UNIVERSIDAD DE INVESTIGACIÓN DE TECNOLOGÍA EXPERIMENTAL YACHAY

Escuela de Ciencias Físicas y Nanotecnología

TÍTULO: GUP corrections to black hole thermodynamics

Trabajo de integración curricular presentado como requisito para la obtención del título de Físico

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Urcuquí, diciembre 2020

UNIVERSIDAD YACHAY TECH

Urcuquí, 4 de diciembre de 2020

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(Vicerrectorado Académico/Cancillería)
ESCUELA DE CIENCIAS FÍSICAS Y NANOTECNOLOGÍA
CARRERA DE FÍSICA
ACTA DE DEFENSA No. UITEY-PHY-2020-00024-AD

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Agradecimiento

Me gustaría expresar mi gran agradecimiento al profesor Ernesto Contreras por sus valiosas y constructivas sugerencias durante la planificación y desarrollo de este trabajo de graduación.

Me gustaría agradecer a los profesores de la Escuela de Ciencias Físicas y Nanotecnología, por su paciencia enseñando y compartiendo conocimientos durante la carrera.

Mi agradecimiento especial a Edgar y Roxana, mis padres por su apoyo y aliento a lo largo de mis estudios y desarrollo profesional.

Eduardo Emilio López Vélez

Resumen

En este trabajo, estudiamos las correcciones a la temperatura y la entropía de un agujero negro en el contexto del principio de incertidumbre generalizado. En particular, obtenemos soluciones de agujero negro asintóticamente anti-de-Sitter corregidas siguiendo el esquema de espacio de fase extendido en la que la constante cosmológica se considera como una presión termodinámica que satisface cierta ecuación de estado. Entre todas las posibilidades, consideramos que la presión cosmológica satisfacía una ecuación de estado de Van der Waals, politrópica o de Chaplying. Se estudió la plausibilidad física de las soluciones en función de las condiciones de energía.

Palabras Clave: Principio de incertidumbre generalizado, termodinámica de agujeros negros, ecuación de estado.

Abstract

In this work, we study corrections to the black hole temperature and entropy in the context of the generalized uncertainty principle. In particular, we obtain corrected asymptotically anti de–Sitter black hole solutions following the extended phase scheme in which the cosmological constant is considered as a thermodynamic pressure satisfying certain equation of state. Among all the possibilities, we considered that the cosmological pressure satisfied either a Van der Waals, polytropic or Chaplying equation of state. The physical plausibility of the solutions was studied based on the energy conditions.

Key Words: Generalized uncertainty principle, black hole thermodynamics, equation of state.

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Chapter 1

Introduction

The very beginning of the study of black hole (BH) lies in the classical realm. In this respect, in the development of black hole physics, some important theorems have arisen, for example, the no hair theorem, positive energy theorem, energy extraction, and area theorem which we will briefly summarize in what follows.

The non–hair theorem establishes that, for a static and asymptotically flat black hole, the system is characterized by its mass, charge and angular momentum only [1] which encodes the energy content of the system. Particularly, for a static, spherically symmetric and vacuum space–time, energy is defined as the gravitating mass as measured at infinity times the speed of light squared of the isolated system. This energy is a conserved quantity in general relativity (GR) and generates time translation symmetry at infinity.

The positive energy theorem states that if the space-time can embrace a non-singular Cauchy surface whose unique boundary is at infinity and the matter has positive energy, the total energy of the space-time must be positive. This was first demonstrated by Schoen and Yau in a geometrical way [2], and a more direct approach by Witten in Ref. [3]. A positive energy theorem has also been proved in the presence of a negative cosmological constant, in anti-de-Sitter space-time [4].

Following the above mentioned results, energy can be extracted from a black hole itself by classical processes if the system is spinning or charged. In this sense, a black hole can be used as an intermediate object to extract rest energy as useful work, for example converting mass in energy, ergoregions or a Penrose process [5–7].

It is worth mentioning that the most efficient extraction of energy occurs when the black hole area is unchanged. However, the area theorem states that the area of an event horizon can never decrease [8,9] which implies that the process of energy extraction is less efficient than in the ideal case [8].

The above mentioned results allow establishing the connection between black holes mechanics and thermodynamics. Indeed, Bekenstein and Hawking give us the first insight into the thermodynamical properties of black holes, elucidating that the entropy of black holes should be proportional to the area [10–15]. Thus, the laws of thermodynamics for black holes emerged and they are stated as follows.

The zeroth law says that the surface gravity, κ, of a stationary black hole is constant over the event horizon. The first law states the relation of central BH and neighboring stationary axisymmetric BH solution under a perfect circular flow. In the same way like entropy is related to the area, the temperature is related to the surface gravity [12]. The Second Law states that the area of the event horizon of each black hole does not decrease with time. If two black holes collide then the area of the final event horizon is greater than the sum of the initial areas. The third law claims the impossibility to reduce by any finite sequence of operations the surface gravity to zero [12].

It is worth mentioning that the thermodynamic laws stated above could be considered as a semi–classical description of BH, in the sense that they include quantum effects specifically the Hawking's radiation [14]. Even more, in the classical theory black holes absorb but cannot emit particles. Furthermore, the quantum mechanical effects yield black holes to create and emit particles as hot bodies do. This thermal emission leads to a slow decrease in the mass of the BH which eventually could evaporate. It is noticeable that, although these quantum effects contradict the classical laws of the area of the event horizon [15], the Generalized Second Law remains [13]. To be more precise, the total entropy of the black hole can never decrease.

In the same way as thermodynamics introduces semi–classical corrections in BH physics as the Hawking's effect, other approaches introduce quantum corrections to the system under study. For example, it is well known that logarithmic corrections to the entropy of BHs have been introduced in the framework of String Theory and Loop Quantum Gravity [16–21]. However, as the description of quantum mechanics (QM) and GR are not unified consistently, it remains an open problem. One possibility is the research of a fundamental theory of quantum gravity (QG), using both theories as a guiding principle [22].

In the same spirit, it is conjecture that the Heisenberg uncertainty principle (HUP) should be violated. As a result, several quantum mechanical systems must require appropriate modification [22]. In this context, a generalized uncertainty relation has been derived, which describes the minimal length as a minimal uncertainty in position measurements [23,24]. The existence of a minimal observable length at the Planck scale is predicted by several quantum theories of gravitation. It is remarkable that, generalized uncertainty principle (GUP) is consistent with String Theory and Loop Quantum Gravity and predicts quantum gravity corrections to several quantum phenomena [25,26].

The effects of the implementation of the GUP have been studied on, Newtonian law of gravity, compact stars physics, cosmic inflation observations, thermodynamics of the early Universe, among others [22]. Nevertheless, the idea of the existence of a minimal length and/or time is not new. The chronon was the first minimum measurable time interval proposed. In 1927, R. Levil proposes the indivisible interval of time as the ratio between the diameter of the electron and the speed of light, as a result this interval conduces to the special relativity (SR) and QM conjecture of a unifying framework the quantum field theory (QFT) [27].

Recently, the introduction of quantum corrections in the BH thermodynamics has been successfully implemented to obtain quantum–corrected BH solutions. In the framework of GUP and doubly special relativity (DSR), the Schwarzschild BH thermodynamics obtain correction for temperature, entropy, and heat capacity [28]. Also, GUP corrections of the topological charged BH in anti–de Sitter (AdS) space–time has been obtained by studying the thermodynamic properties and critical behaviors [29]. Similarly, the topological charged BH conduce to obtain thermodynamic corrections, in this case under the consideration of a special GUP [30]. As well as, thermodynamic effects of the GUP under the DSR on the topological charged AdS BH [31].

The standard viewpoint of the temperature in stationary BH is proportional to its surface gravity. Xiang and Wen propose a heuristic approach which consider a unified expression for the BH temperature in the context of GUP [32,33].

With the aim to investigate a static and spherically symmetric black hole and a Kerr-Newman BH in the context of GUP, in Ref.[34] the authors considered GUP–corrected BH temperature to obtain a modified Van der Waals BH. In this work, we follow an alternative route to obtain the GUP corrected solutions for Polytropic BH[35] and Chaplygin BH[36].

1.1 Overview

This work is organized as follows. In Chapter 2, the theoretical background contains two main sections. On one hand, the section 2.2 is a review of the extended phase space, which introduces negative cosmological constant as thermodynamic pressure. On the other hand, the section 2.3 is a review of the independent result GUP and some possible physical modifications. In Chapter 3, we develop the GUP thermodynamics strategy and we calculate black hole asymptotic GUP corrections for Van der Waals BH, Polytropic BH and Chaplygin BH. Finally, the last Chapter 4 contains final comments and beyond perspectives of this work.

