, it is the weakest one that keeps continuous all the functionals in $J(X)$.

Remark 2.1.3. We say that a Banach space is *reflexive* if J is surjective.

In this way, we have three types of convergences. A sequence $(u_m)_{m \in \mathbb{N}} \in X$ converges *strongly* to an element $u \subseteq X$ iff

$$\lim_{m \to \infty} \|u_m - u\| = 0,$$

i.e. if it converges in the topology $\mathcal{T}_{\|\cdot\|}$. This type of convergence is denoted

$$u_m \to u, \text{ as } m \to +\infty.$$

Convergence of $(u_m)_{m \in \mathbb{N}}$ to u in $\sigma(X, X^*)$ is called *weak convergence* and is denoted by

$$u_m \rightharpoonup u, \text{ as } m \to +\infty.$$

For this type of convergence we have the following characterization.

A sequence $(u_m)_{m \in \mathbb{N}} \subseteq X$ is weakly convergent if and only if

$$\exists u \in X, \forall \phi \in X^* : \quad \lim_{m \to \infty} \phi(u_m) = \phi(u). \tag{2}$$

Remark 2.1.4. If $u_m \to u$, then u_m converge weakly to u. In addition, in the case that X is finite dimensional we have that strong and weak convergence are equivalent.

Finally, a sequence $(\eta_m)_{m \in \mathbb{N}} \subseteq X^*$ converging to u in the weak* topology is said to converge **-weakly* and is denoted by

$$\eta_m \overset{*}{\rightharpoonup} \eta, \text{ as } m \to +\infty.$$

An important concept in normed spaces is *density*. Let X be a normed space and $A, B \subseteq X$. We say that A is *dense* in B iff

$$\forall y \in B, \forall \varepsilon > 0, \exists x \in A : \quad \|x - y\| < \varepsilon.$$

To define the following concepts, we use the Euclidean topology on $\mathbb{R}^N$. The *support* of a continuous function $f : \mathbb{R}^N \to \mathbb{R}$, denoted $\operatorname{supp}(f)$, is

$$\operatorname{supp}(f) := \overline{\{x \in \mathbb{R}^N \,/\, f(x) \neq 0\}}.$$

A more general definition of supp of a function will be given in the next section. For $\Omega \subseteq \mathbb{R}^N$, the space containing all the k-times continuously differentiable functions with compact *support* is denoted $C_c^k(\Omega)$. In this way, we define

$$C_c^\infty(\Omega) := \bigcap_{k \geq 0} C_c^k(\Omega),$$

and we set

$$C_c(\Omega) := C_c^0(\Omega).$$

For two Banach spaces X and Y, with $X \subseteq Y$, we say that X is *continuously embedded* in Y if

$$\exists C > 0, \forall x \in X: \quad \|u\|_Y \leq C\|u\|_X.$$

and it is denoted $X \hookrightarrow Y$. In a similar way, we state that X is *compactly embedded* in Y, denoted $X \subset\subset Y$, if $X \hookrightarrow Y$ and any bounded sequence $(u_m)_{m\in\mathbb{N}} \subseteq X$ is precompact in Y.

2.2 Notions of Measure Theory

We start with some basic notions of measure theory. For more details, we recommend [10], [36], [24], and [23]. For a nice introduction check [29]. Let us denote a *measure space* by $(\Omega, \mathcal{M}, \mu)$, where Ω is a non-void set and

- $\mathcal{M}$ is a σ-algebra in Ω, i.e., $\mathcal{M} \subseteq \mathcal{P}(\Omega)$ is such that
  - $\emptyset \in \mathcal{M}$.
  - $\mathcal{M}$ is closed under complementation:

    $$\forall A \in \mathcal{M}: \quad A^c \in \mathcal{M}.$$

    .
  - $\mathcal{M}$ is closed under countable unions:

    $$\forall (A_n)_{n\in\mathbb{N}} \subseteq \mathcal{M}: \quad \bigcup_{n=1}^{\infty} A_n \in \mathcal{M}.$$

  Any member of $\mathcal{M}$ is called a *measurable set.*
- $\mu : \mathcal{M} \to [0, \infty]$ is a measure, i.e., it satisfies
  - Null empty set:

    $$\mu(\emptyset) = 0.$$
  - Countable additivity:

    $$\forall (A_n)_{n\in\mathbb{N}} \subseteq \mathcal{M} \text{ disjoint family}: \quad \mu\left(\bigcup_{n=1}^{\infty} A_n\right) = \sum_{n=1}^{\infty} \mu(A_n).$$
- Ω is σ-finite, i.e.,

  $$\exists \left(\Omega_n\right)_{n\in\mathbb{N}} \subseteq \mathcal{M} \text{ such that } \Omega = \bigcup_{n=1}^{\infty} \Omega_n \text{ and } \mu\left(\Omega_n\right) < \infty, \forall n.$$

We say that a property holds *"almost everywhere on Ω"*, symbolized *"a.e. Ω"*, if it holds for any element of $\Omega \backslash E$, for any subset E of Ω with $\mu(E) = 0$. For a general function f on a general measure space, we define the *support* of f, $\operatorname{supp}(f)$, by considering the family of open sets $(\omega_i)_{i\in I}$ where $f = 0$, [7]. If we set

$$\omega = \bigcup_{i\in I} \omega_i,$$

then $f = 0$ a.e. Ω and

$$\operatorname{supp}(f) = \Omega\backslash\omega.$$

Moreover, analogously to the concepts of sup and inf, we define ess sup and ess inf by, [40],

$$\operatorname*{ess\,sup}_{x\in\Omega} f(x) = \inf_{E\subseteq\Omega,\mu(E)=0}\left(\sup_{x\in\Omega\setminus E} f(x)\right)$$
$$\operatorname*{ess\,inf}_{x\in\Omega} f(x) = \sup_{E\subseteq\Omega,\mu(E)=0}\left(\inf_{x\in\Omega\setminus E} f(x)\right).$$

Let Ω and Y be two general measurable sets with H and J their respective σ-algebras. Then, we say that a function

$$f:\Omega\to Y,$$

is *measurable* if for any $A\in J$ its preimage $f^{-1}(A)\in H$. For our particular interest, we consider $f:\Omega\to\mathbb{R}$, which is measurable iff for any $t\in\mathbb{R}$, the set

$$\mathcal{A}_t := \{x\in\Omega \,/\, f(x) > t\}$$

belongs to the σ-algebra of Ω. We are interested in the case of $\Omega=\mathbb{R}^N$. Let us consider the Euclidean topology on this space. Thus, we introduce three measures: *Baire, Borel* and *Lebesgue* measures.

To define the Baire measure, let us take into account the set $C_c(\mathbb{R}^N)$. The smallest σ-algebra $\mathcal{B}_1$ for which all elements of $C_c(\mathbb{R}^N)$ are measurable is called the class of *Baire sets*. Any measure μ_1 defined on $\mathcal{B}_1$ is called a *Baire measure* if for any compact set $K\in\mathcal{B}_1$ we have that $\mu_1(K)$ is finite. In this measure space, we define *Baire functions*, which are classified into classes:

- The *Baire class zero* group is the set of continuous functions.
- The *Baire class one* contains all the functions that are pointwise limits of sequences in the class zero.
- In general, we say that a *Baire class* α is the set of all the pointwise limits of sequences in the Baire class $\alpha-1$.

The union of all Baire classes is called the set of Baire functions, [26].

For the Borel measure, we define the Borel σ-algebra $\mathcal{B}_2$ as the intersection of all σ-algebras of $\mathbb{R}^N$ containing all the closed sets (or equivalent all the open sets, due to closed under complementation property).

Then a *Borel measure* is any measure on $\mathcal{B}_2$. For two topological spaces X and Y we say that $f:X\to Y$ is a *Borel measurable* function if for any open set A, $f^{-1}(A)$ is a Borel set. When we have functions $f:\mathbb{R}\to\mathbb{R}$, the set of Borel and Baire functions coincide (see [25], Section 43, Theorem IV).

Now we introduce the concept of *Lebesgue measure* $(\mathbb{R}^N,\mathcal{M},m)$ thanks to the following result, [37].

Theorem 2.2.1 (Existence of Lebesgue Measure)**.** *There exists a positive measure m on a σ-algebra $\mathcal{M}$ such that,*

1. *For any $N-$cell V*
$$V=\{x=(\xi_1,\xi_2,\dots,\xi_N) \,/\, \alpha_i<\xi_i<\beta_i, 1\le i\le N\}$$
we have
$$\mathrm{Vol}(V) := \prod_{i=1}^{N}(\beta_i-\alpha_i) = m(V).$$
2. *Any Borel set is in $\mathcal{M}$.*
3. *For all $E\in\mathcal{M}$ and $x\in\mathbb{R}^N$*
$$m(E+x)=m(E),$$
i.e., m is translation invariant.
4. *For any μ positive translation invariant Borel measure on $\mathbb{R}^N$ such that*
$$\forall K\text{compact}: \quad \mu(K)<+\infty,$$
there exists a number c such that
$$\forall E\subseteq\mathbb{R}^N: \quad \mu(E)=cm(E).$$

5. *For any rotation T we have*
$$m(T(E)) = m(E),$$
for any $E \in \mathcal{M}$.

Note that Theorem 2.2.1 shows that any Borel set is also Lebesgue measurable. In particular, this shows that any Borel measurable function is Lebesgue measurable, but the converse is not always true. However, for any Lebesgue measurable function f there exists a Borel function g such that $f(x) = g(x)$ a.e. Ω respect to the Lebesgue measure (see [24], Section 21). For the Lebesgue measure, the *Lebesgue integral* is defined which, unlike the classical Riemann integral, allows us to work on a set of functions closed under the action of taking pointwise limits of sequences. We shall work with Lebesgue measure and integral along this document.

2.3 Some topics of Functional Analysis

In this section we present definitions and important results about functional spaces on which we will work. We follow the construction in [7] and [22]. We denote by $|A|$ the Lebesgue measure of $A \subseteq \mathbb{R}^N$.

2.3.1 Lebesgue spaces

For a quite complete theory of Lebesgue spaces, we recommend [10] and [11]. Let $\Omega \subseteq \mathbb{R}^N$. Consider the equivalence relation defined by
$$f \sim g \text{ if and only if } f(x) = g(x) \text{ a.e. } \Omega. \tag{3}$$
We denote by $[f]$ an equivalence class given by (3). Let $\mathfrak{L}^1(\Omega)$ be the space of integrable functions, i.e.,
$$\mathfrak{L}^1(\Omega) := \left\{ f : \Omega \to \mathbb{R} \,/\, \int_\Omega |f(x)| dx < +\infty \right\}.$$
The set of equivalence classes obtained by (3) on $\mathfrak{L}^1(\Omega)$ is called the $L^1(\Omega)$ space, so we have that,
$$L^1(\Omega) := \{[f] \,/\, f \in \mathfrak{L}^1(\Omega)\}. \tag{4}$$

In the same manner, we define for $p \geq 1$
$$\mathfrak{L}^p(\Omega) := \{f : \Omega \to \mathbb{R} \,/\, |f|^p \in \mathfrak{L}^1(\Omega)\},$$
and
$$L^p(\Omega) := \{[f] \,/\, f \in \mathfrak{L}^p(\Omega)\}. \tag{5}$$
Also, we have
$$\mathfrak{L}^\infty(\Omega) := \left\{ f : \Omega \to \mathbb{R} \,/\, \operatorname*{ess\,sup}_{x\in\Omega} |f(x)| < +\infty \right\}, \tag{6}$$
and so
$$L^\infty(\Omega) := \{[f] \,/\, f \in \mathfrak{L}^\infty\}. \tag{7}$$
where the equivalence classes are defined by (3). It is common to take a representative element of each equivalence class. In this way, we will say that $f \in L^p(\Omega)$ meaning that $f \in [f]$ for some $[f] \in L^p(\Omega)$. We define the norm
$$\|\cdot\|_{L^p(\Omega)} : L^p(\Omega) \to \mathbb{R},$$
by
$$\|f\|_{L^p(\Omega)} = \begin{cases} \left(\int_\Omega |f(x)|^p dx \right)^{1/p}, & 1 \leq p < +\infty, \\ \operatorname*{ess\,sup}_{x\in\Omega} |f(x)|, & p = +\infty. \end{cases}$$
Remark 2.3.1. $\|\cdot\|_{L^p(\Omega)}$ defines a seminorm on $\mathfrak{L}^p(\Omega)$, but thanks to equivalence relation (3) it defines a norm on $L^p(\Omega)$.

The following is a useful theorem,

Theorem 2.3.2 (Lebesgue's Dominated Convergent Theorem)**.** *Let $(f_n)_{n\in\mathbb{N}} \subseteq L^1(\Omega)$. Assume that there exists $g \in L^1(\Omega)$ such that for any $n \in \mathbb{N}$*

$$|f_n(x)| \leq g(x) \text{ a.e. } \Omega,$$

and that

$$f_n(x) \to f(x) \text{ a.e. } \Omega, \text{ as } n \to +\infty.$$

Then, $f \in L^1(\Omega)$ and

$$\|f_n - f\|_{L^1(\Omega)} \to 0, \text{ as } n \to +\infty.$$

For a proof, see [29].

Theorem 2.3.3. *(Fischer-Riesz) For $p \in [1, \infty]$ the normed space*

$$(L^p(\Omega), \|\cdot\|_{L^p(\Omega)})$$

is a Banach space.

A proof can be found in [7].

Functions $C_c^\infty(\Omega)$ have an important property in $L^p(\Omega)$ spaces, which will be useful later.

Theorem 2.3.4. *For any $p \in [1, \infty)$ the space $C_c^\infty(\Omega)$ is dense in $L^p(\Omega)$.*

In order to prove Theorem 2.3.4 we will use functions called *mollifiers*. We say that a sequence $(\rho_m)_{m\in\mathbb{N}} \subseteq C_c^\infty(\mathbb{R}^N)$ is a sequence of mollifiers iff for every m,

- $\text{supp}(\rho_m) \subseteq \overline{B(0, 1/m)}$,
- $\int_{\mathbb{R}^N} \rho_m = 1$,
- $\rho_m \geq 0$.

Proof of Theorem 2.3.4. Let $f \in L^p(\Omega)$. We have to show that

$$\exists (f_m)_{m\in\mathbb{N}} \subseteq C_c^\infty(\Omega): \quad \lim_{m\to\infty} \|f_m - f\|_{L^p(\Omega)} = 0. \tag{8}$$

Define $\bar{f} : \mathbb{R}^N \to \mathbb{R}$ such that,

$$\bar{f}(x) = \begin{cases} f(x), & x \in \Omega, \\ 0, & \text{otherwise.} \end{cases}$$

So $\bar{f} \in L^p(\mathbb{R}^N)$. Consider a sequence $(K_m)_{m\in\mathbb{N}}$ of compact sets in $\mathbb{R}^N$ such that,

$$\forall m \in \mathbb{N}: \quad \text{dist}(x, \mathbb{R}^N \backslash \Omega) \geq 2/m \quad \text{and} \quad \bigcup_{m=1}^{\infty} K_m = \Omega.$$

For example,

$$K_m = \{x \in \mathbb{R}^N \,/\, |x| \leq m \quad \text{and} \quad \text{dist}(x, \mathbb{R}^N \backslash \Omega) \geq 2/m\}.$$

Define $g_m = \chi_{K_m} \bar{f}$, where χ_{K_m} is the *indicator function* of the set K_m, and $f_m = \rho_m \star g_m$, where

$$\rho_m \star g_m(x) := \int_{\mathbb{R}^N} \rho_m(x - y) g_m(y) dy,$$

is the *convolution* operation. So we get

$$\text{supp}(f_m) \subseteq \overline{B(0, 1/m)} + K_m \subseteq \Omega.$$

Since convolution transfers smoothness from ρ_m to f_m, it follows that $f_m \in C_c^\infty(\Omega)$. Also,

$$\begin{aligned}\|f_m - f\|_{L^p(\Omega)} &= \left\|f_m - \bar{f}\right\|_{L^p(\mathbb{R}^N)} \\ &= \left\|(\rho_m \star g_m) - (\rho_m \star \bar{f}) + (\rho_m \star \bar{f}) - \bar{f}\right\|_{L^p(\mathbb{R}^N)} \\ &\leq \left\|(\rho_m \star g_m) - (\rho_m \star \bar{f})\right\|_{L^p(\mathbb{R}^N)} + \left\|(\rho_m \star \bar{f}) - \bar{f}\right\|_{L^p(\mathbb{R}^N)} \\ &\leq \|\rho_m\|_{L^1(\mathbb{R}^N)}\|g_m - \bar{f}\|_{L^p(\mathbb{R}^N)} + \left\|(\rho_m \star \bar{f}) - \bar{f}\right\|_{L^p(\mathbb{R}^N)} \\ &= \left\|g_m - \bar{f}\right\|_{L^p(\mathbb{R}^N)} + \left\|(\rho_m \star \bar{f}) - \bar{f}\right\|_{L^p(\mathbb{R}^N)}.\end{aligned}$$

Since $g_m = \chi_{K_m}\bar{f}$, by applying the Dominated Convergence Theorem,

$$\lim_{m\to\infty} \left\|g_m - \bar{f}\right\|_{L^p(\mathbb{R}^N)} = 0,$$

Also, since $\bar{f} \in L^p(\mathbb{R}^N)$,

$$\lim_{m\to\infty} = \left\|(\rho_m \star \bar{f}) - \bar{f}\right\|_{L^p(\mathbb{R}^N)} = 0,$$

(see [7], Theorem 4.22.). Therefore we have proven (8). ■

The *conjugate exponent* of $p \in (1, \infty)$ is defined as the real number p' such that

$$\frac{1}{p} + \frac{1}{p'} = 1.$$

For $p = 1$ we define $p' = \infty$. Now we present a very useful inequality that relates p and p' norms.

Theorem 2.3.5 (Hölder inequality)**.** *Assume that $1 \leq p \leq \infty$. Let $u \in L^p(\Omega)$, $v \in L^{p'}(\Omega)$. Then $uv \in L^1(\Omega)$ and*

$$\|uv\|_{L^1(\Omega)} \leq \|u\|_{L^p(\Omega)}\|v\|_{L^{p'}(\Omega)}. \tag{9}$$

A proof can be found in [24].

