UNIVERSIDAD DE INVESTIGACIÓN DE TECNOLOGÍA EXPERIMENTAL YACHAY

Escuela de Ciencias Matemáticas y Computacionales

TÍTULO: Numerical monoids, numerical operads and applications to combinatorics.

Trabajo de integración curricular presentado como requisito para la obtención del título de Matemático

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For those who, without really knowing what a mathematician even is or does, never stopped believing I could become one.

María Cristina Sabando Álvarez

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Resumen

En este trabajo consideramos monoides cancelativos (c-monoides), en la construcción de conjuntos parcialmente ordenados. En particular, aplicamos esta construcción general a los submonoides de $\mathbb{N}$. Asociado a un conjunto parcialmente ordenado localmente finito se estudia clásicamente lo que se denomina el álgebra de incidencia. Toda álgebra de incidencia posee dos elementos especiales e invariantes bajo isomorfismo de conjuntos parcialmente ordenados, la funcion zeta y su inversa, la función de Möbius. Presentamos y probamos resultados usando argumentos combinatorios, funciones generatrices y la función de Möbius asociada a conjuntos parcialmente ordenados. Recientemente, se introdujeron los $^+1$-monoides para el estudio de particiones ordenadas. Usando las propiedades de los $^+1$-monoides construimos una nueva familia de conjuntos parcialmente ordenados. La función generatriz de Möbius de cada uno de estos conjuntos es la inversa (respecto a la composición de series formales) de la función generatriz de su función zeta. Estos resultados nos permiten obtener una nueva derivación para los números de Fuss-Catalán con signos alternantes. Extendemos dicha construcción a c-monoides que surgen del producto ordinal de $\mathscr{L}$ especies y a c-operads, los cuales son también monoides, pero asociados a la sustitución ordinal de $\mathscr{L}$ especies. Finalmente, probamos que la restricción de un operad a los conjuntos con cardinal en un $^+1$-monoide es también un operad, es decir que la ley de composición del operad restringida al $^+1$-monoide está bien definida.

Palabras clave: Monoides numéricos, Operads numéricos, Función de Möbius, L especies, Posets asociados

Abstract

In this work we consider cancellative monoids (c-monoids), in the construction of partially ordered sets (posets). In particular, we apply this general construction to the submonoids of $\mathbb{N}$. In association with a locally finite poset, the incidence algebra is classically studied. Every incidence algebra possesses two elements which are special and invariant under poset isomorphisms, the zeta function and its inverse, the Möbius function. We present and prove results using combinatorial arguments, generating functions and the Möbius function associated to posets. Recently, $^+1$-monoids were introduced for the study of ordered set partitions. Using the properties of $^+1$-monoids, we construct a new family of posets. The Möbius generating function of each of these posets is the inverse (with respect to the composition of formal power series) of its zeta generating function. Those results allow us to obtain new derivation of the Fuss-Catalan numbers with alternating signs. We extend this construction to c-monoids arising from the ordinal product of $\mathscr{L}$-species and to c-operads. Operads are also monoids, but associated to the ordinal substitution of $\mathscr{L}$-species. Finally, we prove that the restriction of an operad to sets whose cardinal is in a $^+1$ monoid is also an operad. That is, we prove that the law of composition of the operad restricted to a $^+1$ monoid is well defined.

Keywords: Numerical monoids, Numerical operads, Generalized Möbius function, L-species, Associated posets

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Notation

Posets

$I(P)$	Incidence algebra of P
P_M	Poset associated to the operad (or monoid) M
μ_P	Möbius function of the poset P
$\text{Möb}P_M[n]$	Möbius cardinal
$\mathbb{C}[\![M]\!]$	Algebra of formal series of the monoid M
ζ_M	Zeta function associated to the c-monoid M
μ_M	Möbius function associated to the c-monoid M
$\zeta_M^{\langle -1\rangle}$	Compositional inverse of the zeta function associated to a c-monoid M
$\mathbf{C}(n)$	Chain of length n
$B(n)$	Boolean poset on n elements

Categories

$\mathbb{F}$	Category of finite sets and arbitrary functions
$\mathscr{L}$	Category of totally ordered sets and order preserving bijections
$\mathrm{Vec}_{\mathbb{K}}$	Category of finite-dimensional vector spaces (over the field $\mathbb{K}$) and linear maps
Set	Category of sets and total functions

Species

0	Empty species
1	Singleton indicator
X	Singleton structures
E	Sets, the uniform species

E_+	Non-empty sets
$\mathbb{S}$	Permutations
$\mathbb{L}$	Linear orders
Π	Partitions
NCP	Non-crossing partitions
$\mathfrak{K}$	Compositions
$\mathscr{C}$	Cyclic permutations
C	Connected permutations
$\mathscr{A}$	Rooted trees
$\mathscr{A}_E$	Ordered trees
$\mathscr{A}_{\mu(n)}$	Weighted rooted trees
$E\langle\mathscr{A}_E\rangle$	Lists of trees in ascending order
M_S	Restriction of M to S
M_+	Positive species for M
$M\langle N\rangle$	Ordinal substitution of $\mathscr{L}$-species
$M \diamond N$	Ordinal product of $\mathscr{L}$-species
$M.N$	Shuffle product $\mathscr{L}$-species
$M(N)$	Shuffle substitution of $\mathscr{L}$-species
$M^{\blacktriangledown}$	Ordinal derivative $\mathscr{L}$-species
M'	Shuffle derivative of $\mathscr{L}$-species
$M^{-1}(x)$	Reciprocal of $M(x)$
$M^{\langle -1\rangle}(x)$	Compositional inverse of $M(x)$

Chapter 1

Introduction

Enumerative combinatorics is a branch of combinatorics concerned with counting the number of combinatorial structures on a finite set such as prime permutations, pattern avoiding permutations[1], and non-crossing partitions. This counting is done, mostly, through calculations and manipulations of generating functions, which are formal power series representing a counting function. The enumeration of these combinatorial structures is fuelled by an interest in their wide range of applications.

Prime permutations, those which don't map intervals into intervals (except the trivial ones), together with pattern avoiding permutations have been helpful for their applications in genetics, as a tool for the comparative study of genomes [BHS02; Bé+05]. In computer science, to check whether a permutation can be sorted by a stack [AA05; Knu97]. Non-crossing partitions are of importance in the field of free probability [NS06; Voi97] on which they are used to recover the moments of non-commutative random variables. Not only are combinatorial objects of interest for their applications but their representations, such as planar trees or paths, are useful in other disciplines.

Perhaps one of the most recurrent, if not the most fascinating, sequence of positive integers in combinatorics is that of Catalan numbers, named after mathematician Eugène Charles Catalan, which are the solution to the problem of counting structures such as binary trees, non-crossing partitions, stack sortable permutations, standard Young tableaux, amongst many others which have been cataloged by the likes of Stanley [Sta11] and many others. A generalization of the Catalan numbers known as the Fuss-Catalan numbers are also of interest in the context of free probability.

As with other branches of mathematics, the study of combinatorics has been aided by powerful tools like partially ordered sets (posets from now on) and combinatorial species. We can associate to a locally finite poset an algebra, called the incidence algebra. Two prominent members of the incidence algebras are the zeta function, and its inverse in the (convolution) product of the algebra, which is called the Möbius function.

The Möbius function used in our environment is a natural generalization of the classical, number theoretical one, associated with the poset of positive integers numbers ordered by the divisibility relation. The classical Riemann's zeta function can be expanded as a series of Dirichlet type. The coefficients of its inverse Dirichlet series are given by the classical Möbius function.

[1] Atk99.

The study of the Möbius function as a tool in combinatorics was systematically developed by Gian-Carlo Rota's in his seminal work on the Foundations of Combinatorial Theory I [Rot64] in the mid 60's, when it caught interest for its properties related to those of the underlying poset. In particular, the Möbius function of a poset gives, as a result, the generating function (generally, alternating in signs) of important structures on finite sets.

Combinatorial species, introduced by Joyal [Joy81], provide a better understanding of the use of generating functions for combinatorial structures and ease their analysis, as calculations on generating functions have a natural analogous to operations in species. $\mathscr{L}$-species are a family of species in which subjacent sets are totally ordered.

The current work is concerned with the building of posets of a special kind of set theoretical monoids called c-monoids. We apply a general procedure, introduced in [MY91], to construct the associated poset to submonoids of $\mathbb{N}$, including numerical monoids, and $^{+}1$ monoids. The Möbius function associated to the posets obtained by this construction gives rise to generating functions with interesting combinatorial meanings.

Numerical monoids are of interest in the study of algebraic curves. While $^{+}1$ monoids, which are a recent construction (see [BMR20]), have applications in the theory of set operads. As such, we extend and generalize our construction in the broader context of Category Theory and that of $\mathscr{L}$-species through monoids and operads.

In Chapter 2 we give a simple overview of monoids in order to define c-monoids and establish the construction of an associated poset P_M to a c-monoid M, with the help of the incidence algebra for a poset P. We establish and prove some properties of P_M and then proceed to go over the Möbius function and how to calculate it in some cases. This construction is then applied to numerical monoids and $^{+}1$ monoids and the results get interpreted combinatorially. Chapter 3 delves into the necessary Category theory content the reader must be familiar with to follow the definition of generating functions and $\mathscr{L}$-species together with its operations. Classical enumerative results are provided together with some figures to help with understanding.

Finally, Chapter 4 deals with the generalization of the construction proposed in Chapter 2 to Category theory. In particular, we make a distinction between monoids arising from the ordinary product of $\mathscr{L}$-species and between operads, those monoids arising from the ordinary substitution of $\mathscr{L}$-species. The extended construction for c-monoids are carried on without any more special consideration. While for operads, we prove that the construction is well defined.

Chapter 2

Monoids

In this chapter, we give a brief review on set monoids and posets in order to introduce the concept of c-monoid and how to construct its associated poset with the help of notions from the incidence algebra $I(P)$. We then extend the construction to numerical monoids and $^+1$ monoids and give a combinatorial interpretation for the Möbius function of these posets.

Definition 2.0.1. A **monoid** is a set S with a law of composition which is associative, and having a unit element e.

Note that by definition, S is not empty. Furthermore, if the law of composition is commutative, we also say that S is a *commutative* (or abelian) monoid.

Definition 2.0.2. A subset H of S, containing the unit element e and closed under the law of composition is called a **submonoid** of S.

This definition implies that a submonoid is also a monoid, under the law of composition induced by that of S. We introduce cancellative monoids, monoids with an additional property, which we will refer to as *c-monoids*

Definition 2.0.3. A monoid $(M, *, e)$ that satisfy the conditions

1. The identity has no proper divisors,i.e.,

$$e = a * b \Longrightarrow a = e = b \tag{2.1}$$

2. The left cancellation law, i.e.,

$$\forall a, b, b' \in S: \; a * b = a * b' \Longrightarrow b = b'. \tag{2.2}$$

3. The finite factorization property, i.e.,

$$\forall a \in M : \left| \{(a_1, a_2) | \; a_1 * a_2 = a\} \right| < \infty \tag{2.3}$$

is called a **c-monoid**.

2.1 Posets and c-monoids

In this section we introduce how to build the associated poset to a c-monoid, based on the work of Méndez and Yang [MY91]. Before doing so, we introduce the necessary notation and preliminary definitions. A *partially ordered set* (poset) P, is a set together with a binary relation denoted by $\leq$ (or $\leq_P$ when the possibility of confusion arises) which is reflexive, transitive and antisymmetric.

Definition 2.1.1. A **(closed) interval** of P is denoted by

$$[p,q] = \{r \in P \mid p \leq r \leq q\}, \; p,q \in P.$$

Open intervals are defined analogously. If every interval of P is finite, we say P is a locally finite poset. As an example, the set of positive integers endowed with the divisibility order relation is locally finite. We say that P has a $\hat{0}$ or $\hat{1}$ if it has a unique minimal or maximal element, respectively. Furthermore, we can think of intervals as posets with a $\hat{0}$ and $\hat{1}$.

Definition 2.1.2. Two posets P_1 and P_2 are **isomorphic**, denoted by $P_1 \cong P_2$, if there exists an order preserving bijection $\phi : P_1 \to P_2$ whose inverse is order-preserving, i.e.,

$$p \leq_{P_1} q \Longleftrightarrow \phi(p) \leq_{P_2} \phi(q).$$

2.1.1 The incidence algebra

Let P be a locally finite poset, let $\operatorname{Int}(P)$ denote the set of all closed intervals of P, and let $\mathbb{C}$ be the field of complex numbers. We define the set $I(P)$ as follows:

$$I(P) := \{f \mid f : \operatorname{Int}(P) \to \mathbb{C}\}$$

In an abuse of notation, for any function f in $I(P)$ and $[x,y] \in \operatorname{Int}(P)$, we write $f(x,y) := f([x,y])$. The set $I(P)$ forms an algebra together with the operations of sum, convolution and scalar multiplication defined in the following way

1. $(f+g)(x,y) = f(x,y) + g(x,y)$.
2. $(f \star g)(x,y) = \sum_{x \leq z \leq y} f(x,z)g(z,y)$
3. $(\lambda f)(x,y) = \lambda f(x,y)$, for $\lambda \in \mathbb{C}$.

Note that the sum in the convolution is finite because P is locally finite. The unit element of our incidence algebra δ is

$$\delta(x,y) = \begin{cases} 1 & if \ x = y \\ 0 & \text{otherwise.} \end{cases}$$

Indeed, for any function f in $I(P)$ we have

$$\begin{aligned}(\delta \star f)(x,y) &= \sum_{x \leq z \leq y} \delta(x,z) f(z,y) \\ &= \delta(x,x) f(x,y) \\ &= 1 f(x,y) = f(x,y).\end{aligned}$$

We define the *zeta function* of $I(P)$ by

$$\zeta(x,y)=1,\text{ for all }(x,y)\in \mathrm{Int}(P).$$

If P is finite and if we use a total order l extending P, the incidence algebra is isomorphic to the algebra of matrices $\mathcal{M}$ with rows and columns indexed by the elements of l, in which the xy entry is $f(x,y)$ if $x\leq_P y$, and 0 otherwise.

