UNIVERSIDAD DE INVESTIGACIÓN DE TECNOLOGÍA EXPERIMENTAL YACHAY

Escuela de Ciencias Matemáticas y Computación

TÍTULO: Minimization of Quantum Free Energy Functionals Defined on *W^p* Sobolev-like cones of Operators

Trabajo de integración curricular presentado como requisito para
la obtención
del título de Matemático

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Dedication

"To Rosita, Tomasito, Amadeo and Constanza. I know all of you have the potential to be great. I only hope that at least one of you joins me in this beautiful and intriguing path of Mathematics. "

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Abstract

We extend the results obtained by Dolbeault, Felmer & Mayorga-Zambrano, and Mayorga-Zambrano & Salinas, for the case $2 \leq p < N$ and an open bounded domain $\Omega \subseteq \mathbb{R}^N$.

Let $V \in \mathrm{L}^\infty(\Omega)$ be a potential such that $V \geq 0$. Consider the set of self-adjoint trace-class operators acting on $\mathrm{L}^2(\Omega)$, which we denote by $\mathscr{S}_1$. By the Hilbert-Schmidt and Riesz-Schauder theorem, we know that there exists a sequence of eigenelements of $T \in \mathscr{S}_1$

$$(\nu_{i,T}, \eta_{i,T})_{i\in\mathbb{N}} \subseteq \mathbb{R} \times \mathrm{L}^2(\Omega)$$

such that $B = \{\eta_{i,T} \ / \ i \in \mathbb{N}\}$ forms a Hilbert basis of $\mathrm{L}^2(\Omega)$. We denote the set of all possible eigenbasis of T by $\mathscr{B}_T$. Our operator setting consists of operators $T \in \mathscr{S}_1$ such that

$$\mathscr{B}_T^p = \left\{ B = \{\eta_{i,T} \ / \ i \in \mathbb{N}\} \in \mathscr{B}_T \ / \ B \subseteq \mathrm{W}_0^{1,p}(\Omega) \right\} \neq \emptyset$$

with *finite energy*, i.e.,

$$\langle\langle T \rangle\rangle_V = \inf_{B\in\mathscr{B}_T^p} \sum_{i\in\mathbb{N}} |\nu_{i,T}| \int_\Omega \left[|\nabla \eta_{i,T}(x)|^p + V(x)|\eta_{i,T}(x)|^p \right] dx < \infty.$$

In this case, we say that T belongs to the *Sobolev-like cone* $\mathscr{W}^p$. Moreover, we denote $\mathscr{W}_+^p = \{T \in \mathscr{W}^p \ / \ T \geq 0\}$.

The two main results of this capstone project provide tools to study free energy functionals acting on $\mathscr{W}_+^p$, which can then be used for certain type of problems in Quantum Mechanics. The first result proves that the embedding

$$\mathscr{W}_+^p \subseteq \mathscr{S}_1$$

when equipped with $\langle\langle \cdot \rangle\rangle_V$ is compact, i.e., if $(T_n)_{n\in\mathbb{N}} \subseteq \mathscr{W}_+^p$ is bounded in $\langle\langle \cdot \rangle\rangle_V$, then there exists subsequence that converges in $\mathscr{S}_1$. Analogous to the embedding $\mathrm{W}_0^{1,p}(\Omega) \subseteq \mathrm{L}^p(\Omega)$, we also obtain some interpolation inequalities in the language of operators. Then, we apply this result to the minimization problem of

$$\mathscr{F}_{0,p,\beta}(T) = \mathrm{Tr}\left[\beta(T)\right] + \langle\langle T \rangle\rangle_0, \quad T \in \mathscr{W}_+^p,$$

where $\beta \colon \mathbb{R} \to \mathbb{R}$ is a convex function called an *entropy seed* such that $\beta(0) = 0$, and $\mathrm{Tr}\left[\cdot\right]$ denotes the trace functional.

Keywords: Sobolev-like cone, compact embeddings, Minimization, free energy functional, generalized entropy functionals, Gagliardo-Nirenberg type inequalities for operators, nuclear operator, Spectral theorem.

Resumen

Extendemos los resultados obtenidos por Dolbeault, Felmer & Mayorga-Zambrano, y Mayorga-Zambrano & Salinas, para el caso $2 \leq p < N$ con un dominio abierto y acotado $\Omega \subseteq \mathbb{R}^N$.

Sea $V \in \mathrm{L}^\infty(\Omega)$ un potencial tal que $V \geq 0$. Considere el conjunto de operadores de traza auto-adjuntos definidos sobre $\mathrm{L}^2(\Omega)$, el cual denotamos por $\mathscr{S}_1$. Por los teoremas de Hilbert-Schmidt Riesz-Schauder, sabemos que existe una sequencia de valores propios y funciones propias de $T \in \mathscr{S}_1$

$$(\nu_{i,T}, \eta_{i,T})_{i\in\mathbb{N}} \subseteq \mathbb{R} \times \mathrm{L}^2(\Omega)$$

tal que $B = \{\eta_{i,T} \ / \ i \in \mathbb{N}\}$ forma una base Hilbertiana de $\mathrm{L}^2(\Omega)$. Denotamos el conjunto de todas las posibles bases de funciones propias de T como $\mathscr{B}_T$. Nuestro marco de trabajo consiste de operadores $T \in \mathscr{S}_1$ tales que

$$\mathscr{B}_T^p = \left\{ B = \{\eta_{i,T} \ / \ i \in \mathbb{N}\} \in \mathscr{B}_T \ / \ B \subseteq \mathrm{W}_0^{1,p}(\Omega) \right\} \neq \emptyset$$

y además con energía finita, es decir

$$\langle\langle T \rangle\rangle_V = \inf_{B\in\mathscr{B}_T^p} \sum_{i\in\mathbb{N}} |\nu_{i,T}| \int_\Omega \left[|\nabla \eta_{i,T}(x)|^p + V(x)|\eta_{i,T}(x)|^p \right] dx < \infty.$$

En este caso, decimos que T pertenece al cono tipo Sobolev $\mathscr{W}_+^p$. Además, denotamos $\mathscr{W}_+^p = \{T \in \mathscr{W}^p \ / \ T \geq 0\}$.

Los dos resultados principales de este proyecto de Titulación proveen herramientas para el estudio de funcionales de energía libre definidas sobre $\mathscr{W}_+^p$, las cuales pueden ser usadas para cierto tipo de problemas en Mecánica Cuántica. El primer resultado prueba que la inmersión

$$\mathscr{W}_+^p \subseteq \mathscr{S}_1$$

cuando es equipada con $\langle\langle \cdot \rangle\rangle_V$, es compacta. Análogo a la inmersión $\mathrm{W}_0^{1,p}(\Omega) \subseteq \mathrm{L}^p(\Omega)$, también obtenemos desigualdades de interpolación, llevadas al lenguaje de operadores. Luego, aplicamos este resultado en el problema de minimización para

$$\mathscr{F}_{0,p,\beta}(T) = \mathrm{Tr}\left[\beta(T)\right] + \langle\langle T \rangle\rangle_0, \quad T \in \mathscr{W}_+^p,$$

donde $\beta\colon \mathbb{R} \to \mathbb{R}$ una función convexa llamada semilla de entropía tal que $\beta(0) = 0$, y $\mathrm{Tr}\left[\cdot\right]$ denota el funcional de traza.

Keywords: Cono tipo Sobolev, inmersión compacta, minimización, funcional de energía libre, funcional de entropía generalizada, desigualdades de tipo Gagliardo-Nirenberg para operadores, operadores nucleares, teorema espectral.

Contents

Chapter 1

Introduction

The advancement of Quantum Mechanics (QM) in the first decades of the twentieth century had at its core the field of mathematics of Functional Analysis. Mathematical objects were developed to explain many of the questions that arise in QM. In particular, positive self-adjoint trace class operators

$$T\colon \mathrm{L}^2(\Omega) \to \mathrm{L}^2(\Omega)$$

were found to naturally describe systems in QM. For example, when investigating a non-relativistic gravitational Hartree system, T can describe a system of gravitating quantum particles, [2]. We denote the set of self-adjoint trace class operators by $\mathscr{S}_1$.

One of the main reasons for this type of operators to be widely used in QM is because of the Riesz-Schauder and Hilbert-Schmidt theorem, which guarantee, for any $T \in \mathscr{S}_1$, the existence of a sequence of eigenvalues $(\nu_{i,T})_{i\in\mathbb{N}} \subseteq \mathbb{R}_+$ and a sequence of eigenfunctions $(\eta_{i,T})_{i\in\mathbb{N}} \subseteq \mathrm{L}^2(\Omega)$ of T such that

$$B = \left\{\eta_{i,T} \;/\; i \in \mathbb{N}\right\}$$

forms a Hilbertian basis of $\mathrm{L}^2(\Omega)$. We denote the set of all possible eigenbasis of T by $\mathscr{B}_T$. In the Schrödinger-Poisson picture of QM, an eigenfuntion $\eta_{i,T}$ is referred to as a *wave function* and $\nu_{i,T}$ as an *occupation number*, [15]. Furthermore, the sequence $(\nu_{i,T}, \eta_{i,T})_{i\in\mathbb{N}}$ is said to be a *mixed state*. In [7], some interpolation inequalities were proved in the study of the stability of mixed states and in [8] and [17] these results were brought to an operator setting. Moreover, in [8] and [17] a compactness theorem was proven at the level of operators that served as a tool for the minimization of a type of *free energy functionals*.

In this work, we extend the results of [8] and [17], for the case $2 \le p < N$ and an open bounded domain $\Omega \subseteq \mathbb{R}^N$. We assume that a potential $V\colon \Omega \to \mathbb{R}$ has the following properties

$$V \ge 0 \text{ and } V \in \mathrm{L}^\infty(\Omega).$$

Our operator setting consists of self-adjoint trace class operators $T \in \mathscr{S}_1$ such that

$$\mathscr{B}_T^p = \left\{ B = \{\eta_{i,T} \;/\; i \in \mathbb{N}\} \in \mathscr{B}_T \;/\; B \subseteq \mathrm{W}_0^{1,p}(\Omega) \right\} \neq \emptyset$$

and

$$\langle\langle T\rangle\rangle_V = \inf_{B\in\mathscr{B}_T^p} \sum_{i\in\mathbb{N}} |\nu_{i,T}| \int_\Omega \left[|\nabla\eta_{i,T}(x)|^p + V(x)|\eta_{i,T}(x)|^p\right] dx < \infty. \tag{1.1}$$

In this case, we say that T belongs to the *Sobolev-like cone* $\mathscr{W}^p$ and we refer to (1.1) as the energy of the operator T. Moreover, we denote $\mathscr{W}_+^p = \{T \in \mathscr{W}^p \ / \ T \geq 0\}$.

Then, we study two type of functionals acting on $\mathscr{W}_+^p$. The first ones are called *entropy functionals*:

$$\mathscr{E}_\beta(T) = \operatorname{Tr}\left[\beta(T)\right] = \sum_{i\in\mathbb{N}} \beta(\nu_{i,T}), \quad T \in \mathscr{W}_+^p,$$

where $\beta : \mathbb{R} \to \mathbb{R} \cup \{\infty\}$ is a convex function such that $\beta(0) = 0$. The second kind of functionals are *(V,p,β)-free energy functionals*,

$$\mathscr{F}_{V,p,\beta}(T) = \mathscr{E}_\beta(T) + \langle\langle T\rangle\rangle_V, \quad T \in \mathscr{W}_+^p.$$

The final result of the present work consists in minimizing a free energy functional in $\mathscr{W}_+^p$. To achieve this, first we prove that $\mathscr{F}_{V,p,\beta}$ is bounded from below. Then we prove our main result: that given a sequence $(T_n)_{n\in\mathbb{N}} \subseteq \mathscr{W}_+^p$ such that its energy is bounded, there exists a subsequence of $(T_n)_{n\in\mathbb{N}}$ that converges in trace norm $\|\cdot\|_1$ to some T in $\mathscr{W}_+^p$. This compactness result shows the similarities between the embeddings $\mathscr{W}_+^p \subseteq \mathscr{S}_1$ and $\mathrm{W}_0^{1,p}(\Omega) \subseteq \mathrm{L}^p(\Omega)$. Although all of our results are proved with the assumption that $2 \leq p < N$, by an appropriate application of the Rellich-Kondrachov theorem, we can extend these results for the case $p \geq N$.

We present a short description of this document

Summary of Chapter 2

In this chapter, we present some basic results of Functional Analysis that are essential for our work. We begin with some definitions and properties of topological, Banach and Hilbert spaces. We also provide the definition of convex functions and some important results about semi-continuity which shall be used for a minimization theorem in Section 4.4.

In Section 2.2, we give a brief overview of Lebesgue and Sobolev spaces and their properties. The most important results are Rellich-Kondrachov theorem and Poincaré's inequality. Then, in Section 2.3, we introduce the theory of bounded linear operators; some basic definitions and important results about the space $\mathscr{L}(\mathscr{H})$, where $\mathscr{H}$ is a Hilbert space. We give the definition of the spectrum of an operator and the definition of several type of bounded operators such as *positive operators*, *self-adjoint operators* and *projection operators*. Furthermore, we present some basic results about them, for instance, the polar decomposition of a bounded linear operator $T \in \mathscr{L}(\mathscr{H})$ and the characterization of a projection operator. The next section is a summary of the main results about *compact operators* and *compact self-adjoint operators*. Two of these results are the Hilbert-Schmidt theorem and the singular value decomposition of compact operators. We finish this section with the definitions of *trace class operators* and *Hilbert-Schmidt operators* along with some fundamental results about them. These kind of operators appear naturally in QM and shall help us define our operator setting $\mathscr{W}_+^p$. We finish this chapter with one of the most important tools for our work: the Spectral Theorem. Specifically, the functional calculus version of this theorem let us define self-adjoint operators of the form $F(T)$, where T is self-adjoint and $F\colon \mathbb{R} \to \mathbb{R}$ is a measurable function.

Summary of Chapter 3

In this chapter, we begin with a historical overview of QM. Then, we present very briefly some operators in QM such as the position and momentum operator. Moreover, we provide some basic but indispensable results such as Heisenberg's uncertainty principle. Finally, in Section 3.3, we consider the Schrödinger operator and some of its properties.

Summary of Chapter 4

In Chapter 4 we present the results of this work. Section 4.1 consists of definitions and preliminary results. We define the Sobolev-like cones $\mathscr{W}^p, \mathscr{W}_+^p$ and the energy of an operator T. Then, we prove some basic facts about $\mathscr{W}^p$. We finish this section with a regularity result about the density of an operator $T \in \mathscr{W}_+^p$. This result states that for every $T \in \mathscr{W}^p$, its associated density function

$$\rho_T(x) = \sum_{i\in\mathbb{N}} |\nu_{i,T}||\eta_{i,T}(x)|^2, \quad x \in \Omega$$

belongs to $\mathrm{W}^{1,r}(\Omega) \cap \mathrm{L}^q(\Omega)$ for certain values of r, q.

In Section 4.2, we define a class of functions referred to as *p-Clasimir functions* that let us define trace-class operators of the form

$$F\left[\left(\frac{-\Delta + V}{\hat{C}}\right)^{p/2}\right]$$

for some constant $\hat{C} > 0$. We also define define (V,p,β)-free energy functionals acting on $\mathscr{W}_+^p$ as the sum of the total energy and entropy of an operator $T \in \mathscr{W}_+^p$. The first result in this section proves that these type of functionals are bounded from below which, at the same time, proves some *Gagliardo-Nirenberg* type inequalities, adapted for our operator setting $\mathscr{W}_+^p$. The next section contains the most difficult result so far: the compactness of the embedding $\mathscr{W}_+^p \subseteq \mathscr{S}_1$ Then, in Section 4.5, we prove the existence of a minimizer for a free energy functional.

Summary of Chapter 5

We present our conclusions and recommendations.

Chapter 2

Theoretical framework

In this chapter we present various concepts and results of Functional Analysis which are fundamental in this work. We start with some basic definitions of mathematical analysis. Then we give some definitions and results about Sobolev spaces. Finally, we study different type of linear operators and their spectral properties.

The terminology and notation used in this work are standard. We shall mostly work over the field $\mathbb{R}$, except in Subsection 2.3.2, where some concepts require the use of $\mathbb{C}$. The main references are [26], [16], [9], [21], [14], [13], [6] and [10].

2.1 Some results and definitions of Functional Analysis

This section provides the standard starting point of Functional Analysis and Operators Theory.

2.1.1 Topological, normed and inner product spaces

The notions of limit and convergence are essential in Functional Analysis. The most general space where we can talk about these concepts are called topological spaces. A *topological space* is a set X having a family of subsets $\mathcal{T}$, called open sets, with the following properties:

(i) The void set $\emptyset$ and the whole space X are in $\mathcal{T}$.

(ii) If $(V_\alpha)_{\alpha\in I}$ is a family of open sets, then $\bigcup_{\alpha\in I} V_\alpha \in \mathcal{T}$.

(iii) If $V, W \in \mathcal{T}$, then $V \bigcap W \in \mathcal{T}$.

$\mathcal{T}$ is called a *topology* on X. We say that $(X, \mathcal{T})$ is a topological space. When the topology on X is obvious in the context, we refer to the topological space only as X.

Remark 2.1. *A set whose complement is open is said to be closed.*

Remark 2.2. *The family of all topologies on a set X is ordered in a natural way: $\mathcal{T}_1 \prec \mathcal{T}_2$ if and only if $\mathcal{T}_1 \subseteq \mathcal{T}_2$. If $\mathcal{T}_1 \prec \mathcal{T}_2$ we say that $\mathcal{T}_1$ is a weaker topology than $\mathcal{T}_2$. As we shall see, the term weaker implies that that convergence in $\mathcal{T}_1$ occurs more often than in $\mathcal{T}_2$.*

The intuition of a sequence of points in a space getting closer and closer to another point can be formalized with the concept of convergence. In order to properly work with convergence, we need the following condition on our topological space X:

$$\forall x, y \in X, x \neq y, \exists A \in \mathcal{O}(x), \exists B \in \mathcal{O}(y): \qquad A \cap B = \emptyset, \tag{T2}$$

where $\mathcal{O}(x), \mathcal{O}(y)$ denote the family of all open sets containing x and y, respectively. This condition is denoted as $(T2)$ and all topological spaces that satisfy this condition are called *Hausdorff spaces*. It is well known that every convergent sequence in a Hausdorff space has a unique limit.

Definition 2.1. *Let $(X, \mathcal{T})$ be a Hausdorff space and $(x_n)_{n\in\mathbb{N}}$ a sequence of points in X. We say that $(x_n)_{n\in\mathbb{N}}$ converges to some $x \in X$ if and only if*

$$\forall A \in \mathcal{O}(x), \exists N \in \mathbb{N}: \quad n > N \implies x_n \in A,$$

x is called the limit of the sequence $(x_n)_{n\in\mathbb{N}}$, and denote

$$\lim_{n\to\infty} x_n = x,$$

or simply $x_n \to x$ if there is no ambiguity.

We introduce the concept of *norm* and *normed space*. The theory of normed spaces and the theory of linear operators defined on them are the standard starting point of functional analysis.

Definition 2.2. *A (real) normed space is a linear space X and a function, $\|\cdot\| : X \to \mathbb{R}$ which satisfies:*

(i) Non-negativity: $\|x\| \geq 0$, for all $x \in X$.

(ii) Point separating: $\|x\| = 0$ if and only if $x = 0$.

(iii) Absolute homogeneity: $\|\alpha x\| = |\alpha|\|x\|$ for all $x \in X$ and all $\alpha \in \mathbb{R}$.

(iv) Triangle inequality: $\|x + y\| \leq \|x\| + \|y\|$, for all $x, y \in X$.

The function $\|\cdot\|$ is called a norm. *We denote the normed space $(X, \|\cdot\|)$. When the norm on X is obvious in a context, we refer to the normed space only as X.*

Note that normed spaces are a richer-in-properties kind of topological spaces. In fact, we can define a topology in $(X, \|\cdot\|)$ as follows

Definition 2.3. *A subset A of a normed space X is open if and only if, for every $x \in X$, there exists $r > 0$ such that $B_r(x) \subseteq A$, where*

$$B_r(x) = \{y \in X / \ \|x - y\| < r\}.$$

These open sets form a topology on X:

$$\mathcal{T} = \{A \subseteq X / \forall x \in X, \exists r > 0 : B_r(x) \subseteq A\}.$$

Remark 2.3. *Every normed space is a Hausdorff space. Therefore, uniqueness of limits of sequences in a normed space is guaranteed.*

In the framework of normed spaces, the concepts of *convergence* and *continuity* are much simpler than in topological spaces. Let X be a normed space. A sequence $(x_n)_{n\in\mathbb{N}} \subseteq X$ converges to $x \in X$ if and only if

$$\forall \epsilon > 0, \exists N \in \mathbb{N} : \quad n > N \implies \|x_n - x\| < \epsilon.$$

Definition 2.4. *Let X, Y be normed spaces. We denote $\|\cdot\|_X$ and $\|\cdot\|_Y$ the norms in X and Y respectively. A function $f : X \to Y$ is* continuous *at a point $x \in X$ if and only if for any convergent sequence $x_n \to x$ we have*

$$f(x_n) \to f(x), \quad \text{as } n \to \infty,$$

in the sense of the norm $\|\cdot\|_Y$. We say that f is continuous on $A \subseteq X$ if and only if it is continuous at every $x \in A$.

The following proposition is useful in some situations

Proposition 2.1. *Let X, Y be normed spaces. A function $f : X \to Y$ is continuous if and only if for every open set V, $f^{-1}(V)$ is open. Similarly, f is continuous if and only if for every closed set C, $f^{-1}(C)$ is closed.*

A proof of this result can be found e.g., in [11].

There exists a particular class of sequences in normed spaces called *Cauchy sequences* which have the property that the points of the tail of the sequence are closer and closer, as $n \to \infty$. To be precise, a sequence of elements $(x_n)_{n\in\mathbb{N}}$ of a normed space X is called a Cauchy sequence if and only if

$$\forall \epsilon > 0, \exists N \in \mathbb{N} : \quad n, m > N \implies \|x_n - x_m\| < \epsilon.$$

It is clear that any convergent sequence is Cauchy. However, there exist normed spaces where the converse is not necessarily true. The particular spaces where the converse statement holds have a special name. A normed space X is said to be a *Banach space* if and only if every Cauchy sequence in X is convergent.

Remark 2.4. *We have omitted the concept of metric spaces. In this case, if every Cauchy sequence in a metric space is convergent, we say this space is* complete.

In the following example, we present an important type of Banach space

Example 2.1. *Let $p \in [1, \infty)$. The space*

$$\ell^p = \ell^p(\mathbb{R}) = \left\{ \alpha = (\alpha_i)_{i \in \mathbb{N}} \subseteq \mathbb{R} \; / \sum_{i \in \mathbb{N}} |\alpha_i|^p < \infty \right\}$$

is a Banach space when equipped with the norm

$$\|\alpha\|_{\ell^p} = \left(\sum_{i \in \mathbb{N}} |\alpha_i|^p \right)^{1/p}.$$

We define the conjugate exponent *p' of a number $1 \leq p \leq \infty$ as follows*

$$\frac{1}{p} + \frac{1}{p'} = 1.$$

In part, this is helpful because of a very practical result called Hölder's inequality, *which states that for any $\alpha = (\alpha_i)_{i \in \mathbb{N}} \in \ell^p$ and $\beta = (\beta_i)_{i \in \mathbb{N}} \in \ell^{p'}$ we have that*

$$\sum_{i \in \mathbb{N}} |\alpha_i \beta_i| \leq \left(\sum_{i \in \mathbb{N}} |\alpha_i|^p \right)^{1/p} \left(\sum_{i \in \mathbb{N}} |\beta_i|^{p'} \right)^{1/p'}.$$

Moreover, there exists a similar result called reversed Hölder's inequality, *which states that if $\alpha, \beta \in \ell^p$ are non-negative and $p \in (0, 1)$, $p' \in (-\infty, 0)$, then*

$$\sum_{i \in \mathbb{N}} \alpha_i \beta_i \geq \left(\sum_{i \in \mathbb{N}} |\alpha_i|^p \right)^{1/p} \left(\sum_{i \in \mathbb{N}} |\beta_i|^{p'} \right)^{1/p'}.$$

Proofs of both inequalities can be found in [16, Prop. 4.1] and [1, Th. 2.6], respectively.

In the study of normed spaces, it is often useful to analyze the behaviour of linear functionals acting on these normed spaces. We denote by X' the *topological dual space* of a normed space X, which consists of all continuous linear functionals acting on X. This space becomes a normed space when equipped with the norm

$$\|\eta\| = \sup_{x \neq 0} \frac{|\eta(x)|}{\|x\|}.$$

X' is in fact a Banach space (see Proposition 2.5). From this definition it is clear that every continuous linear functional is *bounded*, that is

$$\forall \eta \in X', \exists c > 0, \forall x \in X: \quad |\eta(x)| < c\|x\|.$$

It can be proven that the smallest such c is equal to $\|\eta\|$, [21]. Furthermore, it can be proven that every bounded linear functional is continuous. Continuous linear functionals are a special type of bounded linear operators, which are studied in depth in future sections.

We can define a certain type of convergence in a normed space X, using its dual space X'. We say that $(x_n)_{n\in\mathbb{N}} \subseteq X$ *weakly converges* to some $x \in X$ as $n \to \infty$, denoted by

$$x_n \rightharpoonup x, \text{ as } n \to \infty$$

if and only if

$$\forall \phi \in X' : \quad \lim_{n\to\infty} \phi(x_n) = \phi(x).$$

There are many properties that deal with weak convergence. For a more in-depth study, see e.g., [6].

An even richer-in-properties space than normed spaces are the *inner product spaces*. These types of spaces generalize the geometric notions that can be seen in $\mathbb{R}^3$.

Definition 2.5. *A (real) linear space X is called an inner-product space if and only if there is a real-valued function $(\cdot,\cdot)$ on $X \times X$ that satisfies the following conditions, for all $x, y, z \in X$ and $\alpha \in \mathbb{C}$:*

(i) $(x, x) \geq 0$ *and* $(x, x) = 0$ *if and only if* $x = 0$.

(ii) $(x + y, z) = (x, z) + (y, z)$.

(iii) $(\alpha x, y) = \alpha(x, y)$.

(iv) $(x, y) = (y, x)$.

The function $(\cdot,\cdot)$ is called an inner-product on X.

Note that every inner product space is a normed space since $(\cdot,\cdot)$ induces a norm:

$$\|x\| = (x, x)^{1/2}.$$

Furthermore, for a normed space X to be an inner-product space, the following condition is necessary and sufficient:

$$\forall x, y \in X: \quad 2\|x\|^2 + 2\|y\|^2 = \|x + y\|^2 + \|x - y\|^2.$$

This is called the *parallelogram identity*.

2.1.2 Properties of Banach and Hilbert spaces

Previously, we defined Banach spaces, which are very similar to $\mathbb{R}^n$ in the sense that there exists a sense of size and distance. Even though the norm does not always arise from an inner product, it still has some interesting properties.

The next result gives a sufficient and necessary condition for subspaces of a Banach space to be complete.

Proposition 2.2. *Let X be a Banach space and $Y \subseteq X$, then Y is a Banach space if and only if Y is closed.*

A proof of this proposition can be found in [13].

Another important tool that we shall use are *isometries*. We say that two normed spaces X and Y are *isometric* if and only if there exists a bijection $\phi : X \to Y$, called isometry, such that

$$\forall x \in X : \quad \|\phi(x)\| = \|x\|.$$

Two Banach spaces which are isometric have the same Banach space properties and in this context we can consider them as two representations of the same abstract space. Using isometries and the next result, which states that every normed space can be completed in a unique way, we shall be able to work with *completions* of incomplete spaces.

Theorem 2.1. *Let X be a normed space. Then there is a Banach space $\hat{X}$ and an isometry ϕ from X onto a subspace $W \subseteq \hat{X}$ which is dense in $\hat{X}$. The space $\hat{X}$ is unique, except for isometries.*

A proof of this theorem can be found in [13, Th. 2.3.2].

The following result, that is useful when working with normed spaces, is concerned with extending linear functionals while keeping its size. This is called *Hahn-Banach theorem* for normed spaces.

Theorem 2.2. *Let X be a normed linear space, Y a subspace of X, and η an element of Y'. Then there exists a $\overline{\eta} \in X'$ extending η and satisfying*

$$\|\overline{\eta}\|_{X'} = \|\eta\|_{Y'}.$$

There are other important theorems about bounded linear maps on Banach spaces. These results will be given in future sections. For now, we discuss a kind of space which has more properties than Banach spaces.

Inner-product spaces which are complete are called *Hilbert spaces*. We often denote a generic Hilbert space as $\mathcal{H}$. Hilbert spaces are full of geometric properties because of the inner product itself, which is a generalization of the usual dot product on $\mathbb{R}^N$.

First, we give an important inequality of inner product spaces, called the *Cauchy-Schwarz inequality*

Proposition 2.3. *Let x and y in $\mathcal{H}$, then*

$$|(x, y)| \leq \|x\|\|y\|.$$

Proof. Let $x, y \in \mathcal{H}$. Note that

$$0 \leq \left(\frac{x}{\|x\|} - \frac{y}{\|y\|}, \frac{x}{\|x\|} - \frac{y}{\|y\|} \right) \leq 2 \left(1 - \left(\frac{x}{\|x\|}, \frac{y}{\|y\|} \right) \right)$$

and

$$0 \leq \left(\frac{x}{\|x\|} + \frac{y}{\|y\|}, \frac{x}{\|x\|} + \frac{y}{\|y\|} \right) \leq 2 \left(1 + \left(\frac{x}{\|x\|}, \frac{y}{\|y\|} \right) \right).$$

Then

$$\left| \left(\frac{x}{\|x\|}, \frac{y}{\|y\|} \right) \right| \leq 1$$

and the result immediately follows. □

Analogous to $\mathbb{R}^N$, the inner-product $(\cdot, \cdot)$ let us define important concepts such as *orthogonality* of vectors of a Hilbert space $\mathcal{H}$. Two elements $x, y \in \mathcal{H}$ are said to be orthogonal if $(x, y) = 0$. A collection of vectors $\{x_\lambda / \lambda \in \Lambda\}$ in $\mathcal{H}$ is called an *orthonormal set* if

$$\begin{cases} (x_\lambda, x_\beta) = 1, & \text{if } \lambda = \beta, \\ (x_\lambda, x_\beta) = 0, & \text{if } \lambda \neq \beta. \end{cases}$$

Additionally, we can define the *orthogonal complement* of a set contained in $\mathcal{H}$. Let $Y \subseteq \mathcal{H}$, we denote by $Y^\perp$ the set of vectors which are orthogonal to every element in Y, i.e.

$$Y^\perp = \{x \in \mathcal{H} / \forall y \in Y : (x, y) = 0\}.$$

If Y is closed, then $Y^\perp$ is also closed. The following theorem generalizes the idea of projecting a vector onto a closed subspace.

Theorem 2.3. *Let $\mathcal{H}$ be a Hilbert space, Y a closed subspace of $\mathcal{H}$. Then every $x \in \mathcal{H}$ can be uniquely written as $x = z + w$, where $z \in Y$ and $w \in Y^\perp$.*

A proof of this theorem can be found in [21, Th. II.3].

The dual space of a Hilbert space can be easily characterized by the following theorem, which is referred as the *Riesz-Fréchet theorem*

Theorem 2.4. *Let $\mathcal{H}$ be a Hilbert space. For each $\psi \in \mathcal{H}'$, there exists a unique $y_\psi \in \mathcal{H}$ such that*

$$\forall x \in \mathcal{H} : \quad \psi(x) = (x, y_\psi), \text{ and } \|\psi\| = \|y_\psi\|.$$

A proof of this theorem can be found in [21, Th. II.4].

Remark 2.5. *Thanks to Riesz representation theorem, the following notation becomes very helpful. Let X be a normed space and $\psi \in X'$, we denote the action of ψ with the* product in duality *as follows*

$$\psi(x) = \langle \psi, x \rangle, \quad x \in X.$$

The next concept that we generalize from $\mathbb{R}^N$ is that of an *orthonormal basis*. We say that $S \subseteq \mathcal{H}$ is an orthonormal basis for $\mathcal{H}$ if and only if

(i) S is orthonormal.

(ii) $\overline{\langle S \rangle} = H$.

$\langle S \rangle$ denotes the *span* of S, i.e. the set of finite linear combinations of elements of S. Using *Zorn's lemma*, we can prove that every Hilbert space has an orthonormal basis. Most Hilbert spaces that arise in practice have a countable dense subset. We called this kind of spaces separable Hilbert spaces. In this case, every orthonormal basis is countable and it is called a *Hilbert basis*. Furthermore, analogously to the finite dimensional case, every element of $\mathcal{H}$ can be written as a linear combination of elements of this Hilbert basis.

Theorem 2.5. *Let $\mathcal{H}$ be a separable Hilbert space and $S = \{x_n \,/ n \in \mathbb{N}\}$ a Hilbert basis. Then for each $y \in \mathcal{H}$,*

(i) $y = \sum_{n\in\mathbb{N}} (x_n, y)x_n$.

(ii) Parseval identity: $\|y\|^2 = \sum_{n\in\mathbb{N}} |(x_n, y)|^2$.

The coefficients (x_n, y) are often called *Fourier coefficients* of y with respect to the basis S.

2.1.3 Convex functions and semi-continuity

This subsection gives the definition of a convex function along with some important properties, which are very important for minimization problems.

Definition 2.6. *Let A be a linear space. We say that a function $f\colon A \to \mathbb{R}$ is convex function if and only if*

$$\forall x, y \in A, \forall t \in [0,1]: \quad f(tx + (1-t)y) \leq tf(x) + (1-t)f(y).$$

A direct consequence of this definition is the following inequality for convex functions

$$f\left(\sum_{i\in\mathbb{N}} \lambda_i x_i\right) \leq \sum_{i\in\mathbb{N}} \lambda_i f(x_i),$$

where $\sum_{i\in\mathbb{N}} \lambda_i = 1$ and $(x_i)_{i\in\mathbb{N}} \subseteq A$. We can extend this inequality to integrals. A proof of this property can be found in [4].

We recall that a function $f : V \to \mathbb{R} \cup \{\infty\}$ is said to be *lower semi-continuous* if it satisfies

$$\forall \overline{x} \in V, \forall \epsilon > 0, \exists \delta > 0: \quad \|x - \overline{x}\| < \delta \implies f(\overline{x}) - \epsilon < f(x).$$

It is clear that every continuous function is lower semi-continuous.

Lemma 2.1. *Let V be a Banach space and $f\colon X \to \mathbb{R}$ convex and lower semi-continuous. Let $(x_n)_{n\in\mathbb{N}} \subseteq V$ such that*

$$x_n \rightharpoonup x, \text{ as } n \to \infty,$$

for some $x \in V$. Then

$$f(x) \leq \liminf_{n\to\infty} f(x_n).$$

A proof of this result can be found in [6].