Chapter 2

Theoretical Background

2.1 General relativity

In this section, we review the main aspects of General Relativity (GR). In particular, we explain the construction of the Einstein field equations with special emphasis on the aspects and results which allow to understand the following sections. As a complement, we obtain the well known Schwarzschild exterior solution and explore the physics behind the critical points of the solution, namely, the event horizon and the singularity.

2.1.1 Einstein field equations

In 1915 Albert Einstein proposed that gravitation is a manifestation of the curvature of the space–time by the presence of matter[37–41]. In this context, the spacetime can be thought of as curved hypersurface in four dimensions. However, the theory is formulated in such a form that captures the intrinsic aspects of the geometry and not the space in which it is embedded. This generalization is what is known as a manifold. In particular, the manifold is endowed with a Lorentzian metric (of indefinite signature), and it is called pseudo-Riemannian. In summary, space–time is considered as a four dimensional pseudo–Riemannian manifold which properties depend on the matter–energy content. The relationship between geometry and matter is encoded in the celebrated Einstein field equations which read

$$G_{\mu\nu} = \kappa^2 T_{\mu\nu}. \tag{2.1}$$

In the above expression, $g_{\mu\nu}$ is the metric tensor, $\mu, \nu = 0, 1, 2, 3$, $\kappa^2 = 8\pi G/c^4$. and $G_{\mu\nu}$ and $T_{\mu\nu}$ are the Einstein and the energy–momentum, respectively. The Einstein tensor $G_{\mu\nu}$ is defined as,

$$G_{\mu\nu} = R_{\mu\nu} + \frac{1}{2} g_{\mu\nu} R \tag{2.2}$$

where $R_{\mu\nu}$ is the Ricci tensor and R is the Ricci scalar. The Ricci tensor is a symmetric tensor defined by the contraction of the Riemann tensor,

$$R_{\mu\nu} = R^{\alpha}_{\mu\alpha\nu}. \tag{2.3}$$

The Riemann tensor codify the information of the curvature of the manifold is defined as

$$R^{\rho}_{\sigma\mu\nu} = \partial_\mu \Gamma^{\rho}_{\nu\sigma} - \partial_\nu \Gamma^{\rho}_{\mu\sigma} + \Gamma^{\rho}_{\mu\lambda}\Gamma^{\lambda}_{\nu\sigma} - \Gamma^{\rho}_{\nu\lambda}\Gamma^{\lambda}_{\mu\sigma}, \tag{2.4}$$

where $\Gamma^{\alpha}_{\mu\nu}$ corresponds to the metric connections, also known as the Christoffel symbols. In particular, as general relativity (GR) is a free torsion theory, the Christoffel symbols can be expressed in terms of the metric tensor,

$$\Gamma^{\alpha}_{\mu\nu} = \frac{1}{2} g^{\alpha\theta}(\partial_\mu g_{\theta\nu} + \partial_\nu g_{\theta\mu} - \partial_\theta g_{\mu\nu}). \tag{2.5}$$

Finally, the contraction of the Ricci tensor with the metric tensor gives us the Ricci scalar

$$R = g^{\mu\nu} R_{\mu\nu}. \tag{2.6}$$

It is worth mentioning that besides the Ricci scalar, we can define other scalar quantities which in combination with R allow to exploring the regularity of the underlying geometry, namely the Ricci square and the Kretschmann scalar, R^2 and K respectively, defined as

$$R^2 = R^{\mu\nu} R_{\mu\nu} \tag{2.7}$$

$$K = R^{\mu\nu\lambda\sigma} R_{\mu\nu\lambda\sigma}. \tag{2.8}$$

The energy–momentum tensor, $T_{\mu\nu}$, encodes the information of the gravitational source. In particular, for a perfect fluid, the energy–momentum tensor is given by

$$T_{\mu\nu} = \left(\rho + \frac{P}{2}\right) U_\mu U_\nu + P g_{\mu\nu} \tag{2.9}$$

where ρ is the energy density, P is pressure and U_μ corresponds to the four velocity.

Note that both the Einstein and the energy–momentum tensor are symmetric so that Eq. (2.1) corresponds to ten second order, coupled and non–linear differential equations for the metric $g_{\mu\nu}$. In this respect, finding exact solutions of Einstein equations is an extremely difficult task so that assuming extra conditions on the symmetry of the system to obtain solution from (2.1) is mandatory. In the next section, we study the main aspects of the first exact solution of Einstein equations obtained by K. Schwarzschild by assuming a vacuum, static and spherically symmetric system.

2.1.2 Schwarzschild exterior solution

Few months after the publication of the Einstein field equations, the first exact solution was found by Schwarzschild[42]. He was looking for a vacuum, static, spherically symmetric and asymptotically flat solution, which means that for the limit $r \to \infty$ the metric must become in the Minkowski space

$$ds^2_{Minkowski} = -c^2 dt^2 + dr^2 + r^2 d\theta^2 + r^2 \sin(\theta)^2 d\phi^2. \tag{2.10}$$

The first assumption means that

$$T_{\mu\nu} = 0. \tag{2.11}$$

Next, as the solution must be static, the metric is constrained to

$$\partial_t g_{\mu\nu} = 0. \tag{2.12}$$

Then, the spherically symmetric condition can be achieve assuming

$$ds^2 = -A(r)dt^2 + B(r)dr^2 + r^2 d\theta^2 + r^2 \sin(\theta)^2 d\phi^2 \tag{2.13}$$

where A and B are function which only depend on r. Finally, to be asymptotically flat the solution should satisfies

$$\lim_{r\to\infty} A(r) = c^2 \tag{2.14}$$

$$\lim_{r\to\infty} B(r) = 1. \tag{2.15}$$

The main goal now is to introduce the line element (2.13) in (2.1) to obtain A and B. It is worth noticing that, after assuming the vacuum condition, Eq. (2.1) can be written as

$$R_{\mu\nu} = 0. \tag{2.16}$$

Next, the non–vanishing Christoffel symbols are given by

$$\Gamma^0_{10} = \frac{A'(r)}{2A(r)} \qquad \Gamma^1_{00} = \frac{A'(r)}{2B(r)} \qquad \Gamma^1_{11} = \frac{B'(r)}{2B(r)} \tag{2.17}$$

$$\Gamma^1_{22} = -\frac{r}{B(r)} \qquad \Gamma^1_{33} = -\frac{r\sin(\theta)^2}{B(r)} \qquad \Gamma^0_{10} = \frac{1}{r} \tag{2.18}$$

$$\Gamma^0_{10} = -\cos(\theta)\sin(\theta) \qquad \Gamma^0_{10} = \frac{1}{r} \qquad \Gamma^0_{10} = \frac{\cos(\theta)}{\sin(\theta)}, \tag{2.19}$$

from where, the Einstein field equations read

$$R_{11} = \frac{A''(r)}{2B(r)} - \frac{A'(r)B'(r)}{4B(r)^2} - \frac{A'(r)^2}{4A(r)B(r)} + \frac{A'(r)}{rB(r)} = 0 \tag{2.20}$$

$$R_{22} = \frac{[rA'(r) + 4A(r)]B'(r)}{4rA(r)B(r)} + \frac{A'(r)^2 - 2A(r)A''(r)}{4A(r)^2} = 0 \tag{2.21}$$

$$R_{33} = 1 - \frac{rA'(r) + 2A(r)}{2A(r)B(r)} + \frac{rB'(r)}{2B(r)^2} = 0 \tag{2.22}$$

$$R_{44} = \frac{\sin^2(\theta)\{A(r)[rB'(r) + 2B(r)^2 - B(r)] - rB(r)A'(r)\}}{2A(r)B(r)^2} = 0. \tag{2.23}$$

Finally, it can be shown that the solution of the above system is given by

$$A(r) = c^2\left(1 - \frac{2MG}{c^2 r}\right) \tag{2.24}$$

$$B(r) = \left(1 - \frac{2MG}{c^2 r}\right)^{-1}. \tag{2.25}$$

Using (2.24) and (2.25) the line element (2.13) reads

$$ds^2 = -c^2\Big(1 - \frac{2MG}{c^2 r}\Big)dt^2 + \Big(1 - \frac{2MG}{c^2 r}\Big)^{-1} dr^2 + r^2 d\theta^2 + r^2 \sin(\theta)^2 d\phi^2. \tag{2.26}$$

