UNIVERSIDAD DE INVESTIGACIÓN DE TECNOLOGÍA EXPERIMENTAL YACHAY

Escuela de Ciencias Matemáticas y Computacionales

Existence of a standing wave for a p-Laplacian Schrödinger equation

Trabajo de integración curricular presentado como requisito para la obtención
del título de Matemático

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Dedication

"To all my family.
Everything I am and will be
is the outcome of all your love and support,
and for this, I am eternally grateful."

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First of all, I would like to express my profound gratitude and admiration to all the teachers of Yachay Tech. They have taught me what true passion for science and education is all about by believing in us, the Ecuadorian students, and delivering exemplary teaching even through the most discouraging adversities. In particular, I would like to thank my advisor Juan Mayorga-Zambrano for all the knowledge and support he has given me and all his students. His professionalism and passion for mathematics has truly inspired me to pursue an academic and professional career in a field I originally did not realized I had the aptitudes to do. Special thanks to Cédric M. Campos, Juan Carlos López, Antonio Acosta, Hugo Campos, Saba Infante, Rafael Amaro, Raúl Manzanilla, Freddy Cuenca and all the amazing teachers I had the privilege to learn from.

Also, I would like to express all my love and gratitude to my family: my parents, Mariela and Juan Pablo; my grandparents Silvia and Hugo; my stepfather, Santiago; my little brothers María Caridad and Pedro José; and all my beloved family in Quito, Ambato and Riobamba.

My time at Yachay Tech has been a pivotal period of my life and much of this is because of the wonderful people with whom I shared time. I have no doubt that Yachay Tech has been home to some of the most promising professionals and human beings of the country. It is a privilege to call some of these my friends: the group from house # 10, the fourth generation of mathematicians and my roommates from I3-1. I will never forget you.

Finally, to the reader, I truly hope that this text is able help you in any way. I firmly believe that the prime purpose of knowledge is to help new generations in the never-ending mission of creating a better world.

Abstract

In this capstone project we prove the existence of a non-trivial solution to the following quasi linear boundary value problem.

$$\begin{cases} -\varepsilon^2\Delta_p u(x) + V(x)|u(x)|^{p-2}u(x) - |u(x)|^{q-1}u(x) &= 0, \qquad x \in \mathbb{R}^N \\ u(x) &\to 0, \qquad \text{as } |x| \to \infty, \end{cases} \tag{G_ε}$$

where

$$\Delta_p u = \operatorname{div}(|\nabla u|^{p-2}\nabla u)$$

and

$$1 < p < q+1 < p^*, \tag{1}$$

with

$$p^* = \begin{cases} \dfrac{pN}{N-p}, & \text{if } N \geq 3; \\ \infty, & \text{if } N = 1, 2. \end{cases}$$

Additionally, we assume that

$$V \in C(\mathbb{R}^N) \text{ is non-negative and} \tag{C}$$

$$\lim_{|x|\to\infty} V(x) = \infty. \tag{L}$$

By rescaling as $v(x) = u(\varepsilon^\beta x), x \in \mathbb{R}^N, \beta \in \mathbb{R}$, (G_ε) is equivalent to

$$\begin{cases} -\Delta_p u(x) + V_\varepsilon(x)|u(x)|^{p-2}u(x) - |u(x)|^{q-1}u(x) = 0, & x \in \mathbb{R}^N, \\ u(x) \to 0, & \text{as } |x| \to \infty. \end{cases} \tag{E_ε}$$

By working on the manifold defined by

$$\mathcal{M}_\varepsilon = \left\{ \int_{\mathbb{R}^N} |u(x)|^{q+1} dx = 1 \right\}$$

we prove the existence of a non-trivial solution by applying the direct method of Calculus of Variations. We minimize the functional $J_\varepsilon : \mathcal{M}_\varepsilon \subseteq \mathrm{W}_\varepsilon \to \mathbb{R}$, given by

$$J_\varepsilon(u) = \frac{1}{p} \int_{\mathbb{R}^N} \left[|\nabla u(x)|^p + V_\varepsilon(x)|u(x)|^p \right] \, dx. \tag{2}$$

The regularity of the functional, the completeness of $\mathcal{M}_\varepsilon$ and the fact that J_ε satisfies the Palais-Smale condition, allow us to prove the existence of a critical point on the manifold that corresponds to a non-trivial solution for (G_ε).

Keywords: Nonlinear Schrödinger equation, critical frequency, p-Laplacian, existence, Calculus of Variations.

Resumen

En este proyecto de titulación se demuestra la existencia de soluciones no triviales para el siguiente problema cuasi lineal con valores de frontera.

$$\begin{cases} -\varepsilon^2 \Delta_p u(x) + V(x)|u(x)|^{p-2}u(x) - |u(x)|^{q-1}u(x) & = & 0, & x \in \mathbb{R}^N \\ u(x) & \to & 0, & |x| \to \infty, \end{cases} \tag{G_ε}$$

donde

$$\Delta_p u = \operatorname{div}(|\nabla u|^{p-2}\nabla u)$$

y

$$1 < p < q+1 < p^*, \tag{3}$$

con

$$p^* = \begin{cases} \dfrac{pN}{N-p}, & \text{si } N \geq 3; \\ \infty, & \text{si } N = 1, 2. \end{cases}$$

Adicionalmente, asumimos que

$$V \in C(\mathbb{R}^N) \text{ es no negativo y} \tag{C}$$

$$\lim_{|x|\to\infty} V(x) = \infty. \tag{L}$$

Mediante el rescalamiento $v(x) = u(\varepsilon^\beta x), x \in \mathbb{R}^N, \beta \in \mathbb{R}$, (G_ε) es equivalente a

$$\begin{cases} -\Delta_p u(x) + V_\varepsilon(x)|u(x)|^{p-2}u(x) - |u(x)|^{q-1}u(x) = 0, & x \in \mathbb{R}^N, \\ u(x) \to 0, & |x| \to \infty. \end{cases} \tag{E_ε}$$

Al trabajar en una variedad definida por

$$\mathcal{M}_\varepsilon = \left\{ \int_{\mathbb{R}^N} |u(x)|^{q+1} dx = 1 \right\}$$

demostramos la existencia de una solución no trivial por el método directo del Cálculo de Variaciones. Minimizamos el funcional $J_\varepsilon : \mathcal{M}_\varepsilon \subseteq \mathrm{W}_\varepsilon \to \mathbb{R}$, dado por

$$J_\varepsilon(u) = \frac{1}{p} \int_{\mathbb{R}^N} \left[|\nabla u(x)|^p + V_\varepsilon(x)|u(x)|^p \right] \, dx. \tag{4}$$

La regularidad del funcional, la completitud de $\mathcal{M}_\varepsilon$ y el hecho de que J_ε satisface la condición de Palais-Smale, nos permite demostrar la existencia de un punto crítico en la variedad para el funcional que a su vez corresponde a una solución no trivial para (G_ε).

Palabras clave: Ecuación de Schrödinger no lineal, frecuencia crítica, p-Laplaciano, existencia, Cálculo de Variaciones.

Contents

Chapter 1

Introduction

The evolution of physics from understanding the universe through Newton's laws and Classical Mechanics to the introduction of Quantum Mechanics and the famous Schrödinger's equation, required remarkable developments in several fields of mathematics. The motivation of the present project is to prove the existence of a ground state for a quasi linear Schrödinger equation.

We will be particularly interested in a non-linear variant of the Schrödinger equation, which in its original form is written as

$$i\hbar\frac{\partial\Psi}{\partial t}(x,t)+\frac{\hbar}{2}\Delta\Psi(x,t)-V_0(x)\Psi(x,t)+|\Psi(x,t)|^{p-1}\Psi(x,t)=0, \tag{SchrE}$$

where

$$\hbar = 6.62607015\times 10^{-34} J\,Hz^{-1}$$

denotes the Plank constant and i is the imaginary unit. A function

$$\Psi(x,t)=e^{-iEt/\hbar}v(x)$$

is a standing wave solution of (SchrE) if and only if v is such that

$$\frac{1}{2}\hbar^2\Delta v(x)-(V_0(x)-E)v(x)+|v(x)|^{p-1}v(x)=0, \quad x\in\mathbb{R}^N. \tag{1.1}$$

For the study of the semi-classical limit of (SchrE), the behavior of solutions as $\hbar$ approaches zero, is usually rewritten as

$$\begin{array}{rcl} \varepsilon^2\Delta v+V(x)v+|v|^{p-1}v &=& 0 \quad x\in\mathbb{R}^N \\ \lim\limits_{|x|\to\infty} v(x) &=& 0, \end{array} \tag{1.2}$$

where

$$\varepsilon^2=\frac{\hbar^2}{2} \quad \text{and} \quad V(x)=V_0(x)-E.$$

In (1.2) it is also assumed that $N\geq 3$ and $p+1\in(2,2^*)$ with $2^*=\frac{2N}{N-2}$.

There has been numerous studies carried out under the assumption of positivity over the potential V and $p=2$. These works use different approaches based either on the variational method, the Lyapunov-Schmidt reduction or a combination of both. Some of these works are [8], [9], [14], [32], [16] and [1].

The works [4] and [5] change the assumption of a positive potential and consider instead $\{x \in \mathbb{R}^N \,/\, V(x) = 0\} \neq \emptyset$ and $p = 2$. [4] studies the existence and qualitative properties of standing wave solutions of the non-linear Schrödinger equation (SchrE) for small $\hbar$. In [4] it is also shown that there exists a positive standing wave which is trapped in a neighborhood of an isolated component of $\{x \in \mathbb{R}^N \,/\, V(x) = 0\}$ and whose amplitude goes to 0 as $\hbar \to 0$. By rewriting (1.1) as

$$\begin{aligned} \varepsilon^2 \Delta v + V(x)v + v^p &= 0 \quad v > 0, \quad x \in \mathbb{R}^N \\ \lim_{|x| \to 0} v(x) &= 0, \end{aligned} \tag{1.3}$$

the existence of localized solutions for (1.3) under suitable conditions for the potential V is proved in [4] (Sec. 2). This is done by rephrasing the original problem as a minimization problem for the energy functional

$$I^\varepsilon(u) = \int_{\mathbb{R}^N} \varepsilon^2 |\nabla u|^2 + V u^2 \, dx \tag{1.4}$$

under the constraint $\int_{\mathbb{R}^N} |u|^{p+1}\, dx = 1$. In such case, a solution of (1.3) is called a *least-energy* or *ground state* solution if it minimizes (1.4). In order to prove the existence of such a solution, in [4] the concentration-compactness lemma of Lions, [20], is used. Moreover, the authors noticed that the asymptotic profiles of localized solutions depend in a very delicate way on some local properties of an isolated component of the set where V vanishes. Hence, by denoting with A the isolated component of the zero set of V, three cases are identified.

1. **The flat case:** where $\mathrm{int}(A)$ is non-empty.
2. **The finite case:** where A is a single point and V behaves like a finite-order polynomial near A.
3. **The infinite case:** where A is a single point and V is exponentially flat near A.

Alternative approaches to those adopted in [4] have been explored in recent works. For instance, in [12] where the *flat case* is considered, the equation (1.2) together with its limit problem

$$\Delta u + |u|^{p-1} u = 0, \qquad \text{in } \Omega, \tag{1.5}$$

with boundary condition $u = 0$ on $\partial\Omega$, where $\Omega = \mathrm{int}\{x \in \mathbb{R}^N : \; V(x) = \inf V = 0\}$ is assumed to be non-empty, connected and smooth. By considering $V(x) \to \infty$ as $|x| \to \infty$, a Ljusternik-Schnirelman scheme is used to prove the existence of infinitely many solutions. This work motivated several studies that explore similar approaches considering different assumptions. A numerical approach to the one-dimensional flat case was developed in [26]. Moreover, the N-dimensional finite and infinite case studied in [27] and [25] obtained analogous results to those presented in [12] in terms of existence, multiplicity and asymptotic behavior of solutions.

In this work we study a generalization problem (1.3), where we replace the Laplacian operator with the p-Laplacian operator and prove the existence of a non-trivial solution by working as in [10]. Specifically, for $\varepsilon > 0$, we consider the following boundary value problem.

$$\begin{cases} -\varepsilon^2 \Delta_p u(x) + V(x)|u(x)|^{p-2}u(x) - |u(x)|^{q-1}u(x) = 0 \quad x \in \mathbb{R}^N \\ u(x) \to 0 \quad \text{as } |x| \to \infty, \end{cases} \tag{G_ε}$$

where

$$\Delta_p u = \operatorname{div}(|\nabla u|^{p-2}\nabla u)$$

and

$$1 < p < q + 1 < p^*,$$

with

$$p^* = \begin{cases} \frac{pN}{N-p}, & \text{if } N \geq 3; \\ \infty, & \text{if } N = 1, 2. \end{cases}$$

Additionally, we will assume that

$$V \in C(\mathbb{R}^N) \text{ is non-negative and} \tag{C}$$

$$\lim_{|x|\to\infty} V(x) = +\infty. \tag{L}$$

This document is organized as follows:

- In Chapter 2, we present the mathematical framework where we will work on. First, we recall fundamental results from Functional Analysis; namely, the concepts of metric spaces ans their properties together with an introduction of bounded linear operator theory. A review of Lebesgue and Sobolev spaces is also provided. Then, some topics from calculus of variations and its relation with non-linear analysis for PDE's are considered. Here the notions of differentiability and the Euler-Lagrange equation are presented in order to introduce the existence of minimizers, the Palais-Smale condition and the Mountain Pass Theorem in the context of the variational approach for solving PDE's. Lastly, the p-Laplacian operator is introduced.

- In Chapter 3, first we proof some preliminary results about the properties of the space and functional realted to the weak formulation of our problem. Once these properties are obtained, we state the main theorem of this work and present its proof.

- In Chapter 4, we sate some conclusions and recommendations.

Chapter 2

Mathematical framework

2.1 Topics of Functional Analysis

In this section we give a brief overview of some topics of Functional Analysis that are relevant for our work. The section is structured as follows: first, we recall fundamental concepts for Banach spaces; then, we cover the main theorems about Lebesgue spaces; finally, we will introduce Sobolev spaces and embedding theorems.

2.1.1 Preliminaries

In this part we review the notion of ”space” going from general to more specific. For this subsection our main guides are [15] and [24].

Definition 2.1.1 (Metric space)**.** *A metric space is a pair* (X, d)*, where* X *is a non-void set, whose elements* x *will be called points, and* $d : X \times X \longrightarrow \mathbb{R}$ *verifies, for* $x, y, z \in X$*,*

$$0 \leq d(x, y) < \infty \tag{M1}$$

$$d(x, y) = 0 \iff x = y. \tag{M2}$$

$$d(x, y) = d(y, x). \tag{M3}$$

$$d(x, y) \leq d(x, z) + d(z, y). \tag{M4}$$

Let $x_0 \in X$ and $r > 0$, generic and fixed. We define

$$B(x_0, r) = \{x \in X \ / \ d(x, x_0) < r\}, \tag{2.1}$$

$$\overline{B}(x_0, r) = \{x \in X \ / \ d(x, x_0) \leq r\}, \tag{2.2}$$

$$S(x_0, r) = \{x \in X \ / \ d(x, x_0) = r\}, \tag{2.3}$$

referred to as *ball, closed ball* and *sphere* of center x_0 and radius r, respectively.

We say that $U \subset X$ is **open** iff

$$\forall x \in U, \exists r > 0 : \quad B(x, r) \subset U. \tag{2.4}$$

We also say that $K \subset X$ is **closed** iff $K^c = X \setminus K$ is open. It is clear that balls are open sets and closed balls are closed sets. Its important to point out that the terminology of *open sets and closed sets* comes from the more abstract notion of *topological space*. Let's recall this definition.

Definition 2.1.2 (Topological space)**.** *Let $X \neq \emptyset$ and $\mathcal{T}$ a family of subsets of X. We say that $\mathcal{T}$ is a* topology *on X iff the following conditions hold*

$$\emptyset \in \mathcal{T} \quad \wedge \quad X \in \mathcal{T}; \tag{T1}$$

$$\forall A, B \in \mathcal{T}: \quad A \cap B \in \mathcal{T}; \tag{T2}$$

$$\forall (A_\lambda)_{\lambda \in \Lambda} \subset \mathcal{T}: \quad \bigcup_{\lambda \in \Lambda} A_\lambda \in \mathcal{T}. \tag{T3}$$

In this case, the pair $(X, \mathcal{T})$ is called a ***topological space.*** *The elements of $\mathcal{T}$ are referred as* ***open sets*** *and their complements,* ***closed sets****.*

The following theorem states that a the metric induces a topology in the space X. We shall denote the topology of the d-open sets, defined by (2.4), as $\mathcal{T}_d$.

Theorem 2.1.3. *Let (X, d) be a metric space. Then the sets which are open in the metric sense form a topology.*

Proof. Let (X, d) be a metric space and $\mathcal{T}_d$ the collection of sets defined by (2.4).

(T1) By vacuity, $\emptyset \in \mathcal{T}_d$. Also, it is obvious that $X \in \mathcal{T}_d$.

(T2) Let $A, B \in \mathcal{T}_d$, generic. Then, if $A \cap B = \emptyset$ then $A \cap B \in \mathcal{T}_d$ by (T1). Otherwise, let $x \in A \cap B$. Then we have that

$$\exists r_1, r_2: \quad B(x, r_1) \subset A \quad \wedge \quad B(x; r_2) \subset B.$$

Cleary, for $r \leq \min\{r_1, r_2\}, x \subset B(x, r) \subset A \cap B$ and consequently (T2) holds.

(T3) Let $A = \bigcup_{\lambda \in \Lambda} A_\lambda$ with $(A_\lambda)_{\lambda \in \Lambda} \subset \mathcal{T}_d$. Then, for any fixed $x_0 \in A$ we have that there exist λ_0, r_0 such that $x_0 \subset B(x_0, r_0) \subset A_{\lambda_0} \subset A$. This concludes the proof.