We finish this subsection with a useful result

Lemma 2.2. *Let $f : V \to \mathbb{R}$ be a convex function. If f is bounded above by a constant, then f is continuous at V.*

A proof of this lemma can be found in [26, Lemma 2.1].

2.2 Lebesgue and Sobolev spaces

Lebesgue and Sobolev spaces are some of the most used Banach spaces in differential equations and other fields where Functional Analysis plays a role. In this section we study the properties of these spaces. For convenience, throughout this section we shall assume that $\Omega \subseteq \mathbb{R}^N$ is open with boundary $\delta\Omega$ of class C^1. This formally means that Ω is locally very similar to $B_1(0) \subseteq \mathbb{R}^N$. For an exact description of this concept, see e.g., [6].

2.2.1 Lebesgue spaces

The support of a continuous function $f\colon \Omega \to \mathbb{R}$, denoted by $\text{supp}(f)$, is the smallest closed subset of Ω where the function does not vanish, i.e.,

$$\text{supp}(f) = \overline{\{x \in \Omega \,/\, f(x) \neq 0\}}.$$

We denote by $C_0^\infty(\Omega)$ the space of infinite differentiable functions $f : \Omega \to \mathbb{R}$ with compact support. It is well known that $C_0^\infty(\Omega)$ is a normed space when equipped with the norm

$$\|f\|_{L^\infty(\Omega)} = \sup_{x\in\Omega} |f(x)|.$$

For a proof of this result, see e.g., [16].

For $f \in C_0^\infty(\Omega)$, we write

$$\|f\|_{L^1(\Omega)} = \int_\Omega |f(x)|dx. \tag{2.1}$$

(2.1) defines a norm in $C_0^\infty(\Omega)$ and is called the L^1 − norm. The completion of $C_0^\infty(\Omega)$, (see theorem 2.1), denoted by

$$L^1(\Omega) = \overline{\left(C_0^\infty(\Omega), \|\cdot\|_{L^1(\Omega)}\right)}$$

is called the *Lebesgue space* $L^1(\Omega)$.

Remark 2.6. *The space $L^1(\Omega)$ is the set of equivalence classes determined by the equivalence relation*

$$f \sim g \iff \int_\Omega |f(x) - g(x)| dx = 0. \tag{2.2}$$

There are several important results about convergence in $\mathrm{L}^1(\Omega)$. However, we only state Fatou's lemma. For more results about integration, see [6].

Proposition 2.4. *Let $(f_n)_{n\in\mathbb{N}} \subseteq \mathrm{L}^1(\Omega)$ such that for each $n \in \mathbb{N}$, $f_n \geq 0$. Then*

$$\int_\Omega \liminf_{n\to\infty} f_n(x) dx \leq \liminf_{n\to\infty} \int_\Omega f_n(x) dx.$$

A proof of this result can be found in [24].

Let $1 < p < \infty$. Similar to (2.1), for $f \in C_0^\infty(\Omega)$ we use the notation

$$\|f\|_{\mathrm{L}^p(\Omega)} = \left(\int_\Omega |f(x)|^p dx \right)^{1/p}. \tag{2.3}$$

(2.3) represents a norm (see [6]) in $C_0^\infty(\Omega)$ and is called the $\mathrm{L}^\mathrm{p}-$ norm. The completion

$$\mathrm{L}^p(\Omega) = \overline{\left(C_0^\infty(\Omega), \|\cdot\|_{\mathrm{L}^p(\Omega)} \right)}$$

is the *Lebesgue space* $\mathrm{L}^p(\Omega)$ and its elements are equivalence classes given by the equivalence relation (2.2). The next result provides a useful inequality in these spaces.

Theorem 2.6. *For any $f \in L^p(\Omega)$ and $g \in L^{p'}(\Omega)$, fg is in $L^1(\Omega)$ and*

$$\|fg\|_{\mathrm{L}^1(\Omega)} \leq \|f\|_{\mathrm{L}^p(\Omega)} \|g\|_{\mathrm{L}^{p'}(\Omega)}$$

For a proof of Hölder's inequality, see [6, Th. 4.6].

One of the consequences of Hölder's inequality is the following *interpolation inequality*

Corollary 2.1. *Let $f \in \mathrm{L}^p(\Omega) \cap \mathrm{L}^q(\Omega)$, with $1 \leq p \leq q \leq \infty$, then $f \in \mathrm{L}^r(\Omega)$ for all r such that $p \leq r \leq q$. Moreover, we have*

$$\|f\|_{\mathrm{L}^r(\Omega)} \leq \|f\|_{\mathrm{L}^p(\Omega)}^\alpha \|f\|_{\mathrm{L}^q(\Omega)}^{1-\alpha},$$

where

$$\frac{1}{r} = \frac{\alpha}{p} + \frac{1-\alpha}{q}, \quad 0 \leq \alpha \leq 1.$$

Proof. Let $p \leq r \leq q$ and $0 \leq \alpha \leq 1$ such that

$$\frac{1}{r} = \frac{\alpha}{p} + \frac{1-\alpha}{q},$$

which can be rewritten as

$$1 = \frac{1}{p/(r\alpha)} + \frac{1}{q/(r(1-\alpha))}.$$

Then, using Hölder's inequality

$$\begin{aligned}
\|f\|^r_{L^r(\Omega)} &= \int_\Omega |f(x)|^r dx \\
&= \int_\Omega |f(x)|^{r\alpha}|f(x)|^{r-r\alpha} dx \\
&\le \left(\int_\omega ||f(x)|^{r\alpha}|^{\frac{p}{r\alpha}} dx\right)^{\frac{r\alpha}{p}} \left(\int_\Omega ||f(x)|^{r-r\alpha}|^{\frac{q}{r(1-\alpha)}} dx\right)^{\frac{r(1-\alpha)}{q}} \\
&= \left(\int_\Omega |f(x)|^p dx\right)^{\frac{r\alpha}{p}} \left(\int_\Omega |f(x)|^q dx\right)^{\frac{r(1-\alpha)}{q}} \\
&= \|f\|^{r\alpha}_{L^p(\Omega)} \|f\|^{r(1-\alpha)}_{L^q(\Omega)}.
\end{aligned}$$

Thus, since f was chosen arbitrarily, we have proved the interpolation inequality. □

The study of *reflexivity, duality and separability* of $L^p(\Omega)$ needs to be separated into cases depending on the value of p. The most favourable case is when $1 < p < \infty$, since $L^p(\Omega)$ is reflexive, separable and the dual is $L^{p'}(\Omega)$. This last property is due to the fact that every continuous linear functional on $L^p(\Omega)$ can be represented in a unique way as an integral. This result is called the *Riesz representation theorem*:

Theorem 2.7. *Let $1 < p < \infty$ and $\psi \in (L^p(\Omega))'$. Then there exists a unique function $u \in L^{p'}(\Omega)$ such that*

$$\langle \psi, f \rangle = \int_\Omega u(x) f(x) dx, \quad f \in L^p(\Omega).$$

Moreover,

$$\|u\|_{L^{p'}(\Omega)} = \|\psi\|_{(L^p(\Omega))'}.$$

A proof Riesz representation theorem can be found in [6, Th. 4.11]. The mapping $\psi \to u$ given by Riesz theorem is a surjective isometry, and it allows us to make the identification

$$(L^p(\Omega))' \cong L^{p'}(\Omega).$$

In the case where $p = 1$, we have that $L^1(\Omega)$ is separable but not reflexive. Nonetheless, we can make the identification

$$(L^1(\Omega))' \cong L^\infty(\Omega),$$

thanks to the Riesz Representation theorem for $L^1(\Omega)$:

Theorem 2.8. *Let $\psi \in (\mathrm{L}^1(\Omega))'$. Then there exists a unique function $u \in \mathrm{L}^\infty(\Omega)$ such that*

$$\langle \psi, f \rangle = \int_\Omega u(x) f(x) dx, \quad f \in \mathrm{L}^1(\Omega).$$

Moreover,

$$\|u\|_{\mathrm{L}^\infty(\Omega)} = \|\psi\|_{(\mathrm{L}^1(\Omega))'}.$$

A proof of this can be found in [6, Th. 4.14]

Finally, the space $\mathrm{L}^\infty(\Omega)$ is not separable nor is reflexive since this would imply that $\mathrm{L}^1(\Omega)$ is reflexive. A proof of all the properties given until now can be found in [6].

We finish this section by defining the space of *locally integrable functions*. We use the notation χ_K for the characteristic function of a set K:

$$\chi_K(x) = \begin{cases} 1 & , \text{ if } x \in K, \\ 0 & , \text{ if } x \notin K. \end{cases}$$

We say that a function $f : \Omega \to \mathbb{R}$ belongs to $L^p_{loc}(\Omega)$ if $f\chi_K \in L^p(\Omega)$ for every compact set K contained in Ω.

2.2.2 Sobolev spaces and Sobolev embeddings

In the study of partial differential equations (PDE) it is usually difficult to directly find solutions. Nonetheless, we can weaken the notion of derivability and create new spaces that are suitable for the study of PDE's. Therefore we begin by weakening the notion of partial derivatives. In order to achieve this, we use the space of test functions $\mathrm{C}_0^\infty(\Omega)$.

Let $\alpha = (\alpha_1, \ldots, \alpha_n)$ be a multiindex[1] of order $|\alpha| = k$ and $u \in \mathrm{C}^k(\Omega)$. In this case we define

$$D^\alpha u(x) = \frac{\partial^{|\alpha|} u(x)}{\partial x_1^{\alpha_1} \cdots \partial x_n^{\alpha_n}} = \partial_{x_1}^{\alpha_1} \cdots \partial_{x_n}^{\alpha_n} u(x).$$

If $\psi \in \mathrm{C}_0^\infty(\Omega)$, then integration by parts yields

$$\int_\Omega u(x) D^\alpha \psi(x) dx = (-1)^{|\alpha|} \int_\Omega D^\alpha u(x) \psi(x) dx.$$

By changing the space where u lives to a more general one, the equality above motivates the following definition

Definition 2.7. *Let $u \in \mathrm{L}^1_{loc}(\Omega)$. The weak partial derivative of u (if it exists) is a function $v \in \mathrm{L}^1_{loc}(\Omega)$, written*

$$Du = v,$$

[1] A vector of the form $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n)$ where each component α_i is a nonnegative integer is called a multiindex of order $|\alpha| = \alpha_1 + \cdots + \alpha_n$.

such that

$$\int_{\Omega} u(x)D\psi(x)dx = -\int_{\Omega} v(x)\psi(x)dx$$

for all test functions $\psi \in C_0^{\infty}(\Omega)$. *More generally, if* α *is a multiindex, the* α^{th}*-weak partial derivative of* u *is a function* $v \in L^1_{loc}(\Omega)$ *such that*

$$\int_{\Omega} u(x)D^{\alpha}\psi(x)dx = (-1)^{|\alpha|}\int_{\Omega} v(x)\psi(x)dx$$

for all test functions $\psi \in C_0^{\infty}(\Omega)$. *We write* $v = D^{\alpha}u$.

Not all functions have a weak derivative (see [10, Pg. 258, Ex. 2]). Nevertheless, if it exists, the partial weak derivative of a function is unique. In fact, let $u \in L^1_{loc}(\Omega)$ such that there exists $v, \overline{v}$ satisfying

$$\int_{\Omega} uD^{\alpha}\psi(x)dx = (-1)^{|\alpha|}\int_{\Omega} v(x)\psi(x)dx = (-1)^{|\alpha|}\int_{\Omega} \overline{v(x)}\psi(x)dx,$$

for all $\psi \in C_0^{\infty}(\Omega)$. Then

$$\int_{\Omega}(v(x) - \overline{v(x)})\psi(x)dx = 0,$$

whence $v - \overline{v} = 0$ a.e. (see [6, Cor. 4.24]). Hence the partial weak derivative, if it exists, is unique.

Sobolev spaces are built using the notion of weak derivative. Fix $1 \leq p \leq \infty$ and let k be a nonnegative integer. The *Sobolev space* $W^{k,p}(\Omega)$ consists of all functions $u \in L^1_{loc}(\Omega)$ such that for each multiindex α with $|\alpha| \leq k$, $D^{\alpha}u$ exists in the weak sense and belongs to $L^p(\Omega)$, i.e.,

$$W^{k,p}(\Omega) = \{u \in L^1_{loc}(\Omega) / \forall|\alpha| \leq k : D^{\alpha}u \in L^p(\Omega)\}.$$

In the special case when $p = 2$, we write

$$H^k(\Omega) = W^{k,2}(\Omega),$$

where k is a nonnegative integer. We define the norm of an element $u \in W^{k,p}(\Omega)$, for $1 \leq p < \infty$ as

$$\|u\|_{W^{k,p}(\Omega)} = \left(\sum_{|\alpha|\leq k}\int_{\Omega}|D^{\alpha}u(x)|^p\,dx\right)^{1/p}.$$

Similarly, the norm of an element $u \in W^{k,\infty}(\Omega)$ is defined to be

$$\|u\|_{W^{k,\infty}(\Omega)} = \sum_{|\alpha|\leq k}\|D^{\alpha}u\|_{L^{\infty}(\Omega)}.$$

The completion of $C_0^{\infty}(\Omega)$ under this norm is denoted as $W_0^{k,p}(\Omega)$. That is, $u \in W_0^{k,p}(\Omega)$ if and only if

$$\exists (u_m)_{m\in\mathbb{N}} \subseteq C_0^{\infty}(\Omega) : \quad u_m \to u \text{ in } W^{k,p}(\Omega), \quad \text{as } m \to \infty.$$

In the special case where $p = 2$, we write

$$H_0^k(\Omega) = W_0^{k,2}(\Omega).$$

One of the main properties of $W^{k,p}(\Omega)$ inherited from $L^p(\Omega)$ is that it is a Banach space.

Theorem 2.9. *For each $k = 1, 2, 3\dots$ and $1 \leq p \leq \infty$ the Sobolev space $W^{k,p}(\Omega)$ is a Banach space. In the case when $p = 2$, $H^k(\Omega)$ is a Hilbert space with inner product*

$$(u, v)_k = \sum_{0 \leq |\alpha| k} (D^\alpha u, D^\alpha v)_2.$$

Furthermore, $W^{k,p}(\Omega)$ is reflexive for $1 < p < \infty$ and it is separable for $1 \leq p < \infty$.

A proof of this theorem can be found in [6] and [10].

Another advantage of working with Sobolev spaces is that it is not always necessary to use the definition of weak derivative. This strategy consists in approximating a function in a Sobolev space by a sequence of smooth functions. The following theorem is one of several density theorems for Sobolev spaces.

Theorem 2.10. *Assume that Ω is bounded with boundary of class C^1. Suppose $u \in W^{k,p}(\Omega)$ for some $1 \leq p < \infty$. Then*

$$\exists (u_m)_{m \in \mathbb{N}} \subseteq C_0^\infty(\overline{\Omega}) : \quad u_m \to u \text{ in } W^{k,p}(\Omega), \quad \text{as } m \to \infty.$$

A proof of this theorem can be found in [10, Th.3, Sec. 5.3.3]

When studying Sobolev spaces $W^{k,p}(\Omega)$, we define the *Sobolev critical exponent* p^* of $1 \leq p < N$ by

$$p^* = \frac{pN}{N - p}.$$

Now, we want to establish some *Sobolev embeddings* which will depend upon whether we are in one of these cases:

(i) $1 \leq p < N$.

(ii) $p = N$ (the limiting case).

(iii) $N < p \leq \infty$.

In each case we state Sobolev inequalities along with the corresponding embeddings. First, we study the Sobolev space $W^{1,p}(\mathbb{R}^N)$ for the case (i). The Sobolev inequality obtained in this case is called the *Gagliardo-Nirenberg-Sobolev inequality*:

Theorem 2.11. *Assume $1 \leq p < N$. There exists a constant $C = C(p, N)$ such that*

$$\|u\|_{L^{p^*}(\mathbb{R}^N)} \leq C \|Du\|_{W^{1,p}(\mathbb{R}^N)},$$

and

$$W^{1,p}(\mathbb{R}^N) \subseteq L^{p^*}(\mathbb{R}^N).$$

A proof of this result can be found in [6, Th. 9.9]. A direct consequence of this theorem is the following continuous embedding

Corollary 2.2. *Let* $1 \leq p < N$. *Then*

$$\forall q \in [p, p^*] : \quad W^{1,p}(\mathbb{R}^N) \subseteq L^q(\mathbb{R}^N),$$

with continuous injection.

Proof. Given $q \in [p, p^*]$, we write

$$\frac{1}{q} = \frac{\alpha}{p} + \frac{1-\alpha}{p^*},$$

for some $0 \leq \alpha \leq 1$. Let $u \in W^{1,p}(\mathbb{R}^N)$. By Theorem 2.11, $u \in L^p(\mathbb{R}^N) \cap L^{p^*}(\mathbb{R}^N)$. Using the interpolation inequality given in corollary 2.1, we deduce

$$\|u\|_{L^q(\mathbb{R}^N)} \leq \|u\|^{\alpha}_{L^p(\mathbb{R}^N)} \|u\|^{1-\alpha}_{L^{p^*}(\mathbb{R}^N)}.$$

By *Young's inequality*

$$\|u\|_{L^q(\mathbb{R}^N)} \leq \|u\|_{L^p(\mathbb{R}^N)} + \|u\|_{L^{p^*}(\mathbb{R}^N)}.$$

Using theorem 2.11 we get

$$\|u\|_{L^q(\mathbb{R}^N)} \leq C\|u\|_{W^{1,p}(\mathbb{R}^N)}.$$

Since u, q were chosen arbitrarily, we conclude. □

Remark 2.7. *In the proof of the last corollary we use Young's inequality. This inequality states that*

$$ab \leq \frac{a^p}{p} + \frac{b^{p'}}{p'}, \tag{2.4}$$

where $a, b > 0$ *and* p *is such that* $1 < p < \infty$.

The next result derived from Theorem 2.11 gives a Sobolev embedding for the limiting case $p = N$

Corollary 2.3. *Let* $p = N$, *we have*

$$\forall q \in [N, \infty) : \quad W^{1,p}(\mathbb{R}^N) \subseteq L^q(\mathbb{R}^N).$$

The proof of this corollary can be found in [6, Cor. 9.11].

Finally, for the case where $p > N$, The Sobolev embedding is given by *Morrey's theorem*

Theorem 2.12. *Let $p > N$. Then*

$$W^{1,p}(\mathbb{R}^N) \subseteq L^\infty(\mathbb{R}^N)$$

with continuous injection.

A proof of this theorem can be found in [6, Th. 9.12].

Now, in order to obtain the Sobolev embeddings when $\Omega \subseteq \mathbb{R}^N$, we use the extension operator defined in [6, Th.9.7], $P : W^{1,p}(\Omega) \to W^{1,p}(\mathbb{R}^N)$, where

$$\overline{u}(x) = Pu(x) = \begin{cases} u(x), & \text{if } x \in \Omega, \\ 0, & \text{if } x \in \mathbb{R}^N \setminus \Omega. \end{cases}$$

Hence, we can apply all the results that we already have for $\mathbb{R}^N$. We assume that Ω is an open bounded set of class C^1. Thus we have the following result

Corollary 2.4. *Let $1 \leq p \leq \infty$. We have*

(i) Let $p < N$. Then $W^{1,p}(\Omega) \subseteq L^{p^*}(\Omega)$.

(ii) Let $p = N$. Then

$$\forall q \in [p, \infty) : \quad W^{1,p}(\Omega) \subseteq L^q(\Omega).$$

(iii) Let $p > N$. Then $W^{1,p}(\Omega) \subseteq L^\infty(\Omega)$.

All these injections are continuous. Moreover, if $p > N$ we have

$$\forall u \in W^{1,p}(\Omega) : \quad |u(x) - u(y)| \leq C \|u\|_{W^{1,p}(\Omega)} |x - y|^\alpha \quad a.e\ x, y \in \Omega,$$

with $\alpha = 1 - \dfrac{N}{p}$ and $C = C(\Omega, p, N) > 0$. In particular

$$W^{1,p}(\Omega) \subseteq C(\overline{\Omega}).$$

There are similar embeddings for the general case $W^{k,p}(\Omega)$, where $k \geq 1$. These results can be found in [6, Cor. 9.13].

Next theorem gives additional results to Gagliardo-Nirenberg-Sobolev theorem for $W^{1,p}(\Omega)$. For instance, it states that any bounded sequence in $W^{1,p}(\Omega)$, contains a convergent subsequence in $L^p(\Omega)$. This theorem is called *Rellich-Kondrachov theorem*

Theorem 2.13. *Suppose that Ω is bounded with boundary of class C^1. Then we have the following compact injections*

(i) Let $p < N$. Then

$$\forall q \in [1, p^*] : \quad W^{1,p}(\Omega) \subseteq L^q(\Omega).$$

(ii) Let $p = N$. Then

$$\forall q \in [p, \infty) : \quad \mathrm{W}^{1,p}(\Omega) \subseteq \mathrm{L}^q(\Omega)).$$

(iii) Let $p > N$. Then $\mathrm{W}^{1,p}(\Omega) \subseteq \mathrm{C}(\overline{\Omega})$.

In particular, $\mathrm{W}^{1,p}(\Omega) \subseteq \mathrm{L}^p(\Omega)$ with compact injection for all p, N.

A proof of this theorem can be found in [6, Th. 9.16].

Remark 2.8. *The definition of compact operator is given in Definition 2.15*

We finish this section by giving an important inequality for functions in $W_0^{1,p}(\Omega)$. This result is called *Poincaré's inequality*

Theorem 2.14. *Assume that Ω is a bounded open subset of $\mathbb{R}^N$. Suppose $u \in \mathrm{W}^{1,p}(\Omega)$ for some $1 \leq p < N$. Then we have the estimate*

$$\forall q \in [1, p^*] : \quad \|u\|_{\mathrm{L}^q(\Omega)} \leq C \, \|Du\|_{\mathrm{L}^p(\Omega)} \, ,$$

for some constant $C = C(p, q, N, \Omega) > 0$. In particular

$$\forall 1 \leq p < \infty : \quad \|u\|_{\mathrm{L}^p(\Omega)} \leq C \, \|Du\|_{\mathrm{L}^p(\Omega)} \, .$$

A proof of this theorem can be found in [10, Ch.5, Th. 3].

Remark 2.9. *Poincaré's inequality let us use $\|Du\|_{\mathrm{L}^p(\Omega)}$ as an equivalent norm in $\mathrm{W}^{1,p}(\Omega)$, whenever Ω is bounded.*

2.3 Bounded linear operators

At the beginning of this section we give some definitions and fundamental results about bounded linear operators acting on Banach spaces. Then we introduce concepts such as spectral analysis of bounded operators, polar decomposition of bounded operators and projection operators.

2.3.1 Definitions and properties. Adjoint operators

Let X, Y be normed spaces and $T : X \to Y$ a linear operator. We denote by $\mathscr{L}(X, Y)$ the space of all linear operators T such that

$$\exists c > 0, \forall x \in X : \quad \|Tx\|_Y \leq c\|x\|_X \, . \tag{2.5}$$

$T \in \mathscr{L}(X, Y)$ is called a *bounded linear operator* because for any $A \subseteq X$ bounded, $T(A)$ is also bounded.

In this setting, a continuous operator T maps convergent sequences into convergent ones, i.e.

$$x_n \to x \implies Tx_n \to y = Tx, \quad \text{as } n \to \infty.$$

Hence, by the linearity of T, if T is bounded then T is continuous. Moreover, the converse is true as well. A proof of this can be found in [14, Ch. 15, Th. 1].

We define $\|T\|$ as follows

$$\|T\| = \sup_{x \neq 0} \frac{\|Tx\|_Y}{\|x\|_X}.$$

Note that $\|T\|$ is the smallest constant such that (2.5) holds and a proof of this fact can be found in e.g., [16]. It is not difficult to show that the above defines a norm on $\mathscr{L}(X, Y)$. We call this norm the *operator norm*. Additionally, we have the following result

Proposition 2.5. *Let Y be a Banach space. Then $\mathscr{L}(X, Y)$ is a Banach space when equipped with the operator norm.*

Proof. Let $(T_n)_{n\in\mathbb{N}}$ be a Cauchy sequence in $\mathscr{L}(X, Y)$, i.e.

$$\forall \epsilon > 0, \exists N \in \mathbb{N} : \quad n, m > N \implies \|T_n - T_m\| < \epsilon. \tag{2.6}$$

Let $u \in X$, then

$$\|T_n u - T_m u\| \leq \|T_n - T_m\| \|u\|, \tag{2.7}$$

which implies, by (2.6), that the sequence $(T_n(u))_{n\in\mathbb{N}} \subseteq Y$ is Cauchy. Since Y is Banach, we have that there exists $Tu \in Y$ such that

$$\lim_{n\to\infty} T_n u = Tu. \tag{2.8}$$

Since u was chosen arbitrarily, and because the limit is unique, (2.8) defines an operator $T : X \to Y$. Clearly T is linear. Moreover, by (2.7), if we let $m \to \infty$ and by setting $n > N$ and $u \in X$

$$\left\|(T_n - T)u\right\| < \epsilon\|u\|.$$

Since u was arbitrary, the latter shows that $T \in \mathscr{L}(X, Y)$. Moreover, we deduce

$$n > N \implies \|T_n - T\| < \epsilon.$$

This implies that the sequence $(T_n)_{n\in\mathbb{N}}$ is convergent. Furthermore, since T stays in $\mathscr{L}(X, Y)$ and $(T_n)_{n\in\mathbb{N}}$ was an arbitrary Cauchy sequence in $\mathscr{L}(X, Y)$, we have proved that $\mathscr{L}(X, Y)$ is complete. □

Remark 2.10. *Suppose that X, Y are only normed spaces, and T is bounded in the sense of (2.5). Then T can be extended by continuity to a bounded mapping of the completion of X into the completion of Y. This observation is very important since it allows us to construct extensions of bounded maps, now acting on complete spaces. A proof of this fact can be found in [21].*

Remark 2.11. *Given a family of topological spaces $((Y_\lambda, \mathcal{G}_\lambda))_{\lambda\in\Lambda}$ and a family of functions of functions $(f_\lambda)_{\lambda\in\Lambda}$ such that*

$$f_\lambda : X \to Y_\lambda,$$

it is well that known we can construct the smallest topology on X such that all functions f_λ are continuous, [11]. This topology is referred to as the initial topology *determined by the family $(\mathcal{G}_\lambda, f_\lambda)_{\lambda\in\Lambda}$.*

The operator norm induces a topology on $\mathscr{L}(X,Y)$ which is often called *uniform operator topology*. Nonetheless, there are other useful topologies that can be defined for $\mathscr{L}(X,Y)$:

(i) The *strong operator topology* is the initial topology on $\mathscr{L}(X,Y)$ for which all functions

$$\begin{array}{rccl} \phi_x : & \mathscr{L}(X,Y) & \longrightarrow & Y \\ & T & \longmapsto & \phi_x(T) = Tx \end{array}$$

are continuous, for all $x \in X$.

(ii) The *weak operator topology* is the initial topology on $\mathscr{L}(X,Y)$ for which all linear functionals of the form

$$\begin{array}{rccl} \eta_{x,\ell} : & \mathscr{L}(X,Y) & \longrightarrow & \mathbb{R} \\ & T & \longmapsto & \langle \eta_{x,\ell}, T\rangle = \langle \ell, Tx\rangle \end{array}$$

are continuous, for all $x \in X$ and $\ell \in Y'$.

In the strong operator topology, a sequence of operators $(T_n)_{n\in\mathbb{N}}$ converges to an operator T if and only if

$$\forall x \in X : \quad \|T_n x - Tx\| \to 0, \qquad \text{as } n \to \infty,$$

which we denote by $T_n \xrightarrow{s} T$. In the weak operator topology, a sequence of operators $(T_n)_{n\in\mathbb{N}}$ converges to an operator T if and only if

$$\forall x \in X, \forall \ell \in Y' : \quad |\ell(T_n x) - \ell(Tx)| \to 0, \qquad \text{as } n \to \infty,$$

which we denote by $T_n \xrightarrow{w} T$.

Remark 2.12. *Note that convergence in the uniform operator topology implies convergence in the strong operator topology which implies convergence in the weak operator topology.*

Another important concept is the *adjoint* of a bounded linear map $T \in \mathscr{L}(X,Y)$, which is an infinite-dimensional generalization of the transpose of a matrix. Let $\ell \in Y'$. We define an operator $T' : Y' \to X'$ by

$$T'\ell x = \ell(Tx), \quad x \in X.$$

T' is called the adjoint of T. Using the product in duality we can rewrite the definition of the adjoint operator $T' : Y' \to X'$,

$$\langle T'\ell, x\rangle = \langle \ell, Tx\rangle, \quad x \in X.$$

The adjoint of a bounded linear operator is always bounded. Moreover, we have the following proposition:

Proposition 2.6. *Let X, Y be a pair of Banach spaces. The map $T \to T'$ is an isometric isomorphism of $\mathscr{L}(X,Y)$ into $\mathscr{L}(Y', X')$.*

Proof. It is clear that the map $T \to T'$ is linear. Let's prove that T' is bounded. It can be proven that the following norm

$$\|T\| = \sup_{\|x\|=1} \|Tx\| \tag{2.9}$$

is equivalent to the operator norm defined at the beginning of this section. Now, since Y is a Banach space for any $y \in Y$, we have that (see [21, Pg. 77])

$$\|y\| = \sup_{\|\ell\|=1} |\ell(y)|, \quad \ell \in Y'.$$

Hence, using the above and the adjoint operator T', we can rewrite (2.9) as

$$\begin{aligned} \|T\| &= \sup_{\|x\|=1} \left(\sup_{\|\ell\|=1} \left|\ell(Tx)\right| \right) \\ &= \sup_{\|\ell\|=1} \left(\sup_{\|x\|=1} \left|(T'\ell)(x)\right| \right) \\ &= \sup_{\|\ell\|=1} \left\|T'\ell\right\| \\ &= \left\|T'\right\|. \end{aligned}$$

Hence T' is bounded and the mapping $T \to T'$ is an isometry. □

When $X = Y = \mathcal{H}$ is a Hilbert space, we denote the adjoint of T by T^*, and by the Riesz-Fréchet theorem, for any x, y in $\mathcal{H}$,

$$(Tx, y) = (x, T^*y).$$

Remark 2.13. *Strictly speaking, T^* is called the* Hilbert-adjoint *of T.*

The following proposition states some important properties of the adjoint of an operator:

Proposition 2.7. *Let $\mathcal{H}$ be a Hilbert space and $S, T \in \mathscr{L}(\mathcal{H})$. Then:*

(i) $(S+T)^* = S^* + T^*$ *and* $(ST)^* = T^*S^*$.

(ii) $(T^*)^* = T$.

(iii) If T has a bounded inverse, T^{-1}, then T^* has a bounded inverse and $(T^*)^{-1} = (T^{-1})^*$.

(iv) The adjoint of T is bounded and $\|T^*\| = \|T\|$. *Moreover* $\|T^*T\| = \|T\|^2$.

Proof. (i) and (ii) are easily checked. Suppose that T has a bounded inverse T^{-1}, then, by (i) and (ii)

$$\begin{aligned} T^*(T^{-1})^* &= (T^{-1}T)^* \\ &= I^* \\ &= (T^{-1})^*T^*, \end{aligned}$$

and since $I^* = I$, the computations above imply $(T^*)^{-1} = (T^{-1})^*$. The fact that T^* is bounded and that $\|T^*\| = \|T\|$ follows from the last proposition. Note that, for any $x \in X$

$$\begin{aligned} \|T^*Tx\| &\le \|T^*\|\|Tx\| \\ &\le \|T^*\|\|T\|\|x\|, \end{aligned}$$

which implies that

$$\|T^*T\| \le \|T^*\|\|T\| = \|T\|^2.$$

Moreover, by the Cauchy Schwarz inequality

$$\begin{aligned} \|T^*T\| &\ge \sup_{\|x\|=1} (x, T^*Tx) \\ &= \sup_{\|x\|=1} \|Tx\|^2 \\ &= \|T\|^2, \end{aligned}$$

and we conclude that $\|T^*T\| = \|T\|^2$. □

There is an important class of bounded operators called *self-adjoint*. These are all $T \in \mathscr{L}(\mathcal{H})$ such that $T = T^*$, where $\mathcal{H}$ is a separable Hilbert space. We denote this space by $\mathscr{L}_S(\mathcal{H})$. There are several properties of bounded self-adjoint operators that will be studied in the next sections.

Remark 2.14. *If* $\mathcal{H}$ *is a real linear space, then* $\mathscr{L}_S(\mathcal{H})$ *is also a real linear space. This is because when* $\mathcal{H}$ *is a real linear space, its inner-product* $(\cdot,\cdot)$ *is symmetric.*

Example 2.2. *Assume that* $\Omega \subseteq \mathbb{R}^N$ *is an open, bounded domain with boundary of* C^1 *class. In the case where* $\mathcal{H} = L^2(\Omega)$ *we write*

$$\mathscr{L}_S = \mathscr{L}_S(L^2(\Omega)) = \{T \in \mathscr{L} / T \textit{ is self-adjoint }\},$$

where $\mathscr{L} = \mathscr{L}(L^2(\Omega))$.

In the study of linear operators, it is useful to define the following two sets. Let X, Y be normed spaces and $T \in \mathscr{L}(X, Y)$. The *kernel* of T is the set

$$\operatorname{Ker}(T) = \{x \in X / Tx = 0\}.$$

The *range* or *image* of T is the set

$$T(X) = \{y \in Y / \exists x \in X : Tx = y\}.$$

$\operatorname{Ker}(T)$ and $T(X)$ are subspaces of X and Y, respectively. Moreover, we have the following result:

Proposition 2.8. *Let X, Y be normed linear spaces, $T : X \to Y$ a bounded, linear operator. Then $\operatorname{Ker}(T)$ is a closed linear subspace of X.*

Proof. Since T is continuous, the preimage of closed sets are closed sets. Since $\{0\}$ is a singleton in a normed space, it is closed. Hence $T^{-1}(\{0\})$ is closed. But $\operatorname{Ker}(T)$ is precisely $T^{-1}(\{0\})$, which concludes the proof. □

Now, we shall state three very important theorems concerning bounded linear operators on Banach spaces: the *Principle of Uniform Boundedness*, the *Open Mapping Theorem* and the *Closed Graph Theorem*. A proof of these theorems can be found in [13].

The first theorem gives sufficient conditions for $(\|T_n\|)_{n\in\mathbb{N}}$ to be bounded, where the T_n's are bounded linear operators from a Banach space into a normed space.