2.1.3 Event horizon and singularities

By simple inspection, the metric (2.26) has two critical points, namely $r_h = \frac{2GM}{c^2}$ and $r_s = 0$. Indeed,

$$\lim_{r\to r_h}\Big(1 - \frac{2MG}{c^2 r}\Big)^{-1} = \infty. \tag{2.27}$$

$$\lim_{r\to r_s} -c^2\Big(1 - \frac{2MG}{c^2 r}\Big) = \infty. \tag{2.28}$$

Now, this critical behavior could be given by either an inappropriate choice of the coordinate system used to parametrize the metric or to the fact that something is wrong with the manifold. In the last case, the point corresponds to an essential singularity and can not be removed with any choice of the coordinate system. In this work, we will say that a critical point is an essential singularity whenever some of the curvature scalars diverge at this point. In this regard, a straightforward computation reveals that, in the Schwarzschild solution both the Ricci and the Ricci square scalar vanishes but the Kretschmann is given by

$$K = \frac{48G^2M^2}{c^4 r^6}. \tag{2.29}$$

Now, at $r = r_s = 0$ we have

$$\lim_{r\to r_s} K = \infty, \tag{2.30}$$

so that $r = 0$ represents an essential singularity. Next, at $r = r_h$ we have

$$\lim_{r\to r_h} K = \frac{3c^8}{4G^4M^4}. \tag{2.31}$$

so that r_h is not an essential singularity. However, as we shall describe in what follows, the point r_h defines a special null surface. To this end, we consider null geodesics by fixing θ and ϕ and imposing the condition $ds^2 = 0$, as a result

$$\Big(1 - \frac{2GM}{c^2 r}\Big)dt^2 - \Big(1 - \frac{2GM}{c^2 r}\Big)^{-1} dr^2 = 0. \tag{2.32}$$

Solving the differential equation, we obtain

$$t_{\pm} = \pm(r + 2GMln|r - 2GM|), \tag{2.33}$$

where t_+ corresponds to outgoing and the t_- to the ingoing null geodesics. In Figure 2.1, we present the outgoing and the ingoing null geodesics, and in the intersection of these is build a causal null cone. At the intersection is placed the event, the direction of the lines define the causal future and the opposite direction the causal past. It is worth noticing that the null geodesics never reach the surface r_h, instead it approaches asymptotically. This means

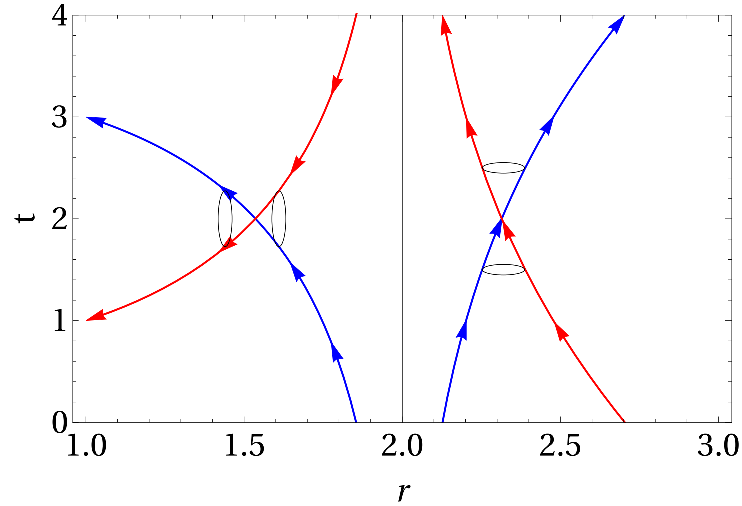

Figure 2.1: Null geodesics for Schwarzschild coordinates. Outgoing (ingoing) null geodesics correspond to blue (red) lines. The null surface is located at $r_h = 2$.

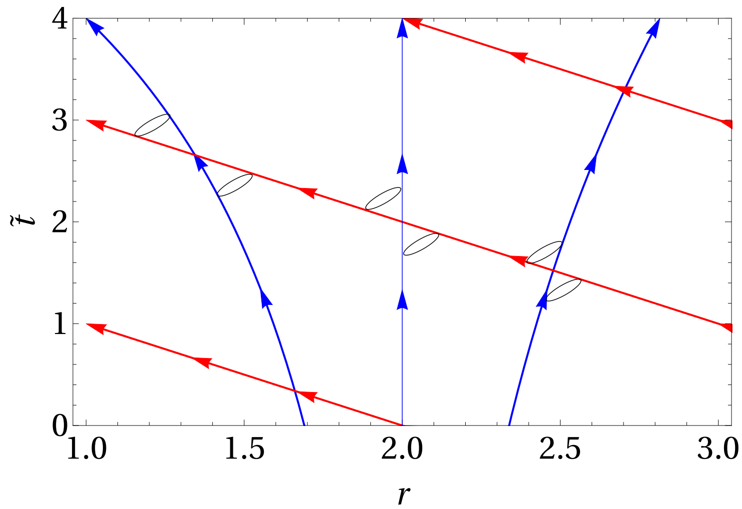

Figure 2.2: Null geodesics for Eddington–Finkelstein coordinates. Outgoing (ingoing) null geodesics correspond to blue (red). The null surface is placed at $r_h = 2$.

that it is not possible to construct a null cone near to this surface. However, in terms of the Eddington–Finkelstein coordinates it can be shown that it is possible to construct a null cone at r_h, which means that it is possible to traverse such a surface. The Eddington–Finkelstein are defined as[43,44]

$$\tilde{t}_{+} = r + 4GMln|r - 2GM| + C, \tag{2.34}$$
$$\tilde{t}_{-} = -r + C, \tag{2.35}$$

where C is an integration constant. In Figure 2.2, it is shown that any test particle that passes through the null surface can never go out. Indeed, all the particles traversing r_h end at the singularity $r = 0$. The surface defined by r_h is the so called Event Horizon of the solution and the point $r_s = 0$ is the singularity. The solutions of the Einstein field equations with a null surface enclosing a singularity are known as Black Hole solutions.

2.2 Extended phase space

As it is well known the basic thermodynamic quantities of a physical system have their counterpart in BH physics. For example, for a Schwarzschild BH the mass is related to the energy, the surface gravity is related to the temperature and the horizon area is associated with the entropy of the system. However, for non–vacuum solutions where pressure and volume terms are introduced[45] the above correspondence is not suitable to describe BH thermodynamics. For example, when considering asymptotically anti–de Sitter (AdS) black holes, the mass must be identified with the enthalpy of system[46,47] and the cosmological constant as the thermodynamic pressure, namely[45,48].

$$P = -\frac{\Lambda}{8\pi} = \frac{3}{8\pi l^2}. \tag{2.36}$$

The set of thermodynamic variables which satisfied the laws of BH thermodynamics is known as extended phase space. In terms of this variables the first law of black hole thermodynamics reads,

$$\delta M = T\delta S + V\delta P + ..., \tag{2.37}$$

where

$$V = \left.\frac{\partial M}{\partial P}\right|_S, \tag{2.38}$$

is the associated thermodynamics volume. It is clear that if we insist in to solve the thermodynamics of a system, an equation of state for the pressure $P = P(V, T)$ must be provided. Indeed, using Eqs. (2.36), (2.37), (2.38) and some equation of state, we can combine the thermodynamic variables of the BH solutions as follows.

First, let us consider the metric

$$ds^2 = -f dt^2 + f^{-1}dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2), \tag{2.39}$$

with

$$f = -\frac{2M}{r} + \frac{r^2}{l^2} - h(r, P), \tag{2.40}$$

to ensure an asymptotically AdS solution of the Einstein field equations,

$$G_{\mu\nu} + \Lambda g_{\mu\nu} = 8\pi T_{\mu\nu}, \tag{2.41}$$

where, the energy momentum tensor is defined as

$$T^{\mu}_{\nu} = diag(-\varrho, p_r, p_\perp, p_\perp). \tag{2.42}$$

Now, by using Eqs. (2.39), (2.41) and (2.42) we obtain

$$\varrho = -p_r = \frac{1 - f - rf'}{8\pi r^2} + P, \tag{2.43}$$

$$p_\perp = \frac{rf'' + 2f'}{16\pi r} - P, \tag{2.44}$$

which results in a system of two equations for three unknowns, $\{f, \varrho, p_\perp\}$.

Furthermore, the physical acceptability of the solution is restricted by the energy conditions (developed in detail in the appendix A), namely the null energy condition (NEC), weak energy condition (WEC), strong energy condition (SEC) and dominant energy condition (DEC),

$$\varrho + p_\perp \geq 0, \tag{2.45}$$

$$\varrho \geq 0, \quad \text{and} \quad \varrho + p_\perp \geq 0, \tag{2.46}$$

$$2p_\perp \geq 0, \quad \text{and} \quad \varrho + p_\perp \geq 0, \tag{2.47}$$

$$\varrho \geq 0, \quad \text{and} \quad \varrho \geq |p_\perp|, \tag{2.48}$$

respectively.