Example 2.1.1. Take the following poset P into account.

$$\begin{tikzcd}[column sep=small]
 & d & \\
b \arrow[ur, dash] & & c \arrow[ul, dash] \\
 & a \arrow[ul, dash] \arrow[ur, dash] &
\end{tikzcd}$$

Let $l=\{a,b,c,d\}$ be the total order extending P. Let $f\in I(P)$, generic. Then f is sent, by this isomorphism, to the following matrix

$$\begin{array}{c c}
 & \begin{array}{cccc} \;\;a\;\; & \;\;b\;\; & \;\;c\;\; & \;\;d\;\; \end{array} \\
\begin{array}{c} a \\ b \\ c \\ d \end{array} &
\begin{pmatrix}
f(a,a) & f(a,b) & f(a,c) & f(a,d)\\
0 & f(b,b) & 0 & f(b,d)\\
0 & 0 & f(c,c) & f(c,d)\\
0 & 0 & 0 & f(d,d)
\end{pmatrix}
\end{array}$$

The zeta function of any interval can be thought of as an upper triangular matrix filled with ones and zeroes. In particular, for the proposed poset we have that its zeta function is sent to (2.4)

$$\begin{array}{c c}
 & \begin{array}{cccc} a & b & c & d \end{array} \\
\begin{array}{c} a \\ b \\ c \\ d \end{array} &
\begin{pmatrix}
1 & 1 & 1 & 1\\
0 & 1 & 0 & 1\\
0 & 0 & 1 & 1\\
0 & 0 & 0 & 1
\end{pmatrix}
\end{array} \tag{2.4}$$

Proposition 2.1.0.1. *A function $f\in I(P)$ is invertible if and only if $f(x,x)\neq 0$ for all $x\in P$.*

The inverse of ζ under convolution is called the *Möbius function*, μ. Thus,

$$\begin{aligned}
(\mu\star\zeta)(x,y) &= \sum_{x\leq z\leq y}\mu(x,z)\zeta(z,y)\\
&= \sum_{x\leq z\leq y}\mu(x,z)=\delta(x,y).
\end{aligned} \tag{2.5}$$

An important feat of $\mu(x,y)$ is that it can be defined recursively, depending only in the poset $[x,y]$. Indeed, by (2.5),

$$(\mu\star\zeta)(x,y)=\sum_{x\leq z\leq y}\mu(x,z)=\sum_{x\leq z<y}\mu(x,z)+\mu(x,y)=0 \tag{2.6}$$

Thus, for any $[x, y] \in \text{Int}(P)$ we have

$$\mu(x, y) = \begin{cases} 1, & x = y; \\ -\sum_{x \leq z \leq y} \mu(x, z), & x < y. \end{cases} \tag{2.7}$$

As an example, take the poset defined on Example 2.1.1 and the matrix (2.4) associated to its zeta function. Then,

Example 2.1.2. The associated matrix to the Möbius function on the intervals of P is

$$\begin{array}{c c} & \begin{array}{cccc} a & b & c & d \end{array} \\ \begin{array}{c} a \\ b \\ c \\ d \end{array} & \begin{pmatrix} 1 & -1 & -1 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 \end{pmatrix} \end{array} \tag{2.8}$$

It can be easily checked that this matrix corresponds to the inverse of (2.4).

Having defined the Möbius function μ, we give the following theorems and definition which will come in handy later on.

Theorem 2.1.1. *Let $[x, y]$ and $[w, z]$ be two isomorphic posets with $\hat{0}$ and $\hat{1}$. Then*

$$\mu(x, y) = \mu(w, z). \tag{2.9}$$

For any two intervals $[a, b]$, $[c, d]$ in locally finite posets P and Q, respectively. We define an order for the Cartesian product$[a, b] \times [c, d]$ in the following way

Definition 2.1.3. For (p_1, q_1),(p_2, q_2) in $[a, b] \times [c, d]$. We have that

$$(p_1, q_1) \leq_\times (p_2, q_2) \;\text{ if }\; p_1 \leq_P p_2 \text{ and } q_1 \leq_Q q_2 \tag{2.10}$$

Hence, the Cartesian product of intervals is equivalent to an interval of the form $[(a, c), (b, d)]$ with the order as in Definition 2.1.3.

Proposition 2.1.1.1. *Let P and Q be locally finite posets, and let $P \times Q$ be their product. If $(p_1, q_1) \leq_\times (p_2, q_2)$ in $P \times Q$ then*

$$\mu_{P\times Q}\big((p_1, q_1), (p_2, q_2)\big) = \mu(p_1, p_2)\mu(q_1, q_2). \tag{2.11}$$

Adapted from [Sta11]. Let $(p_1, q_1) \leq_{P\times Q} (p_2, q_2)$. We have

$$\begin{aligned} \sum_{(p_1,q_1)\leq(p,q)\leq(p_2,q_2)} \mu_P(p_1, p)\mu_Q(q_1, q) &= \Big(\sum_{p_1\leq p\leq p_2} \mu_P(p_1, p)\Big)\Big(\sum_{q_1\leq q\leq q_2} \mu_Q(q_1, q)\Big) \\ &= \delta_{p_1,p_2}\delta_{q_1,q_2} = \delta_{(p_1,q_1),(p_2,q_2)} \end{aligned}$$

The result follows from the fact formula (2.7) determines μ uniquely. □

Denote by $B(n)$, the Boolean poset of n elements which is defined for subsets of $[n]$ under the order of set inclusion in the following way: A subset A of $[n] = \{1,2,\dots,n\}$ is encoded as

$$A \mapsto \left(\delta_{A,i}\right)_{i=1}^{n}$$

with

$$\delta_{A,i} = \begin{cases} 1, & \text{if } i \in A. \\ 0, & \text{otherwise.} \end{cases}$$

That is, we assign to every subset of $[n]$ a tuple of ones and zeroes assigning 1 to the position corresponding to the elements of A and 0 everywhere else. In general, for the set of parts of $[n]$, $\mathscr{P}[n]$ also denoted by $B(n)$, is encoded by ones and zeroes like a characteristic function for subsets assigning 1 to k-th position of the tuple if the k-th element belongs to A. Indeed,

$$B(n) \cong \{0,1\}^n \tag{2.12}$$

Example 2.1.3. To find the Möbius function of $B(n)$ it suffices to notice that, by its definition, $B(n)$ is isomorphic to the product of chains of length 1, denoted by $\mathbf{C}(1)$.

$$\mu(\emptyset,[n]) = \mu(\mathbf{C}(1)^n) \tag{2.13}$$

The graph and associated Möbius function of $\mathbf{C}(1)$

$$\begin{array}{cc} 1 & -1 \\ | & \\ \emptyset & 1 \end{array}$$

together with (2.13) and Proposition 2.1.1.1 gives us the following result

$$\mu(\emptyset,[n]) = (-1)^n \tag{2.14}$$

The proof of Theorem 2.1.1, and more on the incidence algebra and the Möbius function can be found on [Sta11].

2.1.2 The poset associated to a c-monoid

To any c-monoid $(M,*)$ we can always built an associated poset $(P_M,\leq)$. We define a binary relation $\leq$ in the following way:

$$a \leq b \text{ if } \exists b' : a * b' = b. \tag{2.15}$$

Proposition 2.1.1.2. *The relation defined on (2.15) is a partial order in P_M.*

Proof. Let's prove that (2.15) defines an order relation on P_M, i.e., that it is reflexive, transitive and antisymmetric.

- (Reflexivity) Let $a \in M$, we have to prove that

$$a \leq a$$

That is,

$$\exists b \in M : a * b = a$$

But this follows from the existence of the identity element e in M as $a * e = a$.

- (Transitivity) Let a, b, c be elements in M such that

$$a \leq b, \text{ and } b \leq c. \tag{2.16}$$

We have to prove that $a \leq c$. That is,

$$\exists\, c' \in M : a * c' = c. \tag{2.17}$$

(2.16) implies that

$$\exists b' : a * b' = b \tag{2.18}$$

$$\exists c'' : b * c'' = c \tag{2.19}$$

So that taking $c' = b' * c''$, by associativity, we obtain the required result.

- (Antisymmetry) We have to prove that if

$$a \leq b \text{ and } b \leq a \Longrightarrow a = b$$

Let a, b be elements in M such that $a \leq b$ and $b \leq a$. This implies that

$$\exists a' \in M : a * a' = b. \tag{2.20}$$

$$\exists b' \in M : b * b' = a. \tag{2.21}$$

Because M is a c-monoid and by (2.20) we have that

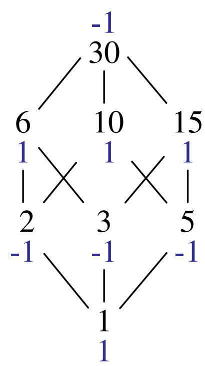
$$a * a' * b' = a * e$$

$$\Longrightarrow a' * b' = e.$$

Since the identity in a c-monoid has no proper divisors, we conclude that $a' = b' = e$ and thus, condition (2.1.2) holds.

Therefore, $\leq$ defines an order relation in M. □

Example 2.1.4. Let $(\mathbb{N}_+, *)$ be the monoid of positive integers with the usual product. Then, the associated poset $P_{\mathbb{N}_+}$ on the interval $[1, 30]$ has the following diagram with associated Möbius function values



As an application of Proposition 2.1.1.1, we may compute the Möbius function for $P_{\mathbb{N}_+}$ on the interval $[1,n]$. For $n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_k^{\alpha_k}$, we have the poset isomorphism

$$[1,n] \cong \mathbf{C}(\alpha_1) \times \mathbf{C}(\alpha_2) \times \cdots \times \mathbf{C}(\alpha_k) \tag{2.22}$$

And thus, as a consequence of Proposition 2.1.1.1,

$$\mu(1,n) = \mu(n) = 0, \text{ if } \exists\ i : \alpha_i \geq 2. \tag{2.23}$$

That is, if the length of a chain is greater or equal than 2, its Möbius function is 0. Otherwise, if a number can be decomposed as the product of non repeated primes, it is isomorphic to the product of chains of length one. Hence,

$$\mu(n) = (-1)^n, \text{ if } \forall\ i : \alpha_i = 1. \tag{2.24}$$

Thus, we get that the Möbius function is given by

$$\mu(n) = \begin{cases} (-1)^k, & n = p_1 p_2 \cdots p_k \\ 0, & \text{otherwise.} \end{cases} \tag{2.25}$$

It is important to remark that while the cancellative property and the lack of proper divisors of identity of a c-monoid M allow us to define the associated poset P_M, the finite factorization property is equivalent to P_M being locally finite. We shall now state some more properties of P_M which are not so immediate.

Proposition 2.1.1.3. *For a c-monoid $(M,*,e)$, the associated poset P_M has the following properties:*

1. *P_M has a minimum element, $\hat{0}$. Furthermore, $\hat{0} = e$.*
2. *For any $[a,b] \in \mathrm{Int}(P_M)$ we have that $[a,b] \cong [e,b']$. With b' being the unique element in P_M such that $a * b' = b$.*

Proof. To prove (1), it suffices to note that for all a in P_M we have that

$$a * e = a.$$

To show that (2) holds, define the mapping $\phi : [a,b] \to [e,b']$ such that to any element x of $[a,b]$ it assigns x' such that $a * x' = x$. This mapping is clearly an order preserving bijection as x' is unique for any $x \in [a,b]$. Indeed, suppose this is not the case and there is some x'' such that $a * x'' = x$. Then by the cancellation law, it must be that $x' = x''$. We've reached a contradiction so it must be that x' is unique. □

To each function in $f \in I(P_M)$ we can assign a function (denoted by abuse of language with the same symbol) $f : M \to \mathbb{C}$, by making $f(a) := f(e,a)$ for $a \in M$.
For $f,g \in \mathbb{C}^M$ we define the convolution product

$$(f \star g)(a) = \sum_{a_1 * a_2 = a} f(a_1) g(a_2),$$

This convolution is well defined as there is a finite number of decompositions of a in two factors. Consider the set of indeterminates (variables) $\{X_a\}_{a\in M}$, with the product $X_aX_b := X_{a*b}$. We can identify a function $f \in \mathbb{C}^M$ with the formal (infinite) sum $f = \sum_{a\in M} f_aX_a$, where $f_a := f(a)$. Then, we have that the set

$$\mathbb{C}[\![M]\!] = \Big\{ \sum_{a\in M} f_aX_a \mid f_a \in \mathbb{C} \Big\}.$$

together with usual sum of formal series, product by a scalar and convolution defined by the extension of the product $X_aX_b := X_{a*b}$, to the whole set $\mathbb{C}[\![M]\!]$,

$$\left(\sum_{a_1\in M} f_{a_1}X_{a_1}\right)\left(\sum_{a_2\in M} g_{a_2}X_{a_2}\right) = \sum_{a\in M}\left(\sum_{a_1*a_2=a} f_{a_1}g_{a_2}\right)X_a.$$

is an algebra. The identity element of this algebra is X_e, which coincides with the delta function of the incidence algebra.

$$X_e = \sum_{a\in M} \delta(e,a)X_a. \tag{2.26}$$

Analogously, we can define the zeta function by

$$\zeta_M(X) = \sum_{a\in M} \zeta(a)X_a = \sum_{a\in M} X_a.$$

An element of $\mathbb{C}[\![M]\!]$ is invertible if its constant term , c_e, is different from zero.

Theorem 2.1.2. *For a monoid M, the inverse of its zeta generating function ,ζ_M, with respect to the product, is given by the Möbius function, μ_M.*

$$\zeta_M^{-1}(X) = \sum_{a\in M} \mu(a)X_a = \mu_M(X)$$

Proof. Indeed, by the definition of the Möbius function on (e,a),

$$\begin{aligned}
\mu_M(X)\zeta_M(X) &= \left(\sum_{a_1\in M} \mu(a_1)X_{a_1}\right)\left(\sum_{a_2\in M} \zeta_M(a_2)X_{a_2}\right)\\
&= \sum_{a\in M}\left(\sum_{a_1*a_2=a} \mu_M(a_1)\right)X_a\\
&= \sum_{a\in M}\left(\sum_{e\le a_1\le a} \mu(e,a_1)\right)X_a = X_e.
\end{aligned}$$

In a similar way, we prove that $\zeta_M(X)\mu_M(X) = 1$, by using the fact that

$$\sum_{a_1*a_2=a} \mu_M(a_2) = \sum_{e\le a_1\le a} \mu_M(a_1,a),$$

since, by Proposition 2.1.1.3, the interval $[e,a_2]$ is isomorphic $[a_1,a]$, whenever $a_1 * a_2 = a$. □

2.2 Numerical Monoids

Denote by $\mathbb{N} = \{0, 1, 2, \ldots\}$ the set of nonnegative integers. From now on, we will focus on additive submonoids of $\mathbb{N}$ and their applications. Notice that, if a submonoid S of $\mathbb{N}$ contains a nonzero element a then it must be an infinite set as $ka \in S$ for all $k \in \mathbb{N}$.

Definition 2.2.1. A numerical monoid S is an additive submonoid of $\mathbb{N}$ with finite complement in $\mathbb{N}$.