$\square$

Given a topological space $(X, \mathcal{T})$, we say that $V \subseteq X$ is a **neighborhood of the point** $x_0 \in X$ iff

$$\exists U \in \mathcal{T}: \quad U \subseteq V.$$

We denote

$$\mathcal{N}(x_0) = \{V \subseteq X \ / \ V \text{ is a neighborhood of } x_0\}.$$

We also say that $\mathcal{F} \subseteq \mathcal{N}(x_0)$ is a *fundamental system of neighborhoods* or *local basis* of x_0 iff

$$\forall V \in \mathcal{N}(x_0), \exists W \in \mathcal{F}: \quad W \subseteq V \tag{2.5}$$

In particular, we say that a fundamental system is *open* iff all its elements are open sets.

Based on the concepts of open and closed sets, we define some concepts that will be constantly used throughout this text. Let (X, d) be a metric space and $A \in X$. We define the **interior** of A, $\text{int}(A)$, as the *biggest* open set contained in A. Analogously, we define the **closure** of A, $\overline{A}$, as the *smallest* closed set that contains A. Using these sets, we also define the **boundary** of A as $\partial A := \overline{A} \setminus \text{int}(A)$. We will now present a fundamental definition for our further study.

Let $M \subset X$, (X, d) metric space. The set M is said to be **dense** in X if

$$\overline{M} = X. \tag{2.6}$$

Moreover, X is said to be **separable** if it has a **countable and dense** subset.

Remark 2.1.4. *(Density) Point* (2.6) *is equivalent to*

$$\forall\, x \in X, \forall\, \varepsilon > 0, \exists\, m \in M : \quad d(x, m) < \varepsilon. \tag{2.7}$$

A concept that is essential to metric spaces is that of *completeness.* Before introducing this concept, we have to provide a couple of concepts related to sequences.

Definition 2.1.5 (Convergence of a sequence. Cauchy sequence)**.** *A sequence $(x_n)_{n\in\mathbb{N}}$ in a metric space (X, d) is said to* ***converge*** *to some limit $x \in X$ iff*

$$\forall \varepsilon > 0,\, \exists N = N(\varepsilon) \in \mathbb{N} : \quad n > N \implies d(x_n, x) < \varepsilon \tag{2.8}$$

In this case we write, $\lim\limits_{n\to\infty} x_n = x$ or simply $x_n \longrightarrow x$. A sequence $(x_n)_{n\in\mathbb{N}}$ is said to be ***Cauchy (or fundamental)*** *iff*

$$\forall \varepsilon,\, \exists N = N(\varepsilon) \in \mathbb{N} : \quad n, m > N \implies d(x_m, x_n) < \varepsilon \tag{2.9}$$

The space X is said to be ***complete*** *if every Cauchy sequence in X converges to some element $x \in X$.*

We are interested in working in complete spaces because they have nicer properties than incomplete spaces (e.g. convergence of Cauchy sequences). A very important fact is that any arbitrary incomplete metric space can be "completed". Before formally understanding the completion process, we need to familiarize ourselves with the concept of *isometric spaces.*

Let $(X, d), (Y, \rho)$ be metric spaces. Then:

(a) A mapping $T : X \to Y$ is said to be **an isometry** if it preserves distances, i.e.,

$$\forall x, y \in X : \quad \rho(Tx, Ty) = d(x, y). \tag{2.10}$$

(b) Moreover, the space X is said to be **isometric** to Y iff T is a bijective isometry. Thus, spaces X, Y are called **isometric spaces**.

It is clear that isometric spaces can be understood as two copies of the same abstract space and differ only by the nature of their elements. As we can notice from Definition 2.1.5, the convergence of a sequence is not an intrinsic property of the sequence itself but depends of the space on which the sequence lies.

Theorem 2.1.6 (Completion)**.** *For a metric space (X, d) there exists a complete metric space $(\tilde{X}, \tilde{d})$ which has a subspace W such that:*

(i) X and W are isometric, and

(ii) $\overline{W} = \tilde{X}$.

Moreover, $\tilde{X}$ is unique except for isometries.

The proof of Theorem 2.1.6 is quite lengthy and can be found e.g. in [15] and [24].

Let's recall that a **linear space** is a set provided with elements called *vectors* and two closed algebraic operations: addition of vectors and multiplication by scalars. A linear space where these operations are considered by the metric produce a richer space, namely a **normed space**. As a result, the following generalization of the *size* of a vector is introduced.

Definition 2.1.7 (Normed space. Banach space.)**.** *Let V be a linear space. We say that $\|\cdot\| : V \to \mathbb{R}$ is a **norm** on V iff*

$$\forall x \in V : \quad \|x\| \geq 0; \tag{N1}$$

$$\forall x \in V : \quad \|x\| = 0 \iff x = 0; \tag{N2}$$

$$\forall x \in V, \forall \alpha \in \mathbb{R} : \quad \|\alpha x\| = |\alpha| \, \|x\| ; \tag{N3}$$

$$\forall x, y \in V : \quad \|x + y\| \leq \|x\| + \|y\| \ \textit{(Triangle inequality)}. \tag{N4}$$

Then we say that $(V, \|\cdot\|)$ is a normed space. A metric d is induced by the by

$$d(x, y) = \|x - y\| .$$

*Whenever the space (X, d) is complete, we say that X is a **Banach space**.*

To finish this subsection, we recall a concept that characterizes norms that can be compared.

Theorem 2.1.8 (Norm equivalence)**.** *Let $\|\cdot\| , \|\cdot\|_0$ be norms on a vector space X. We say that they are equivalent iff*

$$\exists \, c_1, c_2 > 0, \forall x \in V : \quad c_1 \|x\|_0 \leq \|x\| \leq c_2 \|x\|_0 . \tag{2.11}$$

Here, the topologies induced by the norms coincide, i.e., $\mathcal{T}_{\|\cdot\|} = \mathcal{T}_{\|\cdot\|_0}$.

Remark 2.1.9 (Norms on finite dimensional spaces)**.** *Let X be a finite dimensional vector space, i.e., $dim(X) < \infty$. Then, all norms on X are equivalent.*

To conclude we present a useful inequality presented in [19].

Proposition 2.1.10. *Let $1 \leq p \leq 2$. Then*

$$\forall\, a, b \in \mathbb{R}^N : \qquad \Big||b|^{p-2}b - |a|^{p-2}a\Big| \leq 2^{2-p}|b-a|^{p-1}, \tag{2.12}$$

where $|\cdot|$ denotes the usual Euclidean norm on $\mathbb{R}^N$:

$$\forall\, a \in \mathbb{R}^N : \quad |a| = |(a_1, ..., a_2)| = \left(\sum_{i=1}^{N} a_i^2\right)^{1/2}.$$

2.1.2 Bounded linear operators

In the same way as we consider real valued functions over the real line $\mathbb{R}$ in elementary calculus, we would like to consider mappings over more general spaces such as metric and normed spaces. In the specific case where we consider a mapping over a normed space, we would call this mapping an *operator*. For this subsection our main references are [15], [24] and [11].

Let's recall the concept of *linear subspace*. Let X be a linear space and Y a non-empty subset of X. We say that Y is a linear subspace of X iff

$$\forall\, u, v \in Y, \forall\, a, b \in \mathbb{R} : \quad au + bv \in Y.$$

In this context, we will need operators that preserve the operations of the linear space in the sense of the following definition. Let X, Y be real linear spaces and D a linear subspace of X. We say that $T : D \subseteq X \longrightarrow Y$ is a **linear operator** iff

$$\forall\, x, y \in D, \forall\, \alpha \in \mathbb{R} : \quad T(\alpha x + y) = \alpha Tx + Ty. \tag{2.13}$$

Assume now that X, Y are normed spaces. We say that T is **bounded** iff

$$\exists c > 0,\ \forall x \in D : \quad \|Tx\| \leq c\|x\|\,. \tag{2.14}$$

Moreover, we define the norm of the operator T, $\|T\|$, as the infimum of the values c such that (2.14) holds, i.e.,

$$\|T\| = \inf(\mathcal{O}_T), \quad \text{where} \quad \mathcal{O}_T = \{c > 0 \ / \ \forall\, x \in D : \ \|Tx\| \leq c\|x\|\}. \tag{2.15}$$

Remark 2.1.11. *Note that by taking infimum in (2.14) we get*

$$\forall x \in D : \quad \|Tx\| \leq \|T\|\|x\|\,. \tag{2.16}$$

In the following lemma, we recall an alternative formula to calculate the norm of a bounded linear operator and also show that this norm satisfies all conditions from Definition 2.1.7.

Lemma 2.1.12 (Norm of an operator). *Let T be bounded. Then*

(i) An equivalent definition of (2.15) *is given by*

$$\|T\| = \sup_{x \in D \setminus \{0\}} \frac{\|Tx\|}{\|x\|}, \tag{2.17}$$

so that

$$\|T\| = \sup_{\|x\|=1} \|Tx\| \tag{2.18}$$

(ii) The norm defined in (2.15) *satisfies* (N1) - (N4).

Proof. (i) Let $x \in D \setminus \{0\}$ such that $\|x\| = \alpha, \alpha > 0$. Set $y = (1/\alpha)x$. Then, by the linearity of T and (2.17) we have that

$$\|T\| = \sup_{x \in D \setminus \{0\}} \frac{1}{\alpha}\|Tx\| = \sup_{x \in D \setminus \{0\}} \left\| T\left(\frac{1}{\alpha}x\right)\right\| = \sup_{\|y\|=1} \|Ty\|.$$

(ii) By (2.17) we have that:
(N1) is trivial. (N2) holds since we have that $\|0\| = 0$ and if we assume $\|T\| = 0$ then

$$(\forall x \in D : \quad Tx = 0) \quad \implies \quad T = 0.$$

By (i), property (N3) also follows since

$$\|\alpha T\| = \sup_{\|x\|=1} |\alpha|\|Tx\| = |\alpha|\|T\|, \quad \forall \alpha \in \mathbb{R}.$$

Finally by (i), for $x \in D$, (N4) follows from

$$\sup_{\|x\|=1} \|(T_1 + T_2)x\| = \sup_{\|x\|=1} \|T_1x + T_2x\| \leq \sup_{\|x\|=1} \|T_1x\| + \sup_{\|x\|=1} \|T_2x\|.$$

This concludes the proof.

□

Let $T \neq 0$ be a bounded linear operator as in (2.14). Given $x, y \in D$, since D is a subspace of the linear space X, we have that $x - y \in D$. Let $\varepsilon > 0$, generic. Since T is bounded we have that

$$\|Tx - Ty\| \leq \|T\|\|x - y\|. \tag{2.19}$$

Let's choose $\delta = \varepsilon/\|T\|$ and assume that $\|x - y\| < \delta$. Thus, (2.19) implies that

$$\|Tx - Ty\| < \|T\|\,\delta < \varepsilon. \tag{2.20}$$

From (2.20) it is clear that **the boundedness of T implies it's continuity.**

By Lemma 2.1.12, we already know that (2.15) is a norm. We denote $\mathcal{L}(X, Y)$ as the normed space of all bounded linear operators from the normed space X into a normed space Y. The following theorem states a condition for the completeness of such space.

Theorem 2.1.13 (Completeness of $\mathcal{L}(X,Y)$)**.** *If Y is a Banach space, then $\mathcal{L}(X,Y)$ is also a Banach space.*

Proof. We have to prove that every Cauchy sequence of $\mathcal{L}(X,Y)$ is convergent.

(i) Let $(T_n)_{n\in\mathbb{N}} \subseteq \mathcal{L}(X,Y)$ be a generic Cauchy sequence. Then for every $\varepsilon > 0$ there is $N \in \mathbb{N}$ such that $n, m > N$ implies

$$\|T_n - T_m\| < \varepsilon.$$

Therefore, for any $x \in X$,

$$\|T_n x - T_m x\| = \|(T_n - T_m)x\| \leq \|T_n - T_m\|\|x\| < \varepsilon\|x\|\,, \tag{2.21}$$

so that for every $x \in X$, the sequence $(T_n x)_{n\in\mathbb{N}} \subseteq Y$ is of Cauchy. The completeness of Y that

$$\forall\, x \in X, \exists!\, Tx \in Y: \quad \lim_{n\to\infty} T_n x = Tx. \tag{2.22}$$

(ii) By (2.22) we have found an operator $T : X \longrightarrow Y$. Let's prove that T is linear. Let $x, y \in X, \lambda \in \mathbb{R}$ be generic. By the linearity of T_n we have that

$$T(\lambda x + y) = \lim_{n\to\infty}[\lambda T_n x + T_n y] = \lambda \lim_{n\to\infty} T_n x + \lim_{n\to\infty} T_n y = \lambda Tx + Ty.$$

Since x, y, λ were arbitrary, we proved that T is linear.

(iii) If we pass to the limit with $m \to \infty$ in (2.21), then for $n > N$ and any $x \in X$,

$$\|(T_n - T)x\| \leq \varepsilon\|x\|\,, \tag{2.23}$$

which implies $T_n - T \in \mathcal{L}(X,Y), n > N$. Since $\mathcal{L}(X,Y)$ is a linear space, the last provides $T \in \mathcal{L}(X,Y)$.

(iv) By (2.23) we have that for $n > N$

$$\|T_n - T\| < \varepsilon.$$

Since ε was chosen arbitrarily, we have proved that $T_n \to T$ as $n \to \infty$.

□

To finish our overview of operators let's recall the concept of a *functional.* We call **functional** an operator whose range is either on $\mathbb{R}$ (or $\mathbb{C}$). Notice that all the previous theorems also apply to functionals. Specifically, since the space $\mathcal{L}(X,\mathbb{R})$ is so important we will call it **the dual space of X** and use the following notation:

$$X^* = \mathcal{L}(X,\mathbb{R}). \tag{2.24}$$

Notice that (2.24) is always a Banach space by Theorem 2.1.13.

Remark 2.1.14 (Duality product)**.** *Assume X is normed space, $x \in X$ and $\eta \in X^*$. It's usual to find the notation*

$$\eta(x) = \langle \eta, x \rangle \tag{2.25}$$

where $\langle \cdot, \cdot \rangle : X^* \times X \longrightarrow \mathbb{R}$ is called the ***duality product*** *in X.*

One of the most important concepts in Functional Analysis is that of *reflexivity*. Let X be a normed space, $x \in X$ and let's define a mapping

$$\begin{aligned} \varphi_x : X^* &\longrightarrow \quad \mathbb{R} \\ \eta \ &\longmapsto \langle \varphi_x, \eta \rangle = \langle \eta, x \rangle. \end{aligned}$$

Clearly, $\varphi_x \in X^{**}$ and $\|\varphi_x\|_{X^{**}} = \|x\|$. The **cannonical mapping** is

$$\begin{aligned} J : X &\longrightarrow \quad X^{**} \\ x \ &\longmapsto J(x) = \varphi_x. \end{aligned} \tag{2.26}$$

Lemma 2.1.15 (Cannonical embedding)**.** *The cannonical mapping given in* (2.26) *is an embedding of the normed space X into its bidual X^{**}, i.e., its an isomorphism between X and its image through the cannonical mapping $J(X) \subseteq X^{**}$.*

Proof. The linearity of J follows from the linearity of φ_x, i.e.,

$$\forall \alpha, \beta \in \mathbb{R}, \forall x, y \in X : \quad \varphi_{\alpha x + \beta y}(\eta) = \eta(\alpha x + \beta y) = \alpha \eta(x) + \beta \eta(y) = \alpha \varphi_x(\eta) + \beta \varphi_y(\eta).$$

In particular, $\varphi_x - \varphi_y = \varphi_{x-y}$, so that

$$\left\| \varphi_x - \varphi_y \right\| = \left\| \varphi_{x-y} \right\| = \|x - y\| .$$

The previous shows that J is an isometry. Also, from (N2) it follows that it is injective. Hence, since J is bijective if we restrict the codomain to its image $J(X)$, the required result follows. ☐

In general, J will not be surjective. However, when this does happen we say that the space X is **reflexive**, i.e., X is reflexive iff

$$J(X) = X^{**}.$$

Once we have defined the dual space of a normed space X, we can define a new kind of convergence.

Definition 2.1.16 (Weak convergence)**.** *Let X be a real Banach space and $(u_n)_{n \in \mathbb{N}} \subseteq X$ a sequence. We say that $(u_n)_{n \in \mathbb{N}}$* ***converges weakly*** *to some $u \in X$, written*

$$u_n \rightharpoonup u \quad \iff \quad \forall \eta \in X^* : \quad \langle \eta, u_n \rangle \to \langle \eta, u \rangle. \tag{2.27}$$

Additionally,

$$\|u\| \le \liminf_{n \to \infty} \|u_n\| \tag{2.28}$$

Remark 2.1.17 (Weak topology $\sigma(X, X^*)$)**.** *The convergence in* (2.27) *is actually the characterization of the convergence in the* weak topology on X, $\sigma(X, X^*)$. *This is the smallest topology for which all the elements of X^* are still continuous. In particular,* $\sigma(X, X^*) \subseteq \mathcal{T}_{\|\cdot\|}$, *i.e., all the* weak open sets *are open in the topology induced by the norm.*

The following theorem establishes a connection between reflexive Banach spaces and weak convergence.

Theorem 2.1.18 (Weak compactness)**.** *Let X be a reflexive Banach space and suppose the sequence $(x_n)_{n\in\mathbb{N}} \subseteq X$ is* ***bounded****. Then, there exists a subsequence $(x_{n_k})_{k\in\mathbb{N}}$ such that*

$$x_{n_k} \rightharpoonup x.$$

Hence, bounded sequences in a reflexive Banach space are ***weakly precompact.***

The proof of Theorem 2.1.18 can be found in [11].

2.1.3 Lebesgue spaces

In this section we recall some concepts and tools that will be of great help throughout our study. We assume the reader is familiarized with the basic notions of measure theory, namely, the concepts of σ-algebra, measure space, measurable sets, measurable functions, integrable functions and Lebesgue measure (see, for example, [17]). Here our main references are [17],[15] and [3].

Definition 2.1.19 (Space of integrable functions $L^1(\Omega)$)**.** *Let $\Omega \subseteq \mathbb{R}^N$. We define the space of integrable functions on Ω as*

$$L^1(\Omega) := \left\{ f : \Omega \longrightarrow \mathbb{R} \ / \ \|f\|_{L^1(\Omega)} := \int_\Omega |f(x)|dx < \infty \right\}$$

Remark 2.1.20. *The integral symbol in Definition 2.1.19 corresponds to the Lebesgue integral and the symbol dx refers to the Lebesgue measure on $\mathbb{R}^N$.*

Now, we present some fundamental theorems that involve integrable functions. In all of the following, $\Omega \subseteq \mathbb{R}^N$ is measurable.