There exists an important relation between a space $L^p(\Omega)$ and its dual $(L^p(\Omega))^*$, which is stated in the following result

Theorem 2.3.6 (Riesz representation theorem)**.** *For $p \in [1, \infty)$ we have that*

$$(L^p(\Omega))^* \cong L^{p'}(\Omega), \tag{10}$$

that is, for each $\psi \in (L^p(\Omega))^*$ there exists a unique element $v \in L^{p'}(\Omega)$ such that

$$\forall u \in L^p(\Omega): \quad \langle \psi, f\rangle := \psi(f) = \int_\Omega v(x)f(x)dx.$$

Remark 2.3.7. The isomorphism (10) shows that we can identify any linear functional in $(L^p(\Omega))^*$ with a unique element in $L^{p'}(\Omega)$. Then, we can interpret an element in $(L^p(\Omega))^*$ as an element of $L^{p'}(\Omega)$ and vice-versa.

Now, we define the set of *locally integrable functions* denoted by $L^1_{loc}(\Omega)$. A function f belongs to this set if for any compact set $K \subseteq \Omega$

$$\int_K |f(x)|dx < +\infty.$$

Remark 2.3.8. Note that $L^1_{loc}(\Omega)$ is bigger than $L^1(\Omega)$. In addition, if Ω is bounded, by Hölder inequality, we get

$$L^q(\Omega) \subseteq L^p(\Omega) \subseteq \cdots \subseteq L^1(\Omega) = L^1_{loc}(\Omega).$$

for $1 \leq p < q < \infty$ (see also [39] for a deeper explanation).

2.3.2 Sobolev spaces

Based mainly on [7], we introduce Sobolev spaces. Let $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_N) \in \mathbb{N}^N$ be a multiindex whose order is $|\alpha| = \alpha_1 + \alpha_2 + \cdots + \alpha_N = k$. We denote

$$D^\alpha \phi = \frac{\partial^{\alpha_1}}{\partial x_1^{\alpha_1}} \cdots \frac{\partial^{\alpha_N}}{\partial x_N^{\alpha_N}} \phi \tag{11}$$

for $\phi : \mathbb{R}^N \to \mathbb{R}$. Consider the next definition,

Definition 2.3.9 (Weak partial derivative)**.** Let $u, v \in L^1_{loc}(\Omega)$ and α a multiindex. We say that v is the α^{th}-weak partial derivative of u provided

$$\int_\Omega u(x) D^\alpha \phi(x) dx = (-1)^{|\alpha|} \int_\Omega v(x)\phi(x) dx, \quad \forall \phi \in C_c^\infty(\Omega). \tag{12}$$

Remark 2.3.10. Note that when $\Omega \subseteq \mathbb{R}$ and $|\alpha| = 1$, equation (12) is just the formula of integration by parts.

The function v is unique due to the next theorem, usually called the *Fundamental Lemma of the Calculus of Variations.*

Theorem 2.3.11. *Let us assume that $u \in L^1_{loc}(\Omega)$ and*

$$\forall f \in C_c^\infty(\Omega) : \quad \int_\Omega u(x) f(x) dx = 0.$$

Then we get that $u = 0$ a.e. Ω.

We denote the $\alpha^{th}-$weak partial derivative of u as $D^\alpha u$. Then, the *Sobolev space* $W^{k,p}(\Omega)$, for $k \in \mathbb{N}$, $p \in [1, +\infty]$ is

$$W^{k,p}(\Omega) := \left\{ u \in L^p(\Omega) \Big/ \begin{array}{c} \forall |\alpha| \leq k \ , \exists D^\alpha u \in L^p(\Omega), \forall \phi \in C_0^\infty(\Omega) : \\ \int_\Omega u(x) D^k \phi(x) dx = (-1)^{|k|} \int_\Omega \phi(x) D^k u(x) dx \end{array} \right\}. \tag{13}$$

This linear space equipped with the norm $\|\cdot\| : W^{k,p}(\Omega) \to \mathbb{R}$,

$$\|u\|_{W^{k,p}(\Omega)} := \begin{cases} \left(\displaystyle\sum_{|\alpha| \leq k} \int_\Omega \left| D^\alpha u(x) \right|^p dx \right)^{1/p}, & \text{if } 1 \leq p < \infty, \\ \displaystyle\sum_{|\alpha| \leq k} \operatorname*{ess\,sup}_{x \in \Omega} \left| D^\alpha u(x) \right|, & \text{if } p = \infty, \end{cases}$$

becomes a Banach space (for a proof see [22], Chapter 5). In particular, for the case $k = 1$ we have

$$W^{1,p}(\Omega) = \left\{ u \in L^p(\Omega) \Big/ \begin{array}{c} \exists \psi_1, \psi_2, \dots, \psi_N \in L^p(\Omega) \text{ such that} \\ \int_\Omega u(x) \frac{\partial \varphi}{\partial x_i}(x) dx = - \int_\Omega \psi_i(x) \varphi(x), \quad \forall \varphi \in C_c^\infty(\Omega), \quad \forall i = 1, 2, \dots, N \end{array} \right\}.$$

We denote each $\psi_i = \frac{\partial u}{\partial x_i}$, and

$$\nabla u = \left(\frac{\partial u}{\partial x_1}, \frac{\partial u}{\partial x_2}, \dots, \frac{\partial u}{\partial x_N} \right).$$

In this case,

$$\|u\|_{W^{1,p}(\Omega)} = \left(\int_\Omega \left(|u(x)|^p + \left| \nabla u(x) \right|^p \right) dx \right)^{1/p}.$$

Definition 2.3.12. For $1 \leq p < \infty$ we define the space $W_0^{1,p}(\Omega)$ as the closure of $C_c^\infty(\Omega)$ in $W^{1,p}(\Omega)$, i.e.

$$\overline{C_c^\infty(\Omega)} = W_0^{1,p}(\Omega)$$

in norm $W^{1,p}(\Omega)$. When $k = 1$ and $p = 2$ we denote

$$W_0^{k,p}(\Omega) = H_0^1(\Omega).$$

Usually, the dual space of $W_0^{1,p}(\Omega)$ is denoted by $W^{-1,p'}(\Omega)$, i.e.,

$$(W_0^{1,p}(\Omega))^* = W^{-1,p'}(\Omega).$$

$W_0^{1,p}(\Omega)$ with the norm of $W^{1,p}(\Omega)$ is a separable Banach space for $1 \leq p < \infty$ and reflexive for $1 < p < \infty$. Another norm on $W_0^{1,p}(\Omega)$ is $\|\nabla u\|_{L^p(\Omega)}$. Indeed, this norm is equivalent to the norm $W^{1,p}(\Omega)$ on $W_0^{1,p}(\Omega)$ thanks to *Poincaré's inequality*,

Theorem 2.3.13. *There exists a constant $C = C(\Omega, p)$ such that*

$$\forall u \in W_0^{1,p}(\Omega): \quad \|u\|_{L^p(\Omega)} \leq C\|\nabla u\|_{L^p(\Omega)}. \tag{14}$$

2.3.3 Embedding theorems

In this subsection we state an embedding theorem. Grossly speaking, it establishes a relation between Sobolev and L^p spaces. Consider the following notation, [7]. Let $x \in \mathbb{R}^N$, write

$$x = (x', x_N) \text{ with } x' \in \mathbb{R}^{N-1}, \quad x' = (x_1, x_2, \ldots, x_{N-1}).$$

Set

$$\begin{aligned} \mathbb{R}_+^N &= \left\{ x = (x', x_N) \ / \ x_N > 0 \right\} \\ Q &= \left\{ x = (x', x_N) \ / \ |x'| < 1 \text{ and } |x_N| < 1 \right\} \\ Q_+ &= Q \cap \mathbb{R}_+^N \\ Q_0 &= \left\{ x = (x', 0) \ / \ |x'| < 1 \right\}. \end{aligned}$$

We present the following definition,

Definition 2.3.14. An open set Ω is said to be *smooth* or of class C^1 if for every $x \in \partial\Omega$ there exists a neighborhood $U_x \subseteq \mathbb{R}^N$ of x and a bijection $H_x : Q \to U$ such that

$$H_x \in C^1(\bar{Q}), \quad H_x^{-1} \in C^1(\overline{U_x}), \quad H_x(Q_+) = U_x \cap Q, \quad \text{and} \quad H_x(Q_0) = U_x \cap \partial\Omega.$$

Theorem 2.3.15 (Rellich-Kondrachov). *Let $N \in \mathbb{N}$, $1 \leq p \leq \infty$. Assume that $\Omega \subseteq \mathbb{R}^N$ is bounded and of class C^1. If $p^\star$ is given by*

$$p^\star = \begin{cases} \dfrac{Np}{N-p}, & p < N, \\ +\infty, & \text{otherwise.} \end{cases}$$

Then

(1) $W^{1,p}(\Omega) \subset\subset L^q(\Omega) \quad \forall q \in [1, p^\star)$ *, if $p < N$.*

(2) $W^{1,p}(\Omega) \subset\subset L^q(\Omega) \quad \forall q \in [p, +\infty)$ *, if $p = N$.*

(3) $W^{1,p}(\Omega) \subset\subset C(\bar{\Omega})$ *, if $p > N$.*

Also, for $p < N$

$$W^{1,p}(\Omega) \hookrightarrow L^{p^\star}(\Omega).$$

See [7], Chapter 9, for a proof.

Remark 2.3.16. (1) and (2) in Theorem 2.3.15 imply that there exists a constant $C = C(\Omega, p, q)$ such that,

$$\forall u \in W^{1,p}(\Omega): \quad \|u\|_{L^q(\Omega)} \leq C\|u\|_{W^{1,p}(\Omega)}. \tag{15}$$

Remark 2.3.17. Since $W_0^{1,p}(\Omega) \subseteq W^{1,p}(\Omega)$, Theorem 2.3.15 also hold for $W_0^{1,p}(\Omega)$.

2.4 Notions of variational methods

Let us present some results of Calculus of Variations applied to the study of partial differential equations. We start by presenting some concepts about partial differential equations (PDEs) based on [22].

Let $\Omega \subseteq \mathbb{R}^N$ open, $x \in \Omega$, and $u : \Omega \to \mathbb{R}$ a k-times differentiable function. Let's consider a function

$$\mathcal{F} : \mathbb{R}^{N^k} \times \mathbb{R}^{N^{k-1}} \times \cdots \times \mathbb{R}^N \times \mathbb{R} \times \Omega \to \mathbb{R}.$$

We define a PDE of order $k \in \mathbb{N}$ on Ω if it has the form

$$\mathcal{F}\left(D^k u(x), D^{k-1}u(x), \ldots, Du(x), u(x), x\right) = 0$$

PDEs can be classified depending on their linearity properties. We use the notation (11).

(a) We say that a PDE is *linear* if it is linear with respect to u and each derivative of u, i.e., if can be represented as

$$\sum_{|\alpha| \leq k} c_\alpha(x) D^\alpha u(x) = f(x)$$

for given functions f and c_α (with $|\alpha| \leq k$). We say that the equation is linear homogeneous if $f = 0$, and linear inhomogeneous otherwise. Any PDE that is not linear is called *nonlinear*.

Example 2.4.1. Laplace equation

$$\Delta u(x) = \sum_{i=1}^{N} \frac{\partial^2 u(x)}{\partial x_i^2} = 0.$$

(b) A PDE is *semilinear* if it is expressed as

$$\sum_{|\alpha| = k} c_\alpha(x) D^\alpha u(x) + G\left(D^{k-1}u(x), \ldots, Du(x), u(x), x\right) = 0.$$

In this case, the PDE is linear in the highest order derivatives of u.

Example 2.4.2. Nonlinear Poisson equation:

$$-\Delta u(x) = f(u(x)).$$

(c) A PDE is *quasilinear* if it has the structure

$$\sum_{|\alpha| = k} c_\alpha\left(D^{k-1}u(x), \ldots, Du(x), u(x), x\right) D^\alpha u(x) + G\left(D^{k-1}u(x), \ldots, Du(x), u(x), x\right) = 0.$$

This type of equation differs from the semilinear one in the fact that the function c_α depends also on u and its derivatives upon order $k-1$.

Example 2.4.3. p-Laplacian equation:

$$\nabla \cdot (|\nabla u(x)|^{p-2} \nabla u(x)) = 0.$$

(d) In the case that the highest order derivative of u is not linear, we say that the PDE is *fully nonlinear*.

Example 2.4.4. Eikonal equational:

$$|\nabla u(x)| = 1.$$

In the particular case of $k = 2$, we will say that a quasilinear PDE $\mathcal{J}$ is in *divergence form* if it can be written as

$$\mathcal{J}(u) = -\sum_{i,j=1}^{N} \frac{\partial}{\partial x_j}\left(a^{ij}(\nabla u(x), u(x), x)\frac{\partial u(x)}{\partial x_i}\right) + \sum_{i=1}^{N} b^i(x)\frac{\partial u(x)}{\partial x_i} + c(x)u, \tag{16}$$

and it is in *nondivergence form* if for any unknown u we have

$$\mathcal{J}(u) = -\sum_{i,j=1}^{N} a^{ij}(\nabla u(x), u(x),)\frac{\partial u(x)}{\partial^2 x_i x_j} + \sum_{i=1}^{N} b^i(x)\frac{\partial u(x)}{\partial x_i} + c(x)u. \tag{17}$$

2.4.1 Some concepts of Calculus of Variations

In this subsection the main references are [32] and [38]. Let X and Y be normed spaces. We define the *directional derivative* of $f : X \to Y$ at $u_0 \in X$ in the direction $v \in X$ as

$$\partial_v f(u_0) = \frac{d}{d\varepsilon} f(u_0 + \varepsilon v)\Big|_{\varepsilon=0}. \tag{18}$$

By using this concept we get a first definition of differentiability,

Definition 2.4.5 (Gateaux differentiability)**.** We say that the function $f : X \to Y$ is *Gateaux differentiable* (or weak differentiable) at $u_0 \in X$ if for all the directions $v \in X$, $\partial_v f(u_0)$ exists and

$$\exists f'_G(u_0) \in \mathcal{B}(X, Y), \forall v \in X : \quad \partial_v f(u_0) = f'_G(u_0)v. \tag{19}$$

A stronger definition is the following,

Definition 2.4.6 (Fréchet Differentiability)**.** A function $f : X \to Y$ is called *Fréchet differentiable* (or just *differentiable*) at $u_0 \in X$ if for any open set $\mathcal{O} \subseteq X$ such that $u_0 \in \mathcal{O}$ we have that

$$\exists T \in B(X, Y), \forall h \in X : \quad u_0 + h \in \mathcal{O} \implies f(u_0 + h) - f(u_0) = T(h) + g(h), \tag{20}$$

where $g : \mathcal{O} \subseteq X \to Y$ and

$$g(h) = \|h\|\epsilon(h),$$

for some function $\epsilon : B(0, r) \subseteq X \to Y$ such that

$$\lim_{h\to 0} \epsilon(h) = 0.$$

Remark 2.4.7. This behaviour of g around zero is denoted by $g(h) = o(h)$.

Proposition 2.4.8. *The bounded linear operator T of the previous definition is unique.*

Proof. We follow the proof presented in [32]. Assume that there exists a function $\phi \in \mathcal{B}(X, Y)$ for which (20) holds, i.e., for any $h \in X$ such that $u_0 + h \in \mathcal{O}$

$$f(u_0 + h) - f(u_0) = \phi(h) + o(h). \tag{21}$$

We will show that $T = \phi$, i.e.,

$$\forall u \in X : \quad T(u) = \phi(u). \tag{22}$$

Let us take $u \in X$, generic. Since $\mathcal{O}$ is open, there exists a ball $B(u_0, r)$, with $r > 0$ such that

$$B(u_0, r) = u_0 + B(0, r) \subseteq \mathcal{O}.$$

Due to (20) and (21) we have the following identity,

$$\forall h \in B(0, r) : \quad T(h) + \|h\|\epsilon_1(h) = \phi(h) + \|h\|\epsilon_2(h), \tag{23}$$

where for $\epsilon_1, \epsilon_2 : B(0, r) \to X$ we have that

$$\lim_{h\to 0} \epsilon_1(h) = \lim_{h\to 0} \epsilon_2(h) = 0. \tag{24}$$

a) If $u = 0$, by (23) we get $T(u) = \phi(u)$.

b) If $u \neq 0$, then we take $\tilde{N} \in \mathbb{N}$ such that

$$\forall n \in \mathbb{N}, n > \tilde{N} : \quad h_n = \frac{1}{n}\frac{1}{\|u\|}u \in B(0, r).$$

From (23) we get

$$T(h_n) - \phi(h_n) = \|h_n\| \left[\epsilon_2(h_n) - \epsilon_1(h_n)\right].$$

Linearity of T and ϕ provides

$$T(u) - \phi(u) = \|u\| \left[\epsilon_2\left(h_n\right) - \epsilon_1\left(h_n\right)\right].$$

Then, if we take $n \to \infty$, by (24) we conclude that

$$T(u) = \phi(u).$$

By arbitrariness of u, we have shown (22).

■

We denote the Fréchet differential at the point u_0 by $f'(u_0)$.

Remark 2.4.9. Fréchet differentiability implies Gateaux differentiability. In fact, given a Fréchet differentiable function we take

$$f'_G(u_0)v = f'(u_0)v, \text{ for any } v \in X.$$

Let $\mathcal{O} \subseteq X$, open. We say that a function

$$f : \mathcal{O} \subseteq X \to Y$$

is of class $C^1(\mathcal{O})$ if it is a differentiable function (in the Fréchet sense) and

$$f' : \mathcal{O} \to \mathcal{L}(X, Y)$$

is continuous.

Let $I : X \to \mathbb{R}$ be a functional on the topological space $(X, \mathcal{T})$. A point $a \in X$ is called *(global) point of minimum* if

$$\forall u \in X : \quad f(a) \leq f(u).$$

Analogously, we say that $a \in X$ is a *(global) point of maximum* if

$$\forall u \in X : \quad f(a) \geq f(u),$$

and we call $f(a)$ a *(global) maximum* and *minimum* of f, respectively. We say that $a \in X$ is a *point of (global) extremum* if it is a point of either minimum or maximum, and the value $f(a)$ is called an *extremum* of f. When the previous concepts are not true for all u, but for a neighborhood of a, local analogous concepts arise. Thus, we say that $a \in X$ is a *local point of minimum* if

$$\exists G \in \mathcal{N}(a), \forall u \in G : \quad f(a) \leq f(u).$$

In the same way we define *local point of maximum*, *point of local extremum* and *local extremum*.

For the case of a Fréchet differentiable functional we say that u is a *critical point* of I iff

$$I'(u) = 0,$$

and it is a *regular point*, otherwise. When u is a critical point, the value $I(u) = \sigma$ is referred as its corresponding *critical value*. In particular, if u is a local extremum of I, then it is in fact a critical point.

Another type of critical point is given by *saddle points*, which consist in elements $u \in X$ such that

$$\forall U \in \mathcal{N}(u), \exists v, w \in X : \quad I(v) < I(u) < I(w).$$

In other words, a saddle point is a non-extremum critical point.

