UNIVERSIDAD DE INVESTIGACIÓN DE TECNOLOGÍA EXPERIMENTAL YACHAY

Escuela de Ciencias Matemáticas y Computacionales

Synchronization in a class of non-identical master slave systems. Application to the Chua's equations.

Trabajo de integración curricular presentado como requisito para la obtención del título de Matemático

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Dedication

To my beloved grandparents, Rosario y Segundo,
loved mother, sister and little brother.
To my darling aunt and to my beloved Yachay Tech.

Acknowledgments

I would like to thank my mom Rosa Chango and my sister Paulina for being with me in spite of everything, for their effort and unconditional support towards me. I want to thank the first parents I had, my grandparents Rosario Criollo y Segundo Chango, for giving me the best childhood, for being present at every moment, and for their trust in me. To my dear family, uncles, and aunts, I have no words to express my gratitude for your support and your presence at every moment. I especially want to thank my aunt Nancy Chango who had been my example since I was a child and for teaching me the value of constant work and perseverance that is needed to conquer the world.

If I were given the opportunity to go back in time and choose a university I would always choose Yachay Tech. Here I developed mostly my love for science and my passion for mathematics, here I met my second family and my adventure partners. I cannot imagine getting to where I am without my friends, Jonathan, Susana, Dayanara, Camila, and Jennifer, to this last one I owe the friends I have, thank you for not only being my Rommie but my first friend in that new beginning. Thank you guys for being my complement every day, for the laughs, and for not leaving me alone in spite of the circumstances. I would also like to thank the Mathematicians 4G group for not only being my classmates but my tutors, mentors, and friends. Finally, I would like to thank my best friend from High School, Priscila who, despite the distance, was my unconditional support at all times.

I would like to express my sincere gratitude and admiration to my professors for being the inspiration that every student needs, for showing me that professionals are not only those who wear suits and ties but those who know how to dress you with their teachings. Finally, I would like to highlight professors Juan Mayorga-Zambrano, Isidro Amaro, Franklin Camacho, Hugo Fernandes, and Javier García. The experience I had at Yachay would not be the same without you as my teachers.

Among the best teachers, I have had is my advisor, Professor Antonio Acosta. To whom I am infinitely grateful for his patience and guidance during my academic period and during the completion of this work. Thank you for encouraging me with your example and transmitting perseverance and effort in all possible aspects and demonstrating that it is possible to achieve everything you set your mind to with the right guidance and effort. Thank you for being not only my advisor and teacher but also for being my mentor and friend both academically and personally. Thank you for your moral support and your constant work so that young people like me, believe in themselves and see mathematics without fear and that we are all able to understand something so abstract more simply.

Finally, I would like to thank Professor Pedro García for his support in the numerical part of this work, his suggestions, and his important comments to improve this work.

Thank you Yachay Tech!

Abstract

In this work, a master-slave system of autonomous chaotic ordinary differential equations that are not identical is considered. In order to refer to the master-slave system being synchronized, the aim is that the evolution of the solutions of the master and slave equations should coincide as closely as possible. Given a bounded solution for the master equation, we use this as an input in the slave equation. This leads us to a non-autonomous system where the conditions, that are represented by a class of systems illustrated by Chua's equations, are identified. Next, for the non-autonomous system, we are able to show, under suitable conditions, the existence of solutions that ensure synchronization of our master-slave system.

The theoretical results shown are recreated, using Chua's equations, with numerical simulations that effectively show that the solutions of the master and slave equations match as closely as possible.

Keywords: synchronization, master-slave system, Chua's equations, Banach Fixed Point Theorem.

Resumen

En este trabajo se considera un sistema maestro-esclavo de Ecuaciones Diferenciales Ordinarias caóticas autónomas no idénticas. Para referirse a que el sistema maestro-esclavo está sincronizado, el objetivo es que la evolución de las soluciones de las ecuaciones maestro y la ecuación esclavo coincidan lo más posible. Dada una solución acotada para la ecuación maestro, esta es usada como entrada para la ecuación esclavo. Esto nos conduce a un sistema no autónomo donde se identifican las condiciones que representan una clase de sistemas ilustrado por las ecuaciones de Chua. Por consiguiente, para el sistema no autónomo vamos a mostrar, bajo ciertas condiciones, la existencia de soluciones que encierran sincronización por parte de nuestro sistema maestro-esclavo.

Los resultados teóricos mostrados son recreados usando las ecuaciones de Chua con simulaciones numéricas que muestran efectivamente que la solución del sistema maestro-esclavo se ajustan al máximo.

Palabras Clave: sincronización, sistema maestro-esclavo, ecuaciones de Chua, Teorema de Punto Fijo de Banach.

Contents

Chapter 1

Introduction

Synchronization between two or more dynamical systems describes the correlation process of two or more chaotic systems under specific conditions for a time interval [1, 2]. This process is a basis to understand an unknown dynamical system from one or more well-known dynamical systems [1].

This phenomenon has been studied since the century 17th when the scientist Christiaan Huygens gave a detailed description about the inanimated synchronization of dynamical systems by studying a system formed by two pendulum-clocks hanging from simple support and observing phase to phase the synchronized behaviour that this system presented after of a certain time [3]. Huygens' research was a key element in the study of synchronization of dynamical systems until becoming an active research topic.

An important scientific revolution was the chaos in dynamics introduced by Henri Poincaré at the end of the 19th century [4]. Chaotic dynamical systems present an even greater challenge because they defy synchronization, implying that the signals that produce a chaotic system could not synchronize with any other system [5]. In other words, the trajectory of two identical autonomous chaotic systems that begin in the same initial point in the same phase space uncoordinated quickly in time. However, if the two dynamical systems exchange information in the right way, they can synchronize [6].