Now, if we were solving the Einstein field equations in the usual way, the next step should be to propose an equation of state between the variables of the matter sector. However, we can follow an alternative protocol which consist in to propose an equation of state (EoS) for the pressure P and use the equations of BH thermodynamics to solve the system. First note that from the horizon condition, $f(r_+) = 0$, we obtain

$$M = \frac{4\pi}{3} r_+^3 P - \frac{r_+}{2} h(r_+, P). \tag{2.49}$$

Then, form Eq. (2.37) thermodynamic volume reads

$$V = \left(\frac{\partial M}{\partial P}\right)_S = \frac{4\pi}{3} r_+^3 - \frac{r_+}{2} \frac{\partial h(r_+, P)}{\partial P}. \tag{2.50}$$

Similarly, the surface gravity of the associated temperature can be expressed as,

$$T = \frac{f'(r_+, P)}{4\pi} = 2r_+ P - \frac{h(r_+, P)}{4\pi r_+} - \frac{1}{4\pi} \frac{\partial h(r_+, P)}{\partial r_+}. \tag{2.51}$$

Finally, we consider the well establish known relationship between the entropy equal and the area of a BH given by

$$S = \frac{A}{4} = \pi r_+^2. \tag{2.52}$$

Note that in order to close the system an EoS for the pressure P must be provided. In the next section, we shall explore some of the equations of state that have been proposed so far to obtain asymptotically AdS BH solutions.

2.2.1 Van der Waals BH

In this section, we explore how the Van der Waals (VdW) equation of state has been combined with the extended phase space approach to obtain a BH solution[45,49,50]. Let us consider the VdW EoS,

$$P = \frac{T}{v-b} - \frac{a}{v^2}, \tag{2.53}$$

where $a > 0$ is the attraction between particles, $b > 0$ is the particle volume and v is the specific volume defined as

$$v = k\frac{V}{N}. \tag{2.54}$$

Here, the value N is equivalent to the horizon area, $N = \frac{A}{L^2}$ with $A = 4\pi r_+^2$ and $k = \frac{4(d-1)}{d-2}$ such that for $d = 4$ space–times results in $k = 6$. Now, using (2.50) the specific volume reads

$$v = \frac{k}{4\pi r_+^2}\left[\frac{4}{3}\pi r_+^3 - \frac{r_+}{2}\frac{\partial h(r_+, P)}{\partial P}\right]. \tag{2.55}$$

Now, proposing the following ansatz

$$h(r, P) = A(r) - B(r)P, \tag{2.56}$$

and replacing (2.51) and (2.55) in (2.53) we obtain an equation of the form

$$F_1(r) + F_2(r)P = 0, \tag{2.57}$$

where F_1 and F_2 depend on the functions A and B and their derivatives. Imposing $F_2(r) = 0$, we obtain a differential equation for B which solution reads

$$B(r) = 4\pi br + B_0 r^2, \tag{2.58}$$

where B_0 is an integration constant which must be set to zero, in order to preserve the AdS structure. Using the result in equation (2.58) and imposing $F_1(r) = 0$ we obtain

$$A(r) = -2\pi a + \frac{A_0}{r} - \frac{6\pi ab(b+r)}{r(3b+2r)} + \frac{4\pi ab}{r}\log(3b+2r). \tag{2.59}$$

In order to provide a dimensionless logarithmic argument A_0 should be equal to $-4\pi ab\log(2b)$. Finally, using the obtained results (2.58) and (2.59) the solution $h(r, P)$ takes the form

$$h(r, P) = -2\pi a + \frac{3br}{2l^2} - \frac{6\pi ab(b+r)}{r(3b+2r)} + \frac{4\pi ab}{r}\log\left(\frac{3b+2r}{2b}\right). \tag{2.60}$$

2.2.2 Polytropic BH

In analogy with the VdW case, in this section it is proposed a polytropic EoS, namely

$$P = K\rho^{1+\frac{1}{n}} \quad ; \quad \rho = K^{-\frac{n}{n+1}}P^{\frac{n}{n+1}} = CP^{\frac{n}{n+1}}, \tag{2.61}$$

where n is the Polytropic index and K and C are constants, related as

$$C = K^{-\frac{n}{n+1}}. \tag{2.62}$$

Additionally, we assume that the gas satisfies the following integrability condition

$$\frac{\partial^2 S}{\partial T \partial V} = \frac{\partial^2 S}{\partial V \partial T}, \tag{2.63}$$

from where in accordance with the first law of thermodynamics we obtain

$$S = \frac{P+\rho}{T} V. \tag{2.64}$$

In order to proceed, we propose a different anzats, namely

$$h(r,P) = A(r) - B(r)P + D(r)P^{\frac{1}{1+n}}, \tag{2.65}$$

and by using the thermodynamic relations (2.50), (2.51) and (2.52), the polytropic EoS (2.61), and first law (2.64), we obtain

$$F_1(r) + F_2(r)P + F_3(r)P^{\frac{1}{1+n}} + F_4(r)P^{\frac{n}{1+n}} = 0, \tag{2.66}$$

where F_1, F_2, F_3 and F_4 are functions on A, B, D and their derivatives. Now, by setting the condition $F_4(r) = 0$, we obtain

$$B(r) = -\frac{8\pi r^2}{3} + rB_0. \tag{2.67}$$

Then, the condition $F_3(r) = 0$, leads to

$$D(r) = D_0[(n+1)r]^{\frac{1-n}{n+1}}. \tag{2.68}$$

It is worth mentioning that the solution (2.67) is consistent with the remaining equation of $F_2(r) = 0$. Finally, by imposing $F_1(r) = 0$, the function A reads

$$A(r) = \frac{A_0}{r} + D_0 C[(n+1)r]^{\frac{1-n}{n+1}}. \tag{2.69}$$

With these results at hand, the function $h(r,P)$ takes the form

$$h(r,P) = \frac{A_0}{r} + \frac{r^2}{l^2} - \frac{3B_0 r}{8\pi l^2} + D_0\left[C + \left(\frac{3}{8\pi l^2}\right)^{\frac{1}{n+1}}\right](n+1)^{\frac{1-n}{n+1}} r^{\frac{1-n}{n+1}} \tag{2.70}$$

Finally, as stated in[35], we need to impose $A_0 = B_0 = 0$ and $D_0 = \left\{l^2\left[C + \left(\frac{3}{8\pi l^2}\right)^{\frac{1}{n+1}}\right](n+1)^{\frac{1-n}{n+1}}\right\}^{-1}$ for $n = -\frac{1}{3}$, to conserve the asymptotically AdS behaviour.

2.2.3 Chaplygin BH

Another possibility is to consider Chaplygin EoS[36], by following a similar protocol used in the polytropic case. To be more precise, we start from

$$P = A\rho - \frac{B}{\rho^n}, \tag{2.71}$$

where A, B and n are constants. The main difference is the dependence on the density ρ instead of the pressure P. However, the extended phase space formalism remains with a small modification in the volume,

$$V = \left(\frac{\partial M}{\partial P}\right)_S = \frac{4\pi}{3} r_+^3 - \frac{r_+}{2}\frac{\partial h(r_+,\rho)}{\partial \rho}\frac{\partial \rho}{\partial P}. \tag{2.72}$$

In this case, as reported in[36], the ansatz for h is taken as

$$h(r,\rho) = X(r) + Y(r)\rho + Z(r)\rho^{-n}, \tag{2.73}$$

from where, following the steps in the previous case but with the Chaplygin EoS (2.71), we obtain a polynomial equation of the form

$$F_0 + F_1\rho + F_2(r)\rho^{-n} + F_3(r)\rho^{-n-1} + F_4(r)\rho^{-2n-1} = 0 \tag{2.74}$$

Next, we set the condition $F_0(r) = F_3(r) = 0$ which leads to

$$X(r) = \frac{X_0}{r}. \tag{2.75}$$

Now, from $F_4(r) = 0$ we obtain

$$Y(r) = \frac{8}{3}\pi A r^2 + Y_0 r^{\frac{2}{A}+1}. \tag{2.76}$$

Then, setting $F_2(r) = 0$ we have

$$Z(r) = -\frac{8}{3}\pi B r^2 + r Z_0. \tag{2.77}$$

Using (2.76) and (2.77) in the condition $F_1(r) = 0$, results in a polynomial equation with power of r from where

$$A = -\frac{n}{1+n}. \tag{2.78}$$

Finally, Replacing the results (2.75), (2.76), (2.77) and (2.78) in Eq. (2.73), we obtain

$$h(r,\rho) = \frac{X_0}{r} + \frac{r^2}{l^2} + \rho Y_0 r^{-\frac{2}{n}-1} - \frac{3rZ_0}{8\pi B l^2} - \frac{n\rho r Z_0}{B(n+1)}, \tag{2.79}$$

In this expression X_0, Y_0, and Z_0 are integration constants. In order to recover the AdS behavior, we need to impose these conditions[36], $X_0 = Z_0 = 0$, $Y_0 = \frac{8\pi P}{3\rho}$, and $n = -\frac{2}{3}$.