2.4.2 Variational approach for solving PDEs

In this subsection we follow [22] and [38]. Consider the following representation of a PDE

$$\mathcal{A}(u) = 0, \tag{25}$$

where $\mathcal{A}$ could be a nonlinear operator acting on the unknown u. There is no complete theory to find solutions to this kind of PDEs. However, for some partial differential equations, it is possible to use *variational methods* linked to nonlinear functional analysis. To apply these methods, the PDE in (25) must be the "derivative" of a convenient functional I, i.e., $I'(\cdot) = \mathcal{A}(\cdot)$.

To illustrate this approach, let us consider the following case. Assume that Ω is a smooth bounded set of $\mathbb{R}^N$. The smooth function

$$\begin{aligned} \mathcal{L}\colon \mathbb{R}^N \times \mathbb{R} \times \bar{\Omega} &\longrightarrow \mathbb{R} \\ (z, y, x) &\longmapsto \mathcal{L}(z, y, x) = \mathcal{L}\left(z_1, \ldots, z_N, y, x_1, \ldots, x_N\right), \end{aligned} \tag{26}$$

is called *Lagrangian*. Use the following notation for the derivatives of $\mathcal{L}$

$$\begin{cases} D_z\mathcal{L} = \left(\mathcal{L}_{z_1}, \ldots, \mathcal{L}_{z_N}\right) \\ D_y\mathcal{L} = \mathcal{L}_y \\ D_x\mathcal{L} = \left(\mathcal{L}_{x_1}, \ldots, \mathcal{L}_{x_N}\right) \end{cases}$$

and define the functional $I : C^\infty(\Omega) \to \mathbb{R}$ as follows,

$$I(w) := \int_U \mathcal{L}(\nabla w(x), w(x), x)dx. \tag{27}$$

If we suppose that u is a minimum point of I, and that the function $i : \mathbb{R} \to \mathbb{R}$ is

$$i(\tau) := I(u + \tau v) \quad \forall v \in C_c^\infty(\Omega),$$

then we deduce from $i'(0) = 0$ that (see [22], Chapter 8)

$$\int_\Omega \sum_{i=1}^N \mathcal{L}_{z_i}(\nabla u(x), u(x), x))\frac{\partial v(x)}{\partial x_i} + \mathcal{L}_y(\nabla u(x), u(x), x)v(x) = 0 \ \textit{on}\ \Omega. \tag{28}$$

Since this holds for any $v \in C_c^\infty(\Omega)$, then

$$-\sum_{i=1}^N \frac{\partial}{\partial x_i}(\mathcal{L}_{z_i}(\nabla u(x), u(x), x)) + \mathcal{L}_y(\nabla u(x), u(x), x) = 0 \ \textit{on}\ \Omega. \tag{29}$$

Equation (29) is called the *Euler equation* or *Euler-Lagrange equation*, derived from the *energy* functional I. In general, equations of the form

$$I'(u) = 0,$$

where I is a Fréchet differentiable functional on a Banach space V, are called *Euler-Lagrange* equations, and are said to be in *variational form* [38]. For this kind of equations, it is possible to find solutions by identifying critical points of their energy functional I. In this way, functions u satisfying (28) are called *weak solutions* for the PDE (29), while solutions $u \in C^2(\Omega)$ satisfying this PDE are called *classical solutions*.

Depending on the type of critical point we consider different methods. For example, to identify points of local minimum (or maximum, since any problem of maximization can be transformed to a problem of minimization), the so-called *direct methods of Calculus of Variations* are used. Among them, we have the lower semi-continuity, compactness method, duality method of Clarke and Ekeland, and Ekeland's variational principle. To find saddle points, *Minimax methods* are considered. For a detailed study of these techniques see e.g. [38].

In the rest of this section we state some important definitions and results of these variational methods assuming that the *set of admissible functions* X is the space $W_0^{1,p}(\Omega)$, and I is a functional defined by a Lagrangian as in (27). Some results are stated also for any Banach space V.

First, we center our attention on conditions to find minimum values of I . We start by considering a *coercivity property* on the Lagrangian $\mathcal{L}$ of our PDE, which basically says that

$$I(u) \to +\infty, \text{ as } \|u\| \to +\infty. \tag{30}$$

Let $c_1 > 0$, $c_2 \geq 0$, and $0 < q < \infty$. Assume that

$$\forall z \in \mathbb{R}^N, y \in \mathbb{R}, x \in \Omega : \quad \mathcal{L}(z, y, x) \geq c_1|z|^q - c_2.$$

Then by taking the integral in both sides of the inequality (see equation (27)) we get the coercivity condition

$$\forall v \in V : \quad I(v) \geq \alpha\|\nabla v\|^q_{L^q(\Omega)} - \gamma. \tag{31}$$

for some $\alpha > 0$ and $\gamma \geq 0$.

Remark 2.4.10. Note that due to Poincaré's inequality (14), from (31) we derive

$$\forall v \in V : \quad I(v) \geq \alpha C\|v\|^q_{L^q(\Omega)} - \gamma.$$

Then, inequality (31) is equivalent to (30).

However, coercivity is not enough for the existence of a minimum of I. We need I to be *weakly lower semicontinuous (w.l.s.)* on $W^{1,p}_0(\Omega)$; i.e., for $(u_m)_{m\in\mathbb{N}} \subseteq W^{1,p}_0(\Omega)$, and $u \in W^{1,p}_0(\Omega)$,

$$u_m \rightharpoonup u \implies I(u) \leq \liminf_{m\to\infty} I(u_m). \tag{32}$$

Remark 2.4.11. An important example of a lower semicontinuous function is the norm function $\|\cdot\|_V : V \to \mathbb{R}$.

Based on these two properties, it is shown in Theorem 2.4.16 that the functional I attains its infimum. Before state this theorem, we state two classical theorems that will be useful (see [28] and [34]).

Theorem 2.4.12 (Alaoglu's)**.** *Let X be a Banach space. Then the closed unit ball of X^* is weakly* compact.*

From this theorem it follows the next corollary,

Corollary 2.4.13. *If X is a reflexive Banach space, then the closed unit ball is compact in the weak topology.*

Another important result is

Theorem 2.4.14 (Eberlein-Šmulian)**.** *Let X be a Banach space and $A \subseteq B$. Then the following assertions are equivalent*

(i) A is weakly sequentially compact.

(ii) Any countable infinite subset $B \subseteq A$ has a limit point in X.

(iii) A is precompact in the weak topology.

Remark 2.4.15. A is weakly sequentially compact if and only if for any sequence there exists a weakly convergence subsequence.

Theorem 2.4.16. *Assume that $M \subseteq V$ is a weakly closed subset of V. Let us suppose that*

$$I : M \to \mathbb{R} \cup \{+\infty\}$$

satisfies conditions (30) and (32) on M, i.e.,

(1) $I(u) \to +\infty$, as $\|u\| \to +\infty$.

(2) $u_m \rightharpoonup u \implies I(u) \leq \liminf_{m\to\infty} I(u_m)$.

Then, we conclude that I is bounded from below and reaches its infimum in M, i.e.,

$$\inf_{v\in M} I(v) = \min_{v\in M} I(v).$$

Proof. We follow the proof presented in [38], Chapter 1. Let us denote

$$\alpha = \inf_{u \in M} I(u) < +\infty.$$

Let $(u_m)_{m\in\mathbb{N}}$ be a minimizing sequence in M, i.e.,

$$\lim_{m\to\infty} I(u_m) = \alpha. \tag{33}$$

Note that thanks to the coercivity property $(u_m)_{m\in\mathbb{N}}$ is bounded. Indeed, if we suppose the contrary, i.e.,

$$\|u_m\| \to +\infty, \text{ as } m \to +\infty,$$

Then by (*1*), we have

$$I(u_m) \to \infty, \text{ as } m \to +\infty,$$

which contradicts (33).

Since V is a reflexive Banach space, by Corollary 2.4.13, $(u_m)_{m\in\mathbb{N}}$ is compact in the weak topology. Then by Eberlein-Šmulian Theorem, we get that $(u_m)_{m\in\mathbb{N}}$ is weakly sequentially compact, i.e., there exists a subsequence $(u_{m_k})_{k\in\mathbb{N}}$ and $u \in V$ such that

$$u_{m_k} \rightharpoonup u, \text{ as } k \to +\infty.$$

Since M is weakly closed, then $u \in M$. By weak lower semicontinous property we conclude that

$$\alpha \le I(u) \le \liminf_{k\to\infty} I\left(u_{m_k}\right) = \alpha.$$

with

$$\alpha = \inf_{v\in M} I(v) = \min_{v\in M} I(v).$$

■

Remark 2.4.17. Since I reaches its infimum, u is a point of global minimum.

Remark 2.4.18. A set is weakly closed if it is closed in the weak topology $\sigma(V, V^*)$.

Remark 2.4.19. Closed and convex sets of a Banach space are weakly closed. In particular, V is weakly closed.

Now, we briefly introduce minimax methods. In this case, we need a condition of compacity:

Definition 2.4.20 (Palais-Smale)**.** We say that $(u_m)_{m\in\mathbb{N}} \subseteq E$ is a Palais-Smale sequence if $(I(u_m))_{m\in\mathbb{N}} \subseteq \mathbb{R}$ is bounded and

$$\|I'(u_m)\| \to 0, \text{ as } m \to +\infty.$$

We say that I satisfies the *Palais-Smale condition (P.S.)* if any Palais-Smale sequence $(u_m)_{m\in\mathbb{N}}$ has a convergence subsequence. Let us define the following sets. Let $\beta \in \mathbb{R}$, $\delta > 0$, $\rho > 0$

$$\begin{aligned} I_\beta &:= \{u \in V \,/\, I(u) < \beta\}, \\ K_\beta &:= \{u \in V \,/\, I(u) = \beta, I'(u) = 0\}, \\ U_{\beta,\rho} &:= \bigcup_{u\in K_\beta} \{v \in V \,/\, \|v - u\| < \rho\}. \end{aligned}$$

Note that K_β is the set of critical points of I with critical value β, and $\{U_{\beta,\rho}\}_{\rho>0}$ is a *family of neighborhoods* of K_β.

Let's remember that a The next theorem is the basis for many results that allow finding saddle points.

Theorem 2.4.21 (Deformation Lemma)**.** *Assume that $E \in C^1(V)$ and satisfies the Palais-Smale condition. Take any $\beta \in \mathbb{R}, \delta > 0$, and any N neighborhood of K_β. Then there exists $\varepsilon \in]0, \delta[$ and a family of homeomorphisms $(\Phi_t)_{t\in[0,+\infty)}$,*

$$\Phi_t = \Phi(\cdot, t) : V \to V,$$

with the following properties:

1. *If* $t = 0$, *or* $I'(u) = 0$, *or* $|I(u) - \beta| \geq \delta$, *then*

$$\Phi(u,t) = u.$$

2. *For all* $u \in V$: $\quad I(\Phi(u,\cdot))$ *is nonincreasing.*

3. *Deformation property:*

$$\Phi\left(I_{\beta+\varepsilon}\backslash N, 1\right) \subseteq I_{\beta-\varepsilon}, \quad \text{and} \quad \Phi\left(I_{\beta+\varepsilon}, 1\right) \subseteq I_{\beta-\varepsilon} \cup N,$$

4. *Semi-group property of* $\Phi : V \times [0,\infty) \to V$.

$$\forall s,t \geq 0 : \quad \Phi(\cdot,t) \circ \Phi(\cdot,s) = \Phi(\cdot,s+t).$$

We state an existence theorem for saddle points called *Mountain Pass Theorem*, proved in [2] by Ambrosetti and Rabinowitz. First, let us define Γ_a, with $a \in V$, given by

$$\Gamma_a := \{h \in C([0,1],V) \,/\, h(0) = 0, h(1) = a\}.$$

Theorem 2.4.22 (Mountain Pass Theorem)**.** *Let us assume that*

$$I : V \to \mathbb{R} \in C^1(V,\mathbb{R}),$$

and verifies the Palais-Smale condition. Suppose also that

1. $I(0) = 0$,

2. $\exists \delta > 0, \beta > 0 : \quad \|u\| = \delta \implies I(u) \geq \beta$,

3. $\exists e \in V : \quad \|e\| \geq \delta$ *and* $I(e) < \beta$.

Then

$$c = \inf_{h\in\Gamma_e} \sup_{u\in h} I(u) \geq \beta$$

is a critical value.

2.5 Generalized gradient

In this section, we introduce the generalized gradient defined by Clarke, [16], which help to extend the concept of differentiability to locally Lipschitz functionals. It also enlarges the approach of critical points as well as the Palais-Smale condition and Deformation Lemma to PDEs with discontinuities. A complete study of the generalized gradient can be found in [16] or [20]. For the mentioned generalization see [13].

2.5.1 Setup

We start with a Banach space X. We say that a functional $f : X \to \mathbb{R}$ is *locally Lipschitz* if

$$\forall x \in X, \exists U \in \mathcal{N}(x), \exists K = K(U) \geq 0, \forall y \in U : \quad |f(x) - f(y)| \leq K\|x - y\|, \tag{34}$$

that is, f is Lipschitz on some neighborhood of x, for all $x \in X$. For this kind of functions, we define the following concept.

Definition 2.5.1 (Generalized directional derivative)**.** The *generalized directional derivative* of f at a point x with direction v, denoted $f^0(x,v)$, is given by

$$f^0(x,v) = \varlimsup_{\substack{h\to 0\\ \lambda\downarrow 0}} \frac{1}{\lambda}[f(x+h+\lambda v) - f(x+h)]. \tag{35}$$

Note that, thanks to (34), the previous limit is finite for any value $v \in X$. Even more, we have the following inequality.

Proposition 2.5.2. *For any $x, v \in X$ we have*

$$f^0(x,v) \leq K\|v\|. \tag{36}$$

Proof. The proof follows directly from the definition of the generalized directional derivative and the locally Lipschitz property. In fact, we have that

$$\begin{aligned}
|f^0(x,v)| &= \left| \overline{\lim_{\substack{h\to 0\\ \lambda\downarrow 0}}} \frac{1}{\lambda}[f(x+h+\lambda v) - f(x+h)] \right| \\
&= \left| \lim_{\substack{h\to 0\\ \lambda\downarrow 0}} \sup \left(\frac{1}{\lambda}[f(x+h+\lambda v) - f(x+h)] \right) \right| \\
&\leq \lim_{\substack{h\to 0\\ \lambda\downarrow 0}} \sup \left| \frac{1}{\lambda}[f(x+h+\lambda v) - f(x+h)] \right| \\
&\leq \lim_{\substack{h\to 0\\ \lambda\downarrow 0}} \sup \frac{1}{|\lambda|} K\|\lambda v\| = K\|v\|.
\end{aligned}$$

■

In addition, the function

$$f^0(x,\cdot) : X \ni v \longmapsto f^0(x,v) \in \mathbb{R}$$

is positively homogeneous, i.e.,

$$\forall v, w \in X : \quad f^0(x, v+w) \leq f^0(x,v) + f^0(x,w),$$

and subadditive, that is,

$$\forall \alpha > 0, \forall v \in X : \quad f^0(x, \alpha v) = \alpha f^0(x,v).$$

For more properties of the generalized derivative see [13].

We present a main definition.

Definition 2.5.3 (Generalized Gradient)**.** The *generalized gradient* of f at $x \in X$, denoted $\partial f(x)$, is defined as subset of X^* given by

$$\partial f(x) = \left\{ \zeta \in X^* \,/\, \langle \zeta, v \rangle \leq f^0(x,v), \forall v \in X \right\}. \tag{37}$$

Proposition 2.5.4. *The set $\partial f(x)$ is not empty.*

To prove Proposition 2.5.4, we will use Hahn-Banach Theorem, [7],

Theorem 2.5.5 (Hahn-Banach Theorem)**.** *Let $p : X \to \mathbb{R}$ be a subadditive and positively homogeneous functional on the linear space X. Let $Z \subseteq X$ be a linear subspace of X. Let's assume that there exists a linear functional $h : Z \to \mathbb{R}$ such that*

$$\forall x \in Z : \quad h(x) \leq p(x).$$

Then, there exists a linear functional $g : X \to \mathbb{R}$ such that,

i) $\forall x \in Z : \quad h(x) = g(x),$

ii) $\forall x \in X : \quad g(x) \leq p(x).$

Proof of Proposition 2.5.4. Let's apply Hahn-Banach Theorem for

$$p(\cdot) = f^0(x,\cdot), \quad Z = \{0\}, \quad \text{and} \quad h(\cdot) = f^0(x,\cdot).$$

Then there exists a linear functional $\zeta : X \to \mathbb{R}$ such that

$$\forall v \in X : \quad \langle \zeta, v \rangle \leq f^0(x, v). \tag{38}$$

In addition, by equation (36),

$$\forall v \in X : \quad \langle \zeta, v \rangle \leq K \|v\|.$$

Hence,

$$\zeta \in X^*. \tag{39}$$

By (38) and (39) it follows that $\zeta \in \partial f(x)$. ■

Proposition 2.5.6. *The generalized gradient of f at x, $\partial f(x)$, is weak* compact.*

To prove Proposition 2.5.6 we will use the following theorem, which follows from Alaoglu's Theorem 2.4.12, (see [28]),

Theorem 2.5.7. *Let X be a Banach space, and $A \subseteq X$ weak* closed. Then A is weak* compact iff it is bounded.*

Proof of Proposition 2.5.6. This proof is taken from [20]. Let us prove that $\partial f(x)$ is weak* compact. Let $\zeta \in \partial f(x)$. By equation (36), we know that

$$\forall v \in X : \quad \langle \zeta, v \rangle \leq K \|v\|.$$

Since this is true for any $\zeta \in \partial f(x)$, by definition of the dual norm (see Section 2.1),

$$\forall \zeta \in \Sigma : \quad \|\zeta\|_{X^*} \leq K.$$

Hence $\partial f(x)$ is bounded. Also, $\partial f(x)$ is weak* closed (see [16], pag. 54). Then, by Theorem 2.5.7 $\partial f(x)$ is weak* compact. ■

Let us consider the following example, taken from [18], that illustrates how the generalized gradient is calculated.