Theorem 2.15 (Principle of Uniform Boundedness)**.** *Let $(T_n)_{n\in\mathbb{N}} \subseteq \mathscr{L}(X, Y)$, where X is a Banach space and Y is a normed linear space. Suppose that, for any x in X, the sequence $(\|T_n x\|)_{n\in\mathbb{N}}$ is bounded. Then $(\|T_n\|)_{n\in\mathbb{N}}$ is bounded.*

For the second theorem, we need the concept of *open map*. This theorem gives conditions for the inverse of a bounded linear map to be bounded. Let X and Y be topological spaces. We say that an operator

$$T : (X, \mathcal{T}_X) \to (Y, \mathcal{T}_Y)$$

is open if and only if T maps open sets into open sets. That is,

$$\forall A \in \mathcal{T}_X : \quad T(A) \in \mathcal{T}_Y.$$

Theorem 2.16 (Open Mapping Theorem)**.** *A bounded linear operator T from a Banach space X onto a Banach space Y is an open mapping. Hence, if T is bijective, T^{-1} is bounded.*

Thirdly, we have the closed graph theorem. This theorem gives conditions for a *closed operator* to be bounded. As before, we define first what it means for an operator to be closed

Definition 2.8. *Let X, Y be normed spaces and $T : X \to Y$. Then T is closed if and only if its graph*

$$\mathcal{G}(T) = \{(x, y) \in X \times Y / x \in X, y = Tx\},$$

is closed in the normed space $X \times Y$.

Theorem 2.17 (Closed Graph Theorem)**.** *Let X and Y be Banach spaces and*

$$T : \mathrm{Dom}(T) \subseteq X \to Y$$

a closed linear operator. Then, if $\mathrm{Dom}(T)$ is closed in X, the operator T is bounded.

2.3.2 The spectrum

In this section we shall give some tools that will help us in the analysis of bounded linear operators. More specifically, the spectral analysis of elements in $\mathscr{L}(X)$, where X is a Banach space. Throughout this section, we consider complex linear spaces.

Definition 2.9. *Let X be a complex Banach space. The* resolvent set *of $T \in \mathscr{L}(X)$, denoted by $\rho(T)$, consists of those complex numbers λ for which the operator $\lambda I - T$ is invertible, i.e.,*

(i) $T_\lambda = \lambda I - T$ *is injective.*

(ii) $\overline{T_\lambda(X)} = X$.

(iii) $(\lambda I - T)^{-1}$ *is bounded.*

Furthermore, we write $R_\lambda(T) = (\lambda I - T)^{-1}$. $R_\lambda(T)$ is called the resolvent *of T at λ. Further, we refer to λ as a* regular values of T. *If $\lambda \notin \rho(T)$, then λ is said to be in the* spectrum $\sigma(T)$ *of T.*

The next definition divides the spectrum into three: the *point spectrum*, the *residual spectrum* and the *continuous spectrum*. This is necessary since we can extract plenty of information about an operator by just studying its point spectrum.

Definition 2.10. *Let $T \in \mathscr{L}(X)$. An $x \neq 0$ which satisfies $Tx = \lambda x$ for some $\lambda \in \mathbb{C}$ is called an* eigenvector *of T; λ is called the corresponding* eigenvalue*. The set*

$$\sigma_p(T) = \{\lambda \in \mathbb{C} / \exists x \neq 0 : Tx = \lambda x\}$$

is called the point spectrum *of T. Moreover:*

- *If λ is not an eigenvalue and $T_\lambda(X)$ is not dense in X, then λ is said to be in the* residual spectrum *of T, and is denoted by $\sigma_r(T)$.*

- *If λ is not an eigenvalue and $R_\lambda(T)$ is unbounded, then λ is said to be in the* continuous spectrum *of T, and is denoted by $\sigma_c(T)$.*

Note that $\sigma(T) = \sigma_p(T) \cup \sigma_r(T) \cup \sigma_c(T)$.

In order to prove some properties of the resolvent and spectrum of a bounded linear adjoint operator, we need the following result about invertibility of bounded operators

Proposition 2.9. *Let X be a Banach space and $T \in \mathscr{L}(X)$ such that $\|T\| < 1$. Then the series*

$$\sum_{n=0}^{\infty} T^n \tag{2.10}$$

converges in norm and its limit is $(I - T)^{-1}$.

Proof. Since the geometric series $\sum_{n=1}^{\infty} \|T\|^n$ converges for $\|T\| < 1$, and because

$$\forall n \in \mathbb{N}: \quad \left\|T^n\right\| \leq \|T\|^n,$$

the series in (2.10) converges absolutely for $\|T\| < 1$. Since $\mathscr{L}(X)$ is a Banach space, absolute convergence implies convergence, [21]. Hence (2.10) is a convergent series in $\mathscr{L}(X)$. Now, let's prove that (2.10) converges to $(I - T)^{-1}$. We denote the sum of this series by S. Consider the following computation

$$(I - T)(S_n) = (S_n)(I - T) = I - T^{n+1},$$

where S_n denotes the partial sum up to n of our series. Since $\|T\| < 1$, we have that $T^{n+1} \to 0$ as $n \to \infty$. Hence

$$(I - T)S = S(I - T) = I.$$

This shows that $S = (I - T)^{-1}$. □

The last result provides the following representation of the resolvent

$$R_\lambda(T) = (\lambda I - T)^{-1} = \frac{1}{\lambda}\left(I - \frac{1}{\lambda}T\right)^{-1} = \frac{1}{\lambda}\sum_{n=0}^{\infty}\left(\frac{1}{\lambda}T\right)^n,$$

where, by the last proposition, the series converges for all λ such that $|\lambda| > \|T\|$. This implies that

$$\forall \lambda \in \mathbb{C}: \quad |\lambda| > \|T\| \implies \lambda \in \rho(T).$$

Moreover, we deduce that

$$\sigma(T) \subseteq B_{\|T\|}(0).$$

Hence $\sigma(T)$ is bounded in $\mathbb{C}$.

We are ready to give some topological properties about the resolvent and spectrum

Theorem 2.18. *Let $T \in \mathscr{L}(X)$, where X is a Banach space, then:*

(i) The resolvent set $\rho(T)$ is an open susbet of $\mathbb{C}$. Hence $\sigma(T)$ is closed.

(ii) The resolvent of T, defined as a function on $\rho(T)$, is analytic on $\rho(T)$.

(iii) The spectrum $\sigma(T)$ is a non empty bounded subset of $\mathbb{C}$.

Proof. Let's prove (i). Clearly, by the last discussion, $\rho(T)$ is not empty. Let $\lambda_0 \in \rho(T)$, we have to prove there exists $r > 0$ such that $B_r(\lambda_0) \subseteq \rho(T)$. For any $\lambda \in \mathbb{C}$ we have

$$\begin{aligned} \lambda I - T &= -[T - \lambda_0 I - (\lambda - \lambda_0)\, I] \\ &= -(T - \lambda_0 I)\left(I - (\lambda - \lambda_0)\,(T - \lambda_0 I)^{-1}\right) \\ &= (\lambda_0 I - T)V, \end{aligned}$$

where $V = -[I - (\lambda - \lambda_0)R_{\lambda_0}(T)]$. Clearly, V has a bounded inverse for all λ such that $\|(\lambda - \lambda_0)R_{\lambda_0}(T)\| < 1$, i.e.,

$$|\lambda - \lambda_0| < \frac{1}{\|R_{\lambda_0}(T)\|}. \tag{2.11}$$

Hence, since $(\lambda_0 I - T)$ also has a bounded inverse, we see that for all λ satisfying (2.11), $(\lambda I - T)$ also has a bounded inverse. Hence (2.11) represents a disk centered at λ_0 and radius

$$\frac{1}{R_{\lambda_0}(T)}$$

consisting of regular values λ of T. This concludes the proof of (i).

For (ii), we note that since the resolvent can be expanded into a power series around each point λ of $\rho(T)$, assertion (ii) holds.

Finally, let's prove (iii). We already proved that $\sigma(T)$ is bounded. It only remains to prove that it is not empty. The representation for the resolvent

$$R_\lambda(T) = \frac{1}{\lambda}\sum_{n=0}^{\infty}\left(\frac{1}{\lambda}T\right)^n,$$

converges for $|\lambda| > \|T\|$. This representation is a Laurent series for the resolvent around ∞ and its first term is

$$\frac{1}{\lambda}I.$$

Integrating the series with respect to λ around the contour

$$C = \{\lambda \in \mathbb{C} \colon |\lambda| = c\},$$

for a fixed $c > \|T\|$, and using the residual theorem (see [14]), gives

$$\oint_C R_\lambda(T)d\lambda = 2\pi i I. \tag{2.12}$$

We argue by reduction to absurdity. If we assume that $\sigma(T) = \emptyset$, then $\rho(T) = \mathbb{C}$. Hence $R_\lambda(T)$ is an entire function by (ii). Then by the *Cauchy integral theorem* (see [14]), applicable to analytic functions in a Banach space, the integral in (2.12) would vanish, which is a contradiction. This proves that $\sigma(T)$ is not empty. □

Since the spectrum of a bounded linear operator is compact by the last theorem, it is natural to look for the spectral value of maximum size. The absolute value of this eigenvalue is given a particular name and has some interesting properties.

Definition 2.11. *The* spectral radius *of a bounded linear operator T, denoted as $|\sigma(T)|$, is defined as*

$$r_\sigma(T) = \max_{\lambda \in \sigma(T)} |\lambda|.$$

Note that the spectral radius is well defined since $\sigma(T)$ is closed and bounded in $\mathbb{C}$. A very important property of the spectral radius is that it can be computed as follows:

$$r_\sigma(T) = \lim_{n\to\infty} \|T^n\|^{1/n}.$$

A proof of this result can be found in [13, Th. 7.5.5].

The following proposition states that the spectrum of a bounded linear operator is very similar to its adjoint:

Proposition 2.10. *Let $\mathcal{H}$ be a Hilbert space and $T \in \mathscr{L}(\mathcal{H})$. Then*

(i) $\sigma(T^*) = \{\lambda \in \mathbb{C} | \overline{\lambda} \in \sigma(T)\}$;

(ii) $\forall \lambda \in \rho(T): \quad R_{\overline{\lambda}}(T^*) = R_\lambda(T)^*$.

Proof. Let $\lambda \in \rho(T)$. Then

$$\left((\lambda I - T)^{-1}\right)^* = \left((\lambda I - T)^*\right)^{-1} = \left(\overline{\lambda} I - T^*\right)^{-1},$$

which implies (ii). Finally, (i) follows from (ii). □

The spectral analysis of self-adjoint operators on a Hilbert space is quite elegant and very similar to that of transformations in finite dimension. We give three important results: the first result is a summary of the spectral analysis of a self-adjoint operator on a Hilbert space.

Theorem 2.19. *Let $\mathcal{H}$ be a Hilbert space and $T \in \mathscr{L}_S(\mathcal{H})$. Then,*

(i) $\sigma_r(T) = \emptyset$.

(ii) $\sigma(T) \subseteq \mathbb{R}$.

(iii) Eigenvectors corresponding to distinct eigenvalues of T are orthogonal.

A proof of this theorem can be found in [21, Th. VI.8]. The second result sates that the size of a self-adjoint operator on a Hilbert space is precisely the spectral radius of the operator,

Theorem 2.20. *Let $\mathcal{H}$ be a Hilbert space and $T \in \mathscr{L}_S(\mathcal{H})$. Then $r_\sigma(T) = \|T\|$.*

Proof. Since T is self-adjoint, we have

$$\|T\|^2 = \|T^2\|.$$

By induction, this implies that

$$\forall n \in \mathbb{N}: \quad \|T\|^{2n} = \|T^{2n}\|,$$

so

$$r_\sigma(T) = \lim_{k\to\infty} \|T^k\|^{1/k} = \lim_{n\to\infty} \|T^{2n}\|^{1/2n} = \|T\|.$$

□

The third result states the specific bounds the spectrum of a bounded self-adjoint operator,

Theorem 2.21. *Let $\mathcal{H}$ be a complex Hilbert space and $T \in \mathscr{L}_S(\mathcal{H})$. Then*

$$\sigma(T) \subseteq [m, M],$$

where

$$m = \inf_{\|x\|=1} (Tx, x), \quad M = \sup_{\|x\|=1} (Tx, x).$$

Moreover, $m, M \in \sigma(T)$.

Proof. We know that $\sigma(T) \subseteq \mathbb{R}$. Let's prove that

$$\forall c > 0: \quad (M + c) \in \rho(T).$$

Let $x \neq 0$ normalized as

$$v = \frac{x}{\|x\|}, \quad \text{so } x = v\|x\|.$$

A short computation shows $(Tx, x) \leq (x, x)M$. Hence, using Cauchy-Schwarz inequality we obtain

$$\begin{aligned} \|\lambda I - Tx\|\|x\| &\geq -(Tx, x) + \lambda(x, x) \\ &\geq (-M + \lambda)(x, x) \\ &= c\|x\|^2, \end{aligned}$$

where $c = (\lambda - M) > 0$ by assumption. Dividing by $\|x\|$ gives

$$\|\lambda I - Tx\| \geq c\|x\|.$$

Using [13, Th. 9.1.2], we have that $\lambda \in \rho(T)$. Since $c > 0$ was chosen arbitrarily, we have proved that

$$\forall \lambda > M: \quad \lambda \in \rho(T).$$

Similarly, we can prove that any $\lambda < m$ is in $\rho(T)$. Hence $\sigma(T) \subseteq [m, M]$. □

2.3.3 The polar decomposition of a bounded linear operator

Let $\mathcal{H}$ be a Hilbert space. In this section we will show that every bounded operator on a $\mathcal{H}$ can be decomposed into the product of two simpler bounded linear operators. This is analogous to the decomposition of a complex number into two numbers: its module and its argument. First, we give an order relation to the set $\mathscr{L}(\mathcal{H})$.

Definition 2.12. *Let $S, T \in \mathscr{L}(\mathcal{H})$. T is called* positive *if*

$$\forall x \in \mathcal{H}: \quad (Tx, x) \geq 0.$$

We write $T \geq 0$ if T is positive and $T \leq S$ if $S - T \geq 0$.

Similar to $\mathbb{R}$, we can define the square root of a positive bounded linear operator.

Theorem 2.22. *Let $T \in \mathscr{L}(\mathcal{H})$ and $T \geq 0$. Then there is a unique positive $S \in \mathscr{L}(\mathcal{H})$ such that $S^2 = T$. Furthermore, S commutes with every bounded operator which commutes with T.*

A proof of this theorem can be found in [21, Th.VI.9]. We denote

$$B = T^{1/2}.$$

Note that for any $T \in \mathscr{L}(\mathcal{H})$, $T^*T \geq 0$, since $(T^*Tx, x) = \|Tx\|^2 \geq 0$. Hence, by the last theorem, we can define the *absolute value* of T as

$$|T| = (T^*T)^{1/2}.$$

Moreover, we can write any positive bounded operator T as $T = T^{1/2}T^{1/2}$.

Before introducing the polar decomposition of a bounded linear operator, we need the concept of *isometry*:

Definition 2.13. *$U \in \mathscr{L}(\mathcal{H})$ is called an* isometry *if*

$$\forall x \in \mathcal{H}: \quad \|Ux\| = \|x\|.$$

U is called a partial isometry *if U is an isometry when restricted to the closed subspace $(\mathrm{Ker}(U))^\perp$.*

Note that U is a unitary operator between $(\text{Ker}(U))^{\perp}$ and $U(\mathcal{H})$. That is,

$$\forall x, y \in (\text{Ker } U)^{\perp}: \quad (Ux, Uy) = (x, y).$$

We can prove that U^* is a map from $U(\mathcal{H})$ to (Ker U)$^{\perp}$ that acts as the inverse of U . In fact, let $x, y \in (\text{Ker } U)^{\perp}$, then

$$(x, y) = (Ux, Uy) = (U^*Ux, y).$$

It follows that $(U^*Ux - x, y) = 0$. Since y is an arbitrary element of (Ker U)$^{\perp}$:

$$(U^*Ux - x) \in \text{Ker}(U).$$

However, U^* maps $U(\mathcal{H})$ into (Ker U)$^{\perp}$. Hence

$$(U^*Ux - x) \in \text{Ker}(U) \cap (\text{Ker}(U))^{\perp}.$$

Therefore $U^*Ux = x$. Since x was chosen arbitrary, we have proved $U^*U|_{(\text{Ker}(U))^{\perp}} = I$. $UU^*|_{(\text{Ker}(U))^{\perp}} = I$ can be proved in a similar manner.

We are ready to state the following result:

Theorem 2.23. *Let $T \in \mathscr{L}(\mathcal{H})$. Then there is a partial isometry U such that $T = U|T|$. U is uniquely determined by the conditions*

$$\text{Ker}(U) = \text{Ker}(T), \quad U(\mathcal{H}) = \overline{T(\mathcal{H})}.$$

Proof. Define $U \colon |T|(\mathcal{H}) \to T(\mathcal{H})$ by

$$U(|T|u) = Tu, \quad u \in \mathcal{H}.$$

It is well defined since

$$\|Tu\|^2 = (u, T^*Tu) = (u, |T|^2u) = \||T|u\|^2,$$

so that if $Tu = Tv$, then $|T|u = |T|v$. Define U to be zero on the orthogonal complement of $|T|(\mathcal{H})$:

$$Uz = o, \quad z \in |T|(\mathcal{H}).$$

Since

$$\forall z \in (|T|(\mathcal{H}))^{\perp}, \forall u \in \mathcal{H}: \quad (Uz, u) = (z, U^*u) = 0,$$

it follows that U^* maps $\mathcal{H}$ into the orthogonal complement of $(|T|(\mathcal{H}))^{\perp}$; thus the range of U^* lies in $\overline{|T|(\mathcal{H})}$. Since $|T|$ is self-adjoint, $(|T|(\mathcal{H}))^{\perp} = \text{Ker}(|T|)$. Furthermore,

$$|T|u = 0 \iff Tu = 0$$

so that $\text{Ker}(|T|) = \text{Ker}(T)$ and thus $\text{Ker}(U) = \text{Ker}(T)$. Hence U is a partial isometry. Since

$$\forall w \in \overline{|T|(\mathcal{H})}: \quad U^*Uw = w,$$

it is clear that $T = U|T|$. □

Remark 2.15. *When T is a linear transformation on $\mathbb{R}^n$, it is well known that it can be decomposed into the product of an orthogonal matrix and a hermitian matrix. This is similar to Theorem 2.23 but in the finite-dimensional case.*

2.3.4 Projection operators

Projection operators are a very simple type of bounded operators. The fact that every self-adjoint operator can be decomposed using a special family of projection operators called spectral family allow us to obtain more tools to work with self-adjoint operators.

Recall that for every closed subset Y of a Hilbert space, $\mathcal{H}$ can be decomposed as $\mathcal{H} = Y \bigoplus Y^{\perp}$, i.e.

$$\forall x \in \mathcal{H}, \exists! y \in Y, \exists! z \in Y^{\perp}: \quad x = y + z.$$

This let us define a linear operator $P\colon \mathcal{H} \to \mathcal{H}$ such that $Px = y$. P is called an *orthogonal projection* of $\mathcal{H}$ onto Y. Similarly, the projection of $\mathcal{H}$ onto $Y^{\perp}$ is $I - P$. P has several properties:

(i) $P(\mathcal{H}) = Y$.

(ii) $\text{Ker}(P) = Y^{\perp}$.

(iii) $P|_Y = I|_Y$.

Furthermore, we have the following equivalent definition of an orthogonal projection. A proof of this equivalence can be found in [13, Th. 9.5.1].

Proposition 2.11. *If $P \in \mathscr{L}(\mathcal{H})$ is self-adjoint and idempotent, i.e., $P^2 = P$, then P is an orthogonal projection.*

Orthogonal projections have many properties. Nonetheless, we state only a few. For a deeper study, see [13].

Theorem 2.24. *Let P be an orthogonal projection of a Hilbert space $\mathcal{H}$ onto a closed subspace $Y = P(\mathcal{H})$. Then:*

(i) $\|Px\|^2 = (Px, x)$.

(ii) $P \geq 0$.

(iii) $\|P\| \leq 1$.

Proof. (i) and (ii) are direct since P is self-adjoint and idempotent. (iii) follows from the Cauchy-Schwarz inequality. Indeed,

$$\|Px\|^2 = (Px, x) \leq \|Px\| \|x\|,$$

for any $x \neq 0$ in $\mathcal{H}$. This implies that $\|P\| \leq 1$. □

In finite dimensional space, self-adjoint operators can be decomposed as the sum of orthogonal projections onto different eigenspaces. This motivates the following definition

Definition 2.14. *A* real spectral family *is a one-parameter family $\mathcal{E} = (E_\lambda)_{\lambda\in\mathbb{R}}$ of projections E_λ defined on a Hilbert space $\mathcal{H}$ which depends on a real parameter λ and verifies*

(i) If $\lambda < \mu$, then: $E_\lambda \leq E_\mu$, hence $E_\lambda E_\mu = E_\mu E_\lambda = E_\lambda$.

(ii) $\forall x \in \mathcal{H}$: $\quad E_\lambda x \to 0$ as $\lambda \to -\infty$.

(iii) $\forall x \in \mathcal{H}$: $\quad E_\lambda x \to x$ as $\lambda \to +\infty$.

(iv) The mapping $\lambda \to E_\lambda$ is strongly operator continuous from the right. That is, $E_\mu x \to E_\lambda x$ as $\mu \to \lambda$ from the right.

Remark 2.16. *We say that $\mathcal{E}$ is a* spectral family on an interval $[a, b]$ *if*

$$E_\lambda \equiv \begin{cases} 0 & , \text{ if } \lambda < a, \\ I & , \text{ if } \lambda \geq b. \end{cases}$$

In Section 2.5, we shall associate a spectral family to any given bounded self-adjoint operator on any Hilbert space. This can be used for representing T by a *Riemann-Stieltjes integral* (see Appendix A). This is known as a spectral representation.

2.4 Compact linear operators

The theory of integral equations plays a major role in mathematical physics, and compact operators are a essential part of this theory. Their properties are very similar to those of operators in finite dimensional spaces. In this section we will study these properties.

Definition 2.15. *Let X, Y be a pair of normed spaces. A linear operator $T : X \to Y$ is called* compact *if,*

$$\forall A \subseteq X, \text{ bounded}: \quad T(A) \subseteq Y \text{ is relatively compact}.$$

Equivalently, T is compact if and only if

$$\forall (x_n)_{n\in\mathbb{N}} \subseteq X \text{ bounded}: \quad (Tx_n)_{n\in\mathbb{N}} \text{ has a convergent subsequence}.$$

We write

$$\mathscr{I}_\infty(X, Y) = \{T \in \mathscr{L}(X, Y) / T \text{ is compact}\}.$$

Remark 2.17. *Compact operators are also called* completely continuous operators*. This is due to the fact that all compact operators are bounded.*

Example 2.3. *Similar to Example 2.2, we define the linear spaces*

$$\mathscr{I}_\infty = \mathscr{I}_\infty(\mathrm{L}^2(\Omega)) = \{T \in \mathscr{L} / T \text{ is compact}\}$$

and

$$\mathscr{S}_\infty = \mathscr{S}_\infty(\mathrm{L}^2(\Omega)) = \{T \in \mathscr{I}_\infty / T \text{ is self-adjoint }\}.$$

Some tools that help to prove that an operator is compact are given in the following result:

Theorem 2.25. *Let X, Y be Banach spaces. Then*

(i) Let $(T_n)_{n\in\mathbb{N}} \subseteq \mathscr{I}_\infty(X,Y)$ and $T_n \to T$ as $n \to \infty$, in the uniform operator topology. Then $T \in \mathscr{I}_\infty(X,Y)$.

(ii) Let $S \in \mathscr{L}(Y,Z)$ with Z a Banach space, and T or S is compact. Then their product ST and TS are compact.

(iii) Let $T \in \mathscr{I}_\infty(X,Y)$. Then $T' \in \mathscr{I}_\infty(Y',X')$.

Proof. We prove (i) and (ii). The proof of (iii) can be found in [13, Th. 8.2-5].

For (i), we will show that for any bounded sequence $(x_m)_{m\in\mathbb{N}} \subseteq X$, their image under T, that is, $(Tx_m)_{m\in\mathbb{N}} \subseteq Y$, has a convergent subsequence. In order to find this subsequence, we use a diagonalization method. Since T_1 is compact, $(x_m)_{m\in\mathbb{N}}$ has a subsequence $(x_{1,m})_{m\in\mathbb{N}}$ such that $(T_1x_{1,m})_{m\in\mathbb{N}}$ is convergent, and since Y is complete, it is Cauchy. Now, from the subsequence $(x_{1,m})_{m\in\mathbb{N}}$ we extract a new subsequence $(x_{2,m})_{m\in\mathbb{N}}$ such that $(T_2x_{2,m})_{m\in\mathbb{N}}$ is Cauchy. Note that $(T_1x_{2,m})_{m\in\mathbb{N}}$ is also Cauchy since $(T_1x_{2,m})_{m\in\mathbb{N}}$ is a subsequence of the convergent sequence $(T_1x_{1,m})_{m\in\mathbb{N}}$. By induction, we see that that the diagonal sequence $(x_{m,m})_{m\in\mathbb{N}}$ is a subsequence of $(x_m)_{m\in\mathbb{N}}$ such that

$$\forall n \in \mathbb{N}: \quad (T_nx_{m,m})_{m\in\mathbb{N}} \subseteq Y \text{ is Cauchy.}$$

Now we will show that the subsequence $(Tx_{m,m})_{m\in\mathbb{N}}$ is Cauchy using the properties we gave to $(x_{m,m})_{m\in\mathbb{N}}$ in its construction. It is clear that $(x_{m,m})_{m\in\mathbb{N}}$ is bounded, say

$$\exists c \in \mathbb{R}, \forall m \in \mathbb{N}: \quad \|x_{m,m}\| \leq c.$$

This is the same bound that the sequence $(x_m)_{m\in\mathbb{N}}$ has. Let $\epsilon > 0$. Since $T_m \to T$,

$$\exists n = p \in \mathbb{N}: \quad \left\|T - T_p\right\| < \frac{\epsilon}{3c}.$$

By construction, $(T_p x_{m,m})_{m\in\mathbb{N}}$ is Cauchy. Hence

$$\exists N \in \mathbb{N}, \forall j,k > N: \quad \left\| T_p x_{j,j} - T_p x_{k,k} \right\| < \frac{\epsilon}{3}.$$

Therefore, we deduce

$$\begin{aligned}
\|Tx_{j,j} - Tx_{k,k}\| &\leq \|Tx_{j,j} - T_p x_{j,j}\| + \|T_p x_{j,j} - T_p x_{k,k}\| + \|T_p x_{k,k} - Tx_{k,k}\| \\
&\leq \|T - T_p\| \|x_{j,j}\| + \frac{\epsilon}{3} + \|T_p - T\| \|x_{k,k}\| \\
&< \frac{\epsilon}{3c} c + \frac{\epsilon}{3} + \frac{\epsilon}{3c} c \\
&= \epsilon.
\end{aligned}$$

Since ϵ was arbitrary, this shows that $(Tx_{m,m})_{m\in\mathbb{N}}$ is Cauchy, thus convergent in Y. We conclude that T is compact since the sequence $(x_m)_{m\in\mathbb{N}}$ was chosen arbitrarily.

For the second part of this theorem, we first prove that the product ST is compact. Let $(x_n)_{n\in\mathbb{N}}$ be a bounded sequence in X. Since T is compact, $(Tx_n)_{n\in\mathbb{N}}$ has a convergent subsequence $(Tx_{n_k})_{k\in\mathbb{N}}$. Since S is continuous, the sequence $(STx_{n_k})_{k\in\mathbb{N}}$ converges. Hence ST is compact. To prove that TS is compact, note that the image of any bounded set under a bounded operator is bounded, and since T is compact, the image of any bounded set under TS must be precompact. Hence TS is compact. □

In the case of $T \in \mathscr{I}_\infty(\mathcal{H})$, where $\mathcal{H}$ is a separable Hilbert space, we have more interesting results. The first result states that compact operators can be approximated by operators of finite rank. The second one is the *Fredholm alternative*, which itself reveals further properties of compact operators.

Theorem 2.26. *Let $\mathcal{H}$ be a separable Hilbert space. Then every compact operator T on $\mathcal{H}$ is the uniform operator limit of a sequence of operators of finite rank.*

A proof of this result can be found in [21, Th. VI.13].

Remark 2.18. *We can explicitly construct the sequence mentioned in Theorem 2.26. Consider a Hilbert basis $B = \{u_i \ / \ i \in \mathbb{N}\} \subseteq \mathcal{H}$. We define the sequence of finite rank operators $(T_n)_{n\in\mathbb{N}}$ by*

$$T_n(u) = \sum_{i=1}^{n} (u, u_i) T(u_n), \quad u \in \mathcal{H}.$$

Clearly, for each $n \in \mathbb{N}$, T_n is bounded since T is bounded. Moreover, this sequence converges in the operator norm to T as $n \to \infty$, [21].

The next result is a short version of the analytic Fredholm theorem

Theorem 2.27. *Let $T \in \mathscr{I}_\infty(\mathcal{H})$. Then either $(I - T)^{-1}$ exists or $Tu = u$ has a solution.*

The following theorem, called the *Riesz-Schauder theorem*, uses the last result to prove some properties about the spectrum of a compact operator

Theorem 2.28. *Let A be a compact operator on $\mathcal{H}$, then $\sigma(A)$ is a discrete set and its only possible point of accumulation is $\lambda = 0$. Further, any nonzero $\lambda \in \sigma(A)$ is an eigenvalue of finite multiplicity in the sense that the corresponding eigenspace is finite dimensional.*

A proof of the Fredholm alternative and the Riesz-Schauder theorem can be found in [21].

2.4.1 Compact self-adjoint linear operators

In finite dimensional space, it is well known that a Hermitian symmetric matrix has a complete set of orthogonal eigenvectors. This property can be generalized in Hilbert spaces for compact self-adjoint operators. In this section we give that result, which is often called *Hilbert-Schmidt theorem*. Further, we state a theorem that let us compare the eigenvalues of two compact self-adjoint operators on a Hilbert space.

We define the linear space (see Ex. 2.3)

$$\mathscr{S}_\infty(\mathcal{H}) = \{T \in \mathscr{I}_\infty(\mathcal{H}) / T \text{ is self-adjoint } \},$$

where $\mathcal{H}$ is a separable Hilbert space. The following theorem, called the *Hilbert-Schmidt theorem*, gives a very important property to all the elements in this space.

Theorem 2.29. *Let $T \in \mathscr{S}_\infty(\mathcal{H})$. Then there is a Hilbert basis for $\mathcal{H}$ formed by eigenvectors of T. Moreover,*

$$\lambda_n \to 0, \text{ as } n \to \infty,$$

where $(\lambda_n)_{n\in\mathbb{N}}$ is the sequence of eigenvalues of T.

Proof. Let $(\lambda_n)_{n\in\mathbb{N}}$ be the sequence of all distinct nonzero eigenvalues of T. Set $\lambda_0 = 0$ and we denote each eigenspace as follows

$$E_0 = \text{Ker}(T), \quad E_n = \text{Ker}(T - \lambda_n I).$$

By Theorem 2.28, the dimension of E_n is finite, for any $n \in \mathbb{N}$. We claim that $\mathcal{H}$ is the Hilbert sum of these eigenspaces, including E_0. After this result is proven, we extract an Hilbert basis from each E_n in order to construct an Hilbert basis of eigenvectors for the entire space $\mathcal{H}$.

Let's prove first that $(E_n)_{n\geq 0}$ are mutually orthogonal. This is direct since T is self-adjoint. Indeed, let $u \in E_m$ and $v \in E_n$, with $n \neq m$, then

$$(\lambda_m u, v) = (Tu, v) = (u, Tv) = (u, \lambda_n v),$$

and since λ_m, λ_n are real and distinct, we deduce $(u, v) = 0$. Since u, v were chosen arbitrarily, we have proved that

$$\forall n \neq m: \quad E_n \perp E_m.$$

Let's denote F as the linear space spanned by the $(E_n)_{n \geq 0}$. We have to prove that $\overline{F} = \mathcal{H}$, which is equivalent to proving that $F^{\perp} = \{0\}$. First, recall that the eigenspaces of T are invariant under T, that is, $T(F) \subseteq F$. Moreover, we claim that this implies $T(F^{\perp}) \subseteq F^{\perp}$. Indeed, let $y \in T(F^{\perp})$. Then there exists some $x \in \mathcal{H}$ such that $y = Tx$ and $(x, f) = 0$ for any $f \in F$. Hence

$$(y, f) = (Ax, f) = (x, Af) = 0,$$

since $T(F) \subseteq F$. Therefore $T(F^{\perp}) \subseteq F^{\perp}$. Let's denote by T_p the restriction of T under $T^{\perp}$. Clearly, $T_p \in \mathscr{S}_{\infty}(F^{\perp})$ since $T \in \mathscr{S}_{\infty}(\mathcal{H})$. We claim that $\sigma(T_p) = \{0\}$. In fact, let's argue by contradiction. Since

$$\exists \lambda \in \sigma(T_p) \setminus \{0\} \implies \lambda \in \sigma_p(T_p)$$

by the Riesz-Schauder theorem, we have

$$\exists u \in F^{\perp}: \quad T_p u = \lambda u.$$

This implies that λ is an eigenvalue of T, say $\lambda = \lambda_n$ for some $n \in \mathbb{N}$. Thus $u \in E_n \subseteq F$. Then $u = 0$ since

$$u \in E_n \cap (E_n)^{\perp}.$$

This is a contradiction since $u \neq 0$. Using a result found in [6, Cor. 6.10], we conclude that $T_p = 0$, i.e., T is the zero operator on $F^{\perp}$. It follows that

$$F^{\perp} \subseteq \text{Ker}(T) \subseteq F.$$

Hence $F^{\perp} \subseteq F$. This implies $F^{\perp}$={0}, and so F is dense in $\mathcal{H}$.

Finally, since every separable Hilbert space has an Hilbert basis, we can extract a Hilbert basis from E_0. Additionally, we can extract a Hilbert basis from each E_n, since they are of finite dimension. The union of these bases form a Hilbert basis for $\mathcal{H}$. Clearly, this basis is composed of eigenvectors of A. □

Recall that in Definition 2.12 we provided an order to the set $\mathscr{L}(\mathcal{H})$. The following theorem let us use this definition in order to compare eigenvalues of two compact self-adjoint operators.

Theorem 2.30. *Let S and T be two compact self-adjoint operators such that $S \leq T$. Denote their positive eigenvalues, indexed in decreasing order by α_k and β_k, $k = 1, 2, \ldots,$ respectively. Then*

$$\forall k \in \mathbb{N}: \quad \alpha_k \leq \beta_k.$$

For negative eigenvalues, the opposite inequality holds.

A proof of this theorem can found in [14, Ch. 28, Th. 6]. The proof uses a concept that we will not discuss here called Rayleigh quotient.

Remark 2.19. *When T is compact, so is its absolute value $|T|$. The nonzero eigenvalues of $|T|$, denoted as $(s_n(T))_{n \in I}$, are positive numbers that tend to zero by the Riesz-Schauder theorem (see th. 2.28); we index them in decreasing order. The numbers $s_n(T)$ are called the* singular values *of T.*

We finish this section with a property of compact operators which is proved using the Hilbert-Schmidt theorem and the concept of singular values.