Over time, the general interest in synchronization has increased, since this phenomenon has presented relevant manifestation in fields such as technology, physics, biology and engineering [2]. Synchronization, besides being a fundamental element to understand the natural phenomena, describes a spontaneous transition to order due to the interaction between different processes in a time interval [4].

Reference [7] shows that it has not been possible to establish a single definition of synchronization that encompasses each and every example of synchronization known and yet to be known. A lot of forms of synchronization have been established over the last decade. One of them is classified on the basis of the unidirectional or bidirectional nature of the coupling process.

At the end of the century 20th, Pecora and Carroll introduced in [5] a new phenomenon where the synchronized systems present a unidirectional interconnection known as master-slave synchronization [8]. This synchronization implies that one subsystem flows freely and directs the flow of another one.

Master-slave synchronization consists of two non-identical subsystems which are coupled such that the solution of one of them (slave system) always converges, or at least stay close, to the solution of the other (master system), independently of the initial condition [9]. There exist some forms to represent a master−slave system. In our case, let us consider a master system in the following form

$$\dot{\mathbf{x}} = f(\mathbf{x}, \bar{\mu}), \tag{1.1}$$

where $\mathbf{x} \in \mathbb{R}^n$ is the state variable and $\bar{\mu}$ is a constant vector in $\mathbb{R}^2$

The slave system is written in the following form

$$\dot{\mathbf{y}} = f(\mathbf{y}, \mu) + \nu(\mathbf{y} - \mathbf{x}), \tag{1.2}$$

where $\mathbf{y} \in \mathbb{R}^n$ and μ is a constant vector in $\mathbb{R}^2$. The system (1.1)-(1.2) is called the master-slave system. The solution of (1.2) follows the evolution of a given bounded solution of the system (1.1).

In this work, we will discuss the conditions and results for master−slave synchronization of chaotic systems, more specifically, let us apply the last unidirectional synchronization in a specific example, that is, in Chua's equations. This review is inspired by [10], where, using exponential dichotomies, it is established specific conditions on non-identical chaotic dynamical systems to obtain master-slave synchronization. It is also important to note that in our work the main interest, when it comes to thinking about applications, is directed to chaotic systems. Certainly if the systems considered are not chaotic and evolve freely, are not coupled, they could synchronize. The same is not true for chaotic systems.

This work is organized as follows.

- In chapter 2, we present some basic definitions related to normed spaces and complete normed spaces. Next, we give some important examples of complete normed spaces startling the set of bounded continuous functions on $\mathbb{R}^n$ with the infinite norm. Then, we present the Banach Fixed Point Theorem, also known as the *Principle of contraction mappings* which states sufficient conditions for the existence and uniqueness of a fixed point. Later, we summarize some general properties of differential equations as the basic existence theorem and uniqueness of solutions. Also, we recall some important facts concerning linear systems. The chapter closes with a brief discussion of eigenvalues and eigenvectors as well as some very simple propositions relating to these concepts. The concrete case of one type of matrix, which appears when considering Chua's equations is considered.

- In chapter 3, we present the problem to be considered. Specifically the type of master−slave system mentioned above. Through a transformation, the master−slave system is brought to a non-autonomous system on which our study is performed. Thus our main theoretical result, which of course guarantees synchronization of the master-slave system, is obtained on the non-autonomous system and provides conditions that guarantee synchronization. Finally, a simple application of Gronwall's lemma allows us to say more about theoretical results.

- In chapter 4, we present the application of the results in Chua's equations. We begin highlighting important aspects about the circuit from which these equations derive and establishing explicitly the system to discuss. Next, we propose a transformation on Chua's equation such that the conditions given in *Chapter 2* are satisfied and we can apply the theoretical results. In the end, we work with a Computer Algebra System, Maxima, to expose the numerical results of master-slave synchronization in Chua's equations.

- In Chapter 5, we present some conclusions and recommendations related to this work. There we highlight the characteristics that the systems considered in our work have, and also mention considerations that may lead us to carry out further research on what has been developed in this work.

Chapter 2

Mathematical Framework

In this chapter we present a number of basic facts from analysis, ordinary differential equations and linear algebra, which are fundamental in this work. For the development of this chapter, the main references are the following [11], [12], [13], and [14].

2.1 Concepts and Definitions

This section introduces some conventions, notions and theorems related to Vector Spaces.

2.1.1 Banach spaces and examples

We start recalling that a **linear vector space** (or **linear space**) X over $\mathbb{R}$ (or $\mathbb{C}$) is a set $\{x,\ y,\ z,\dots\}$ such that

1. for each $x,\ y,\ z \in X$, the sum $x+y$ is defined, $x+y \in X$, as

$$x+y = y+x,\ \ (x+y)+z = x+(y+z),$$

2. there is an element $0 \in X$ such that for every $x \in X$,

$$x+0 = x.$$

3. For a given $x \in X$ there is an element $\bar{x} \in X$ such that

$$x+\bar{x} = 0.$$

4. Also, for each $\alpha,\ \beta \in \mathbb{R}$ (or $\mathbb{C}$) and for each $x,\ y \in X$, scalar multiplication αx is defined, $\alpha x \in X$ and $1x = x$,

$$\begin{aligned}(\alpha\beta)x &= \alpha(\beta x) = \beta(\alpha x)\\ (\alpha+\beta)x &= \alpha x + \beta x\\ \alpha(x+y) &= \alpha x + \alpha y.\end{aligned}$$

From now on will be enough for us to pay attention to linear spaces over $\mathbb{R}$ (real linear spaces). Let X be a real linear space, a **norm** on X is a map $\|\cdot\| : X \to [0, \infty)$ which satisfies

i) $\forall x \in X : \|x\| > 0 \text{ if } x \neq 0, \ \|0\| = 0.$

ii) $\forall \alpha \in \mathbb{R}, \forall x \in X : \|\alpha x\| =| \alpha | \, \|x\|.$

iii) $\forall x, y \in X : \|x + y\| \leq \|x\| + \|y\|$ (triangle inequality).