Example 2.5.8. We calculate the generalized gradient for $f : \mathbb{R} \to \mathbb{R}$ where $f(x) = |x|$. Assume that $x > 0$. By definition we have that,

$$\begin{aligned} f^0(x, v) &= \overline{\lim_{\substack{h \to 0 \\ \lambda \downarrow 0}}} \frac{1}{\lambda}[|x + h + \lambda v| - |x + h|] \\ &= \overline{\lim_{\substack{h \to 0 \\ \lambda \downarrow 0}}} \frac{1}{\lambda}[x + h + \lambda v - x - h] \\ &= v. \end{aligned}$$

Le's recall that, by Riesz representation theorem, $\mathbb{R} \cong \mathbb{R}^*$. Since $\partial f(x)$ is the set of ξ such that $f^0(x, v) = v \geq \xi v$. Then $\partial f(x) = \{1\}$ for $x > 0$. In the same way $\partial f(x) = \{-1\}$ for any $x < 0$. For $x = 0$ we have that $f^0(x, v) = |v|$. By definition of the generalized derivative we get that $\partial f(0) = [-1, 1]$.

Let's list some useful properties of the generalized gradient:

1. $\forall \beta \in \mathbb{R}$:
$$\partial f(\beta x) = \beta \, \partial f(x).$$
2. If x is a point of local minimum or maximum of f, then $0 \in \partial f(x)$.
3. Let f and g be locally Lipschitz functions, then $f + g$ is locally Lipschitz and
$$\forall x \in X : \quad \partial(f + g)(x) \subseteq \partial f(x) + \partial g(x). \tag{40}$$
4. $\lambda : X \to \mathbb{R}$ given by
$$\lambda(x) = \min_{v \in \partial f(x)} \|v\|_{X^*}$$
is a well-defined lower semicontinuous function.

5. For any $v \in X$,
$$f^0(x, v) = \max\{\langle \xi, v\rangle \,/\, \xi \in \partial f(x)\}. \tag{41}$$

6. When the function f is Gateaux differentiable at $x \in X$, we have that
$$f'_G(x) \in \partial f(x).$$
Moreover, if
   i) For any point y of some neighborhood of x, $f'_G(y)$ exists.
   ii) The function
   $$f'_G : X \to X^*$$
   is continuous.

   Then
$$\partial f'_G(x) = \{f'_G(x)\}.$$

The proofs of properties 1 to 5 can found in [18], and of property 6 in [15]. We end this subsection with the definition of a critical point, [13],

Definition 2.5.9. [Generalized critical point] Let X be a Banach space, $f : X \to \mathbb{R}$ be a locally Lipschitz function. An element $u_0 \in X$ is called a *generalized critical point* of f if and only if $0 \in \partial f(u_0)$.

Remark 2.5.10. Any point of local minimum or point of local maximum is a generalized critical point.

2.5.2 Applications

In this section, the main references are [13] and [9]. We recommend checking the concepts introduced in Section 2.2. We focus on a functional g defined on a real Banach functional space X where the following formula makes sense,
$$g(u) := \int_\Omega \int_0^{u(x)} \phi(x, t)dtdx. \tag{42}$$
The function ϕ is measurable on $\Omega \times \mathbb{R}$ and such that $\Omega \subseteq \mathbb{R}^N$ is a bounded *domain*, i.e. an open connected subset in $\mathbb{R}^N$, and $u : \Omega \to \mathbb{R}$ is in X. Our goal is to determine $\partial g(u)$. With this intention, we give some conditions for ϕ. First, let us assume that
$$\exists C_1 > 0, C_2 > 0, \alpha > 0, \forall (x, t) \in \Omega \times \mathbb{R} : \quad |\phi(x, t)| \le C_1 + C_2|t|^\alpha. \tag{43}$$
Hence the functional
$$g : L^{\alpha+1}(\Omega) \to \mathbb{R}$$
given in (42) is locally Lipschitz.

Second, consider the functions $\underline{\phi}$ and $\bar{\phi}$ from $\Omega \times \mathbb{R}$ to $\mathbb{R}$ given by, (see definition of $\underline{\lim}$ and $\overline{\lim}$ in Section 2.1,
$$\begin{aligned}\underline{\phi}(x, t) &= \underline{\lim}_{s\to t}\, \phi(x, t)\\ &= \lim_{s\to t} \operatorname{ess\,inf} \phi(x, t)\\ &= \lim_{\varepsilon\to 0} \operatorname{ess\,inf}\{\phi(x, t) \,/\, |t - s| < \varepsilon\},\end{aligned}$$
and
$$\begin{aligned}\bar{\phi}(x, t) &= \overline{\lim}_{s\to t}\, \phi(x, t)\\ &= \lim_{s\to t} \operatorname{ess\,sup} \phi(x, t)\\ &= \lim_{\varepsilon\to 0} \operatorname{ess\,sup}\{\phi(x, t) \,/\, |t - s| < \varepsilon\}.\end{aligned}$$

To use the generalized gradient for the functional (42), ϕ, $\bar{\phi}$, and $\underline{\phi}$ have to be superpositionally measurable.

Remark 2.5.11. Remember that a function φ is *superpositionally measurable*, also called *N-measurable*, if for any measurable function u, $\varphi(\cdot, u(\cdot))$ on Ω is a measurable function. Since $\overline{\phi}$ and $\underline{\phi}$ are finite-valued upper semi-continuous and lower semi-continuous functions, respectively, we have that they are both Baire functions (see [23], pag. 83). Considering that any Baire function is superpositionally measurable, [12], we have that ϕ, $\overline{\phi}$, and $\underline{\phi}$ are superpositionally measurable.

By this discussion, we state the following theorem

Theorem 2.5.12. *Let us assume that (43) holds for ϕ, i.e.,*

$$\exists C_1 > 0, C_2 > 0, \alpha > 0, \forall (x,t) \in \Omega \times \mathbb{R} : \quad |\phi(x,t)| \leq C_1 + C_2|t|^\alpha.$$

and that $\overline{\phi}$ and $\underline{\phi}$ are superpositionally measurable functions. Then for each $u \in L^{\alpha+1}(\Omega)$ we have that

$$\partial g(u) \subseteq [\underline{\phi}(x, u(x)), \overline{\phi}(x, u(x))] \text{ a.e. } \Omega. \tag{44}$$

Remark 2.5.13. Formula (44) means that given

$$w \in \partial g(u) \subseteq (L^{\alpha+1}(\Omega))^* \cong L^{(\alpha+1)/\alpha}(\Omega),$$

we have

$$\underline{\phi}(x, u(x)) \leq w(x) \leq \overline{\phi}(x, u(x)) \text{ a.e. } \Omega, \tag{45}$$

considering w as an element of $L^{(\alpha+1)/\alpha}(\Omega)$.

Proof of Theorem 2.5.12. This proof is based on [13]. Consider the following notation,

$$\{v > 0\} := \left\{x \in \Omega \,/\, v(x) > 0\right\},$$
$$\{v < 0\} := \left\{x \in \Omega \,/\, v(x) < 0\right\}.$$

i) First, we show that,

$$\forall u, v \in L^{\alpha+1}(\Omega) : \quad g^0(u, v) \leq \int_{\{v>0\}} \overline{\phi}(x, u(x))v(x)dx + \int_{\{v<0\}} \underline{\phi}(x, u(x))v(x)dx. \tag{46}$$

Let $u, v \in L^{\alpha+1}(\Omega)$, and $(h_n)_{n\in\mathbb{N}} \in L^{\alpha+1}(\Omega)$ such that

$$\|h_n\|_{L^{\alpha+1}(\Omega)} \to 0, \text{ as } n \to +\infty.$$

By definition,

$$\begin{aligned} g^0(u, v) &= \varlimsup_{\substack{n\to\infty \\ \lambda\downarrow 0}} \frac{1}{\lambda}[g(u + h_n + \lambda v) - g(u + h_n)] \\ &= \varlimsup_{\substack{n\to\infty \\ \lambda\downarrow 0}} \frac{1}{\lambda} \int_\Omega \int_{(u+h_n)(x)}^{(u+h_n+\lambda v)(x)} \phi(x, \xi)d\xi \\ &= \varlimsup_{\substack{n\to\infty \\ \lambda\downarrow 0}} \frac{1}{\lambda} \int_\Omega \int_{h_n(x)}^{(h_n+\lambda v)(x)} \phi(x, u(x) + \xi)d\xi. \end{aligned} \tag{47}$$

Moreover, by the computations in [13], the generalized gradient of

$$\Phi(x, t) = \int_0^t \phi(x, \xi)d\xi$$

respect to t is

$$\partial_t\Phi(x, t) = [\underline{\phi}(x, t), \overline{\phi}(x, t)].$$

So,

$$\partial_t\Phi(x, u(x)) = [\underline{\phi}(x, u(x)), \overline{\phi}(x, u(x))].$$

We also have that,

$$\begin{aligned}\Phi^0(x,t;z) &= \overline{\lim_{\substack{k\to 0\\ \lambda\downarrow 0}}} \frac{1}{\lambda}\left[\Phi(x,t+k+\lambda z)-\Phi(x,t+k)\right]\\ &= \overline{\lim_{\substack{k\to 0\\ \lambda\downarrow 0}}} \frac{1}{\lambda}\int_{t+k}^{t+k+\lambda z}\phi(x,\xi)d\xi\\ &= \overline{\lim_{\substack{k\to 0\\ \lambda\downarrow 0}}} \frac{1}{\lambda}\int_{k}^{k+\lambda z}\phi(x,t+\xi)d\xi.\end{aligned}$$

Hence,

$$\Phi^0(x,u(x);v(x)) = \overline{\lim_{\substack{k\to 0\\ \lambda\downarrow 0}}} \frac{1}{\lambda}\int_{k}^{k+\lambda v(x)}\phi(x,u(x)+\xi)d\xi. \tag{48}$$

Without loss of generality, we may assume that

$$h_n(x)\to 0 \quad \text{a.e.} \quad \Omega.$$

By considering these kind of sequences $(h_n)_{n\in\mathbb{N}}$, due to (47) and (48) we get

$$\begin{aligned}g^0(u,v) &= \overline{\lim_{\substack{n\to +\infty\\ \lambda\downarrow 0}}} \frac{1}{\lambda}\int_\Omega\int_{h_n(x)}^{(h_n+\lambda v)(x)}\phi(x,u(x)+\varepsilon)d\xi\\ &\leq \int_\Omega \overline{\lim_{\substack{k\to 0\\ \lambda\downarrow 0}}} \frac{1}{\lambda}\int_{k}^{k+\lambda v(x)}\phi(x,u(x)+\xi)d\xi dx\\ &= \int_\Omega \Phi^0(x,u(x);v(x))dx \end{aligned}\tag{49}$$

By property 5 of the generalized gradient, equation (41), we have

$$\int_\Omega \Phi^0(x,u(x);v(x))dx = \int_\Omega \max\left\{\varphi(x)v(x) \,/\, \varphi\in\partial_t(\cdot,u(\cdot))\subseteq L^{(\alpha+1)/\alpha}(\Omega)\right\}dx. \tag{50}$$

Since

$$\partial_t\Phi(x,u(x)) = [\underline{\phi}(x,u(x)),\overline{\phi}(x,u(x)],$$

it follows, for any $\varphi\in\partial_t(\cdot,u(\cdot))\subseteq L^{(\alpha+1)/\alpha}(\Omega)$, that

$$\underline{\phi}(x,u(x))\leq\phi(x)\leq\overline{\phi}(x,u(x)).$$

Then, for $v\in L^{\alpha+1}(\Omega)$

$$\begin{aligned}&\text{For } v(x)<0: \quad \underline{\phi}(x,u(x))v(x)\geq\phi(x)v(x)\geq\overline{\phi}(x,u(x))v(x).\\ &\text{For } v(x)>0: \quad \underline{\phi}(x,u(x))v(x)\leq\phi(x)v(x)\leq\overline{\phi}(x,u(x))v(x).\end{aligned}$$

Therefore the right-hand-side term in (50) is equal to

$$\int_{\{v>0\}}\overline{\phi}(x,u(x))v(x)dx + \int_{\{v<0\}}\underline{\phi}(x,u(x))v(x)dx,$$

which proves (46).

ii) Second, we prove (44), i.e.,

$$\forall\omega\in\partial g(u): \quad \underline{\phi}(x,u(x))\leq w(x)\leq\bar{\phi}(x,u(x)) \text{ a.e. } \Omega. \tag{51}$$

Let $\omega\in\partial g(u)$. We will prove that

$$\underline{\phi}(x,u(x))\leq\omega(x) \text{ a.e. } \Omega. \tag{52}$$

Suppose by contradiction that there exits a set $E \subseteq \Omega$, $|E| > 0$, such that

$$\forall x \in E: \quad \omega(x) < \underline{\phi}(x, u(x)). \tag{53}$$

If we take

$$v = -\chi_E \in L^{\alpha+1}(\Omega),$$

in (46), we get

$$-\int_E \omega(x)dx \leq -\int_E \underline{\phi}(x, u(x))dx,$$

which contradicts (53). Similarly we prove

$$\omega(x) \leq \overline{\phi}(x, u(x)) \text{ a.e. } \Omega. \tag{54}$$

By (52) and (54) we have shown (51).

■

If we assume that ϕ is a Borel measurable function and it is nondecreasing in the variable t, then $\underline{\phi}$ and $\bar{\phi}$ are superpositionally measurable and

$$\begin{aligned} \forall (x,t) \in \Omega \times \mathbb{R}: \quad \underline{\phi}(x,t) &= \min\{\phi(x,t^+), \phi(x,t^-)\}, \\ \forall (x,t) \in \Omega \times \mathbb{R}: \quad \bar{\phi}(x,t) &= \max\{\phi(x,t^+), \phi(x,t^-)\}. \end{aligned} \tag{55}$$

In order to extend Theorem 2.5.12 to $W_0^{1,p}(\Omega)$, we consider the following result.

Lemma 2.5.14. *Let X, Y be two Banach spaces, such that*

$$X \hookrightarrow Y \text{ and } \overline{X} = Y,$$

i.e, X is continuously embedded and dense in Y. Let

$$g : Y \to \mathbb{R} \text{ and } f = g|_X.$$

Therefore, we have that

$$\forall x \in X: \quad \partial f(x) \subseteq \partial g(x), \tag{56}$$

where the inclusion is interpreted as follows: for each $w \in \partial f(x)$ there is a unique extension $v \in \partial g(x)$.

Remark 2.5.15. Take $X = W_0^{1,p}(\Omega)$ and $Y = L^{\alpha+1}(\Omega)$. Since $C_c^\infty(\Omega)$ is dense in both $W_0^{1,p}(\Omega)$ and $L^{\alpha+1}(\Omega)$, it follows that $W_0^{1,p}(\Omega)$ is dense in $L^{\alpha+1}(\Omega)$. Let us consider Theorem 2.3.15. Let $p \in [1, +\infty]$. First, suppose that $p < N$. Let

$$1 + \alpha \leq p^\star = \frac{pN}{N-p}$$

which is equivalent to say that

$$\alpha \leq \frac{pN}{N-p} - 1 = \frac{(p-1)N + p}{N-p},$$

then point (1) of the Theorem 2.3.15 holds. Second, suppose that $p = N$ and take $\alpha + 1 \geq p$. So point (2) of Theorem 2.3.15 holds. Third, if $p > N$ note that $C(\bar{\Omega}) \subseteq L^p(\bar{\Omega}) = L^p(\Omega)$. So point (3) of Theorem 2.3.15 also holds. Hence, if we take

$$p - 1 \leq \alpha \leq \frac{(p-1)N + p}{N-p},$$

by Remark 2.3.17 we get

$$W_0^{1,p}(\Omega) \hookrightarrow L^{\alpha+1}(\Omega).$$

Therefore Lemma 2.5.14 can be applied. In our case, we will have

$$\forall u \in W_0^{1,p}(\Omega): \quad \partial f(u) \subseteq \partial g(u),$$

where the domains of f and g are $W_0^{1,p}(\Omega)$ and $L^{\alpha+1}(\Omega)$, respectively, and g is given by (42). Consequently, given a functional

$$w \in W^{-1,p'}(\Omega) \subseteq \partial f(u),$$

there exists a unique functional $v \in (L^{\alpha+1}(\Omega))^*$ such that

$$w = v|_{W_0^{1,p}(\Omega)}.$$

By combining Theorem 2.5.12 and Lemma 2.5.14 we get the following Corollary

Corollary 2.5.16. *Consider the same assumptions of Theorem 2.5.12, i.e., assume that*

$$\exists C_1 > 0, C_2 > 0, \alpha > 0, \forall (x,t) \in \Omega \times \mathbb{R}: \quad |\phi(x,t)| \leq C_1 + C_2|t|^\alpha.$$

and that $\overline{\phi}$ and $\underline{\phi}$ are superpositionally measurable functions. Additionally, suppose also that

$$p - 1 \leq \alpha \leq \frac{(p-1)N + p}{N - p}.$$

Then we have that g is locally Lipschitz in $W_0^{1,p}(\Omega)$, and

$$\partial g(u) \subseteq [\underline{\phi}(x, u(x)), \overline{\phi}(x, u(x))] \text{ a.e. } \Omega. \tag{57}$$

Remark 2.5.17. If in addition ϕ is Borel measurable and it is nondecreasing respect to t, then $\underline{\phi}$ and $\overline{\phi}$ are given as in (55).

When the function g is convex we get a stronger result. Recall that the *subdifferential*, [18], of a convex function f at a point x is the set of all $\zeta \in X^*$ satisfying

$$\forall y \in X: \quad f(y) - f(x) \geq \langle \zeta, y - x \rangle.$$

Theorem 2.5.18. *Let ϕ be a Baire-measurable function on $\Omega \times \mathbb{R}$ satisfying (43), with $\phi(x,\cdot)$ nondecreasing for each x, and*

$$p - 1 \leq \alpha \leq \frac{(p-1)N + p}{N - p}.$$

Then g is convex on $W_0^{1,p}(\Omega)$ and,

$$\partial g(u) = [\phi(x, u(x)^-), \phi(x, u(x)^+)] \text{ a.e. } \Omega. \tag{58}$$

for $u \in L^{\alpha+1}(\Omega)$ and $W_0^{1,p}(\Omega)$.

The proof can be found in [13] and [9].

2.5.3 Variational approach for nondifferentiable functionals

In this section we present some results analogous to those of Section 2.4.2. Let f be a locally Lipschitz function. We start by defining

$$\begin{aligned} A_c &= \{x \in X \,/\, f(x) \leq c\}, \\ K_c &= \{x \in X \,/\, 0 \in \partial f(x), f(x) = c\}. \end{aligned}$$

As in the case of variational methods for C^1 functionals, we have the next concept analogous to Definition 2.4.20.