Numerical monoids, despite having a very simple definition as mathematical objects, have applications on important areas such as algebraic geometry, cryptography and error-correcting codes [BA13]. It is important to remark that numerical monoids can also be found denoted as numerical semigroups, semimodules, or demimodules in the available literature. See [Sae12; AGS16]. We will denote them numerical monoids on the present work.
To any numerical monoid S we associate its set of gaps, $G(S)$, defined as

$$G(S) = \mathbb{N} \setminus S \tag{2.27}$$

Note that by definition, the set of gaps of any numerical monoid must be finite. This cardinality, denoted by $g(S)$, is called the genus or degree of singularity of S. We focus particularly in numerical monoids because they classify the set of all submonoids of $(\mathbb{N}, +)$ up to isomorphism as we shall prove. Let us introduce some notation and more conditions for numerical monoids which will be useful to prove the isomorphism and further results.
For a submonoid S of $\mathbb{N}$ we denote by

$$\mathrm{Gr}(S) = \{x - y \mid x, y \in S\}$$

the subgroup of $\mathbb{Z}$ generated by S. A submonoid S of $\mathbb{N}$ is a numerical monoid if $1 \in \mathrm{Gr}(S)$. Therefore, the finite complement property of numerical monoids implies $gcd(S) = 1$ for any numerical monoid M.

Proposition 2.2.0.1. *Let S be a submonoid of $\mathbb{N}$. Then S is isomorphic to a numerical monoid.*

Adapted from [AGS16]. Let $d = \gcd(S)$, i.e., d is the generator of the group generated by S in $\mathbb{Z}$. Define $S_1 = \{s/d \mid s \in S\}$, which is a numerical monoid as it inherits the zero element from S and clearly has finite complement in $\mathbb{N}$. The map $\phi : S \to S_1$ given by $\phi(s) = s/d$ is a monoid isomorphism. Thus, S is isomorphic to the numerical monoid S_1. □

Another important quality of numerical monoids is the fact that they are finitely generated, meaning that the elements of a numerical monoid can be described as a linear combination of a finite number of them.

All the submonoids of $\mathbb{N}$ are c-monoids. Hence, we can define the algebra of formal series for any numerical monoid S, denoted by $\mathbb{C}[\![S]\!]$, in a similar way as that for the general case of a c-monoid M and the product extension of $X_a.X_b = X_{a+b}$. This leads us to represent $\mathbb{C}[\![S]\!]$ as the (isomorphic) algebra of power series $\mathbb{C}[\![\{x^s | s \in S\}]\!]$. Take for example the numerical monoid $S = \{\langle 2, 3\rangle\} = \left\{2k_1 + 3k_2 \mid k_1, k_2 \in \mathbb{N}\right\} = \mathbb{N} \setminus \{1\}$. Then,

$$\zeta_S(x) = \sum_{k=0,\ k\neq 1}^{\infty} x^k = \frac{x}{1-x} - x = \frac{1 - x + x^2}{1-x}. \tag{2.28}$$

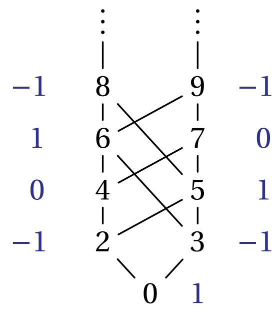
And, by Theorem 2.1.2, its associated Möbius function

$$\mu_S(x) = \zeta_S^{-1}(x) = \frac{1-x}{1-x+x^2}. \tag{2.29}$$

Expanding (2.29) up to the first eighteen terms we obtain.

$$\mu_S(x) = 1 - x^2 - x^3 + x^5 + x^6 - x^8 - x^9 + x^{11} + x^{12} - x^{14} - x^{15} + x^{17} + x^{18} \cdots \tag{2.30}$$

So that, the coefficient of the k-th power of x would be the corresponding value of the Möbius function for $k \in \{\langle 2,3\rangle\}$, which can be checked on the diagram of the poset.



2.3 $^+1$ Monoids

We define $^+1$ monoids, introduced in [BMR20], as their properties make them of interest in the study of operads and their associated Möbius function.

Definition 2.3.1. A subset S^+ of $\mathbb{N}_+$ such that $S^+ - 1$ is an additive submonoid of $\mathbb{N}_+$ will be called a $^+1$ monoid.

The set S^+ of odd integers is a $^+1$ monoid as, $S^+ - 1$, the set of even integers is an additive monoid. The set of multiples of a positive integer m plus one is always a $^+1$ monoid.

Proposition 2.3.0.1. *Let S^+ be a $^+1$ monoid. Then:*

1. *If $s_1, s_2, \ldots, s_t$ and t are elements of S^+, then $s_1 + s_2 + \cdots + s_t$ is in S^+.*

2. *If $s_1, s_2, \ldots, s_{t-1}$ are elements of S^+ and t is also in S^+, then $s_1 + s_2 + \cdots + s_{t-1}$ is in $S^+ - 1$.*

A proof for Proposition 2.3.0.1 can be found on [BMR20]. We describe a family of posets associated to a $^+1$ monoid. They have as underlying sets compositions whose parts are in S^+ (S^+-restricted compositions from now on). We define an order relation with the help of Proposition 2.3.0.1 in the following way.

Definition 2.3.2. Let $\mathfrak{c} = (s_1, s_2, \ldots, s_l)$, and $\mathfrak{c}' = (s'_1, s'_2, \ldots, s'_r)$ be two S^+-restricted numerical compositions of a positive integer n. Then we say that

$$(s_1, s_2, \ldots, s_l) \leq (s'_1, s'_2, \ldots, s'_r)$$

if there are r numbers $k_i \in S^+$, for $i = 1,2,3,\dots,r$, such that

$$s'_i = s_{\sigma_{i-1}+1} + s_{\sigma_{i-1}+2} + \cdots + s_{\sigma_{i-1}+k_i},$$

where σ_0 is defined to be zero, and for $i = 1,2,3,\dots,r$, σ_i is defined as the sum $\sigma_i = k_1 + k_2 + \cdots + k_i$.

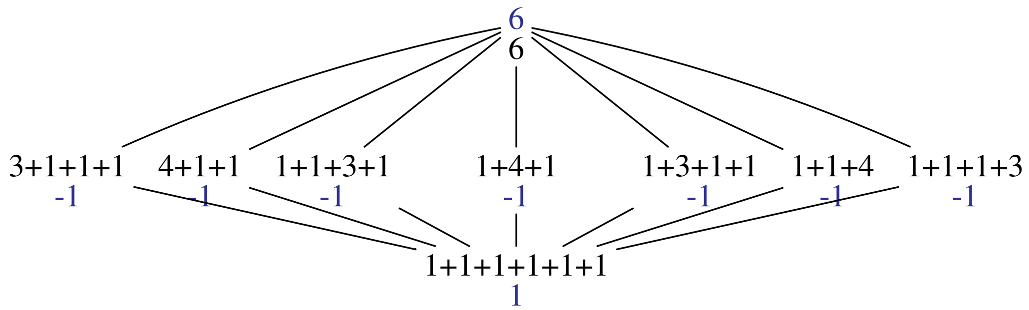
We denote by $P_{S^+}(n)$ the poset of compositions of n with components in S^+. It has a zero, the compositions with n parts, each part equal to 1. It also has a unique maximal element n if, and only if, n is an element of S^+. It can be shown that, for any $^+1$ monoid S^+, the poset $P_{S^+}(n)$ is isomorphic to the poset $P_{E_{S^+}}[n]$, obtained from restricting the operad E_+ to S^+, which will be discussed further in the operadic context of the following chapter.
Indeed, $P_{E_{+S}^{+}}[n]$ is an order on the compositions of the linear order $1,2,\dots,n$ having as the $\hat{0}$ of the poset compositions with all its segments having length one. The order compares the compositions obtained by concatenating, at each level, k neighbouring compositions, with k in S^+. This, is analogous to adding k neighbouring parts of the S^+ restricted numerical compositions of n, with $k \in S^+$ as we have just introduced.

This fact, makes possible the computing of the compositional inverse of the ζ function of the poset in the following way

$$\zeta^{\langle -1\rangle}(x) = \sum_{s\in S^+} \mu(0,s)x^s. \tag{2.31}$$

We provide some examples of the proposed poset.

Example 2.3.1. Denote by $\mathbb{N}\setminus 2$ the $^+1$ monoid obtained by adding 1 to the numerical monoid generated by 2 and 3. Then $\mathrm{P}_{\mathbb{N}\setminus 2}(6)$ has the following diagram



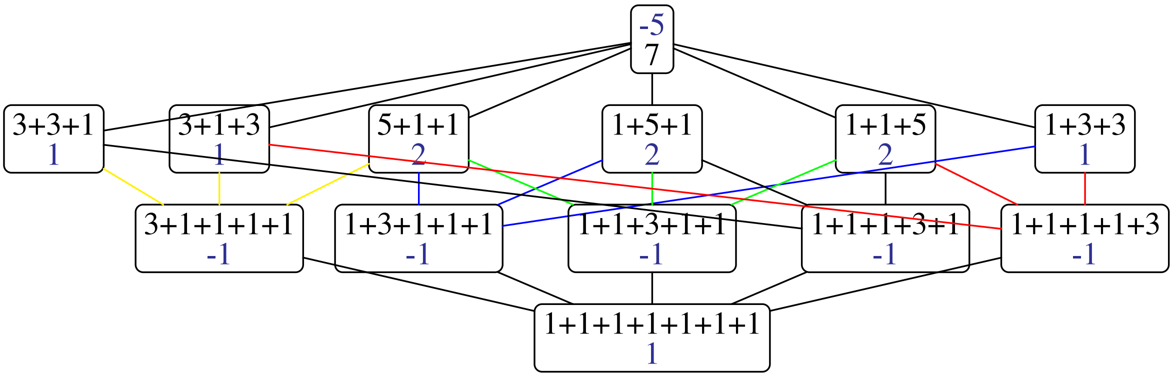
The compositional inverse of the ζ function for $\mathbb{N}\setminus 2$,

$$\zeta_{\mathbb{N}\setminus 2}^{\langle -1\rangle} = \Big(\sum_{k=1,k\neq 2}^{\infty} x^k\Big)^{\langle -1\rangle} = \Big(\frac{x}{1-x} - x^2\Big)^{\langle -1\rangle} = \Big(\frac{x-x^2+x^3}{1-x}\Big)^{\langle -1\rangle}$$

expanded up to the first terms with the help of Mathematica,

$$x-x^3-x^4+2x^5+6x^6-x^7-28x^8-31x^9+98x^{10}+288x^{11}-131x^{12}-1730x^{13}-1638x^{14}\cdots \tag{2.32}$$

Example 2.3.2. Let Odd denote the $^{+}1$ monoid of odd numbers, then the interval $[\hat{0},7]$ of the associated poset $P_{\text{Odd}}(\hat{0},7) := P_{\text{Odd}}(7)$, has the following diagram



The compositional inverse of the ζ function for Odd,

$$\zeta_{\text{Odd}}^{\langle -1\rangle} = \Big(\sum_{s\in S} x^s\Big)^{\langle -1\rangle} = \Big(\sum_{k=0}^{\infty} x^{2k+1}\Big)^{\langle -1\rangle} = \Big(x\sum_{k=0}^{\infty} x^{2k}\Big)^{\langle -1\rangle}.$$

After some calculations in Mathematica, and expanded up to the first terms

$$\zeta_{\text{Odd}}^{\langle -1\rangle} = \frac{-1-\sqrt{1+4x^2}}{2x} = x-x^3+2x^5-5x^7+14x^9-42x^{11}+132x^{13}-429x^{15}+1430x^{17}-4862x^{19}\dots \tag{2.33}$$

Note that the coefficients of the inverse function are exactly the Catalan numbers with alternating sign. This is no coincidence, as we will see later on.

We give the following proposition to help understand the combinatorial meaning behind the compositional inverse of $^{+}1$ monoids.

Proposition 2.3.0.2. *Let $F(x)$ be a formal power series of the form $F(x) = xM(x)$ where $M(x)$ has 1 as constant term, $M(0) = 1$. Then*

$$F^{\langle -1\rangle} = \mathscr{A}_{M^{-1}}(x),$$

Where

$$\mathscr{A}_{M^{-1}}(x) = xM^{-1}\big(\mathscr{A}_{M^{-1}}(x)\big) \tag{2.34}$$

Proof. Condition (2.34) implies

$$\mathscr{A}_{M^{-1}}(x)M\big(\mathscr{A}_{M^{-1}}(x)\big) = x.$$

Which is equivalent to,

$$(xM(x))\circ\big(\mathscr{A}_{M^{-1}}(x)\big) = x.$$

Then, $xM(x)$ is the left compositional inverse of $\mathscr{A}_{M^{-1}}(x)$. By associativity of the composition of generating functions, it is also its right inverse. We have

$$\big(xM(x)\big)^{\langle -1\rangle} = \mathscr{A}_{M^{-1}}(x).$$

□

In general for $^{+}1$ monoids of the form $n\mathbb{N}+1$, its zeta function is given by

$$\zeta_{n\mathbb{N}+1}(x)=\sum_{k=0}^{\infty}x^{nk+1}=x\sum_{k=0}^{\infty}x^{nk}=\frac{x}{1-x^{n}}=x\zeta_{n\mathbb{N}}. \tag{2.35}$$

Equation(2.35) clearly follows the suppositions needed for Proposition 2.3.0.2 to hold. Hence, the compositional inverse for the zeta function of $^{+}1$ monoids of the form $n\mathbb{N}+1$ gives us the generating function of trees enriched with the multiplicative inverse of $\zeta_{n\mathbb{N}}$, that is,

$$\zeta_{n\mathbb{N}+1}^{\langle -1\rangle}=\mathscr{A}_{\zeta_{n\mathbb{N}+1}^{-1}}=\mathscr{A}_{\mu_{n\mathbb{N}}} \tag{2.36}$$

With Möbius function given by

$$\mu_{n\mathbb{N}}(0,jn)=\begin{cases}1, & \text{if } j=0\\ -1, & \text{if } j=1\\ 0, & \text{otherwise.}\end{cases} \tag{2.37}$$

From (2.37), we can deduce that the resulting trees must be n-ary with weight -1 on each internal vertex. Hence, equation (2.36) can be interpreted combinatorially as the generating function of weighted n-ary trees on k vertexes, including its root. The n-ary trees are trees in which every internal node has exactly n sons. The weight, $w(T)$, is given in function of the internal vertexes, denoted Iv, of the n-ary tree and their inner degree, denoted id, the number of nodes leaving the vertex, as evidenced by its generating function

$$\zeta_{n\mathbb{N}+1}^{\langle -1\rangle}=\sum_{k=1}^{\infty}\Big(\sum_{T:n-\text{ary tree}}w(T)\Big)x^{k}. \tag{2.38}$$

Where the weight of an n-ary tree is given by

$$w(T)=\prod_{v\in \mathrm{Iv}(T)}\mu(\mathrm{id}(v)).$$

We provide some examples to try and establish some patterns

Example 2.3.3. Set $n=3$. Then, the compositional inverse for the zeta function of the $^{+}1$ monoid of multiples of 3 plus one is given by

$$\zeta_{3\mathbb{N}+1}^{\langle -1\rangle}=\Big(\sum_{k=0}^{\infty}x^{3k+1}\Big)^{\langle -1\rangle}=\Big(x\sum_{k=0}^{\infty}x^{3k}\Big)^{\langle -1\rangle}$$

which expanded to the first terms, with the help of Mathematica, gives us

$$x-x^{4}+3x^{7}-12x^{10}+55x^{13}-273x^{16}+1428x^{19}\cdots$$

That is, for seven vertexes we have 3 ternary trees with positive weight as we can see in Fig. 2.1