Proposition 2.12. *Let T be a compact operator on $\mathcal{H}$. Then there exists orthonormal sets $\{\psi_n\}_{n=1}^N = \{\psi_n / n \in I_N\}$, $\{\phi_n\}_{n=1}^N = \{\phi_n / n \in I_N\}$ such that*

$$T = \sum_{n=1}^{N} s_n(T)(\psi_n, \cdot)\phi_n.$$

This sum converges in norm. Note that this sum may be finite or infinite.

Proof. We know that the adjoint of a compact operator is compact. Moreover, the product of compact operators is compact. Hence T^*T is compact. Further, it is self-adjoint. By the Hilbert-Schmidt theorem, there is an orthonormal set $\{\psi_n\}_{n=1}^N$ of eigenvectors of T^*T, that is

$$T^*T\psi_n = \lambda_n \psi_n, \quad \lambda_n \neq 0.$$

Then T^*T is the zero operator on the subspace orthogonal to $\{\psi_n\}_{n=1}^N$. Since $T^*T \geq 0$, each $\lambda_n > 0$. Clearly, $s_n(T) = \sqrt{\lambda_n}$. Set

$$\phi_n = \frac{T\psi_n}{s_n(T)}.$$

We claim that the ϕ_n's are orthonormal. Indeed, the computation

$$(\phi_n, \phi_m) = \left(\frac{T\psi_n}{s_n(T)}, \frac{T\psi_m}{s_m(T)} \right) = \left(\frac{\psi_n}{s_n(T)}, \frac{T^*T\psi_m}{s_m(T)} \right) = \left(\frac{\psi_n}{s_n(T)}, s_m(T)\psi_m \right)$$

shows that

$$(\phi_n, \phi_m) = \begin{cases} 1 & , \text{ if } n = m, \\ 0 & , \text{ if } n \neq m. \end{cases}$$

Let $\psi \in \mathcal{H}$. Since $\psi = \sum_{n=1}^{N}(\psi_n, \psi)\psi_n$, we have

$$\begin{aligned} T\psi &= T\left(\sum_{n=1}^{N}(\psi_n, \psi)\psi_n\right) \\ &= \sum_{n=1}^{N}(\psi_n, \psi)T\psi_n \\ &= \sum_{n=1}^{N} s_n(T)(\psi_n, \psi)\phi_n. \end{aligned}$$

□

2.4.2 Trace-class and Hilbert-Schmidt operators

The trace formula for square matrix states that the sum of its eigenvalues equals the trace of the matrix. In this section we give a generalization of this result. As a consequence, we define a new class of linear operators acting on a Hilbert space.

Let $\mathcal{H}$ be a separable Hilbert space, $T \in \mathscr{L}(\mathcal{H})$ and $B = \{\psi_i / i \in \mathbb{N}\}$ any Hilbert basis in $\mathcal{H}$. The first step is to provide T sufficient properties so that its *trace*

$$\mathrm{Tr}\,[T] = \sum_{n=1}^{\infty}\left(\psi_n, T\psi_n\right) \tag{2.13}$$

converges and does not depend on the choice of basis. Analogous to the construction of the Lebesgue integral, where one defines it first for non negative functions, we will define the *trace* of a positive bounded operator. The trace has values in $[0, \infty]$. The following result let us define a trace for positive bounded operators and states some properties.

Theorem 2.31. *Let $\mathcal{H}$ be a separable Hilbert space, $B = \{\psi_i / i \in \mathbb{N}\}$ a Hilbert basis. Then for any positive operator $T \in \mathscr{L}(\mathcal{H})$, (2.13) is independent of the chosen basis. Let $S \in \mathscr{L}(\mathcal{H})$ be a positive operator. The* trace *has the following properties*

(i) $\mathrm{Tr}\,[S + T] = \mathrm{Tr}\,[S] + \mathrm{Tr}\,[T]$.

(ii) $\forall \lambda \geq 0: \quad \mathrm{Tr}\,[\lambda T] = \lambda \mathrm{Tr}\,[T]$.

(iii) $\mathrm{Tr}\left[UTU^{-1}\right] = \mathrm{Tr}\,[T]$ *for all unitary operator U.*

(iv) If $0 \leq T \leq S$, *then* $\mathrm{Tr}\,[T] \leq \mathrm{Tr}\,[S]$.

Proof. Let's first prove that the trace is independent of the basis chosen. Given a Hilbert basis $B = \{\varphi_n / n \in \mathbb{N}\}$, define $\mathrm{Tr}_{\varphi}[\mathrm{T}] = \sum_{n=1}^{\infty}\left(\varphi_n, T\varphi_n\right)$. Suppose $B' = \{\psi_n / n \in \mathbb{N}\}$ is a different Hilbert basis, then

$$\begin{aligned}\mathrm{Tr}_{\varphi}[T] &= \sum_{n=1}^{\infty} (\varphi_n, T\varphi_n) \\ &= \sum_{n=1}^{\infty} \left(T^{1/2}\varphi_n, T^{1/2}\varphi_n\right) = \sum_{n=1}^{\infty} \left\|T^{1/2}\varphi_n\right\|^2 .\end{aligned} \tag{2.14}$$

Since $B = \{\varphi_n / n \in \mathbb{N}\}$ is a Hilbert basis for $\mathcal{H}$, we obtain

$$\left\|T^{1/2}\varphi_n\right\|^2 = \sum_{m=1}^{\infty} \left|\left(\psi_m, T^{1/2}\varphi_n\right)\right|^2, \quad T^{1/2}\varphi_n = \sum_{m=1}^{\infty} \left(\psi_m, T^{1/2}\varphi_n\right) \psi_m. \tag{2.15}$$

Hence, we have in (2.14)

$$\begin{aligned}\mathrm{Tr}_{\varphi}[T] &= \sum_{n=1}^{\infty} \left(\sum_{m=1}^{\infty} \left|\left(\psi_m, T^{1/2}\varphi_n\right)\right|^2 \right) \\ &= \sum_{m=1}^{\infty} \left(\sum_{n=1}^{\infty} \left|\left(T^{1/2}\psi_m, \varphi_n\right)\right|^2 \right).\end{aligned}$$

Applying (2.15) with ψ_m, we deduce from the above

$$\begin{aligned}\mathrm{Tr}_{\varphi}[T] &= \sum_{m=1}^{\infty} \left\|T^{1/2}\psi_m\right\| \\ &= \sum_{m=1}^{\infty} (\psi_m, T\psi_m) \\ &= \mathrm{Tr}_{\psi}[T].\end{aligned}$$

Since $B = \{\varphi_n / n \in \mathbb{N}\}$ and $B' = \{\psi_n / n \in \mathbb{N}\}$ were chosen arbitrarily, we conclude that the trace is independent of the Hilbert basis chosen and therefore it is well defined.

Now we are ready to prove the properties listed in the theorem. The first two properties are trivial. In order to prove *(iii)*, we note that if $B = \{\varphi_n / n \in \mathbb{N}\}$ is a Hilbert basis, then so is $\{U\varphi_n \ / \ n \in \mathbb{N}\}$. Hence,

$$\begin{aligned}\mathrm{Tr}\left[UTU^{-1}\right] &= \mathrm{Tr}_{(U\varphi)}\left[UTU^{-1}\right] \\ &= \sum_{n=1}^{\infty} \left(U\varphi_n, UTU^{-1}U\varphi_n\right) \\ &= \sum_{n=1}^{\infty} (U\varphi_n, UT\varphi_n) \\ &= \sum_{n=1}^{\infty} (\varphi_n, T\varphi_n) \\ &= \mathrm{Tr}_{\varphi}[T].\end{aligned}$$

In order to prove (iv), recall that $T \leq S$ if and only if $(x, Tx) \leq (x, Sx)$, for all $x \in \mathcal{H}$. Hence, in particular

$$\forall n \in \mathbb{N}: \quad (\varphi_n, T\varphi_n) \leq (\varphi_n, S\varphi_n),$$

where $B = \{\varphi_n / n \in \mathbb{N}\}$ is a Hilbert basis. Then

$$\text{Tr}\,[T] = \sum_{n=1}^{\infty} (\varphi_n, T\varphi_n) \leq \sum_{n=1}^{\infty} (\varphi_n, S\varphi_n) = \text{Tr}\,[S]\,.$$

□

Considering all these properties, we define the family of trace class operators:

Definition 2.16. *A bounded linear operator T is in* trace class *if and only if Tr $[|T|] < \infty$. The set of all trace class operators is denoted by $\mathscr{I}_1(\mathcal{H})$.*

Remark 2.20. *The set $(\mathscr{L}(\mathcal{H}), +, \cdot, \circ)$ is an associative* Banach algebra *with unit.*

The following theorem states some basic properties of trace class operators

Theorem 2.32. *Let $\mathcal{H}$ be a separable Hilbert space. Then $\mathscr{I}_1(\mathcal{H})$ is an $*-$ideal in $\mathscr{L}(\mathcal{H})$, that is,*

(i) $\mathscr{I}_1(\mathcal{H})$ is a linear space.

(ii) Let $T \in \mathscr{I}_1(\mathcal{H})$ and $S \in \mathscr{L}(\mathcal{H})$. Then $TS \in \mathscr{I}_1(\mathcal{H})$ and $ST \in \mathscr{I}_1(\mathcal{H})$.

(iii) Let $T \in \mathscr{I}_1(\mathcal{H})$. Then $T^* \in \mathscr{I}_1(\mathcal{H})$.

A proof of this result can be found in [21, Th.VI.19].

The relation between trace class operators and compact operators is elegant and very important. The following theorem states this relation and gives a new definition for trace class operators

Theorem 2.33. *Every $T \in \mathscr{I}_1$ is compact. A compact operator T is a trace class operator if and only if*

$$\sum_{n=1}^{\infty} s_n(T) < \infty, \tag{2.16}$$

where $(s_n(T))_{n\in\mathbb{N}}$ are the singular values of T.

Proof. Since $T \in \mathscr{I}_1(\mathcal{H})$, then $|T|^2 = T^*T \in \mathscr{I}_1(\mathcal{H})$ by theorem 2.32, hence

$$\text{Tr}\left[|T|^2\right] = \sum_{n=1}^{\infty} (\psi_n, |T|^2\psi_n) = \sum_{n=1}^{\infty} \left\|T\psi_n\right\|^2 < \infty,$$

for any Hilbert basis $B = \{\psi_n / n \in \mathbb{N}\}$. Suppose $\psi \in [\psi_1, \ldots, \psi_N]^\perp$ and $\|\psi\| = 1$, then we have

$$\|T\psi\|^2 \leq \mathrm{Tr}\left[|T|^2\right] - \sum_{n=1}^{N} \|T\psi_n\|^2$$

since $\{\psi_1, \ldots, \psi_N, \psi\}$ can always be completed to a Hilbert basis. Thus

$$\sup\{\|T\psi\| : \psi \in [\psi_1, \ldots, \psi_N]^\perp, \|\psi\| = 1\} \to 0, \quad \text{as } N \to \infty.$$

Therefore $\sum_{n=1}^{N} (\psi_n, \cdot) T\psi_n$ is norm convergent to T. Thus T is compact.

For the second part of the theorem, we use the canonical form of compact operators. Suppose that T is a trace class operator. Since T is compact, then $|T|$ is compact. Moreover it is self-adjoint. Hence by the Hilbert-Schmidt theorem there exists $B = \{\psi_n / n \in \mathbb{N}\}$ is a Hilbert basis of eigenvectors of $|T|$. Thus

$$\mathrm{Tr}\left[|T|\right] = \sum_{n=1}^{\infty} (\psi_n, |T|\psi_n) = \sum_{n=1}^{\infty} s_n(T) < \infty,$$

Now suppose that T is compact and that $\sum_{n=1}^{\infty} s_n(T) < \infty$. Let $B = \{\psi_n / n \in \mathbb{N}\}$ be any Hilbert basis, we have to prove that

$$\sum_{n=1}^{\infty} (\psi_n, |T|\psi_n) < \infty,$$

which follows from the canonical form of T given in proposition 2.12 □

The sum (2.16) is called the *trace norm* of T and is denoted by $\|T\|_1$. It's clear that

$$\forall T \in \mathscr{I}_1(\mathcal{H}): \quad \|T\|_1 = \mathrm{Tr}\left[|T|\right].$$

The next result gives some properties of the trace norm. Moreover, it shows that $\|\cdot\|_1$ is indeed a norm in $\mathscr{I}_1$.

Remark 2.21. *Thanks to theorem 2.33 and the Hilbert-Schmidt theorem, a trace class operator that is also self-adjoint has a Hilbert basis of eigenvectors. Therefore, every result shown in this section can be proved more easily if we add the assumption of self-adjointness.*

Proposition 2.13. *Let T be a trace class operator, S any bounded operator. Then*

(i) $\|T\|_1 = \|T^*\|_1$.

(ii) $\|ST\|_1 \leq \|S\| \|T\|_1$.

(iii) $\|TS\|_1 \leq \|S\| \|T\|_1$.

(iv) If S is also a trace class operator, then $\|T + S\|_1 \leq \|T\|_1 + \|S\|_1$.

Proof. In order to prove (i), we have to show that the singular values of T and T^* coincide. The singular values of T^* are the positive eigenvalues of

$$(T^{**}T^*)^{1/2} = (TT^*)^{1/2}.$$

We claim that T^*T and TT^* have the same positive eigenvalues. In fact, let ψ be an eigenvector and λ the corresponding eigenvalue of T^*T, i.e.,

$$T^*T\psi = \lambda\psi, \quad \lambda \neq 0.$$

Applying T on both sides in the equality above yields

$$TT^*T\psi = \lambda T\psi,$$

which implies that λ is an eigenvalue of TT^* with eigenvector $T\psi \neq 0$. Since $|T| = (T^*T)^{\frac{1}{2}}$, the eigenvectors of $|T|$ are those of T^*T, and the eigenvalues the square root of those of T^*T. Hence we have proved that $s_n(T) = s_n(T^*)$ for every singular value. Therefore $\|T\| = \|T^*\|_1$.

Let's prove (ii). We claim that $s_n(ST) \leq \|S\| \|T\|_1$. In fact, notice that

$$|ST|^2 = (T^*S^*ST) \leq \|S\|^2 |T|^2 = \|S\|^2 (T^*T).$$

Indeed, let $u \in \mathcal{H}$. Then

$$(T^*S^*STu, u) = (STu, STu) = \|STu\|^2 \leq \|S\| \|Tu\|^2 = \|S\|^2 (T^*Tu, u).$$

Since the n-th eigenvalue is a monotonic function,

$$s_n^2(ST) \leq \|S\|^2 s_n^2(T). \tag{2.17}$$

Taking the square root and summing over n, we obtain inequality (ii). By (i), the singular values of adjoint operators are the same. Then, by (2.17) we have that

$$s_n(TS) = s_n(S^*T^*) \leq \|S^*\|^2 s_n(T^*) = \|S\| s_n(T).$$

Summing over n we obtain (iii).

Finally, to prove (iv) we introduce the following characterization of trace norm

$$\|T\|_1 = \sup \sum_{n=1}^{\infty} |(\psi_n, T\phi_n)|, \tag{2.18}$$

where the supremum is taken over all pairs of orthonormal bases $B = \{\psi_n / n \in \mathbb{N}\}$, $B' = \{\phi_n / n \in \mathbb{N}\}$. We have to prove that the right side of (2.18) never exceeds $\|T\|_1$, and equals it for the appropriate choice of ψ_n and ϕ_n. Since $|T| \in \mathscr{S}_\infty(\mathcal{H})$, then the

Hilbert-Schmidt theorem gives a Hilbert basis $\{\xi_n / n \in \mathbb{N}\}$ of eigenvectors of $|T|$. For any $\psi \in \mathcal{H}$, we can expand,

$$\psi = \sum_{n=1}^{\infty} (\xi_n, \psi)\xi_n,$$

so that applying $|T|$ above yields

$$|T|\psi = \sum_{n=1}^{\infty} s_n(T)(\xi_n, \psi)\xi_n.$$

Applying U on both sides above and using the polar decomposition $T = U|T|$, we get

$$T\psi = \sum_{n=1}^{\infty} s_n(T)(\xi_n, \psi)U\xi_n. \tag{2.19}$$

Since $U(\mathcal{H}) = |T|(\mathcal{H})$ and U is unitary, then $\{U\xi_n / n \in \mathbb{N}\}$ forms a Hilbert basis on $|T|(\mathcal{H})$. We take the scalar product of 2.19 with some $\phi \in \mathcal{H}$:

$$(\phi, T\psi) = \sum_{n=1}^{\infty} s_n(T)(\xi_n, \psi)(\phi, U\xi_n). \tag{2.20}$$

We set now $\psi = \psi_m$ and $\phi = \phi_m$ in 2.20 and sum over m:

$$\sum_{m=1}^{\infty} (\phi_m, T\psi_m) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} s_n(T)(\xi_n, \psi_m)(\phi_m, U\xi_n). \tag{2.21}$$

We can estimate the right side of (2.21) as follows: sum first with respect to m and apply the Cauchy-Schwarz inequality,

$$\sum_{n=1}^{\infty} s_n(T) \left(\sum_{m=1}^{\infty} |(\xi_n, \psi_m)|^2 \sum_{m=1}^{\infty} |(\phi_m, U\xi_n)|^2 \right)^{1/2}.$$

By Parseval:

$$\forall n \in \mathbb{N}\colon \sum_{m=1}^{\infty} |(\xi_n, \psi_m)|^2 = \|\xi_n\|^2 = 1, \quad \sum_{m=1}^{\infty} |(\phi_m, U\xi_n)|^2 = \|U\xi_n\| = 1.$$

This shows that

$$\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} s_n(T)(\xi_n, \psi_m)(\phi_m, U\xi_n) \leq \sum_{n=1}^{\infty} s_n(T) = \|T\|_1.$$

This holds for any Hilbert basis B, B'. Hence

$$\sup \sum_{n=1}^{\infty} |(\psi_n, T\phi_n)| \leq \|T\|_1.$$

To complete the proof we choose $\psi_n = \xi_n$ and $\phi_n = U\xi_n$, supplemented by a Hilbert basis on the orthogonal complement of $|T|(\mathcal{H})$. Setting $\psi = \xi_n$ and $\phi = U\xi_n$ in (2.20) we get, since U is an isometry on $|T|(\mathcal{H})$, that

$$(U\xi_n, T\xi_n) = (U\xi_n, U|T|\xi_n) = (\xi_n, |T|\xi_n) = s_n(T)(\xi_n, \xi_n) = s_n(T),$$

summing over n, we get equality in (2.18). Now, using this characterization and the triangle inequality:

$$\begin{aligned} \|S+T\|_1 &= \sup \sum_{n=1}^{\infty} |(\psi_n, (S+T)\phi_n)| \\ &\leq \sup \left(\sum_{n=1}^{\infty} (|(\psi_n, S\phi_n)| + |(\psi_n, T\phi_n)|) \right) = \|S\|_1 + \|T\|_1 . \end{aligned}$$

□

We have that $\mathscr{I}_1(\mathcal{H})$ is a normed space under the trace norm. Now, we can state the completeness of the space of trace class operators.

Theorem 2.34. *$\mathscr{I}_1$ is a Banach space when equipped with the norm $\|\cdot\|_1$. Moreover,*

$$\forall T \in \mathscr{I}_1(\mathcal{H}) : \quad \|T\| \leq \|T\|_1 .$$

Proof. Let $T \in \mathscr{I}_1$. Then $|T| \in \mathscr{I}_\infty(\mathcal{H})$. Since $|T|$ is also self-adjoint, there exists a Hilbert basis $B = \{\psi_n / n \in \mathbb{N}\}$ of eigenvectors of $|T|$. Then, by Parseval

$$\forall u \in \mathcal{H}: \quad u = \sum_{n=1}^{\infty} (\psi_n, u)\psi_n,$$

The above implies

$$|T|u = \sum_{n=1}^{\infty} (\psi_n, u)|T|\psi_n = \sum_{n=1}^{\infty} (\psi_n, u)s_n(T)\psi_n.$$

Decomposing T yields

$$\begin{aligned} \|Tu\|^2 &= |(Tu, Tu)| \\ &= |(U|T|u, U|T|u)| \\ &= \||T|u\|^2 \\ &= \left\| \sum_{n=1}^{\infty} (\psi_n, u)s_n(T)\psi_n \right\|^2 \end{aligned}$$

Hence, using the Cauchy-Schwarz inequality and since the singular values are positive, we deduce

$$\|Tu\| \leq \sum_{n=1}^{\infty} s_n(T)\|u\| = \|T\|_1 \|u\| .$$

Since u was chosen arbitrary, we conclude. Completeness of the space follows from this inequality. □

The next result is derived from Theorem 2.33 and it gives a necessary and sufficient condition for a positive compact self-adjoint operator to be in $\mathscr{I}_1$.

Corollary 2.5. *Let $T \in \mathscr{L}(\mathcal{H})$ be a positive compact self-adjoint operator. Then T is a trace class operator if and only if*

$$\sum_{n=1}^{\infty} \lambda_n < \infty,$$

where $(\lambda_n)_{n\in\mathbb{N}}$ is the sequence of eigenvalues of T.

Proof. This is direct since $|T| = T$. Therefore the singular values of T are precisely the eigenvalues of T, thus by Theorem 2.33, we conclude. □

Until this point, the trace of an operator was only defined for some operators. The following theorem let us define the trace $\mathrm{Tr} : \mathscr{I}_1 \to \mathbb{C}$ for all trace class operators.

Theorem 2.35. *If T is a trace class operator. Then*

$$\mathrm{Tr}\,[T] = \sum_{n=1}^{\infty} \left(\varphi_n, T\varphi_n\right)$$

converges absolutely and the limit is independent of the choice of basis.

Proof. In order to prove this theorem, we use some results found in the proof of Proposition 2.13. That is, inequality (2.21):

$$\sum_{m=1}^{\infty} (\phi_n, T\psi_n) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} s_n(T)(\xi_n, \psi_m)(\phi_m, U\xi_n),$$

where $B = \{\psi_n / n \in \mathbb{N}\}$, $B' = \{\phi_n / n \in \mathbb{N}\}$ is a pair of arbitrary orthonormal bases, U is the unitary operator such that $T = U|T|$ and $\{\xi_n / n \in \mathbb{N}\}$ is a Hilbert basis of eigenvectors of T. Setting $\phi_n = \psi_n$ in this inequality yields

$$\sum_{m=1}^{\infty} (\psi_n, T\psi_n) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} s_n(T)(\xi_n, \psi_m)(\psi_m, U\xi_n). \tag{2.22}$$

As we already shown in the proof of proposition 2.13, the double series on the right side converges and its value is bounded by $\|T\|_1$.

Let's now prove that the trace is independent of the choice of basis. Summing in (2.22) over m first:

$$\mathrm{Tr}\,[T] = \sum_{n=1}^{\infty} s_n(T) \sum_{m=1}^{\infty} (\xi_n, \psi_m)(\psi_m, U\xi_n). \tag{2.23}$$

Moreover, by Parseval relation

$$\sum_{m=1}^{\infty}(\xi_n,\psi_m)(\psi_m,U\xi_n)=(\xi_n,U\xi_n). \tag{2.24}$$

Hence, using (2.24) in (2.23) gives

$$\operatorname{Tr}[T]=\sum_{n=1}^{\infty}s_n(T)(\xi_n,U\xi_n),$$

which is basis independent. □

Finally, we are ready to give one of the most important properties of the trace, which was proved by Lidskii in 1959:

Theorem 2.36. *The trace of a trace class operator T is the sum of its eigenvalues:*

$$\operatorname{Tr}[T]=\sum_{n=1}^{\infty}\lambda_n,$$

where $(\lambda_n)_{n\in\mathbb{N}}$ is the sequence of eigenvalues of T.

The proof of this theorem requires various lemmas and is given in [14, Ch. 30, Th.5].

Remark 2.22. *Note that, by Theorems 2.35 and 2.36, the sum of the eigenvalues of a trace class operators converges absolutely.*

Example 2.4. *Following the notation given in Example 2.3, we define the space of* nuclear class *operators*

$$\mathscr{S}_1=\mathscr{S}_1(\mathrm{L}^2(\Omega))=\{T\in\mathscr{I}_1/T \text{ is self-adjoint }\}.$$

By the Hilbert-Schmidt theorem, we have that for every $T\in\mathscr{S}_1$

$$\operatorname{Tr}[T]=\sum_{n=1}^{\infty}\lambda_n,$$

where $(\lambda_n)_{n\in\mathbb{N}}$ is the sequence of eigenvalues of T. Furthermore, this sum converges absolutely. The space of nuclear class operators becomes a Banach space when equipped with the norm

$$\|T\|_1=\operatorname{Tr}\left[|T|\right]=\sum_{n=1}^{\infty}|\lambda_n|.$$

By Proposition 2.13, we note that $\mathscr{S}_1$ is an ideal of $\mathscr{L}_S$ and

$$\|RT\|_1\le\|R\|\|T\|_1 \quad \text{and } \|TR\|_1\le\|R\|\|T\|_1$$

for a given $R\in\mathscr{L}_S$ and $T\in\mathscr{S}_1$.

Analogous to the Lebesgue space of square integrable functions $L^2(\Omega)$, we shall define a second class of operators, called *Hilbert-Schmidt operators*.

Definition 2.17. *An operator $T \in \mathscr{L}(\mathcal{H})$ is called* Hilbert-Schmidt *if $T^*T \in \mathscr{I}_1(\mathcal{H})$, i.e*

$$\operatorname{Tr}\left[T^*T\right] = \sum_{n=1}^{\infty} \left(\varphi_n, T^*T\varphi_n\right) < \infty.$$

The family of all Hilbert-Schmidt operators is denoted by $\mathscr{I}_2(\mathcal{H})$.

$\mathscr{I}_2(\mathcal{H})$ is rich in properties that can be proven analogously to $\mathscr{I}_1(\mathcal{H})$. However, we focus in the case when $\mathcal{H} = L^2(\Omega)$. For a general study of $\mathscr{I}_2(\mathcal{H})$, see e.g., [21].

In this case, the space of Hilbert-Schmidt operators

$$\mathscr{I}_2 = \mathscr{I}_2(L^2(\Omega)) = \{T \in \mathscr{I}_\infty / |T| = T^*T \in \mathscr{I}_1\}$$

has an elegant characterization. An operator $T \in \mathscr{L}(L^2(\Omega))$ is in $\mathscr{I}_2$ if and only if there is a function

$$D_T \in L^2(\Omega \times \Omega)$$

called the kernel of T, such that

$$(Tf)(x) = \int_\Omega D_T(x,y) f(y) dy, \quad f \in L^2(\Omega).$$

Moreover,

$$\|T\|_2^2 = \int_{\Omega\times\Omega} |D_T(x,y)|^2 dxdy = \|D_T\|_{L^2(\Omega\times\Omega)}.$$

A proof of this result can be found in [21, Th. VI.23]. Analogous to Example 2.4, we have the Hilbert space

$$\mathscr{S}_2 = \mathscr{S}_2(L^2(\Omega)) = \{T \in \mathscr{I}_2 / T \text{ is self-adjoint }\}$$

with inner product

$$(T,R)_2 = \operatorname{Tr}\left[R^*T\right].$$

Using the Hilbert-Schmidt theorem, we deduce

$$\|T\|_2^2 = \operatorname{Tr}\left[|T|^2\right] = \sum_{n=1}^{\infty} |\lambda_n|^2,$$

where $(\lambda_n)_{n\in\mathbb{N}}$ is the sequence of eigenvalues of T. Similar to $\mathscr{I}_2$, we have that $T \in \mathscr{S}_2$ if and only if

$$D_T(x,y) = D_T(y,x) \quad \text{a.e } x,y \in \Omega.$$

Finally, we note that $\mathscr{S}_2$ is an ideal of $\mathscr{L}_S$ by a result found in [21, Th. VI.22].

2.5 Spectral theorem

In this section, we present a structure theorem in the sense that it depicts a concrete description of self-adjoint operators. There are many formulations of the spectral theorem. They are similar as they give a representation of self-adjoint operators that makes them simpler and easier to study. Furthermore, we shall use this results in order to present additional ideals $\mathscr{S}_p$ of the space $\mathscr{L}_S$.

2.5.1 Spectral representation of a bounded self-adjoint operator

In Section 2.3.4 we introduced the concept of spectral family. Now, we shall present a result that states that every bounded self-adjoint operator has a spectral family $\mathcal{E}$ such that $\mathcal{E}$ can be used for a spectral representation of our bounded self-adjoint operator. But before that, we define the positive part and the negative part of a bounded self-adjoint operator respectively

$$T^+ = \frac{1}{2}(|T| + T), \quad T^- = \frac{1}{2}(|T| - T).$$

Note that $T = T^+ - T^-$ and $|T| = T^+ + T^-$. The properties of these operators and their behaviour can be found in [13]. In the following proposition we only state few of them:

Proposition 2.14. *Let $\mathcal{H}$ be a separable Hilbert space and $T \in \mathscr{L}_S(\mathcal{H})$. Then*

(i) T^+ and T^- are bounded and self-adjoint.

(ii) T^+ and T^- commute with every bounded linear operator that commutes with T.

(iii) $T^+T^- = 0$ and $T^+, T^- \geq 0$.

All these properties are also true if we replace T by T_λ.

A proof of this proposition can be found in [13, Lemma 9.8-1].

Theorem 2.37. *Let $\mathcal{H}$ be a complex Hilbert space and $T \in \mathscr{L}_S(\mathcal{H})$. Furthermore, let E_λ be the projection of $\mathcal{H}$ onto $Y_\lambda = \mathrm{Ker}(T_\lambda^+)$, where $\lambda \in \mathbb{R}$. Then $\mathcal{E} = (E_\lambda)_{\lambda \in \mathbb{R}}$ is a spectral family on the interval $[m, M] \subseteq \mathbb{R}$, where m, M are given by Theorem 2.21.*

The proof of this theorem requires various lemmas and it is developed in [13, Th. 9.8.3].

Now, with this result, we can obtain a integral representation of any bounded self-adjoint operator on a complex Hilbert space, which involves the spectral family constructed in Theorem 2.37.

Theorem 2.38. *Let $T \in \mathscr{L}_S(\mathcal{H})$. Then:*

(i) T has the spectral representation

$$T = \int_{m^-}^{M} \lambda dE_\lambda,$$

where $\mathcal{E} = (E_\lambda)_{\lambda \in \mathbb{R}}$ is the spectral family associated with T given by Theorem 2.37. This integral is to be understood in the sense of uniform operator convergence, and for all $x, y \in \mathcal{H}$,

$$(Tx, y) = \int_{m^-}^{M} \lambda d(E_\lambda x, y).$$

This integral is a Riemann-Stieltjes integral *(see Appendix A).*

(ii) *More generally, if p is a polynomial in λ with real coefficients, say*

$$p(\lambda) = \alpha_n \lambda^n + \cdots + \alpha_0,$$

then the operator $p(T)$ defined by

$$p(T) = \alpha_n T^n + \cdots + \alpha_0 I$$

has the spectral representation

$$p(T) = \int_{m^-}^{M} p(\lambda) dE_\lambda.$$

Moreover, for all $x, y \in \mathcal{H}$,

$$(p(T)x, y) = \int_{m^-}^{M} p(\lambda) d(E_\lambda x, y).$$

A proof of this result can be found in [13, Th. 9.9.2]. In this theorem we used the notation m^-. This indicates that one must take into account a contribution at $\lambda = m$, which occurs if $E_m \neq 0$ and if $m \neq 0$. This means that for any $a < m$, we have

$$\int_{a}^{M} p(\lambda) dE_\lambda = \int_{m^-}^{M} p(\lambda) dE_\lambda = p(m)E_m + \int_{m}^{M} p(\lambda) dE_\lambda.$$

2.5.2 Functional calculus

Let $\mathcal{H}$ be a Hilbert space and $T \in \mathscr{L}_S(\mathcal{H})$. The next formulation of the spectral theorem, called functional calculus, gives enough conditions so that for any given function

$$f : \sigma(T) \to \mathbb{C},$$

the expression $f(T)$ is well defined as an operator. Furthermore, $f(T)$ has very important properties that are necessary for our work.

The first step is to prove our theorem when f is a polynomial with real coefficients. We already know that in this case $f(T)$ is well defined. Furthermore, we have information about the spectrum of $f(T)$, as the following lemma states:

Proposition 2.15. *Let $p(\lambda) = \sum_{n=0}^{N} a_n \lambda^n$ and $p(T) = \sum_{n=0}^{N} a_n T^n$, Then*

$$\sigma(p(T)) = \{p(\lambda)/\lambda \in \sigma(T)\}.$$

A proof of this proposition can be found in [21, Lemma 1,Ch. VII]

Another very important result that we need is a generalization of Theorem 2.20

Proposition 2.16. *Let $p(\lambda) = \sum_{n=0}^{N} a_n \lambda^n$. Then,*

$$\|p(T)\| = \sup_{\lambda \in \sigma(T)} |p(\lambda)|.$$

Proof. Note that

$$\|p(T)\|^2 = \|p(T)^* p(T)\| = \|(\overline{p}p)(T)\|.$$

Since $\overline{p}p(T)$ is self-adjoint, by theorem 2.20 and proposition 2.15, we have

$$\begin{aligned}\|p(T)\|^2 = \|(\overline{p}p)(T)\| &= \sup_{\lambda \in \sigma\left(\overline{p}p(T)\right)} |\lambda| \\ &= \sup_{\lambda \in \sigma(T)} |\overline{p}p(T)| \\ &= \left(\sup_{\lambda \in \sigma(T)} |p(\lambda)| \right)^2.\end{aligned}$$

□

We are ready to present the functional calculus for continuous functions on $\sigma(T)$. A generalization for any measurable function can be found in [21].

Theorem 2.39. *There is a unique map $\phi : C(\sigma(T)) \to \mathscr{L}(\mathcal{H})$ with the following properties.*

*(i) ϕ is an algebraic *-homomorphism, that is,*

$$\phi(fg) = \phi(f)\phi(g), \quad \phi(\lambda f) = \lambda\phi(f),$$
$$\phi(1) = I, \quad \phi(\overline{f}) = \phi(f)^*.$$

(ii) ϕ is continuous and $\|\phi(f)\| = \|f\|_\infty$.

(iii) If f is the function such that $f(x) = x$, then $\phi(f) = T$.

(iv) If $T\psi = \lambda\psi$, then $\phi(f)\psi = f(\lambda)\psi$.

(v) $\sigma(\phi(f)) = \{f(\lambda) : \lambda \in \sigma(T)\}$.

(vi) If $f \geq 0$, then $\phi(f) \geq 0$.