The pair $(X, \|\cdot\|)$ is called a **normed linear space**. When confusion may arise, we will write $\|\cdot\|_X$ for the norm function on X. The resulting space is called a normed space.

Some well known facts and definitions related with normed linear spaces are the following

- The $\varepsilon-$**neighborhood** of an element x of a normed linear space X is
$$\{y \in X : \|y - x\| < \varepsilon\}.$$
- A set S in X is **open** if for every $x \in S$, there exists an $\varepsilon-$neighborhood of x which is contained in S.
- S is **closed** if and only if $X - S$ is open.
- A set S in X is **bounded** if there exists $r > 0$ such that $S \subset \{x \in V : \|x\| < r\}$.
- An element x is a **limit point** of a set S in X if each $\varepsilon-$neighborhood of x contains points of S.
- A set S in X is **closed** if it contains its limit points.
- A sequence $\{x_n\}$ in a normed linear space X **converges** to x in X if
$$\lim_{n\to\infty} \|x_n - x\| = 0.$$
We shall write this as
$$\lim_{n\to\infty} x_n = x.$$
- A sequence $\{x_n\}$ in X is a **Cauchy sequence** if for every $\varepsilon > 0$, there is an $N(\varepsilon) \in \mathbb{N}$ such that if $n, m \geq N(\varepsilon)$ then
$$\|x_n - x_m\| < \varepsilon.$$
- The space X is **complete** if every Cauchy sequence in X converges to an element of X. A complete normed linear space is a **Banach space**.
- A Cauchy sequence $\{x_n\}$, in a Banach space X, which is contained in a closed set S converges to an element of S.
- The real line and the complex plane are complete normed spaces.

- A subset A of X is complete if (A, d) is complete.

Examples

1. Let $X = \mathbb{R}^n$ be the space of real $n-$dimensional column vectors. For a particular coordinate system, elements x in $\mathbb{R}^n$ will be written as $x = (x_1, \ldots, x_n)$, where each x_j is in $\mathbb{R}$. If $x = (x_1, \ldots, x_n)$, $y = (y_1, \ldots, y_n)$ are in $\mathbb{R}^n$, then $\alpha x + \beta y$ for α, β in $\mathbb{R}$ is defined to be $(\alpha x_1 + \beta y_1, \ldots, \alpha x_n + \beta y_n)$. The space $\mathbb{R}^n$ is clearly a linear space. Moreover, it is a Banach space if we choose $\|x\|$ to be either
$$\sup\{|x_i| : i = 1, \ldots, n\} \ , \ \sum_{i=1}^{n} |x_i| \text{ or } \left(\sum_{i=1}^{n} x_i^2\right)^{\frac{1}{2}} .$$
Each of these norms are equivalent in the sense that a sequence converging in one norm converges in any of the other norms. The fact that $X = \mathbb{R}^n$ is complete follows because convergence implies coordinate wise convergence and $\mathbb{R}$ is complete. In particular when $n = 1$ all the above norms coincide for any $x \in \mathbb{R}$.
Also, an important consideration, which we frame in this example, corresponds to the set of real matrices of size $n \times n$ and which will be denoted by $\mathcal{M}_{n\times n}$. Certainly $\mathcal{M}_{n\times n}$ can be identified with $\mathbb{R}^{n^2}$ and one has that $X = \mathcal{M}_{n\times n}$ is a Banach space if we choose $\|A\|$, where $A = \left(a_{ij}\right)$,$i, j = 1, \ldots, n$, to be either
$$\sup\{|a_{ij}| : i, j = 1, \ldots, n\} \ , \ \sum_{i,j=1}^{n} |a_{ij}| \text{ or } \left(\sum_{i,j=1}^{n} a_{ij}^2\right)^{\frac{1}{2}} .$$
We close this example by pointing out that throughout this work, unless otherwise specified, we will use the Euclidean norm, i.e., $\|x\| = \left(\sum_{i=1}^{n} x_i^2\right)^{\frac{1}{2}}$, $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$. Of course if $A \in \mathcal{M}_{n\times n}$ it corresponds to $\left(\sum_{i,j=1}^{n} a_{ij}^2\right)^{\frac{1}{2}}$.

2. Let $X = C([a, b], \mathbb{R}^n)$ be the linear space of continuous functions which maps the closed interval $[a, b]$ into $\mathbb{R}^n$. If we define for a given $x \in X$
$$\|x\|_\infty = \sup_{t\in[a,b]} \|x(t)\|,$$
then $\|\cdot\|_\infty$ is a norm on X and also X is complete with this norm. Thus, the pair $\left(C([a, b], \mathbb{R}^n), \|\cdot\|_\infty\right)$ is a Banach space.