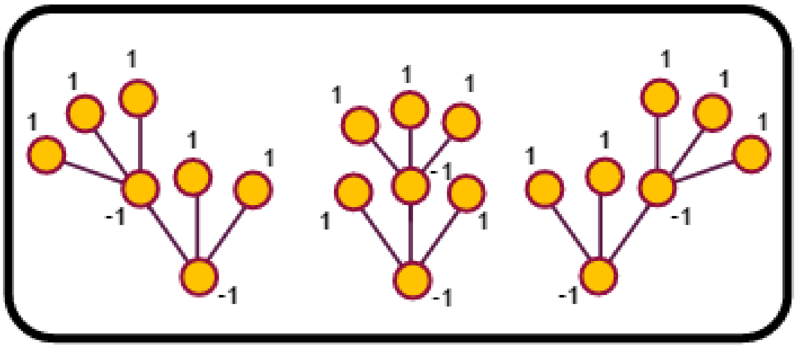


Figure 2.1: Weighted Ternary trees with seven vertexes

For $n = 4$, the inverse of the zeta function expanded is

$$x - x^5 + 4x^9 - 22x^{13} + 140x^{17} - 969x^{21} + 7084x^{25} - 53820x^{29} + 420732x^{33} - 3362260x^{37} \cdots$$

And, for $n = 5$, we obtain

$$x - x^6 + 5x^{11} - 35x^{16} + 285x^{21} - 2530x^{26} + 23751x^{31} - 231880x^{36} \cdots .$$

These sequences of coefficients coincide with those of the Fuss-Catalan numbers for $p = n$ and $r = 1$.
The Fuss-Catalan numbers, denoted by $A_m(p, r)$, are numbers of the form

$$A_m(p,r) \equiv \frac{r}{mp+r}\binom{mp+r}{m} = \frac{r}{m!}\prod_{i=1}^{m-1}(mp+r-i) = r\frac{\Gamma(mp+r)}{\Gamma(1+m)\Gamma(m(p-1)+r+1)}. \tag{2.39}$$

This family of generalized Catalan numbers are of interest for their numerous applications in combinatorics [MP15; Ava07; Lin11] and non-commutative probability [Mł10]. In general, the compositional inverse of the zeta function for $^+1$ monoids of the form $n\mathbb{N}+1$ is given by

$$\zeta^{\langle -1\rangle}_{n\mathbb{N}+1} = \sum_{k=0}^{\infty}\left((-1)^k A_k(n,1)\right)x^{kn+1} \tag{2.40}$$

Chapter 3

Preliminaries

In this chapter, we introduce the reader to basic category theory concepts needed to properly define combinatorial species. Then, we proceed to introduce combinatorial species together with their operations, providing some classical examples for a better understanding of the notions introduced.

3.1 Category Theory Fundamentals

We begin by defining the notion of category and then, presenting some examples which will be used later on.

Definition 3.1.1. A **category C** consists of

- a collection of **objects**;
- a collection of **morphisms**

so that:

- Each morphism has specified **domain** and **codomain** objects; the notation $f : X \to Y$ (or $X \xrightarrow{f} Y$) means that f is a morphism with domain X and codomain Y. The collection of all morphisms with domain X and codomain Y is denoted $\mathbf{C}(X, Y)$;
- For any pair of morphisms f, g with codomain of f equal to the domain of g, exists a **composite morphism** $g \circ f : \mathrm{dom} f \to \mathrm{cod} g$, satisfying the following *associative law*:
  for any morphisms $f : X \to Y$, $g : Y \to Z$, $h : Z \to W$ (with X, Y, Z, and W not necessarily distinct),
  $$h \circ (g \circ f) = (h \circ g) \circ f;$$
- Each object X has an **identity morphism** $id_X : X \to X$ satisfying the following *identity law*:
  for any morphism $f : X \to Y$
  $$\mathrm{id}_Y \circ f = f \text{ and } f \circ \mathrm{id}_X = f$$

Example 3.1.1. The category **Set** has as objects sets and total functions between sets as morphisms.

Example 3.1.2. The category $\mathbb{F}$ has finite sets as objects and arbitrary functions as morphisms.

Example 3.1.3. The category $\mathbf{Vec}_{\mathbb{K}}$ has as objects finite-dimensional vector spaces (over the field $\mathbb{K}$) and linear maps as morphisms.

Example 3.1.4. The category $\mathscr{L}$ has totally ordered sets as objects and order preserving bijections as morphisms.

Example 3.1.5. The category **Poset** has as objects all partially-ordered sets and as morphisms all order-preserving total functions

Example 3.1.6. The category **Top** has as objects topological spaces and continuous functions as morphisms

As we have seen now, categories can be thought of as universes categorizing mathematical objects but they also constitute a mathematical domain in their own right. It makes sense then to ask: What is a morphism between categories?

Definition 3.1.2. Let **C** and **D** be categories. A **covariant functor** $F : \mathbf{C} \to \mathbf{D}$ is a map taking each **C**-object X to a **D**-object $F(X)$ and each **C**-morphism $f : X \to Y$ to a **D**-morphism $F(f) : F(X) \to F(Y)$, such that for all **C**-objects X and composable **C**-morphisms f and g:

1. $F(\mathrm{id}_X) = \mathrm{id}_{F(X)}$
2. $F(g \circ f) = F(g) \circ F(f)$

When F reverses the direction of arrows, i.e., every morphism $f : X \to Y$ is sent to a morphism $F(f) : Y \to X$, and condition (2) is changed by

$$F(f \circ g) = F(g) \circ F(f),$$

the functor is called *contravariant*.

In particular, a (covariant) functor consists of a mapping on objects and a mapping on morphisms preserving all of the category structure, namely domains and codomains, composition, and indentities. Having defined functors as mappings from one category to another we now proceed to define structure-preserving mappings from one *functor* to another called natural transformations.

Definition 3.1.3. Let C and D be categories and let F and G be functors from C to D. A **natural transformation** α from F to G, written $\alpha : F \to G$ consists of:

- an arrow $\alpha_c : F(c) \to G(c)$ in D for every object $c \in C$, the collection of which define the *components* of the natural transformation, so that, for any morphism $f : c \to c'$ in C, the following square of morphisms in D commutes.

$$\begin{tikzcd} F(c) \arrow[r, "\alpha_c"] \arrow[d, "F(f)"'] & G(c) \arrow[d, "G(f)"] \\ F(c') \arrow[r, "\alpha_{c'}"] & G(c') \end{tikzcd} \tag{3.1}$$

The commutativity of the diagram above is referred to as the **naturality condition** on α. Also, if each component α_c of α is an isomorphism in D then α is called a **natural isomorphism**.

3.2 $\mathscr{L}$-Species

$\mathscr{L}$-Species are introduced to justify the combinatorial structures that use a linear order on an underlying set. A linear order, or totally ordered set, is equivalent to a bijective word, or a list, of a subjacent set V. For example, acb is a total order over the set $\{a,b,c\}$.
More generally, a totally ordered set can be defined as a pair (V,l) where V is a finite set and $l:[n]\to V$ is a bijection, with $n=|V|$. For simplicity, we denote by $|l|$ the cardinality of the subjacent set V.

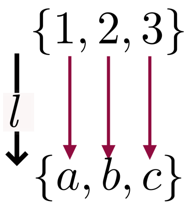


Figure 3.1: A totally ordered set.

Definition 3.2.1. Let l be a totally ordered set. A **permutation** of l is a bijection $\sigma: l\to l$, not necessarily order preserving.

Note that a permutation σ of a linear order l turns l into another linear order l' over the same subjacent set. For example, the bijection $\sigma(a)=b$, $\sigma(b)=d$, $\sigma(c)=a$, $\sigma(d)=c$ sends the linear order $l=\{a<b<c<d\}$ to $l'=\{b<d<a<c\}$, as can be seen in figure 3.2.

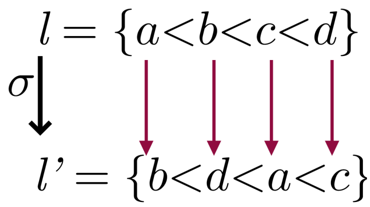


Figure 3.2: A permutation of the linear order abcd

Definition 3.2.2. An **$\mathscr{L}$-Species N** is a covariant functor from $\mathscr{L}$ to the category $\mathbb{F}$ of finite sets and arbitrary functions. In the case when $N:\mathscr{L}\to \mathbf{Vec}_{\mathbb{K}}$, we say that N is a $\mathscr{L}$-vector species.

We denote by $N[l]$ the image of the object l under the functor N. In an abuse of notation, when $l=[n]$, we denote $N[n]=N[\{1,2,\dots,n\}]$. Intuitively, by this definition we mean that to any finite linear order l, N associates a finite set $N[l]$ which elements are known as **N-structures on l**.

It should be noted that $\mathscr{L}$-species with natural transformations as morphisms form a category. We say two $\mathscr{L}$-species M and N are isomorphic, denoted $M=N$, if there exists a natural

isomorphism between them. Let us proceed to define some useful $\mathscr{L}$-Species.
The *singleton* especies is defined by

$$X[l] = \begin{cases} \{l\} & , \text{if } |l| = 1 \\ \emptyset & , \text{otherwise.} \end{cases}$$

The *uniform* species defined by

$$E[l] = \{l\}, \text{ for every linear order } l.$$

$E[l] = \{l\}$ can be interpreted as the identity permutation on the elements of l. We can denote E with the symbol $E^{\lhd}$ to differentiate it from its isomorphic version $E^{\rhd}[l] = \{l^r\}$, where l^r denotes the reverse of the linear order l. In order to simplify our notation we will use E from now on and will explicitly state the change to $E^{\rhd}$ to denote the isomorphic version.

The species of *permutations* defined by

$$\begin{aligned} \mathbb{S}[l] &= \{\sigma|\ \sigma : l \to l,\ \sigma \text{ a bijection}\}. \\ &= \{l'|l' \text{ a linear order over the same subjacent set}\}. \end{aligned}$$

3.2.1 Generating Functions

There are two kind of generating functions associated to a $\mathscr{L}$-Species N. The ordinary generating function

$$\widetilde{N}(x) = \sum_{k=0}^{\infty} |N[n]|x^n, \tag{3.2}$$

and the exponential one

$$N(x) = \sum_{k=0}^{\infty} \frac{|N[n]|}{n!} x^n \tag{3.3}$$

where $N[n]$ is the set that N assigns to the totally ordered set $[n]$. By functoriality, $|N[l]| = |N[n]|$ for any totally ordered set l with cardinality n. An $\mathscr{L}$-Species is then equivalent to an $\mathbb{N}$-graded set, or to a sequence of sets $\{N[n]\}_{n=0}^{\infty}$. We won't go into the technicalities of this but will give a way of recovering one from the other:
The functor N is recovered from the sequence of sets $\{N[n]\}_{n=0}^{\infty}$ via a correspondence G that defines $N[l] := N[n] \times \{l\}$, with $l \in \mathscr{L}$. For the other way around, there is a correspondence F which sends N to $\{N[n]\}_{n=0}^{\infty}$.
From this we can see that a morphism between $\mathscr{L}$-species is then equivalent to a sequence of functions between $\mathbb{N}$-graded sets.
It is important to remark that in equations (3.2) and (3.3), whenever $|N[n]|$ is replaced by $\dim N[n]$, N is a vector species. Let us now define the generating functions for some $\mathscr{L}$-Species.
The singleton species have as generating function

$$X(x) = \sum_{n=0}^{\infty} \delta_{n,1} \frac{x^n}{n!} = x \tag{3.4}$$

The uniform species have as generating functions

$$E(x) = e^x \tag{3.5}$$

$$\widetilde{E}(x) = \frac{1}{1-x} \tag{3.6}$$

The species of permutations have as generating functions

$$\mathbb{S}(x) = \sum_{n=0}^{\infty} n!\frac{x^n}{n!} = \frac{1}{1-x} \tag{3.7}$$

$$\widetilde{\mathbb{S}}(x) = \sum_{n=0}^{\infty} n!x^n \tag{3.8}$$

3.2.2 Combinatorial Operations

The functorial notation is useful for defining operations amongst $\mathscr{L}$-Species, and it is fundamental to the understanding of the combinatorial meaning of them. These set-theoretical operations mimic operations between their generating functions. Because we have two types of generating functions for $\mathscr{L}$-Species, we will have two kinds of products, substitutions, and derivatives.

Sum

The nth coefficient of the sum of two formal power series $M(x)+N(x)$ is the sum of the respective coefficients $|M[n]|+|N[n]|$. Let M and N be $\mathscr{L}$-Species, thus we define their sum as

$$(M+N)[l] := M[l] \biguplus N[l] \tag{3.9}$$

with $\uplus$ standing for disjoint union.

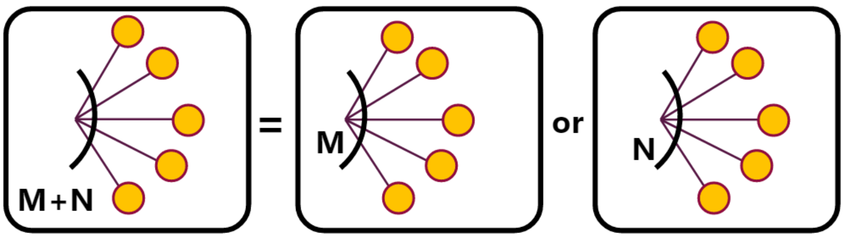


Figure 3.3: Sum of M and N

Ordinal Product and Substitution

The nth coefficient of the product of ordinary formal power series is given by

$$(\widetilde{M}\widetilde{N})[n] = \sum_{k+j=n} |M[k]||N[n-k]|,$$

where $\sum$ denotes the disjoint union. From this we define the ordinal product of M and N

$$(M \diamond N)[l] := \sum_{l_1+l_2=l} M[l_1] \times N[l_2] \tag{3.10}$$

Where $l_1 + l_2$ means ordinal sum of the ordered sets l_1 and l_2.