Definition 2.5.19 (Generalized Palais-Smale condition)**.** Let f be a locally Lipschitz functional on a Banach space X. We say that f satisfies the (generalized) Palais-Smale condition (PS) if any sequence $(x_n)_{n\in\mathbb{N}} \subseteq X$ with properties

- $(f(x_n))_{n\in\mathbb{N}} \subseteq$ bounded,

- $\lambda(x) = \min_{v \in \partial f(x)} \|v\|_{X^*} \to 0$,

has a convergent subsequence. In addition, we say that f satisfies (PS)$^+$ (respectively (PS)$^-$) if this condition is true just for regions of X where $f \geq c > 0$ (respectively $f \leq c < 0$) for any $c > 0$.

By using the last definition we can state a generalized Deformation Lemma

Theorem 2.5.20 (Deformation Lemma)**.** *Assume that f is a locally Lipschitz functional on a reflexive Banach space X satisfying the Palais-Smale condition. Let $c \in \mathbb{R}, \alpha > 0$, and N any neighborhood of K_c. Then, there exists $\epsilon \in (0, \alpha)$ and a homeomorphism $\Phi : X \to X$ such that*

1. $\forall x \notin A_{c+\alpha} \backslash A_{c-\alpha} : \quad \Phi(x) = x$,
2. $\Phi(A_{c+\alpha} \backslash N) \subseteq A_{c-\alpha}$,
3. *If $K_c = \emptyset$, then $\Phi(A_{c+\alpha}) \subseteq A_{c-\alpha}$.*

From this theorem, it is deduced a generalized version of the Mountain Pass theorem

Theorem 2.5.21 (Generalized Mountain Pass Theorem)**.** *Suppose that X is a reflexive Banach space and f is a locally Lipschitz function satisfying (PS)$^+$. Let*

1. $f(0) = 0$,
2. $\exists \delta > 0, \beta > 0$ *such that*
$$\forall x \in B(0, \delta) \backslash \{0\} : \quad f(x) > 0,$$
and
$$\forall x \in S(0, \delta) \backslash \{0\} : \quad f(x) > \beta.$$
3. $\exists e \in X, e \neq 0 : \quad f(e) = 0$.

Then there exists a critical value $c > 0$ of f.

2.6 The p-Laplacian operator in short

In this Section, we briefly describe the p-Laplacian operator. We also present the eigenvalue problem for the p-Laplacian and focus on its first eigenvalue. We follow [30], [14] Chapter 5, and [31]. For a complete study of this topic, we recommend [35]. Let us start with the Laplace equation

$$\Delta u = \frac{\partial^2 u}{\partial x_1^2} + \frac{\partial^2 u}{\partial x_2^2} + \cdots + \frac{\partial^2 u}{\partial x_N^2} = 0. \tag{59}$$

Equation (59) is the Euler-Lagrange equation of the functional,

$$H(u) = \int_\Omega |\nabla u(x)|^2 dx = \int \cdots \int \left[\left(\frac{\partial u(x)}{\partial x_1} \right)^2 + \cdots + \left(\frac{\partial u(x)}{\partial x_N} \right)^2 \right] dx_1 \ldots dx_N.$$

By changing the square by any p power, we get

$$B(u) = \int_\Omega |\nabla u(x)|^p dx = \int \cdots \int \left[\left(\frac{\partial u(x)}{\partial x_1} \right)^2 + \cdots + \left(\frac{\partial u(x)}{\partial x_N} \right)^2 \right]^{\frac{p}{2}} dx_1 \ldots dx_N.$$

Its corresponding Euler-Lagrange equation is the p-Laplacian equation, which is a quasilinear equation in divergence form, given by

$$\nabla \cdot (|\nabla u(x)|^{p-2} \nabla u(x)) = 0.$$

Thus, the p-Laplacian operator is

$$\Delta_p u = \nabla \cdot (|\nabla u|^{p-2} \nabla u).$$

Consider some special cases:

a) For $p = 1$ we get the *Mean Curvature* operator H,

$$H = -\Delta_1 u = -\nabla \left(\frac{\nabla u}{|\nabla u|} \right).$$

b) For $p = 2$ we have the Laplacian operator,

$$\Delta u = \sum_{i=1}^{N} \frac{\partial u}{\partial x_i}.$$

c) If we let $p \to +\infty$, the following equation arises

$$\Delta_\infty u = \sum_{i,j=1}^{N} \frac{\partial u}{\partial x_i} \frac{\partial u}{\partial x_j} \frac{\partial^2 u}{\partial x_i \partial x_j} = 0.$$

Remark 2.6.1. The p-Laplacian operator appears in many phenomena in physics, e.g., non-Newtonian fluids, elasticity, reaction-diffusion problems, radiation of heat, and rheology; and recently in development of technology, e.g., in image processing and machine learning (see [21]).

An important problem concerning the p-Laplacian operator is the following,

$$\begin{cases} -\Delta_p u(x) = \lambda |u(x)|^{p-2} u(x), & x \in \Omega, \\ u(x) = 0, & x \in \partial\Omega. \end{cases} \tag{60}$$

where $\Omega \subseteq \mathbb{R}^N$ is a bounded domain and $p > 1$. This is called the *eigenvalue problem* for the p-Laplacian and consists in finding nontrivial solutions in the weak sense for (60), that is, find $u \in W_0^{1,p}(\Omega) \backslash \{0\}$ and $\lambda \in \mathbb{R}$ such that

$$\forall v \in W_0^{1,p}(\Omega) : \quad \int_\Omega |\nabla u(x)|^{p-2} \nabla u(x) \cdot \nabla v(x) dx = \lambda \int_\Omega |u(x)|^{p-2} u(x) v(x) dx. \tag{61}$$

If (λ, u) is a solution of (60), we call λ an *eigenvalue* and u an *eigenfunction* associated with λ.

Remark 2.6.2. For $p = 2$, problem (60) is actually an eigenvalue problem for the linear operator $-\Delta$, that is

$$\begin{cases} -\Delta u(x) = \lambda u(x), & x \in \Omega, \\ u(x) = 0, & x \in \partial\Omega. \end{cases}$$

There exists an infinite number of eigenvalues. In fact, there is an unbounded sequence $(\lambda_k)_{k \in \mathbb{N}}$ of eigenvalues of the p-Laplacian problem. One process to find such a sequence is described briefly in the next lines.

First, let us define some concepts. We say that a subset A of a group $(G, +)$ is *symmetric* if

$$\forall v \in A : \quad -v \in A.$$

Let B be a Banach space. Let C be a symmetric closed subset of B. Then, we define the *genus* of C, denoted $\gamma(C)$, as the smallest integer k for which there exists a function

$$\varphi : C \to \mathbb{R}^k \backslash \{0\},$$

continuous and odd. When such number k does not exist, we define

$$\gamma(C) = +\infty.$$

Let us work on the Banach space $W_0^{1,p}(\Omega)$. Let $D \subseteq W_0^{1,p}(\Omega)$ be a symmetric subset. Denote

$$\Sigma_D = \{v \in D \,/\, \|v\|_{W_0^{1,p}(\Omega)} = 1\}.$$

Consider the collection

$$\mathcal{D}_k = \{C \subseteq W_0^{1,p}(\Omega) \,/\, C \text{ is symmetric, } \Sigma_C \text{ is compact, and } \gamma(C) \geq k\}.$$

Then, the values

$$\lambda_k = \inf_{C\in\mathcal{D}_k} \max_{v\in C} \frac{\displaystyle\int_\Omega |\nabla u(x)|^p dx}{\displaystyle\int_\Omega |v(x)|^p dx},$$

define a sequence of infinitely many eigenvalues for the problem (60), [14]. Moreover,

$$\lambda_k \to +\infty, \text{ as } k \to +\infty.$$

The element λ_1 is the first eigenvalue of the p-Laplacian and their associated eigenfunctions are called *first eigenfunctions*. One can see also that, [31],

$$\lambda_1 = \inf_{u\in W_0^{1,p}(\Omega), u\neq 0} \left\{ \frac{\displaystyle\int_\Omega |\nabla u(x)|^p dx}{\displaystyle\int_\Omega |u(x)|^p dx} \right\}. \tag{62}$$

The first eigenvalue has many properties, among them:

i) $\lambda_1 > 0$.

ii) *Isolated*: There exists $a > \lambda_1$ such that λ_1 is the unique eigenvalue in $[0, a]$.

iii) *Simple*: For any u, v, eigenfunctions of λ_1, there exits $\alpha \in \mathbb{R}$ such that $u = \alpha v$.

3 Results

3.1 Preliminaries

Our problem extend the results of three previous works. In the first, Ambrosetti and Badiale [1] studied the problem

$$\begin{cases} -\Delta u(x) = f(u(x)) + q(x), & x \in \Omega, \\ u(x) = 0, & x \in \partial\Omega. \end{cases} \tag{P1}$$

where f has upward discontinuities, and

($\mathbf{F_2'}$) There exists $\nu : \mathbb{R} \to \mathbb{R}$ such that $\nu(s) = ms + f(s)$, strictly increasing, for some $m \geq 0$.

By using Clarke's Dual Action Principle, see [17] and [19], they found the dual functional $\mathfrak{I} \in C^1(L^2(\Omega), \mathbb{R})$ associated with (P1), given by

$$\mathfrak{I}(u) = \int_\Omega \left(G(u(x)) - \frac{1}{2} u(x) K(u(x)) - u(x) K(q(x)) \right) dx,$$

where $K : L^2(\Omega) \to L^2(\Omega)$ is a linear operator and $G : L^2(\Omega) \to \mathbb{R}$ is a functional build from ν (see [1] for more details). They search for solutions of (P1) by finding critical points of the smooth functional $\mathfrak{I}$.

In order to expand the work of [1], Arcoya and Calahorrano [4] generalized the problem for the p-Laplacian operator $\Delta_p u = \nabla \cdot (|\nabla u|^{p-2} \nabla u)$, for $p > 1$, by using the concept of generalized gradient. To use the results obtained by [13] on functional spaces, they considered that $\Omega \subseteq \mathbb{R}^N$ is a bounded domain, $q \in L^{p'}(\Omega)$, and that the discontinuous function f verifies the following conditions:

($\mathbf{F_1}$) There exists $a \in \mathbb{R}$, such that

a) $f \in C(\mathbb{R} \backslash \{a\}, \mathbb{R})$,

b) $f(a^-) < f(a^+)$,

c) $f(a) \in [f(a^-), f(a^+)]$.

($\mathbf{F_2}$) There exist $\alpha, C_1, C_2 > 0$, with $1 + \alpha \in [p, p^\star]$, such that

$$\forall s \in \mathbb{R} : \quad |f(s)| \leq C_1 + C_2 |s|^\alpha. \tag{63}$$

where $p^\star$ is the same as in Theorem 2.3.15, i.e.,

$$p^\star = \begin{cases} \dfrac{Np}{N-p}, & p < N \\ +\infty, & \text{otherwise.} \end{cases} \tag{64}$$

Remark 3.1.1. Note that condition (64) is the same given in (43) for ϕ.

Next, Mayorga-Zambrano and Calahorrano, [8], considered the problem

$$\begin{cases} -\Delta_p u(x) = h(x) f(u(x)) + q(x), & x \in \Omega, \\ u(x) = 0, & x \in \partial\Omega, \end{cases} \tag{PP}$$

assuming ($\mathbf{F_1}$), ($\mathbf{F_2}$) and

($\mathbf{H_1}$) $h \in L^\infty(\Omega)$ and $h > 0$.

By following the approach of [4] and [8], we use the generalized gradient for functional spaces developed in [13], to study (PP) considering ($\mathbf{F_1}$), ($\mathbf{F_2}$), and assuming that

($\mathbf{H_2}$) $h \in L^\infty(\Omega)$.

Therefore, we extend the results of [8] by removing the condition of $h > 0$. In order to study (PP), we will work with the functional

$$I : W_0^{1,p}(\Omega) \to \mathbb{R},$$

given by

$$I(u) = \frac{1}{p}\int_\Omega |\nabla u(x)|^p dx - \int_\Omega q(x)u(x)dx - \int_\Omega F(u(x))h(x)dx, \tag{65}$$

where

$$F(t) = \int_0^t f(s)ds.$$

I is the energy functional of (PP) as we show next.

Lemma 3.1.2. *Consider $\mathcal{L}$ as in (26) given by*

$$\mathcal{L}(z, y, x) = \frac{1}{p}|z|^p - q(x)y - F(y)h(x).$$

Then

$$-\Delta_p u(x) = h(x)f(u(x)) + q(x), \tag{66}$$

is the Euler equation of the functional I in (65),

$$I(u) = \int_\Omega \mathcal{L}(\nabla u(x), u(x), x),$$

Proof. We follow the approach of [5].

(i) First let's check that I is well-defined. Recall, from Remark 2.5.15, that

$$W_0^{1,p}(\Omega) \hookrightarrow L^{\alpha+1}(\Omega),$$

i.e., there exists a constant $C > 0$ such that,

$$\forall u \in W_0^{1,p}(\Omega): \quad \|u\|_{L^{\alpha+1}(\Omega)} \leq C\|u\|_{W_0^{1,p}(\Omega)}.$$

Take any $u \in W_0^{1,p}(\Omega)$. Denote

$$M = \operatorname*{ess\,sup}_{x\in\Omega} |h(x)| = \|h\|_{L^\infty(\Omega)}.$$

By Hölder's inequality, (9), we have

$$\begin{aligned} I(u) &= \frac{1}{p}\int_\Omega |\nabla u(x)|^p dx - \int_\Omega F(u(x))h(x)dx - \int_\Omega q(x)u(x)dx \\ &\leq \frac{1}{p}\|u\|_{W_0^{1,p}(\Omega)} + \int_\Omega |F(u(x))h(x)| + \int_\Omega \left|q(x)u(x)\right| dx \\ &\leq \frac{1}{p}\|u\|_{W_0^{1,p}(\Omega)} + M\int_\Omega |F(u(x))|dx + \|u\|_{L^p(\Omega)}\|q\|_{L^{p'}(\Omega)}. \end{aligned}$$

By condition $(\mathbf{F_2})$,

$$|F(u(x))| = \left|\int_0^{u(x)} f(s)ds\right| \leq \int_0^{u(x)} |f(s)|ds \leq C_1|u(x)| + \frac{C_2}{\alpha+1}|u(x)|^{\alpha+1}.$$

So, it follows that

$$\begin{aligned} \int_\Omega |F(u(x))|dx &\leq C_1\int_\Omega |u(x)|dx + \frac{C_2}{\sigma+1}\int_\Omega |u(x)|^{\alpha+1}dx \\ &\leq C_1(\Omega)^{1/p'}\|u\|_{L^p(\Omega)} + \frac{C_2}{\alpha+1}\|u\|^{\alpha+1}_{L^{\alpha+1}(\Omega)}. \end{aligned}$$

Then I is well-defined.

(ii) Now, we compute the directional derivative of I at u in the direction v. Let $u, v \in W_0^{1,p}(\Omega)$. Define

$$\kappa_x : \mathbb{R} \to \mathbb{R}$$

given by

$$\kappa_x(t) := \mathcal{L}(\nabla(u+tv)(x), (u+tv)(x), x),$$

The directional derivative at u in the direction of v is

$$\begin{aligned}
\partial_v I(u) := \frac{d}{dt} I(u+tv)\Big|_{t=0} &= \lim_{t\to 0} \int_\Omega \frac{1}{t}[\mathcal{L}(\nabla(u+tv)(x), (u+tv)(x), x) - \mathcal{L}(\nabla u(x), u(x), x)] \\
&= \lim_{t\to 0} \int_\Omega \frac{\kappa_x(t) - \kappa_x(0)}{t} dx \\
&= \lim_{t\to 0} \frac{1}{t} \int_\Omega \frac{1}{p} |\nabla(u+tv)(x)|^p - q(x)(u+tv)(x) - F((u+tv)(x))h(x) \\
&\quad - \frac{1}{p}|\nabla u(x)|^p + q(x)u(x) + F(u(x))h(x)dx.
\end{aligned}$$

By taking $t \to 0$ we have

$$\begin{aligned}
\kappa_x'(0) &= \lim_{t\to 0} \frac{1}{t}[\mathcal{L}(\nabla(u+tv)(x), (u+tv)(x), x) - \mathcal{L}(\nabla u(x), u(x), x)] \\
&= \frac{d}{dt} \mathcal{L}(\nabla(u+tv)(x), (u+tv)(x), x)\Big|_{t=0} \\
&= \frac{d}{dt}\left(Q(t) - J(t) + R(t)\right)\Big|_{t=0},
\end{aligned}$$

where

$$\begin{aligned}
Q(t) &= \frac{1}{p}|\nabla(u+tv)(x)|^p, \\
R(t) &= -(u+tv)(x)q(x), \\
J(t) &= F((u+tv)(x))h(x).
\end{aligned} \tag{67}$$

By direct computation we get

$$\begin{aligned}
\frac{d}{dt} Q(t)\Big|_{t=0} &= \frac{d}{dt}\frac{1}{p}|\nabla(u+tv)(x)|^p\Big|_{t=0} \\
&= \frac{d}{dt}\left(\frac{1}{p}|\nabla(u+tv)(x)|^2\right)^{p/2}\Bigg|_{t=0} \\
&= \frac{d}{dt}\left(\frac{1}{p}|\nabla u(x)|^2 + 2t\nabla u(x)\cdot\nabla v(x) + t^2|\nabla v(x)|^2\right)^{p/2}\Bigg|_{t=0} \\
&= \frac{p}{2}\left(\frac{1}{p}|\nabla u(x)|^2 + 2t\nabla u(x)\cdot\nabla v(x) + t^2|\nabla v(x)|^2\right)^{\frac{p-2}{2}}\left(2\nabla u(x)\cdot\nabla v(x) + 2t|\nabla v(x)|^2\right)\Bigg|_{t=0} \\
&= |\nabla u(x)|^{p-2}\nabla u(x)\cdot\nabla v(x).
\end{aligned} \tag{68}$$

$$\begin{aligned}
\frac{d}{dt} R(t)\Big|_{t=0} &= -\frac{d}{dt} q(x)(u+tv)(x)\Big|_{t=0} \\
&= -q(x)v(x).
\end{aligned} \tag{69}$$

$$\begin{aligned}\frac{d}{dt}J(t)\Big|_{t=0} &= \frac{d}{dt}h(x)F((u+tv)(x))\Big|_{t=0}\\ &= \frac{d}{dt}h(x)\int_0^{(u+tv)(x)} f(s)ds\Big|_{t=0}\\ &= h(x)f((u+tv)(x))v(x)\big|_{t=0}\\ &= h(x)f(u(x))v(x).\end{aligned} \tag{70}$$

And so

$$\kappa_x'(0) = |\nabla u(x)|^{p-2}\nabla u(x)\cdot\nabla v(x) - q(x)v(x) - h(x)f(u(x))v(x).$$