3. Let $X = C_b([0, \infty), \mathbb{R}^n)$ be the linear space of bounded continuous functions which maps the interval $[0, \infty)$ into $\mathbb{R}^n$. If we define, for a given $x \in X$,
$$\|x\|_\infty = \sup_{t\in[0,\infty)} \|x(t)\|,$$
the pair $\left(C_b([0, \infty), \mathbb{R}^n), \|\cdot\|_\infty\right)$ is a Banach space. Let us prove that X is complete. We start by considering a Cauchy sequence $\{x_n\}$ in X. Given $t \in [0, \infty)$, we have

that $\{x_n(t)\}$ is a Cauchy sequence in $\mathbb{R}^n$ and since $\mathbb{R}^n$ is complete its limit is in $\mathbb{R}^n$, i.e, $\lim_{n\to\infty} x_n(t)$ exists. Now, define $x : [0,\infty) \to \mathbb{R}^n$ as

$$x(t) = \lim_{n\to\infty} x_n(t) \quad (\text{ pointwise limit}).$$

We claim that

$$x \in X \text{ and } \lim_{n\to\infty} \|x_n - x\|_\infty = 0 \ .$$

Given an arbitrary $t \in [0,\infty)$ and a closed interval $[a,b]$ that contains t, the limit function $\lim_{n\to\infty} x_n(t)$ is, with the norm $\|\cdot\|_\infty$, in $C([a,b],\mathbb{R}^n)$. Thus, x is continuous at t.

To prove that x is bounded, we proceed by contradiction. Suppose that x is not bounded, then there exists a sequence $\{t_m\}$ in the interval $[0,\infty)$ such that $t_m < t_{m+1}$, $m = 1,2,\ldots$, $\lim_{m\to\infty} t_m = \infty$ and $\|x(t_m)\| \geqslant m$. Now, given $m \geqslant 1$ we have that $\lim_{n\to\infty} x_n(t_m) = x(t_m)$. Next, given $0 < \varepsilon < 1$, if we choose N and m such that $\|x_N\|_\infty < m-1$ and $\|x_N(t_m) - x(t_m)\| < \varepsilon$, then

$$\begin{aligned} m \leqslant \|x(t_m)\| &= \left\|(x(t_m) - x_N(t_m)) + x_N(t_m)\right\| \\ &< \varepsilon + m - 1. \end{aligned}$$

This is a contradiction. Now, we have to prove that x_n converges to x in norm. Given $\varepsilon > 0$, there exists an N such that for $n,m \geq N$, $\|x_n - x_m\|_\infty < \frac{\varepsilon}{2}$. Now, for $t \in [0,\infty)$ and $n,m \geqslant N$ we have that

$$\begin{aligned} \|x_n(t) - x(t)\| &= \left\|(x_n(t) - x_m(t)) + (x_m(t) - x(t))\right\| \\ &\leqslant \|x_n - x_m\|_\infty + \|x_m(t) - x(t)\| \\ &< \frac{\varepsilon}{2} + \|x_m(t) - x(t)\|. \end{aligned}$$

Next, choose $m = m(t) \geqslant N$ such that

$$\|x_m(t) - x(t)\| < \frac{\varepsilon}{2} \ .$$

This implies that $\|x_n(t) - x(t)\| < \varepsilon$ for all $t \in [0,\infty)$. Thus, $\|x_n - x\|_\infty \leqslant \varepsilon$. Hence, $\lim_{n\to\infty} \|x_n - x\|_\infty = 0$.

2.1.2 Banach Fixed Point Theorem

A function taking a set A of some linear space into a set B of some linear space is called a **transformation** or **mapping** of A into B. A will be called the **domain** of the transformation and the set of values of the transformation will be called the **range** of the transformation. The notation $T : A \to B$ corresponds to the case where T is a transformation of A into B. We are interested in transformations that act from A to A, i.e., $A = B$, where $A \subset X$ and X is a Banach space. In this case, a **fixed point** of a transformation $T : A \to A$ is a point x in A such that

$$Tx = x.$$

Among the many results on existence of fixed points, for a given transformation, we highlight that three important results are Banach's fixed point theorem, Brouwer's fixed point theorem and Schauder's fixed point theorem. These results play an important role concerning the existence of solutions for differential equations. In this work a fundamental tool is Banach's fixed point theorem also known as the contraction mapping theorem. Before stating the contraction mapping theorem we need to introduce the following

Definition 1 (Lipschitz continuity, contraction)**.** *If A is a subset of a Banach space X and T is a transformation mapping A into a Banach space Y (written as $T : A \to Y$), then T is a* **contraction** *on A if there is a λ, $0 \leqq \lambda < 1$, such that*

$$\|Tx - Ty\|_Y \leq \lambda \|x - y\|_X, \quad \textit{with } x,\ y \in A.$$

The constant λ is called the **contraction constant** *for T on A.*

The Banach fixed point theorem to be stated below gives a constructive procedure for obtaining better and better approximations to the solution of the practical problem. This procedure is called an **iteration**. Moreover, iteration procedures are used in nearly every branch of applied mathematics, and convergence proofs and error estimates are very often obtained by an application of Banach's fixed point theorem.

Theorem 1 (Banach Fixed Point Theorem)**.** *If A is a closed subset of a Banach space X and $T : A \to A$ is a contraction on A, then T has one and only one fixed point, i.e.,*

$$\exists!\ \bar{x} \in X : T(\bar{x}) = \bar{x}.$$

Moreover, if x_0 in X is arbitrary, then the sequence $\{x_{n+1} = Tx_n,\ n = 0, 1, 2, \ldots\}$ converges to $\bar{x}$ as $n \to \infty$ and

$$\|x_n - \bar{x}\| \leq \frac{\lambda^n \|x_1 - x_0\|}{1 - \lambda},$$

where λ is the contraction constant for T on A.