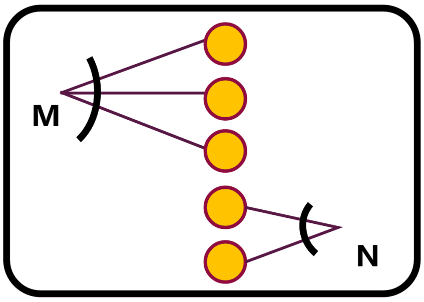


Figure 3.4: The ordinal product of $\mathscr{L}$-Species M an N

Let N be a positive $\mathscr{L}$-Species, the nth coefficient of the substitution of ordinary generating functions is given by

$$\widetilde{M}(\widetilde{N})[n] = \sum_k \sum_{n_1+n_2+\cdots+n_k=n} |M[k]||N[n_1]||N[n_2]|...|N[n_k]|,$$

where $n_1 + n_2 + \cdots + n_k = n$ goes over the strong compositions of n. With that in mind we define the ordinal substitution of $\mathscr{L}$-Species. Let M, N be $\mathscr{L}$-Species such that N is positive then we have that

$$M\langle N\rangle[l] = \sum_{\mathfrak{c}\in\mathfrak{K}[l]} N\left[l_1\right] \times N\left[l_2\right] \times \cdots \times N\left[l_{|\mathfrak{c}|}\right] \times M[\mathfrak{c}] \tag{3.11}$$

where $\mathfrak{c}$ goes over the set $\mathfrak{K}[l]$ of *strong compositions* of l. Notice that $\mathfrak{c}$ is a totally ordered set itself, and thus the expression $M[\mathfrak{c}]$ makes sense.

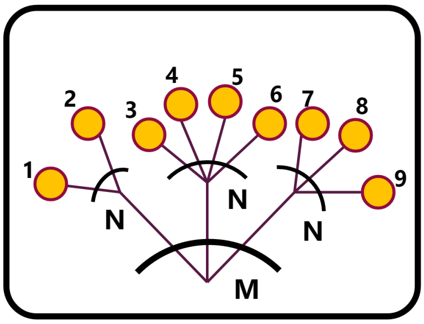


Figure 3.5: Ordinal Substitution of M and N on [9].

Remark 3.2.1. The elements of $M\langle N\rangle[l]$ are pairs $(a, m_{\mathfrak{c}})$ with $a = (n_{l_1}, n_{l_2}, \ldots, n_{l_{|\mathfrak{c}|}})$ being segmented assemblies of N-structures for which n_{l_i} is an element of $N[l_i]$ for every $l_i \in \mathfrak{c}$, and $m_{\mathfrak{c}}$ an M-structure on the composition $\mathfrak{c}$ of l which is also a linear order.

Shuffle Product and Substitution

The shuffle product and substitution of $\mathscr{L}$-Species imitate the product of exponential generating series. Let us introduce some notation beforehand.
If l is a linear order on a set V, and $B \subseteq V$, we denote as l_B the restriction of l to B. It is evident that l_B is a total order over B.
Given two $\mathscr{L}$-Species M and N, their shuffle product is defined as

$$(M.N)[l] = \sum_{V_1 \uplus V_2 = V} M[l_{V_1}] \times N[l_{V_2}]. \tag{3.12}$$

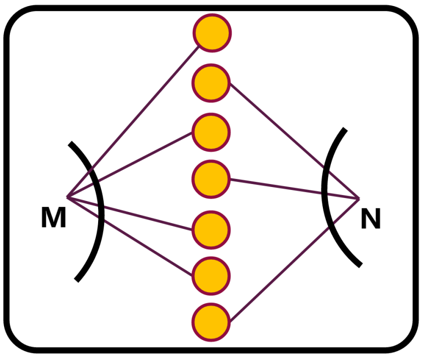


Figure 3.6: Shuffle Product of M and N

Consider a partition π on V, the subjacent set of the linear order l. It is clear that on each block B of the partition π, l induces by restriction a linear order l_B and a total order on π by making

$$\pi_l = \{B_1 < B_2 < \cdots < B_k\},\ B_i < B_{i+1} \text{ if } \min l_{B_i} < \min l_{B_{i+1}}$$

Let M and N be $\mathscr{L}$-Species, such that N is positive, then their shuffle substitution is defined as

$$M(N)[l] = \sum_{\pi_l \in \Pi[l]} (\prod_{B \in \pi} N[l_B]) \times M[\pi_l]. \tag{3.13}$$

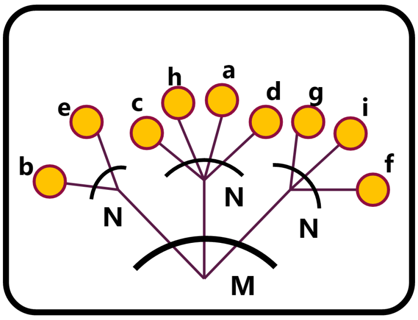


Figure 3.7: Shuffle substitution of M and N on $\{a, b, c, d, e, f, g, h, i\}$

Ordinal Derivative

The derivative of a generating function $\widetilde{N}(x)$ is equal to

$$\widetilde{N}'(x) = \sum_{n=1}^{\infty} |N[n]| n x^{n-1} = \sum_{n=0}^{\infty} |N[n+1]|(n+1)x^n.$$

Notice that there are $n+1$ ways of adding a 'ghost' element $*$ in a linear order l of length n, from which we obtain a linear order of length $n+1$ every time. We denote by $l \overset{(k)}{\leftarrow} *$ the linear order obtained from inserting the 'ghost' vertex $*$ in the kth position of l, for example

$$v_1 v_2 \overset{(1)}{\leftarrow} * = * v_1 v_2, \quad v_1 v_2 \overset{(2)}{\leftarrow} * = v_1 * v_2, \quad v_1 v_2 \overset{(3)}{\leftarrow} * = v_1 v_2 *$$

Whence, the (ordinal) derivative of an $\mathscr{L}$-Species N is defined as

$$N^{\blacktriangledown}[l] = \sum_{k=1}^{|l|+1} N[l \overset{(k)}{\leftarrow} *]. \tag{3.14}$$

Shuffle Derivative

Just as in the case of the ordinal derivative, the shuffle derivative is obtained by adding a 'ghost' vertex. In this case, the 'ghost' vertex is added as the first element of the linear order. Let N be an $\mathscr{L}$-Species then its shuffle derivative is defined as

$$N'[l] = DN[l] = N[* + l]. \tag{3.15}$$

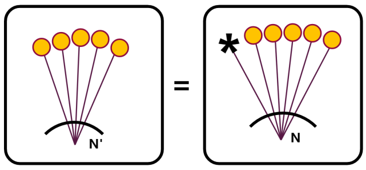


Figure 3.8: Shuffle derivative of N

All the elemental properties, associativity, commutativity, distributivity, linearity, etc., of the operations are true at the combinatorial level, including the product rule and the chain rule for the derivative:

$$(MN)' = MN' + M'N, \tag{3.16}$$

$$M(N)' = M'(N)N' \tag{3.17}$$

where the equality means isomorphism of $\mathscr{L}$-species.

3.2.3 Some classical enumerative results

Having defined operations between $\mathscr{L}$-Species, we show an application of them as means to find some classic enumerative results. In order to ease the reader with the notation we have defined and will later use, we emphasize the difference between ordinal and shuffle operations. It is of utmost importance to keep in mind that for ordinal operations we work with ordinary generating functions while we use the exponential generating functions for the shuffle ones.

Results pertaining shuffle operations

Let's introduce the species $\mathscr{C}$ of *cyclic permutations*, we have that

$$\mathscr{C}[l] := \{l' | l' \text{is a permutation with } l'(1) = \min(l)\}$$

Note that every permutation l' of a linear order can be uniquely partitioned as an ordinal sum of cyclic permutations $l' = l'_1 + l'_2 + \cdot + l'_k$, with $\min(l'_{i+1}) < \min(l'_i)$. Take for example

$$579468312 = 579 + 468 + 3 + 12$$

Example 3.2.1. In order to find the generating series $\mathscr{C}(x)$ we use the following identity

$$\mathbb{S} = E(\mathscr{C}) \tag{3.18}$$

Thus, we have that

$$\begin{aligned}
&\mathbb{S}(x) = E(\mathscr{C}(x)). \\
&\frac{1}{1-x} = e^{\mathscr{C}(x)}. && \text{Replacing the known exponential generating functions.} \\
&\log(\frac{1}{1-x}) = \mathscr{C}(x). && \text{Taking the logarithm on both sides.}
\end{aligned}$$

Example 3.2.2. The following identity: $\mathscr{C}(x)' = \mathbb{S}(x)$, which can be easily verified taking the derivative on the exponential generating function for cyclic permutations obtained in Example 3.2.1. We provide a pictorial proof below

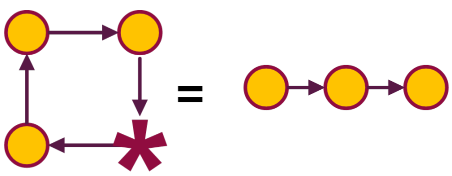


Figure 3.9: Shuffle derivative of a cyclic permutation

Example 3.2.3. Let us denote by Π the species of *partitions* of a set, then we have that $\Pi = E(E_+)$. Indeed, we obtain the generating series for the *Bell numbers*.

$$\begin{aligned}
\Pi(x) &= E(E_+(x)) \\
&= e^{e^x - 1}
\end{aligned}$$

Results pertaining ordinal operations

Example 3.2.4. To find the generating series for *compositions*, $\mathfrak{K}$, given by

$$\mathfrak{K} = E\langle E_+ \rangle$$

we proceed in the following way

$$\widetilde{\mathfrak{K}}(x) = E\langle E_+ \rangle(x) = \widetilde{E}(\widetilde{E_+}(x)) = \frac{1}{1 - \frac{x}{1-x}} = \frac{1-x}{1-2x} \tag{3.19}$$

Example 3.2.5. In a similar way, one can find the ordinary generating function for non-empty compositions given by the identity

$$\mathfrak{K}_+ = E_+\langle E_+\rangle$$

Thus,

$$\widetilde{\mathfrak{K}_+}(x) = \widetilde{E}_+\langle\widetilde{E_+}\rangle = \frac{\frac{x}{1-x}}{1-\frac{x}{1-x}} = \frac{\frac{x}{1-x}}{\frac{1-x-x}{1-x}} = \frac{x}{1-2x} = \sum_{k=1}^{\infty} 2^{k-1}x^k \tag{3.20}$$

giving us (the well known) result that $|\mathfrak{K}_+[n]| = 2^{n-1}$.

Some species can be defined by implicit equations and thanks to Joyal's [Joy81] combinatorial formulation of the implicit function theorem we can find their generating functions.

Example 3.2.6. Let $\mathscr{A}_E$ be the species of *ordered trees* defined by

$$\mathscr{A}_E = X \diamond E\langle\mathscr{A}_E\rangle \tag{3.21}$$

This implicit equation can be interpreted in the following way: each structure of $\mathscr{A}_E[l]$ is constructed by choosing as root the first (minimum) element of l (denoted by m_l) and then placing a structure of $\mathscr{A}_E$ in each segment of a composition on the rest of the linear order of l^+, $l = m_l + l^+$, $\mathfrak{c} = l_1^+ + l_2^+ + \cdots + l_n^+$,

$$\begin{aligned}\mathscr{A}_E[l] = X \diamond E\langle\mathscr{A}_E\rangle[l] &= X[\{m_l\}] \times \sum_{\mathfrak{c}\in\mathfrak{K}[l^+]} \mathscr{A}_E\left[l_1^+\right] \times \mathscr{A}_E\left[l_2^+\right] \times \cdots \times \mathscr{A}_E\left[l_{|\mathfrak{c}|}^+\right] \times E[\mathfrak{c}] \\ &= \{m_l\} \times \sum_{\mathfrak{c}\in\mathfrak{K}[l^+]} \mathscr{A}_E\left[l_1^+\right] \times \mathscr{A}_E\left[l_2^+\right] \times \cdots \times \mathscr{A}_E\left[l_{|\mathfrak{c}|}^+\right]\end{aligned} \tag{3.22}$$

We drop $E[\mathfrak{c}]$ in (3.22) because it gives $E[\mathfrak{c}] = \{\mathfrak{c}\}$ which does not add any information because this composition is already encoded in the increasing order of the roots attached to m_l. Because this is a recursive definition, it then takes the first member in every segment of the new linear order as the father to the remaining members in it and so on until we have gone through l. As we have seen in sub-subsection (3.2.2), every segment of l^+ is ordered and we have that $l_i^+ < l_j^+$, for $i < j$.

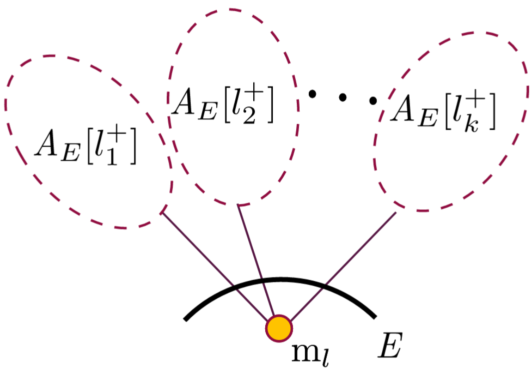


Figure 3.10: Recursion of ordered trees.

Note that the construction gives all the possible ordered trees taking into account all the possible compositions of l. But, by the definition of the ordinal operations used, the order is always respected so that the label of any son of $\mathscr{A}_E[l_i^+]$ is always less than that on any son of $\mathscr{A}_E[l_j^+]$ if $i < j$. By induction in the length of l, we easily obtain that the labels of the resulting plane tree are in preorder (reading first the root and then from left to right) according to the order of l. For a better understanding of the construction and the ordinal operations behind it, we provide a concrete example for $n = 13$ on which we separate the internal compositions with $|$ and the external ones with $\|$. The sequence is as follows: In the first step the minimum element 1 is chosen as the root, and a composition is created, in this case 234|567|8910|111213, with an element of $\mathscr{A}_E$ in each of the segments of this composition. In the second stage, by the implicit equation that defines $\mathscr{A}_E$, the minimum of each segment of the composition is chosen as the root of the respective tree, an internal composition is taken in the remaining elements of each segment, and an element of $\mathscr{A}_E$ if placed in each of them. The two level of composition is represented as follows. The vertices in red are the roots of each of the second stage subtrees,

$$23|4\|567\|89|10\|1112|13.$$

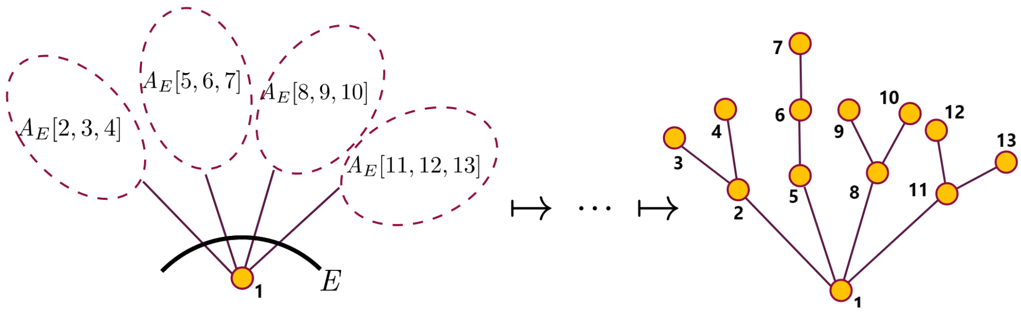


Figure 3.11: An element of $\mathscr{A}_E[13]$
in $X[\{1\}] \times \mathscr{A}_E[\{2,3,4\}] \times \mathscr{A}_E[\{5,6,7\}] \times \mathscr{A}_E[\{8,9,10\}] \times \mathscr{A}_E[\{11,12,13\}]$