By applying the Dominated Convergence Theorem, we will show that

$$\lim_{t\to 0}\int_\Omega \frac{\kappa_x(t)-\kappa_x(0)}{t}dx = \int_\Omega \lim_{t\to 0}\kappa_x'(0). \tag{71}$$

From the Mean Value Theorem, for each t there exists a φ, $|\varphi|<|t|$ such that

$$\frac{\kappa_x(t)-\kappa_x(0)}{t} = \kappa_x'(\varphi).$$

Note that from (68), (69), and (70), we get,

$$\kappa_x'(\varphi) = \nabla(u+\varphi v)(x)|^{p-2}\nabla(u+\varphi v)(x)\cdot\nabla v(x) - (h(x)f((u+\varphi v)(x)) + q(x))v(x).$$

We will use the following inequality that comes from the fact that $j:\mathbb{R}_+\ni x\to x^q\in\mathbb{R}_+$ is convex for $q\geq 1$,

$$\forall a,b\in\mathbb{R}_+: \quad |a+b|^q \leq 2^q\left(|a|^q+|b|^q\right). \tag{72}$$

So, by using Cauchy-Schwarz's inequality several times and applying (72) on

$$|\nabla(u+v)(x)|^{p-1} \leq ||\nabla u(x)| + |\nabla v(x)||^{p-1}$$

and

$$|u(x)+v(x)|^{p-1} \leq ||u(x)|+|v(x)||^{p-1},$$

we get

$$\begin{aligned}\left|\frac{\kappa_x(t)-\kappa_x(0)}{t}\right| &= \left|\frac{\mathcal{L}(\nabla(u+tv)(x),(u+tv)(x),x) - \mathcal{L}(\nabla u(x),u(x),x)}{t}\right|\\ &= \Big||\nabla(u+\varphi v)(x)|^{p-2}\nabla(u+\varphi v)(x)\cdot\nabla v(x) - (h(x)f((u+\varphi v)(x)) + q(x))v(x)\Big|\\ &\leq |\nabla(u+\varphi v)(x)|^{p-1}|\nabla v(x)| + |h(x)f(u+\varphi v)(x)v(x)| + |q(x)v(x)| \qquad (\varphi << 1)\\ &\leq |\nabla(u+v)(x)|^{p-1}|\nabla v(x)| + M|f((u+v)(x))v(x)| + |q(x)v(x)|\\ &\leq |\nabla(u+v)(x)|^{p-1}|\nabla v(x)| + \left(MC_1 + MC_2|(u+v)(x)|^\alpha\right)|v(x)| + |q(x)||v(x)|\\ &\leq 2^{p-1}|\nabla u(x)|^{p-1}|\nabla v(x)| + 2^{p-1}|\nabla v(x)|^p + MC_1|v(x)| + 2^{p-1}MC_2|u(x)|^\alpha|v(x)|\\ &+ 2^{p-1}MC_2|v(x)|^{\alpha+1} + |q(x)||v(x)|.\end{aligned}$$

By Hölder's inequality, equation (15), and Theorem 2.3.15

$$\begin{aligned}\int_\Omega |\nabla u(x)|^{p-1}|\nabla v(x)|dx &\leq \left(\int_\Omega \left(|\nabla u(x)|^{p-1}\right)^{p/(p-1)}dx\right)^{(p-1)/p}\left(\int_\Omega |\nabla v(x)|^p dx\right)^{1/p}\\ &= \|u\|^{p-1}_{W_0^{1,p}(\Omega)}\|v\|_{W_0^{1,p}(\Omega)} < \infty\end{aligned} \tag{73}$$

$$\begin{aligned}\int_\Omega |u(x)|^\alpha |v(x)|dx &\leq \left(\int_\Omega \left(|u(x)|^\alpha\right)^{(\alpha+1)/\alpha} dx\right)^{\alpha/(\alpha+1)} \left(\int_\Omega |v(x)|^{\alpha+1} dx\right)^{1/(\alpha+1)} \\ &= \left(\|u\|_{L^{\alpha+1}(\Omega)}\right)^\alpha \|v\|_{L^{\alpha+1}(\Omega)} \\ &\leq \left(C_1\|u\|_{W_0^{1,p}(\Omega)}\right)^\alpha C\|v\|_{W_0^{1,p}(\Omega)} < \infty,\end{aligned}$$

where C_1 and C are the constants from Theorem 2.3.15 and Poincaré's inequality, respectively. This shows that $\frac{k(t)-k(0)}{t} \in L^1(\Omega)$. So by applying the Dominated Convergence Theorem we obtain (71), i.e.,

$$\partial_v I(u) = \frac{d}{dt} I(u+tv)\bigg|_{t=0} = \int_\Omega |\nabla u(x)|^{p-2}\nabla u(x)\cdot\nabla v(x) - q(x)v(x) - h(x)f(u(x))v(x).$$

(iii) Finally, we show

$$-\Delta_p u(x) = h(x)f(u(x)) + q(x)$$

is the Euler equation of I. Suppose that u is a critical point. By using integration by parts and the fact that v has compact support, we formally obtain

$$\begin{aligned}\int_\Omega |\nabla u(x)|^{p-2}\nabla u(x)\cdot\nabla v(x)dx &= |\nabla u(x)|^{p-2}\nabla u(x)v(x)\bigg|_{\partial\Omega} - \int_\Omega \nabla\cdot(|\nabla u(x)|^{p-2}\nabla u(x))v(x)dx \\ &= -\int_\Omega \Delta_p u(x)v(x)dx.\end{aligned} \tag{74}$$

Since u is a critical point,

$$\begin{aligned}\partial_v I(u) &= \int_\Omega (-\Delta_p u(x)v(x) - q(x)v(x) - h(x)f(u(x))v(x))dx \\ &= -\int_\Omega (\Delta_p u(x) + q(x) + h(x)f(u(x)))v(x)dx \\ &= 0.\end{aligned}$$

Because it is true for any $v \in W_0^{1,p}(\Omega)$, we get weakly,

$$\Delta_p u(x) + q(x) + h(x)f(u(x)) = 0 \text{ a.e. } \Omega. \tag{75}$$

■

Remark 3.1.3. Equation (74) makes sense if, e.g., $u \in W_0^{1,p}(\Omega) \cap W_(^{2,p}\Omega)$.

To prove our main result, we will use the following Lemma,

Lemma 3.1.4. *Consider $Q : W_0^{1,p}(\Omega) \to \mathbb{R}$, and $R : W_0^{1,p}(\Omega) \to \mathbb{R}$, with*

$$\begin{aligned}Q(u) &= \frac{1}{p}\int_\Omega \left|\nabla u(x)\right|^p dx, \\ R(u) &= -\int_\Omega q(x)u(x)dx.\end{aligned}$$

The generalized gradient of Q and R at u are given by,

$$\partial Q(u) = \left\{Q'_G(u)\right\} \text{ and } \partial R(u) = \left\{R'_G(u)\right\}. \tag{76}$$

Proof. To prove the Lemma we will use property 6 of the generalized gradient presented in Section 2.5.

i) First, let us check that $Q'_G(u)$ and $R'_G(u)$ exist for each $u \in W_0^{1,p}(\Omega)$. From (ii) in the proof of Lemma 3.1.2 their directional derivatives at u are

$$\begin{aligned}\forall v \in W_0^{1,p}(\Omega) : \quad \partial_v Q(u) &= \int_\Omega |\nabla u(x)|^{p-2}\nabla u(x)\cdot\nabla v(x)dx. \\ \forall v \in W_0^{1,p}(\Omega) : \quad \partial_v R(u) &= -\int_\Omega q(x)v(x)dx.\end{aligned}$$

By equation (73),

$$\forall v \in W_0^{1,p}(\Omega): \quad |\partial_v Q(u)| \leq \|u\|_{W_0^{1,p}(\Omega)}^{p-1} \|v\|_{W_0^{1,p}(\Omega)},$$

i.e., $\partial_v Q$ is bounded. Therefore, Q is Gateaux differentiable for any $u \in W_0^{1,p}(\Omega)$, and it defines a map

$$Q'_G : W_0^{1,p}(\Omega) \to W^{-1,p'}(\Omega), \tag{77}$$

given by

$$\forall v \in W_0^{1,p}(\Omega): \quad \langle Q'_G(u), v\rangle = \int_\Omega |\nabla u(x)|^{p-2} \nabla u(x) \cdot \nabla v(x) dx.$$

Moreover, since $q \in L^{p'}(\Omega)$,

$$\forall v \in W_0^{1,p}(\Omega): \quad |\partial_v R(u)| \leq \|q\|_{L^{p'}(\Omega)} \|u\|_{W_0^{1,p}(\Omega)}.$$

Then $\partial_v R$ is also bounded, so that R is Gateaux differentiable on $W_0^{1,p}(\Omega)$. In this way, the map

$$R'_G : W_0^{1,p}(\Omega) \to W^{-1,p'}(\Omega), \tag{78}$$

given by,

$$\forall v \in W_0^{1,p}(\Omega): \quad \langle R'_G(u), v\rangle = -\int_\Omega q(x) v(x) dx.$$

is well defined.

ii) Second, let us prove that the maps (77) and (78) are continuous. The fact that $Q'_G : W_0^{1,p}(\Omega) \to W^{-1,p'}(\Omega)$ is continuous follows from the proof presented in [5], pag. 100. Therefore, by property 6,

$$\partial Q(u) = \left\{ Q'_G(u) \right\}.$$

To prove that

$$R'_G : W_0^{1,p}(\Omega) \to W^{-1,p'}(\Omega) \tag{79}$$

is continuous, we will show that

$$u_m \to u \text{ in } W_0^{1,p}(\Omega) \quad \Longrightarrow \quad R'_G(u_m) \to R'_G(u) \text{ in } W^{-1,p'}(\Omega), \tag{80}$$

Let us consider a sequence $(u_m)_{m\in\mathbb{N}} \subseteq W_0^{1,p}(\Omega)$, such that

$$u_m \to u \text{ in } W_0^{1,p}(\Omega).$$

We have

$$|\langle R'_G(u_m) - R'_G(u), v\rangle| = |R(v) - R(v)| \leq \frac{1}{n} \|v\|_{W_0^{1,p}(\Omega)}, \quad \forall n \in \mathbb{N}.$$

Since,

$$|\langle R'_G(u_m) - R'_G(u), v\rangle| \leq \|R'_G(u_m) - R'_G(u)\|_{W^{-1,p'}(\Omega)} \|v\|_{W_0^{1,p}(\Omega)},$$

then by definition of the dual norm (see Section 2.1),

$$\|R'_G(u_m) - R'_G(u)\|_{W^{-1,p'}(\Omega)} \leq \frac{1}{n}, \quad \forall n \in \mathbb{N}.$$

i.e.,

$$\|R'_G(u_m) - R'_G(u)\|_{W^{-1,p'}(\Omega)} = 0,$$

which proves (80). Then the map (79) is continuous. By property 6 of the generalized gradient,

$$\partial R(u) = \left\{ R'_G(u) \right\}.$$

■

3.2 Main result

Let $p > 1$ and $\Omega \subseteq \mathbb{R}^N$ be a smooth bounded domain. Consider

$$\Omega_+ := \left\{x \in \Omega \,/\, h(x) \geq 0\right\},$$
$$\Omega_- := \left\{x \in \Omega \,/\, h(x) < 0\right\};$$

$I : W_0^{1,p}(\Omega) \to \mathbb{R}$ given by

$$I(u) = \frac{1}{p}\int_\Omega |\nabla u(x)|^p dx - \int_\Omega q(x)u(x)dx - \int_\Omega F(u(x))h(x)dx,$$

where

$$F(u(x)) = \int_0^{u(x)} f(s)ds,$$

$f : \mathbb{R} \to \mathbb{R}$, $q \in L^{p'}(\Omega)$, and consider the following conditions:

$(\mathbf{F_1})$ There exists $a \in \mathbb{R}$ such that

a) $f \in C(\mathbb{R}\backslash\{a\}, \mathbb{R})$,
b) $f(a^-) < f(a^+)$,
c) $f(a) \in [f(a^-), f(a^+)]$.

$(\mathbf{F_2})$ There exist $\alpha, C_1, C_2 > 0$, with $1 + \alpha \in [p, p^\star]$, such that

$$\forall s \in \mathbb{R} : \quad |f(s)| \leq C_1 + C_2|s|^\alpha,$$

where,

$$p^\star = \begin{cases} \dfrac{np}{n-p}, & p < n, \\ +\infty, & \text{otherwise.} \end{cases}$$

$(\mathbf{H_2})$ $h \in L^\infty(\Omega)$.

We will show the following theorem.

Theorem 3.2.1. *Assume that $(\mathbf{F_1})$, $(\mathbf{F_2})$ and $(\mathbf{H_2})$ hold. Let $\hat{\phi}$ be the multivalued function given by,*

$$\hat{\phi}(x, s) = \begin{cases} \{h(x)f(s)\}, & s \neq a, \\ [h(x)f(a^-), h(x)f(a^+)], & s = a, x \in \Omega_+, \\ [h(x)f(a^+), h(x)f(a^-)], & s = a, x \in \Omega_-. \end{cases} \tag{81}$$

Then,

(1) An element $u \in W_0^{1,p}(\Omega)$ is a generalized critical point of the functional I if and only if

$$-\Delta_p u(x) - q(x) \in \hat{\phi}(x, u(x)), \text{ a.e. } \Omega.$$

(2) Let $u \in W_0^{1,p}(\Omega)$ be a generalized critical point of I. Suppose that

$$-q(x) \notin \left[\alpha^- \alpha^+\right], \text{ a.e. } \Omega,$$

where

$$\alpha^- = \min\{\alpha^-_{<0}, \alpha^-_{\geq 0}\},$$
$$\alpha^+ = \max\{\alpha^+_{<0}, \alpha^+_{\geq 0}\},$$

with

$$\alpha^-_{\geq 0} := \min\left\{m_+ f\left(a^-\right), M_+ f\left(a^-\right)\right\},$$
$$\alpha^+_{\geq 0} := \max\left\{m_+ f\left(a^+\right), M_+ f\left(a^+\right)\right\},$$
$$\alpha^-_{<0} := \min\left\{m_- f\left(a^+\right), M_- f\left(a^+\right)\right\},$$
$$\alpha^+_{<0} := \max\left\{m_- f\left(a^-\right), M_- f\left(a^-\right)\right\},$$

and

$$m_+ := \operatorname*{ess\,inf}_{x\in\Omega_+}(h(x)) \quad M_+ := \operatorname*{ess\,sup}_{x\in\Omega_+}(h(x))$$
$$m_- := \operatorname*{ess\,inf}_{x\in\Omega_-}(h(x)) \quad M_- := \operatorname*{ess\,sup}_{x\in\Omega_-}(h(x)).$$

Then we have that

$$\left|\{x \in \Omega \,/\, u(x) = a\}\right| = 0$$

and

$$-\Delta_p u(x) - q(x) = h(x)f(u(x)) \ \textit{a.e. } \Omega. \tag{82}$$

(3) Suppose that

(i) $|\Omega_-| = 0$ *and* I *has a point of local minimum at* $u \in W_0^{1,p}(\Omega)$ *or,*

(ii) $|\Omega_+| = 0$ *and* I *has a point of local maximum at* $u \in W_0^{1,p}(\Omega)$*,*

then the results in point (2) also hold.

Before the proof of Theorem 3.2.1, we establish the conditions to apply the generalized gradient to our problem. Recall ϕ and g as in Section 2.5.2. Take

$$\phi(x, s) := h(x)f(s),$$

and $g = J : L^{\alpha+1}(\Omega) \to \mathbb{R}$ such that

$$\begin{aligned} J(u) &= \int_\Omega \int_0^{u(x)} \phi(x, s)\,ds\,dx \\ &= \int_\Omega \int_0^{u(x)} h(x)f(s)\,ds\,dx \\ &= \int_\Omega h(x) \int_0^{u(x)} f(s)\,ds\,dx \\ &= \int_\Omega h(x)F(u(x))\,dx. \end{aligned} \tag{83}$$

Let

$$M = \|h\|_{L^\infty(\Omega)}.$$

Then, by $(\mathbf{F_2})$, for $(x, s) \in \Omega \times \mathbb{R}$ it holds

$$|\phi(x, s)| = |h(x)f(s)| \leq M|f(s)| \leq M\left(C_1 + C_2|s|^\alpha\right) = \tilde{C}_1 + \tilde{C}_2|s|^\alpha \text{ a.e. } \Omega,$$

i.e., condition (43) holds. From this, the conditions of Theorem 2.5.12 are verified. Also, by $(\mathbf{F_2})$,

$$p - 1 \leq \alpha \leq \frac{(p-1)N + p}{N - p}.$$

Hence, we can apply Corollary 2.5.16 to J. Thus, we have that J is locally Lipschitz on $L^{\alpha+1}(\Omega)$ and $W_0^{1,p}(\Omega)$, and

$$\partial J(u) \subseteq [\underline{\phi}(x, u(x)), \overline{\phi}(x, u(x))] \text{ a.e. } \Omega. \tag{84}$$

where (84) is understood as in Remark 2.5.13, i.e., given

$$w \in \partial g(u) \subseteq (L^{\alpha+1}(\Omega))^* \cong L^{(\alpha+1)/\alpha}(\Omega),$$

we have

$$\underline{\phi}(x, u(x)) \leq w(x) \leq \bar{\phi}(x, u(x)) \text{ for a.e. } x \in \Omega,$$

considering w as an element of $L^{(\alpha+1)/\alpha}(\Omega)$.

Now consider f at the point a.

(a) Let $x \in \Omega_+$. By condition b) of $(\mathbf{F_1})$, $f(a^-) < f(a^+)$. Since $h(x) \geq 0$, we have the following inequality

$$h(x)f(a^-) \leq h(x)f(a^+),$$

i.e., $h(x)f(\cdot)$ is nondecreasing near a.

(b) Let $x \in \Omega_-$. Let us define the function $\mathfrak{r} : \mathbb{R} \to \mathbb{R}$ such that

$$\mathfrak{r}(s) = -f(-s + 2a).$$

Note that

$$\begin{aligned}\mathfrak{r}(a^-) &= \lim_{\varepsilon \to 0^+} \mathfrak{r}(a - \varepsilon) = \lim_{\varepsilon \to 0^+} -f(-(a - \varepsilon) + 2a) = -f(a^+),\\ \mathfrak{r}(a^+) &= \lim_{\varepsilon \to 0^+} \mathfrak{r}(a + \varepsilon) = \lim_{\varepsilon \to 0^+} -f(-(a + \varepsilon) + 2a) = -f(a^-).\end{aligned}$$

then by condition b) of $(\mathbf{F_1})$ we have that

$$\mathfrak{r}(a^-) < \mathfrak{r}(a^+), \tag{85}$$

Also, since $h(x) < 0$

$$\begin{aligned}h(x)f(a^-) &= -|h(x)|f(a^-)\\ &= |h(x)|(-f(a^-))\\ &= |h(x)|\mathfrak{r}(a^+).\\ h(x)f(a^+) &= -|h(x)|f(a^+)\\ &= |h(x)|(-f(a^+))\\ &= |h(x)|\mathfrak{r}(a^-).\end{aligned} \tag{86}$$

So, by (85) we get

$$h(x)f(a^+) = |h(x)|\mathfrak{r}(a^-) < |h(x)|\mathfrak{r}(a^+) = h(x)f(a^-),$$

that is, $|h(x)|\mathfrak{r}(\cdot)$ is nondecreasing near a.