Proof. Let T be a contraction on A. Then, there is some $\lambda \in [0, 1)$ such that, for any $x, y \in X$

$$\|T(x) - T(y)\| \leq \lambda \|x - y\|. \tag{2.1}$$

Let x, y are elements of A such that $Tx = x$ and $Ty = y$, then

$$\|x - y\| = \|Tx - Ty\| \leqslant \lambda \|x - y\|.$$

Thus,

$$\begin{aligned} (1 - \lambda)\|x - y\| &\leqslant 0 \\ \|x - y\| &= 0 \\ x &= y. \end{aligned}$$

(Uniqueness) Let $x_0 \in A$ be arbitrary and define the "iteration sequence" $(x_n)_{n\in\mathbb{N}}$ by

$$x_{n+1} = T(x_n).$$

Given $n > 1$, let us see how close x_{n+1} and x_n are

$$\begin{aligned}
\|x_{n+1} - x_n\| =& \|Tx_n - Tx_{n-1}\| \\
\leqslant& \lambda\|x_n - x_{n-1}\| \\
=& \|Tx_{n-1} - Tx_{n-2}\| \\
\leqslant& \lambda^2\|x_{n-1} - x_{n-2}\| \\
&\vdots \\
\leqslant& \lambda^n\|x_1 - x_0\|.
\end{aligned}$$

Thus,

$$\|x_{n+1} - x_n\| \leq \lambda^n\|x_1 - x_0\|. \tag{2.2}$$

Now, let us prove that $(x_n)_{n\in\mathbb{N}}$ is a Cauchy sequence so that it converges in the complete space X. This limit $\bar{x}$ will be the fixed point we are looking for. In fact, for $m > n > 1$ we have, using the triangle inequality and (2.2), that

$$\begin{aligned}
\|x_m - x_n\| &= \|(x_m - x_{m-1}) + (x_{m-1} - x_{m-2}) + \cdots + (x_{n+1} - x_n)\| \\
&\leq \|x_m - x_{m-1}\| + \|x_{m-1} - x_{m-2}\| + \cdots + \|x_{n+1} - x_n\| \\
&\leq \lambda^{m-1}\|x_1 - x_0\| + \lambda^{m-2}\|x_1 - x_0\| + \cdots + \lambda^n\|x_1 - x_0\| \\
&= \lambda^n\left(\lambda^{m-1-n} + \lambda^{m-2-n} + \cdots + 1\right)\|x_1 - x_0\| \\
&= \lambda^n\left(\frac{1 - \lambda^{m-n}}{1 - \lambda}\right)\|x_1 - x_0\| \\
&\leq \frac{\lambda^n}{1 - \lambda}\|x_1 - x_0\|.
\end{aligned}$$

Then, giving $\varepsilon > 0$ pick $N = N(\varepsilon) \geqslant 1$ such that $\frac{\lambda^N}{1-\lambda}\|x_1 - x_0\| < \varepsilon$. For $m > n \geqslant N$ we have,

$$\|x_m - x_n\| \leq \frac{\lambda^n}{1 - \lambda}\|x_1 - x_0\| \leqslant \frac{\lambda^N}{1 - \lambda}\|x_1 - x_0\|| < \varepsilon.$$

Since $\{x_n\}$ converges, $\{x_n\} \subset A$ and A is closed, there exists $\bar{x} \in A$ such that

$$\lim_{n\to\infty} x_n = \bar{x}.$$

Finally to show that $T(\bar{x}) = \bar{x}$ (existence) we use that T is continuous. In fact, if $x_n \to \bar{x}$, then $T(x_n) \to T(\bar{x})$. But, $x_{n+1} = T(x_n)$ and then uniqueness of the limit implies that

$$T(\bar{x}) = \bar{x}.$$

□

Remark 1. *The above theorem can be formulated in a more general framework which corresponds to metric spaces. However, due to our interests in this work, we restrict attention to a framework corresponding to Banach spaces.*

2.2 Aspects of Ordinary Differential Equations

In this section we recall some general definitions and results of ordinary differential equations. We begin with the definition of solutions and theorems concerning with existence and uniqueness of solution for first order ordinary differential equations in a general form. Next, we pay attention to linear equations.

2.2.1 Existence and Uniqueness

Let t be a scalar, let D be an open set in $\mathbb{R}^{n+1}$ with an element of D written as (t,x); let $f : D \longrightarrow \mathbb{R}^n$, be continuous and $\dot{x} = \frac{dx}{dt}$. Consider a system of d first order differential equations

$$\dot{x} = f(t,x). \tag{2.3}$$

For the most part, it will be assumed that f is continuous. We say x is a **solution** of (2.3) on an interval $I \subset \mathbb{R}$ if x is a continuously differentiable function defined on I, $(t,\ x(t)) \in\ D$, $t \in I$ and x satisfies (2.3) on I. Suppose $(t_0, x_0) \in D$ is given. An *initial value problem for the equation* (2.3) consists of finding an interval I containing t_0 and a **solution** x of (2.3) satisfying $x(t_0) = x_0$. We write this problem as

$$\begin{cases} \dot{x}(t) = f(t, x(t)), & \\ x(t_0) = x_0, & t \in I. \end{cases} \tag{2.4}$$

If there exists an interval I containing t_0 and an x satisfying (2.4), we refer to this as a solution of (2.3) passing through (t_0, x_0). These requirements on x are equivalent to the following lemma

Lemma 1. *Problem* (2.4) *is equivalent to*

$$x(t) = x_0 + \int_{t_0}^{t} f(\tau, x(\tau))d\tau, \tag{2.5}$$

provided $f(t,x)$ is continuous.