Theorem 2.1.21 (Monotone convergence theorem)**.** *Let $(f_n)_{n\in\mathbb{N}} \subseteq L^1(\Omega)$ be an increasing sequence. Then there exists $f \in L^1(\Omega)$ such that*

$$\lim_{n\to\infty} f_n(x) = f(x), \qquad a.e.\ x \in \Omega$$

and

$$\lim_{n\to\infty} \int_\Omega f_n(x)dx = \int_\Omega f(x)dx.$$

Lemma 2.1.22 (Fatou's Lemma). *Let* $(f_n)_{n\in\mathbb{N}} \subseteq L^1(\Omega)$ *be a sequence of nonnegative functions. Then* $f(x) := \liminf_{n\to\infty} f_n(x)$ *is integrable and*

$$\liminf_{n\to\infty} \int_\Omega f_n(x)dx \geq \int_\Omega f(x)dx.$$

Theorem 2.1.23 (Dominated convergence theorem). *Let* $(f_n)_{n\in\mathbb{N}} \subseteq L^1(\Omega)$ *and assume that it converges to some* f *pointwise a.e. If there exists* $G \in L^1(\Omega)$ *such that*

$$\forall n \in \mathbb{N}: \qquad |f_n(x)| \leq G(x),$$

then

$$|f(x)| \leq G(x) \quad \text{and} \quad \lim_{n\to\infty} \int_\Omega f_n(x)dx = \int_\Omega f(x)dx.$$

Theorem 2.1.24 (Fubini). *Consider* $\Omega_1, \Omega_2 \subseteq \mathbb{R}^N$*, measurable, and let* $f \in L^1(\Omega_1 \times \Omega_2)$*. If* $f \geq 0$*, then the following 3 integrals are equal:*

$$\int_{\Omega_1\times\Omega_2} f(x,y)d(x,y) = \int_{\Omega_2} \left(\int_{\Omega_1} f(x,y)dx \right) dy = \int_{\Omega_1} \left(\int_{\Omega_2} f(x,y)dy \right) dx.$$

For a function $f \in \mathrm{C}(\mathbb{R}^N)$ we define its *support* as

$$\mathrm{supp}(f) = \{x \in \mathbb{R}^N \,/\, f(x) = 0\}$$

We denote the *space of continuous functions with compact support* as

$$\mathrm{C}_0(\mathbb{R}^N) = \{f \in \mathrm{C}(\mathbb{R}^N) \,/\, \mathrm{supp}(f) \text{ is compact }\}. \tag{2.29}$$

Theorem 2.1.25 (Dense subspace of $L^1(\mathbb{R}^n)$). *The space* $\mathrm{C}_0(\mathbb{R}^N)$ *defined in* (2.29) *is dense in* $L^1(\mathbb{R}^N)$*, i.e.,*

$$\forall f \in L^1(\mathbb{R}^N), \forall \varepsilon > 0, \exists g \in C_c(\mathbb{R}^N): \qquad \|f - g\|_{L^1(\Omega)} \leq \varepsilon.$$

Theorem 2.1.25 says that elements of $L^1(\mathbb{R}^N)$ can be approximated by continuous and compactly supported functions in $\mathbb{R}^N$. The previous results are the most important ones from Measure Theory and their proofs can be found e.g. in [17] and [3].

Now we introduce the Lebesgue spaces, also known as L^p spaces. It is necessary to review the properties of these spaces since they will be used for further results at the end of this section. Our main guide to the study of these spaces is [3].

Definition 2.1.26 ($L^p(\Omega)$). *Let $p \in \mathbb{R}, 1 < p < \infty$; we set*

$$L^p(\Omega) := \left\{ f : \Omega \longrightarrow \mathbb{R} \ / \ f \text{ is measurable and } |f|^p \in L^1(\Omega) \right\}$$

with

$$\|f\|_{L^p} = \|f\|_p := \left(\int_\Omega |f(x)|^p dx \right).$$

The case when $p = \infty$ is special and requires a different definition.

Definition 2.1.27 ($L^\infty(\Omega)$). *We define*

$$L^\infty(\Omega) := \left\{ f : \to \mathbb{R} \;\middle|\; \begin{array}{l} f \text{ is measurable and} \\ \exists C > 0 \ : \ |f(x)| \leq C \text{ a.e. on } \Omega \end{array} \right\}$$

with

$$\|f\|_{L^\infty(\Omega)} = \|f\|_\infty = \inf\{C \ / \ |f(x)| \leq C > 0 \text{ a.e. on } \Omega\}$$

For proving that $\|\cdot\|_p, 1 < p < \infty$, is indeed a norm, we will need *Hölder's inequality.* Before providing the proof of this very useful result, we first recall a couple of important concepts.

Remark 2.1.28 (Conjugate exponent). *Let $1 \leq p \leq \infty$. We define p', the conjugate exponent of p, by*

$$\frac{1}{p} + \frac{1}{p'} = 1.$$

Remark 2.1.29 (Young's inequality). *Let $1 < p < \infty$. Then we have that*

$$\forall a, b \geq 0 : \qquad ab \leq \frac{a^p}{p} + \frac{b^{p'}}{p'} \tag{2.30}$$

Inequality 2.30 follows directly from the concavity of the function ln *on* $(0, \infty)$*:*

$$\ln\left(\frac{a^p}{p} + \frac{b^{p'}}{p'} \right) \geq \frac{1}{p} \ln(a^p) + \frac{1}{p'} \ln(b^{p'}) = \ln(ab).$$

Theorem 2.1.30 (Hölder's inequality). *Let $\Omega \subseteq \mathbb{R}^N$, measurable. Assume that $f \in L^p(\Omega)$ and $g \in L^{p'}(\Omega)$ with $1 \leq p \leq \infty$. Then $fg \in L^1(\Omega)$ and*

$$\|fg\|_1 \leq \|f\|_p \|g\|_{p'}. \tag{2.31}$$

Proof. Let $f \in L^p(\Omega), g \in L^{p'}(\Omega)$, generic. If $f = 0$ or $g = 0$, (2.31) immediately follows. Hence, lets assume that $f \neq 0$ and $g \neq 0$.

i) Let's first assume that $p = 1$. Thus,

$$\int_\Omega |f(x)g(x)|dx \leq \|g\|_\infty \|f\|_1 .$$

This immediately implies $fg \in L^1(\Omega)$ and (2.31) holds. An analogous reasoning provides the result if $p = \infty$.

ii) Now, let's assume that $1 < p < \infty$. From (2.30), we have that

$$\begin{aligned} |f(x)g(x)| &\leq \frac{1}{p}|f(x)|^p + \frac{1}{p'}|g(x)|^{p'} \\ \Longrightarrow \int_\Omega |f(x)g(x)|dx &\leq \frac{1}{p}\|f\|_p^p + \frac{1}{p'}\|g\|_{p'}^{p'} \end{aligned} \tag{2.32}$$

Inequality (2.32) implies that $fg \in L^1(\Omega)$. Now, by replacing f with $\lambda f, (\lambda > 0)$ in (2.32) it follows that

$$\begin{aligned} \int_\Omega |\lambda f(x)g(x)|dx &\leq \frac{1}{p}\|\lambda f\|_p^p + \frac{1}{p'}\|g\|_{p'}^{p'} , \\ \int_\Omega |f(x)g(x)|dx &\leq \frac{\lambda^{p-1}}{p}\|f\|_p^p + \frac{1}{\lambda p'}\|g\|_{p'}^{p'} . \end{aligned} \tag{2.33}$$

By choosing $\lambda = \|f\|_p^{-1}\|g\|_p^{p'/p}$ and recalling Remark 2.1.28, inequality (2.33) becomes

$$\begin{aligned} \int_\Omega |f(x)g(x)|dx &\leq \frac{1}{p}\|f\|_p^{p+1-p}\|g\|_{p'}^{p'(1-1/p)} + \frac{1}{p'}\|f\|_p\|g\|_{p'}^{p'(1-1/p)} \\ &\leq \left(\frac{1}{p} + \frac{1}{p'}\right)\|f\|_p\|g\|_{p'} \\ &\leq \|f\|_p\|g\|_{p'} \end{aligned}$$

Since f, g were chosen arbitrarily, the required result follows. □

The following remarks are extensions of Theorem 2.1.30. They shall be used in the following chapter.

Remark 2.1.31 (Extension of Hölder's inequality)**.** *Assume* $(f_i)_{i=1}^k$ *s.t.* $f_i \in L^{p_i}(\Omega), 1 \leq i \leq k$ *with*

$$\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2} + ... + \frac{1}{p_k} \leq 1.$$

Then the product $f = f_1 f_2 ... f_k$ *belongs to* $L^p(\Omega)$ *and*

$$\|f\|_p \leq \|f_1\|_{p_1}\|f_2\|_{p_2} ... \|f_k\|_{p_k} .$$

Theorem 2.1.32. *(Interpolation inequality) Let* $\Omega \subseteq$ *and* $f \in L^p(\Omega) \cap L^q(\Omega), 1 \leq p \leq q \leq \infty$. *Then* $f \in L^r(\Omega), \forall r \in [p, q]$. *Moreover,*

$$\|f\|_r \leq \|f\|_p^{\alpha} \|f\|_q^{1-\alpha} \quad \text{where } \frac{1}{r} = \frac{\alpha}{p} + \frac{1-\alpha}{q}, \quad \alpha \in [0,1] \tag{2.34}$$

The next result can be found in [29]. It will be crucial in a future proof.

Theorem 2.1.33 (Hölder's inequality for $0 < p < 1$)**.** *Let* $0 < p < 1$ *and let* $f, g > 0$ *be functions in* $\mathrm{L}^p(\Omega)$ *and* $\mathrm{L}^{p'}(\Omega)$, *respectively. Then we have*

$$\int_\Omega f(x)g(x)\,dx \geq \left(\int_\Omega f(x)^p\,dx\right)^{1/p} \left(\int_\Omega g(x)^{p'}\,dx\right)^{1/p'}, \tag{2.35}$$

unless $\int_\Omega g(x)^{p'}\,dx = 0$. *Moreover, note that* $p' < 0$.

Now we have all the necessary tools to prove that the functionals given in Definitions 2.1.26 and 2.1.27, are actually norms. The next theorem then states that for $1 \leq p \leq \infty$ the space $L^p(\Omega)$ is a normed space.

Theorem 2.1.34 ($L^p(\Omega)$ is a normed space)**.** *Assume* $1 \leq p \leq \infty$, *then* $L^p(\Omega)$ *is a vector space and* $\|\cdot\|_p$ *is a norm.*

Proof. i) Cases $p = 1$ and $p = \infty$ are trivial.

ii) Assume $p \in (1, \infty)$ and $f, g \in L^p(\Omega)$, arbitrary. We will only show that the triangle inequality holds since the other norm properties can be easily verified. Since $|\cdot|$ is a norm on $\mathbb{R}$, by (N4) we have that

$$|f(x) + g(x)|^p \leq (|f(x)| + |g(x)|)^p \leq 2^p \left(|f(x)|^p + |g(x)|^p\right).$$

This implies that $f + g \in L^p(\Omega)$. Moreover, we notice that

$$\begin{aligned}
\|f+g\|_p^p &= \int_\Omega |f(x)+g(x)|^{p-1}|f(x)+g(x)|dx \\
&\leq \int_\Omega |f(x)+g(x)|^{p-1}\left(|f(x)|+|g(x)|\right)dx \\
&\leq \int_\Omega |f(x)+g(x)|^{p-1}|f(x)|dx + \int_\Omega |f(x)+g(x)|^{p-1}|g(x)|dx, \qquad (2.36)
\end{aligned}$$

and by Remark 2.1.28 we observe that

$$|f+g|^{p-1} \in L^{p'} \quad \text{since} \quad p'(p-1) = p.$$

Therefore, by Theorem 2.1.30 and inequality (2.36) we have that

$$\|f+g\|_p^p \leq \|f+g\|_p^{p-1}\left(\|f\|_p + \|g\|_p\right).$$

This concludes the proof.

□

The following theorem states that the Lebesgue spaces with the norms in Definitions 2.1.26 and 2.1.27 are complete normed spaces, i.e., Banach spaces. The proof of the following theorem can be found in [3], Sec. 4.2.

Theorem 2.1.35 (Fischer-Riesz). *$L^p(\Omega)$ is a Banach space for any $p, 1 \leq p \leq \infty$.*

Analogously to the case of the space of integrable functions $L^1(\Omega)$ in Theorem 2.1.25, we would like to have a dense subspace of $L^p(\Omega)$ whose elements are easier to handle. To conclude with our short overview of $L^p(\Omega)$ spaces and their properties, we recall the following density theorem.

Theorem 2.1.36 (Dense subspace). *Let $\Omega \subset \mathbb{R}^N$ be an open set. Then, $C_c^\infty(\Omega)$ is dense in $L^p(\Omega)$ for any $1 \leq p < \infty$.*

We conclude with the following summary of the properties of Lebesgue spaces. The detailed study of each of the following cases and the corresponding proofs can be found in [3].

	Reflexive	Separable	Dual space
L^p with $1 < p < \infty$	YES	YES	$L^{p'}$
L^1	NO	YES	L^∞
L^∞	NO	NO	Strictly bigger than L^1

Table 2.1: Summary of the main properties of the L^p spaces.

2.1.4 Sobolev spaces

Once we have defined the Lebesgue spaces (L^p spaces), we are in position to define the Sobolev spaces. These spaces are of particular interest as a common framework for the study of partial differential equations. In this section, we will present some basic concepts and theorems about these spaces that will be useful for the development of our work. Our main reference is [3].

Definition 2.1.37 (Sobolev space $W^{1,p}(\Omega)$). *Let $\Omega \subset \mathbb{R}^N$ be an open set and let $1 \leq p \leq \infty$. The Sobolev space $W^{1,p}(\Omega)$ is defined by*

$$W^{1,p}(\Omega) := \left\{ u \in L^p(\Omega) \;\middle|\; \begin{array}{l} \exists g_1, g_2, ..., g_n \in L^p(\Omega) \text{ such that} \\ \int_\Omega u \dfrac{\partial \varphi}{\partial x_i} = -\int_\Omega g_i \varphi, \;\; \forall \varphi \in C_c^\infty(\Omega), \;\; \forall i = 1, 2, ..., N \end{array} \right\}$$

In the context of Definition 2.1.37, for $u \in W^{1,p}(\Omega)$ we denote $\frac{\partial u}{\partial x_i} := g_i$ and

$$\nabla u = \left(\frac{\partial u}{\partial x_i}, ..., \frac{\partial u}{\partial x_N} \right).$$

The space $W^{1,p}(\Omega)$ is equipped with the norm

$$\|u\|_{1,p} = \|u\|_p + \sum_{i=1}^{N} \left\| \frac{\partial u}{\partial x_i} \right\|_p$$

or the equivalent norm

$$\|u\|_{1,p}^* = \left(\|u\|_p^p + \sum_{i=1}^{N} \left\| \frac{\partial u}{\partial x_i} \right\|_p^p \right)^{1/p}, \qquad 1 \leq p < \infty.$$

In the following proposition, we state some properties of the Sobolev space $W^{1,p}(\Omega)$ that depend on p.

Proposition 2.1.38. *Let $\Omega \subset \mathbb{R}^N$ be an open set. Then, we have that:*

i) $1 \leq p \leq \infty \implies W^{1,p}(\Omega)$ *is a Banach Space.*

ii) $1 < p < \infty \implies W^{1,p}(\Omega)$ *is reflexive.*

iii) $1 \leq p < \infty \implies W^{1,p}(\Omega)$ *is separable.*

Once we have defined the Sobolev space $W^{1,p}(\Omega)$, we can define more general Sobolev spaces. Let $m \geq 2$ be an integer and $1 \leq p \leq \infty$. We inductively define

$$W^{m,p}(\Omega) = \left\{ u \in W^{m-1,p}(\Omega); \frac{\partial u}{\partial x_i} \in W^{m-1,p}(\Omega) \ \forall i = 1, 2, ..., N \right\}. \tag{2.37}$$

Remark 2.1.39. *We also state a characterization of Sobolev spaces $W^{m,p}(\Omega)$ using the standard* multi-index notation*,*

$$W^{m,p}(\Omega) = \left\{ u \in L^p(\Omega) \;\middle|\; \begin{array}{l} \forall \alpha \text{ with } |\alpha| \leq m, \exists g_\alpha \in L^p(\Omega) \text{ such that} \\ \\ \displaystyle\int_\Omega u D^\alpha \varphi = (-1)^{|\alpha|} \int_\Omega g_\alpha \varphi, \ \ \forall \varphi \in C_c^\infty(\Omega) \end{array} \right\}$$

where $\alpha = (\alpha_1, ..., \alpha_N)$, $\alpha_i \in \mathbb{N}_*$, $i = 1, ..., N$, *and*

$$|\alpha| = \sum_{i=1}^{N} \alpha_i \quad \text{and} \quad D^\alpha = \frac{\partial^{|\alpha|} \varphi}{\partial x_1^{\alpha_1} ... \partial x_N^{\alpha_N}}.$$

Analogously as we did for $m = 1$, we define $D^\alpha u := g_\alpha$. Moreover, the space $W^{m,p}(\Omega)$ equipped with the norm

$$\|u\|_{W^{m,p}} = \sum_{0 \leq |\alpha| \leq m} \|D^\alpha u\|_p$$

is a Banach space.

From the definition of the Sobolev spaces, it is clear that they are subspaces of Lebesgue spaces. Hence, similarly to the case of a Lebesgue space (Theorems 2.1.25, 2.1.36), we would like to find a dense subspace in a Sobolev space whose elements are easier to handle. In order to do this, it is often convenient to work first in $W^{1,p}(\mathbb{R}^N)$. The following theorem and proposition will provide such space.