Hence, by (a) and (b), it follows from Remark 2.5.17 that,

$$\begin{aligned}\underline{\phi}(x, u(x)) &= \min\{h(x)(f(u(x)^-)), h(x)(f(u(x)^+))\}\\ &= \begin{cases} h(x)(f(u(x)^-)), & x \in \Omega_+,\\ h(x)(f(u(x)^+)), & x \in \Omega_-.\end{cases}\end{aligned}$$

and

$$\begin{aligned}\overline{\phi}(x, u(x)) &= \max\{h(x)(f(u(x)^-)), h(x)(f(u(x)^+))\}\\ &= \begin{cases} h(x)(f(u(x)^+)), & x \in \Omega_+,\\ h(x)(f(u(x)^-)), & x \in \Omega_-.\end{cases}\end{aligned}$$

Therefore

$$\partial J(u) \subseteq \hat{\phi}(x, u(x)) \text{ for a.e. } x \in \Omega.$$

Proof of Theorem 3.2.1. (I) Let's prove (1). Let $u \in W_0^{1,p}(\Omega)$. Consider

$$I(u) = Q(u) - J(u) + R(u),$$

where

$$Q(u) = \frac{1}{p}\int_\Omega \left|\nabla u(x)\right|^p dx,$$
$$J(u) = \int_\Omega h(x)F(u(x))dx,$$
$$R(u) = -\int_\Omega q(x)u(x)dx.$$

By properties 1 and 3 of the generalized gradient presented in Section 2.5 and Lemma 3.1.4, we have that,

$$\partial I(u) = \{A(u)\} - \partial J(u) + \{B(u)\}. \tag{87}$$

where we have denoted $Q'_G = A$ and $R'_G = B$.

By definition, $u \in W_0^{1,p}(\Omega)$ is a generalized critical point of I if and only if $0 \in \partial I(u)$ which, in its turn, it is equivalent to the existence of a function $\omega \in \partial J(u)$ such that,

$$A(u) - \omega + B(u) = 0,$$

or equivalently, i.e.,

$$A(u) + B(u) = \omega, \tag{88}$$

and

$$\omega(x) \in \hat{\phi}(x, u(x)) \text{ a.e. } \Omega. \tag{89}$$

Note that, due to Remark 2.5.15, we can consider ω both as a function in $L^{(\alpha+1)/\alpha}(\Omega) \cong (L^{\alpha+1}(\Omega))^*$ and as an element of $W^{-1,p'}(\Omega)$. Thus, by (88), for any $v \in W^{-1,p'}(\Omega)$ it holds

$$\langle A(u) + B(u), v\rangle = \langle \omega, v\rangle.$$

In consequence,

$$\int_\Omega |\nabla u(x)|^{p-2}\nabla u(x)\cdot\nabla v(x)dx - \int_\Omega q(x)v(x)dx = \int_\Omega w(x)v(x)dx, \quad \forall v \in W_0^{1,p}(\Omega).$$

Therefore,

$$\int_\Omega |\nabla u(x)|^{p-2}\nabla u(x)\cdot\nabla v(x)dx = \int_\Omega (q(x) + w(x))v(x)dx, \quad \forall v \in W_0^{1,p}(\Omega).$$

By (75), it holds formally

$$-\int_\Omega \Delta_p u(x)v(x)dx = \int_\Omega (q(x) + w(x))v(x)dx, \quad \forall v \in W_0^{1,p}(\Omega).$$

By the arbitrariness of v we have

$$-\Delta_p u(x) = w(x) + q(x) \text{ a.e. } \Omega.$$

Finally, by (89),

$$-\Delta_p u(x) - q(x) \in \hat{\phi}(x, u(x)) \text{ a.e. } \Omega.$$

Remark 3.2.2. By considering

$$I(u) = \frac{1}{p}\int_\Omega |\nabla u(x)|^p dx - \int_\Omega q(x)u(x)dx - \int_\Omega \int_0^{u(x)} \phi(x,s)dsdx,$$

and

$$\hat{\phi}(x,s) = \begin{cases} \{\phi(x,s)\}, & s \neq a, \\ [\underline{\phi}(x,s), \overline{\phi}(x,s)], & s = a, x \in \Omega, \end{cases}$$

with $\underline{\phi}$ and $\overline{\phi}$ given as in (55), and following the proof of part (I), we get that (1) in Theorem 3.2.1 is true for any function $\phi : \Omega \times \mathbb{R} \to \mathbb{R}$ satisfying

($\mathbf{PH_1}$) For some $a \in \mathbb{R}$,

a) $\phi(x, \cdot) \in C(\mathbb{R}\backslash\{a\}, \mathbb{R})$.

b) For each $x \in \Omega$:
$$\phi(x, a^-) \leq \phi(x, a^+).$$

c) For each $x \in \Omega$:
$$\phi(x, a) \in [\phi(x, a^-), \phi(x, a^+)].$$

($\mathbf{PH_2}$) There exist $\alpha, C_1, C_2 > 0$, with $1 + \alpha \in [p, p^\star]$, such that
$$\forall s \in \mathbb{R} : \quad |\phi(x, s)| \leq C_1 + C_2|s|^\alpha.$$

(II) Let's prove (2). Let $u \in W_0^{1,p}(\Omega)$ be a generalized critical point of I and
$$-q(x) \notin \left[\alpha^-, \alpha^+\right] \text{ a.e. } \Omega.$$

Let
$$\Gamma = \{x \in \Omega \,/\, u(x) = a\}.$$

Assume that $|\Gamma| > 0$. By part (1)
$$-\Delta_p u(x) - q(x) \in \hat{\phi}(x, u(x)) \text{ a.e. } \Omega.$$

Since $u(x) = a$ for $x \in \Gamma$, $\Delta_p u(x) = 0$ on Γ. Therefore, by definition of $\hat{\phi}$,
$$-q(x) \in \begin{cases} [h(x)f(a^-), h(x)f(a^+)], & \text{a.e. } \Omega_+ \cap \Gamma, \\ [h(x)f(a^+), h(x)f(a^-)], & \text{a.e. } \Omega_- \cap \Gamma. \end{cases}$$

Since
$$[h(x)f(a^-), h(x)f(a^+)] \subseteq \left[\alpha^-, \alpha^+\right] \text{ a.e. } \Omega_+ \cap \Gamma,$$

and
$$[h(x)f(a^+), h(x)f(a^-)] \subseteq \left[\alpha^-, \alpha^+\right] \text{ a.e. } \Omega_- \cap \Gamma,$$

then
$$-q(x) \in \left[\alpha^-, \alpha^+\right] \text{ a.e. } \Gamma. \tag{90}$$

Note that by condition $-q(x) \notin \left[\alpha^-, \alpha^+\right]$ a.e. Ω we have
$$\forall \mathcal{N} \subseteq \Omega : \quad |\mathcal{N}| = 0 \Rightarrow -q(\Omega \setminus \mathcal{N}) \subseteq \mathbb{R} \setminus \left[\alpha^-, \alpha^+\right]. \tag{91}$$

By equation (90),
$$\forall \mathcal{M} \subseteq \Gamma : \quad |\mathcal{M}| = 0 \Rightarrow -q(\Gamma \setminus \mathcal{M}) \subseteq \left[\alpha^-, \alpha^+\right]. \tag{92}$$

Since we assumed that $|\Gamma| > 0$ we have that
$$\forall \mathcal{M} \subseteq \Gamma, |\mathcal{M}| = 0 : \quad \Gamma \setminus \mathcal{M} \neq \emptyset.$$

Also, since
$$\Gamma \setminus \mathcal{M} \subseteq \Omega \setminus \mathcal{M},$$

we have, by (91), that
$$\emptyset \neq -q(\Gamma \setminus \mathcal{M}) \subseteq -q(\Omega \setminus \mathcal{M}) \subseteq \mathbb{R} \setminus \left[\alpha^-, \alpha^+\right],$$

which contradicts (92). Therefore, we conclude that $|\Gamma| = 0$. By definition of $\hat{\phi}$ we have
$$-\Delta_p u(x) - q(x) = h(x)f(u(x)) \text{ a.e. } \Omega \setminus \Gamma.$$

Since $|\Gamma| = 0$,
$$-\Delta_p u(x) - q(x) = h(x)f(u(x)) \text{ a.e. } \Omega.$$

(III) Let's prove (3). Consider

$$\Gamma = \{x \in \Omega \,/\, u(x) = a\}.$$

By Remark 2.5.10 in Section 2.5 we have that any point of local minimum or point of local maximum is a generalized critical point. Then, by following the same process of part (II) we get that,

$$-q(x) \in \begin{cases} [h(x)f(a^-), h(x)f(a^+)], & \text{a.e. } \Gamma \cap \Omega_+, \\ [h(x)f(a^+), h(x)f(a^-)], & \text{a.e. } \Gamma \cap \Omega_-. \end{cases} \tag{93}$$

Note that,

$$|\Gamma| = |\Gamma \cap \Omega| = |\Gamma \cap \Omega_+| + |\Gamma \cap \Omega_-|. \tag{94}$$

i) Let us prove (*3i*). Assume that $|\Omega_-| = 0$ and that $u \in W_0^{1,p}(\Omega)$ is a point of local minimum of I. We will show that

$$|\Gamma| = 0.$$

Since $|\Omega_-| = 0$, by (94),

$$|\Gamma| = |\Gamma \cap \Omega_+|, \tag{95}$$

and by (93),

$$-q(x) \in [h(x)f(a^-), h(x)f(a^+)] \text{ a.e. } \Gamma \cap \Omega_+. \tag{96}$$

Consider the set

$$\mathcal{E} = \left\{x \in \Omega \,/\, -q(x) \in [h(x)f(a^-), h(x)f(a^+)]\right\}.$$

Then

$$\Gamma \cap \Omega_+ \subseteq \mathcal{E} \cup \Lambda_1,$$

where $\Lambda_1 \subseteq (\Gamma \cap \Omega_+)$ is a set of measure zero such that relation (96) possibly does not hold. So, we have

$$\Gamma \cap \Omega_+ \subseteq ((\Gamma \cap \Omega_+) \cap \mathcal{E}) \cup \Lambda_1,$$

and then,

$$\Gamma \cap \Omega_+ = \left\{x \in \Gamma \cap \Omega_+ \,/\, -q(x) \in [h(x)f(a^-), h(x)f(a^+)]\right\} \cup \Lambda_1.$$

Let's observe that

$$\left\{x \in \Gamma \cap \Omega_+ \,/\, -q(x) \in [h(x)f(a_{k_0}^-),\, h(x)f(a_{k_0}^+)]\right\} \subseteq \Gamma_1 \cup \Gamma_2, \tag{97}$$

where

$$\Gamma_1 = \left\{x \in \Gamma \cap \Omega_+ \,/\, -q(x) > h(x)f(a^-)\right\},$$
$$\Gamma_2 = \left\{x \in \Gamma \cap \Omega_+ \,/\, -q(x) < h(x)f(a^+)\right\}.$$

(**a**) Let us show that $|\Gamma_1| = 0$. By the purpose of contradiction, assume that $|\Gamma_1| > 0$. Let ψ be a positive bounded function in $C_c^\infty(\Omega)$. Let

$$I_1(\varepsilon) = \frac{1}{p}\frac{|\nabla u(x) + \varepsilon \nabla\psi(x)|^p - |\nabla u(x)|^p}{\varepsilon}.$$

Therefore,

$$\lim_{\varepsilon\to 0^+} I_1(\varepsilon) = \frac{1}{p}\lim_{\varepsilon\to 0^+}\frac{|\nabla u(x) + \varepsilon\nabla\psi(x)|^p - |\nabla u(x)|^p}{\varepsilon} = \frac{1}{p}\frac{d}{d\varepsilon}|\nabla u(x)+\varepsilon\nabla\psi(x)|^p\bigg|_{\varepsilon=0} = |\nabla u(x)|^{p-2}\nabla u(x)\nabla\psi(x).$$

Let

$$I_2(\varepsilon) = \frac{F(u(x)+\varepsilon\psi(x))h(x) - F(u(x))h(x)}{\varepsilon}.$$

Hence,

$$\begin{aligned}\lim_{\varepsilon\to 0^+} I_2(\varepsilon) &= \lim_{\varepsilon\to 0^+} \frac{F(u(x)+\varepsilon\psi(x))-F(u(x))}{\varepsilon} h(x)\\ &= \lim_{\varepsilon\to 0^+} \frac{d}{d\varepsilon}\left(F(u(x)+\varepsilon\psi(x))\right) h(x)\\ &= \lim_{\varepsilon\to 0^+} f(u(x)+\varepsilon\psi(x))\psi(x)h(x)\\ &= f(u(x)^+)\psi(x)h(x).\end{aligned}$$

Since u is a point of local minimum and ψ is positive, by applying the Dominated Convergence Theorem, as in Lemma 3.1.2, we get

$$\begin{aligned}0 \le \lim_{\varepsilon\to 0^+} \frac{I(u+\varepsilon\psi)-I(u)}{\varepsilon} &= \lim_{\varepsilon\to 0^+} \frac{1}{\varepsilon}\left[\frac{1}{p}\int_\Omega \left(\left|\nabla u(x)+\varepsilon\nabla\psi(x)\right|^p - |\nabla u(x)|^p\right)dx\right.\\ &\quad \left.-\int_\Omega (F(u(x)+\varepsilon\psi(x))h(x)-F(u(x))h(x))dx - \int_\Omega \varepsilon q(x)\psi(x)dx\right]\\ &= \lim_{\varepsilon\to 0^+}\left(\int_\Omega I_1(\varepsilon)dx - \int_\Omega I_2(\varepsilon)dx - \int_\Omega q(x)\psi(x)dx\right)\\ &= \int_\Omega |\nabla u(x)|^{p-2}\nabla u(x)\cdot\nabla\psi(x)dx - \int_\Omega f(u(x)^+)h(x)\psi(x)dx - \int_\Omega q(x)\psi(x)dx\\ &= \int_\Omega -\Delta_p u(x)\psi(x)dx - \int_\Omega f(u(x)^+)h(x)\psi(x)dx - \int_\Omega q(x)\psi(x)dx\\ &= -\int_\Omega (\Delta_p u(x) + f(u(x)^+)h(x) + q(x))\psi(x)dx. \end{aligned} \tag{98}$$

In addition, f is continuous for $x \in \Omega\setminus\Gamma$ by condition a) of $(\mathbf{F_1})$, then

$$f(u(x)^+) = \lim_{\varepsilon\to 0^+} f(u(x)+\varepsilon) = f(u(x)).$$

It follows that,

$$-\Delta_p u(x) - q(x) = f(u(x))h(x) \qquad \text{a.e. } \Omega\backslash\Gamma.$$

So, integral (98) is zero on $\Omega\setminus\Gamma$. Hence,

$$\begin{aligned}\lim_{\varepsilon\to 0^+} \frac{I(u+\varepsilon\psi)-I(u)}{\varepsilon} &= -\int_\Omega \left(\Delta_p u(x) + f(u(x)^+)h(x) + q(x)\right)\psi(x)dx\\ &= -\int_\Gamma (f(a^+)h(x)+q(x))\psi(x)dx\\ &= -\left(\int_{\Gamma\cap\Omega_+} (f(a^+)h(x)+q(x))\psi(x)dx + \int_{\Gamma\cap\Omega_-} (f(a^+)h(x)+q(x))\psi(x)dx\right) \end{aligned} \tag{99}$$

$$= -\int_{\Gamma\cap\Omega_+} (f(a^+)h(x)+q(x))\psi(x)dx.$$

Since $|\Gamma_1| > 0$ and $0 < q(x) + h(x)f(a^+)$ on Γ_1, we conclude that

$$0 \le -\int_{\Gamma\cap\Omega_+} (f(a^+)h(x)+q(x))\psi(x)dx < 0,$$

which is a contradiction. Therefore, $|\Gamma_1| = 0$.