To obtain the explicit ordinary generating function of ordered trees we carry on the following calculations:
Taking generating functions in (3.21), we get

$$\widetilde{\mathscr{A}_E}(x) = x\widetilde{E}(\widetilde{\mathscr{A}_E}(x)) = \frac{x}{1-\widetilde{\mathscr{A}_E}(x)}. \tag{3.23}$$

Making the change of variable $y = \widetilde{\mathscr{A}_E}(x)$ and replacing in (3.23), we obtain

$$\begin{aligned} y &= \frac{x}{1-y} \\ \Longrightarrow y(1-y) &= x \\ \Longrightarrow y^2 - y + x &= 0 \end{aligned}$$

So that, solving the quadratic equation and taking our variable change back we get

$$\widetilde{\mathscr{A}_E}(x) = \frac{1-\sqrt{1-4x}}{2} \tag{3.24}$$

We choose the root with negative sign to ensure the positivity of the species and can properly carry out the substitution. Getting back to (3.21) we can obtain the generating function for lists of planar trees in ascending order, solving (3.23) for $\widetilde{E}(\widetilde{\mathscr{A}_E}(x))$.

$$\begin{aligned}\widetilde{E}(\widetilde{\mathscr{A}_E}(x)) &= \frac{\widetilde{\mathscr{A}_E}(x)}{x} \\ &= \frac{1-\sqrt{1-4x}}{2x}\end{aligned} \tag{3.25}$$

Which is the generating function for the Catalan numbers, a sequence of positive integers appearing in many counting problems in combinatorics, given by the formula

$$C_n = \frac{1}{n+1}\binom{2n}{n} = \binom{2n}{n} - \binom{2n}{n+1}, \text{ for} n \geq 0. \tag{3.26}$$

Furthermore, we can establish a bijection between lists of trees in ascending order (elements of $E\langle\mathscr{A}_E$) and non-crossing partitions, another member of the Catalan family, which is a collection of discrete objects whose enumeration can be done using (3.25). The reader interested in getting a wider background on Catalan numbers and their applications is referred to [Sta11; Sta15]. Before stating and constructing the bijection, let us define non-crossing partitions.

Definition 3.2.3. A set partition of $[n]$ is called **non-crossing** if whenever four elements, $1 \leq a < b < c < d \leq n$, are such that if a, c are in the same class and b, d are in the same class, then the classes must be the same.

To illustrate Definition 3.2.3.

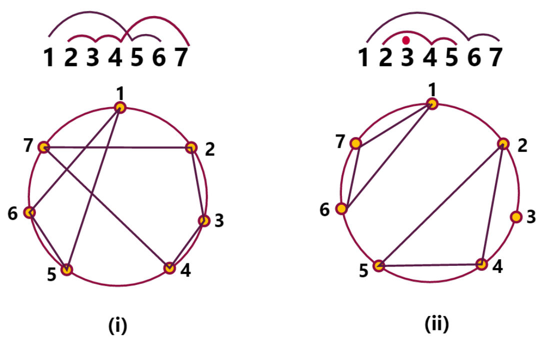


Figure 3.12: Two partitions of [7], in linear and circular representation: (i)crossing 156/2347, (ii) non-crossing 167/245/3

Theorem 3.2.1. *There is a bijection between the species of non-crossing partitions, $NCP[n]$, and lists of trees in ascending order $E\langle\mathscr{A}_E\rangle[n]$.*

Proof. Given a non-crossing partition of n in k blocks, the corresponding list of trees in ascending order is built as follows: The roots of each tree in the list are labelled, from left to right, with the labels in the block containing 1. We then find the block that contains the smallest remaining element, say m, and label with them the sons of the node labelled $m-1$. Repeat this procedure until all k blocks are used.

For the reverse transformation, given a list of trees in ascending order with n edges and k internal nodes, we can recover the blocks of the corresponding non-crossing partition in the following way: The list of roots form a block always and then for each tree in the list, the blocks are formed by nodes on the same level. □

This bijective proof was adapted from that on [DZ86]. To give the reader a better understanding of the presented bijection we do so with the following graph.

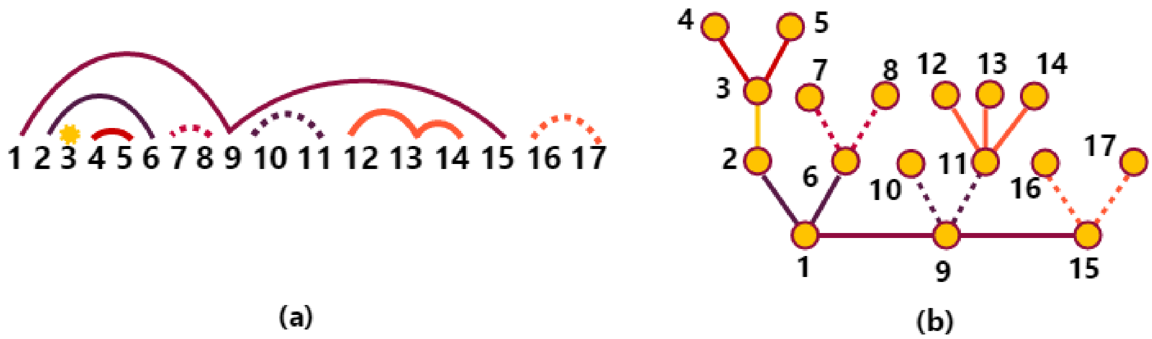


Figure 3.13: (a) a non-crossing partition of [17], (b) the list of ascending ordered trees associated to it.

3.3 The category of $\mathscr{L}$-structures

The description of a species in particular is done frequently by specifying the conditions structures must fit to belong to the species. A significant part of the concept is the transport of structures, i.e., the ability to change labels without changing the underlying structure. To enrich $\mathscr{L}$-Species as a category in itself we turn to Joyal's categorification of combinatorial species [Joy81] and adapt it to our frame of reference.

If l is a linear order, $M[l]$ is the set of all *structures* of the species M on l. We say that l is the subjacent linear order of $m \in M[l]$. We also say, in an abuse of notation, that m is an *element* of M and that it is an M-structure. In particular, M sends a linear order of *labels* to *labelled ordered structures*.

If $u : l \to l'$ is a bijection, the element $t = M[u](m)$ is the structure on l' obtained by *transport along u*. The bijection u is an isomorphism between $m \in M[l]$ and $t \in M[l']$ which can be thought of as a relabelling which, via transport, sends the structure s to the structure t.

$$u : s \to t$$

We denote by $el(\mathscr{L})$ the category whose objects are pairs of the form (l, m) such that $m \in M[l]$, and whose morphisms are order preserving bijections; it is the grupoid of *elements* of $\mathscr{L}$, constructions that can be performed on linear orders. A *grupoid* is a category where all morphisms have an inverse.

Definition 3.3.1. Let $P : l \to l$ and $Q : l' \to l'$ be permutations. Then $f : l \to l'$ is a permutation isomorphism if and only if the following diagram commutes:

$$\begin{tikzcd} l \arrow[r, "P"] \arrow[d, "f"'] & l \arrow[d, "f"] \\ l' \arrow[r, "Q"] & l' \end{tikzcd} \tag{3.27}$$

The concept of isomorphism of structures defines an equivalence relation whose classes are the *types* of structures of the species M, that is, those species related by relabelling. We denote by $\pi_0(M)$ the set of types of the species M. Additionally, if $m \in M$ we denote the type of m by $|m| \in \pi_0(M)$. We introduce the concept of subspecies.

Definition 3.3.2. A **subspecies** S of a species M is a subset of M which is also a species.

Subspecies result from restricting a species to any subset defined with a property that is relabelling invariant.

Example 3.3.1. Rooted trees is a subspecies of forests

Example 3.3.2. Permutations is a subspecies of endofunctions.

Chapter 4

Monoids and Operads in Species

In this chapter we generalize the results from Chapter 2 to the setting of species. For this, we introduce c-monoids and c-operads together with some additional considerations needed to ensure our construction is well defined. We also apply the construction and give a combinatorial interpretation of the results.

4.1 Monoids and c-monoids

Monoids can be defined in the general context of monoidal categories, see [ML78]. In Chapter 2, we defined monoids in the familiar monoidal category of Sets with respect to the Cartesian product. By simplicity, we shall avoid the technical details about monoids in the general context of monoidal categories. It suffices to say that the binary operations of product and substitution among species defines each of them a monoidal category of its own (as the Cartesian product defines a monoidal category on Sets). This fact enables us to define monoids with respect to the product (we keep calling them monoids), and monoids with respect to the substitution (that are called operads). As the backbone of this work is $\mathscr{L}$-species, we shall define monoids and operads on $\mathscr{L}$-species with respect to the operations of product substitution respectively. Operads with respect to the ordinal substitution are found in the literature with the name of non-symmetric operads.

Definition 4.1.1. A **monoid** $(M, \mathfrak{e}, \nu)$ with respect to the ordinal product is an $\mathscr{L}$-species M together with the morphism $\mathfrak{e}: 1 \to M$, and the product $\nu: M \diamond M \to M$ satisfying the following properties:

1. (Associativity) For $m_{l_1}, m_{l_2}, m_{l_3}$:

$$\nu(\nu(m_{l_1}, m_{l_2}), m_{l_3}) = \nu(m_{l_1}, \nu(m_{l_2}, m_{l_3})).$$

   With m_{l_i} in $M[l_i]$ for all $i = 1, 2, 3$ and $l_i \cap l_j = \emptyset$ for $i \neq j$.

2. (Existence of identity element) $\exists \mathfrak{e} \in M[\emptyset],\ \forall\ m \in M[l] : \nu(m, \mathfrak{e}) = m = \nu(\mathfrak{e}, m)$.

Equivalently, in diagram form we have the existence of the identity element

$$\begin{tikzcd} M\diamond 1 \arrow[r, "M\diamond\mathfrak{e}"] \arrow[dr, "\tilde{\varrho}"'] & M\diamond M \arrow[d, "\nu"] & 1\diamond M \arrow[l, "\mathfrak{e}\diamond M"'] \arrow[dl, "\tilde{\lambda}"] \\ & M & \end{tikzcd} \tag{4.1}$$

and associativity of the product

$$\begin{tikzcd} M\diamond(M\diamond M) \arrow[r, "M\diamond\nu"] \arrow[d, "\tilde{\alpha}"] & M\diamond M \arrow[r, "\nu"] & M \\ (M\diamond M)\diamond M \arrow[r, "\nu\diamond M"] & M\diamond M \arrow[ur, "\nu"] & \end{tikzcd} \tag{4.2}$$

As category theory is only a mean to and end in the present work, we expand on some of the presented morphisms used to define the monoid structure

- $M\diamond M[l] = \sum_{l_1+l_2=l} M[l_1]\times M[l_2] \xrightarrow{\nu} M[l]$.
- $M\diamond 1[l] = \sum_{l_1+l_2=l} M[l_1]\times 1[l_2] \xrightarrow{\tilde{\varrho}} M[l]$.
- $1\diamond M[l] = \sum_{l_1+l_2=l} 1[l_1]\times M[l_2] \xrightarrow{\tilde{\alpha}} M[l]$.

The morphisms $\tilde{\lambda},\tilde{\varrho}$ correspond to the left and right identity isomorphisms, while the morphism $\tilde{\alpha}$ is the associativity isomorphism.

Definition 4.1.2. A monoid $(M,\nu,\mathfrak{e})$ for which the following properties hold

1. The left cancellation law, i.e.,

$$\nu(m_{l_1}, m_{l_2'}) = \nu(m_{l_1}, m_{l_2}) \Rightarrow m_{l_2'} = m_{l_2}. \tag{4.3}$$

2. $|M[\emptyset]| = 1$.

is called a **c-monoid**. Since M has as identity an element of $M[\emptyset]$, what Property 2 really says is that the only M-structure on the empty set is the identity, $M[\emptyset] = \{\mathfrak{e}\}$.

Example 4.1.1. E is a c-monoid with the empty segment as identity and product $\nu : E\diamond E[l]\to E[l]$ decomposing the subjacent linear order into two and then putting it back together by means of concatenation.

In a similar way as the one proposed previously, we introduce how to build a poset P_M over a totally ordered set l associated to a c-monoid M. The subjacent set of $P_M[l]$ is defined as

$$\begin{aligned} M\diamond E[l] &= \{(m_{l_1}, l_2) \mid l_1 + l_2 = l\} \\ &= \{m_{l_1} \mid m_l \in M \ \&\ l_1 \text{is an initial segment of } l\} \end{aligned}$$

We define the relation $\leq$ as follows:

$$m_{l_1} \leq m_{l_2} \iff \exists m'_{l_2} : \nu(m_{l_1}, m'_{l_2}) = m_{l_2}. \tag{4.4}$$

The definition of ordinal product implies that,

$$(m_{l_1}, m'_{l_2}) \in M \diamond M[l_2].$$

Thus, (4.4) implies that

$$m_{l_1} \leq m_{l_2} \implies l_1 \text{ is an initial segment of } l_2. \tag{4.5}$$

Proposition 4.1.0.1. *The order relation $\leq$ defined on 4.4 is a partial order in $M \diamond E[l]$*

The proof of Proposition 4.1.0.1 is similar to that in the case on monoids in the category Set, presented in the previous chapter. As the monoids treated in this chapter are a generalization in the context of species of those in Chapter 2, it comes as no surprise that some results get carried on to more general settings.

Proposition 4.1.0.2. *For a c-monoid M, the associated poset P_M has the following properties*

1. *$P_M[l]$ has a minimum element, $\hat{0}$. Furthermore, $\mathfrak{e} \in M[\emptyset]$ is the $\hat{0}$ of $P_M[l]$.*
2. *For any $(m_{l_1}, m_{l_2}) \in P_M[l]$ we have that $(m_{l_1}, m_{l_2}) \cong (\mathfrak{e}, m_{l'_2})$. With $m_{l'_2}$ the element such that $\nu(m_{l_1}, m'_{l_2}) = m_{l_2}$.*
3. *The elements of $M[n]$ are maximal on $P_M[n]$.*

Proof. Point 1 is immediate as the empty linear order is an initial segment of any linear order. In the case of 2, we define a mapping such that to any pair $(\mathfrak{e}, m'_{l_2})$ it assigns the m structure on l needed to obtain (m_{l_1}, m_{l_2}) which is an order preserving bijection. For 3, we have to show that for any element $m_n \in P_M$ such that for m in $M[n]$, $m \leq m_n$ then it must be that $m = m_n$. Note that the length of m and m_n must be n, this with the definition of $\leq$ imply that $m = m_n$. □

We can restrict a c-monoid to any submonoid S of $\mathbb{N}$ by restricting the product ν to pairs of M-structures of the form (m_{l_1}, m_{l_2}), where the lengths of the linear orders l_1 and l_2 are both in S.