The next theorem to be introduced drops the assumption of Lipschitz continuity and the assertion of uniqueness.

Theorem 2 (Peano)**.** *If f is continuous in D, then for any $(t_0, x_0) \in D$, there is at least one solution of* (2.3) *passing through (t_0, x_0).*

A proof of this theorem is given in [11].

If $f(t,x)$ is continuous in a domain D, then the fundamental existence theorem implies the existence of at least one solution of (2.3). The basic **existence** and **uniqueness** theorem under the hypothesis that $f(t,x)$ is locally lipschitzian in x is usually referred to as the *Picard-Lindelöf* theorem. This result as well as additional information is contained in

Theorem 3 (Picard- Lindelöf Theorem)**.** *If $f(t,x)$ is continuous in D and locally lipschitzian with respect to x in D, then for any (t_0,x_0) in D, there exists a unique solution $x(t,t_0,x_0)$, $x(t_0,t_0,x_0)=x_0$, of* (2.3) *passing through (to, xo). Furthermore, the domain D in $\mathbb{R}^{n+2}$ of definition of the function $x(t,t_0,x_0)$ is open and $x(t,t_0,x_0)$ is continuous in D.*

The proof of this theorem can be found in [11].

2.2.2 Linear Systems

A linear system of n first order equations is a particular case of (2.3) that is expressed by

$$\dot{x}_j=\sum_{k=1}^{n}a_{jk}(t)x_k+h_j(t),\quad j=1,2,\ldots,n,$$

where the a_{jk} and h_j for $j,k=1,2,\ldots,n$ are continuous real valued functions on the interval $(-\infty,+\infty)$. In matrix notation this equation can be written in more compact form as

$$\dot{x}=A(t)x+h(t)\ , \tag{2.6}$$

where $A=\left(a_{jk}\right), j,k=1,2,\ldots,n;$ $h=(h_1,\ldots,\ h_n)^T$. When $h(t)=0$ for all $t\in(-\infty,+\infty)$, we obtain the system

$$\dot{x}=A(t)x\ . \tag{2.7}$$

We will refer to (2.7) as a homogeneous linear system. In another case, when h is not the null function, we will say that equation (2.6) is a non-homogeneous linear system.

To continue our discussion of the systems (2.6) and (2.7), we recall some aspects of linear algebra. A set of vectors $x^1,\ldots,x^n$ in $\mathbb{R}^n$ are said to be linearly independent if $\sum_{j=1}^{n}c_jx^j=0$ for any real constants c_j implies $c_j=0$ for $j=1,\ldots,n$. The vectors $x^1,\ldots,x^n$ are said to be linearly dependent if they are not linearly independent. A useful and very known criterion for deciding on linear dependence is the following:

The vectors $x^1,\ldots,x^n$ are linearly independent if and only if $\det\left[x^1,\ldots,x^n\right]\neq 0$.

Now, an $n\times n$ matrix $X(t), t>0$, is said to be an $n\times n$ **matrix solution** of (2.7) if each column of $X(t)$ satisfies (2.7). A **fundamental matrix solution** of (2.7) is an $n\times n$ matrix solution $X(t)$ of (2.7) such that $\det X(t)\neq 0$. A **principal matrix solution** of (2.7) at initial time t_0 is a fundamental matrix solution such that $X\left(t_0\right)=I$, where I denotes the identity matrix. The principal matrix solution at t_0 will be designated by $X\left(t,\ t_0\right)$.

From the above definition of a fundamental matrix solution it is clear that a fundamental matrix solution is simply a matrix solution of (2.7) such that the n columns of $X(t)$ are linearly independent.

Lema 1. *If $X(t)$ is an $n \times n$ matrix solution of (2.7), then either* $\det X(t) \neq 0$ *for all* t *or* $\det X(t) = 0$ *for all* t.

Proof. A proof of this lemma is found in [11] page 80. $\square$

Lema 2. *If $X(t)$ is any fundamental matrix solution of (2.7), then a general solution of (2.7) is $X(t)v$ where v is an arbitrary vector in $\mathbb{R}^n$.*

Proof. A proof of this lemma is found in [11] page 80. $\square$

Theorem 4. *If X is a fundamental matrix solution of (2.7) then every solution of (2.6) is given by formula*

$$x(t) = X(t)\left[X^{-1}(\tau)x(\tau) + \int_{\tau}^{t} X^{-1}(s)h(s)ds\right], \tag{2.8}$$

for any real number $\tau \in (-\infty, +\infty)$.

Proof. A proof of this theorem is found in [11] page 81. $\square$

2.2.3 Linear Systems with Constant Coefficients

In this subsection, we consider the homogeneous equation

$$\dot{x} = Ax, \tag{2.9}$$

and the non-homogeneous equation

$$\dot{x} = Ax + h(t), \tag{2.10}$$

where A is an $n \times n$ real constant matrix and $h : (-\infty, \infty) \to \mathbb{R}^n$ is continuous.