(**b**) Let us prove that $|\Gamma_2| = 0$. Again by contradiction, consider $|\Gamma_2| > 0$ and ψ a positive bounded function in $C_c^\infty(\Omega)$. Let

$$\overline{I_1}(\varepsilon) = \frac{1}{p}\frac{\left|\nabla u(x) - \varepsilon\nabla\psi(x)\right|^p - |\nabla u(x)|^p}{\varepsilon}.$$

Therefore,

$$\lim_{\varepsilon\to 0^+} \overline{I_1}(\varepsilon) = \frac{1}{p}\lim_{\varepsilon\to 0^+}\frac{|\nabla u(x) - \varepsilon\nabla\psi(x)|^p - |\nabla u(x)|^p}{\varepsilon} = -|\nabla u(x)|^{p-2}\nabla u(x)\nabla\psi(x).$$

Let

$$\overline{I_2}(\varepsilon) = \frac{F(u(x) - \varepsilon\psi(x))h(x) - F(u(x))h(x)}{\varepsilon}.$$

Hence,

$$\lim_{\varepsilon\to 0^+} \overline{I_2}(\varepsilon) = -f(u(x)^-)\psi(x)h(x).$$

Similar to the case of Γ_1 we have

$$\begin{aligned}
0 \le \lim_{\varepsilon\to 0^+}\frac{I(u-\varepsilon\psi) - I(u)}{\varepsilon} &= \lim_{\varepsilon\to 0^+}\left(\int_\Omega \overline{I_1}(\varepsilon)dx - \int_w \overline{I_2}(\varepsilon)dx + \int_\Omega q(x)\psi(x)dx\right)\\
&= \int_\Omega (\Delta_p u(x) + f(u(x)^-)h(x) + q(x))\psi(x)dx\\
&= \int_\Gamma (f(a^-)h(x) + q(x))\psi(x)dx\\
&= \int_{\Gamma\cap\Omega_+} (f(a^-)h(x) + q(x))\psi(x)dx + \int_{\Gamma\cap\Omega_-} (f(a^-)h(x) + q(x))\psi(x)dx \qquad (100)\\
&= \int_{\Gamma\cap\Omega_+} (f(a^-)h(x) + q(x))\psi(x)dx.
\end{aligned}$$

Since $-q(x) > f(a^-)h(x)$ on Γ_2 and $|\Gamma_2| > 0$

$$0 \le \int_{\Gamma\cap\Omega_+} (f(a^-)h(x) + q(x))\psi(x)dx < 0.$$

So $|\Gamma_2| = 0$. Therefore by (97), $|\Gamma| = 0$. By following the same procedure of part (II), it follows that,

$$-\Delta_p u(x) - q(x) = h(x)f(u(x)) \text{ a.e. } \Omega.$$

ii) Let us prove (*3ii*). Assume that $|\Omega_+| = 0$ and that $u \in W_0^{1,p}(\Omega)$ is a point of local maximum of I. We follow the same process as for (*3i*). We want to prove that $|\Gamma| = 0$. Since $|\Omega_+| = 0$, by (93) and (94) it follows

$$|\Gamma| = |\Gamma\cap\Omega_-|, \tag{101}$$

and

$$-q(x) \in [h(x)f(a^+), h(x)f(a^-)] \text{ a.e. } \Gamma\cap\Omega_-. \tag{102}$$

Consider the set,

$$\mathcal{C} = \left\{x\in\Omega \,/\, -q(x) \in [h(x)f(a^+), h(x)f(a^-)]\right\}.$$

As in (*3i*) we get that,

$$\Gamma\cap\Omega_- = \left\{x\in\Gamma\cap\Omega_- \,/\, -q(x) \in [h(x)f(a^+), h(x)f(a^-)]\right\}\cup\Lambda_2,$$

with $\Lambda_2 \subseteq \Gamma\cap\Omega_-$ begin the set of null measure containing the points where the relation (102) possibly does not hold. So,

$$\left\{x\in\Gamma\cap\Omega_- \,/\, -q(x) \in [h(x)f(a^+), h(x)f(a^-)]\right\} \subseteq \Gamma_3\cup\Gamma_4, \tag{103}$$

where

$$\Gamma_3 = \left\{x \in \Gamma \cap \Omega_- /\ -q(x) > h(x)f(a^+)\right\},$$
$$\Gamma_4 = \left\{x \in \Gamma \cap \Omega_- /\ -q(x) < h(x)f(a^-)\right\}.$$

(**c**) Let us proof that $|\Gamma_3| = 0$. By purpose of contradiction we suppose $|\Gamma_3| > 0$. From equation (99) and since u is a point of local maximum, we have

$$0 \geq \lim_{\varepsilon \to 0^+} \frac{I(u+\varepsilon\psi) - I(u)}{\varepsilon} = -\int_{\Gamma \cap \Omega_-} (f(a^+)h(x) + q(x))\psi(x)dx.$$

Since $f(a^+)h(x) + q(x) < 0$ on $|\Gamma_3| > 0$, then

$$0 \geq -\int_{\Gamma \cap \Omega_-} (f(a^+))h(x) + q(x))\psi(x)dx > 0.$$

Thus $|\Gamma_3| = 0$.

(**d**) By applying the same reasoning using equation (100), we prove that $|\Gamma_4| = 0$. By (103) we get that $|\Gamma| = 0$. Therefore,

$$-\Delta_p u(x) - q(x) = h(x)f(u(x)) \text{ a.e. } \Omega.$$

■

Remark 3.2.3. The proof of Theorem 3.2.1 considered the following steps:

1) In point (1) of Theorem 3.2.1 we computed the generalized gradient of I by using Chang's machinery,

$$\partial I(u) = \{A(u)\} - \partial J(u) + \{B(u)\}.$$

Then, since $u \in W_0^{1,p}(\Omega)$ was a generalized critical point and $\partial J(u) \subseteq \hat{\phi}(x, u(x))$, after some computations, we got that u is a multivalued solution of (PP) and vice-versa.

Moreover, we realized that the proof of point (1) can be applied to any function ϕ under the conditions ($\mathbf{PH_1}$) and ($\mathbf{PH_2}$).

2) In point (2) of Theorem 3.2.1 we computed the interval $\left[\alpha^-, \alpha^+\right]$ by realizing that

$$\min\{m_+ f\left(a^-\right), M_+ f\left(a^-\right)\} \leq h(x)f(a^-) \leq h(x)f(a^+) \leq \max\{m_+ f\left(a^+\right), M_+ f\left(a^+\right)\}, \text{ a.e. } \Omega_+$$
$$\min\{m_- f\left(a^+\right), M_- f\left(a^+\right)\} \leq h(x)f(a^+) < h(x)f(a^-) \leq \max\{m_- f\left(a^-\right), M_- f\left(a^-\right)\}, \text{ a.e. } \Omega_-,$$

i.e.,

$$\alpha^-_{\geq 0} \leq h(x)f(a^-) \leq h(x)f(a^+) \leq \alpha^+_{\geq 0}, \text{ a.e. } \Omega_+$$
$$\alpha^-_{<0} \leq h(x)f(a^+) < h(x)f(a^-) \leq \alpha^+_{<0}, \text{ a.e. } \Omega_-,$$

By a proof based in the purpose of contradiction and by using facts of measure theory, we proved that $|\Gamma| = 0$, which implies that u is an almost everywhere weak solution of (PP).

3) Point (3) of Theorem 3.2.1 had to be divided into two problems, i) and ii), since $-q(x)$ does not belong to a unique interval on Γ as in [4] and [8]. In fact,

$$\begin{aligned} -q(x) &\in [h(x)f(a^-), h(x)f(a^+)], \quad \text{a.e. } \Gamma \cap \Omega_+, \\ -q(x) &\in [h(x)f(a^+), h(x)f(a^-)], \quad \text{a.e. } \Gamma \cap \Omega_-. \end{aligned} \tag{104}$$

To prove i), we showed that $|\Gamma| = 0$. Since $|\Omega_-| = 0$, to prove that $|\Gamma| = 0$ is equivalent to prove that $|\Gamma_1| = 0$ and $|\Gamma_2| = 0$. We proceed to show that $|\Gamma_1| = 0$ by the purpose of contradiction, i.e., by assuming that $|\Gamma_1| > 0$. From this assumption and since u was a point of local minimum, we got the following contradiction

$$0 \leq -\int_{\Gamma \cap \Omega_+} (f(a^+)h(x) + q(x))\psi(x)dx < 0,$$

which implies that $|\Gamma_1| = 0$. We followed the same process to show that $|\Gamma_2| = 0$.

Next, we stated ii) by realizing that it was possible to get a result analogous to i) by changing the condition of point of local minimum by the one of point of local maximum. To prove ii), we followed the same approach given in i).

Something important to mention is that we had to assume $|\Omega_-| = 0$ in i) and $|\Omega_+| = 0$ in ii) to get the desired contradictions from equations (99) and (100).

3.3 Application

In this subsection we present an application of our main result. Consider the following conditions, which are particular cases of $(\mathbf{H_2})$ and $(\mathbf{F_2})$, respectively.

$(\mathbf{H_3})$ $h \in L^\infty(\Omega)$ such that $|\Omega_-| = 0$ and $\|h\|_{L^\infty(\Omega)} = M \neq 0$.

$(\mathbf{F_3})$ $\exists \delta, \rho > 0, \forall s \in \mathbb{R}: \quad |f(s)| \leq \delta |s|^{p-1} + \rho$,

where

$$\delta < \frac{\lambda_1}{M},$$

with λ_1 being the first eigenvalue of the p-Laplacian (see Section 2.6).

Remark 3.3.1. In [8] it was assumed that $\operatorname{ess\,inf}_{x\in\Omega} h(x) > 0$, which is a particular case of $(\mathbf{H_3})$.

We have the following result.

Theorem 3.3.2. *If conditions* $(\mathbf{F_1})$, $(\mathbf{F_3})$ *and* $(\mathbf{H_3})$ *are verified, then*

$$\begin{cases} -\Delta_p u(x) = h(x) f(u(x)) + q(x), & x \in \Omega, \\ u(x) = 0, & x \in \partial\Omega, \end{cases}$$

has an a.e. weak solution $u \in W_0^{1,p}(\Omega)$.

Proof. (i) By the characterization of λ_1, formula (62),

$$0 < \lambda_1 = \inf_{u \in W_0^{1,p}(\Omega), u \neq 0} \left\{ \frac{\displaystyle\int_\Omega |\nabla u(x)|^p dx}{\displaystyle\int_\Omega |u(x)|^p dx} \right\}. \tag{105}$$

So we have

$$\forall u \in W_0^{1,p}(\Omega)\backslash\{0\}: \quad \lambda_1 \leq \frac{\displaystyle\int_\Omega |\nabla u(x)|^p dx}{\displaystyle\int_\Omega |u(x)|^p dx}.$$

Then

$$\lambda_1 \int_\Omega |u(x)|^p dx \leq \int_\Omega |\nabla u(x)|^p dx. \tag{106}$$

By Hölder inequality we have

$$\begin{aligned} \int_\Omega |q(x)u(x)| dx &\leq \left(\int_\Omega |q(x)|^{p'} dx \right)^{1/p'} \left(\int_\Omega |u(x)|^p dx \right)^{1/p} \\ &= \|q\|_{L^{p'}(\Omega)} \|u\|_{L^p(\Omega)}. \end{aligned} \tag{107}$$

Also

$$\int_\Omega h(x) F(u(x)) dx \leq M \int_\Omega |F(u(x))| dx. \tag{108}$$

From (107) and (108) we have

$$\begin{aligned} I(u) &= \frac{1}{p}\int_\Omega \left|\nabla u(x)\right|^p dx - \int_\Omega h(x)F(u(x))dx - \int_\Omega q(x)u(x)dx \\ &\geq \frac{1}{p}\|\nabla u\|^p_{L^p(\Omega)} - \|q\|_{L^{p'}(\Omega)}\|u\|_{L^p(\Omega)} - M\int_\Omega |F(u(x))|dx. \end{aligned} \tag{109}$$

Now consider $(\mathbf{F_3})$. By integrating we get

$$|F(u(x))| = \int_0^{u(x)} |f(s)|ds \leq \delta\int_0^{u(x)} |s|^{p-1}ds + \rho\int_0^{u(x)} ds = \frac{\delta}{p}|u(x)|^p + \rho u(x). \tag{110}$$

Hence

$$\begin{aligned} \int_\Omega |F(u(x))|dx &\leq \int_\Omega \left|\frac{\delta}{p}|u(x)|^p + \rho u(x)\right| dx \\ &\leq \frac{\delta}{p}\int_\Omega |u(x)|^p dx + \int_\Omega \rho|u(x)|dx \\ &\leq \frac{\delta}{p\lambda_1}\int_\Omega |\nabla u(x)|^p dx + \rho\int_\Omega |u(x)|dx \quad (\text{ by (106)}). \end{aligned} \tag{111}$$

Applying once more Hölder inequality

$$\int_\Omega |u(x)|dx \leq (\Omega)^{1/p'}\|u\|_{L^p(\Omega)}.$$

So, for

$$\tilde{k} = M\rho(\Omega)^{1/p'},$$

we get

$$M\int_\Omega |F(u(x))|dx \leq \frac{M\delta}{p\lambda_1}\int_\Omega |\nabla u(x)|^p dx + \tilde{k}\|u\|_{L^p(\Omega)}. \tag{112}$$

From (109) and (112)

$$I(u) \geq \frac{1-(M\delta)/\lambda_1}{p}\|\nabla u\|^p_{L^p(\Omega)} - k\|u\|_{L^p(\Omega)},$$

where $k = \|q\|_{L^{p'}(\Omega)} + \tilde{k} > 0$. Since

$$\begin{aligned} 0 < \delta < \frac{\lambda_1}{M} \quad &\Longrightarrow \quad \frac{M\delta}{\lambda_1} < 1 \\ &\Longrightarrow \quad \frac{1 - M\alpha/\lambda_1}{p} > 0, \end{aligned}$$

then conditions presented in (31) hold for I, i.e. I is coercive.

(ii) Let's prove that I is weakly lower semicontinuous. Let $(u_m)_{m\in\mathbb{N}} \subseteq W_0^{1,p}(\Omega)$, $u \in W_0^{1,p}(\Omega)$ such that $u_m \rightharpoonup u$. We have

$$I(u) = Q(u) - J(u) + R(u),$$

where

$$\begin{aligned} Q(u) &= \frac{1}{p}\int_\Omega \left|\nabla u(x)\right|^p dx, \\ J(u) &= \int_\Omega h(x)F(u(x))dx, \\ R(u) &= -\int_\Omega q(x)u(x)dx. \end{aligned}$$

Since Q is a norm, it is weakly lower semicontinuous ($w.l.s$), by Remark 2.4.11. Also, since

$$q \in L^{p'}(\Omega) \cong (L^p(\Omega))^* \subseteq W^{-1,p'}(\Omega),$$

by the characterization of weakly convergence, presented in Section 2.1, equation (2), we get that

$$R(u_m) \to R(u), \text{ as } m \to +\infty.$$

For the case of J, note that

$$-h(x)f(s) \leq |h(x)f(s)| \leq \delta M|s|^{p-1} + \rho M, \text{ a.e. } \Omega. \tag{113}$$

Hence by (110) and (113)

$$\begin{aligned}
-J(u) = -\int_\Omega h(x)F(u(x))dx &\leq \frac{\delta M}{p}\int_\Omega |u(x)|^p dx + \rho M\int_\Omega u(x)dx \\
&= \frac{\delta M}{p}\|u\|^p_{L^p(\Omega)} + \rho M\int_\Omega u(x)dx.
\end{aligned}$$

Note that $P : W_0^{1,p}(\Omega) \to \mathbb{R}$, given by

$$P(u) = \int_\Omega u(x)dx,$$

belongs to $W^{-1,p'}(\Omega)$, due to Hölder inequality. So, by characterization of weakly convergence, equation (2),

$$\int_\Omega u_m(x)dx \to \int_\Omega u(x)dx, \text{ as } m \to +\infty.$$

Then J is $w.l.s.$. It follows that

$$I(u) \leq \lim_{m\to\infty} \inf I(u_m).$$

By Theorem 2.4.16 we conclude that u is a point of global minimum. Finally, by applying (*3i*) of our main result, we get the desired result.

■

4 Conclusions and recommendations

4.1 Conclusions

In this work we studied the following boundary value problem involving the p-Laplacian:

$$\begin{cases} -\Delta_p u(x) = h(x)f(u(x)) + q(x), & x \in \Omega, \\ u(x) = 0, & x \in \partial\Omega, \end{cases} \tag{PP}$$

where $p > 1$, $\Omega \subseteq \mathbb{R}^N$ is a smooth bounded domain, $q \in L^{p'}(\Omega)$, $h \in L^\infty(\Omega)$, and $f : \mathbb{R} \to \mathbb{R}$ is a discontinuous function at a point $a \in \mathbb{R}$, satisfying:

($\mathbf{F_1}$) a) $f \in C(\mathbb{R}\backslash\{a\}, \mathbb{R})$,
b) $f(a^-) < f(a^+)$,
c) $f(a) \in [f(a^-), f(a^+)]$,

($\mathbf{F_2}$) there exist $\alpha, C_1, C_2 > 0$, with $1 + \alpha \in [p, p^\star]$, such that

$$\forall s \in \mathbb{R} : \quad |f(s)| \leq C_1 + C_2|s|^\alpha,$$

where

$$p^\star = \begin{cases} \dfrac{Np}{N-p}, & p < N, \\ +\infty, & \text{otherwise.} \end{cases}$$

By calculating the generalized gradient to the energy functional $I : W_0^{1,p}(\Omega) \to \mathbb{R}$ of (PP),

$$I(u) = \frac{1}{p}\int_\Omega |\nabla u(x)|^p dx - \int_\Omega q(x)u(x)dx - \int_\Omega \int_0^{u(x)} f(s)h(x)dsdx,$$

and applying Chang's machinery, we have shown that:

1. Any generalized critical point u of I is a multivalued solution of (PP), i.e., $0 \in \partial I(u)$ is equivalent to

$$-\Delta_p u(x) - q(x) \in \hat{\phi}(x, u(x)), \text{ a.e. } \Omega.$$

Moreover, this fact is true if, instead of considering the particular case $h(x)f(s)$, we consider $\phi(x, s)$, where $\phi : \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$ is any function such that:

($\mathbf{PH_1}$) For some $a \in \mathbb{R}$,
a) $\phi(x, \cdot) \in C(\mathbb{R}\backslash\{a\}, \mathbb{R})$.
b) For each $x \in \Omega$:
$$\phi(x, a^-) \leq \phi(x, a^+).$$
c) For each $x \in \Omega$:
$$\phi(x, a) \in [\phi(x, a^-), \phi(x, a^+)].$$

($\mathbf{PH_2}$) There exist $\alpha, C_1, C_2 > 0$, with $1 + \alpha \in [p, p^\star]$, such that

$$\forall s \in \mathbb{R} : \quad |\phi(x, s)| \leq C_1 + C_2|s|^\alpha.$$

hold.

2. A generalized critical point u of I is an almost everywhere weak solution of (PP) if the function q does not take values in the interval $[\alpha^-, \alpha^+]$ a.e. on the set Ω.

3. If u is a point of local minimum and $|\Omega_-| = 0$, or if u is a point of local maximum and $|\Omega_+| = 0$, then u is an almost everywhere weak solution of (PP).

4. By considering that

   ($\mathbf{H_3}$) $h \in L^{\infty}(\Omega)$ such that $|\Omega_-| = 0$ and $\|h\|_{L^{\infty}(\Omega)} = M \neq 0$.

   ($\mathbf{F_3}$) $\exists \delta, \rho > 0, \forall s \in \mathbb{R}: \quad |f(s)| \leq \delta |s|^{p-1} + \rho,$

   we have shown that I is weakly lower semicontinuous and coercive. Therefore we concluded that I has a global point of minimum and then (PP) has an almost everywhere weak solution.

The development of this work used many topics and tools studied during the career of mathematics of Yachay Tech: Calculus of Variations, Functional Analysis, Partial Differential Equations, and Measure Theory. It is important to note that a lot of the material necessary to produce the results goes beyond the standard curriculum. Concepts like generalized gradient, topics in Nonlinear Analysis, and some concepts of Measure theory were studied independently to understand and solve the problem.

4.2 Recommendations

a) In my opinion, the authorities of the School should organize periodically seminars related to the work that each professor is doing. This would help students to determine which fields of mathematics attract them, and even to start working on research projects before entering to the last two semesters of the career.

b) Also, I think that it is important to offer at least one more class related to the field of Algebra between the optional subjects. I consider that it is a key field for students that have more interest in pure mathematics.

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