Proposition 4.1.0.3. *A c-monoid M restricted to a submonoid of $\mathbb{N}$, denoted by M_S, is also a c-monoid.*

The proof for Prop. 4.1.0.3 is analogous to that in the case of restricting of c-operads to $^+1$ monoids, which will be shown in the following section.

We define the following Möbius analogue of the cardinal,

$$\text{Möb}P_M[n] = \sum_{m \in M[n]} \mu(\hat{0}, m)$$

which gives us the multiplicative inverse of the ordinary generating function of M in the following way

$$\widetilde{M}^{-1}(x) = \sum_{n=0}^{\infty} \text{Möb}P_M[n] x^n. \tag{4.6}$$

Example 4.1.2. Let us consider the c-monoid E. The associated poset $P_E[3]$, and its Möbius function are as follows,

$$
\begin{array}{cc}
123 & 0 \\
| & \\
12 & 0 \\
| & \\
1 & -1 \\
| & \\
\emptyset & 1
\end{array}
$$

Since $P_E[n]$ is a chain of length $n+1$, Möb $P_E[n] = 0$ for $n \geq 2$. And we obtain the multiplicative inverse, which according to 4.6 is

$$\widetilde{E}^{-1}(x) = 1 - x. \tag{4.7}$$

Restricting E to any submonoid S of $\mathbb{N}$ in the following way

$$E_S[l] = \begin{cases} E[l] = \{l\}, & \text{if } |l| \in S. \\ \emptyset, & \text{otherwise.} \end{cases}$$

gives rise to another c-monoid with generating function

$$\widetilde{E}_S(x) = \sum_{s \in S} x^s.$$

And with multiplicative inverse given by

$$\widetilde{E}_S^{-1}(x) = \left(\sum_{s \in S} x^s \right)^{-1} = \sum_{s \in S} \text{Möb} P_{E_S}[s] x^n. \tag{4.8}$$

Example 4.1.3. $\mathbb{S}$ is a c-monoid with the empty permutation as identity and with product $\nu : \mathbb{S}[l_1] \times \mathbb{S}[l_2] \longrightarrow \mathbb{S}[l]$.

We construct the associated poset $P_{\mathbb{S}}$ to the c-monoid $\mathbb{S}$, with subjacent set $(\mathbb{S} \diamond E[l], \leq_\nu)$ in the following way:

$$\mathbb{S} \diamond E[l] = \{(\sigma_{l_1}, l_2) | l_1 + l_2 = l\}$$

Which is equivalent to the set of permutation on initial segments of l,

$$\mathbb{S} \diamond E[l] = \{(\sigma_{l_1}) | l_1 \text{ an initial segment of } l\ \}$$

The elements of $\mathbb{S} \times \mathbb{S}[l]$ are of the form (σ_1, σ_2) where σ_1 is a permutation on an initial segment of l_1 of l, σ_2 is a permutation on l_2 and $l_1 + l_2 = l$. Thus, ν acts as the concatenation of permutations on the initial and final segment of l in the following way

$$\begin{aligned} \mathbb{S}[l_1] \times \mathbb{S}[l_2] &\longrightarrow \mathbb{S}[l] \\ (\sigma_1, \sigma_2) &\xrightarrow{\nu} \sigma_1 + \sigma_2 \end{aligned}$$

The order relation is defined by

$$\sigma_1 \leq_v \sigma \iff \sigma|_{l_1} = \sigma_1. \tag{4.9}$$

That is, if σ_1 is a permutation on an initial segment of l and σ restricted to this segment is equal to σ_1. The poset $P_{\mathbb{S}}[l]$ is always a tree with the empty set as its root and with permutations of l as leaves. To illustrate we provide an example.

Example 4.1.4. The poset $P_{\mathbb{S}}[3]$ is represented in the following tree

$$\begin{tikzcd}
 & 123 \arrow[d, dash] & & & & \\
132 \arrow[dr, dash] & 12 \arrow[d, dash] & 213 \arrow[d, dash] & & & \\
 & 1 \arrow[drr, dash] & 21 \arrow[dr, dash] & 231 \arrow[d, dash] & 312 \arrow[dl, dash] & 321 \arrow[dll, dash] \\
 & & & \emptyset & &
\end{tikzcd}$$

Note that in any tree-like poset, with the root being the zero of the poset, every interval is a chain. In Example 4.1.4 we have that for leaves σ in $P_{\mathbb{S}}[n]$, if the chain $[\hat{0},\sigma]$ is of length 2 or more, $\mu(\hat{0},\sigma) = 0$. Otherwise, when a leaf σ covers the root, $\mu(\hat{0},\sigma) = -1$. Hence,

$$\text{Möb} P_{\mathbb{S}}[n] = -\sum_{\sigma\in\mathbb{S},\ \hat{0}\lessdot\sigma} \mu(\hat{0},\sigma)$$

A permutation $\sigma \in \mathbb{S}[n]$ covers the empty set if, and only if, the image by σ of every proper initial segment $[k]$ of $[n]$ is not $[k]$, $1 \leq k < n$,

$$\sigma([k]) \neq [k], \text{ for every } k,\ 1 \leq k < n. \tag{4.10}$$

Otherwise, we would have $\emptyset < \sigma_1 < \sigma$, with $\sigma_1 = \sigma|_{[k]}$. Permutations satisfying (4.10) have been called *connected* in the literature. Denote by C the species of connected permutations. Then, we have

$$\text{Möb} P_{\mathbb{S}}[n] = \begin{cases} 1 & \text{if } n = 0 \\ -|C[n]| & \text{otherwise.} \end{cases} \tag{4.11}$$

So, the Möbius function counts the primes on the poset $P_{\mathbb{S}}$ (elements covering $\hat{0}$), as connected permutations are indeed the primes in this case. Furthermore, all maximal elements which are not prime have a Möbius function value of 0. Then, by a simple computation of the Möbius function we recover the result by Comtet [Com74].

$$\widetilde{\mathbb{S}}^{-1}(x) = \Big(\sum_{n=0}^{\infty} n!x^n\Big)^{-1} = 1 - \widetilde{C}(x). \tag{4.12}$$

Another way of obtaining the result is to observe that every permutation $\sigma_l \in \mathbb{S}[l]$ can be decomposed in blocks of connected permutations. Thus,

$$\mathbb{S} = \sum_{k=0}^{\infty} C^k. \tag{4.13}$$

where,

$$C^k[l] = \sum_{l_1+l_2+\cdots+l_k=l} C[l_1] \times C[l_2] \times \cdots \times C[l_k]. \tag{4.14}$$

We can rewrite $\mathbb{S}$ as $E\langle C\rangle$ and then (4.13), as shown by Gessel and Stanley [GS95], is equivalent to

$$\widetilde{\mathbb{S}}(x) = \frac{1}{1-\widetilde{C}(x)} \tag{4.15}$$

Then we have that,

$$1-\widetilde{C}(x) = (\widetilde{\mathbb{S}}(x))^{-1}. \tag{4.16}$$

Expanding the right hand side of (4.16), up to the first 10 terms, we obtain

$$\widetilde{C}(x) = x - 2x^2 + 2x^3 - 4x^4 - 4x^5 - 48x^6 - 336x^7 - 2928x^8 - 28144x^9 - 298528x^{10}\ldots \tag{4.17}$$

4.2 Operads

In this section we introduce operads in the simpler and familiar context of sets. We start by giving an informal and natural description of what a set operad is.

A set operad is a collection of:

- A family of labeled combinatorial structures.
- An "associative" mechanism η that produces larger structures from smaller ones, using as an assembler an external structure within the family.
- Identity structures over the singleton sets.

Operads are monoids in the general categorical sense and live in an environment that is called a *monoidal category*. In this work we consider only positive species, that is, species assigning the empty set to the empty set.

Operads are assembling mechanisms given by a pair $(a,\theta') \in \mathscr{O}\langle\mathscr{O}\rangle[l]$ where a is a segmented assembly of l and θ' is a structure on the subjacent composition of a. Hence, $\eta(a,\theta')$ is a structure θ on $\mathscr{O}[l]$ which is associative and has a neutral element.

Positive $\mathscr{L}$-species together with the operation of ordinal substitution, the identity $e : X \longrightarrow \mathscr{O}$ choosing the identity in $\mathscr{O}[1]$, and the isomorphisms α, ϱ, and λ constitute a monoidal category ($\mathscr{L}_+$,$\circ$, e, α,ϱ, λ). For arbitrary, positive $\mathscr{L}$-species M, N, and R we have the canonical identity and associativity isomorphisms ϱ,λ, and α given by

$$M\langle X\rangle \xrightarrow{\varrho} M \xleftarrow{\lambda} X\langle M\rangle \tag{4.18}$$

and

$$M\langle N\langle R\rangle\rangle \xrightarrow{\alpha} \langle M\langle N\rangle\rangle\langle R\rangle \tag{4.19}$$

We proceed to formally define operads

Definition 4.2.1. An **non-symmetric operad** ($\mathscr{O}$, η, e) is a monoid in the monoidal category of positive species with respect to the operation of substitution. This means:

1. The morphism $\mathscr{O}\langle\mathscr{O}\rangle \xrightarrow{\eta} \mathscr{O}$ is an associative product and the following diagram commutes

$$\begin{tikzcd}
\mathscr{O}\langle\mathscr{O}\langle\mathscr{O}\rangle\rangle \arrow[r, "\mathscr{O}\langle\eta\rangle"] \arrow[d, "\alpha"] & \mathscr{O}\langle\mathscr{O}\rangle \arrow[dr, "\eta"] & \\
\langle\mathscr{O}\langle\mathscr{O}\rangle\rangle\mathscr{O} \arrow[r, "\eta\langle\mathscr{O}\rangle"] & \mathscr{O}\langle\mathscr{O}\rangle \arrow[r, "\eta"] & \mathscr{O}
\end{tikzcd} \tag{4.20}$$

2. The morphism $X \xrightarrow{e} \mathscr{O}$ is the operadic identity and the following diagram commutes

$$\begin{tikzcd}
\mathscr{O}\langle X\rangle \arrow[r, "\mathscr{O}\langle e\rangle"] \arrow[dr, "\varrho"'] & \mathscr{O}\langle\mathscr{O}\rangle \arrow[d, "\eta"] & X\langle\mathscr{O}\rangle \arrow[l, "e\langle\mathscr{O}\rangle"'] \arrow[dl, "\lambda"] \\
& M &
\end{tikzcd} \tag{4.21}$$

In both (4.20) and (4.21), $\mathscr{O} = 1_{\mathscr{O}}$ stands for the identity morphism of $\mathscr{O}$. We define also *shuffle operads*

Definition 4.2.2. A **shuffle operad** $\mathscr{O}$ is a monoid in the monoidal category of positive $\mathscr{L}$-species with respect to the shuffle substitution with a product η and identity e satisfying the associativity and identity properties akin to those in Definition 4.2.1.

As we have seen, it is important to establish mechanisms to operate between mathematical objects, we proceed to do so by defining homomorphims between operads.

Definition 4.2.3. Let $(\mathscr{O}_1, \eta_1, e_1)$ and $(\mathscr{O}_2, \eta_2, e_2)$ be two operads. A natural transformation $\mathscr{O}_1 \xrightarrow{\Psi} \mathscr{O}_2$ is called an **operad homomorphism** if Ψ preserves products and identities, and the following diagrams commute

$$\begin{tikzcd}
\mathscr{O}_1\langle\mathscr{O}_1\rangle \arrow[r, "\Psi\langle\Psi\rangle"] \arrow[d, "\eta_1"] & \mathscr{O}_1\langle\mathscr{O}_2\rangle \arrow[d, "\eta_2"] \\
\mathscr{O}_1 \arrow[r, "\Psi"] & \mathscr{O}_2
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathscr{O}_1 \arrow[r, "\Psi"] & \mathscr{O}_2 \\
X \arrow[u, "e_1"] \arrow[ur, "e_2"'] &
\end{tikzcd} \tag{4.22}$$

4.3 Posets associated to c- Operads

The understanding of the combinatorial interpretation of the ordinal substitution (3.11) of $\mathscr{L}$-species provided in Remark 3.2.1 is of utmost importance in the generalization of our construction. With that in mind, we get on with it. As in the two previous monoidal categories where we have defined c-monoids, a c-operad is an operad with non proper divisor of unity, which in this context means $|\mathscr{O}|[1] = 1$.

Definition 4.3.1. A c-operad is an operad that satisfies

1. The left cancellation law,
$$\eta(a,\theta) = \eta(a,\theta') \Rightarrow \theta = \theta'.$$

2. There is a single element in $\mathscr{O}[1]$,
$$|\mathscr{O}[1]| = 1.$$

We can also establish a poset $P_{\mathscr{O}}$ associated to any c-operad $\mathscr{O}$. Posets in the operadic sense differ to those coming from ordinal c-monoids as they come from different underlying operations.
The subjacent set of $P_{\mathscr{O}}$ is defined as $(E_+\langle\mathscr{O}\rangle[l], \leq_\eta)$. With

$$\begin{aligned}
E_+\langle\mathscr{O}\rangle[l] &= \sum_{l_1+l_2+\cdots+l_k=l} \mathscr{O}[l_1] \times \mathscr{O}[l_2] \times \cdots \times \mathscr{O}[l_k] \times E[\mathfrak{c}] \\
&= (\theta_{l_1}, \theta_{l_2}, \ldots, \theta_{l_k}) \times \mathfrak{c} \\
&= (\theta_{l_1}, \theta_{l_2}, \ldots, \theta_{l_k})
\end{aligned}$$

Hence, elements in $E_+\langle\mathscr{O}\rangle[l]$ are what we call segmented assemblies of operad structures restricted to segments l_i of the composition $\mathfrak{c}$ of l, with $\mathfrak{c} = (l_1, l_2, \ldots, l_k)$. In order to define the general order relation on $P_{\mathscr{O}}$, we introduce some notation necessary to avoid confusion.

1. $a = (\theta_{l_1}, \theta_{l_2}, \ldots, \theta_{l_k})$ a segmented assembly of $\mathscr{O}$-structures such that $\theta_{l_i} \in \mathscr{O}[l_i]$. Recall that $\mathfrak{c} = (l_1, l_2, \ldots, l_k)$ is a composition of l, that we call *the subjacent composition of a* .

2. Given the composition $\mathfrak{c}$ we shall consider compositions of $\mathfrak{c}$, i.e., ordered segments $\mathfrak{c}^{(i)}$ of $\mathfrak{c}$ such that $\mathfrak{c}^{(1)} + \mathfrak{c}^{(2)} + \cdots + \mathfrak{c}^{(r)} = \mathfrak{c}$.

3. We denote by $a_{\mathfrak{c}^{(i)}}$ the restriction of a to the segments in $\mathfrak{c}^{(i)}$, and by l_i' the ordinal sum of linear orders in $\mathfrak{c}^{(i)}$.