2.3 Generalized uncertainty principle

2.3.1 Heisenberg Uncertainty Principle

As it is well known, the Heisenberg uncertainty principle (HUP) represents one of the fundamental properties of quantum systems. It says that should be a fundamental limit for the measurement accuracy between certain pairs of physical observables, for example between position and momentum, or energy and time. In other words, they cannot be measured simultaneously; if one quantity is measured with high precision the accuracy associated with the other increases considerably[22,27].

In quantum mechanics (QM), the physical observables are described by operators in Hilbert space. More precisely, given an observable A, we define the standard deviation of A,

$$\Delta A = A - \langle A \rangle, \tag{2.80}$$

where the expectation value is given by

$$\langle (\Delta A)^2 \rangle = \langle A^2 \rangle - \langle A \rangle^2. \tag{2.81}$$

Now, using Schwartz inequality

$$\langle \alpha|\alpha \rangle \langle \beta|\beta \rangle \geq |\langle \alpha|\beta \rangle|^2, \tag{2.82}$$

we obtain

$$(\Delta A)^2 (\Delta B)^2 \geq \frac{1}{4} |\langle \Delta A \Delta B \rangle|^2, \tag{2.83}$$

which is the well known Cauchy-Schwartz inequality. In other words,

$$\Delta A \Delta B \geq \frac{1}{2} |\langle [\hat{A}, \hat{B}] \rangle|. \tag{2.84}$$

Now, as it is well known, the position $\hat{x}$ and momentum $\hat{p}$ satisfy the canonical commutation relation

$$[\hat{x}, \hat{p}] = \hat{x}\hat{p} - \hat{p}\hat{x} = i\hbar \tag{2.85}$$

from where

$$\Delta x \Delta p \geq \frac{\hbar}{2}. \tag{2.86}$$

2.3.2 Minimal Length Uncertainty

In this section, we explore how the HUP is generalized to take into account the existence of a minimal length. As it is well known the generalized uncertainty principle (GUP) states that the fundamental algebra in QM modifies as

$$[x_i, p_j] = i\hbar[\delta_{ij}(1 + \alpha p^2) + 2\alpha p_i p_j], \tag{2.87}$$

with

$$[x_i, x_j] = 0 = [p_i, p_j]. \tag{2.88}$$

Now, we can define

$$\begin{aligned} x_i &= x_{0i} && (2.89) \\ p_j &= p_{0j}(1 + \alpha p_0^2), && (2.90) \end{aligned}$$

where $[x_{0i}, p_{0j}] = i\hbar\delta_{ij}$. Furthermore, p_{0j} is interpreted as the momentum at low energy scale which is represented by $p_{0j} = -i\hbar\frac{\partial}{\partial x_{0j}}$, while p_j is considered as the momentum a high energy scales. Now the GUP reads

$$\Delta x \Delta p \geq \hbar + \frac{\alpha}{\hbar}\Delta p^2, \tag{2.91}$$

where α is a positive constant called the GUP parameter. Note that, after solving the quadratic equation for Δp we obtain

$$\Delta p \geq \frac{\hbar}{2\alpha}\left(\Delta x - \sqrt{\Delta x^2 - 4\alpha}\right), \tag{2.92}$$

from where

$$\Delta p \geq \frac{\hbar}{\Delta x} + \frac{\hbar\alpha}{\Delta x^3} + O(\alpha^2). \tag{2.93}$$

The above result corresponds to the quadratic minimal length approach.

2.4 Minimal length GUP–corrected BH

In this section, we introduce a generalization of the BH temperature and entropy modified by GUP as reported in Refs.[32–34]. To this end, we expand (2.93) up to second order, namely

$$\Delta A \geq \Delta x \Delta p \simeq \hbar\left(1 + \frac{\alpha}{\Delta x^2}\right) = \hbar'. \tag{2.94}$$

Now, as it is well known, the thermodynamic black hole (BH) temperature associated with the surface gravity κ, is given by

$$T = \frac{dA}{dS} \times \frac{\kappa}{8\pi}, \tag{2.95}$$

where

$$\frac{dA}{dS} \simeq \frac{(\Delta A)_{min}}{(\Delta S)_{min}} = \frac{\gamma}{ln2}\hbar\left(1 + \frac{\alpha}{\Delta x^2}\right) = \frac{\hbar'\gamma}{ln2}. \tag{2.96}$$

Then, replacing the relation (2.96) into (2.95), we obtain

$$T = \frac{\hbar'\gamma}{ln2} \times \frac{\kappa}{8\pi} = \frac{\hbar'\kappa}{2\pi}, \tag{2.97}$$

where we have taken $\gamma = 4ln2$. Finally, after replacing the surface gravity κ in the above expression, the GUP extended temperature reads

$$T = \left(1 + \frac{\alpha}{4r_+^2}\right)\left(2r_+P - \frac{h(r_+,P)}{4\pi r_+} - \frac{1}{4\pi}\frac{\partial h(r_+,P)}{\partial r_+}\right). \tag{2.98}$$

Note that, in the limit $\alpha \to 0$ the expression for the temperature reduces to $T = \frac{\kappa}{2\pi}$, as expected. In consequence, the entropy calculated with (2.98) provide logarithmic correction, associated with quantum corrections[34,51]

$$S = \int \frac{dM}{T} = \pi r_+^2 - \frac{\alpha\pi}{4} ln\left(\frac{4r_+^2 + \alpha}{\alpha}\right). \tag{2.99}$$

In the following sections, we use the GUP corrected temperature (2.98) and entropy (2.99) with the extended phase space formalism, in order to obtain GUP corrected asymptotically AdS BH solutions.

Chapter 3

Results & Discussion

3.1 Minimal length GUP–corrected Van der Waals BH

This section is devoted to implement the strategy developed previously to obtain a GUP corrected VdW BH solution. Let us start by introducing the ansatz (2.56) into the GUP corrected temperature (2.98) to obtain a polynomial equation of the form $F_1(r) + F_2(r)P = 0$, where the associated functions are

$$F_1(r) = \frac{16\pi^2 ar^2}{(3B + 8\pi r^2)^2} + \frac{(\alpha + 4r^2)(rA' + A)}{4r^2[3B - 4\pi r(b - 2r)]} \tag{3.1}$$

$$F_2(r) = 1 - \frac{(\alpha + 4r^2)(rB' + 8\pi r^2 + B)}{4r^2[3B - 4\pi r(b - 2r)]}. \tag{3.2}$$

Now, by imposing the condition $F_2 = 0$, we obtain

$$B(r) = 4\pi br - \frac{8\pi r^2}{3} + \frac{2\pi\alpha b}{3r} + B_0\sqrt{\alpha + 4r^2}\left(4r + \frac{\alpha}{r}\right). \tag{3.3}$$

Then, by setting the integration constant $B_0 = \frac{\pi}{3}$ and expanding in series the GUP parameter α around zero up to second order, the expression reads

$$B(r) = 4\pi br + \alpha\pi\left(1 + \frac{2b}{3r}\right). \tag{3.4}$$

Now after expanding F_1 up to second order in α and setting the condition $F_1 = 0$ we obtain,

$$\begin{aligned} A(r) = {} & \frac{A_0}{r} + \frac{\pi a 81 b^4(3b^2 - 6br - 4r^2)}{r(\alpha + 9b^2)^2(3b + 2r)} - \frac{\pi a\alpha^2(57b^3 + 61b^2 r + 24br^2 + 8r^3)}{r(\alpha + 9b^2)^2(3b + 2r)^2} \\ & - \frac{\pi a 9\alpha b^2(18b^3 + 53b^2 r + 48br^2 + 16r^3)}{r(\alpha + 9b^2)^2(3b + 2r)^2} + \frac{\pi a\alpha^{3/2}(5\alpha^2 - 27b^4 + 2\alpha b^2)}{2r(\alpha + 9b^2)^3}\arctan\left(\frac{2r}{\sqrt{\alpha}}\right) \\ & + \frac{7\alpha^2 + 108b^4 + 45\alpha b^2}{\frac{r}{\pi ab\alpha}(\alpha + 9b^2)^3}\log(\alpha + 4r^2) + \frac{1458b^6 + 378\alpha b^4 + 9\alpha^2 b^2 - 5\alpha^3}{\frac{r}{2\pi ab}(\alpha + 9b^2)^3}\log(3b + 2r), \end{aligned} \tag{3.5}$$

from where, in order to provide a dimensionless logarithmic argument A_0 should be

$$A_0 = -\frac{\pi ab\alpha(7\alpha^2+108b^4+45\alpha b^2)}{(\alpha+9b^2)^3}\log(4b^2) - \frac{2\pi ab(1458b^6+378\alpha b^4+9\alpha^2b^2-5\alpha^3)}{(\alpha+9b^2)^3}\log(2b). \quad (3.6)$$