Proof. Let $\phi(p) = p(T)$. Then by Proposition 2.16

$$\|\phi(p)\| = \|p\|_{C(\sigma(T))}.$$

Hence ϕ is bounded in $\sigma(T)$, so ϕ has a unique continuous linear extension

$$\hat{\phi} : \overline{\mathrm{P}(\sigma(T))} = \mathrm{C}(\sigma(A)) \to \mathbb{C},$$

where $\mathrm{P}(\sigma(T))$ denotes the set of polynomials on $\sigma(T)$. Let's denote again this extension as ϕ for simplicity.

Property (i), (iii) is obvious since ϕ is bounded and equality in (ii) follows from proposition 2.16. Property (iv) is proved using the continuity of the extension of ϕ and that

$$\forall \lambda \in \sigma(T): \quad \phi(p)\psi = p(\lambda)\psi.$$

We give the scheme of the proof of (v). Suppose $\lambda \notin f(\sigma(T))$, and define $g = (f - \lambda)^{-1}$, then $\phi(g) = (\phi(f) - \lambda)^{-1}$, i.e, $g(T) = (f(T) - \lambda I)^{-1}$. Hence $\lambda \in \rho(T)$ and

$$\sigma(\phi(f)) \subseteq \{f(\lambda)/\lambda \in \sigma(T)\}.$$

Now, if $\lambda \in f(\sigma(T))$, we can prove that there are $\psi \in \mathcal{H}$, with $\|\psi\| = 1$ and $\|(\phi(f) - \lambda)\psi\|$ arbitrarily small so that $\lambda \in \sigma(\phi(f))$. Therefore

$$\{f(\lambda)/\lambda \in \sigma(T)\} \subseteq \sigma(\phi(f))$$

and we conclude (v). To prove (vi), note that

$$f \geq 0 \implies \exists g \in \mathrm{C}(\sigma(T), \mathbb{R}): \quad f = g^2.$$

Thus $\phi(f) = \phi(g^2) = \phi(g)^2$ with $\phi(g)$ self-adjoint, hence $\phi(f) \geq 0$. □

Remark 2.23. *Item (v) in Theorem 2.39 is often called the* spectral mapping theorem. *If we denote $\phi(f)$ by $f(T)$, then the spectral mapping theorem has a more familiar look:*

$$\sigma(f(T)) = \{f(\lambda)/\lambda \in \sigma(T)\}.$$

2.5.3 Other ideals

In Subsection 2.4.2 we defined the space of trace class operators $\mathscr{I}_1$ and the space of Hilbert-Schmidt operators $\mathscr{I}_2$. Similarly, for $p \geq 1$ we define the linear space

$$\mathscr{I}_p = \mathscr{I}_p(\mathrm{L}^2(\Omega)) = \{T \in \mathscr{I}_\infty / |T|^p \in \mathscr{I}_1\}.$$

Note that $|T|^p$ is well-defined by theorem 2.39. Moreover, $\mathscr{I}_p$ is a Banach space when equipped with the norm

$$\|T\|_p = \sqrt[p]{\mathrm{Tr}\left[|T|^p\right]}.$$

Remark 2.24. *Using Parseval relation, we know that for any $u \in \mathrm{L}^2(\Omega)$*

$$Tu = \sum_{n=1}^{\infty} \lambda_n (\psi_n, u)_{\mathrm{L}^2(\Omega)} \psi_n,$$

where $\{\psi_n\}_{n=1}^{\infty}$ is a Hilbert basis of eigenvectors of T. Moreover, by the spectral theorem we have the following representation of $|T|^p$

$$|T|^p u = \sum_{n=1}^{\infty} |\lambda_n|^p (\psi_n, u)_{\mathrm{L}^2(\Omega)} \psi_n.$$

In the last section, we introduced two ideals of $\mathscr{L}_S$: $\mathscr{S}_1$ and $\mathscr{S}_2$. In the same manner, we define for $p \geq 1$ the Banach space $\mathscr{S}_p$

$$\mathscr{S}_p = \mathscr{S}_p(\mathrm{L}^2(\Omega)) = \{T \in \mathscr{I}_p \ / \ T \text{ is self-adjoint}\}. \tag{2.25}$$

In this case, by the Hilbert-Schmidt theorem

$$\|T\|_p = \left(\sum_{n=1}^{\infty} |\lambda_n|^p\right)^{1/p},$$

where $(\lambda_n)_{n\in\mathbb{N}}$ is the sequence of eigenvalues of T. Hence

$$1 \leq p \leq q \leq \infty \implies \mathscr{S}_1 \subseteq \mathscr{S}_p \subseteq \mathscr{S}_q \subseteq \mathscr{S}_\infty.$$

Furthermore, we have the following inequality, which is similar to Hölder's inequality in Lebesgue spaces.

Proposition 2.17. *Let $1 \leq p \leq \infty$. Then, for all $T \in \mathscr{S}_p$ and all $R \in \mathscr{S}_{p'}$, we have that $TR \in \mathscr{S}_1$, and*

$$\|TR\|_1 \leq \|T\|_p \|R\|_{p'}.$$

A proof of this theorem can be bound in [23, Th. 2.8].

Chapter 3

An introduction to Quantum Mechanics

Quantum Mechanics (QM) is a branch of Physics so powerful that it can describe virtually everything, from subatomic particles to galaxies, [3]. Nonetheless, it was in the study of microphysical phenomena at the end of the nineteenth and early twentieth century that QM was born. The present chapter is a short introduction of the main concepts and results in QM that are necessary for our work. We begin with a historical overview of QM. Then we introduce briefly the formulation of QM and fundamental results such as the Heisenberg uncertainty principle. Finally, we give some properties of the Schrödinger operator. The principal references used for this chapter are: [13], [18], [3], [20], [22], [25] and [12].

3.1 The birth of Quantum Mechanics

The main concepts of Classical Physics (CP) can be simplified into two fundamental ideas: the concept of particle and the concept of an electromagnetic wave, [20]. CP described the world through this ideas; the laws of particle motion were used to understand the dynamics of material bodies and Maxwell's electromagnetism provided the proper framework to study radiation, in particular light. The connection between matter and radiation and how they interact where explained either by the Lorentz force or by thermodynamics, [20] and [18].

The apparent success of describing the world through CP, classical theory of electromagnetism and thermodynamics crumbled in 1900 when Max Planck published his theory of black-body radiation, [20]. Planck stated that the observed properties of black-body radiation can be explained by assuming that atoms emit and absorb discrete *quanta* of radiation, each with energy

$$e = hv, \tag{3.1}$$

where v is the frequency of radiation and h is what would be later known as Planck's constant, [20]. Equation (3.1) suggests that the energy exchanged between an electromagnetic wave with frequency v and matter occurs only in integer multiples of hv, which Planck called the energy of *quantum*, [18].

Planck's work started a domino effect of new discoveries that led to the solutions of the most important problems of the time. In 1905, a solution of the photoelectric problem was proposed by Einstein; inspired by Planck's "quantization" scheme, he proposed that the light itself is made of discrete bits of energy, called *photons*, [18]. The evidence for the existence of photons became compelling in 1923 when Compton showed that the wavelength of an X-ray increases when it is scattered by an atomic electron. If we assume that the scattering process is a photon-electron collision in which energy and momentum are conserved, then the particle behavior of the X-ray photons becomes more apparent, [20] and [18].

The introduction of the hydrogen atom model by Bohr in 1913 successfully explained atomic stability and atomic spectroscopy, among other outstanding problems of the time. The main idea of Bohr's work is that the emission or absorption of radiation by atoms can only take place in discrete amounts since it results from transitions of the atom between its discrete energy states, [18]. The first direct evidence for discrete atomic energy levels was provided by Franck and Hertz in 1914, [3].

In 1923 de Broglie proposed that material particles display a wave-like behaviour; in 1927 it was confirmed experimentally by Davisson and Germer, [18]. Specifically, de Broglie proposed a concept called the de Broglie wavelength, which can be understood as the wavelength of a particle of matter with momentum p, and can be calculated by

$$\lambda = \frac{h}{p}.$$

All of these theoretical and experimental results in the early twentieth century proved that at a microscopical level, waves exhibit a particle-like behaviour, [20] .

There have been several experiments that showed that particles of matter such as photons, do not behave like classical particles with well defined trajectories, one of this type of experiments are the two slit experiments. It is well known, that when presented with two possible trajectories (the two slits), photons seem to pass along both of them and arrive at a random point on the screen, building up an interference pattern. Particles with both particle and wave-like properties are often called quantum particles, [20]. Quantum particles are indeed particles, but their behaviour is very different from what CP would have predicted about them, [3]. It was the search for a theoretical foundation underlying all these new ideas and results that led to Heisenberg and Schrödinger to construct a consistent theory that could explained all Planck's ideas and Bohr's postulates into one refined theory: QM, [18].

Heisenberg and Schrödinger, independently came up with two formulations of QM: matrix mechanics and wave mechanics. Later, it was shown that these two formulations are equivalent. Finally, it was Dirac that suggested a more general formalization of QM which deal more abstract objects such as functions, state vectors and operators, [18].

In summary, QM is the branch of physics that studies the consequences of wave-particle duality of objects, which occurs at the microscopical level. The main phenomena of QM can be summarized under three headings:

(i) discreetness,

(ii) diffraction and

(iii) coherence.

An explanation of this phenomena can be found in [3].

3.2 Operators in Quantum Mechanics

In this section we explain the connection between Hilbert spaces, self-adjoint operators and QM. First, we explain the basics of QM using a single particle system in one dimension. Then, we present two important observables in QM: the position and momentum operators. Furthermore, we give two postulates of QM. Finally, we state the Heisenberg uncertainty principle and the principle of superposition. The main reference for this section is [13].

3.2.1 Basic concepts. The position and momentum operator

In order to explain a number of basic ideas and concepts in QM, we begin by studying a simple system: a single particle in one dimensional space, that is, $\mathbb{R}$.

In QM, the state of a system is described by a function $\psi : \mathbb{R} \to \mathbb{C}$. Since we consider the system at an arbitrary time t, ψ is not time dependent. ψ is related to the probability of a particle being in a given subset $J \subset \mathbb{R}$: we assume that $\psi \in L^2(\mathbb{R})$. More precisely, the probability is

$$\int_J |\psi(q)|^2 dq.$$

This physical interpretation of ψ suggest that it is a normalized vector in $L^2(\mathbb{R})$. Note that the norm of ψ remains unchanged up to a complex rotation. Hence, by defining a state as an element of unit norm in the space $L^2(\mathbb{R})$, we can define an equivalence relation as follows

$$\psi_1 \sim \psi_2 \iff \psi_1 = \alpha\psi_2, \quad |\alpha| = 1.$$

Remark 3.1. *Every state in our system generates a one dimensional space $Y = \{\varphi : \varphi = \beta\psi, \beta \in \mathbb{C}\}$. Hence, a state can also be defined as a 1 dimensional subspace Y of $L^2(\mathbb{R})$. In this case, for our system, we choose an element Y of unit norm in order the define the probability of finding a particle in a particular location on the line.*

Note that $|\psi(\cdot)|^2$ is the density function of a probability distribution on $\mathbb{R}$. Hence, we can define the expected value

$$\mu_\psi = \int_{\mathbb{R}} q|\psi(q)|^2 dq,$$

which characterizes the *average position* of the particle for a given state ψ.

The next natural definition is the *variance* parameter of the distribution

$$\text{var}_\psi = \int_{\mathbb{R}} (q - \mu_\psi)^2 |\psi(q)|^2 dq,$$

which characterizes the *spread of the distribution* for a given state ψ. From this we define the standard deviation as

$$\text{sd}_\psi = \sqrt{\text{var}_\psi}.$$

By defining the operator $Q : \text{Dom}(Q) \to L^2(\mathbb{R})$ such that

$$Q\psi(q) = q\psi(q),$$

we can write the expected value as follows:

$$\mu_\psi(Q) = (Q\psi, \psi) = \int_{\mathbb{R}} Q\psi(q)\overline{\psi(q)}dq.$$

Q is called the *position operator* because of the interpretation of $\mu_\psi(Q)$. Note that $\text{Dom}(Q)$ consists of all $\psi \in L^2(\mathbb{R})$ such that $Q\psi$ stays in $L^2(\mathbb{R})$.

Remark 3.2. *Q is an unbounded, closed, self-adjoint operator whose domain is dense in $L^2(\mathbb{R})$. A more extensive study of this operator can be found in [13].*

We say that Q is an *observable* of our system since it gives quantifiable information about the system at state ψ that we can observe experimentally. Note that in the case of the position operator, it is a self-adjoint linear operator. This suggests that for other observable variables, such as momentum, we should be able to define a self-adjoint operator in the same manner as the position operator. Hence we have the definition

Definition 3.1. *An operator $T : \text{Dom}(T) \to L^2(\mathbb{R})$ is called an observable if and only if T is self-adjoint and* $\text{Dom}(T)$ *is dense in* $L^2(\mathbb{R})$.

Analogous to the position operator, we can define the *expected value* of T

$$\mu_\psi(T) = (T\psi, \psi) = \int_{\mathbb{R}} T\psi(q)\overline{\psi(q)}dq$$

and its *variance*

$$\text{var}_\psi(T) = ((T - \mu I)^2\psi, \psi) = \int_{\mathbb{R}} (T - \mu I)^2\psi(q)\overline{\psi(q)}.$$

A very important observable is the momentum, the corresponding *momentum operator* $D : \text{Dom}(D) \to L^2(\mathcal{R})$ is defined by

$$D\psi = \frac{h}{2\pi i}\frac{d\psi}{dq},$$

where h is Planck's constant. A deeper discussion of the momentum operator can be found in [13].

Example 3.1. *The* energy observable *in QM is described by an operator called the* Hamiltonian operator *and is denoted by $\hat{H}$. If we assume that the energy behaves similarly to that of* *energy in CP, then the Hamiltonian operator for a particle with mass m in a potential energy field V is given by*

$$\hat{H} = -\frac{\hbar}{2m}\Delta + V.$$

The corresponding expected value of the energy of a particle at a given state ψ is

$$\mu_\psi(\hat{H}) = \int_{\mathbb{R}} \hat{H}\psi(q)\overline{\psi(q)}dq.$$

We are ready to present strengthened versions of the first two postulates of QM. The postulates can be found e.g., in [3].

Postulate 1. *To each* observable magnitude *there corresponds a self-adjoint linear operator, and the possible eigenvalues of the observable variable are the eigenvalues of the operator.*

Postulate 2. *To each* state *there corresponds a unique statistical operator, which must be self-adjoint, positive and of unit trace.*

3.2.2 Heisenberg uncertainty principle

Heisenberg uncertainty principle states that in an experiment, we cannot take simultaneous measurements of position and momentum of a particle with an unlimited accuracy. This very important theorem in QM requires some basic concepts. First, we begin by defining the *commutator* of two operators:

Definition 3.2. *Let S, T be any two self-adjoint linear operators defined on a complex Hilbert space. We define the* commutator operator *of S and T by*

$$C = ST - TS$$

with $\mathrm{Dom}(C) = \mathrm{Dom}(ST) \cap \mathrm{Dom}(TS)$.

The following theorem about commutators is a generalization of Heisenberg uncertainty principle:

Theorem 3.1. *Let S, T be any two self-adjoint linear operators with domain and range in $L^2(\mathbb{R})$. Then the commutator C of S, T satisfies*

$$|\mu_\psi(C)| \leq 2\mathrm{sd}_\psi(S)\mathrm{sd}_\psi(T),$$

for every $\psi \in \mathrm{Dom}(C) = \mathrm{Dom}(ST) \cap \mathrm{Dom}(TS)$.

Proof. Let $\psi \in \mathcal{D}(C)$. Denote $\mu_1 = \mu_\psi(S)$ and $\mu_2 = \mu_\psi(T)$. We write $A = S - \mu_1 I$ and $B = T - \mu_2 I$. A simple computation gives

$$C = ST - TS = AB - BA.$$

Since S and T are self-adjoint and μ_1, μ_2 are real, then A and B are self-adjoint as well. Moreover, we have

$$\begin{aligned}\mu_\psi(C) &= \left(C\psi, \psi\right) \\ &= \left(AB\psi, \psi\right) - \left(BA\psi, \psi\right) \\ &= \left(B\psi, A\psi\right) - \left(A\psi, B\psi\right).\end{aligned}$$

Since $\mid \left(B\psi, A\psi\right) \mid = \mid \left(A\psi, B\psi\right) \mid$, by the triangle inequality we have

$$|\mu_\psi(C)| \leq \mid \left(B\psi, A\psi\right) \mid + \mid \left(A\psi, B\psi\right) \mid = 2 \mid \left(B\psi, A\psi\right) \mid.$$

Using the Cauchy-Schwarz inequality yields

$$|\mu_\psi(C)| \leq 2 \|B\psi\| \|A\psi\|.$$

Since B and A are self adjoint,

$$\|B\psi\| = \left(B^2\psi, \psi\right) = \sqrt{\mathrm{var}_\psi(T)}, \quad \|A\psi\| = \left(A^2\psi, \psi\right) = \sqrt{\mathrm{var}_\psi(S)}.$$

This concludes the proof. □

It is straightforward that the commutator operator of the position and the momentum is

$$DQ - QD = \frac{h}{2\pi i}\bar{I},$$

where $\bar{I}$ is the identity operator on the domain of $DQ - QD$. Hence,

$$|\mu_\psi(DQ - QD)| = \frac{h}{2\pi i},$$

since $\|\psi\|_{L^2(\mathbb{R})} = 1$. Therefore, by the last theorem, we have *Heisenberg uncertainty principle*:

Corollary 3.1. *For the position operator Q and momentum operator D,*

$$\mathrm{sd}_\psi(D)\mathrm{sd}_\psi(Q) \geq \frac{h}{4\pi}.$$

Hence, at a given time, we cannot know with precision the momentum and the position of a particle simultaneously.

There is another important principle in QM, the principle of superposition. Mathematically, it is direct if we consider that states of a system are elements of a linear space. This principle asserts that given two states of a quantum system ψ_1, ψ_2 and two complex numbers α_1, α_2, then

$$\alpha_1\psi_1 + \alpha_2\psi_2 \tag{3.2}$$

is also a state of the system. Experimentally, the principle of superposition explains the existence of the interference pattern in the double slit experiment: the interference results from the superposition of the waves emitted by slits 1 and 2. That is, if ψ_1, ψ_2 denote the waves reaching the screen emitted by slit 1 and 2 respectively and each represent two physically possible states of the system, then any linear superposition (3.2) also represents a physically possible outcome of the system, [18] and [12].

3.3 Schrödinger operators

Quantum systems are well described by linear differential operators called *Schrödinger operators*. Specifically, these are operators acting on $L^2(\mathbb{R}^n)$ of the form

$$H = \frac{-h^2}{2m}\Delta + V,$$

where $h = 2\pi\hbar$ is Planck's constant, m is the mass of the particle in the system and V is a real-valued function called the potential. A function in $L^2(\mathbb{R}^N)$ that represents a state of the system is called a *wave function*, [12]. The time evolution of a wave function for a quantum system with Schrödinger H is controlled by Schrödinger's equation

$$i\hbar\psi_t = H\psi_t. \tag{3.3}$$

The role of Schrödinger equation in QM is analogous to that of Newton's law in CP, [20]. Equation (3.3) and example 3.1 suggests that the energy operator governs the time evolution of the wave function. Therefore, in QM there is a fundamental connection between energy and time, [20] and [12] .

Let us now state some properties of Schrödinger operators. Assume that $V \in \mathrm{L}^2_{loc}(\mathbb{R}^N)$. We define the Schrödinger operator $H = -\Delta + V$ on

$$\mathrm{Dom}(H) = \mathrm{Dom}(\Delta) \cap \mathrm{Dom}(V),$$

where $\mathrm{Dom}(\Delta) = \mathrm{H}^2(\Omega)$ and

$$\mathrm{Dom}(V) = \{f \in \mathrm{L}^2(\mathbb{R}^N) / \int_{\mathbb{R}^N} |V(x)f(x)|^2 dx < \infty\}.$$

We take this domain since H is symmetric on $\mathrm{Dom}(H)$, that is

$$(Hu, v) = (u, Hv)$$

and $\mathrm{Dom}(H) \subset \mathrm{Dom}(H^*)$. Moreover, note that $\mathrm{C}_0^\infty(\Omega) \subset \mathrm{Dom}(\Delta)$, so H is densely defined.

The next theorem gives a necessary condition for H to be essentially self-adjoint, that is, its smallest closed extension, denoted by $\overline{H}$, is self-adjoint.

Theorem 3.2. *Let $V \in \mathrm{L}^2_{loc}(\mathbb{R}^N)$ and $V \geq 0$. Then H is essentially self-adjoint on $\mathrm{C}_0^\infty(\Omega)$.*

A proof of this theorem can be found in [12, Th. 8.14].

The final result that we state simplifies the spectrum of H when V has certain properties

Theorem 3.3. *Assume that $v \geq 0$ and $V \in \mathrm{L}^2_{loc}(\mathbb{R}^N)$. Furthermore, if*

$$V(x) \to \infty \text{ as } \|x\| \to \infty$$

then H has purely discrete spectrum.

A proof of this theorem can be found in [12, Th. 10.7].

Chapter 4

Results

In this section we shall extend the results proven in [8] and [17], when $p \geq 2$. Mainly, we shall prove some compactness properties of our operator setting to minimize a *free energy functional*.

4.1 Sobolev-like cones

Throughout this work, we shall assume that

$$2 \leq p < N, \quad N \geq 3$$

and

(VΩ) $\Omega \subseteq \mathbb{R}^N$ is open and bounded with boundary C^1, and $V \in L^\infty(\Omega)$ is non-negative, i.e.,

$$V(x) \geq 0, \qquad \text{for a.e. } x \in \Omega.$$

Let us recall the following spaces of operators

$$\begin{array}{rcl} \mathscr{L} & = & \mathscr{L}(L^2(\Omega)) = \{T : L^2(\Omega) \to L^2(\Omega) \ / \ T \text{ is linear and bounded}\}, \\ \mathscr{L}_S & = & \mathscr{L}_S(L^2(\Omega)) = \{T \in \mathscr{L} \ / \ T \text{ is self-adjoint}\}, \\ \mathscr{I}_\infty & = & \mathscr{I}_\infty(L^2(\Omega)) = \{T \in \mathscr{L} \ / \ T \text{ is compact}\}, \\ \mathscr{S}_\infty & = & \mathscr{S}_\infty(L^2(\Omega)) = \{T \in \mathscr{I}_\infty \ / \ T \text{ is self-adjoint}\} \end{array}$$

which were defined and studied in previous chapters. We shall deal with nuclear class operators $T \in \mathscr{S}_1(L^2(\Omega))$, defined in Example 2.4.

Remark 4.1. *By the Riesz-Schauder theorem (Theorem 2.28) and Hilbert-Schmidt theorem (Theorem 2.29), given* $T \in \mathscr{S}_\infty$, *there exists* $(\nu_{i,T}, \eta_{i,T})_{i \in \mathbb{N}} \subseteq \mathbb{R} \times L^2(\Omega)$ *such that*

$$\forall i \in \mathbb{N}: \quad T\eta_i = \nu_{i,T}\, \eta_{i,T},$$

and

$$B = \{\eta_{i,T} \ / \ i \in \mathbb{N}\}$$

is a Hilbert basis of $\mathrm{L}^2(\Omega)$*. In this context it is assumed that*

$$\forall i,j \in \mathbb{N}: \quad i<j \implies |\nu_{i,T}| \geq |\nu_{j,T}|.$$

Also, if for some $\nu > 0$ *both* ν *and* $-\nu$ *are eigenvalues, then the index of* $-\nu$ *is less than that of* ν*. When there is no confusion we shall write* ν_i *instead of* $\nu_{i,T}$ *and* η_i *instead of* $\eta_{i,T}$*.*

We denote by $\mathrm{W}^{1,p}_V(\Omega)$ the completion of $\mathrm{C}^\infty_0(\Omega)$ in the norm

$$\|\psi\|_{V,p} = \left(\int_\Omega |\nabla\psi(x)|^p dx + \int_\Omega V(x)|\psi(x)|^p dx\right)^{1/p}.$$

Remark 4.2. *From* $(V\Omega)$ *we deduce that*

$$\int_\Omega V(x)|\psi(x)|^p dx \leq \|V\|_{\mathrm{L}^\infty(\Omega)} \|\psi\|^p_{\mathrm{L}^p(\Omega)}.$$

Since $\mathrm{L}^p(\Omega)$ *is complete and* $\|\cdot\|_{(\mathrm{L}^p(\Omega);V(x)dx)}$, $\|\cdot\|_{\mathrm{L}^p(\Omega)}$ *are comparable, then they are equivalent norms. This result comes from a corollary of the Open Mapping Theorem (see e.g., [13]). Then,* $\|\cdot\|_{V,p}$ *is equivalent to the* $\mathrm{W}^{1,p}_0(\Omega)$*- norm. Thus*

$$\mathrm{W}^{1,p}_V(\Omega) = \left(\mathrm{W}^{1,p}_0(\Omega), \|\cdot\|_{V,p}\right).$$

Hence, we have the following results,

1. ***Rellich-Kondrachov embedding.*** *For all* $q \in [1,p^*[$*, the embedding*

$$\mathrm{W}^{1,p}_V(\Omega) \subseteq \mathrm{L}^q(\Omega)$$

holds and it's compact. Then, it follows that there exists $S_{p,q} > 0$ *such that*

$$\forall u \in \mathrm{W}^{1,p}_V(\Omega): \quad \|u\|_{\mathrm{L}^q(\Omega)} \leq S_{p,q}\|u\|_{V,p}. \tag{4.1}$$

2. ***Poincaré's inequality.*** *For all* $q \in [1,p^*]$ *and* $2 \leq p < N$*, we have that there exists* $C_{p,q} > 0$ *such that*

$$\forall u \in \mathrm{W}^{1,p}_V(\Omega): \|u\|_{\mathrm{L}^q(\Omega)} \leq C_{p,q} \|\nabla u\|_{\mathrm{L}^p(\Omega)}.$$

In particular, for all $2 \leq p \leq \infty$

$$\forall u \in \mathrm{W}^{1,p}_V(\Omega): \|u\|_{\mathrm{L}^p(\Omega)} \leq C_p \|\nabla u\|_{\mathrm{L}^p(\Omega)}.$$

Note that these results are Theorem 2.13 and Theorem 2.14 written in the context of $\mathrm{W}^{1,p}_V(\Omega)$*.*

Remark 4.3. *By the embedding* $L^p(\Omega) \subseteq L^2(\Omega)$, *Hölder's inequality and Poincaré's inequality, we see that*

$$\begin{aligned}
&\int_\Omega |\nabla\psi(x)|^2 + V(x)|\psi(x)|^2 dx \\
&\leq |\Omega|^{(p-2)/p}\left(\int_\Omega |\nabla\psi(x)|^p dx\right)^{2/p} + |\Omega|^{(p-2)/p} C_p^2 \|V\|_{L^\infty(\Omega)} \left(\int_\Omega |\nabla\psi(x)|^p dx\right)^{2/p} \\
&\leq |\Omega|^{(p-2)/p} \max\{1, C_p^2 \|V\|_{L^\infty(\Omega)}\} \left(\int_\Omega |\nabla\psi(x)|^p dx\right)^{2/p} \\
&\leq \hat{C}^{2/p}\left(\int_\Omega |\nabla\psi(x)|^p dx + \int_\Omega V(x)|\psi(x)|^p dx\right)^{2/p}
\end{aligned}$$

where,

$$\hat{C}^{2/p} = \begin{cases} |\Omega|^{(p-2)/p} \max\{1, C_p^2 \|V\|_{L^\infty(\Omega)}\} & , \quad \text{if } p > 2. \\ 1 & , \quad \text{if } p = 2. \end{cases}$$

Then we have proved that

$$\forall \psi \in W_0^{1,p}(\Omega): \quad \|\psi\|_{V,2} \leq \hat{C}^{1/p} \|\psi\|_{V,p}.$$

Note that $\hat{C}$ *only depends on* $2, p, V, \Omega$.

We denote by $\mathscr{B}_T$ the set of eigenbasis of $L^2(\Omega)$ associated to an operator $T \in \mathscr{S}_\infty$ (Remark 4.1). So that

$$\forall T \in \mathscr{S}_\infty: \quad \mathscr{B}_T \neq \emptyset.$$

We write

$$\mathscr{B}_T^p = \left\{ B = \{\eta_i \ / \ i \in \mathbb{N}\} \in \mathscr{B}_T \ / \quad (\eta_i)_{i\in\mathbb{N}} \subseteq W_0^{1,p}(\Omega) \right\}.$$

With all of these concepts in mind, we are ready to define our operator setting. This definition is a generalization of [8, Def. 2.1] and [17, Def. 2.1] for $p \geq 2$.

Definition 4.1. *We say that an operator* $T \in \mathscr{S}_1$ *is in the* Sobolev-like cone $\mathscr{W}^p$ *if and only if* $\mathscr{B}_T^p \neq \emptyset$ *and*

$$\inf_{B\in\mathscr{B}_T^p} \sum_{i\in\mathbb{N}} |\nu_{i,T}| \|\eta_i\|_{V,p}^p < \infty. \tag{4.2}$$

Given $T \in \mathscr{W}^p$ and $B = \{\eta_i \ / \ i \in \mathbb{N}\} \in \mathscr{B}_T^p$, we refer to

$$\langle T \rangle_{V,B} = \sum_{i\in\mathbb{N}} |\nu_{i,T}| \cdot \|\eta_i\|_{V,p}^p$$

as the *B-energetic value of* T. We say that

$$\langle\langle T \rangle\rangle_V = \inf_{B\in\mathscr{B}_T^p} \langle T \rangle_{V,B}, \tag{4.3}$$

is the *energy of the operator* T.

Remark 4.4. *In the case $p = 2$, it is not necessary to take infimum over $\mathscr{B}_T^2$ since*

$$\sum_{i\in\mathbb{N}} |\nu_{i,T}| \cdot \|\eta_i\|_{V,2}^2$$

is independent of the choice of Hilbert basis, [8]. Nonetheless, note that (4.2) implies that there exists $B \in \mathscr{B}_T^p$ such that

$$\langle T\rangle_{V,B} < \infty.$$

The following proposition presents some basic results about $\mathscr{W}^p$ that justify the term cone.

Proposition 4.1. *Let $T \in \mathscr{W}^p$ and $\alpha \in \mathbb{R}$. Then $\alpha T \in \mathscr{W}^p$,*

$$\langle\langle \alpha T\rangle\rangle_V = |\alpha| \cdot \langle\langle T\rangle\rangle_V, \tag{4.4}$$

and

$$\langle\langle \alpha T\rangle\rangle_V = 0 \quad \Longleftrightarrow \quad (\alpha = 0 \vee T = 0). \tag{4.5}$$

Proof. If $T \in \mathscr{W}^p$, then $\mathscr{B}_T^p \neq \emptyset$. Hence, there exists $B = \{\eta_i \ / \ i \in \mathbb{N}\} \in \mathscr{B}_T^p$. Moreover, for any $i \in \mathbb{N}$:

$$\alpha T\eta_{i,T} = \alpha\nu_{i,T}\eta_{i,T},$$

which implies that

$$\forall i \in \mathbb{N}: \quad \nu_{i,\alpha T} = \alpha\nu_{i,T}, \quad \eta_{i,\alpha T} = \eta_{i,T}.$$

Thus, $\alpha T \in \mathscr{W}^p$. Let $B \in \mathscr{B}_T^p$ such that $\langle T\rangle_{V,B} < \infty$. Then

$$\begin{aligned}\langle \alpha T\rangle_{V,B} &= \sum_{i\in\mathbb{N}} |\alpha\nu_{i,T}| \cdot \|\eta_i\|_{V,p}^p \\ &= |\alpha| \sum_{i\in\mathbb{N}} |\nu_{i,T}| \cdot \|\eta_i\|_{V,p}^p \\ &= |\alpha|\langle T\rangle_{V,B}.\end{aligned}$$

Since this holds for all $B \in \mathscr{B}_T^p$, we have that

$$\langle\langle \alpha T\rangle\rangle_V = |\alpha|\langle\langle T\rangle\rangle_V.$$

We conclude the proof of (4.4) by the arbitrariness of T and α.

Now, by (4.4) it is clear that

$$(\alpha = 0 \vee T = 0) \implies \langle\langle \alpha T\rangle\rangle_V = 0$$

and

$$\langle\langle \alpha T\rangle\rangle_V = 0 \implies (\alpha = 0 \vee \langle\langle T\rangle\rangle_V = 0).$$

We have to prove that

$$\langle\langle T\rangle\rangle_V = 0 \implies T = 0$$

in order to conclude our proof. Indeed, setting $\langle\langle T\rangle\rangle_V = 0$ yields

$$\inf_{B\in\mathscr{B}_T^p} \sum_{i\in\mathbb{N}} |\nu_{i,T}| \cdot \|\eta_i\|_{V,p}^p = 0.$$

so that for all $\epsilon > 0$, there exists some $B = \{\eta_i \ / \ i \in \mathbb{N}\} \in \mathscr{B}_T^p$ such that

$$\sum_{i\in\mathbb{N}} |\nu_{i,T}| \cdot \|\eta_i\|_{V,p}^p < \epsilon.$$

Since this holds for any $\epsilon > 0$, the above implies

$$\forall i \in \mathbb{N}: \quad |\nu_{i,T}| \cdot \|\eta_i\|_{V,p}^p = 0.$$

The η_i's are eigenfunctions, so that $\eta_i \neq 0$. We deduce that

$$\forall i \in \mathbb{N}: \quad |\nu_{i,T}| = 0$$

whence

$$\|T\|_1 = \sum_{i\in\mathbb{N}} |\nu_{i,T}| = 0 \implies T = 0.$$

□

Next, we present an estimate of the trace norm of an operator $T \in \mathscr{W}^p$ by its energy which shall be very useful.

Proposition 4.2. *Assume ($V\Omega$). Then*

$$\forall T \in \mathscr{W}^p: \quad \|T\|_1 \leq K \cdot C_p \langle\langle T\rangle\rangle_V, \tag{4.6}$$

where C_p is Poincaré's constant and K is given by the embedding $\mathrm{L}^p(\Omega) \subseteq \mathrm{L}^2(\Omega)$.