Theorem 2.1.40 (Linear Extension Operator)**.** *Assume* $\Omega \subset \mathbb{R}^N$ *of class* C^1 *with* Γ *bounded (or else* $\Omega = \mathbb{R}^N_+$*).Then, there exists a linear extension operator*

$$P : W^{1,p}(\Omega) \longrightarrow W^{1,p}(\mathbb{R}^N) \ \ (1 \leq p \leq \infty)$$

s.t. that for all $u \in W^{1,p}(\Omega)$*,*

$$\begin{aligned} &\textit{(i)} \ \ Pu|_{\Omega} = u, \\ &\textit{(ii)} \ \ \|u\|_{L^p(\mathbb{R}^n)} \leq C\|u\|_{L^p(\Omega)}, \\ &\textit{(iii)} \ \ \|u\|_{W^{1,p}(\mathbb{R}^N)} \leq C\|u\|_{W^{1,p}(\Omega)}, \end{aligned}$$

where $C = C(\Omega)$*.*

The proof of Theorem 2.1.40 is classical and can be found on [3], Sec.9.2. It uses extensions by reflection, local charts and the partition of unity lemma. A very useful and direct result from the previous theorem says that smooth and compactly supported functions in $\mathbb{R}^N$ are dense in $W^{1,p}(\Omega)$.

Corollary 2.1.41 (Density in $W^{1,p}(\Omega)$)**.** *Assume* $\Omega \subseteq \mathbb{R}^N$ *of class* C^1 *and let* $u \in W^{1,p}(\Omega)$ *with* $1 \leq p < \infty$*. Then*

$$\exists (u_n) \subset C_c^\infty(\mathbb{R}^N) : \qquad u|_{\Omega} \longrightarrow u \qquad \textit{in } W^{1,p}(\Omega),$$

i.e., $C_c^\infty(\mathbb{R}^N)$ *functions restricted to* Ω *are dense in* $W^{1,p}(\Omega)$*.*

Embedding theorems

In this section we deal with the following question: if we have some $u \in W^{m,p}(\Omega)$, for what range of values of q does one have $u \in L^q(\Omega)$? The next couple of theorems answer this important query. We advise the reader to pay special attention to the nature of the domain in the following results.

Theorem 2.1.42 (Sobolev, Gagliardo, Nirenberg)**.** *Let* $1 \leq p < N$*. Then*

$$W^{1,p}(\mathbb{R}^N) \subset L^{p^\star}, \ \textit{where } p^\star \textit{ is given by } \frac{1}{p^\star} = \frac{1}{p} - \frac{1}{N}, \ \textit{and} \tag{2.38}$$

$$\exists C = C(p,N), \forall u \in W^{1,p}(\mathbb{R}^N) : \ \ \|u\|_{p^\star} \leq C\|\nabla u\|_p . \tag{2.39}$$

A detailed proof of Theorem 2.1.42 can be found in [3]. The following Corollary extends the previous result.

Corollary 2.1.43. *Let* $1 \leq p < N$*. Then*

$$\forall q \in [p, p^\star] : \ \ W^{1,p}(\mathbb{R}^N) \subset L^q(\mathbb{R}^N)$$

with continuous injection, i.e.,

$$\exists C = C(p,N), \forall \|u\|_q \leq C\|u\|_{W^{1,p}}$$

Proof. Let $q \in [p, p^\star]$, generic. Then for some $\alpha \in [0,1]$ we have that

$$\frac{1}{q} = \frac{\alpha}{p} + \frac{1-\alpha}{p^\star}.$$

Let $u \in W^{1,p}(\mathbb{R}^N)$, generic. From (2.30) and (2.34) it follows that

$$\|u\|_q \leq \|f\|_p^\alpha \|f\|_{p^\star}^{1-\alpha} \leq \|u\|_p + \|u\|_{p^\star}.$$

Theorem 2.1.42 implies that

$$\|u\|_q \leq C\|u\|_{W^{1,p}}.$$

This concludes the proof. □

Before presenting the next result, we recall the following definitions.

Definition 2.1.44 (Compact linear operator. Compact embedding)**.** *Let X and Y be normed spaces. An operator $T : X \to Y$ is called a **compact linear operator** if T is linear and if for every $M \subseteq X$ bounded, the image $T(M)$ is relatively compact, i.e.,*

$$\overline{T(M)} \textit{ is compact.}$$

Moreover if $X \subseteq Y$ and Y is Banach, we say that $X \subset Y$ with compact embedding if the identity operator $I : X \to Y$ is compact.

Now, we have the necessary concepts to conclude this section with a very important embedding theorem.

Theorem 2.1.45 (Rellich-Kondrachov)**.** *Assume that $\Omega \subseteq \mathbb{R}^N$ is of C^1 class and bounded. Then, we have the following compact embeddings:*

$$\begin{aligned} &W^{1,p}(\Omega) \subset L^q(\Omega) \quad \forall q \in [1, p^\star), \ \textit{where } \frac{1}{p^\star} = \frac{1}{p} - \frac{1}{N}, && \textit{if } p < N \\ &W^{1,p}(\Omega) \subset L^q(\Omega) \quad \forall q \in [p, \infty), && \textit{if } p = N \\ &W^{1,p}(\Omega) \subset C(\overline{\Omega}) && \textit{if } p > N \end{aligned}$$

In particular, for any combination of N and p we have that $W^{1,p}(\Omega) \subset L^p(\Omega)$ with compact embedding.

The proof of Theorem 2.1.45 can be found e.g. in [3] and [11]. In other words, for the appropriate exponents, a bounded sequence in a Sobolev space has a Lebesgue convergent subsequence. This will be of critical importance for applications in the study of linear and non-linear partial differential equations.

2.2 Some topics from Calculus of Variations

In the last two sections we briefly studied normed spaces and their properties on a very general context. We also defined the Banach spaces $L^p, W^{m,p}$ and studied some properties of these specific spaces. In this section, we present concepts that will allow us to study functions over Banach spaces in a very simple way. We will first recall some basic calculus of variations theory. After this, we will present some more sophisticated results that shall be used later in this text. For this section we used [24, 23, 7, 13] as our main sources.

2.2.1 Fundamentals of calculus on normed spaces

In this subsection we want to extend the concept of derivative of a real valued function over $\mathbb{R}$ and consider analogous concepts for mappings between normed spaces. These notions of calculus on normed spaces is necessary for understanding some important results from the area of calculus of variations. All the ideas and concepts that follow can be found in greater detail in [23, 7, 13].

Since the following definitions share a very similar context, from now on we consider E, F to be normed spaces, $\mathcal{O} \subset E$ open and $f : \mathcal{O} \subset E \longrightarrow F$. Let $a \in \mathcal{O}$ a point and $u \in E$ a direction. If the limit

$$\partial_u f(a) = \lim_{t \to 0} \frac{1}{t}[f(a + tu) - f(a)] \tag{2.40}$$

exists, then we call (2.40) the **directional derivative of f at a in the direction u.**

Assume that $\partial_u f(a)$ exists for any direction $u \in E$. If

$$\exists\, f'_G(a) \in \mathcal{L}(E, F),\, \forall\, u \in E : \quad \partial_u f(a) = f'_G(a)u \tag{2.41}$$

then we say that f is **Gateaux differentiable at a**. Moreover, since $f'_G(a)$ in (2.41) is unique, then it is referred to as the **Gateaux differential of f at a.**

For the next kind of differentiability we need to introduce the concept of a small o. Let $\mathcal{O} \subset E \longrightarrow F$ such that $g(0) = 0$. If there exists a mapping $\epsilon : B(0, r) \subset E \longrightarrow F$ such that

$$\lim_{h \to 0} \epsilon(h) = 0, \tag{o1}$$

$$g(h) = \|h\|\, \epsilon(h), \tag{o2}$$

then we write $g(h) = o(h)$ and say that g **is a small o of h**. Notice that from the previous definition, if $g(h) = o(h)$ then by (o1) and (o2) it follows that

$$\lim_{h \to 0} \frac{1}{\|h\|} g(h) = \lim_{h \to 0} \frac{\|g(h)\|}{\|h\|} = 0. \tag{o3}$$

Since the quotients in (o3) converge, we know that $g(h)$ converges faster to zero than the norm of h when $h \to 0$. This is an important fact to define strong differentiability.

Definition 2.2.1 (Fréchet differential). *Let $a \in \mathcal{O}$ be a point. If*

$$\exists\, \phi \in \mathcal{L}(E,F), \forall\, h \in E : \quad a+g \in \mathcal{O} \implies f(a+h) - f(a) = \phi(h) + o(h), \tag{2.42}$$

then we say that f is ***differentiable (or Fréchet or strongly differentiable) at*** *a. If it is differentiable at all points of its domain, we simply say that it is differentiable.*

Proposition 2.2.2. *The bounded linear operator in* (2.42) *is unique.*

Proof. Let $\varphi \in \mathcal{L}(E,F)$ such that

$$f(a+h) - f(a) = \varphi(h) + o(h) \tag{2.43}$$

for any $h \in E$ such that $a+h \in \mathcal{O}$. Since $\mathcal{O}$ is open, by 2.4 we have that

$$\exists r > 0 : \quad B(a,r) = a + B(0,r) \subset \mathcal{O}$$

Hence, from (2.42) and (2.43) we have that

$$\forall h \in B(0,r) : \quad \phi(h) + \|h\|\, \epsilon_1(h) = \varphi(h) + \|h\|\, \epsilon_1(h), \tag{2.44}$$

where

$$\lim_{h \to 0} \epsilon_i(h) = 0, \quad i = 1,2. \tag{2.45}$$

Let $u \in E$, generic. If $u = 0$, then by linearity the required result immediately follows. Let's consider $u \neq 0$ and choose some $N \in \mathbb{N}$ such that

$$\forall n \in \mathbb{N} : \quad n > N \implies h_n := \frac{1}{n} \cdot \frac{1}{\|u\|} u \in B(0;r).$$

By (2.44) it follows that

$$\phi(h_n) - \varphi(h_n) = \|h_n\| \left(\epsilon_2(h_n) - \epsilon_1(h_n)\right).$$

Since $\phi, \varphi \in \mathcal{L}(E,F)$ we have that

$$\left(\frac{1}{n} \cdot \frac{1}{\|u\|}\right)(\phi(u) - \varphi(u)) = \left(\frac{1}{n} \cdot \frac{1}{\|u\|}\right) \|u\| \left(\epsilon_2(h_n) - \epsilon_1(h_n)\right).$$

By passing to the limit with $n \to 0$ and considering (2.45), we obtain

$$\phi(u) = \varphi(u).$$

Since u was chosen arbitrarily, we have proved 2.2.2. □

Remark 2.2.3 (Fréchet differential). *By Proposition 2.2.2, we rewrite* (2.42) *as*

$$f(a+h) - f(a) = f'(a)h + o(h)$$

where $f'(a) \in \mathcal{L}(E,F)$ is the Fréchet differential of a *at* u.

Remark 2.2.4 (Variation)**.** *Whenever f is a functional, i.e., $F = \mathbb{R}$ then by* (2.24) *we have that $f'(a) \in E^*$ and it is sometimes referred to as **the variation of f at** a.*

From (2.40), (2.41) and Definition 2.2.1 we have the following implications:

$$\text{Fréchet differentiable} \implies \text{Gateaux differentiable} \implies \text{Existence of partial derivatives.}$$

This motivates the denomination of strong differential (Fréchet differential) since the implications above are not reversible in general.

For real valued functions on $\mathbb{R}^N$, we say that a function is of *class* C^1 if it is differentiable and all its partial derivatives are continuous. In the context of functionals, we have the following definition.

Definition 2.2.5 (Mapping of class C^1)**.** *We say that $f : \mathcal{O} \to F$ belongs to the class $C^1(\mathcal{O}, F)$ iff f is differentiable and the function*

$$f' : \mathcal{O} \subseteq E \longrightarrow \mathcal{L}(E, F) \quad \text{is continuous.}$$

Alternatively, we simply say that f is of class C^1 if there is no confusion.

Now let us consider again the case when $F = \mathbb{R}$. The next proposition presented in [31] states a condition under which a Gateaux differentiable functional is of class C^1.

Proposition 2.2.6. *If $f : \mathcal{O} \to \mathbb{R}$ has a **continuous** Gateaux derivative on $\mathcal{O}$ then $f \in \mathrm{C}^1(\mathcal{O}, \mathbb{R})$.*

If $f : \mathcal{O} \subseteq E \longrightarrow \mathbb{R}$, and there exists a point $x \in \mathcal{O}$ such that

$$\forall y \in \mathcal{O} : \quad f(x) \leq f(y) \quad (f(x) \geq f(y)),$$

then we say that x is a *point of local minimum (maximum).* Whenever, $\mathcal{O} = E$ we say it is a *global point of minimum (maximum)* and in the case the inequality above is strict we say x is a *strict point of minimum (maximum).* A point of either minimum or maximum is called a point of **extremum**. If f is differentiable at $z \in \mathcal{O}$, then we say that z is a **critical point of** f. The following theorem relates the concepts of extremum and critical point.

Theorem 2.2.7 (Extremums and critical points)**.** *Let E be a normed space, $\mathcal{O} \subseteq E$ and $f : \mathcal{O} \longrightarrow \mathbb{R}$. Assume that*

(i) f has a local extremum at $x \in \mathcal{O}$, and

(ii) f is differentiable at x.

Then x is a critical point of f, i.e.,

$$f'(x) = 0.$$

The proof for Theorem 2.2.7 can be found e.g. in [7]. From what we have just seen, a point of local extremum is also a critical point. However, the converse is false.

2.2.2 The elementray problem of calculus of variations and the Euler-Lagrange equation

For many applications, one is interested in minimizing (or maximizing) the value of functionals defined over some functional space. As we mentioned in the last section, we can find such points by studying the variation of the functional in question. Before formally introducing the main concepts of this section, let us first look at a concrete example of a classical variational problem.

Example 2.2.8 (The brachistochrone problem). *This problem consists in finding the minimum-time path that follows a particle moving between two fixed points under the influence of gravity without friction. The following scheme illustrates the problem at hand.*

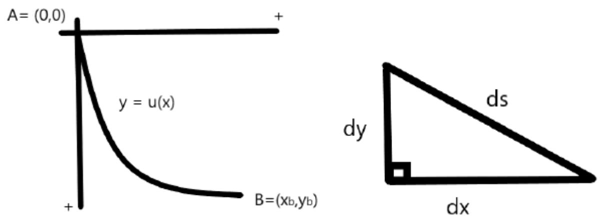

Figure 2.1: A simple scheme for the brachistochrone problem, [23]

Since we want to minimize the time it takes to travel from A to B, we need to find the "curve" that describes the optimal path. This curve will be represented by a function $u \in \mathrm{C}^1(\mathbb{R}^N)$*. Taking in consideration the law of conservation of energy and the Pythagoras theorem, the time functional for the situation shown in Figure 2.1 will be given by*

$$\begin{aligned} T: \quad E &\longrightarrow \mathbb{R}^N \\ u &\longmapsto T(u) = \frac{1}{2g}\int_0^{x_b} \sqrt{\frac{1+u'(x)^2}{u(x)}}dx, \end{aligned}$$

where g *denotes the gravitational acceleration constant and*

$$E = \{u \in \mathrm{C}^1([0,x_b]) \,/\, u(0) = 0 \text{ and } u(x_b) = y_b\}. \tag{2.46}$$

Finally, observe that the time functional T *is bounded from bellow:*

$$\inf_{u \in E} T(u) \geq 0.$$

Let us generalize the problem shown in Example 2.2.8. Let $F \in C^2([a,b] \times \mathbb{R} \times \mathbb{R})$. A generic element of the domain will be denoted by (x, u, ξ). The partial derivatives of F will be denoted as

$$F_x = \frac{\partial F}{\partial x}, \quad F_u = \frac{\partial F}{\partial u}, \quad F_\xi = \frac{\partial F}{\partial \xi}, \quad \text{etc.}$$

We will be particularly interested in functionals of the form

$$\begin{aligned} J : \mathscr{M} \subseteq C^1([a,b]) &\longrightarrow \mathbb{R} \\ y \quad &\longmapsto J(y) = \int_a^b F(x, y(x), y'(x))dx. \end{aligned} \tag{2.47}$$

A functional of the form (2.47) is known as the *Lagrangian functional* and $\mathscr{M}$ is the *set of admissible functions* and is defined as

$$\mathscr{M} = \{u \in C^1([a,b]) \quad / \quad u(a) = A \wedge u(b) = B\}, \tag{2.48}$$

Notice the resemblance of $\mathscr{M}$ to E from (2.46) in Example 2.2.8.