Thus, using the obtained results (3.4) and (3.5), we can construct the solution for $h(r,P)$, which reads

$$\begin{aligned} h &= -\frac{12\pi br}{8\pi l^2} - \frac{3\alpha\pi}{8\pi l^2}\left(1+\frac{2b}{3r}\right) + \frac{\pi a 81 b^4(3b^2-6br-4r^2)}{r(\alpha+9b^2)^2(3b+2r)} - \frac{\pi a\alpha^2(57b^3+61b^2r+24br^2+8r^3)}{r(\alpha+9b^2)^2(3b+2r)^2} \\ &\quad -\frac{\pi a 9\alpha b^2(18b^3+53b^2r+48br^2+16r^3)}{r(\alpha+9b^2)^2(3b+2r)^2} + \frac{\pi a\alpha^{3/2}(5\alpha^2-27b^4+2\alpha b^2)}{2r(\alpha+9b^2)^3}\arctan\left(\frac{2r}{\sqrt{\alpha}}\right) + \\ &\quad \frac{7\alpha^2+108b^4+45\alpha b^2}{\frac{r}{\pi ab\alpha}(\alpha+9b^2)^3}\log\left(\frac{\alpha+4r^2}{4b^2}\right) + \frac{1458b^6+378\alpha b^4+9\alpha^2b^2-5\alpha^3}{\frac{r}{2\pi ab}(\alpha+9b^2)^3}\log\left(\frac{3b+2r}{2b}\right). \end{aligned} \quad (3.7)$$

Finally, replacing the equation (3.7) in (2.40), we arrive to the metric function f. In Figure 3.1 we show f as function of r. It is noticeable that as the values of the cosmological pressure increase, the location of the horizon radius shift to the right.

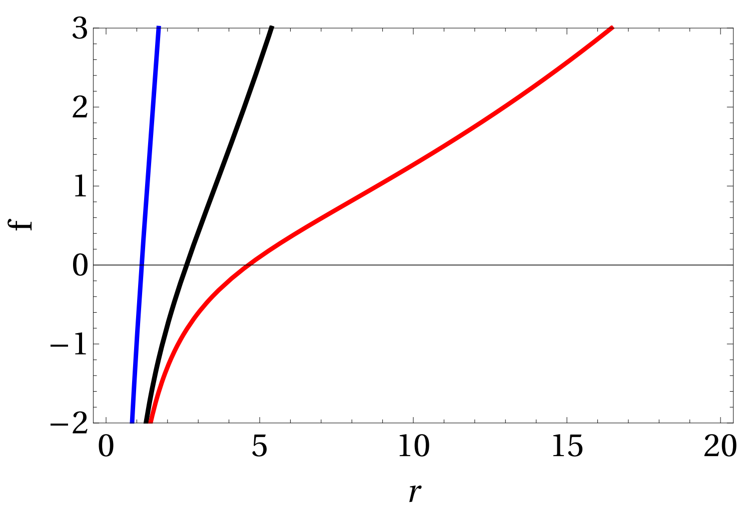

Figure 3.1: f as a function of r, with the following parameters $M = 1$, $a = \frac{1}{2\pi}$, $b = 1$, $\alpha = 0.1$ and $P = 0.1$ (blue line), $P = 0.01$ (black line), and $P = 0.001$ (red line).

Using the previous result, the matter sector reads

$$\varrho = -p_r = \frac{1}{8\pi r^2} - \frac{P(\alpha+8br)}{8r^2} + \frac{a\alpha(b+2r)(2b+3r)-8ar^2(b+r)(3b+2r)}{2r(3b+2r)^3(\alpha+4r^2)} \quad (3.8)$$

$$p_\perp = \frac{bP}{2r} + \frac{8abr^3(3b+4r)}{(3b+2r)^3(\alpha+4r^2)^2} + \frac{a\alpha\{8r^2[15b^3+34b^2r+26br^2+10r^3]-\alpha b(b+4r)(3b+5r)\}}{2r(3b+2r)^4(\alpha+4r^2)^2}. \quad (3.9)$$

In Figure 3.2, we show the weak energy condition (WEC) is violated. It is worth mentioning that the WEC is also

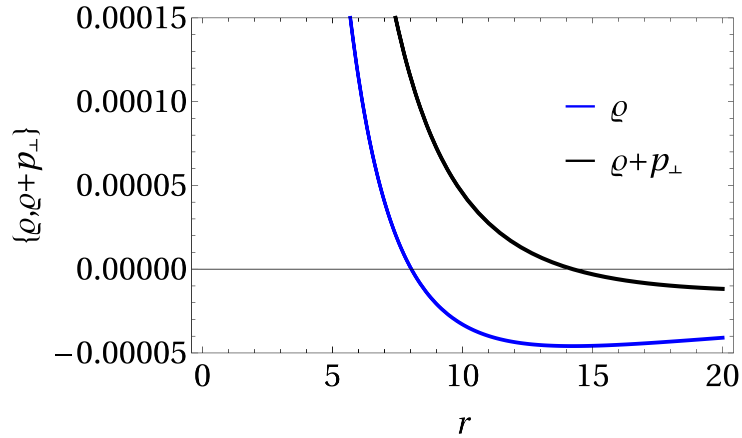

Figure 3.2: Weak energy condition. The energy density ϱ (blue line) and energy density plus the perpendicular pressure $\varrho + p_\perp$ (black line), by setting $a = \frac{1}{2\pi}$, $P = 0.001$, $b = 1$ and $\alpha = 0.1$.

violated in the standard case[50] so that, the introduction of GUP does not allow to remove the ill behavior of the solution.

3.2 Minimal length GUP–corrected Polytropic BH

In this section we obtain a GUP corrected polytropic BH solution. In order to do so, we replace the corrected entropy (2.99) and temperature (2.98) in the first law (2.64) to obtain a polynomial equation of the form $F_1(r) + F_2(r)P + F_3(r)P^{\frac{1}{1+n}} + F_4(r)P^{\frac{n}{1+n}} = 0$, where

$$F_1(r) = \frac{CrD(r)}{2(n+1)} - \frac{\alpha + 4r^2}{16}\left(A' + \frac{A}{r}\right). \tag{3.10}$$

$$F_2(r) = \frac{B'(\alpha + 4r^2)}{16} + \frac{B(\alpha - 4r^2)}{16r} + \frac{\pi r(3\alpha + 4r^2)}{6}. \tag{3.11}$$

$$F_3(r) = -\frac{1}{16}\left(D'(\alpha + 4r^2) + D\left(\frac{4r(n-1)}{n+1} - \frac{\alpha}{r}\right)\right). \tag{3.12}$$

$$F_4(r) = -\frac{1}{2}KrB - \frac{4}{3}\pi K r^3. \tag{3.13}$$

In order to solve the system of equations, lets us impose $F_4 = 0$ to obtain

$$B(r) = -\frac{8\pi r^2}{3}. \tag{3.14}$$

Alternatively, we solve for the function B, but by setting the condition $F_2 = 0$,

$$B(r) = -\frac{8\pi r^2}{3} + \frac{B_0(\alpha + 4r^2)}{r}. \tag{3.15}$$

Thus, we conclude that $B_0 = 0$ in order to avoid inconsistencies. Next, by taking the $F_2 = 0$ and solving the differential equation we obtain

$$D(r) = \frac{D_0(\alpha + 4r^2)^{\frac{1}{n+1}}}{r}. \tag{3.16}$$

Finally, after imposing $F_1 = 0$ we obtain

$$A(r) = \frac{A_0}{r} + \frac{D_0 C(\alpha + 4r^2)^{\frac{1}{n+1}}}{r}. \tag{3.17}$$

Then, replacing (3.14), (3.16), and (3.17) into the ansatz (2.65) the function $h(r, P)$ reads

$$h(r, P) = \frac{A_0}{r} + \frac{r^2}{l^2} - \frac{3B_0 r}{2\pi l^2}\left(1 + \frac{\alpha}{4r^2}\right) + \frac{D_0(\alpha + 4r^2)^{\frac{1}{n+1}}}{r}\left[\left(\frac{3}{8\pi l^2}\right)^{\frac{1}{n+1}} + C\right]. \tag{3.18}$$

Finally, after replacing (3.18) in equation (2.40) we arrive to the final result. As a particular case we set $n = -\frac{1}{3}$, $K = 1$, $A_0 = 0$ and $D_0 = -\frac{\pi P}{3}(K + P^{\frac{1}{n+1}})^{-1}$ to obtain

$$f = -\frac{2M}{r} + \frac{P\pi(4r^2 + \alpha)^{3/2}}{3r}. \tag{3.19}$$

Note that the anti–de Sitter (AdS) structure can be recovered by taking the limit $\alpha \to 0$. In Figure 3.3 it is shown the function f for certain parameters. Note that, as in the previous case, the horizon is located at higher radius as the the cosmological pressure decreases.