Proof. Let $T \in \mathscr{W}^p$ and $B = \{\eta_i \ / \ i \in \mathbb{N}\} \in \mathscr{B}_T^p$. Then, by the embedding $\mathrm{L}^p(\Omega) \subseteq \mathrm{L}^2(\Omega)$

$$\forall i \in \mathbb{N}: \quad |\nu_{i,T}| \leq K|\nu_{i,T}| \left\|\eta_i\right\|_{\mathrm{L}^p(\Omega)}^p.$$

Using Poincaré's inequality

$$\begin{aligned} \exists C_p > 0, \forall i \in \mathbb{N}: \quad |\nu_{i,T}| &\leq KC_p|\nu_{i,T}| \cdot \left\|\nabla\eta_i\right\|_{\mathrm{L}^p(\Omega)}^p \\ &\leq KC_p|\nu_{i,T}| \left(\left\|\nabla\eta_i\right\|_{\mathrm{L}^p(\Omega)}^p + \|\eta_i\|_{(\mathrm{L}^p(\Omega);V(x)dx)}^p\right) \end{aligned}$$

whence,

$$\begin{aligned} \|T\|_1 &= \sum_{i\in\mathbb{N}} |\nu_{i,T}| \\ &\leq \sum_{i\in\mathbb{N}} KC_p|\nu_{i,T}| \left(\|\nabla\eta_i\|^p_{\mathrm{L}^p(\Omega)} + \|\eta_i\|^p_{(\mathrm{L}^p(\Omega);V(x)dx)} \right) \\ &= KC_p\langle T\rangle_{V,B}, \end{aligned}$$

which, by the arbitrariness of B and T, implies (4.6). □

In QM positive nuclear operators are important since they can represent a physical state of a system. For example, [2] uses a positive nuclear operator acting on $\mathrm{L}^2(\mathbb{R}^3)$ to investigate a *non-relativistic gravitational Hartree system*. In this case, the operator represented a system of gravitating quantum particles.

In our context, let's consider the cone

$$\mathscr{W}^p_+ = \{T \in \mathscr{W}^p \ / \ T \geq 0\}.$$

We define the *B-kinetic energetic value* of T for $B = \{\eta_i \ / \ i \in \mathbb{N}\} \in \mathscr{B}^p_T$,

$$\mathscr{K}_{p,B}(T) = \sum_{i\in\mathbb{N}} \nu_{i,T} \int_\Omega |\nabla\eta_i(x)|^p dx$$

and the *p-kinetic energy functional* of T, $\mathscr{K}_p(\cdot) : \mathscr{W}^p_+ \to \mathbb{R}$, by

$$\mathscr{K}_p(T) = \inf_{B\in\mathscr{B}^p_T} \mathscr{K}_{p,B}(T).$$

Furthermore, we define the *(p,V,B)-potential energetic value* of T as follows

$$\mathscr{P}_{p,V,B}(T) = \sum_{i\in\mathbb{N}} \nu_{i,T} \int_\Omega V(x)|\eta_i(x)|^p dx$$

and similarly to the p-kinetic energy, we define the *(p,V)-potential energy* of T, $\mathscr{P}_{p,V}(\cdot) : \mathscr{W}^p_+ \to \mathbb{R}$, by

$$\mathscr{P}_{p,V}(T) = \inf_{B\in\mathscr{B}^p_T} \mathscr{P}_{p,V,B}(T),$$

Remark 4.5. *Let's observe that for $T \in \mathscr{W}^p_+$ and $B = \{\eta_i \ / \ i \in \mathbb{N}\} \in \mathscr{B}^p_T$, we formally have*

$$\begin{aligned} \mathrm{Tr}_B[-\Delta_p T] &= \sum_{i\in\mathbb{N}} (\eta_i, -\Delta_p T\eta_i)_{\mathrm{L}^2(\Omega)} \\ &= \sum_{i\in\mathbb{N}} \nu_{i,T} \int_\Omega \eta_i(x) \cdot [-\Delta_p\eta_i(x)]dx \\ &= \sum_{i\in\mathbb{N}} \nu_{i,T} \int_\Omega |\nabla\eta_i(x)|^p dx \\ &= \mathscr{K}_{p,B}(T). \end{aligned} \tag{4.7}$$

Recall that the trace is a linear functional defined only for linear operators. In this case we are using a non-linear extension *of the trace because $\Delta_p T$ is not a linear operator. If $V \in \mathrm{L}^1_{loc}(\Omega)$ and $T \in \mathscr{W}^p_+$ is such that $\rho_T V \in \mathrm{L}^1(\Omega)$, then we have the following expression for the* 2-potential energy of *T:*

$$\mathscr{P}_{2,V}(T) = \mathrm{Tr}\left[VT\right]. \tag{4.8}$$

In this case, V is understood as the multiplication operator:

$$V : \mathrm{L}^2(\Omega) \to \mathrm{L}^2(\Omega)$$

with $V(u) = V \cdot u$.

Remark 4.6. *For any $T \in \mathscr{W}^p_+$, we can write the energy of the operator T as*

$$\langle\langle T \rangle\rangle_V = \inf_{B \in \mathscr{B}^p_T} \left[\mathscr{K}_{p,B}(T) + \mathscr{P}_{p,V,B}(T)\right]$$

Moreover, by properties of the infimum we have that

$$\begin{aligned} \langle\langle T \rangle\rangle_V &\geq \inf_{B \in \mathscr{B}^p_T} \mathscr{K}_{p,B}(T) + \inf_{B \in \mathscr{B}^p_T} \mathscr{P}_{p,V,B}(T) \\ &= \mathscr{K}_p(T) + \mathscr{P}_{p,V}(T). \end{aligned} \tag{4.9}$$

We feel that that (4.9) *is a manifestation of loss of energy due to the generalization we are accomplishing. In the case $p = 2$, the conservation of energy (equality in (4.9)) is preserved since the case $p = 2$ has strong connection with bilineal forms.*

For each $T \in \mathscr{S}_1$ such that $T \geq 0$ we can associate a function function $\rho_T : \Omega \to \mathbb{R}$ given by

$$\rho_T(x) = \sum_{i \in \mathbb{N}} |\nu_{i,T}||\eta_i(x)|^2.$$

ρ_T is called the *density function associated to T* and it does not depend on the choice of Hilbert basis. Moreover, we have that $\rho \in \mathrm{L}^1(\Omega)$:

$$\int_\Omega \rho_T(x)dx = \|T\|_1.$$

Our first result gives regularity properties to ρ_T when $T \in \mathscr{W}^p$. The first step is to prove the following integrability result about $\nabla\rho_T$:

Theorem 4.1. *Let $N \geq 3$, $2 \leq p < N$ and $T \in \mathscr{W}^p$. Then*

$$\forall r \in \left[p/2, pN/2(N-1)\right] : \quad \left\|\nabla\rho_T\right\|_{\mathrm{L}^r(\Omega)} \leq C\langle\langle T \rangle\rangle_V$$

where C only depends on p, N, r, B and Ω. Therefore $\nabla\rho_T \in \mathrm{L}^r(\Omega)$.

Proof. Let $B = \{\eta_i \,/\, i \in \mathbb{N}\} \in \mathscr{B}_T^p$ such that $\langle T \rangle_{V,B} < \infty$. Clearly,

$$\nabla \rho_T(x) = 2 \sum_{i \in \mathbb{N}} \nu_i |\eta_i(x)| \nabla \eta_i(x).$$

Then

$$\begin{aligned}
\int_\Omega |\nabla \rho_T(x)|^r dx &\leq 2^r \int_\Omega \left(\sum_{i \in \mathbb{N}} |\nu_i| |\eta_i(x)| |\nabla \eta_i(x)| \right)^r dx \\
&= 2^r \int_\Omega \left[\sum_{j \in \mathbb{N}} |\nu_j| \sum_{i \in \mathbb{N}} \left(\frac{|\nu_i|}{\sum_{j \in \mathbb{N}} |\nu_j|} \right) |\eta_i(x)| |\nabla \eta_i(x)| \right]^r dx \\
&= 2^r \|T\|_1^r \int_\Omega \left[\sum_{i \in \mathbb{N}} \left(\frac{|\nu_i|}{\sum_{j \in \mathbb{N}} |\nu_j|} \right) |\eta_i(x)| |\nabla \eta_i(x)| \right]^r dx.
\end{aligned}$$

Using the convexity of $s \mapsto s^r$ and Jensen's inequality:

$$\begin{aligned}
\int_\Omega |\nabla \rho_T(x)|^r dx &\leq 2^r \|T\|_1^r \int_\Omega \sum_{i \in \mathbb{N}} \left(\frac{|\nu_i|}{\sum_{j \in \mathbb{N}} |\nu_j|} \right) |\eta_i(x)|^r |\nabla \eta_i(x)|^r dx. \\
&= 2^r \|T\|_1^{r-1} \sum_{i \in \mathbb{N}} |\nu_i| \int_\Omega |\eta_i(x)|^r |\nabla \eta_i(x)|^r dx.
\end{aligned}$$

By assumption, we have that $p/r \geq 1$ and $p/(p-r) \geq 1$. Furthermore:

$$\frac{1}{p/r} + \frac{1}{p/(p-r)} = 1$$

whence, by Hölder's inequality we formally have that

$$\int_\Omega |\nabla \rho_T(x)|^r dx \leq 2^r \|T\|_1^{r-1} \sum_{i \in \mathbb{N}} |\nu_i| \left(\int_\Omega |\nabla \eta_i(x)|^p dx \right)^{r/p} \left(\int_\Omega |\eta_i(x)|^{pr/(p-r)} dx \right)^{1-r/p}. \tag{4.10}$$

In order to justify this inequality, we need to show that

$$\int_\Omega |\eta_i(x)|^{pr/(p-r)} dx < \infty.$$

In fact, using Poincaré's inequality since $p \leq q = pr/(p-r) \leq p^*$:

$$\exists C_{p,q} > 0, \forall i \in \mathbb{N}: \quad \|\eta_i\|_{\mathrm{L}^{pr/(p-r)}(\Omega)} dx \leq C_{p,q} \|\nabla \eta_i\|_{\mathrm{L}^p(\Omega)}. \tag{4.11}$$

This implies that there exists $K_1 > 0$ such that

$$\forall i \in \mathbb{N}: \quad \int_\Omega |\eta_i(x)|^{pr/(p-r)} dx \leq K_1. \tag{4.12}$$

In fact, by (4.11), we have that

$$\exists C_{p,q} > 0, \forall i \in \mathbb{N}: \quad \left(\int_\Omega |\eta_i(x)|^{pr/(p-r)} dx \right)^{(p-r)/r} \leq C_{p,q}^p \int_\Omega |\nabla \eta_i(x)|^p dx. \tag{4.13}$$

Multiplying both sides in (4.13) by $|\nu_i|$ gives

$$\exists C_{p,q} > 0, \forall i \in \mathbb{N}: \quad |\nu_i| \left(\int_\Omega |\eta_i(x)|^{pr/(p-r)} dx \right)^{(p-r)/r} \leq C_{p,q}^p |\nu_i| \int_\Omega |\nabla \eta_i(x)|^p dx. \tag{4.14}$$

Since $T \in \mathscr{W}^p$, we have that

$$\sum_{i\in\mathbb{N}} |\nu_i| \int_\Omega |\nabla \eta_i(x)|^p dx < \infty \implies \lim_{i\to\infty} |\nu_i| \int_\Omega |\nabla \eta_i(x)|^p dx = 0.$$

Therefore, the sequence $(|\nu_i| \, \|\nabla \eta_i\|_{L^p(\Omega)}^p)_{i\in\mathbb{N}}$ is bounded. Hence, by (4.14)

$$\exists M > 0, \exists C_{p,q} > 0, \forall i \in \mathbb{N}: \quad |\nu_i| \left(\int_\Omega |\eta_i(x)|^{pr/(p-r)} dx \right)^{(p-r)/r} \leq C_{p,\Omega}^p \cdot M.$$

Furthermore, since $T \in \mathscr{S}_1$, $\left(|\nu_i|\right)_{i\in\mathbb{N}}$ is bounded as well. Then

$$\exists M > 0, \exists C_{p,q} > 0, \forall i \in \mathbb{N}: \quad C' \left(\int_\Omega |\eta_i(x)|^{pr/(p-r)} dx \right)^{(p-r)/r} \leq C_{p,\Omega}^p \cdot M,$$

where $C' = \sup_{i\in\mathbb{N}} |\nu_i|$. Therefore, by setting $K_1 = \left[M \cdot C_{p,q}^p / C' \right]^{r/(p-r)}$ we have (4.12).

Now, since $(N-2)/N \leq (p-r)/r \leq 1$ and by (4.12), we have that

$$\frac{1}{K_1} \int_\Omega |\eta_i(x)|^{pr/(p-r)} dx \leq \left(\frac{1}{K_1} \int_\Omega |\eta_i(x)|^{pr/(p-r)} dx \right)^{(p-r)/r},$$

whence, by (4.13)

$$K_1^{(p/r-2)} \int_\Omega |\eta_i(x)|^{pr/(p-r)} dx \leq C_{p,q}^p \int_\Omega |\nabla \eta_i(x)|^p dx.$$

Hence, we have that

$$\left(\int_\Omega |\eta_i(x)|^{pr/(p-r)} dx \right)^{1-r/p} \leq \gamma \left(\int_\Omega |\nabla \eta_i(x)|^p dx \right)^{1-r/p} < \infty, \tag{4.15}$$

where $\gamma = \left[C_{p,q}^p K_1^{2-p/r} \right]^{1-r/p}$. We can use (4.15) in (4.10) to obtain

$$\begin{aligned} &\int_\Omega |\nabla \rho_T(x)|^r dx \\ &\leq 2^r \gamma \|T\|_1^{r-1} \sum_{i\in\mathbb{N}} |\nu_i| \left(\int_\Omega |\nabla \eta_i(x)|^p dx \right)^{r/p} \left(\int_\Omega |\nabla \eta_i(x)|^p dx \right)^{1-r/p} \\ &= 2^r \gamma \|T\|_1^{r-1} \sum_{i\in\mathbb{N}} |\nu_i| \int_\Omega |\nabla \eta_i(x)|^p dx \\ &\leq 2^r \|T\|_1^{r-1} \gamma \langle T \rangle_{V,B}. \end{aligned} \tag{4.16}$$

Since $B = \{\eta_i \ / \ i \in \mathbb{N}\} \in \mathscr{B}_T^p$ was chosen arbitrary, by (4.16) we have that

$$\int_\Omega |\nabla \rho_T(x)|^r dx \leq 2^r \|T\|_1^{r-1} \gamma \langle\langle T \rangle\rangle_V. \tag{4.17}$$

Finally, by Proposition 4.2 there exists constants K, C_p such that

$$\|T\|_1^{r-1} \leq (KC_p)^{r-1} \langle\langle T \rangle\rangle_V^{r-1}. \tag{4.18}$$

Then, by (4.16) and (4.18):

$$\left\| \nabla \rho_T \right\|_{\mathrm{L}^r(\Omega)} \leq 2\gamma^{1/r} (KC_p)^{1-1/r} \langle\langle T \rangle\rangle_V.$$

Since T and r were chosen arbitrary, we conclude the proof. □

This theorem shall help us proof the following result:

Theorem 4.2. *Let $N \geq 3$, $2 \leq p < N$ and $T \in \mathscr{W}^p$. Then*

$$\forall r \in \left[p/2, pN/2(N-1)\right], \forall q \in \left[1, pN/2(N-(1+p/2))\right]: \quad \rho_T \in \mathrm{W}^{1,r}(\Omega) \cap \mathrm{L}^q(\Omega).$$

Proof. Let's prove

$$\forall q \in \left[1, pN/2(N-(1+p/2))\right]: \quad \rho_T \in \mathrm{L}^q(\Omega). \tag{4.19}$$

Using the interpolation inequality, we only need to prove that

$$\rho_T \in \mathrm{L}^1(\Omega) \cap \mathrm{L}^{pN/2(N-(1+p/2))}(\Omega).$$

We already know that $\rho_T \in \mathrm{L}^1(\Omega)$. Then it remains to prove that $\rho_T \in \mathrm{L}^{pN/2(N-2)}(\Omega)$. Indeed, by setting

$$b = \frac{p}{2}\left(\frac{N}{N-1}\right) < N, b^* = \frac{p}{2}\left(\frac{N}{N-(1+p/2)}\right)$$

we have by Poincaré's inequality and Theorem 4.1 that

$$\left\|\rho_T\right\|_{\mathrm{L}^{b^*}(\Omega)} \leq C_p \left\|\nabla \rho_T\right\|_{\mathrm{L}^b(\Omega)} \leq C_p C \langle\langle T \rangle\rangle_V$$

where C_p is Poincaré constant and $C = 2\gamma^{1/r}(KC_p)^{1-1/r}$ is given in the proof of the same Theorem. Then, by the interpolation inequality, we have proved (4.19). Finally, by (4.19) and Theorem 4.1, we conclude the proof. □

4.2 Free energy functionals

Similar to [8] and [17] we define Casimir classes of functions. As we shall see, these functions generate *entropy functionals* which subsequently shall help us define *free energy functionals*.

For a potential $V \in L^{\infty}(\Omega)$ and $\alpha > 0$, consider the following kind of conditions:

(V_α) The operator $-\alpha\Delta + V$, with Dirichlet boundary conditions, has a sequence of eigenelements

$$\left\{ (\hat{\lambda}_{V,i}^{(\alpha)}, \hat{\phi}_{V,i}^{(\alpha)}) \right\}_{i \in \mathbb{N}} \subseteq \mathbb{R} \times H_0^1(\Omega) \tag{4.20}$$

such that $\{\hat{\phi}_{V,i}^{(\alpha)} \ / \ i \in \mathbb{N}\}$ is a Hilbert basis of $L^2(\Omega)$ and $(\hat{\lambda}_{V,i}^{(\alpha)})_{i \in \mathbb{N}}$ verifies

$$0 < \hat{\lambda}_{V,1}^{(\alpha)} < \hat{\lambda}_{V,2}^{(\alpha)} \leq \cdots \leq \hat{\lambda}_{V,i}^{(\alpha)} \leq \cdots \tag{4.21}$$

and

$$\lim_{i \to \infty} \hat{\lambda}_{V,i}^{(\alpha)} = +\infty. \tag{4.22}$$

For the sake of simplicity, we shall write $\hat{\lambda}_{V,i}$ and $\hat{\phi}_{V,i}$ instead of $\hat{\lambda}_{V,i}^{(1)}, \hat{\phi}_{V,i}^{(1)}$, respectively.

Remark 4.7. *It is well known, (see e.g., [6, Th. 9.31]) that there exists a sequence of eigenelements of the* Laplacian operator $-\Delta$, *which we denote by*

$$\left\{ (\hat{\lambda}_{0,\ell}, \hat{\phi}_{0,\ell}) \right\}_{\ell \in \mathbb{N}}$$

such that $\{\hat{\phi}_{0,\ell} \ / \ \ell \in \mathbb{N}\} \subseteq H_0^1(\Omega)$ *is a Hilbert basis of* $L^2(\Omega)$ *and* (4.21), (4.22) *hold with* $\alpha = 1, V = 0$. *Furthermore, since* $V \geq 0$, *we have that*

$$-\Delta \leq -\Delta + V$$

in the operator sense. Therefore, we see that

$$\forall i \in \mathbb{N}: \quad \hat{\lambda}_{0,i} \leq \hat{\lambda}_{V,i}.$$

Then, the sequence $(\hat{\lambda}_{V,i})_{i \in \mathbb{N}}$ *diverges since* $(\hat{\lambda}_{0,i})_{i \in \mathbb{N}}$ *diverges.*

Definition 4.2. *Assume that* V *and* α *satisfy* (V_α). *A function* $F : \mathbb{R} \to \mathbb{R} \cup \{\infty\}$ *belongs to the* p-Casimir class $\mathscr{C}_{p,V}^{(\alpha)}$ *if it is convex, non-increasing on* $(0, \infty)$ *and*

$$\sum_{i \in \mathbb{N}} F([\hat{C}^{-1} \hat{\lambda}_{V,i}^{(\alpha)}]^{p/2}) < \infty, \tag{4.23}$$

where $\hat{C}$ *is the constant defined on Remark 4.3. If* $\alpha = 1$, *we write* $\mathscr{C}_{p,V}$ *instead of* $\mathscr{C}_{p,V}^{(1)}$.

Example 4.1. *Let $\gamma > N/p$. We claim that the function $F : \mathbb{R} \to \mathbb{R} \cup \{\infty\}$ given by*

$$F(s) = \begin{cases} s^{-\gamma} & , \quad \text{if } s \geq 0, \\ \infty & , \quad \text{if } s < 0 \end{cases}$$

belongs to the Casimir class $\mathscr{C}_{p,0}$ for all $2 < p < \infty$. In fact, by [19, Th. 1.3.1] we have that there exists constants $c(\Omega), C(\Omega) > 0$ depending only on the domain Ω such that

$$\forall i \in \mathbb{N} : \quad c(\Omega) i^{2/N} \leq \lambda_{0,i} \leq C(\Omega) i^{2/N}.$$

This implies that

$$\sum_{i \in \mathbb{N}} (\lambda_0, i)^{-\gamma p/2} \leq [c(\Omega)]^{-\gamma p/2} \sum_{i \in \mathbb{N}} i^{-p\gamma/N} \tag{4.24}$$

and, since $\gamma > N/p$

$$\sum_{i \in \mathbb{N}} i^{-p\gamma/N} < \infty.$$

Then, we have that

$$\sum_{i \in \mathbb{N}} (\hat{C}^{-1} \lambda_0, i)^{-\gamma p/2} < \infty.$$

Since F is convex and decreasing on $(0, +\infty)$, we have proved our claim. It is worth mention that this results is also true if Ω is not bounded. This was proven in [17].

Remark 4.8. *Sufficient conditions for (4.23) to hold are the following:*

$$\exists M' > 0, \forall \alpha > M' : \quad F(\alpha) \leq M' |\alpha|^{-pq/2}, \tag{4.25}$$

and

$$\sum_{i \in \mathbb{N}} (\hat{\lambda}_{V,i}^{(\alpha)})^{-q} < \infty, \tag{4.26}$$

for some $q \geq N/p$. Indeed, by (4.25), (4.26), we have, for $N_0 \in \mathbb{N}$ large enough, that

$$\sum_{i=N_0}^{\infty} F\left(\left(\hat{C}^{-1} \hat{\lambda}_{V,i}^{(\alpha)} \right)^{p/2} \right) \leq \hat{C}^{-pq/2} M' \sum_{i \in \mathbb{N}} (\hat{\lambda}_{V,i}^{(\alpha)})^{-q} < \infty.$$

Now we introduce the concept of *entropy functionals*, which can be generated by elements in the class $\mathscr{C}_{p,V}^{(\alpha)}$. This was already defined by [8] and [17]. Nonetheless, we give this definition for completeness.

Definition 4.3. *Given $T \in \mathscr{W}_+^p$ and a convex function $\beta : \mathbb{R} \to \mathbb{R} \cup \{\infty\}$ such that $\beta(0) = 0$. The value*

$$\mathscr{E}_\beta(T) = \mathrm{Tr}\left[\beta(T)\right]$$

is called the $\beta-$entropy of T provided $\mathscr{E}_\beta(T) \in (-\infty, \infty]$. In this case we say that β is an entropy seed. *Moreover, we define the $(V, p, \beta)-$* free energy functional *of T by*

$$\begin{aligned}\mathscr{F}_{V,p,\beta}(T) &= \mathscr{E}_\beta(T) + \inf_{B \in \mathscr{B}_T^p} \left[\mathscr{K}_{p,B}(T) + \mathscr{P}_{p,V,B}(T)\right] \\ &= \mathscr{E}_\beta(T) + \langle\langle T \rangle\rangle_V.\end{aligned}$$

We say that an entropy seed β is generated by the convex function $F : \mathbb{R} \to \mathbb{R} \cup \{\infty\}$ if

$$\beta(s) = F^*(-s) = \sup_{\lambda \in \mathbb{R}} \{-s\lambda - F(\lambda)\},$$

where F^* denotes the *Legendre-Fenchel transform* of F, (see Appendix C).

Remark 4.9. *By the Spectral Theorem, we know that, for any $B \in \mathscr{B}_T^p$:*

$$\forall i \in \mathbb{N}: \quad \beta(T)\eta_i = \beta(\nu_{i,T})\eta_i.$$

Then

$$\mathrm{Tr}\left[\beta(T)\right] = \sum_{i \in \mathbb{N}} \beta(\nu_{i,T}).$$

Moreover, this value does not depend on the basis of T.

Example 4.2. *The function F defined in Example 4.1 generates the seed*

$$\beta_m(s) = \begin{cases} \infty & , \quad if\ s < 0, \\ -(1-m)^{m-1} m^{-m} s^m & , \quad if s \geq 0, \end{cases}$$

with

$$m = \frac{\gamma}{\gamma + 1} \in \left(\frac{N}{N+p}, 1\right).$$

Indeed, Let $s > 0$. We shall compute $F^*(-s)$ by finding max values of the function $f : \mathbb{R}^+ \setminus \{0\} \to \mathbb{R}$ defined by

$$f(\lambda) = -s\lambda - \lambda^{-\gamma}.$$

It is straightforward that f is concave and that $f'(\lambda^*) = 0$ where

$$\lambda^* = \left(\frac{\gamma}{s}\right)^{1/(\gamma+1)}.$$

Therefore, it is clear that

$$\begin{aligned}F^*(-s) = \sup_{\lambda \in \mathbb{R} \setminus \{0\}} f(\lambda) &= f(\lambda^*) \\ &= -s\left(\frac{\gamma}{s}\right)^{1/(\gamma+1)} - \left(\frac{\gamma}{s}\right)^{-\gamma/(\gamma+1)} \\ &= -\gamma^{-m} s^m (\gamma + 1) \\ &= -(1-m)^{m-1} m^{-m} s^m.\end{aligned}$$

Recall that if $s < 0$, then $F(s) = +\infty$. Moreover, if $s = 0$, then $F^*(0) = \beta_m(0) = 0$. Hence, we have proved that

$$\forall s \in \mathbb{R}: \quad F^*(-s) = \beta_m(s).$$

The following proposition gives a lower bound for $\mathscr{F}_{V,p,\beta}$. This result is very important as it allows us to prove some *Gagliardo-Nirenberg* inequalities in the context of our Sobolev-like cone $\mathscr{W}_+^p$. Furthermore, this result also helps us prove the compactness theorem and minimization theorem from the next sections.

Proposition 4.3. *Consider an entropy seed β generated by $F \in \mathscr{C}_{p,V}$, then*

$$\forall T \in \mathscr{W}_+^p: \quad \mathscr{F}_{V,p,\beta}(T) \geq -\mathrm{Tr}\left[F\left(\left(\frac{-\Delta + V}{\hat{C}}\right)^{p/2}\right)\right],$$

where $\hat{C}$ is defined as in Remark 4.3.

Proof. Let $T \in \mathscr{W}_+^p$. Let $B \in \mathscr{B}_T^p$. Using Remark 4.3, we know that that there exists $\hat{C} > 0$ such that

$$\forall i \in \mathbb{N}: \quad \|\eta_i\|_{V,2}^p \leq \hat{C}\|\eta_i\|_{V,p}^p.$$

Since F is non-increasing on $(0, +\infty)$:

$$F(\hat{C}^{-1}\|\eta_i\|_{V,2}^p) \geq F(\|\eta_i\|_{V,p}^p). \tag{4.27}$$

Moreover, consider the expansion

$$\eta_i = \sum_{j\in\mathbb{N}} \left(\hat{\phi}_{V,j}, \eta_i\right)_{\mathrm{L}^2(\Omega)} \hat{\phi}_{V,j}$$

and

$$\sum_{j\in\mathbb{N}} |\left(\hat{\phi}_{V,j}, \eta_i\right)_{\mathrm{L}^2(\Omega)}|^2 = 1.$$

Then, as in [8] we have that

$$\begin{aligned}
&\hat{C}^{-1}\int_\Omega \left(|\nabla\eta_i(x)|^2 + V(x)|\eta_i(x)|^2\right)dx \\
=&\hat{C}^{-1}\sum_{j\in\mathbb{N}} |\left(\hat{\phi}_{V,j}, \eta_i\right)_{\mathrm{L}^2(\Omega)}|^2 \left(\int_\Omega |\nabla\hat{\phi}_{V,j}(x)|^2 dx + \int_\Omega V(x)|\hat{\phi}_{V,j}(x)|^2 dx\right) \\
=&\sum_{j\in\mathbb{N}} |\left(\hat{\phi}_{V,j}, \eta_i\right)_{\mathrm{L}^2(\Omega)}|^2 \hat{C}^{-1}\hat{\lambda}_{V,j}.
\end{aligned}$$

Since F and $s \to s^{p/2}$ are convex functions, we get

$$\begin{aligned}
F(C^{-1}\|\eta_i\|_{V,2}^p) &= F\left(\left(\sum_{j\in\mathbb{N}} |\left(\hat{\phi}_{V,j}, \eta_i\right)_{\mathrm{L}^2(\Omega)}|^2 \hat{C}^{-1}\hat{\lambda}_{V,j}\right)^{p/2}\right) \\
&\leq \sum_{j\in\mathbb{N}} |\left(\hat{\phi}_{V,j}, \eta_i\right)_{\mathrm{L}^2(\Omega)}|^2 F\left(\left(\hat{C}^{-1}\hat{\lambda}_{V,j}\right)^{p/2}\right).
\end{aligned} \tag{4.28}$$

Moreover, using the Spectral Theorem, we have that

$$\forall j \in \mathbb{N}: \quad F\left(\left(\frac{-\Delta + V}{\hat{C}}\right)^{p/2}\right)\hat{\phi}_{V,j} = F\left(\left(\hat{C}^{-1}\hat{\lambda}_{V,j}\right)^{p/2}\right)\hat{\phi}_{V,j}.$$

Then

$$\begin{aligned}
&\sum_{j\in\mathbb{N}} |\left(\hat{\phi}_{V,j}, \eta_i\right)_{L^2(\Omega)}|^2 F\left(\left(\hat{C}^{-1}\hat{\lambda}_{V,j}\right)^{p/2}\right) \\
&= \sum_{j\in\mathbb{N}} |\left(\hat{\phi}_{V,j}, \eta_i\right)_{L^2(\Omega)}|^2 F\left(\left(\hat{C}^{-1}\hat{\lambda}_{V,j}\right)^{p/2}\right)\left(\hat{\phi}_{V,j}, \hat{\phi}_{V,j}\right)_{L^2(\Omega)} \\
&= \left(\sum_{j\in\mathbb{N}} \left(\hat{\phi}_{V,j}, \eta_i\right)_{L^2(\Omega)}\hat{\phi}_{V,j}, \sum_{j\in\mathbb{N}} \left(\hat{\phi}_{V,j}, \eta_i\right)_{L^2(\Omega)} F\left(\left(\hat{C}^{-1}\hat{\lambda}_{V,j}\right)^{p/2}\right)\hat{\phi}_{V,j}\right)_{L^2(\Omega)} \\
&= \left(\eta_i, F\left(\left(\frac{-\Delta + V}{\hat{C}}\right)^{p/2}\right)\eta_i\right)_{L^2(\Omega)}.
\end{aligned} \tag{4.29}$$

Therefore, by (4.27), (4.28) and (4.29)

$$F(\|\eta_i\|_{V,p}^p) \leq \left(\eta_i, F\left(\left(\frac{-\Delta + V}{\hat{C}}\right)^{p/2}\right)\eta_i\right)_{L^2(\Omega)}.$$

Adding over $i \in \mathbb{N}$ yields

$$\begin{aligned}
\sum_{i\in\mathbb{N}} F(\|\eta_i\|_{V,p}^p) &\leq \sum_{i\in\mathbb{N}} \left(\eta_i, F\left(\left(\frac{-\Delta + V}{\hat{C}}\right)^{p/2}\right)\eta_i\right)_{L^2(\Omega)} \\
&= \operatorname{Tr}\left[F\left(\left(\frac{-\Delta + V}{\hat{C}}\right)^{p/2}\right)\right]
\end{aligned} \tag{4.30}$$

Since β is an entropy seed generated by F, we have that

$$\forall \nu, \lambda \in \mathbb{R}: \quad \beta(\nu) + \nu\lambda \geq -F(\lambda).$$

Therefore, by using 4.30 and choosing $\nu = \nu_{i,T}, \lambda = \|\eta_i\|_{V,p}^p$ and adding over $i \in \mathbb{N}$

$$\begin{aligned}
\sum_{i\in\mathbb{N}} \beta(\nu_{i,T}) + \sum_{i\in\mathbb{N}} \nu_{i,T}\|\eta_i\|_{V,P}^p &\geq -\sum_{i\in\mathbb{N}} F(\|\eta_i\|_{V,P}^p) \\
&\geq -\operatorname{Tr}\left[F\left(\left(\frac{-\Delta + V}{\hat{C}}\right)^{p/2}\right)\right].
\end{aligned}$$

Since B was chosen arbitrary, by taking infimum with respect to $\mathscr{B}_T^p$ in the inequality above we get

$$\sum_{i\in\mathbb{N}} \beta(\nu_{i,T}) + \langle\langle T\rangle\rangle_V \geq -\mathrm{Tr}\left[F\left(\left(\frac{-\Delta+V}{\hat{C}}\right)^{p/2}\right)\right]. \tag{4.31}$$

Here we use the fact that

$$\sum_{i\in\mathbb{N}} \beta(\nu_{i,T}); \quad -\mathrm{Tr}\left[F\left(\left(\frac{-\Delta+V}{\hat{C}}\right)^{p/2}\right)\right].$$

do not depend on the choice of basis $B \in \mathscr{B}_T^p$. Since T was chosen arbitrary, we conclude. □

Remark 4.10. *Proposition 4.3 can be written in terms of sums as follows:*

$$\begin{aligned}&-\sum_{i\in\mathbb{N}} F\left(\left(\hat{C}^{-1}\hat{\lambda}_{V,i}\right)^{p/2}\right)\\ &\leq \sum_{i\in\mathbb{N}} \beta(\nu_{i,T}) + \inf_{B\in\mathscr{B}_T^p}\sum_{i\in\mathbb{N}} |\nu_{i,T}|\left(\int_\Omega |\nabla\eta_i(x)|^p dx + \int_\Omega V(x)|\eta_i(x)|^p dx\right)\end{aligned}$$

The following result applies the same ideas used in the proof of Proposition 4.3.

Proposition 4.4. *Let $\alpha > 0$ and V a potential verifying (V_α). Consider an entropy seed generated by $F \in \mathscr{C}_{p,V}^{(\alpha)}$. Then, for any $T \in \mathscr{W}_+^p$, we have*

$$\mathscr{E}_\beta(T) + (\alpha+1)^{p/2}\langle\langle T\rangle\rangle_V \geq -\mathrm{Tr}\left[F\left(\left(\frac{-\alpha\Delta+V}{\hat{C}}\right)^{p/2}\right)\right]. \tag{4.32}$$

Moreover, if $V \equiv 0$, we have that

$$\mathscr{E}_\beta(T) + \alpha\mathscr{K}_p(T) \geq -\mathrm{Tr}\left[F\left(\left(\frac{-\alpha\Delta}{\hat{C}}\right)^{p/2}\right)\right] \tag{4.33}$$

Proof. Consider the Hilbert basis $\left\{\hat{\phi}_{V,i}^{(\alpha)} \,/\, i \in \mathbb{N}\right\}$. By the Spectral Theorem we know that

$$\forall j \in \mathbb{N}: \quad F\left(\left(\frac{-\alpha\Delta+V}{\hat{C}}\right)^{p/2}\right)\hat{\phi}_{V,j}^{\alpha} = F\left(\left(\hat{C}^{-1}\hat{\lambda}_{V,j}^{(\alpha)}\right)^{p/2}\right)\hat{\phi}_{V,j}^{(\alpha)}.$$

Then the proof follows exactly as in the proof of Proposition 4.3. □

A direct consequence of Proposition 4.4 is the following corollary. This result is very useful for the minimization of a free energy functional.