Remark 2.2.9 (Admissible increments)**.** *Notice that* (2.48) *is not a linear space in general. Let's define the set of* ***admissible increments*** *as*

$$\mathscr{J}^1([a,b]) = \{h \in C^1([a,b]) \quad / \quad h(a) = h(b) = 0\} \tag{2.49}$$

It is clear that $\mathscr{J}^1([a,b])$ *is a linear subspace of* $C^1([a,b])$ *and that*

$$\mathscr{M} + \mathscr{J}^1 \subseteq \mathscr{M},$$

i.e., if we add an admissible increment h *to an admissible function* u *then* $u + h \in \mathscr{M}$*.*

Returning to our initial goal of minimizing or maximizing a functional, let's recall that any maximization problem can be stated as a minimization one. Hence, the **elementary problem of the Calculus of Variations** is mathematically posed as follows:

$$\begin{cases} \text{Find } y_0 \in \mathscr{M} \text{ such that} \\ J(y_0) = \inf\limits_{u \in \mathscr{M}} J(u), \end{cases}$$

or in short,

$$\inf\{J(y) \ / \ y \in \mathscr{M}\}. \tag{EPCV}$$

Let J be a Lagrangian functional of the form (2.47) and let's assume that $y \in \mathscr{M}$ is a point of minimum of J. Then, by Theorem 2.2.7 it follows that

$$\forall\, h \in \mathscr{J}^1 : \quad J'(y)h = 0. \tag{2.50}$$

Any point that satisfies (2.50) is called an **extremal**. Notice that an extremum is also an extremal but an extremal is not necessarily a point of extremum. This condition leads us to a differential equation known as the *Euler-Lagrange equation.*

Theorem 2.2.10 (Euler-Lagrange Equation)**.** *Let* $y_0 \in \mathscr{M}$ *be a point of extremum of the Lagrangian functional* (2.47)*. Then,* y_0 *is the solution of the* ***Euler-Lagrange*** *equation*

$$F_u(x, y(x), y'(x)) - \frac{d}{dx}F_\xi(x, y(x), y'(x)) = 0, \quad x \in [a,b] \tag{E-L}$$

Proof. Let $y \in \mathscr{M}, h \in \mathscr{J}^1$, generic. Let's find $J'(y)h$. Since F is of class C^2, we have that for any $x \in [a,b]$ and any $u, \xi, \varepsilon_1, \varepsilon_2 \in \mathbb{R}$

$$F(x, u+\varepsilon_1, \xi+\varepsilon_2) - F(x,u,\xi) = F_u(x,u,\xi)\varepsilon_1 + F_\xi(x,u,\xi)\varepsilon_2 + o(\varepsilon_1) + o(\varepsilon_2).$$

Hence, we have that

$$J'(y)h = \int_a^b [F_u(x, y(x), y'(x))h(x) + F_\xi(x, y(x), y'(x))h'(x)]dx \tag{2.51}$$

Now, since $y_0 \in \mathscr{M}$ is a point of extremum of J, it is also an extremal and (2.50) implies that

$$\forall h \in \mathscr{J}^1([a,b]) : \quad \int_a^b [F_u(x, y_0(x), y_0'(x))h(x) + F_\xi(x, y_0(x), y_0'(x))h'(x)]dx = 0. \tag{2.52}$$

Lemma 4 of Chapter 1 in [13] and (2.52) imply that

$$\forall\, x \in [a,b] : \quad F_u(x, y_0(x), y_0'(x)) = \frac{d}{dx} F_\xi(x, y_0(x), y_0'(x)),$$

i.e., y_0 is a solution of (E-L). □

To conclude this section, we present the *isoperimetric problem.* In many applications of the calculus of variations, we face problems that not only impose boundary conditions but also additional *subsidiary conditions* or *side constraints.* Thus we will consider a new set of admissible functions,

$$\mathscr{M} = \{w \in C^1([a,b]) \ / \ w(a) = A \wedge w(b) = B \wedge K(w) = l\} \tag{2.53}$$

with $l \in \mathbb{R}$ prescribed and

$$K(w) = \int_a^b G(x, w(x), w'(x))dx \tag{2.54}$$

where $G \in C^2([a,b] \times \mathbb{R} \times \mathbb{R})$. In this new case, and admissible increment will be any $h \in C^1([a,b])$ such that

$$h(a) = h(b) = 0 \quad \wedge \quad K(w+h) = l.$$

The next result states a necessary condition for the constrained problem to have a solution and its proof can be found in [13] (Sec.12, Theorem 1).

Theorem 2.2.11 (Euler)**.** *Let J be as in* (2.47)*. Assume that*

(i) $y \in \mathscr{M}$ is a point of extremum of J, and

(ii) $y \in \mathscr{M}$ is not an extremal of K.

Then there exists a Lagrange multiplier, $\lambda \in \mathbb{R}$, such that y is an extremal of the functional $L : \mathscr{M} \longrightarrow \mathbb{R}$ given by

$$L(w) = J(w) + \lambda K(w)$$

i.e., it satisfies

$$\Phi_u(x, y(x), y'(x)) - \frac{d}{dx}\Phi_\xi(x, y(x), y'(x))$$

where

$$\Phi = F + \lambda G$$

2.3 Variational and non-linear topics for PDE's

In this section, we shall see how the previous tools are used to solve partial differential equations. A *partial differential equation* (PDE) is an equation that involves a function of two or more independent variables and some of the function's partial derivatives in respect to said variables. Using the standard *multi-index notation*, already used in the Sobolev spaces section for denoting partial derivatives, we can represent a PDE as

$$F(D^k u(x), D^{k-1}u(x), ..., Du(x), u(x), x) = 0, \quad x \in \Omega, \tag{2.55}$$

where $k \geq 1$, $\Omega \subseteq \mathbb{R}^N$, and

$$F : \mathbb{R}^{n^k} \times \mathbb{R}^{n^{k-1}} \times ... \times \mathbb{R}^N \times \mathbb{R} \times \Omega \longrightarrow \mathbb{R}$$

is called a k^{th}-order PDE. The function $u : \Omega \longrightarrow \mathbb{R}$ that satisfies (2.55) is a *solution of the PDE.*

We classify a PDE according to its linearity as follows. Given a PDE of the form (2.55), we say that it is:

(i) **Linear** if

$$\sum_{|\alpha| \leq k} a_\alpha(x) D^\alpha u = f(x), \tag{2.56}$$

for given functions $a_\alpha(|\alpha \leq k|), f$, i.e., F is linear with respect to u and its derivatives. Moreover, if $f = 0$ then the linear PDE is **homogeneous**.

(ii) **Semilinear** if

$$\sum_{|\alpha| = k} a_\alpha(x) D^\alpha u + a_0(D^{k-1}u, ..., Du, u, x) = 0, \tag{2.57}$$

i.e., F is non-linear with respect to u but linear for its derivatives.

(iii) **Quasilinear** if

$$\sum_{|\alpha| = k} a_\alpha(D^{k-1}u, ..., Du, u, x) D^\alpha u + a_0(D^{k-1}u, ..., Du, u, x) = 0, \tag{2.58}$$

i.e., F is linear for the highest order derivatives of u.

(iv) **Fully non-linear** if it depends non-linearly upon the highest order of derivatives.

There is no general method for solving PDE's. The ideas and methods applied vary depending on the structure of the problem. One thing that can be said in most occasions is that non-linear PDE's are harder to solve than linear PDE's. This section will be mainly concerned with the necessary ideas used to study non-linear PDE's following the ideas from [11].

Let's assume that we wish to find the solution u of the PDE represented by

$$F(u) = 0, \tag{2.59}$$

where F denotes a (possibly non-linear) partial differential operator. Now let's assume that the operator F is the "derivative" of an appropriate "energy" functional J, i.e.,

$$F = J'. \tag{2.60}$$

Then solving the PDE (2.59) is equivalent to finding the critical points of J as defined in (2.60). Let's assume now that J has a form similar to

$$J(w) = \int_\Omega L(x, w(x), Dw(x))dx, \tag{2.61}$$

where $L = L(x, u, \xi) = L(x_1, ..., x_N, u, \xi_1, ..., \xi_N)$. If y is a minimizer of J, then y will also be an extremal, i.e., a solution of the Euler-Lagrange equation associated to J,

$$L_u(x, y, Dy) - \sum_{i=1}^{N} (L_{\xi_i}(x, y, Dy))_{x_i} = 0, \quad x \in \Omega.$$

In order to elucidate this approach, we provide some examples of the previous process.

Example 2.3.1 (Minimal surfaces)**.** *Let*

$$L(x, u, \xi) = (1 + |\xi|^2)^{1/2}, \quad \textit{so that} \quad J(w) = \int_\Omega (1 + |Dw|^2)^{1/2} dx$$

is the area of the graph of the function $w : \Omega \longrightarrow \mathbb{R}$. The associated Euler-Lagrange equation is

$$\sum_{i=1}^{n} \left(\frac{u_{x_i}}{(1 + |Du|^2)^{1/2}} \right)_{x_i} = 0 \textit{ in } \Omega. \tag{2.62}$$

This partial differential equation is the ***minimal surface equation.*** *The expression on the left side of* (2.62) *is n times the mean curvature of the graph of u. Thus, a minimal surface has zero mean curvature.*

Example 2.3.2 (Non-linear Schrödinger equation)**.** *Let $p \geq 1, N \in \mathbb{N}, \Omega \subseteq \mathbb{R}^N$ open connected with smooth boundary, and $T \in C(\overline{\Omega})$ non-negative. We look for solutions of*

$$\begin{cases} -\Delta u(x) + T(x)u(x) - |u(x)|^{p-1}u(x) &= 0 \quad x \in \Omega \\ u(x) &= 0 \quad x \in \partial\Omega, \end{cases} \tag{2.63}$$

which is the Euler-Lagrange equation associated to the functional

$$J(u) = \frac{1}{2} \int_\Omega |\nabla u(x)|^2 + T(x)|u(x)|^2 dx$$

with the restriction

$$\|u\|_{L^{p+1}(\Omega)} = 1.$$

Thus, a critical point of J weakly verifies (2.63)*, which serves e.g. to model systems of a very large number of particles interacting at very low temperatures, like Bose-Einstein condensates.*

We conclude this section with the concept of *manifold*.

Definition 2.3.3 (Manifold). *Let X be a Banach space and $\mathcal{I}$ a set of indices. A topological space M is a C^k* ***manifold*** *modelled on X if*

$$\exists \{U_i\}_{i\in\mathcal{I}} \text{ open covering of } M \text{ and}$$
$$\psi_i : U_i \to X \text{ family of mappings}$$

such that the following conditions hold:

$$V_i = \psi_i(U_i) \text{ is open in } X,$$
$$\psi_i \text{ is a homeomorphism between } U_i \text{ and } V_i, \text{ and}$$
$$\psi_j \circ \psi_i^{-1} : \psi_i(U_i \cap U_j) \to \psi_j(U_i \cap U_j) \text{ is of class } C^k.$$

For further information on manifolds we refer the reader to [2].

2.3.1 Ground state solutions

In this subsection we state some conditions that will ensure that J as in (2.61) has indeed a minimizer, at least within an appropriate Sobolev space. Let $1 < p < \infty$. Let's define

$$\mathcal{A} := \{w \in W^{1,p}(\Omega) \ / \ \forall x \in \partial\Omega : \ \ w(x) = g(x)\}, \tag{2.64}$$

for some function g, as the class of *admissible functions* (notice the similarity to (2.48)). Clearly, if $g = 0$ then $\mathcal{A} = W_0^{1,p}(\Omega)$. We are interested in ensuring that J is bounded and that it attains its infimum through some compactness property. The most effective way to ensure boundedness from bellow is to hypothesize that $J(w)$ "grows rapidly as $|w| \to \infty$". This motivates the following definition.

Definition 2.3.4 (Coercivity). *Let's assume that*

$$\exists \alpha > 0, \beta \geq 0, \forall (x, u, \xi) \in \Omega \times \mathbb{R} \times \mathbb{R}^N : \quad L(x, u, \xi) \geq \alpha|\xi|^p - \beta.$$

Therefore,

$$\exists \delta > 0 : \quad J(w) \geq \delta \|Dw\|_p^p - \gamma, \tag{2.65}$$

where $\gamma := \beta|\Omega|$. Thus $I(w) \to \infty$ as $\|Dw\|_p \to \infty$.

In general, the previous condition is not enough to ensure that our integral functional J attains its infimum. To further develop on this, let's assume that

$$m := \inf_{w\in\mathcal{A}} J(w), \tag{2.66}$$

and choose functions $u_n \in \mathcal{A}, n \in \mathbb{N}$, such that

$$J(u_n) \to m \text{ as } n \to \infty. \tag{2.67}$$

A sequence verifying (2.67) is called a **minimizing sequence.** Let's assume that J is coercive. Then, from (2.65) it follows that the minimizing sequence lies in a bounded set of the infinite dimensional space $W^{1,p}(\Omega)$. Notice that from our previous assumption of $1 < p < \infty$ and Proposition 2.1.38 we have that $W^{1,p}(\Omega)$ is reflexive. Hence, from the boundedness of $(u_n)_{n\in\mathbb{N}}$ and Theorem 2.27 we conclude that there is some subsequence $(u_{n_k})_{k\in\mathbb{N}}$ and $u \in W^{1,p}(\Omega)$ such that

$$u_{n_k} \rightharpoonup u \quad \text{in } W^{1,p}(\Omega). \tag{2.68}$$

Moreover, from (2.68) we also have that $u \in \mathcal{A}$. Notice that from (2.68) we cannot deduce that

$$J(u) = \lim_{k\to\infty} J(u_{n_k}) \tag{2.69}$$

since J is not continuous with respect to weak convergence. However, the strength of (2.69) is not really needed. Instead, the next condition suffices.

Definition 2.3.5 (Weakly lower semicontinuity)**.** *We say that J is* ***(sequentially) weakly lower semicontinuous*** *on $W^{1,p}(\Omega)$, provided*

$$J(u) \leq \liminf_{n\to\infty} J(u_n)$$

whenever

$$u_n \rightharpoonup u \text{ in } W^{1,p}(\Omega).$$

Thus, if we assume that J is weakly lower semicontinuous then from (2.68) it follows that $J(u) \leq m$. Since (2.66) implies that $m \leq J(u)$, we can conclude that u is indeed a minimizer.

2.3.2 Palais-Smale condition

Let J be a functional on a Banach space X. In this subsection we will assume the more restrictive condition of J being of class C^1 in the Frechét sense. We will also impose the compactness assumption mentioned earlier: the *Palais-Smale* condition. For this, we follow the ideas presented in [30](Chap. II, Sec. 2).

In the original work of Palais and Smale, this condition is stated as follows:

Definition 2.3.6 (Original Palais-Smale condition)**.** *Let $S \subset X$ on which*

(i) J is bounded and

(ii) $\|J'\|$ is ***not*** *bounded away from zero.*

Then, there is a critical point in $\overline{S}$.

In order to work with a more convenient condition, slightly stronger than Definition 2.3.6, Struwe introduces the following concept.

Definition 2.3.7 (Palais-Smale sequence)**.** *A sequence* $(u_n) \subset X$ *is a* ***Palais-Smale sequence*** *for* J *if*

(i) $\exists c > 0, \forall n \in \mathbb{N}: \quad |J(u_n)| \leq c,$ *and*

(ii) $\|J'(u_n)\| \to 0$ *as* $n \to \infty$.

In terms of the previous definition, the modified compactness condition states:

Any Palais-Smale sequence has a (strongly) convergent subsequence. (P.S.)

Note that (P.S.) implies that any set of critical points of uniformly bounded energy is relatively compact. In fact, if we were to strengthen condition 2.3.6 by this requirement, this new condition would be equivalent to (P.S.). We have the following proposition.

Proposition 2.3.8. *Assume that* J *has the following properties.*

(i) Any Palais-Smale sequence for J *is bounded in* X.

(ii) It is possible to decompose J' *as*

$$\forall u \in X: \quad J'(u) = L + K(u),$$

where $L : X \to X^*$ *is a fixed boundedly invertible linear map and the operator* K *maps bounded sets in* X *to relatively compact sets in* X^*.

Then J *satisfies* (P.S.).

Proof. Notice that if (u_n) is a (P.S.) sequence of J then

$$J'(u_n) = Lu_n + K(u_n) \to 0$$

implies that

$$u_n = o(1) - L^{-1}K(u_n)$$

where

$$o(1) \to 0 \text{ in } X \text{ as } n \to \infty.$$

By boundedness of (u_n) and compactness of K, the sequences $(L^{-1}K(u_n))$, and hence (u_n), are relatively compact. □

The (P.S.) condition allows us to identify a certain family of neighbourhoods of critical points of a functional J. For $\beta \in \mathbb{R}, \delta > 0, \rho > 0$ let

$$\begin{aligned}
J_\beta &= \{u \in X \ / \ J(u) < \beta\},\\
K_\beta &= \{u \in X \ / \ J(u) = \beta \ \wedge \ J'(u) = 0\},\\
N_{\beta,\delta} &= \{u \in X \ / \ |J(u) - \beta| < \delta \ \wedge \ \left\|J'(u)\right\| < \delta\},\\
U_{\beta,\delta} &= \bigcup_{u \in K_\beta} \{v \in X \ / \ \|u - v\| < \rho\}.
\end{aligned}$$

Lemma 2.3.9 (PS and neighbourhoods)**.** *Assume J satisfies* (P.S.)*. Let $\beta \in \mathbb{R}$ generic. Thus, the following holds:*

(i) K_β is compact,

(ii) $\{U_{\beta,\rho}\}_{\rho>0}$ is a fundamental system of neighborhoods of K_β, and

(iii) $\{N_{\beta,\delta}\}_{\delta>0}$ is a fundamental system of neighborhoods of K_β.

Proof. (i) From (P.S.) we have that any sequence $(u_n) \subset K_\beta$ has a convergent subsequence. Since both J and J' are continuous, the limit of such subsequence lies in K_β. Therefore, K_β is compact.

(ii) Clearly, any $U_{\beta,\rho}, \rho > 0$, is a neighbourhood of K_β. Let V any open neighbourhood of K_β. For the purpose of contradiction assume that

- $\exists (\rho_n) \subset \mathbb{R}: \quad \rho_n \to 0 \quad \text{as} \quad n \to \infty$, and
- $\exists (u_n) \subset X, \forall n \in \mathbb{N}: \quad u_n \in U_{\beta,\rho_n} \setminus V$.

Let $(v_n) \subset K_\beta$ such that $\|u_n - v_n\| \leq \rho_n$. Point (i) implies that there is some $v \in K_\beta$ such that $v_n \to v$. Hence, $u_n \to v$ and $u_n \in V$ for a sufficiently large n. This is a contradiction.

(iii) Evidently, $N_{\beta,\delta}, \delta > 0$, is a neighborhood of K_β. Once again for the purpose of contradiction, assume that there is some neighbourhood W of K_β such that

- $\delta_n \to 0$, and
- $\exists (u_n) \subset X, \forall n \in \mathbb{N}: \quad u_n \in N_{\beta,\delta_n} \setminus W$.

From (P.S.) we have that $u_n \to u \in K_\beta \subset W$. The result follows by contradiction.

□

The Palais-Smale condition is a useful technical assumption that occurs frequently in critical point theory. In the next subsection we will see how it is a key ingredient for one of the most important minimax methods.

2.3.3 Mountain Pass Theorem

This subsection is mainly based on [28]. Minimax methods are those that characterize a critical value c of a functional J as a minimax over a suitable class of sets.

$$c = \inf_{A \in S} \max_{u \in A} J(u) \tag{2.70}$$

Assume that J is Fréchet differentiable and that J' is continuous on X, i.e., $J \in C^1(X)$ where X is a real Banach space. Let $B_\rho := B(0; \rho) \subset X, \rho > 0$. We will now present the usual version of the *Mountain Pass Theorem*. An important result used in the proof of this theorem is the Deformation Theorem ([28] Appendix A.4). However, given its length and technicalities, it will suffice to consider the following special case.