Given any fundamental matrix solution $X(t)$ of (2.9) it is well known that

$$e^{At} := X(t)X^{-1}(0)\ , \tag{2.11}$$

where $X^{-1}(0)$ denotes the inverse matrix of $X(0)$, is the only matrix solution of (2.9) that evaluates at $t = 0$ gives the identity matrix; see for instance discussions given in [11], [12] and [13]. Some important properties of the principal matrix solution e^{At}, which provide the reason for this choice of notation, are the following:

$$\begin{aligned} e^{\mathbf{A}(t+s)} &= e^{\mathbf{A}t}e^{\mathbf{A}s} \\ \left(e^{\mathbf{A}t}\right)^{-1} &= e^{-\mathbf{A}t} \\ \frac{d}{dt}e^{\mathbf{A}t} = \mathbf{A}e^{\mathbf{A}t} &= e^{\mathbf{A}t}\mathbf{A}, \end{aligned}$$

and

$$e^{\mathbf{A}t} = \sum_{n=0}^{+\infty} \frac{1}{n!}\mathbf{A}^n t^n = \mathbf{I} + \mathbf{A}t + \frac{1}{2!}\mathbf{A}^2 t^2 + \cdots . \tag{2.12}$$

Particularly, the formula (2.12) is useful in some cases where A decomposes as $A = PJP^{-1}$, where P is an invertible matrix and for J its powers are known.

We close this subsection by noting that, according to the previous subsection, a general solution of (2.9) is $e^{At}v$ where v is an arbitrary vector in $\mathbb{R}^n$ and, for the equation (2.10), (2.8) lead us to the formula

$$x(t) = e^{At}\left[x(0) + \int_0^t e^{-As}h(s)ds\right] . \tag{2.13}$$

2.3 Eigenvalues and Eigenvectors

A real or complex number λ is called an **eigenvalue** of an $n \times n$ matrix A, if there exists a non-zero vector v such that

$$Av = \lambda v, \quad \text{or} \quad (A - \lambda I)v = 0, \tag{2.14}$$

where I is the $n \times n$ identity matrix. If λ is an eingenvalue of the matrix A and v is any non-zero solution of equation (2.14), then v is called an **eigenvector** associated with the eigenvalue λ. Hence, λ is an eigenvalue of the matrix A if λ is a solution of the polynomial equation

$$\det(A - \lambda I) = 0. \tag{2.15}$$

We refer to equation (2.15) as the **characteristic equation** of the matrix A. This equation has n solutions, with possible repetitios

Now, let us consider an $n \times n$ matrix A and define a matrix B as

$$B := A + \nu I, \tag{2.16}$$

where ν is a real or complex number.

Proposition 1. *λ is an eigenvalue of A if and only if $\lambda + \nu$ is an eigenvalue of B.*

Proof. If λ is an eigenvalue of A, there exists a non-zero vector v such that $Av = \lambda v$. Now,

$$\begin{aligned} Bv = (A + \nu I)v &= Av + \nu Iv \\ &= (\lambda + \nu)v \ . \end{aligned}$$

Thus, $\lambda + \nu$ is an eigenvalue of B.

If $\lambda+\nu$ is an eigenvalue of B, then there exists a non-zero vector v such that $Bv = (\lambda+\nu)v$. The fact that B is given by (2.16) implies $Av = \lambda v$. Hence, λ is an eigenvalue of A. □

Proposition 2. *v is an eigenvector of A if and only if v is an eigenvector of B. Moreover,*

$$Av = \lambda v \Longleftrightarrow Bv = (\lambda + \nu)v \ .$$

Proof. If $Av = \lambda v$,$v \neq 0$, then

$$\begin{aligned} Bv = (A + \nu I)v &= \lambda v + \nu v \\ &= (\lambda + \nu)v \ . \end{aligned}$$

Thus, v is an eigenvector of B.
If $Bv = (\lambda + \nu)v$, $v \neq 0$, then $(A + \nu I)v = (\lambda + \nu)v$. Therefore $Av = \lambda v$. □

The following propositions will be of great importance for the discussion of chapter 4 of this work.

Let A be a real matrix in $\mathcal{M}_{3\times 3}$ and suppose that A has two eigenvalues that are complex conjugates and one real eigenvalue, i.e., $\lambda_1 = \alpha + i\beta$, $\lambda_2 = \alpha - i\beta$ and λ_3, with $\alpha, \beta \in \mathbb{R}$ and $\beta > 0$. Let B be a real matrix in $\mathcal{M}_{3\times 3}$ defined as in (2.16), with $\nu \in \mathbb{R}$.

Proposition 3. *There exists a real invertible matrix P in $\mathcal{M}_{3\times 3}$ such B can be decomposed as*

$$B = PJP^{-1} \ ,$$

where J is given by

$$J = \begin{pmatrix} \alpha + \nu & \beta & 0 \\ -\beta & \alpha + \nu & 0 \\ 0 & 0 & \lambda_3 + \nu \end{pmatrix} . \tag{2.17}$$

Moreover, P does not depend on ν.

Proof. Let v be an eigenvector of A associated to the eigenvalue λ_1. Since λ_1 is complex, v could be expressed as $v = v_1 + iv_2$, where v_1 and v_2 are two non−zero real vectors. Now, from Proposition 1, we have that

$$B\left(v_1 + iv_2\right) = \left(\alpha + \nu + i\beta\right)\left(v_1 + iv_2\right) \ . \tag{2.18}$$

We claim that vectors v_1 and v_2 are linearly independent. Indeed, suppose that they were linearly dependent. Then there would be two non−zero real constants c_1 and c_2 such that $c_1v_1 + c_2v_2 = 0$, or equivalently $v_1 = -\left(c_2/c_1\right)v_2$. Using this in equation (2.18) we obtain

$$B\Big(-\left(c_2/c_1\right)v_2 + iv_2\Big) = \left(\alpha + \nu + i\beta\right)\Big(-\left(c_2/c_1\right)v_2 + iv_2\Big) .$$