Corollary 4.1. *Let β be an entropy seed generated by a function $F \in \mathscr{C}^{(1/2)}_{p,0}$. If $(T_\sigma)_{\sigma\in\Sigma}$ is a family in $\mathscr{W}^p_+$ such that $(\mathscr{F}_{0,p,\beta}(T_\sigma))_{\sigma\in\Sigma}$ is bounded, then the families*

$$(\mathscr{K}_p(T_\sigma))_{\sigma\in\Sigma}; \quad (\mathscr{E}_\beta(T_\sigma))_{\sigma\in\Sigma}$$

are also bounded.

Proof. Assume that $(\mathscr{F}_{0,p,\beta}(T_\sigma))_{\sigma\in\Sigma}$ is bounded, i.e,

$$\exists K_1 > 0, \forall \sigma \in \Sigma: \quad \mathscr{F}_{0,p,\beta}(T_\sigma) = \mathscr{E}_\beta(T_\sigma) + \mathscr{K}_p(T_\sigma) < K_1.$$

By Proposition 4.4, we have that

$$\mathscr{E}_\beta(T) + \frac{1}{2}\mathscr{K}_p(T) \geq -\mathrm{Tr}\left[F\left(\left(\frac{-\Delta}{2\hat{C}}\right)^{p/2}\right)\right]. \tag{4.34}$$

Furthermore, since $F \in \mathscr{C}^{(1/2)}_{p,0}$, there exists $K_2 > 0$ such that

$$\mathrm{Tr}\left[F\left(\left(\frac{-\Delta}{2\hat{C}}\right)^{p/2}\right)\right] < K_2. \tag{4.35}$$

Therefore, by (4.34) and (4.35), we get

$$-\mathrm{Tr}\left[F\left(\left(\frac{-\Delta}{2\hat{C}}\right)^{p/2}\right)\right] + \frac{1}{2}\mathscr{K}_p(T_\sigma) \leq \mathscr{E}_\beta(T_\sigma) + \frac{1}{2}\mathscr{K}_p(T_\sigma) + \frac{1}{2}\mathscr{K}_p(T_\sigma) < K_1,$$

which implies

$$\mathscr{K}_p(T_\sigma) \leq 2(K_1 + K_2).$$

Then, we have proved that the sequence $(\mathscr{K}_p(T_\sigma))_{\sigma\in\Sigma}$ is bounded. The boundedness of $(\mathscr{E}_\beta(T_\sigma))_{\sigma\in\Sigma}$ immediately follows. □

Remark 4.11. *We can extend the definition of ρ_T for a general $p \geq 2$. Indeed, we say that*

$$\rho_{p,B,T}(\cdot) = \sum_{i\in\mathbb{N}} \nu_{i,T}|\eta_{i,T}(\cdot)|^p$$

is the B-density value *for a given $T \in \mathscr{W}^p_+$ and $B \in \mathscr{B}^p_T$. We avoid using this definition since we lose the physical interpretation of ρ_T. However, we can prove similar regularity results for this extension.*

Now that we have proved that $\mathscr{F}_{V,p,\beta}$ is bounded from below, we shall obtain some Gagliardo-Nirenberg inequalities for our operator setting. The following result is an extension of [8, Theorem 3.2] to the Sobolev-like cone $\mathscr{W}^p_+$.

Theorem 4.3. *Let β be an entropy seed generated by $F \in \mathscr{C}_{p,V}$ and $G : \mathbb{R} \to \mathbb{R}$ a strictly convex function such that*

$$\sum_{i\in\mathbb{N}} F\left(\left(\hat{C}^{-1}\hat{\lambda}_{V,i}\right)^{p/2}\right) \leq \int_\Omega G(V(x))dx. \tag{4.36}$$

Moreover, suppose that τ is a function such that

$$\forall s \in \mathbb{R}: \quad (-G)^*(-s) = -\tau(s),$$

then

$$\forall T \in \mathscr{W}_+^p: \quad \mathscr{E}_\beta(T) + \inf_{B\in\mathscr{B}_T^p} \mathscr{K}_{p,B}(T) \geq \inf_{B\in\mathscr{B}_T^p} \int_\Omega \tau(\rho_{p,B,T}(x))dx.$$

Proof. Let $T \in \mathscr{W}_+^p$. Let $B \in \mathscr{B}_T^p$. Then, by Proposition 4.3

$$\begin{aligned}\mathscr{E}_\beta(T) + \mathscr{K}_{p,B}(T) + \mathscr{P}_{p,V,B}(T) &\geq \mathscr{F}_{V,p,\beta} \\ &\geq -\sum_{i\in\mathbb{N}} F\left(\left(\hat{C}^{-1}\hat{\lambda}_{V,i}\right)^{p/2}\right).\end{aligned}$$

Thus, by (4.36)

$$\begin{aligned}\mathscr{E}_\beta(T) + \mathscr{K}_{p,B}(T) &\geq -\int_\Omega G(V(x))dx - \mathscr{P}_{p,V,B}(T) \\ &= \int_\Omega \left[-G(V(x)) - V(x)\sum_{i\in\mathbb{N}} \nu_{i,T}|\eta_i(x)|^p\right]dx.\end{aligned} \tag{4.37}$$

Let $s \in \mathbb{R}$. Since $(-G)^*(s) = \sup_{\lambda\in\mathbb{R}}\{\lambda s - (-G)(\lambda)\}$, it is clear that

$$\forall \lambda \in \mathbb{R}: \quad (-G)^*(s) \geq \lambda s + G(\lambda).$$

Then, in particular

$$(-G)^*\left(\sum_{i\in\mathbb{N}} \nu_{i,T}|\eta_i(x)|^p\right) \geq V(x)\sum_{i\in\mathbb{N}} \nu_{i,T}|\eta_i(x)| + G(V(x)).$$

Multiplying the inequality above by (-1) and integrating over Ω yields

$$-\int_\Omega (-G)^*\left(\sum_{i\in\mathbb{N}} \nu_{i,T}|\eta_i(x)|^p\right) \leq \int_\Omega \left[-V(x)\sum_{i\in\mathbb{N}} \nu_{i,T}|\eta_i(x)| - G(V(x))\right]dx.$$

Hence, by (4.37) we get

$$\begin{aligned}\mathscr{E}_\beta(T) + \mathscr{K}_{p,B}(T) &\geq \int_\Omega -(-G)^*\left(\sum_{i\in\mathbb{N}} \nu_{i,T}|\eta_i(x)|^p\right)dx \\ &= \int_\Omega \tau\left(\sum_{i\in\mathbb{N}} \nu_{i,T}|\eta_i(x)|^p\right)dx.\end{aligned} \tag{4.38}$$

Since B was chosen arbitrary, by taking infimum over $\mathscr{B}_T^p$ on (4.38) we have that

$$\mathscr{E}_\beta(T) + \mathscr{K}_p(T) \geq \inf_{B \in \mathscr{B}_T^p} \int_\Omega \tau \left(\sum_{i \in \mathbb{N}} \nu_{i,T} |\eta_i(x)|^p \right) dx.$$

We conclude by the arbitrariness of T. □

Remark 4.12. *The inequality in Theorem 4.3 can be written in terms of sums as follows*

$$\sum_{i \in \mathbb{N}} \beta(\nu_{i,T}) + \inf_{B \in \mathscr{B}_T^p} \sum_{i \in \mathbb{N}} |\nu_{i,T}| \int_\Omega |\nabla \eta_i(x)|^p dx \geq \inf_{B \in \mathscr{B}_T^p} \int_\Omega \tau \left(\sum_{i \in \mathbb{N}} \nu_{i,T} |\eta_i(x)|^p \right) dx$$

Example 4.3. *Let $g : \mathbb{R}^+ \to \mathbb{R}^+$ such that*

$$\int_0^\infty g(t)(1 + t^{-N/2}) \frac{dt}{t} < \infty.$$

Moreover, consider the convex non-increasing functions $F, G : \mathbb{R} \to \mathbb{R} \cup \{+\infty\}$ given by

$$F(s) = \int_0^\infty e^{-ts} g(t) \frac{dt}{t}, \qquad G(s) = \int_0^\infty e^{-ts} (4\pi t)^{-N/2} g(t) \frac{dt}{t}.$$

For the case $p = 2$, it was proven by [8] that F, G satisfy the conditions of Theorem 4.3.

4.3 Compactness results

In this section, we shall extend the compactness of the Sobolev embedding $\mathrm{W}_0^{1,p}(\Omega) \subseteq \mathrm{L}^p(\Omega)$ up to our operator setting

$$\mathscr{W}_+^p \subseteq \mathscr{S}_1.$$

Similar to $\mathrm{W}_0^{1,p}(\Omega) \subseteq \mathrm{L}^p(\Omega)$, it is expected to have interpolation inequalities associated with this compactness result. Indeed, Theorem 4.3 already proved some Gagliardo-Nirenberg inequalities in the context of our Sobolev-like cone $\mathscr{W}_+^p$.

Before presenting the main result of this section, we need some notation. Let $(T_n)_{n \in \mathbb{N}} \subseteq \mathscr{W}_+^p$. For each $n \in \mathbb{N}$, we shall denote by

$$(\nu_i^{(n)})_{i \in \mathbb{N}}$$

the sequence of eigenvalues of T_n. Moreover, we shall denote by

$$(\eta_i^{(n)})_{i \in \mathbb{N}}$$

a sequence of eigenfunctions of T_n such that

$$B^{(n)} = \{\eta_i^{(n)} \ / \ i \in \mathbb{N}\} \in \mathscr{B}_{T_n}^p.$$

Theorem 4.4 (Compactness theorem). *Let $(T_n)_{n\in\mathbb{N}} \subseteq \mathscr{W}_+^p$ be a sequence such that*

$$K_\infty = \sup_{n\in\mathbb{N}} \left\{ \inf_{B^{(n)}\in\mathscr{B}_{T_n}^p} \left[\mathscr{K}_{p,B^{(n)}}(T_n) + \mathscr{P}_{p,V,B^{(n)}}(T_n) \right] \right\} < \infty.$$

Then, there exists a subsequence of $(T_n)_{n\in\mathbb{N}}$ that converges in trace norm $\|\cdot\|_1$ to some T in $\mathscr{W}_+^p$.

The proof of Theorem 4.4 requires the following technical results.

Lemma 4.1. *Assume the conditions of Theorem 4.4. Then*

(i) $(\|T_n\|_1)_{n\in\mathbb{N}}$ *is uniformly bounded and*

$$\sup_{n\in\mathbb{N}} \sum_{i\in\mathbb{N}} (\nu_i^{(n)})^m < \infty,$$

where m is given by Example 4.2.

(ii) *Up to a subsequence*

$$\forall i \in \mathbb{N}: \quad \lim_{n\to\infty} \nu_i^{(n)} = \bar{\nu}_i, \quad \text{in } \mathbb{R}^+ \cup \{0\}. \tag{4.39}$$

Proof. Let us proof (i). By Proposition 4.2, we know that

$$\begin{aligned} \exists C_p > 0, \forall n \in \mathbb{N}: \quad \|T_n\|_1 &\leq C_p \langle\langle T_n \rangle\rangle_V \\ &= C_p \inf_{B^{(n)}\in\mathscr{B}_{T_n}^p} \left\{ \mathscr{K}_{p,B^{(n)}}(T_n) + \mathscr{P}_{p,V,B^{(n)}}(T_n) \right\} \\ &\leq C_p \sup_{n\in\mathbb{N}} \inf_{B^{(n)}\in\mathscr{B}_{T_n}^p} \left\{ \mathscr{K}_{p,B^{(n)}}(T_n) + \mathscr{P}_{p,V,B^{(n)}}(T_n) \right\} \\ &= C_p K_\infty < \infty. \end{aligned}$$

That is,

$$\sup_{n\in\mathbb{N}} \sum_{i\in\mathbb{N}} \nu_i^{(n)} < C_p K_\infty. \tag{4.40}$$

Then $(\|T_n\|_1)_{n\in\mathbb{N}}$ is uniformly bounded. Now, for each $n \in \mathbb{N}$, consider the entropy seed β_m defined in Example 4.2 and the $(0, p, \beta_m)$-free energy functional

$$\mathscr{F}_{0,p,\beta_m}(T_n) = -\sum_{i\in\mathbb{N}} (1-m)^{m-1} m^{-m} (\nu_i^{(n)})^m + \inf_{B^{(n)}\in\mathscr{B}_{T_n}^p} \left[\mathscr{K}_{p,B^{(n)}}(T_n) \right].$$

By Proposition 4.3, we have that

$$\forall n \in \mathbb{N}: \quad \mathscr{F}_{0,p,\beta_m}(T_n) \geq -\sum_{i\in\mathbb{N}} F\left(\left(\hat{C}^{-1} \hat{\lambda}_{0,i} \right)^{p/2} \right).$$

Hence,

$$(1-m)^{m-1}m^{-m}\sum_{i\in\mathbb{N}}(\nu_i^{(n)})^m \le \sum_{i\in\mathbb{N}} F\left(\left(\hat{C}^{-1}\hat{\lambda}_{0,i}\right)^{p/2}\right) + \inf_{B^{(n)}\in\mathscr{B}^p_{T_n}}\left[\mathscr{K}_{p,B^{(n)}}(T_n)\right]$$
$$\le \sum_{i\in\mathbb{N}} F\left(\left(\hat{C}^{-1}\hat{\lambda}_{0,i}\right)^{p/2}\right) + K_\infty < \infty.$$

Since n was arbitrary, this implies that

$$\sup_{n\in\mathbb{N}}\sum_{i\in\mathbb{N}}(\nu_i^{(n)})^m < \infty.$$

Let us prove (ii). Let $i\in\mathbb{N}$. By (4.40), we have that

$$\forall n\in\mathbb{N}: \nu_i^{(n)} < C_p K_\infty.$$

Therefore, for each $i\in\mathbb{N}$ the sequence $(\nu_i^{(n)})_{n\in\mathbb{N}}$ is bounded in $\mathbb{R}$. Since every bounded sequence in $\mathbb{R}$ has a convergent subsequence and by the arbitrariness of i, we have proved (4.39). □

Lemma 4.2. *Following the conditions of Theorem* 4.4*, assume that*

$$\forall i\in\mathbb{N}: \quad \overline{\nu}_i \neq 0.$$

Then, up to a subsequence,

$$\forall i\in\mathbb{N}: \quad \lim_{n\to\infty}\eta_i^{(n)} = \overline{\eta}_i, \quad \text{in } \mathrm{L}^p(\Omega)\subseteq \mathrm{L}^2(\Omega). \tag{4.41}$$

Moreover,

$$\forall i\in\mathbb{N}: \quad \overline{\eta}_i \in \mathrm{W}_0^{1,p}(\Omega)$$

and $\{\overline{\eta}_i \ / \ i\in\mathbb{N}\}$ *is a Hilbert basis of* $\mathrm{L}^2(\Omega)$.

Proof. Assume that

$$\forall i\in\mathbb{N}: \quad \overline{\nu}_i \neq 0.$$

Let $i\in\mathbb{N}$. We claim that there exists a sequence $(\eta_i^{(n)})_{n\in\mathbb{N}}$ of eigenfunctions that is bounded in $\mathrm{W}_V^{1,p}(\Omega)$, i.e.,

$$\exists C>0, \forall n\in\mathbb{N}: \quad \int_\Omega |\nabla\eta_i^{(n)}(x)|^p dx + \int_\Omega V(x)|\eta_i^{(n)}(x)|^p \le C.$$

Indeed, note that

$$\|\eta_i^{(n)}\|_{V,p}^p = \frac{\nu_i^{(n)}}{\nu_i^{(n)}}\|\eta_i^{(n)}\|_{V,p}^p$$
$$\le \frac{1}{\nu_i^{(n)}}\sum_{j\in\mathbb{N}}\nu_j^{(n)}\|\eta_j^{(n)}\|_{V,p}^p.$$

Moreover, since $(\nu_i^{(n)})_{n\in\mathbb{N}}$ is bounded, we have that there exists $K_1 > 0$ such that

$$\|\eta_i^{(n)}\|_{V,p}^p \leq K_1 \sum_{j\in\mathbb{N}} \nu_j^{(n)} \|\eta_j^{(n)}\|_{V,p}^p.$$

Then,

$$\begin{aligned}\inf_{B^{(n)}\in\mathscr{B}_{T_n}^p} \|\eta_i^{(n)}\|_{V,p}^p &\leq K_1 \inf_{B^{(n)}\in\mathscr{B}_{T_n}^p} \sum_{j\in\mathbb{N}} \nu_j^{(n)} \|\eta_i^{(n)}\|_{V,p}^p \\ &\leq K_1 K_\infty.\end{aligned}$$

This implies that

$$\forall n \in \mathbb{N}, \forall \epsilon > 0, \exists B_\epsilon^{(n)} \in \mathscr{B}_{T_n}^p : \quad \|\eta_i^{(n)}\|_{V,p}^p - \epsilon \leq K_1 K_\infty.$$

Therefore, in particular for $\epsilon = K_1 K_\infty$, we get

$$\forall n \in \mathbb{N}, \exists B_\epsilon^{(n)} \in \mathscr{B}_{T_n}^p : \quad \|\eta_i^{(n)}\|_{V,p}^p \leq 2K_1 K_\infty$$

Then, we have proved our claim. Hence, by the compactness of the embedding

$$\mathrm{W}_0^{1,p}(\Omega) \subseteq \mathrm{L}^p(\Omega),$$

we have proved that there exists a subsequence, which we denote $(\eta_i^{(n)})_{n\in\mathbb{N}}$ for convenience, such that (4.41) holds. It is clear that $\{\overline{\eta}_i \ / \ i \in \mathbb{N}\}$ is a Hilbert basis of $\mathrm{L}^2(\Omega)$. Moreover, we have that, since $\mathrm{L}^p(\Omega) \subseteq \mathrm{L}^2(\Omega)$

$$\forall i \in \mathbb{N}: \quad \lim_{n\to\infty} \int_\Omega |\eta_i^{(n)}(x)|^2 dx = \int_\Omega |\overline{\eta}_i(x)|^2 dx.$$

Then

$$\forall i \in \mathbb{N}: \quad \|\overline{\eta}_i\|_{\mathrm{L}^2(\Omega)} = 1.$$

Then, it remains to prove that for each $i \in \mathbb{N}$, $\overline{\eta}_i \in \mathrm{W}_0^{1,p}(\Omega)$. This is equivalent to the following statement for any $i \in \mathbb{N}$, (see [6, Prop. 9.18]).

$$\exists C > 0, \forall \psi \in \mathrm{C}_0^\infty(\mathbb{R}^N), \forall j \in \{1, ..., N\}: \quad \left| \int_\Omega \overline{\eta}_i(x) \frac{\partial \psi(x)}{\partial x_j} dx \right| \leq C \left\|\psi\right\|_{\mathrm{L}^{p'}(\Omega)}. \tag{4.42}$$

Indeed, let $\psi \in \mathrm{C}_0^\infty(\mathbb{R}^N)$. Let $j \in \{1, ..., N\}$. Since for each $n \in \mathbb{N}$ we have $\eta_i^{(n)} \in \mathrm{W}_0^{1,p}(\Omega)$, by [6, Prop. 9.18] we get

$$\left| \int_\Omega \eta_i^{(n)}(x) \frac{\partial \psi(x)}{\partial x_j} dx \right| \leq M \left\|\psi\right\|_{\mathrm{L}^{p'}(\Omega)}, \tag{4.43}$$

where M depends only on $\eta_i^{(n)}$. Now, we claim that

$$\lim_{n\to\infty} \eta_i^{(n)} \frac{\partial\psi}{\partial x_j} = \overline{\eta}_i \frac{\partial\psi}{\partial x_j}, \quad \text{in } \mathrm{L}^1(\Omega). \tag{4.44}$$

Let $\epsilon > 0$. By (4.41), we have that there exists $N_0 \in \mathbb{N}$ such that

$$n \geq N_0 \colon \quad \int_\Omega |\eta_i^{(n)} - \overline{\eta}_i|^p dx < \epsilon \left\| \frac{\partial\psi}{\partial x_j} \right\|_{\mathrm{L}^{p'}(\Omega)}^{-1}.$$

Then, for $n \geq N_0$, we deduce

$$\begin{aligned}\int_\Omega \left| \eta_i^{(n)}(x) \frac{\partial\psi(x)}{\partial x_j} - \overline{\eta}_i(x) \frac{\partial\psi(x)}{\partial x_j} \right| dx &\leq \left\| \eta_i^{(n)} - \overline{\eta}_i \right\|_{\mathrm{L}^p(\Omega)} \left\| \frac{\partial\psi}{\partial x_j} \right\|_{\mathrm{L}^{p'}(\Omega)} \\ &< \epsilon.\end{aligned}$$

Since ϵ was chosen arbitrary, we have proved claim (4.44). Finally, by (4.43) and (4.44), we get

$$\left| \int_\Omega \overline{\eta}_i(x) \frac{\partial\psi(x)}{\partial x_j} dx \right| \leq M' \left\| \psi \right\|_{\mathrm{L}^{p'}(\Omega)},$$

where M' only depends on $\overline{\eta}_i$. Whence, by choosing $C = M'$ and by the arbitrariness of ψ, j and i, we have proved

$$\forall i \in \mathbb{N} \colon \quad \overline{\eta}_i \in \mathrm{W}_0^{1,p}(\Omega).$$

□

Lemma 4.3. *Assume the conditions of Theorem 4.4 and Lemma 4.2. Then,*

$$\forall \epsilon > 0, \exists M_0 \in \mathbb{N}, \forall n \in \mathbb{N} \colon \quad \sum_{i=M_0}^{\infty} (\nu_i^{(n)})^m \leq \epsilon. \tag{4.45}$$

Moreover, up to a subsequence

$$\lim_{n\to\infty} \sum_{i\in\mathbb{N}} |\nu_i^{(n)}|^m = \sum_{i\in\mathbb{N}} |\overline{\nu}_i|^m. \tag{4.46}$$

Proof. Let us prove (4.45). Let $\epsilon > 0$. Let $\gamma > N/p$, by Example 4.1, we know that

$$\forall \epsilon_0 > 0, \exists N' \in \mathbb{N} \colon \quad \sum_{\ell=N'}^{\infty} (\hat{\lambda}_{0,\ell})^{-\gamma p/2} < \epsilon_0. \tag{4.47}$$

For each $i, n \in \mathbb{N}$ and $B^{(n)} \in \mathscr{B}_{T_n}^p$, consider the expansion

$$\eta_i^{(n)} = \sum_{k\in\mathbb{N}} \left(\eta_i^{(n)}, \hat{\phi}_{0,k} \right)_{\mathrm{L}^2(\Omega)} \hat{\phi}_{0,k} \tag{4.48}$$

and

$$\left\|\eta_i^{(n)}\right\|_{L^2(\Omega)}^2 = \sum_{k\in\mathbb{N}} \left|\left(\eta_i^{(n)}, \hat{\phi}_{0,k}\right)_{L^2(\Omega)}\right|^2 \tag{4.49}$$

According to the reverse Hölder inequality, (see e.g., [8]), we have that, for an arbitrary $N_0 \in \mathbb{N}$,

$$\sum_{i=N_0}^{\infty} \nu_i^{(n)} \left\|\eta_i^{(n)}\right\|_{H_0^1(\Omega)}^p \geq \left(\sum_{i=N_0}^{\infty} (\nu_i^{(n)})^m\right)^{1/m} \left(\sum_{i=N_0}^{\infty} \left\|\eta_i^{(n)}\right\|_{H_0^1(\Omega)}^{-\gamma p}\right)^{-1/\gamma}.$$

Then, since $L^p(\Omega) \subseteq L^2(\Omega)$, we have that there exists $K > 0$ depending only on Ω, such that

$$\left(\sum_{i=N_0}^{\infty} (\nu_i^{(n)})^m\right)^{1/m} \leq K \left(\sum_{i=N_0}^{\infty} \nu_i^{(n)} \|\eta_i^{(n)}\|_{V,p}^p\right) \left(\sum_{i=N_0}^{\infty} \left\|\eta_i^{(n)}\right\|_{H_0^1(\Omega)}^{-\gamma p}\right)^{1/\gamma}. \tag{4.50}$$

The inequality above holds for all $B^{(n)} \in \mathscr{B}_{T_n}^p$. Furthermore, from [8, Th.3.4], we have that

$$\left\|\eta_i^{(n)}\right\|_{H_0^1(\Omega)}^2 = \sum_{\ell\in\mathbb{N}} \left|\left(\eta_i^{(n)}, \hat{\phi}_{0,\ell}\right)_{L^2(\Omega)}\right|^2 \hat{\lambda}_{0,\ell}$$

and by (4.49)

$$\sum_{k\in\mathbb{N}} \left|\left(\eta_i^{(n)}, \hat{\phi}_{0,k}\right)_{L^2(\Omega)}\right|^2 = 1,$$

Then, by the convexity of the function $s \to s^{-\gamma p/2}$ and Jensen's inequality, we have

$$\left\|\eta_i^{(n)}\right\|_{H_0^1(\Omega)}^{-\gamma p} \leq \sum_{\ell\in\mathbb{N}} \left|\left(\eta_i^{(n)}, \hat{\phi}_{0,\ell}\right)_{L^2(\Omega)}\right|^2 \hat{\lambda}_{0,\ell}^{-\gamma p/2}. \tag{4.51}$$

Thus, adding over $i \in \mathbb{N}$, we get, for a fixed $M \in \mathbb{N} \setminus \{1\}$

$$\begin{aligned}
\sum_{i=N_0}^{\infty} \left\|\eta_i^{(n)}\right\|_{H_0^1(\Omega)}^{-\gamma p} &\leq \sum_{i=N_0}^{\infty} \sum_{\ell\in\mathbb{N}} \left|\left(\eta_i^{(n)}, \hat{\phi}_{0,\ell}\right)_{L^2(\Omega)}\right|^2 \hat{\lambda}_{0,\ell}^{-\gamma p/2} \\
&= \sum_{\ell=1}^{M-1} \sum_{i=N_0}^{\infty} \left|\left(\eta_i^{(n)}, \hat{\phi}_{0,\ell}\right)_{L^2(\Omega)}\right|^2 \hat{\lambda}_{0,\ell}^{-\gamma p/2} \\
&+ \sum_{\ell=M}^{\infty} \sum_{i=N_0}^{\infty} \left|\left(\eta_i^{(n)}, \hat{\phi}_{0,\ell}\right)_{L^2(\Omega)}\right|^2 \hat{\lambda}_{0,\ell}^{-\gamma p/2} \\
&\leq \frac{M-1}{(\hat{\lambda}_{0,1})^{\gamma p/2}} \sum_{i=N_0}^{\infty} \left|\left(\eta_i^{(n)}, \hat{\phi}_{0,\ell}\right)_{L^2(\Omega)}\right|^2 \\
&+ \sum_{\ell=M}^{\infty} \sum_{i=N_0}^{\infty} \left|\left(\eta_i^{(n)}, \hat{\phi}_{0,\ell}\right)_{L^2(\Omega)}\right|^2 \hat{\lambda}_{0,\ell}^{-\gamma p/2}.
\end{aligned} \tag{4.52}$$

where in the last step we use the fact that $(\hat{\lambda}_{0,\ell})_{\ell\in\mathbb{N}}$ is a non-decreasing sequence of eigenvalues. Since $\{\overline{\eta}_i \ / \ i \in \mathbb{N}\}$ is a Hilbert basis of $\mathrm{L}^2(\Omega)$, we have that, for each $\ell \in \mathbb{N}$

$$\sum_{i=1}^{\infty} \left| (\overline{\eta}_i, \hat{\phi}_{0,\ell})_{\mathrm{L}^2(\Omega)} \right|^2 = \|\hat{\phi}_{0,\ell}\|_{\mathrm{L}^2(\Omega)} = 1$$

which implies that, for any $\epsilon_1 > 0$, there exists $N_1 \in \mathbb{N}$ large enough such that for some $n_0 \in \mathbb{N}$ and $\ell \in \{1, ..., N_1 - 1\}$,

$$\forall n \geq n_0 : \quad \sum_{i=N_1}^{\infty} \left| \left(\eta_i^{(n)}, \hat{\phi}_{0,\ell} \right)_{\mathrm{L}^2(\Omega)} \right|^2 < \epsilon_1. \tag{4.53}$$

Then, by taking $M = N'$ and $N_0 = N_1$ in (4.52) we get

$$\sum_{i=N_1}^{\infty} \left\| \eta_i^{(n)} \right\|_{\mathrm{H}_0^1(\Omega)}^{-\gamma p} \leq \frac{M-1}{(\hat{\lambda}_{0,1})^{\gamma p/2}} \epsilon_1 + \epsilon_0 \epsilon_1.$$

Hence, by taking ϵ_0, ϵ_1 such that

$$\epsilon_1 \left(\frac{M-1}{(\hat{\lambda}_{0,1})^{\gamma p/2}} + \epsilon_0 \right) < \epsilon.$$

We have that

$$\sum_{i=N_1}^{\infty} \left\| \eta_i^{(n)} \right\|_{\mathrm{H}_0^1(\Omega)}^{-\gamma p} < \epsilon. \tag{4.54}$$

Then, by (4.50) and (4.54)

$$\begin{aligned} \left(\sum_{i=N_1}^{\infty} (\nu_i^{(n)})^m \right)^{1/m} &\leq K \inf_{B^{(n)} \in \mathscr{B}_{T_n}^p} \left(\sum_{i=N_1}^{\infty} \nu_i^{(n)} \|\eta_i^{(n)}\|_{V,p}^p \right) \epsilon^{1/\gamma} \\ &\leq K K_\infty \epsilon^{1/\gamma} \end{aligned}$$

which concludes the proof of (4.45). Then we have that, by uniform convergence

$$\begin{aligned} \lim_{n\to\infty} \sum_{i\in\mathbb{N}} (\nu_i^{(n)})^m &= \sum_{i\in\mathbb{N}} \lim_{n\to\infty} (\nu_i^{(n)})^m \\ &= \sum_{i\in\mathbb{N}} (\overline{\nu}_i)^m. \end{aligned}$$

□

Lemma 4.4. *Assume the conditions of Theorem 4.4 and Lemma 4.2. Then, up to a subsequence,*

$$\forall m' \in (m, 1] : \quad \lim_{n\to\infty} \sum_{i\in\mathbb{N}} |\nu_i^{(n)}|^{m'} = \sum_{i\in\mathbb{N}} |\overline{\nu}_i|^{m'}. \tag{4.55}$$

Proof. Let $m' \in (m, 1]$. Let $\epsilon > 0$. Recall that, for any $n \in \mathbb{N}$, the sequence $(\nu_i^{(n)})_{i \in \mathbb{N}}$ is non-increasing. Then, since the function $s \to s^{m'-m}$ is non-decreasing on $(0, +\infty)$, we get

$$\forall i \leq j: \quad (\nu_i^{(n)})^{m'-m} \geq (\nu_j^{(n)})^{m'-m}. \tag{4.56}$$

For each $n \in \mathbb{N}$, we write

$$Z_{N_0} = \{i \in \mathbb{N} \ / \ i \geq N_0 \wedge \nu_i^n \neq 0\},$$

where $N_0 \in \mathbb{N}$ is fixed. Then, by (4.56)

$$\begin{aligned}
\sum_{i=N_0}^{\infty} (\nu_i^{(n)})^{m'} &= \sum_{i \in Z_{N_0}} (\nu_i^{(n)})^{m'} \\
&= \sum_{i \in Z_{N_0}} \frac{(\nu_i^{(n)})^m}{(\nu_i^{(n)})^m} (\nu_i^{(n)})^{m'} \\
&= \sum_{i \in Z_{N_0}} (\nu_i^{(n)})^m (\nu_i^{(n)})^{m'-m} \\
&\leq (\nu_{N_0}^{(n)})^{m'-m} \sum_{i=N_0}^{\infty} (\nu_i^{(n)})^m
\end{aligned} \tag{4.57}$$

We know that, by Lemma 4.3

$$\exists N_1 \in \mathbb{N}, \forall n \in \mathbb{N}: \quad \sum_{i=N_1}^{\infty} (\nu_i^{(n)})^m < \frac{\epsilon}{(\nu_{N_1}^{(n)})^{m'-m}}.$$

Hence, by taking $N_0 = N_1$ in (4.57) and by the arbitrariness of ϵ, we have proved that

$$\forall \epsilon > 0, \exists N_1 \in \mathbb{N}, \forall n \in \mathbb{N}: \quad \sum_{i=N_1}^{\infty} (\nu_i^{(n)})^{m'} < \epsilon.$$

Then, by uniform convergence,

$$\begin{aligned}
\lim_{n \to \infty} \sum_{i \in \mathbb{N}} (\nu_i^{(n)})^{m'} &= \sum_{i \in \mathbb{N}} \lim_{n \to \infty} (\nu_i^{(n)})^{m'} \\
&= \sum_{i \in \mathbb{N}} (\overline{\nu}_i)^{m'}.
\end{aligned}$$

□

Proof of Theorem 4.4. By point (ii) of Lemma 4.1 and Lemma 4.2 with $m' = 1$, we have that

$$\sum_{i \in \mathbb{N}} \overline{\nu}_i < \infty. \tag{4.58}$$

Note that, for any $\eta \in \mathrm{L}^2(\Omega)$, by the Cauchy-Schwarz inequality and the embedding $\mathrm{L}^p(\Omega) \subseteq \mathrm{L}^2(\Omega)$,

$$\left\| \sum_{i\in\mathbb{N}} (\eta, \overline{\eta}_i)_{\mathrm{L}^2(\Omega)} \overline{\nu}_i \overline{\eta}_i \right\|_{\mathrm{L}^2(\Omega)} \leq K \|\eta\|_{\mathrm{L}^2(\Omega)} \sum_{i\in\mathbb{N}} \overline{\nu}_i < \infty,$$

where $K > 0$ only depends on Ω. Therefore, the operator $\overline{T} \colon \mathrm{L}^2(\Omega) \to \mathrm{L}^2(\Omega)$ given by

$$\overline{T}\eta = \sum_{i\in\mathbb{N}} (\eta, \overline{\eta}_i)_{\mathrm{L}^2(\Omega)} \overline{\nu}_i \overline{\eta}_i$$

is well defined. Moreover,

$$\forall i \in \mathbb{N} \colon \quad \overline{T}\overline{\eta}_i = \overline{\nu}_i \overline{\eta}_i.$$

Thus, we have that $(\overline{\nu}_i, \overline{\eta}_i)_{i\in\mathbb{N}} \subseteq \mathbb{R} \times \mathrm{W}_0^{1,p}(\Omega)$ is a sequence of eigenelements of $\overline{T}$. Moreover, by Lemma 4.2, we have that $B = \{\overline{\eta}_i \ / \ i \in \mathbb{N}\} \in \mathscr{B}^p_{\overline{T}}$. Then $\mathscr{B}^p_{\overline{T}} \neq \emptyset$.