4. We shall consider segmented assemblies of the form $a" = (\theta"_{\mathfrak{c}^{(1)}}, \theta"_{\mathfrak{c}^{(2)}}, \ldots, \theta"_{\mathfrak{c}^{(r)}})$ where each $\theta''_{\mathfrak{c}^{(i)}}$ is an element of $\mathscr{O}[\mathfrak{c}^{(i)}]$. Recall that each $\mathfrak{c}^{(i)}$ is itself a linear order.

Definition 4.3.2. We define a partial order relation on the set on the segmented assemblies of $\mathscr{O}$, $E\langle\mathscr{O}\rangle[l]$ as follows

$$a \leq_\eta a' \iff \exists a" : \eta(a_{\mathfrak{c}^{(i)}}, \theta''_{\mathfrak{c}^{(i)}}) = \theta'_{l_i'}, \text{ for } i = 1, 2, \ldots, r.$$

That is, an assembly a is comparable to a' if there is an assembly $a"$ such that a' is the result of assembling the segments of a with respect to the order structure on the segments of $a"$. Figure 4.1 provides a scheme of the proposed assembling mechanism.

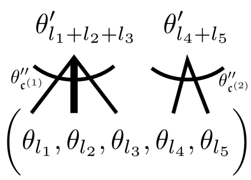


Figure 4.1: Order on the segmented assembly $(\theta_{l_1}, \theta_{l_2}, \theta_{l_3}, \theta_{l_4}, \theta_{l_5})$.

We provide some particular examples to clarify the proposed construction of the poset associated to a c-operad $\mathscr{O}$.

Example 4.3.1. E_+ is a c-operad. Then $E_+\langle E_+\rangle[l]$ defines a partial order over the compositions of l with $\eta : E_+\langle E_+\rangle[l] \longrightarrow E_+[l]$ joining the segments of the linear order. And with the identity being the linear order with one element segments.

Take $E_+\langle E_+\rangle[3]$, with graph

$$\begin{tikzcd} & 123 \arrow[dl, dash] \arrow[dr, dash] & \\ 12|3 \arrow[dr, dash] & & 1|23 \arrow[dl, dash] \\ & 1|2|3 & \end{tikzcd}$$

Notice that this poset is isomorphic to the Boolean poset $P(n-1)$ if we identify it with the complement of the position of the bars separating the segments in the following way

$$\begin{tikzcd}[row sep=small] & & 1 & & \\ & & \{1,2\} \arrow[dl, dash] \arrow[dr, dash] & & \\ -1 & \{1\} \arrow[dr, dash] & & \{2\} \arrow[dl, dash] & -1 \\ & & \varnothing & & \\ & & 1 & & \end{tikzcd}$$

Thus, since $\mu(B(n)) = (-1)^{n-1}$,

$$\text{Möb}P_{E_+}[n] = \mu(\hat{0}, 12\ldots n) = (-1)^{n-1} \tag{4.23}$$

The inverse of the ordinary generating function with respect to the substitution for E_+ can be found solving

$$\widetilde{E_+}(x) = \sum_{n=1}^{\infty} x^n = \frac{x}{1-x}$$

for x, which yields

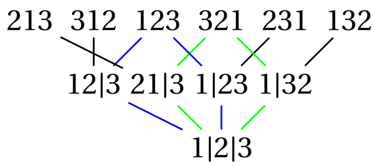
$$\widetilde{E_+}^{<-1>}(x) = \frac{x}{1-x} = x\frac{1}{1-x} = x\sum_{n=0}^{\infty}(-1)^n x^n = \sum_{n=1}^{\infty}(-1)^{n-1}x^n. \tag{4.24}$$

Example 4.3.2. $\mathbb{S}_+$ is a c-operad. The elements of $\mathbb{S}_+\langle\mathbb{S}_+\rangle$ are pairs (a,$\tau_{\mathfrak{c}}$) where a is a segmented assembly of permutations together with a permutation rearrangement for each segment, and $\tau_{\mathfrak{c}}$ an external permutation on the composition $\mathfrak{c}$. The product $\eta(a,\sigma_{\mathfrak{c}})$ is obtained by concatenating the internal permutations after rearranging them using the external permutation $\tau_{\mathfrak{c}}$. For example,

$$\eta(\{12,43,5,867\}, l_4 l_2 l_1 l_3) = 86743125$$

Here $l_1 = 12,\ l_2 = 34,\ l_3 = 5, l_4 = 678,\ \mathfrak{c} = 12|34|5|678$, and $\tau_{\mathfrak{c}} = l_4 l_2 l_1 l_3$

Consider the poset $P_{\mathbb{S}_+}[3]$, with graph



Theorem 4.3.1. *The Möbius function of the intervals* $[\hat{0},\sigma]$ *of* $P_{\mathbb{S}_+}[n]$*,* $\sigma \in \mathbb{S}_+[n]$ *is equal to*

$$\mu(\hat{0},\sigma) = \begin{cases} (-1)^n & \textit{either if } \sigma \textit{ is the identity or its reverse} \\ -1 & \textit{if } \sigma \textit{ is prime.} \\ 0 & \textit{any other case.} \end{cases} \tag{4.25}$$

As in the case of monoids, we can restrict any c-operad $\mathscr{O}$ to a $^+1$ monoid S^+, denoted by $\mathscr{O}_{S^+}$. For this construction we take into consideration properties of the ordinal substitution and of $^+1$ monoids to ensure it is well defined, that is, that our restricted operadic structure is closed under the assembling mechanism η.

Proposition 4.3.1.1. *A c-operad* $\mathscr{O}$ *restricted to a* $^+1$ *monoid* S^+*, denoted by* $\mathscr{O}_{S^+}$*, is also a c-operad.*

Proof. Let $\mathscr{O}$ be a c-operad, S^+ be a $^+1$ monoid. Then, the elements of $\mathscr{O}_S\langle\mathscr{O}_S\rangle$ are of the form

$$\big((\theta_{l_1},\theta_{l_2},\dots,\theta_{l_s}),\theta_{\mathfrak{c}}\big) \tag{4.26}$$

By the definition of ordinal substitution, for $i = 1,2,\dots,s$, it follows that

$$\theta_{l_i} \in \mathscr{O}_S^+[l_i] \Longrightarrow |l_i| \in S^+.$$

And

$$\theta_{\mathfrak{c}} \in \mathscr{O}_S^+[\mathfrak{c}] \Longrightarrow |\mathfrak{c}| = s \in S^+. \tag{4.27}$$

So that, applying η in (4.26) we obtain

$$\eta\big((\theta_{l_1},\theta_{l_2},\dots,\theta_{l_s}),\theta_{\mathfrak{c}}\big) = \theta'_{l_1+l_2+\cdots+l_s} \tag{4.28}$$

By (4.27) and Proposition 2.3.0.1, we have that $|l_1 + l_2 + \cdots + l_s| \in S^+$, and thus $\theta'_{l_1+l_2+\cdots+l_s} \in \mathscr{O}_{S^+}[l]$. Hence, η is closed and our operadic structure is well defined. Thus, an operad restricted to a $^+1$ monoid is also an operad. □

Furthermore, a c-operad restricted to any $^+1$ monoid is also a c-operad because the restricted set inherits the properties of the initial one, including those neeeded for constructing $P_{\mathscr{O}_{S^+}}$, the poset associated to $\mathscr{O}_{S^+}$.
In particular, for any $^+1$ monoid and the positive uniform species E_+, we have that $E_{S^+}[l]$ is a c-operad with associated poset $P_{E_{S^+}}[l]$ which follows an order relation on the compositions of l with cardinality in S^+, as we have seen in Section 2.3.

Chapter 5

Conclusions and future work

In this work we built posets from numerical monoids and $^{+}1$ monoids. For $^{+}1$ monoids, this construction hints to the fact that they are operadic structures as we have proven that, due to their clausure properties, the restriction of an operad to a $^{+}1$ monoid it's still an operad. The restriction of other c-operads, such as $\mathbb{S}_{+}$, gives rise to different families of operads whose Möbius function can lead to results with combinatorial interpretations of potential interest.

Another aspect to possibly study further is the meaning behind the Möbius function associated to posets arising from different classes of numerical monoids, such as Arf and symmetrical semigroups, and the $^{+}1$ monoids we can get from them.

Algebras associated to additive submonoids of $\mathbb{N}$ can be studied for their Koszulness and their dual algebras. The zeta function for this kind of algebras gives their Hilbert series as a graded algebra. In turn, the Möbius function gives the Hilbert series of the graded dual algebra, with alternating signs. Connections between these algebras and algebraic curves associated to numerical monoids can also be studied in depth later on.

References

[AA05] M. H. Albert and M. D. Atkinson. "Simple permutations and pattern restricted permutations". In: *Discrete Mathematics* 300 (2005), pp. 1–15.

[AAK03] M. H. Albert, M. D. Atkinson, and M. Klazar. "The enumeration of simple permutations". In: *Journal of Integer Sequences* (2003).

[AGS16] Abdallah Assi and Pedro A. García-Sánchez. *Numerical Semigroups and Applications*. First. Vol. 1. RSME Springer Series. Springer International Publishing, 2016. DOI: 10.1007/978-3-319-41330-3.

[Atk99] M. D. Atkinson. "Restricted permutations". In: *Discrete Mathematics* 195 (1999), pp. 27–38. ISSN: 0012-365X. DOI: 10.1016/s0012-365x(98)00162-9.

[Ava07] Jean-Christophe Aval. "Multivariate Fuss-Catalan numbers". In: *Discrete Mathematics 308 (2008) 4660-4669* (2007).

[BA13] Maria Bras-Amorós. "Numerical semigroups and codes". In: *Algebraic Geometry Modeling in Information Theory*. World Scientific, 2013. Chap. 5, pp. 167–218. DOI: https://doi.org/10.1142/9789814335768_0005.

[BHS02] A. Bergeron, S. Heber, and J. Stoye. "Common intervals and sorting by reversals: a marriage of necessity". In: *Bioinformatics* 18 (2002), S54–S63. DOI: 10.1093/bioinformatics/18.suppl_2.s54.

[BLL97] François Bergeron, Gilbert Labelle, and Pierre Leroux. *Combinatorial Species and Tree-like Structures*. Cambridge University Press, 1997. DOI: 10.1017/CBO9781107325913.

[BMR20] Beáta Bényi, Miguel Méndez, and José L. Ramirez. "Generalized Ordered Set Partitions". In: *Australasian Journal of Combinatorics* 77(2) (2020), pp. 157–179. ISSN: 2202-3518.

[Bé+05] Sèverine Bérard et al. "Perfect Sorting by Reversals Is Not Always Difficult". In: *Algorithms in Bioinformatics*. Lecture Notes in Computer Science. Springer, 2005, pp. 228–238. DOI: 10.1007/11557067_19.

[Com74] Louis Comtet. *Advanced Combinatorics*. Springer Netherlands, 1974. DOI: 10.1007/978-94-010-2196-8.

[DZ86] Nachum Dershowitz and Shmuel Zaks. "Ordered trees and non-crossing partitions". In: *Discrete mathematics* 62.2 (1986), pp. 215–218.

[Gri04] R. P. Grimaldi. *Discrete and Combinatorial Mathematics*. Pearson College Division, 2004.

[GS95] Ira M. Gessel and Richard P. Stanley. “Algebraic Enumeration”. In: *Handbook of combinatorics* 2 (1995), pp. 1021–1061.

[Joy81] André Joyal. “Une théorie combinatoire des séries formelles”. In: *Advances in Mathematics* 42 (1981), pp. 1–82. DOI: 10.1016/0001-8708(81)90052-9.

[Knu97] Donald E. Knuth. *The art of computer programming*. Vol. 1. Addison-Wesley, 1997.

[Lan02] Serge Lang. *Algebra*. Third. Graduate Texts in Mathematics 211. Springer-Verlag New York, 2002. DOI: 10.1007/978-1-4613-0041-0.

[Lin11] Chin-Hung Lin. “Some Combinatorial Interpretations and Applications of Fuss-Catalan Numbers”. In: *ISRN Discrete Mathematics* (2011). DOI: https://doi.org/10.5402/2011/534628.

[Mén15] Miguel A. Méndez. *Set Operads in Combinatorics and Computer Science*. Springer International Publishing, 2015. DOI: 10.1007/978-3-319-11713-3.

[ML78] Saunders Mac Lane. “Monoids”. In: *Categories for the Working Mathematician*. Graduate Texts in Mathematics. Springer-Verlag, 1978. Chap. VII, p. 161. ISBN: 9780387984032. DOI: 10.1007/978-1-4757-4721-8.

[MP15] Wojciech Młotkowski and Karol A. Penson. “A Fuss-type family of positive definite sequences”. In: *Colloquium Mathematicum, 151(2), 2018, 289-304* (2015).

[MY91] Miguel Méndez and Julia Yang. “Möbius Species”. In: *Advances in Mathematics* 85.1 (1991), pp. 83–128.

[Mł10] Wojciech Młotkowski. “Fuss-Catalan Numbers in Noncommutative Probability”. In: *Documenta Mathematica* 15 (2010), pp. 939–955.

[NS06] Alexandru Nica and Roland Speicher. *Lectures on the Combinatorics of Free Probability*. Cambridge University Press, 2006.

[Pie91] Benjamin C. Pierce. *Basic Category Theory for Computer Scientists*. MIT press, 1991.

[Rie17] Emily Riehl. *Category Theory in Context*. Dover Publications, 2017.

[Rot64] Gian-Carlo Rota. “On the foundations of combinatorial theory I. Theory of Möbius Functions”. en. In: *Zeitschrift für Wahrscheinlichkeitstheorie und Verwandte Gebiete* 2.4 (1964), pp. 340–368. ISSN: 1432-2064. DOI: 10.1007/BF00531932.

[Sae12] K. Saengsura. “On Apéry Sets of Symmetric Numerical Semigroups with e(S)=p”. In: *International Mathematical Forum*. Vol. 7. 37. 2012, pp. 1805–1809.

[SS18] S. Samchuck-Schnarch. “An introduction to operad theory”. 2018. URL: https://alistairsavage.ca/pubs/Samchuck-Schnarch-Operads.pdf.

[Sta11] Richard P. Stanley. *Enumerative Combinatorics*. Vol. 1. Cambridge Studies in Advanced Mathematics. Cambridge University Press, 2011. DOI: 10.1017/CBO9781139058520.

[Sta15] Richard P. Stanley. *Catalan Numbers*. Cambridge University Press, 2015. ISBN: 9781139871495. DOI: 10.1017/CBO9781139871495.

[Voi97] Dan V. Voiculescu. *Free Probability Theory*. American Mathematical Soc., 1997.

[Yor14] Brent Yorgey. "Combinatorial species and labelled structures". 2014. URL: http://ozark.hendrix.edu/~yorgey/pub/series-formelles.pdf.