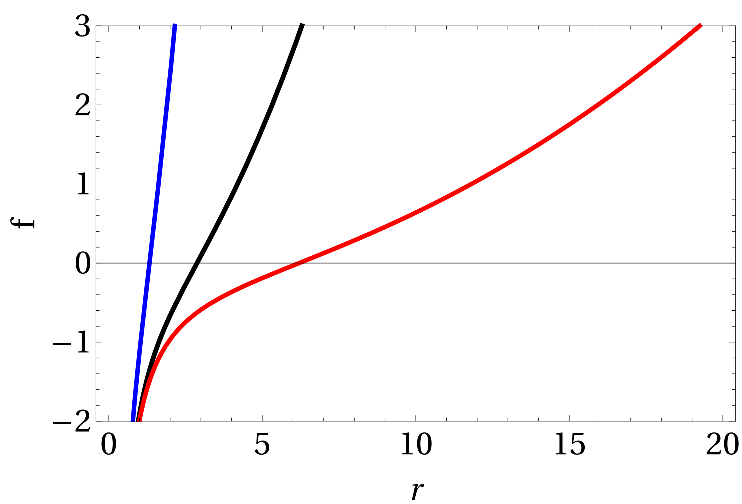

Figure 3.3: f as a function of r, with $M = 1$, $\alpha = 0.1$ and $P = 0.1$ (blue line), $P = 0.01$ (black line), and $P = 0.001$ (red line).

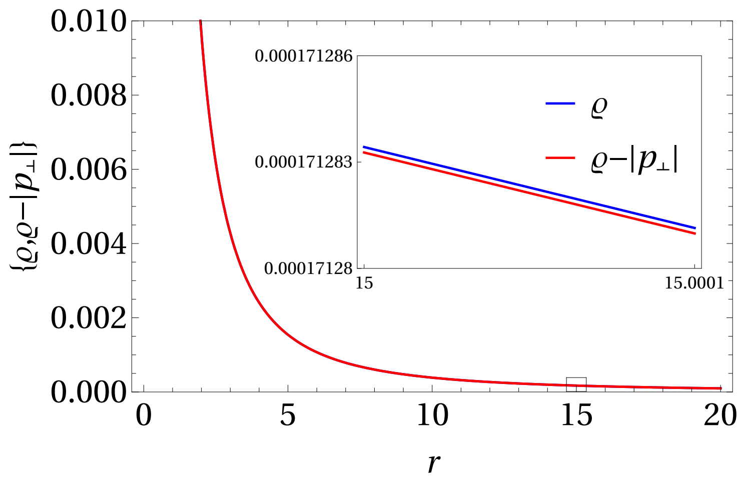

Figure 3.4: Dominant energy condition. The energy density ϱ (blue line) and energy density minus the absolute values of perpendicular pressure $\varrho - |p_\perp|$ (red line) are plotted by setting $P = 0.1$ and $\alpha = 0.1$.

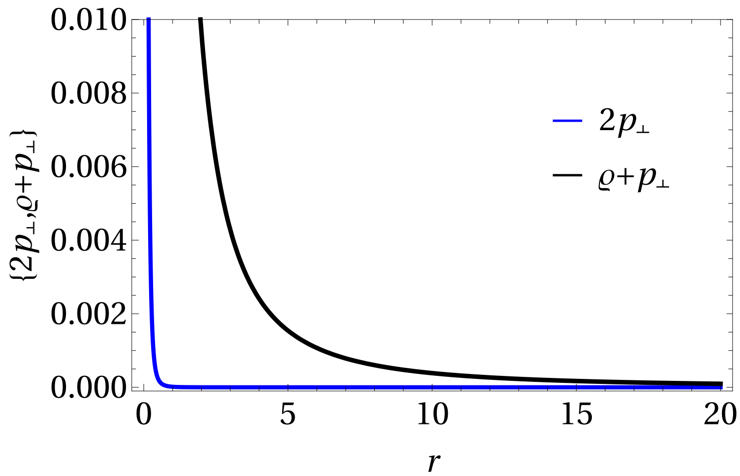

Figure 3.5: Strong energy condition. The double perpendicular pressure $2p_\perp$ (blue line) and energy density plus the perpendicular pressure $\varrho + p_\perp$ (black line) are plotted by setting $P = 0.1$ and $\alpha = 0.1$.

Using the above results, the matter sector reads

$$\varrho = -p_r = P + \frac{1}{8\pi r^2} - \frac{P\sqrt{\alpha + 4r^2}}{2r}. \tag{3.20}$$

$$p_\perp = P\left(\frac{\alpha + 8r^2}{4r\sqrt{\alpha + 4r^2}} - 1\right). \tag{3.21}$$

In figures 3.4 and 3.5 it is shown that the solutions satisfies both DEC and SEC. It is worth noticing that the introduction of GUP corrections does not alter the behavior of the standard solution in the sense that the model satisfies all the energy conditions as its unperturbed counterpart reported in[35].

3.3 Minimal length GUP–corrected Chaplygin BH

In this section, we follow the same strategy previously implemented but this time we assume

$$P = -\frac{B}{\rho^n}, \tag{3.22}$$

which corresponds to set $A = 0$ in (2.71). It is worth mentioning that generalized uncertainty principle (GUP) corrections of Chaplygin equation of state (EoS) (2.71) result in very complex differential equations and for this reason, we work with the simplified version. The resulting polynomial equation takes the form $F_0(r) + F_1(r)\rho + F_2(r)\rho^{-n} + F_3(r)\rho^{n+2} = 0$, with the corresponding associated functions defined as,

$$F_0(r) = -\frac{\alpha(rX' + X)(\alpha + 4r^2)}{64r^3}\left[\frac{4r^2}{\alpha} - \log\left(1 + \frac{4r^2}{\alpha}\right)\right]. \tag{3.23}$$

$$F_1(r) = -\frac{4\pi r^3}{3} - \frac{Y'(\alpha + 4r^2)}{16} - \frac{Y}{16}\left[\frac{4r(n+2)}{n} + \frac{\alpha}{r}\right] - \frac{Zr}{2B} + \frac{\alpha(rY' + Y)(\alpha + 4r^2)}{64r^3}\log\left(1 + \frac{4r^2}{\alpha}\right). \tag{3.24}$$

$$F_2(r) = -\frac{Z'(\alpha + 4r^2)}{16} + \frac{B\pi r(3\alpha + 4r^2)}{6} + \frac{4r^2 - \alpha}{16r} + \frac{\alpha(Z'r + Z + B8\pi r^2)(\alpha + 4r^2)}{64r^3}\log\left(1 + \frac{4r^2}{\alpha}\right). \tag{3.25}$$

$$F_3(r) = \frac{rY}{2Bn}. \tag{3.26}$$

Solving for the function X by setting the state $F_0 = 0$, we obtain

$$X(r) = \frac{X_0}{r}. \tag{3.27}$$

Next, imposing the condition $F_2 = 0$ we arrive to

$$Z(r) = -\frac{8}{3}\pi B r^2 + Z_0\left[4r - \frac{\alpha}{r}\log\left(1 + \frac{4r^2}{\alpha}\right)\right]. \tag{3.28}$$

Using (3.28) in (3.24), and replacing the result to solve for the function Y by imposing $F_1 = 0$, we have

$$Y(r) = -\frac{nZ_0}{B(n+1)}\left[4r - \frac{\alpha}{r}\log\left(1 + \frac{4r^2}{\alpha}\right)\right] + \frac{Y_0}{r}\left[4r^2 - \alpha\log\left(1 + \frac{4r^2}{\alpha}\right)\right]^{-\frac{1}{n}}. \tag{3.29}$$