Proposition 2.3.10 (Special case)**.** *Suppose J satisfies* (P.S.)*. For $s, c \in \mathbb{R}$ we define*

$$K_c = \{u \in X \ / \ J(u) = c \ \wedge \ J'(u) = 0\} \tag{2.71}$$

$$A_s = \{u \in X \ / \ J(u) \leq s\} \tag{2.72}$$

If c ***is not a critical value of*** *J then*
$\forall \bar{\varepsilon} > 0, \exists \varepsilon \in (0, \bar{\varepsilon}), \exists \eta \in C([0,1] \times X, X):$

$$J(u) \notin [c - \bar{\varepsilon}, c + \bar{\varepsilon}] \implies \eta(1, u) = u, \tag{2.73}$$

$$\eta(1, A_{c+\varepsilon}) \subset A_{c-\varepsilon}. \tag{2.74}$$

From this, we can now prove

Theorem 2.3.11 (Mountain Pass Theorem (MPT))**.** *Let X be a real Banach space and $J \in C^1(X)$ satisfying* (P.S.)*. Assume:*

$$J(0) = 0, \tag{J_0}$$

$$\exists \rho, \alpha > 0: \quad J|_{\partial B_\rho} \geq \alpha, \tag{J_1}$$

$$\exists e \in X \setminus B_\rho: \quad J(e) \leq 0. \tag{J_2}$$

Then J possess a critical value $c \geq \alpha$. Moreover c can be characterized as

$$c = \inf_{g \in \Gamma} \max_{u \in g([0,1])} J(u), \tag{2.75}$$

where

$$\Gamma = \{g \in C([0,1], X) \ / \ g(0) = 0 \wedge g(1) = e\}.$$

Proof. From the definition of c it is clear that $c < \infty$. Also, from the definition of Γ, (J_0) and (J_2), we have that if $g \in \Gamma$, then $g([0,1]) \cap \partial B_\rho \neq \emptyset$. Thus, by (J_1) it follows that

$$\max_{u \in g([0,1])} J(u) \geq \inf_{w \in \partial B_\rho} J(w) \geq \alpha,$$

and consequently $c \geq \alpha$. For the purpose of contradiction, let's assume that c is not a critical value of J. Then, by Proposition 2.3.10 with $\bar{\varepsilon} = \alpha/2$ yields $\varepsilon \in (0, \alpha/2)$ and η as in the result. Choose $g \in \Gamma$ such that

$$\max_{u \in g([0,1])} J(u) \leq c + \varepsilon \tag{2.76}$$

and consider $h(t) := \eta(1, g(t))$. Clearly, $h \in C([0,1], X)$. Notice that $g(0) = 0$ and (J_0), $J(0) = 0 < \alpha/2 \leq c - \bar{\varepsilon}$, imply $h(0) = 0$ by (2.73). By an analogous reasoning we have that

$$g(1) = e \ \wedge \ J(e) \leq 0 \quad \Longrightarrow \quad h(1) = e.$$

Consequently, $h \in \Gamma$ and by (2.75)

$$c \leq \max_{u \in h([0,1])} J(u). \tag{2.77}$$

However, by (2.76), $g([0,1]) \subset A_{c+\varepsilon}$ so (2.74) implies

$$h([0,1]) \subset A_{c-\varepsilon} \quad \Longleftrightarrow \quad \max_{u \in h([0,1])} J(u) \leq c - \varepsilon, \tag{2.78}$$

contradicting (2.77). This contradiction implies that c must be a critical value of J. □

The intuition behind the MPT is that if a pair of points in the graph of J are separated by a "mountain range", then there must be a mountain pass containing a critical point between them (see Figure 2.2).

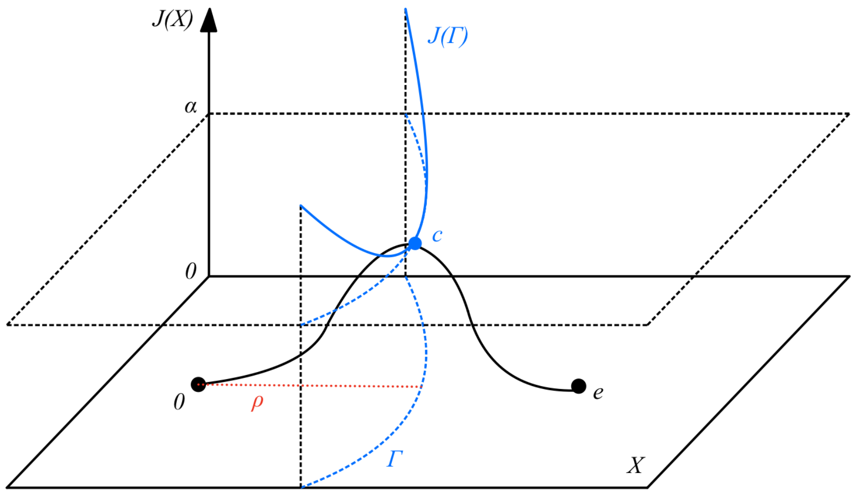

Figure 2.2: Diagram of Theorem 2.3.11 with $\Gamma = \partial B_\rho$.

2.4 p-Laplacian operator

In this section we shall present some properties of the $p-$*Laplacian operator*, a generalization of the well-known Laplacian operator. The classical theory developed for the regular Laplace equation involves areas such as: Calculus of Variations, Partial Differential Equations, Calculus of Probability, etc. In a similar way, the $p-$*Laplace equation* occupies a similar role when it comes to non-linear diffusion phenomena. Since our main problem involves the $p-$Laplacian operator, we will present some important concepts and results taken from [19] and [18].

2.4.1 Definition

The usual Laplacian operator appears in the classical Laplace equation

$$\Delta u = \frac{\partial^2 u}{\partial x_1^2} + \frac{\partial^2 u}{\partial x_2^2} + \cdots + \frac{\partial^2 u}{\partial x_N^2} = 0.$$

This is the Euler-Lagrange equation of the functional,

$$D(u) = \int_\Omega |\nabla u(x)|^2 dx = \int \cdots \int \left[\left(\frac{\partial u(x)}{\partial x_1} \right)^2 + \cdots + \left(\frac{\partial u(x)}{\partial x_N} \right)^2 \right] dx_1 \ldots dx_N.$$

This is just the particular case $p = 2$, for the more general Dirichlet integral given by

$$I(u) = \int_\Omega |\nabla u(x)|^p dx = \int \cdots \int \left[\left(\frac{\partial u(x)}{\partial x_1} \right)^2 + \cdots + \left(\frac{\partial u(x)}{\partial x_N} \right)^2 \right]^{\frac{p}{2}} dx_1 \ldots dx_N. \quad (2.79)$$

Definition 2.4.1. *We call p-Laplace equation to the Euler-Lagrange equation corresponding to* (2.79)*, i.e.,*

$$\operatorname{div}(|\nabla u(x)|^{p-2} \nabla u(x)) = 0.$$

*Thus, we define the **p-Laplace operator** as*

$$\Delta_p u := \operatorname{div}(|\nabla u|^{p-2} \nabla u). \quad (2.80)$$

Notice that the value of p can change. Then we have that:

(i) For $p = 1$ we get the *Mean Curvature* operator H,

$$H = -\Delta_1 u = -\nabla \left(\frac{\nabla u}{|\nabla u|} \right).$$

(ii) For $p = 2$ we have the Laplace operator,

$$\Delta u = \sum_{i=1}^{N} \frac{\partial^2 u}{\partial x_i^2}.$$

(iii) If we let $p \to +\infty$, the following equation arises

$$\Delta_\infty u = \sum_{i,j=1}^{N} \frac{\partial u}{\partial x_i} \frac{\partial u}{\partial x_j} \frac{\partial^2 u}{\partial x_i \partial x_j} = 0.$$

The p-Laplace operator appears in many contexts. Some examples are:

(i) The non-linear eigenvalue problem

$$-\Delta_p u = \lambda |u|^{p-2} u = 0,$$

which generalizes the eigenvalue problem $-\Delta u = \lambda u$.

(ii) The p-Poisson equation

$$\Delta_p u = f(x).$$

(iii) Equations similar to

$$\Delta_p u + |u|^\alpha u = 0,$$

that are of interest when the exponent α is "critical".

2.4.2 Eigenvalue problem

In this section we are interested in the following eigenvalue problem for the $p-$Laplacian.

$$\begin{cases} -\Delta_p u - \lambda |u|^{p-2}u &= 0 \quad \text{in } \Omega \\ u &= 0 \quad \text{on } \partial\Omega, \end{cases} \tag{2.81}$$

with $\Omega \subset \mathbb{R}^N$. In [21], the first eigenvalue is the nonlinear Rayleigh quotient

$$\lambda_1(\Omega) = \min_{\phi \in W_0^{1,p}(\Omega), \phi \neq 0} \frac{\int_\Omega |\nabla \phi(x)|^p dx}{\int_\Omega |\phi(x)|^p dx} = \frac{\int_\Omega |\nabla u_1(x)|^p dx}{\int_\Omega |u_1(x)|^p dx}, \tag{2.82}$$

where the minimum is achieved at some u_1. Notice that u_1 also is a weak solution to the Euler-Lagrange equation in (2.81).The first eigenvalue has many special properties:

(i) $\lambda_1(\Omega) > 0$,

(ii) $\lambda_1(\Omega)$ is simple, i.e., its algebraic multiplicity is one, for all Ω bounded and connected, and

(iii) u_1, the associated eigenfunction, is the only positive eigenfunction for the p-Laplacian.

In order to describe how higher eigenvalues are produced, from [30] we present the concept of *genus of Krasnoselskii.*

Definition 2.4.2 (Krasnoselskii Genus)**.** *Let X be a Banach space and define*

$$\mathcal{A}(X) := \{A \in \mathcal{P}(X) \setminus \{\emptyset\} \ / \ \overline{A} = A \ \wedge \ A = -A\},$$

the class of non-void ***closed symmetric*** *subsets of X. Let $A \in \mathcal{A}$. Then $\gamma(A)$ is called* ***the Krasnoselskii's genus of*** *A. It corresponds to the infimum integer k such that there exists an odd continuous mapping from A to $\mathbb{R}^k \setminus \{0\}$, i.e.,*

$$\gamma(A) = \begin{cases} \inf\{k \ / \ \exists\, h \in C(A, \mathbb{R}^k \setminus \{0\}) : h(u) = -h(-u)\}, & \{...\} \neq \emptyset \\ \infty, & \{...\} = \emptyset \ \vee \ 0 \in A \\ 0, & A = \emptyset. \end{cases} \tag{KG}$$

The notion of genus generalizes the notion of dimension of a linear space.

Now let Σ_k denote the collection of all symmetric subsets A of $W_0^{1,p}(\Omega)$ such that

(i) $\lambda(A) \geq k$, and

(ii) $\{u \in A \ / \ \|u\|_{L^p(\Omega)} = 1\}$ is compact.

Then according to [6], the higher eigenvalues of the p-Laplacian are given by

$$\lambda_k(\Omega) = \inf_{A \in \Sigma_k} \max_{\phi \in A} \frac{\int_\Omega |\nabla \phi(x)|^p dx}{\int_\Omega |\phi(x)|^p dx} \tag{2.83}$$

and there are infinitely many of them. The fact that this minimax procedure provides eigenvalues is explained thorugh the (P.S.) condition.

Chapter 3

Results

3.1 Preliminaries

Let $\varepsilon > 0$. Consider the following quasilinearboundary value problem.

$$\begin{cases} -\varepsilon^2 \Delta_p u(x) + V(x)|u(x)|^{p-2}u(x) - |u(x)|^{q-1}u(x) &= 0, \qquad x \in \mathbb{R}^N \\ u(x) &\to 0, \qquad \text{as } |x| \to \infty, \end{cases} \tag{G_ε}$$

where

$$\Delta_p u = \operatorname{div}(|\nabla u|^{p-2}\nabla u)$$

and

$$1 < p < q + 1 < p^*, \tag{3.1}$$

with

$$p^* = \begin{cases} \dfrac{pN}{N-p}, & \text{if } N \geq 3; \\ \infty, & \text{if } N = 1, 2. \end{cases}$$

Additionally, we will assume that

$$V \in C(\mathbb{R}^N) \text{ is non-negative and} \tag{C}$$

$$\lim_{|x| \to \infty} V(x) = \infty. \tag{L}$$

Let's assume that $u \in \mathrm{C}_0^\infty(\mathbb{R}^N)$ verifies (G_ε). Then, multiplying in (G_ε) by u and integrating we get

$$-\int_{\mathbb{R}^N} \varepsilon^2 [\Delta_p u(x)] u(x) dx + \int_{\mathbb{R}^N} V(x)|u(x)|^{p-2}u(x)^2 dx - \int_{\mathbb{R}^N} |u(x)|^{q-1}u(x)^2 dx = 0.$$

Integration by parts in the first integral yields

$$\int_{\mathbb{R}^N} \varepsilon^2 |\nabla u(x)|^{p-2}(\nabla u(x) \cdot \nabla u(x)) dx + \int_{\mathbb{R}^N} V(x)|u(x)|^{p-2}|u(x)|^2 dx - \int_{\mathbb{R}^N} |u(x)|^{q-1}|u(x)|^2 dx = 0,$$

and consequently

$$\int_{\mathbb{R}^N} \varepsilon^2 |\nabla u(x)|^p dx + \int_{\mathbb{R}^N} V(x)|u(x)|^p dx - \int_{\mathbb{R}^N} |u(x)|^{q+1} dx = 0. \tag{3.2}$$

Remark 3.1.1. *Equation* (3.2) *corresponds to the* ***variational formulation*** *for* (G_ε).

From the previous, let's observe that the standard functional associated to (G_ε), written

$$\Gamma_\varepsilon : X \to \mathbb{R},$$

is given by

$$\Gamma_\varepsilon(u) = \int_{\mathbb{R}^N} \left[\frac{\varepsilon^2}{p} |\nabla u(x)|^p + \frac{1}{p} V(x) |u(x)|^p - \frac{1}{q+1} |u(x)|^{q+1} \right] dx, \tag{3.3}$$

where

$$X = \left\{ u \in \mathrm{W}_0^{1,p}(\mathbb{R}^N) \,/\, V^{1/p}\, u \in \mathrm{L}^p(\mathbb{R}^N) \right\}. \tag{3.4}$$

Remark 3.1.2. *Notice that thanks to* (3.4), (3.3) *makes sense and is well defined since by Corollary 2.1.43 we have that*

$$\mathrm{W}_0^{1,p}(\mathbb{R}^N) \subseteq \mathrm{L}^\alpha(\mathbb{R}^N), \quad \forall \alpha \in [p, p^*],$$

so that, by condition 3.1,

$$\mathrm{W}_0^{1,p}(\mathbb{R}^N) \subseteq \mathrm{L}^{q+1}(\mathbb{R}^N).$$

Remark 3.1.3. *If we had* $V \in \mathrm{C}(\mathbb{R}^N) \cap \mathrm{L}^\infty(\mathbb{R}^N)$, *then it would suffice*

$$X = \mathrm{W}_0^{1,p}(\mathbb{R}^N)$$

In what follows, we want to derive a problem equivalent to (G_ε) by introducing a scaling that will allow us to remove the ε^2 factor in (G_ε). Let $\varepsilon > 0$ and let's assume that $v \in \mathrm{C}^2(\mathbb{R}^N)$ verifies (G_ε). We shall use the scaling

$$v(x) = u(\varepsilon^\beta x), \quad x \in \mathbb{R}^N, \beta \in \mathbb{R}. \tag{3.5}$$

Hence, by (3.5) it follows that

$$\begin{aligned} \nabla v(x) &= \varepsilon^\beta \nabla u(\varepsilon^\beta x), \\ |\nabla v(x)|^{p-2} &= \varepsilon^{\beta(p-2)} |\nabla u(\varepsilon^\beta x)|^{p-2}, \\ |\nabla v(x)|^{p-2} \nabla v(x) &= \varepsilon^{\beta(p-1)} |\nabla u(\varepsilon^\beta x)|^{p-2} \nabla u(\varepsilon^\beta x) \end{aligned}$$

which implies

$$\begin{aligned} \Delta_p v(x) &= \operatorname{div}(|\nabla v(x)|^{p-2} \nabla v(x)) \\ &= \varepsilon^{\beta(p-1)} \varepsilon^\beta \operatorname{div}\left(|\nabla u(\varepsilon^\beta x)|^{p-2} \nabla u(\varepsilon^\beta x) \right) \\ &= \varepsilon^{\beta p} \Delta_p u(\varepsilon^\beta x). \end{aligned} \tag{3.6}$$

Together (3.5) and (3.6), provide

$$-\varepsilon^{\beta p+2} \Delta_p u(\varepsilon^\beta x) + V(x) |u(\varepsilon^\beta x)|^{p-2} u(\varepsilon^\beta x) - |u(\varepsilon^\beta x)|^{q-1} u(\varepsilon^\beta x) = 0, \quad x \in \mathbb{R}^N.$$

Then, by choosing,

$$\beta = -\frac{2}{p},$$

$$\begin{aligned} y &= \varepsilon^{\beta} x \\ &= \varepsilon^{-2/p} x, \end{aligned} \tag{3.7}$$

we get

$$-\Delta_p u(y) + V_\varepsilon(y)|u(y)|^{p-2}u(y) - |u(y)|^{q-1}u(y) = 0, \qquad y \in \mathbb{R}^N, \tag{3.8}$$

where

$$V_\varepsilon(x) = V(\varepsilon^{-2/p}x). \tag{3.9}$$

As a result of (3.5) and (3.6), we have that the problem (G_ε) is equivalent to

$$\begin{cases} -\Delta_p u(x) + V_\varepsilon(x)|u(x)|^{p-2}u(x) - |u(x)|^{q-1}u(x) = 0, & x \in \mathbb{R}^N, \\ \qquad u(x) \to 0, & \text{as } |x| \to +\infty. \end{cases} \tag{E_ε}$$

Now, we present the following proposition.

Proposition 3.1.4. *The functional* $\|\cdot\|_\varepsilon : \mathrm{C}_0^\infty(\mathbb{R}^N) \longrightarrow \mathbb{R}$ *given by*

$$\|u\|_\varepsilon = \left(\int_{\mathbb{R}^N} |\nabla u(x)|^p + V_\varepsilon(x)|u(x)|^p \, dx \right)^{1/p} \tag{3.10}$$

is a norm.