(i) We have to prove that $\overline{T} \in \mathscr{W}^p_+$. Let us prove first that $\overline{T}$ is self-adjoint. Let $\eta, \psi \in \mathrm{L}^2(\Omega)$. By lemma 4.2, we know that $\{\overline{\eta}_i \ / \ i \in \mathbb{N}\}$ is a Hilbert basis of $\mathrm{L}^2(\Omega)$. Then, we have the following expansions

$$\eta = \sum_{i\in\mathbb{N}} (\eta, \overline{\eta}_i)_{\mathrm{L}^2(\Omega)} \overline{\eta}_i; \qquad \psi = \sum_{i\in\mathbb{N}} (\psi, \overline{\eta}_i)_{\mathrm{L}^2(\Omega)} \overline{\eta}_i.$$

Then

$$\begin{aligned}
\left(\overline{T}\eta, \psi\right)_{\mathrm{L}^2(\Omega)} &= \left(\sum_{i\in\mathbb{N}} (\eta, \overline{\eta}_i)_{\mathrm{L}^2(\Omega)} \overline{T}\overline{\eta}_i, \sum_{i\in\mathbb{N}} (\psi, \overline{\eta}_i)_{\mathrm{L}^2(\Omega)} \overline{\eta}_i \right)_{\mathrm{L}^2(\Omega)} \\
&= \left(\sum_{i\in\mathbb{N}} (\eta, \overline{\eta}_i)_{\mathrm{L}^2(\Omega)} \overline{\nu}_i\overline{\eta}_i, \sum_{i\in\mathbb{N}} (\psi, \overline{\eta}_i)_{\mathrm{L}^2(\Omega)} \overline{\eta}_i \right)_{\mathrm{L}^2(\Omega)} \\
&= \sum_{i\in\mathbb{N}} \overline{\nu}_i (\eta, \overline{\eta}_i)_{\mathrm{L}^2(\Omega)} (\psi, \overline{\eta}_i)_{\mathrm{L}^2(\Omega)}
\end{aligned}$$

and

$$\begin{aligned}
\left(\eta, \overline{T}\psi\right)_{\mathrm{L}^2(\Omega)} &= \left(\sum_{i\in\mathbb{N}} (\eta, \overline{\eta}_i)_{\mathrm{L}^2(\Omega)} \overline{\eta}_i, \sum_{i\in\mathbb{N}} (\psi, \overline{\eta}_i)_{\mathrm{L}^2(\Omega)} \overline{T}\overline{\eta}_i \right)_{\mathrm{L}^2(\Omega)} \\
&= \left(\sum_{i\in\mathbb{N}} (\eta, \overline{\eta}_i)_{\mathrm{L}^2(\Omega)} \overline{\eta}_i, \sum_{i\in\mathbb{N}} (\psi, \overline{\eta}_i)_{\mathrm{L}^2(\Omega)} \overline{\nu}_i\overline{\eta}_i \right)_{\mathrm{L}^2(\Omega)} \\
&= \sum_{i\in\mathbb{N}} \overline{\nu}_i (\eta, \overline{\eta}_i)_{\mathrm{L}^2(\Omega)} (\psi, \overline{\eta}_i)_{\mathrm{L}^2(\Omega)}
\end{aligned}$$

Therefore,

$$\left(\overline{T}\eta, \psi\right)_{\mathrm{L}^2(\Omega)} = \left(\eta, \overline{T}\psi\right)_{\mathrm{L}^2(\Omega)}.$$

Since η, ψ were chosen arbitrary, we have proved that $\overline{T}$ is self-adjoint. Note that the sequence of eigenvalues of $\overline{T}$ is non-negative. Then, $\overline{T} \geq 0$. Therefore, by (4.58), we have that $T \in \mathscr{S}_1$. Since $\mathscr{B}^p_{\overline{T}} \neq \emptyset$, it only remains to prove that

$$\langle\langle \overline{T} \rangle\rangle_V < \infty.$$

First, fix $N_0 \in \mathbb{N}$. For each $n \in \mathbb{N}$, we fix $B^{(n)} \in \mathscr{B}^p_{T_n}$ to define

$$f_n(x) = \sum_{i=1}^{N_0} \nu_i^{(n)} (|\nabla \eta_i^{(n)}(x)|^p + V(x)|\eta_i^{(n)}(x)|^p), \quad x \in \Omega.$$

Then, we have that

$$\sup_{n\in\mathbb{N}} \int_\Omega f_n(x)dx \geq K_\infty. \tag{4.59}$$

Furthermore, it is clear that for each $n \in \mathbb{N}$, $f_n \geq 0$. Hence, by Fatou's lemma (see e.g., [9, Pg. 19]), Lemma 4.1 and Lemma 4.2, we get

$$\begin{aligned} &\int_\Omega \sum_{i=1}^{N_0} \left[\overline{\nu}_i (|\nabla \overline{\eta}_i|^p + V(x)|\overline{\eta}_i|^p)\right] dx \\ &\leq \liminf_{n\to\infty} \int_\Omega \left[\sum_{i=1}^{N_0} \nu_i^{(n)} (|\nabla \eta_i^{(n)}(x)|^p + V(x)|\eta_i^{(n)}(x)|^p)\right] dx. \end{aligned} \tag{4.60}$$

Note that the above holds for all possible basis. By properties of the infimum, we have that,

$$\liminf_{n\to\infty} \inf_{B^{(n)}\in\mathscr{B}^p_{T_n}} \left[\sum_{i=1}^{N_0} \nu_i^{(n)} \|\eta_i^{(n)}\|^p_{V,p}\right] \leq \liminf_{n\to\infty} \sum_{i=1}^{N_0} \nu_i^{(n)} \|\eta_i^{(n)}\|^p_{V,p}. \tag{4.61}$$

Moreover, for all $\epsilon > 0$, we have that up to a choice of basis

$$\liminf_{n\to\infty} \left[\sum_{i=1}^{N_0} \nu_i^{(n)} \|\eta_i^{(n)}\|^p_{V,p} - \epsilon\right] \leq \liminf_{n\to\infty} \inf_{B^{(n)}\in\mathscr{B}^p_{T_n}} \left[\sum_{i=1}^{N_0} \nu_i^{(n)} \|\eta_i^{(n)}\|^p_{V,p}\right]. \tag{4.62}$$

Thus, by (4.60), (4.61), (4.62), we have that

$$\begin{aligned} &\int_\Omega \sum_{i=1}^{N_0} \left[\overline{\nu}_i (|\nabla \overline{\eta}_i(x)|^p + V(x)|\overline{\eta}_i(x)|^p)\right] dx \\ &\leq \liminf_{n\to\infty} \inf_{B^{(n)}\in\mathscr{B}^p_{T_n}} \int_\Omega \left[\sum_{i=1}^{N_0} \nu_i^{(n)} (|\nabla \eta_i^{(n)}(x)|^p + V(x)|\eta_i^{(n)}(x)|^p)\right] dx. \end{aligned}$$

Since N_0 was chosen arbitrary, we have proved that

$$\langle \overline{T} \rangle_{V,B} = \int_\Omega \sum_{i\in\mathbb{N}} \left[\overline{\nu}_i (|\nabla \overline{\eta}_i(x)|^p + V(x)|\overline{\eta}_i(x)|^p)\right] dx \leq \liminf_{n\to\infty} \langle\langle T_n \rangle\rangle_V < K_\infty, \tag{4.63}$$

which implies

$$\langle\langle \overline{T} \rangle\rangle_V \leq \liminf_{n\to\infty} \langle\langle T_n \rangle\rangle_V < \infty.$$

Thus, $\overline{T} \in \mathscr{W}_+^p$.

(ii) Let us prove that $(T_n)_{n\in\mathbb{N}}$ converges to $\overline{T}$ in $\mathscr{S}_1$, i.e.,

$$\forall \epsilon > 0, \exists n_0 \in \mathbb{N} : \quad n \geq n_0 \implies \|T_n - \overline{T}\|_1 < \epsilon. \tag{4.64}$$

Let $\epsilon > 0$. For each $N_0 \in \mathbb{N}$ we define

$$F_{N_0}^{(n)} = \operatorname{span}\{\eta_i^{(n)} \ / \ i = 1, ..., N_0 - 1\}$$

and

$$F_{N_0} = \operatorname{span}\{\overline{\eta}_i \ / \ i = 1, ..., N_0 - 1\}.$$

We denote by $P_{N_0}^{(n)} \colon \mathrm{L}^2(\Omega) \to F_{N_0}^{(n)}$ the orthogonal projection onto $F_{N_0}^{(n)}$, which is defined by

$$P_{N_0}^{(n)}(\eta) = \sum_{i=1}^{N_0} \left(\eta, \eta_i^{(n)}\right)_{\mathrm{L}^2(\Omega)} \eta_i^{(n)}, \quad \eta \in \mathrm{L}^2(\Omega).$$

Similarly, we denote by $P_{N_0} \colon \mathrm{L}^2(\Omega) \to F_{N_0}$ the orthogonal projection onto F_{N_0}, which is defined by

$$P_{N_0}(\eta) = \sum_{i=1}^{N_0} (\eta, \overline{\eta}_i)_{\mathrm{L}^2(\Omega)} \overline{\eta}_i, \quad \eta \in \mathrm{L}^2(\Omega).$$

Moreover, we denote by $Q_{N_0}^{(n)} = I - P_{N_0}^{(n)}$ and $Q_{N_0} = I - P_{N_0}$ the orthogonal projections onto $(F_{N_0}^{(n)})^\perp$, $(F_{N_0})^\perp$ respectively. Now, for any $n \in \mathbb{N}$, note that

$$\begin{aligned}
\left\| T_n - \overline{T} \right\|_1 &= \left\| (T_n - \overline{T}) I \right\|_1 \\
&= \left\| (T_n - \overline{T})(P_{N_0} + Q_{N_0}) \right\|_1 \\
&= \left\| (T_n - \overline{T}) P_{N_0} + T_n Q_{N_0} - \overline{T} Q_{N_0} \right\|_1 \\
&= \left\| (T_n - \overline{T}) P_{N_0} + T_n Q_{N_0} - \overline{T} Q_{N_0} + T_n Q_{N_0}^{(n)} - T_n Q_{N_0}^{(n)} \right\|_1 \\
&\leq \|(T_n - \overline{T}) P_{N_0}\|_1 + \|T_n Q_{N_0}^{(n)}\|_1 + \left\| \overline{T} Q_{N_0} \right\|_1 + \left\| T_n (Q_{N_0}^{(n)} - Q_{N_0}) \right\|_1 .
\end{aligned} \tag{4.65}$$

Note that $\operatorname{Dom}((T_n - \overline{T}) P_{N_0}) = F_{N_0}$. Then, by Lemma 4.1 and Lemma 4.2, the first $N_0 - 1$ eigenelements $\{(\nu_i^{(n)}, \eta_i^{(n)})\}_{i=1,...,N_0-1}$ strongly converge in $\mathbb{R}$ and $\mathrm{L}^2(\Omega)$, respectively. Then,

$$\exists n_1 \in \mathbb{N} \colon \quad n \geq n_1 \implies \|(T_n - \overline{T}) P_{N_0}\|_1 < \frac{\epsilon}{4}. \tag{4.66}$$

The second term and third term in (4.65) converges to 0 by Lemma 4.3 with $m' = 1$ and N_0 large enough. Then

$$\exists N_1 \in \mathbb{N}: \quad \left\| T_n Q_{N_1}^{(n)} \right\|_1 < \frac{\epsilon}{4} \tag{4.67}$$

and

$$\exists N_2 \in \mathbb{N}: \quad \left\| \overline{T} Q_{N_2} \right\|_1 < \frac{\epsilon}{4}. \tag{4.68}$$

Now, for the last term note that

$$\begin{aligned} Q_{N_0}^{(n)} - Q_{N_0} &= I - P_{N_0}^n - I + P_{N_0} \\ &= P_{N_0} - P_{N_0}^n. \end{aligned}$$

Therefore, by Lemma 4.1 and 4.2, we have that

$$\exists n_2 \in \mathbb{N}: \quad n \geq n_2 \implies \left\| T_n (Q_{N_0}^{(n)} - Q_{N_0}) \right\|_1 \leq \|T_n\|_1 \left\| Q_{N_0}^{(n)} - Q_{N_0} \right\|_1 \leq \frac{\epsilon}{4}. \tag{4.69}$$

Hence, collecting the estimates (4.66), (4.67), (4.68), (4.69) with $N_0 = \max\{N_1, N_2\}$ and $n_0 = \max\{n_1, n_2\}$, we obtain

$$n \geq n_0 \implies \|T_n - \overline{T}\|_1 < \epsilon.$$

Since ϵ was chosen arbitrary, we have proved that the sequence $(T_n)_{n\in\mathbb{N}}$ converges to $\overline{T}$ in $\mathscr{S}_1$. □

4.4 Minimization of a free energy functional

In this section, we present a minimization problem of a free energy functional. The main tool that we shall use for this minimization problem is the compactness theorem proved in the previous section.

Theorem 4.5. *Let β be an entropy seed generated by $F \in \mathscr{C}_{p,0}^{(1/2)} \cap \mathscr{C}_{p,0}$. Then there exists a unique $T_\infty \in \mathscr{W}_+^p$ such that*

$$\mathscr{F}_{0,p,\beta}(T_\infty) = \inf_{T \in \mathscr{W}_+^p} \mathscr{F}_{0,p,\beta}(T) \tag{4.70}$$

Proof. Proposition 4.3 gives a lower bound for $\mathscr{F}_{0,p,\beta}$. Then, we know there exists a minimizing sequence $(T_n)_{n\in\mathbb{N}} \subseteq \mathscr{W}_+^p$ of $\mathscr{F}_{0,p,\beta}$, i.e.,

$$\lim_{n\to\infty} \mathscr{F}_{0,p,\beta}(T_n) = \inf_{T \in \mathscr{W}_+^p} \mathscr{F}_{0,p,\beta}(T). \tag{4.71}$$

By (4.71), we have that $(\mathscr{F}_{0,p,\beta}(T_n))_{n\in\mathbb{N}}$ is bounded. Then, by Corollary 4.1, we have that

$$(\mathscr{K}_p(T_n))_{n\in\mathbb{N}}; \quad (\mathscr{E}_\beta(T_n))_{n\in\mathbb{N}}$$

are bounded sequences as well. This implies that

$$K_\infty = \sup_{n\in\mathbb{N}} \mathscr{K}_p(T_n) < \infty.$$

Then we can apply Theorem 4.4 and extract a subsequence, denoted by $(T_n)_{n\in\mathbb{N}}$ for simplicity, such that

$$\lim_{n\to\infty} T_n = \overline{T} \quad \text{in } \mathscr{S}_1, \tag{4.72}$$

for some $\overline{T} \in \mathscr{W}_+^p$. Moreover, by (4.63), we have that

$$\mathscr{K}_p(\overline{T}) \leq \liminf_{n\to\infty} \mathscr{K}_p(T_n).$$

We claim that

$$\mathscr{E}_\beta(\overline{T}) \leq \liminf_{n\to\infty} \mathscr{E}_\beta(T_n). \tag{4.73}$$

In fact, consider the set

$$\mathscr{A}_+ = \{\mu = (\mu_i)_{i\in\mathbb{N}} \in \ell^1 \;/\; \sum_{i\in\mathbb{N}} \beta(\mu_i) < A\},$$

where $A = \sup_{n\in\mathbb{N}} \mathscr{E}_\beta(T_n) < \infty$. Since β is a convex function, we have that $\mathscr{A}_+$ is convex. Furthermore, the convexity of β also implies that the function $D\colon \mathscr{A}_+ \to \mathbb{R}$ given by

$$D(\mu) = \sum_{i\in\mathbb{N}} \beta(\mu_i), \quad \mu = (\mu_i)_{i\in\mathbb{N}} \in \ell^1$$

is convex. Note that, by definition D is bounded from above by A. Then, by [26, Lemma 2.1], we have in particular that D is *lower semi-continuous*. By Lemma 4.1, we know that, up to a subsequence

$$\lim_{n\to\infty} \nu_i^{(n)} = \overline{\nu}_i$$

which implies

$$\nu_n \rightharpoonup \overline{\nu}, \quad \text{as } n \to \infty$$

where $\overline{\nu} = (\overline{\nu}_i)_{i\in\mathbb{N}}$ and $\nu_n = (\nu_i^{(n)})_{i\in\mathbb{N}}$. Since D is lower semi-continuous, (see [6, Pg. 10])

$$D(\overline{\nu}) \leq \liminf_{n\to\infty} D(\nu_n)$$

and we have proved (4.73). Then

$$\begin{aligned} \mathscr{F}_{0,p,\beta}(\overline{T}) &= \mathscr{E}_\beta(\overline{T}) + \mathscr{K}_p(\overline{T}) \\ &\leq \liminf_{n\to\infty} \mathscr{K}_p(T_n) + \liminf_{n\to\infty} \mathscr{E}_\beta(T_n) \\ &\leq \liminf_{n\to\infty} \mathscr{F}_{0,p,\beta}(T_n) \\ &= \inf_{T\in\mathscr{W}_+^p} \mathscr{F}_{0,p,\beta}(T) \end{aligned} \tag{4.74}$$

whence, since $\overline{T} \in \mathscr{W}_{+}^{p}$, we have that

$$\mathscr{F}_{0,p,\beta}(\overline{T}) = \inf_{T \in \mathscr{W}_{+}^{p}} \mathscr{F}_{0,p,\beta}(T).$$

That is, $\overline{T}$ is a minimizer for $\mathscr{F}_{0,p,\beta}$. $\square$

Chapter 5

Conclusions and recommendations

5.1 Conclusions

We extended the results obtained by Dolbeault, Felmer and Mayorga-Zambrano, [8] and Mayorga-Zambrano, Salinas, [17], for the case $2 \leq p < N$ and an open bounded domain $\Omega \subseteq \mathbb{R}^N$.

We considered a potential $V \in \mathrm{L}^\infty(\Omega)$ such that $V \geq 0$ and $T \in \mathscr{S}_1$ such that its eigenelements

$$(\nu_{i,T}, \eta_{i,T})_{i\in\mathbb{N}} \subseteq \mathbb{R} \times \mathrm{L}^2(\Omega)$$

form a Hilbert basis $B = \{\eta_{i,T} \ / \ i \in \mathbb{N}\}$ of $\mathrm{L}^2(\Omega)$. We denoted the set of all possible eigenbasis of T by $\mathscr{B}_T$.

The Sobolev-like cone $\mathscr{W}^p$ was defined as the set of nuclear operators $T \in \mathscr{S}_1$ such that

$$\mathscr{B}_T^p = \left\{ B = \{\eta_{i,T} \ / \ i \in \mathbb{N}\} \in \mathscr{B}_T \ / \ B \subseteq \mathrm{W}_0^{1,p}(\Omega) \right\} \neq \emptyset$$

and

$$\langle\langle T \rangle\rangle_V = \inf_{B\in\mathscr{B}_T^p} \sum_{i\in\mathbb{N}} |\nu_{i,T}| \int_\Omega \left[|\nabla \eta_{i,T}(x)|^p + V(x)|\eta_{i,T}(x)|^p \right] dx < \infty.$$

$\langle\langle T \rangle\rangle_V$ is called the energy of T. We denoted $\mathscr{W}_+^p = \{T \in \mathscr{W}^p \ / \ T \geq 0\}$.

The two main results of this work consisted in a compactness result for $\mathscr{W}_+^p$ and a minimization problem. The first result proved that the embedding

$$\mathscr{W}_+^p \subseteq \mathscr{S}_1$$

when equipped with $\langle\langle \cdot \rangle\rangle_V$ is compact, i.e., if $(T_n)_{n\in\mathbb{N}} \subseteq \mathscr{W}_+^p$ is bounded in $\langle\langle \cdot \rangle\rangle_V$, then there exists subsequence that converges in $\mathscr{S}_1$. Then, analogous to the embedding $\mathrm{W}_0^{1,p}(\Omega) \subseteq \mathrm{L}^p(\Omega)$, we also obtained some interpolation inequalities in the language of operators. This compactness result was applied to the minimization problem:

$$\begin{cases} \text{Find } T_\infty \in \mathscr{W}_+^p \text{ such that} \\ \inf_{T\in\mathscr{W}_+^p} \mathscr{F}_{0,p,\beta}(T) = \inf_{T\in\mathscr{W}_+^p} \left[\mathrm{Tr}\left[\beta(T)\right] + \langle\langle T \rangle\rangle_0 \right] = \mathrm{Tr}\left[\beta(T_\infty)\right] + \langle\langle T_\infty \rangle\rangle_0 \end{cases}$$

where $\beta\colon \mathbb{R} \to \mathbb{R}$ is a convex function called an entropy seed such that $\beta(0) = 0$, and $\operatorname{Tr}[\cdot]$ denotes the trace functional.

These results were proven under the conditions that $2 \leq p < N$ and $N \geq 3$, with an open bounded domain $\Omega \subseteq \mathbb{R}^N$. However, with an appropriate application of *Rellich-Kondrachov* theorem, we can prove results similar to ours for $p \geq N$. Moreover, by a scheme similar to [17], we may prove similar results for an unbounded domain $\Omega \subseteq \mathbb{R}^N$.

5.2 Recommendations

1. In this capstone project, the $p-$Casimir class of functions was constructed around the eigenvalue problem of the laplacian operator. However, we believe that another approach such as the use of the eigenvalue problem for $p-$laplacian could be more rewarding, yet more difficult. The main problem arises when trying to prove a lower bound for the $(V, p, \beta)-$free energy functional. If one chooses to take this path, [5], [27] and [19] are very good sources.
2. I believe that an introductory course in mathematical physics should be a part of the mathematics program offered at Yachay Tech because it provides classical perspectives to mathematics.
3. Mathematics lies at the heart of every branch of science, which at the same time is fundamental for innovation, something that Ecuador desperately needs. Therefore, I believe that the right step for Ecuador to become a developed country is to push for universities to improve their mathematical departments. This is because this would create a chain reaction in the superior education of Ecuador that would create better professionals.

Bibliography

[1] R. Adams, *Sobolev Spaces*, vol. 64, Academic Press, 1 ed., 1975.

[2] G. L. Aki, J. Dolbeault, and C. Sparber, *Thermal effects in gravitational hartree systems*, Annales Henri Poincaré, 12 (2011), p. 1055–1079.

[3] L. Ballentine, *Quantum Mechanics A Modern Development*, World Scientific Publishing, 1998.

[4] H. H. Bauschke and P. L. Combettes, *Lower Semicontinuous Convex Functions*, Springer New York, New York, NY, 2011, pp. 129–141.

[5] J. F. Bonder and L. M. D. Pezzo, *An optimization problem for the first eigenvalue of the p-laplacian plus a potential*, Communications on Pure & Applied Analysis, 5 (2006), pp. 675–690.

[6] H. Brezis, *Function Analysis, Sobolev Spaces and Partial Differential Equations*, Springer, 01 2010.

[7] J. Dolbeault, P. Felmer, M. Loss, and E. Paturel, *Lieb–thirring type inequalities and gagliardo–nirenberg inequalities for systems*, Journal of Functional Analysis, 238 (2006), pp. 193 – 220.

[8] J. Dolbeault, P. Felmer, and J. Mayorga-Zambrano, *Compactness properties for trace-class operators and applications to quantum mechanics*, Monatshefte fur Mathematik, 155 (2008).

[9] L. Evans and R. Gariepy, *Measure Theory and Fine Properties of Functions, Revised Edition (1st ed.)*, Chapman and Hall/CRC, 2015.

[10] L. C. Evans, *Partial Differential Equations*, American Mathematical Society, 2010.

[11] W. Fairchild and C. I. Tulcea, *Topology*, W.B. Saunders Co,. Philadelphia, Pa., 1971.

[12] P. Hislop and I. Sigal, *Introduction to Spectral Theory With Applications to Schrödinger Operators*, Springer, 1996.

[13] E. Kreyszig, *Introductory Functional Analysis with Applications*, John Wiley and Sons, 1978.

[14] P. Lax, *Functional Analysis*, Wiley Interscience, 2002.

[15] P. A. Markowich, G. Rein, and G. Wolansky, *Existence and nonlinear stability of stationary states of the schrödinger–poisson system*, Journal of Statistical Physics, 106 (2002), pp. 1221 – 1239.

[16] J. Mayorga-Zambrano, *An Introduction to Functional Analysis*, Amarun, (Working book), 2021.

[17] J. Mayorga-Zambrano and Z. Salinas, *Sobolev-like cones of trace-class operators on unbounded domains: Interpolation inequalities and compactness properties*, Nonlinear Analysis: Theory, Methods & Applications, 93 (2013), pp. 78 – 89.

[18] Z. Nouredine, *Quantum Mechanics Concepts and Applications*, John Wiley and Sons, 2009.

[19] I. Peral, *Multiplicity of solutions for the p-laplacian*, Lecture Notes, Second School on Nonlinear Functional Analysis and Applications to Differential Equations (ICTP), 1997.

[20] A. C. Phillips, *Introduction to Quantum Mechanics*, John Wiley and Sons, 2003.

[21] M. Reed and B. Simon, *Methods of Modern Mathematical Physics*, vol. I,Functional Analysis, Academic Press, 1980.

[22] R. Shankar, *Principles of Quantum Mechanics*, Plenum press, 1994.

[23] B. Simon, *Trace Ideals and their Applications*, American Mathematical Society, 2005.

[24] H. R. van der Vaart and E. H. Yen, *Weak sufficient conditions for fatou's lemma and lebesgue's dominated convergence theorem*, Mathematics Magazine, 41 (1968), pp. 109–117.

[25] S. Weinberg, *Lectures on Quantum Mechanics*, Cambridge University Press, 2013.

[26] I. E. . H. Weinert, *Convex analysis and variational problems*, ZAMM - Journal of Applied Mathematics and Mechanics / Zeitschrift für Angewandte Mathematik und Mechanik, 59 (1979).

[27] M. Ôtani and T. Teshima, *On the first eigenvalue of some quasilinear elliptic equations*, Proc. Japan Acad. Ser. A Math. Sci., 64 (1988), pp. 8–10.

Appendices

Appendix A

The Riemann-Stieltjes integral

The *Riemann-Stieltjes integral* is a generalization of the Riemann integral. In order to understand this concept, we need to define the space of functions of bounded variation.

Let $w \in C([a,b])$, where $a, b \in \mathbb{R}$. We denote

$$\mathscr{M} = \{p_n = (t_0, \dots, t_n) \ / \ a = t_0 < t_1 < \cdots < t_n = b\}$$

the set of all finite meshings of the interval $[a,b]$. Here, $n \in \mathbb{N}$ is arbitrary. The *total variation* of w on $[a,b]$ is defined by

$$\operatorname{Var}(w) = \sup_{p \in \mathscr{M}} \sum_{j=1}^{n} |w(t_j) - w(t_{j-1})|.$$

We say that w is of *bounded variation* on $[a,b]$ if its total variation $\operatorname{Var}(w)$ on $[a,b]$ is finite. Note that the set of all functions of bounded variation on $[a,b]$, which we denote by $BV([a,b])$ forms a linear space. Moreover,

$$\|w\| = |w(a)| + \operatorname{Var}(w), \quad w \in BV([a,b])$$

is a norm on $BV([a,b])$, [13].

Now, consider $x \in C([a,b])$ and $w \in BV([a,b])$. Let $p_n \in \mathscr{M}$, we write

$$\eta(p_n) = \max\{(t_1 - t_0), \dots, (t_n - t_{n-1})\}.$$

Consider a sequence of partitions $(p_n)_{n \in \mathbb{N}} \subseteq \mathscr{M}$ satisfying

$$\lim_{n \to \infty} \eta(p_n) = 0.$$

Then, the *Riemann-Stieljtes integral* of x over $[a,b]$ is defined by

$$\int_a^b x(t)dw(t) = \lim_{n \to \infty} \sum_{j=1}^{n} x(t_j)[w(t_j) - w(t_{j-1})].$$

The Riemann-Stieltjes integral of a function x over $[a,b]$ is a generalization of the familiar Riemann integral of x over $[a,b]$ since we can choose $w \in BV([a,b])$ as the identity to obtain

$$\int_a^b x(t)dw(t) = \int_a^b x(t)dt.$$

Moreover, if we choose $w \in \mathrm{BV}([a,b])$ to be differentiable on $[a,b]$, then we have

$$\int_a^b x(t)dw(t) = \int_a^b x(t)w'(t)dt.$$

This integral is linear both on $x \in C([a,b])$ and on $w \in \mathrm{BV}([a,b])$. A useful inequality for these type of integrals is the following

$$\left| \int_a^b x(t)dw(t) \right| \leq \max_{t\in[a,b]} |x(t)| \mathrm{Var}(w).$$

We finish this section with an important representation theorem proved by Riesz in 1909

Theorem A.1. *Every bounded linear functional f on $\mathrm{C}([a,b])$ can be represented by a Riemann-Stieltjes integral*

$$f(x) = \int_a^b x(t)dw(t), \quad x \in \mathrm{C}([a,b]),$$

where $w \in \mathrm{BV}([a,b])$ is such that

$$\mathrm{Var}(w) = \|f\|.$$

A proof of this result can be found in [13, Th. 4.4.1]

Appendix B

The eigenvalue problem for the p-Laplacian operator

Let $p \geq 1$ and $\Omega \subseteq \mathbb{R}^N$. The $p-$laplacian operator is defined by

$$\Delta_p u \equiv -div(|\nabla u|^{p-2}\nabla u|),$$

where $u \in W_0^{1,p}(\Omega)$. Here *div* denotes the usual divergence operator. We give a brief overview of the eigenvalue problem for the $p-$laplacian.

Defining the eigenvalue problem presents some issues. Mainly, the non-linearity of the p-Laplacian operator Δ_p. However, we can still deal with its generalized eigenvectors ϕ and eigenvalues λ as stated in [19]. That is, $(\lambda, \phi) \in \mathbb{R} \times W_0^{1,p}(\Omega)$ is an eigenelement of Δ_p if and only if

$$\begin{cases} \Delta_p \phi(x) = -\nabla \cdot (|\nabla \phi(x)|^{p-2}\nabla \phi(x)) = \lambda |\phi(x)|^{p-2}\phi(x), & x \in \Omega, \\ u(x) = 0, & x \in \partial\Omega \end{cases} \tag{B.1}$$

in the weak sense. Furthermore, we shall state two results involving the existence of an infinite sequence of eigenvalues of Δ_p. For each $k \in \mathbb{N}$ and $\alpha \in \mathbb{R}$ consider,

$$\mathcal{C}_k = \{C \subset \mathcal{M} \;/\; C \text{ compact }, C = -C \wedge \gamma(C) \geq k\} \tag{B.2}$$

where γ is *Krasnolseskii genus* (see [19]) and $\mathcal{M}$ is the manifold

$$\mathcal{M} = \left\{ u \in W_0^{1,p}(\Omega) \;/\; \frac{1}{p}\int_\Omega |\nabla u(x)|^p dx = \alpha \right\}.$$

Theorem B.1. *Consider $B : W_0^{1,p}(\Omega) \to \mathbb{R}$ defined by $B(\phi) = \frac{1}{p}\int_\Omega |\phi(x)|^p dx$ and let β_k be defined by*

$$\beta_k = \sup_{C \in \mathcal{C}_k} \min_{\phi \in C} B(\phi).$$

Then, $\beta_k > 0$ and there exists $\phi_k \in \mathcal{M}$ such that $B(\phi_k) = \beta_k$, and ϕ_k is an eigenvector of Δ_p for $\lambda_k = \alpha/\beta_k$.

A proof of this theorem can be found in [19]. The next lemma completes this result by stating that we have a infinite sequence of eigenvalues that explode at infinity:

Lemma B.1. *Let β_k defined in Theorem B.1. Then $\lim_{k\to\infty} \beta_k = 0$. Hence, $\lim_{k\to\infty} \lambda_k = \infty$.*

A proof of this result can be found in [19].

Remark B.1. *We can extend this results for the operator $-\Delta_p + V$, where $V \colon \Omega \to \mathbb{R}$ is a potential. We denote the sequence of eigenelements of the operator $-\Delta_p + V$ with Dirichlet boundary conditions by $\{(\lambda_{V,i}, \phi_{V,i})\}_{i\in\mathbb{N}} \subset \mathbb{R} \times W_0^{1,p}(\Omega)$. That is, for all $i \in \mathbb{N}$:*

$$\begin{cases} -\Delta_p \phi_{V,i}(x) + V(x)|\phi_{V,i}(x)|^{p-2}\phi_{V,i}(x) = \lambda_{V,i}|\phi_{V,i}(x)|^{p-2}\phi_{V,i}(x) & , x \in \Omega, \\ -\Delta_p \phi_{V,i}(x) + V(x)|\phi_{V,i}(x)|^{p-2}\phi_{V,i}(x) = 0 & , x \in \partial\Omega \end{cases}$$

in the weak sense. Bonder and del Pezzo proved in [5] that there exists an increasing, unbounded sequence of eigenvalues for $-\Delta_p + V$, i.e.

$$\lambda_{V,1} \leq \lambda_{V,2} \leq \cdots \leq \lambda_{V,i} \leq \cdots, \tag{B.3}$$

and

$$\lim_{i\to\infty} \lambda_{V,i} = \infty. \tag{B.4}$$

Moreover, it was also proven in [27] that the first eigenvalue is positive and isolated, i.e.,

$$\lambda_{V,1} < \lambda_{V,2} \leq \cdots \leq \lambda_{V,i} \leq \cdots. \tag{B.5}$$

Appendix C

Legendre-Fenchel transform

Let $\mathscr{H}$ be a Hilbert space. Let $f\colon \mathscr{H} \to \mathbb{R} \cup \{\infty\}$. The Legendre-Fenchel transform of f is the function $f^*\colon \mathscr{H} \to \mathbb{R} \cup \{\infty\}$ defined by

$$f^*(s) = \sup_{\lambda \in \mathbb{R}} \{ s\lambda - f(\lambda) \}, \quad s \in \mathscr{H}.$$

Example C.1. *The Legendre-Fenchel transform of the exponential functions is given by*

$$\exp^*(u) = \begin{cases} u\ln(u) - u & , \text{ if } u > 0, \\ 0 & , \text{ if } u = 0, \\ \infty & , \text{ if } u < 0. \end{cases}$$

The following proposition is a compilation of various properties of the Legendre-Fenchel transform.

Proposition C.1. *Let $f, g : \mathscr{H} \to \mathbb{R} \cup \{\infty\}$. Then*

(i) $f^{**} \leq f$.

(ii) If $f \leq g$. Then $f^* \geq g^*$ and $f^{**} \leq g^{**}$.

(iii) If f is even, then f^* is even.

(iv) Let $\alpha > 0$. Then $(\alpha f)^* = \alpha f^*(\cdot/\alpha)$.

(v) f^* is lower semi-continuous.

A proof of these results can be found in [4].