Thus, we can replace (3.27), (3.28), and (3.29) in (2.73), to obtain

$$\begin{aligned} h(r,P) = {} & \frac{X_0}{r} + \frac{r^2}{l^2} - \frac{3Z_0}{B8\pi l^2}\left[4r - \frac{\alpha}{r}\log\left(1 + \frac{4r^2}{\alpha}\right)\right] + \\ & \left(-\frac{3}{B8\pi l^2}\right)^{-\frac{1}{n}}\left\{-\frac{nZ_0}{B(n+1)}\left[4r - \frac{\alpha}{r}\log\left(1 + \frac{4r^2}{\alpha}\right)\right] + \frac{Y_0}{r}\left[4r^2 - \alpha\log\left(1 + \frac{4r^2}{\alpha}\right)\right]^{-\frac{1}{n}}\right\}. \end{aligned} \tag{3.30}$$

Finally, we determine the metric function by introducing the solution (3.30) in the equation (2.40). In order to conserve the AdS behavior, we define the values for the constants as $X_0 = Z_0 = 0$, $Y_0 = \frac{1}{3}\pi P\left(-\frac{P}{B}\right)^{\frac{1}{n}}$ and $n = -\frac{2}{3}$, as a result

$$f = -\frac{2M}{r} + \frac{P\pi}{3r}\left[4r^2 - \alpha\log\left(\frac{4r^2}{\alpha} + 1\right)\right]^{3/2}. \tag{3.31}$$

In this expression, if we take the limit $\alpha \to 0$ then the AdS structure is recovered, as expected. In Figure 3.6, we display the solution f as a function of r for some values of P

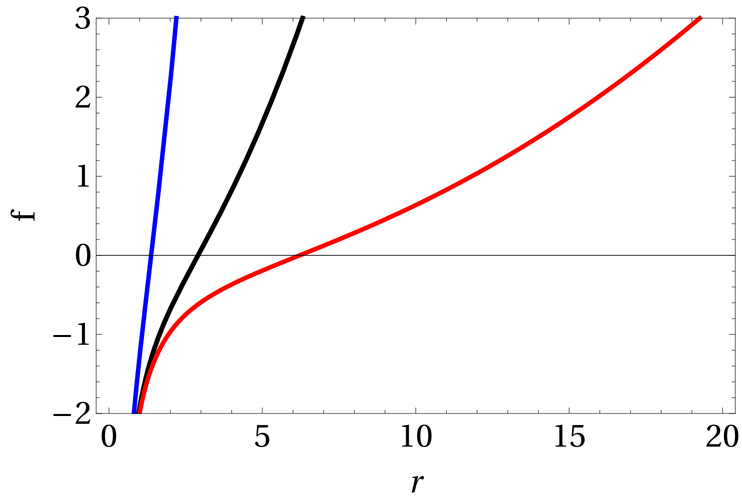

Figure 3.6: f as a function of r, with $M = 1$, and $\alpha = 0.1$ and $P = 0.1$ (blue line), $P = 0.01$ (black line), and $P = 0.001$ (red line).

For this solution the matter sector reads

$$\varrho = -p_r = P + \frac{1}{8\pi r^2} - \frac{2Pr\sqrt{4r^2 - \alpha\log\left(\frac{4r^2}{\alpha} + 1\right)}}{\alpha + 4r^2} \tag{3.32}$$

$$p_\perp = -P + \frac{4P\left(8r^5 + 3\alpha r^3\right)}{(\alpha + 4r^2)^2\sqrt{4r^2 - \alpha\log\left(\frac{4r^2}{\alpha} + 1\right)}} - \frac{\alpha Pr\left(3\alpha + 4r^2\right)\log\left(\frac{4r^2}{\alpha} + 1\right)}{(\alpha + 4r^2)^2\sqrt{4r^2 - \alpha\log\left(\frac{4r^2}{\alpha} + 1\right)}}. \tag{3.33}$$

In Figure 3.7 we show that the solution satisfies the DEC

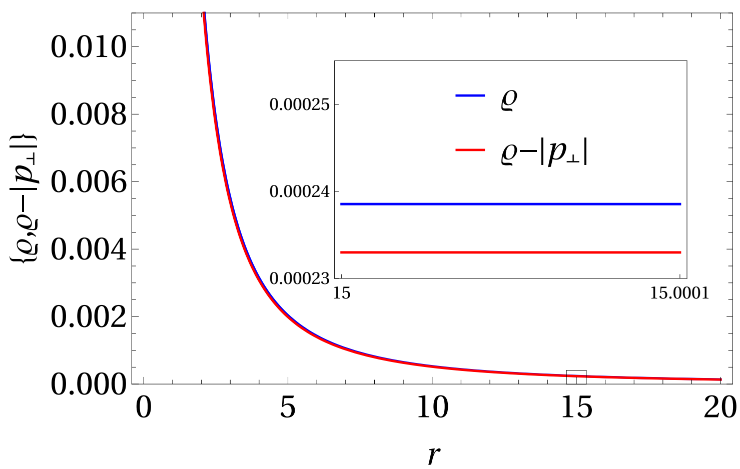

Figure 3.7: Dominant energy condition. The energy density ϱ (blue line) and energy density minus the absolute values of perpendicular pressure $\varrho - |p_\perp|$ (red line), by setting $P = 0.1$ and $\alpha = 0.1$.

Chapter 4

Conclusions & Outlook

In this work we obtained asymptotically anti–de Sitter solutions in the context of extended phase space corrected by the generalized uncertainty principle. To be more precise, we used the minimal length corrections provided by the generalized uncertainty principle to modify the black hole temperature and entropy in the first law of the thermodynamics and we close the system by assuming a suitable equation of state for the cosmological pressure which arises in the context of the extended phase space. In particular, we introduced the formalism of generalized uncertainty principle to the so called extended phase space for the Polytropic and Chaplygin black holes, inspired by the corrected Van der Waals solution proposed the last year. First, we solved the corrected extended phase space equations using the Van der Waals equation of state. In this case, all the energy conditions are violated. Second, by using the first law of thermodynamics and the Polytropic equation of state, a solution was found. In this case, all the energy conditions are satisfied. Finally, adapting the previous protocol to the Chaplygin equation of state, we found another solution which, in contrast to the Polytropic case, satisfies only the dominant energy condition. In conclusion, the corrected Polytropic and Chaplygin black holes can be considered as physical acceptable solutions.

It is worth mentioning that for this work we only review the quadratic minimal length corrections. The combination of other possible corrections could result in unexplored families of black hole (BH) solutions. This and other features are left for future works.

Appendix A

Energy conditions

In this section we define the null, weak, strong, and dominant energy conditions[52,53].
The null energy condition (NEC) states that the energy density of any matter distribution, measured by any observer must be non negative, namely

$$T_{\mu\nu}k^{\mu}k^{\nu} \geq 0, \tag{A.1}$$

where k^{μ} is an arbitrary null vector. Thus, for a general anisotropic fluid, the NEC implies

$$\forall i, \quad \varrho + p_i \geq 0 \tag{A.2}$$

The weak energy condition (WEC) makes the same statement as the null condition, with the difference that k^{μ} is replaced by V^{μ}.

$$T_{\mu\nu}V^{\mu}V^{\nu} \geq 0 \tag{A.3}$$

In this case, V^{μ} is an arbitrary timelike vector, which represents the four velocity of an observer in space–time. The weak condition conduce to

$$\varrho \geq 0, \quad \text{and} \quad \forall i, \quad \varrho + p_i \geq 0, \tag{A.4}$$

with respect to the principal pressures. Notice that, by continuity also imply the NEC. The physical meaning of this condition is that for any timelike observer the local energy density measured must be positive.
The strong energy condition (SEC) is the assertion that

$$\left(T_{\mu\nu} - \frac{1}{2}T g_{\mu\nu}\right)V^{\mu}V^{\nu} \geq 0. \tag{A.5}$$

Here T is the trace of the stress-energy tensor,

$$T = T_{\mu\nu}g^{\mu\nu} = -\varrho + \sum_i p_i. \tag{A.6}$$

This energy condition therefore in principal pressures terms means that

$$\varrho + \sum_i p_i \geq 0, \quad \text{and} \quad \forall i, \quad \varrho + p_i \geq 0. \tag{A.7}$$

By continuity, we can notice that the SEC implies the NEC, but it does not imply the WEC.
The dominant energy condition (DEC) embraces the notion that locally the energy density measured is always positive, and the energy flow along timelike or null world lines.

$$T_{\mu\nu} V^\mu V^\nu \geq 0, \quad \text{and} \quad T_{\mu\nu} V^\nu \quad \text{is not spacelike} \tag{A.8}$$

This expression in terms of the energy density and principal pressures results,

$$\varrho \geq 0, \quad \text{and} \quad \forall i, \varrho \geq |p_i|. \tag{A.9}$$

The DEC implies the WEC, and thus also the NEC, nevertheless it does not necessarily imply the SEC.

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Abbreviations