Proof. Let $u, v \in \mathrm{C}_0^\infty(\mathbb{R}^N)$, generic. It is clear by (3.10) that (N1) and (N3) hold. Let's prove that the triangle inequality holds, i.e.,

$$\|u+v\|_\varepsilon \leq \|u\|_\varepsilon + \|v\|_\varepsilon \,. \tag{3.11}$$

By the triangle inequality in Theorem 2.1.34 and Minkowski's inequality for finite sums we have that

$$\begin{aligned} \|u+v\|_\varepsilon &= \left(\int_{\mathbb{R}^N} |\nabla u(x) + \nabla v(x)|^p + V_\varepsilon |u(x)+v(x)|^p dx \right)^{1/p} \\ &= \left(\int_{\mathbb{R}^N} |\nabla u(x) + \nabla v(x)|^p dx + \int_{\mathbb{R}^N} V_\varepsilon |u(x)+v(x)|^p dx \right)^{1/p} \\ &= \left(\|\nabla u + \nabla v\|^p_{\mathrm{L}^p(\mathbb{R}^N)} + \left\| V_\varepsilon^{1/p} u + V_\varepsilon^{1/p} v \right\|^p_{\mathrm{L}^p(\mathbb{R}^N)} \right)^{1/p} \\ &\leq \left(\|\nabla u\|^p_{\mathrm{L}^p(\mathbb{R}^N)} + \left\| V_\varepsilon^{1/p} u \right\|^p_{\mathrm{L}^p(\mathbb{R}^N)} \right)^{1/p} + \left(\|\nabla v\|^p_{\mathrm{L}^p(\mathbb{R}^N)} + \left\| V_\varepsilon^{1/p} v \right\|^p_{\mathrm{L}^p(\mathbb{R}^N)} \right)^{1/p} \\ &= \left(\int_{\mathbb{R}^N} |\nabla u(x)|^p + V_\varepsilon(x)|u(x)|^p \right)^{1/p} + \left(\int_{\mathbb{R}^N} |\nabla v(x)|^p + V_\varepsilon(x)|v(x)|^p \right)^{1/p} \\ &= \|u\|_\varepsilon + \|v\|_\varepsilon \,. \end{aligned}$$

Since u, v were chosen arbitrarily, we have proved (3.11).

Finally, let's prove that (N2) holds for (3.10). If $u = 0$ then clearly $\|u\|_\varepsilon = 0$. Conversely, let's assume that $\|u\|_\varepsilon = 0$. Thus, it follows that

$$\int_{\mathbb{R}^N} |\nabla u(x)|^p dx + \int_{\mathbb{R}^N} V_\varepsilon |u(x)|^p dx = 0,$$

and consequently

$$0 \leq \int_{\mathbb{R}^N} |\nabla u(x)|^p dx = - \int_{\mathbb{R}^N} V_\varepsilon(x) |u(x)|^p dx.$$

The last inequality implies that $\left\| V_\varepsilon^{1/p} u \right\|_{\mathrm{L}^p(\mathbb{R}^N)} = 0$. Hence, it must be the case that either

$$u(x) = 0 \qquad \vee \qquad V_\varepsilon(x) = 0, \qquad \text{a.e. } x \in \mathbb{R}^N.$$

This, together with (C)-(L), provides that $u = 0$. This concludes the proof. $\square$

Remark 3.1.5. *By Theorem 2.1.6 the space W_ε, that results of completing $\mathrm{C}_0^\infty(\mathbb{R}^N)$ in the norm $\|\cdot\|_\varepsilon$, is a Banach space.*

The following theorem will be useful to achieve our results.

Theorem 3.1.6. *Assume that $V_\varepsilon \in \mathrm{C}(\mathbb{R}^N)$ is non-negative and such that $V_\varepsilon(x) \to +\infty$ as $|x| \to +\infty$. Let W_ε be the Banach space that results of completing $\mathrm{C}_0^\infty(\mathbb{R}^N)$ whenever is equipped with the norm given by*

$$\|u\|_\varepsilon = \left(\int_{\mathbb{R}^N} |\nabla u(x)|^p + V_\varepsilon(x) |u(x)|^p \right)^{1/p}.$$

Then, the embedding

$$\mathrm{W}_\varepsilon \subseteq \mathrm{L}^q(\mathbb{R}^N),$$

is compact for all $q \in [p, p^*[$, where $p^* = \dfrac{pN}{N-p}$. For $q = p^*$ the embedding is continuous.

Remark 3.1.7. *For $p = 2$ the previous result has been used in [12] and [27]. Theorem 3.1.6 is obtained by an application of the theorems on criteria for strong compactness presented in [3]. It is obtained by compensating the non-boundedness of the domain with the property of the potential exploding at infinity.*

Let's consider the manifold

$$\mathcal{M}_\varepsilon = \left\{ u \in \mathrm{W}_\varepsilon \ / \ \int_{\mathbb{R}^N} |u(x)|^{q+1} dx = 1 \right\}, \tag{3.12}$$

and the functional $J_\varepsilon : \mathcal{M}_\varepsilon \subseteq \mathrm{W}_\varepsilon \to \mathbb{R}$, given by

$$\begin{aligned} J_\varepsilon(u) &= \frac{1}{p} \|u\|_\varepsilon^p \\ &= \frac{1}{p} \int_{\mathbb{R}^N} \left[|\nabla u(x)|^p + V_\varepsilon(x) |u(x)|^p \right] dx. \end{aligned} \tag{3.13}$$

Now, we have the following result

Proposition 3.1.8. *The functional J_ε is of class C^1. Moreover, for all $u, w \in \mathcal{M}_\varepsilon$*

$$\langle J'_\varepsilon(u), w\rangle = \int_{\mathbb{R}^N} |\nabla u(x)|^{p-2}\nabla u(x)\cdot\nabla w(x) + V_\varepsilon(x)|u(x)|^{p-2}u(x)w(x)\,dx \tag{3.14}$$

Proof. 1. Let's first compute the directional derivative of J_ε at $u \in \mathcal{M}_\varepsilon$, in the direction $h \in \mathcal{M}_\varepsilon$.

$$\begin{aligned}
J_\varepsilon(u+th) &= \frac{1}{p}\int_{\mathbb{R}^N} |\nabla u + t\nabla h|^p + V_\varepsilon(x)|u+th|^p\,dx,\\
\frac{d}{dt}J_\varepsilon(u+th) &= \int_{\mathbb{R}^N} \Big(|\nabla u|^2 + t^2|\nabla h|^2 + 2t\nabla u\cdot\nabla h\Big)^{(p-2)/2}\Big(t|\nabla h|^2 + \nabla u\cdot\nabla h\Big)\\
&\qquad + V_\varepsilon(x)\Big(|u|^2 + t^2|h|^2 + 2tuh\Big)^{(p-2)/2}\Big(t|h|^2 + uh\Big)\,dx,\\
\frac{d}{dt}J_\varepsilon(u+th)\bigg|_{t=0} &= \int_{\mathbb{R}^N} |\nabla u|^{p-2}\nabla u\cdot\nabla h + V_\varepsilon(x)|u|^{p-2}uh\,dx.
\end{aligned}$$

Arbitrariness of u, h implies that the Gateaux differential of J_ε exists at every $u \in \mathcal{M}_\varepsilon$. We will denote it by $J'_{G,\varepsilon}(u)$.

2. Let $u \in \mathcal{M}_\varepsilon$, generic and fixed. Let's define $g : \mathcal{M}_\varepsilon \to \mathbb{R}$ as

$$g(h) = J_\varepsilon(u+h) - J_\varepsilon(u) - \langle J'_{G,\varepsilon}(u), h\rangle. \tag{3.15}$$

Let's also define $L_\alpha : \mathbb{R} \to \mathbb{R}$ by $L_\alpha := z^\alpha$ with $\alpha \in \mathbb{R}$. A first order Taylor expansion of L_α around some $z \in \mathbb{R}$ provides

$$\begin{aligned}
L_\alpha(z+\epsilon) &= L_\alpha(z) + L'_\alpha(z)\epsilon + g_\alpha(\epsilon)\\
&= z^\alpha + \alpha z^{\alpha-1}\epsilon + g_\alpha(\epsilon),
\end{aligned} \tag{3.16}$$

where $g_\alpha(\epsilon) \to 0$ as $\epsilon \to 0$, i.e. $g_\alpha(\epsilon) = o(\epsilon)$. Let $h \in \mathcal{M}_\varepsilon$, generic. If we denote

$$z_1 = |\nabla u|^2, \qquad \epsilon_1 = |\nabla h|^2 + 2\nabla u\cdot\nabla h \tag{3.17}$$

$$z_2 = |u|^2, \qquad \epsilon_2 = |h|^2 + 2uh \tag{3.18}$$

then it follows from (3.16) - (3.18) that

$$\begin{aligned}
J_\varepsilon(u+h) &= \frac{1}{p}\int_{\mathbb{R}^N} |\nabla u + \nabla h|^p\,dx + \frac{1}{p}\int_{\mathbb{R}^N} V_\varepsilon(x)|u+h|^p\,dx\\
&= \frac{1}{p}\int_{\mathbb{R}^N} L_{p/2}(z_1+\epsilon_1)\,dx + \frac{1}{p}\int_{\mathbb{R}^N} L_{p/2}(z_2+\epsilon_2)\,dx\\
&= \frac{1}{p}\int_{\mathbb{R}^N} |\nabla u|^p + \frac{p}{2}|\nabla u|^{p-2}|\nabla h|^2 + p|\nabla u|^{p-2}\nabla u\cdot\nabla h + g_{p/2}(\epsilon_1)\,dx\\
&\quad + \frac{1}{p}\int_{\mathbb{R}^N} V_\varepsilon(x)\Big(|u|^p + \frac{p}{2}|u|^{p-2}|h|^2 + p|u|^{p-2}uh + g_{p/2}(\epsilon_2)\Big)\,dx.
\end{aligned} \tag{3.19}$$

Therefore, from (3.15), (3.19) and the the Gateaux differential of J_ε we have that

$$g(h) = \frac{1}{2}\int_{\mathbb{R}^N} |\nabla u|^{p-2}|\nabla h|^2 + V_\varepsilon(x)|u|^{p-2}|h|^2\,dx + \frac{1}{p}\int_{\mathbb{R}^N} \tilde{g}(\epsilon_1,\epsilon_2)\,dx, \tag{3.20}$$

where $\tilde{g}(\epsilon_1,\epsilon_2) = g_{p/2}(\epsilon_1) + V_\varepsilon g_{p/2}(\epsilon_2)$. Clearly, $g(h) = o(h)$ and therefore the functional J_ε is Fréchet differentiable. Thus, we will denote it by J'_ε.

3. Let's prove that J'_ε is continuous. Let $u_0 \in \mathcal{M}_\varepsilon$, generic and fixed. For any $u, v \in \mathcal{M}_\varepsilon$ we have to consider two cases:

   (a) If $1 < p \leq 2$, by (2.12) and (2.31) we have that

$$\begin{aligned}
\left|\langle J'_\varepsilon(u_0) - J'_\varepsilon(u), v\rangle\right| &= \left|\langle J'_\varepsilon(u_0), v\rangle - \langle J'_\varepsilon(u), v\rangle\right| \\
&\leq \int_{\mathbb{R}^N} \left||\nabla u_0|^{p-2}\nabla u_0 - |\nabla u|^{p-2}\nabla u\right| |\nabla v| dx \\
&\quad + \int_{\mathbb{R}^N} V_\varepsilon(x) \left||u_0|^{p-2}u_0 - |u|^{p-2}u\right| |v|\,dx \\
\text{(by (2.12))} &\leq 2^{2-p}\left(\int_{\mathbb{R}^N} |\nabla u_0 - \nabla u|^{p-1}|\nabla v|\,dx \right.\\
&\quad \left. + \int_{\mathbb{R}^N} V_\varepsilon(x)|u_0 - u|^{p-1}|v|\,dx\right) \\
\text{(by (2.31))} &\leq 2^{2-p}\left[\left(\int_{\mathbb{R}^N} |\nabla u_0 - \nabla u|^{\frac{p(p-1)}{(p-1)}}\,dx\right)^{\frac{p-1}{p}} \|\nabla v\|_{\mathrm{L}^p(\mathbb{R}^N)}\right. \\
&\quad \left. + \left(\int_{\mathbb{R}^N} V_\varepsilon(x)|u_0 - u|^{\frac{p(p-1)}{(p-1)}}\,dx\right)^{\frac{p-1}{p}} \left\|V_\varepsilon^{1/p} v\right\|_{\mathrm{L}^p(\mathbb{R}^N)}\right] \\
\text{(by (3.10))} &\leq 2^{2-p}\left(\|\nabla u_0 - \nabla u\|^{p-1}_{\mathrm{L}^p(\mathbb{R}^N)} + \left\|V_\varepsilon^{1/p}(u_0 - u)\right\|^{p-1}_{\mathrm{L}^p(\mathbb{R}^N)}\right)\|v\|_\varepsilon \\
&\leq 2^{3-p}\,\|u_0 - u\|_\varepsilon^{p-1}\,\|v\|_\varepsilon\,.
\end{aligned}$$

   (b) Moreover, if $p > 2$, then (2.31) and Theorem 2.1.31 with p'' such that

$$\frac{1}{p} + \frac{1}{p} + \frac{1}{p''} = 1,$$

   where clearly

$$p'' = \frac{p}{p-2},$$

   yields

$$
\begin{aligned}
&\left|\langle J_\varepsilon'(u_0) - J_\varepsilon'(u), v\rangle\right| = \left|\langle J_\varepsilon'(u_0), v\rangle - \langle J_\varepsilon'(u), v\rangle\right| \\
&= \left|\int_{\mathbb{R}^N} |\nabla u_0|^{p-2}\nabla u_0 \nabla v + V_\varepsilon(x)|u_0|^{p-2}u_0 v\, dx - \int_{\mathbb{R}^N} |\nabla u|^{p-2}\nabla u \nabla v + V_\varepsilon(x)|u|^{p-2}uv\, dx\right| \\
&= \left|\int_{\mathbb{R}^N} |\nabla u_0|^{p-2}\nabla u_0 \nabla v + V_\varepsilon(x)|u_0|^{p-2}u_0 v\, dx - \int_{\mathbb{R}^N} |\nabla u_0|^{p-2}\nabla u \nabla v + V_\varepsilon(x)|u_0|^{p-2}uv\, dx\right. \\
&\quad \left. + \int_{\mathbb{R}^N} |\nabla u_0|^{p-2}\nabla u \nabla v + V_\varepsilon(x)|u_0|^{p-2}uv\, dx - \int_{\mathbb{R}^N} |\nabla u|^{p-2}\nabla u \nabla v + V_\varepsilon(x)|u|^{p-2}uv\, dx\right| \\
&= \left|\int_{\mathbb{R}^N} |\nabla u_0|^{p-2}(\nabla u_0 - \nabla u)\nabla v + V_\varepsilon(x)|u_0|^{p-2}(u_0 - u)v\, dx\right. \\
&\quad \left. + \int_{\mathbb{R}^N} \left(|\nabla u_0|^{p-2} - |\nabla u|^{p-2}\right)\nabla u \nabla v + V_\varepsilon(x)\left(|u_0|^{p-2} - |u|^{p-2}\right)uv\, dx\right| \\
&\le \int_{\mathbb{R}^N} |\nabla u_0|^{p-2}|\nabla u_0 - \nabla u||\nabla v|\, dx + \int_{\mathbb{R}^N} V_\varepsilon(x)|u_0|^{p-2}|u_0 - u||v|\, dx \\
&\quad + C\int_{\mathbb{R}^N} |\nabla u_0 - \nabla u|^{p-2}|\nabla u||\nabla v|\, dx + C\int_{\mathbb{R}^N} V_\varepsilon(x)|u_0 - u|^{p-2}|u||v|\, dx \\
&\underset{\le}{\big(\text{by } (2.1.31)\big)} \left\|(\nabla u_0)^{p-2}\right\|_{p''}\|\nabla u_0 - \nabla u\|_p\|\nabla v\|_p + \left\|V_\varepsilon^{1/p''}u_0^{p-2}\right\|_{p''}\left\|V_\varepsilon^{1/p}(u_0 - u)\right\|_p\left\|V_\varepsilon^{1/p}v\right\|_p \\
&\quad + C\left\|(\nabla u_0 - \nabla u)^{p-2}\right\|_{p''}\|\nabla u\|_p\|\nabla v\|_p + C\left\|V^{1/p''}(u_0 - u)^{p-2}\right\|_{p''}\left\|V_\varepsilon^{1/p}u\right\|_p\left\|V_\varepsilon^{1/p}v\right\|_p \\
&\underset{\le}{\big(\text{by } (3.10)\big)} \left[\left(\int_{\mathbb{R}^N} |\nabla u_0|^{\frac{p(p-2)}{p-2}}\right)^{\frac{p-2}{p}} + \left(\int_{\mathbb{R}^N} V_\varepsilon(x)|u_0|^{\frac{p(p-2)}{p-2}}\right)^{\frac{p-2}{p}}\right]\|u_0 - u\|_\varepsilon\|v\|_\varepsilon \\
&\quad + C\left[\left(\int_{\mathbb{R}^N} |\nabla u_0 - \nabla u|^{\frac{p(p-2)}{p-2}}\right)^{\frac{p-2}{p}} + \left(\int_{\mathbb{R}^N} V_\varepsilon(x)|u_0 - u|^{\frac{p(p-2)}{p-2}}\right)^{\frac{p-2}{p}}\right]\|u\|_\varepsilon\|v\|_\varepsilon \\
&= \left(\|\nabla u_0\|_p^{p-2} + \left\|V_\varepsilon^{1/p}u_0\right\|_p^{p-2}\right)\|u_0 - u\|_\varepsilon\|v\|_\varepsilon \\
&\quad + C\left(\|\nabla u_0 - \nabla u\|_p^{p-2} + \left\|V_\varepsilon^{1/p}(u_0 - u)\right\|_p^{p-2}\right)\|u\|_\varepsilon\|v\|_\varepsilon \\
&\underset{\le}{\big(\text{by } (3.10)\big)} \left(\|u_0\|_\varepsilon^{p-2}\|u_0 - u\|_\varepsilon + C\|u\|_\varepsilon\|u_0 - u\|_\varepsilon^{p-2}\right)\|v\|_\varepsilon .
\end{aligned}
$$

Therefore, by parts (a) and (b), by the arbitrariness of $v \in \mathcal{M}_\varepsilon$ we know that

$$\left\|J_\varepsilon'(u_0) - J_\varepsilon'(u)\right\|_{\mathcal{M}_\varepsilon'} \to 0 \quad \text{as} \quad \|u_0 - u\|_\varepsilon \to 0.$$

Since $u \in \mathcal{M}_\varepsilon$ was also generic, it follows that the functional in (3.14) is continuous.

□

Let's show that the critical points of J_ε provide weak solutions for (E_ε) and, by (3.5) and (3.6), also for (G_ε). The formula of the corresponding Lagrangian functional is given by

$$L_\lambda(v) = J_\varepsilon(v) + \lambda\left(\int_{\mathbb{R}^N} |v(x)|^{q+1}dx - 1\right),$$

where $\lambda \in \mathbb{R}$ is the Lagrange multiplier. Let's recall that

$$J'_\varepsilon(v) = 0 \quad \iff \quad \begin{cases} L'_\lambda(v) = 0, \\ \displaystyle\int_{\mathbb{R}^N} |v(x)|^{q+1} dx = 1. \end{cases}$$

Let's observe that for $v, w \in \mathcal{M}_\varepsilon$,

$$\begin{aligned} L_\lambda(v)w = &\int_{\mathbb{R}^N} \left[|\nabla v(x)|^{p-2} \nabla v(x) \nabla w(x) + V_\varepsilon(x)|v(x)|^{p-2} v(x)w(x) \right] dx \\ &-\lambda(q+1) \int_{\mathbb{R}^N} |v(x)|^{q-1} v(x)w(x)dx. \end{aligned} \tag{3.21}$$

Now if $L'_\lambda(v) = 0$, then we get, by choosing $w = v$ in (3.21),

$$-\lambda(q+1) = pc,$$

where

$$c = J_\varepsilon(v)$$

is the corresponding critical value. From this it follows that v is a weak solution of

$$\Delta_p v(x) + V_\varepsilon(x)|v(x)|^{p-2} v(x) - pc|v(x)|^{q-1} v(x) = 0.$$

Therefore, the function u defined by

$$v = \gamma u,$$

with

$$|\gamma| = (pc)^{1/(p-q-1)},$$

is a weak solution of (E_ε).

3.2 Main results

Now we state the main result of this work.

Theorem 3.2.1. *Let $p > 1$ and assume*

$$V \in C(\mathbb{R}^N) \text{ is non-negative and} \tag{C}$$

$$\lim_{|x| \to \infty} V(x) = \infty. \tag{L}$$

Then, the problem (G_ε) possesses at least one nontrivial solution.

For the proof of Theorem 3.2.1, we will need the following result that can be found on [22].

Theorem 3.2.2 (Mawhin & Willem)**.** *Let X be a real Banach space and $I \in \mathrm{C}^1(X)$ satisfies the* (P.S.)*-condition. If I is bounded from below, then*

$$c = \inf_X I$$

is a critical value of I.

Proceeding as in [10], since in Proposition 3.1.8 we already proved that J_ε is of class C^1 on $\mathcal{M}_\varepsilon$, in order to prove Theorem 3.2.1 using Theorem 3.2.2 it will suffice to show that the conditions of boundedness from below and the (P.S.) condition hold for J_ε. From (3.13) it immediately follows that

Remark 3.2.3. *The functional $J_\varepsilon : \mathcal{M}_\varepsilon \to \mathbb{R}$, defined by $J_\varepsilon(u) = \frac{1}{p}\|u\|_\varepsilon^p$ is bounded from below, i.e., $J_\varepsilon(u) \to \infty$ as $\|u\|_\varepsilon \to \infty$.*

As in the proof from Proposition 3.1.8, we will consider the following cases: $p \geq 2$ and $1 < p < 2$

Lemma 3.2.4. *The functional J_ε from* (3.13) *satisfies the* (P.S.) *condition for $p \geq 2$.*

Proof. Let $(u_n)_{n\in\mathbb{N}} \subseteq \mathcal{M}_\varepsilon$ a sequence such that

$$(J_\varepsilon(u_n))_{n\in\mathbb{N}} \quad \text{is bounded, and} \tag{3.22}$$

$$J'_\varepsilon(u_n) \to 0 \quad \text{as} \quad n \to \infty. \tag{3.23}$$

We have to prove that $(u_n)_{n\in\mathbb{N}}$ has a converging subsequence. By Remark 3.2.3, there exists some $k_1 \in \mathbb{R}$ and, by (3.22), some $k_2 \in]\max\{0, |k_1|\}, +\infty[$ such that

$$\forall\, n \in \mathbb{N} : \qquad k_1 \leq J_\varepsilon(u_n) \leq k_2. \tag{3.24}$$

By Theorem 3.1.6 and (3.24) we get

$$\exists\, C_1, C_2 > 0, \forall\, n \in \mathbb{N} : \qquad \|u_n\|_{\mathrm{L}^p(\mathbb{R}^N)} \leq C_1 \|u_n\|_\varepsilon \leq C_2. \tag{3.25}$$

Therefore, up to a subsequence, by (3.25) it follows that

$$u_n \rightharpoonup u_0, \qquad \text{weakly in } \mathcal{M}_\varepsilon, \text{ i.e.,}$$

$$\forall\, \eta \in \mathcal{M}'_\varepsilon : \qquad \langle \eta, u_n - u_0 \rangle \to 0, \quad \text{as } n \to \infty. \tag{3.26}$$

In the view of weak convergence, we have that

$$\langle J'_\varepsilon(u_n) - J'_\varepsilon(u_0), u_n - u_0 \rangle \to 0 \quad \text{as} \quad n \to 0. \tag{3.27}$$

Moreover, by (3.14) we know that

$$\begin{aligned}
&\langle J'_\varepsilon(u_n) - J'_\varepsilon(u_0), u_n - u_0 \rangle = \langle J'_\varepsilon(u_n), u_n - u_0 \rangle - \langle J'_\varepsilon(u_0), u_n - u_0 \rangle \\
&= \int_{\mathbb{R}^N} |\nabla u_n|^{p-2}\nabla u_n \cdot (\nabla u_n - \nabla u_0) + V_\varepsilon(x)|u_n|^{p-2}u_n(u_n - u_0)\, dx \\
&\quad - \int_{\mathbb{R}^N} |\nabla u_0|^{p-2}\nabla u_0 \cdot (\nabla u_n - \nabla u_0) + V_\varepsilon(x)|u_0|^{p-2}u_0(u_n - u_0)\, dx \\
&= \int_{\mathbb{R}^N} \langle |\nabla u_n|^{p-2}\nabla u_n - |\nabla u_0|^{p-2}\nabla u_0, \nabla u_n - \nabla u_0 \rangle dx \\
&\quad + \int_{\mathbb{R}^N} V_\varepsilon(x)\langle |u_n|^{p-2}u_n - |u_0|^{p-2}u_0, u_n - u_0 \rangle dx,
\end{aligned} \tag{3.28}$$

where in (3.28) the symbol $\langle \cdot, \cdot \rangle$ denotes the euclidean inner product on $\mathbb{R}^N$. Let us recall from [19] the following identity. Let $a, b \in \mathbb{R}^N$, generic. Then we have that

$$\langle |b|^{p-2}b - |a|^{p-2}a, b-a \rangle = \frac{|b|^{p-2} + |a|^{p-2}}{2}|b-a|^2 + \frac{(|b|^{p-2} - |a|^{p-2})(|b|^2 - |a|^2)}{2}. \tag{3.29}$$

Since $p \geq 2$, from (3.29) we have that

$$\langle |b|^{p-2}b - |a|^{p-2}a, b-a \rangle \geq 2^{2-p}|b-a|^p. \tag{3.30}$$

By (3.28) and (3.30) we obtain

$$\begin{aligned} \langle J_\varepsilon'(u_n) - J_\varepsilon'(u_0), u_n - u_0 \rangle &\geq 2^{2-p} \int_{\mathbb{R}^N} |\nabla u_n - \nabla u_0|^p + V_\varepsilon(x)|u_n - u_0|^p \, dx \\ &= 2^{2-p} \left\| u_n - u_0 \right\|_\varepsilon^p . \end{aligned} \tag{3.31}$$

Together with (3.27) and (3.31) we have

$$\|u_n - u_0\|_\varepsilon^p \leq 2^{p-2} \langle J_\varepsilon'(u_n) - J_\varepsilon'(u_0), u_n - u_0 \rangle \to 0 \quad \text{as } n \to \infty,$$

which implies that

$$u_n \to u_0, \quad \text{in } \mathcal{M}_\varepsilon.$$

Therefore, J_ε satisfies the (P.S.)-condition. □

Considering the second case we present the following analogous result.

Lemma 3.2.5. *The functional J_ε from* (3.13) *satisfies the* (P.S.) *condition for* $1 < p < 2$.

Proof. We proceed exactly the same as in Lemma 3.2.4 in steps (3.22)-(3.28). Since $1 < p < 2$, the identity (3.29) provides

$$\langle |b|^{p-2}b - |a|^{p-2}a, b-a \rangle \geq (p-1)(1 + |b|^2 + |a|^2)^{\frac{p-2}{2}} |b-a|^2. \tag{3.32}$$

From (3.28) and (3.32) it follows that

$$\begin{aligned} &\langle J_\varepsilon'(u_n) - J_\varepsilon'(u_0), u_n - u_0 \rangle \\ &\geq (p-1) \int_{\mathbb{R}^N} |\nabla u_n - \nabla u_0|^2 \left(1 + |\nabla u_n|^2 + |\nabla u_0|^2\right)^{\frac{p-2}{2}} \, dx \\ &\quad + (p-1) \int_{\mathbb{R}^N} V_\varepsilon(x)|u_n - u_0|^2 \left(1 + |u_n|^2 + |u_0|^2\right)^{\frac{p-2}{2}} \, dx. \end{aligned} \tag{3.33}$$

Since $1 < p < 2$, we now define

$$q := \frac{p}{2} \quad \text{and} \quad q' := \frac{p}{p-2} \qquad \text{so that} \quad \frac{1}{q} + \frac{1}{q'} = 1. \tag{3.34}$$

Notice that since $0 < q < 1$, we can apply the variant of Hölder's inequality (Theorem 2.1.33) for q and q' as in (3.34) for (3.33). Then by the parallelogram identity we have that

$$\begin{aligned}
&(p-1)\int_{\mathbb{R}^N} |\nabla u_n - \nabla u_0|^2 \left(1 + |\nabla u_n|^2 + |\nabla u_0|^2\right)^{\frac{p-2}{2}} dx \\
&\quad + (p-1)\int_{\mathbb{R}^N} V_\varepsilon(x)|u_n - u_0|^2 \left(1 + |u_n|^2 + |u_0|^2\right)^{\frac{p-2}{2}} dx. \\
&\geq (p-1)\left(\int_{\mathbb{R}^N} |\nabla u_n - \nabla u_0|^{\frac{2p}{2}} dx\right)^{\frac{2}{p}} \left(\int_{\mathbb{R}^N} \left(1 + |\nabla u_n|^2 + |\nabla u_0|^2\right)^{\frac{(p-2)p}{2(p-2)}} dx\right)^{\frac{p-2}{p}} \\
&\quad + (p-1)\left(\int_{\mathbb{R}^N} V_\varepsilon(x)|u_n - u_0|^{\frac{2p}{2}} dx\right)^{\frac{2}{p}} \left(\int_{\mathbb{R}^N} V_\varepsilon(x)\left(1 + |u_n|^2 + |u_0|^2\right)^{\frac{(p-2)p}{2(p-2)}} dx\right)^{\frac{p-2}{p}} \\
&\geq (p-1)\left(\int_{\mathbb{R}^N} |\nabla u_n - \nabla u_0|^p dx\right)^{\frac{2}{p}} \left(\int_{\mathbb{R}^N} \left(|\nabla u_n|^2 + |\nabla u_0|^2\right)^{\frac{p}{2}} dx\right)^{\frac{p-2}{p}} \\
&\quad + (p-1)\left(\int_{\mathbb{R}^N} V_\varepsilon(x)|u_n - u_0|^p dx\right)^{\frac{2}{p}} \left(\int_{\mathbb{R}^N} V_\varepsilon(x)\left(|u_n|^2 + |u_0|^2\right)^{\frac{p}{2}} dx\right)^{\frac{p-2}{p}} \\
&\geq \frac{p-1}{2}\left(\int_{\mathbb{R}^N} |\nabla u_n - \nabla u_0|^p dx\right)^{\frac{2}{p}} \left(\int_{\mathbb{R}^N} |\nabla u_n - \nabla u_0|^p dx\right)^{\frac{p-2}{p}} \\
&\quad + \frac{p-1}{2}\left(\int_{\mathbb{R}^N} V_\varepsilon(x)|u_n - u_0|^p dx\right)^{\frac{2}{p}} \left(\int_{\mathbb{R}^N} V_\varepsilon(x)|u_n - u_0|^p dx\right)^{\frac{p-2}{p}} \\
&= \frac{p-1}{2}\int_{\mathbb{R}^N} |\nabla u_n - \nabla u_0|^p + V_\varepsilon(x)|u_n - u_0|^p dx \\
&= \frac{p-1}{2}\|u_n - u_0\|_\varepsilon^p
\end{aligned} \tag{3.35}$$

Together with (3.27) and (3.35) we have

$$\|u_n - u_0\|_\varepsilon^p \leq \frac{2}{p-1}\langle J'_\varepsilon(u_n) - J'_\varepsilon(u_0), u_n - u_0\rangle \to 0 \quad \text{as } n \to \infty,$$

which implies that

$$u_n \to u_0, \quad \text{in } \mathcal{M}_\varepsilon.$$

Therefore, J_ε satisfies the (P.S.)-condition. $\square$

By Theorem 3.2.2, Lemmas 3.2.4, 3.2.5, and Remark 3.2.3,

$$c = \inf_{\mathcal{M}_\varepsilon} J_\varepsilon(u)$$

is a critical value of J_ε, i.e., there exists a critical point $u^* \in \mathcal{M}_\varepsilon$ such that $J_\varepsilon(u^*) = c$. Thus, we have proved Theorem 3.2.1.

Chapter 4

Conclusions and recommendations

4.1 Conclusions

We have proved the existence of a non-trivial ground-state solution for the following quasi linear boundary value problem.

$$\left\{ \begin{array}{rcll} -\varepsilon^2 \Delta_p u(x) + V(x)|u(x)|^{p-2}u(x) - |u(x)|^{q-1}u(x) & = & 0, & x \in \mathbb{R}^N \\ u(x) & \to & 0, & \text{as } |x| \to \infty, \end{array} \right. \tag{G_ε}$$

where

$$\Delta_p u = \operatorname{div}(|\nabla u|^{p-2} \nabla u)$$

and

$$1 < p < q + 1 < p^*, \tag{4.1}$$

with

$$p^* = \left\{ \begin{array}{ll} \dfrac{pN}{N-p}, & \text{if } N \geq 3; \\ \infty, & \text{if } N = 1, 2. \end{array} \right.$$

Additionally, we assumed that

$$V \in C(\mathbb{R}^N) \text{ is non-negative and} \tag{C}$$

$$\lim_{|x| \to \infty} V(x) = +\infty. \tag{L}$$

By rescaling, (G_ε) is equivalent to

$$\left\{ \begin{array}{ll} -\Delta_p u(x) + V_\varepsilon(x)|u(x)|^{p-2}u(x) - |u(x)|^{q-1}u(x) = 0, & x \in \mathbb{R}^N, \\ u(x) \to 0, & \text{as } |x| \to +\infty. \end{array} \right. \tag{E_ε}$$

Also, we consider the manifold

$$\mathcal{M}_\varepsilon = \left\{ u \in \mathrm{W}_\varepsilon \;/\; \int_{\mathbb{R}^N} |u(x)|^{q+1} dx = 1 \right\}, \tag{4.2}$$

and the functional $J_\varepsilon : \mathcal{M}_\varepsilon \subseteq \mathrm{W}_\varepsilon \to \mathbb{R}$, given by

$$\begin{aligned} J_\varepsilon(u) &= \frac{1}{p}\|u\|_\varepsilon^p \\ &= \frac{1}{p}\int_{\mathbb{R}^N} \left[|\nabla u(x)|^p + V_\varepsilon(x)|u(x)|^p\right] dx. \end{aligned} \tag{4.3}$$

We summarize the main results as follows:

1. In the preliminaries section, in Proposition 3.1.8 we proved that the functional (3.13) is of class C^1 and that its Fréchet differential is given by (3.14). In order to determine J'_ε, we used the Gateaux differentiability of J_ε and a first order Taylor expansion. For the continuity of J'_ε we had to consider two cases for p: $1 < p \leq 2$ and $2 < p$. Some inequalities from [19] and the extension of Hölder's inequality from [3] were applied for this purpose.
2. The main results section was dedicated exclusively to our main Theorem 3.2.1. The proof of this theorem required the use of three partial results:
    - Theorem 3.2.2 from [22] provided the necessary conditions for the existence of a critical value for our functional. The completeness of $\mathcal{M}_\varepsilon$ was already given by the fact that $\|\cdot\|_\varepsilon$ is a norm, proved in Proposition 3.1.4, and Theorem 2.1.6. The properties of the functional J_ε of being C^1 and bounded from bellow readily followed by Proposition 3.1.8 and Remark 3.2.3, respectively.
    - Lemma 3.2.4 shows that J_ε satisfies the (P.S.) condition when $p \geq 2$. Theorem 3.1.6, weak convergence and an inequality from [19] provided the required result. This proceeding was done based on [10].
    - Lemma 3.2.5 shows that J_ε satisfies the (P.S.) condition when $1 < p < 2$. The proof is analogous to that of Lemma 3.2.4 with the exception that Theorem 2.1.33, the parallelogram identity and another inequality from [19] were also required.

4.2 Recommendations

1. This work can be the first step in a more in depth study of the solutions for problem (G_ε). Additional assumptions over the zero set of the potential could lead to the study of asymptotic profiles for solutions as in [4], [5] and [27].
2. An additional interesting problem that could benefit from what has been just presented would be a non-stationary version of problem (G_ε).
3. In my opinion, the mathematics students from Yachay Tech could improve their undergraduate research output by the inclusion in the academic plan of specific topics dedicated to the reading and understanding of mathematics papers and publications.

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