Now, straightforward computations allow us to conclude

$$Bv_2 = \Big(\alpha + \nu - \left(c_2/c_1\right)\beta\Big)v_2 \ \text{ and } \ Bv_2 = \Big(\alpha + \nu + \left(c_1/c_2\right)\beta\Big)v_2 \ .$$

Therefore, $\alpha + \nu - \left(c_2/c_1\right)\beta = \alpha + \nu + \left(c_1/c_2\right)\beta$ and from this we obtain the contradiction $c_1^2 + c_2^2 = 0$. So, the vectors v_1 and v_2 must be linearly independent.
Let us now consider an eigenvector v_3 of A corresponding to the eigenvalue λ_3. Defining

P as the matrix whose columns consist of the real vectors v_1, v_2 and v_3; it is denoted by $P = (v_1|v_2|v_3)$, we have that P is invertible and satisfies $BP = PJ$, where J is given by (2.17).

In fact, observe that by equating its real and imaginary parts, complex equation (2.18) is equivalent to the following pair of real equations

$$Bv_1 = (\alpha + \nu)\, v_1 - \beta v_2 \ , \ Bv_2 = \beta v_1 + (\alpha + \nu)\, v_2.$$

Now, routine matrix multiplication yield

$$BP = (Bv_1|Bv_2|Bv_3) = ((\alpha + \nu)\, v_1 - \beta v_2|\beta v_1 + (\alpha + \nu)\, v_2|\, (\lambda_3 + \nu)\, v_3) = PJ.$$

If we multiply both sides of this matrix equation, we obtain the desired result. Finally, since the matrix A depends on α, β and λ_3, the same is true for P. □

Proposition 4. *For the matrix B as in the previous proposition we have that*

$$e^{Bt} = P \begin{pmatrix} e^{(\alpha+\nu)t}\cos\beta t & e^{(\alpha+\nu)t}\sin\beta t & 0 \\ -e^{(\alpha+\nu)t}\sin\beta t & e^{(\alpha+\nu)t}\cos\beta t & 0 \\ 0 & 0 & e^{(\lambda_3+\nu)t} \end{pmatrix} P^{-1}. \tag{2.19}$$

Proof. From (2.12) and the fact that $B^n = PJ^nP^{-1}$, it is obtained that

$$e^{Bt} = Pe^{Jt}P^{-1} \ .$$

Our task is to obtain e^{Jt}. We write the matrix J given by (2.17) as the sum of two commuting matrices

$$J = \begin{pmatrix} \alpha+\nu & 0 & 0 \\ 0 & \alpha+\nu & 0 \\ 0 & 0 & \lambda_3+\nu \end{pmatrix} + \begin{pmatrix} 0 & \beta & 0 \\ -\beta & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} .$$

In this scenario, commuting matrices, $e^{Jt} = e^{Dt}e^{Ct}$, where

$$D = \begin{pmatrix} \alpha+\nu & 0 & 0 \\ 0 & \alpha+\nu & 0 \\ 0 & 0 & \lambda_3+\nu \end{pmatrix} \text{ and } C = \begin{pmatrix} 0 & \beta & 0 \\ -\beta & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} .$$

Now, by using (2.12) and Maclaurin's knowledge of series it is obtained

$$e^{Dt} = \begin{pmatrix} e^{(\alpha+\nu)t} & 0 & 0 \\ 0 & e^{(\alpha+\nu)t} & 0 \\ 0 & 0 & e^{(\lambda_3+\nu)t} \end{pmatrix} \text{ and } e^{Ct} = \begin{pmatrix} \cos\beta t & \sin\beta t & 0 \\ -\sin\beta t & 1\cos\beta t & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Finally, $e^{Jt} = e^{Dt}e^{Ct}$ and this allows to obtain the desired result.

□

Chapter 3

Setting of the problem and Theoretical results

This chapter is divided into two sections. The first part shows the problem statement where we establish conditions on the master−slave system to obtain synchronization in chaotic systems. In the last part, we present the main theoretical results obtained.

3.1 Setting of the Problem

Let us consider the master−slave system as follows

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, \bar{\mu}), \tag{3.1}$$

$$\dot{\mathbf{y}} = \mathbf{f}(\mathbf{y}, \mu) + \nu(\mathbf{y} - \mathbf{x}), \tag{3.2}$$

where ν is a real constant, $\bar{\mu}, \mu$ are vector parameters in $\mathbb{R}^m$ and $\mathbf{f} : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ is a continuous function.

To pose the problem from which we will be able to guarantee conditions that imply the synchronization of the system (3.1) − (3.2), we will begin by considering a bounded solution $\mathbf{x}(t, \mathbf{x}_0, \bar{\mu})$ of (3.1). Here $\mathbf{x}(t, \mathbf{x}_0, \bar{\mu})$ stands by a solution such that at $t = 0$ gives $\mathbf{x}(0, \mathbf{x}_0, \bar{\mu}) = \mathbf{x}_0$. Now, consider the following transformation

$$\mathbf{z} = \mathbf{y} - \mathbf{x}(t, \mathbf{x}_0, \bar{\mu}). \tag{3.3}$$

If we consider $\mathbf{y}$ as a slave solution, i.e., solution of (3.2) with input $\mathbf{x}(t, \mathbf{x}_0, \bar{\mu})$, then the previous transformation yields the non-autonomous equation

$$\dot{\mathbf{z}} = \nu\mathbf{z} + \mathbf{f}(\mu, \mathbf{z} + \mathbf{x}(t, \mathbf{x}_0, \bar{\mu})) - \mathbf{f}(\bar{\mu}, \mathbf{x}(t, \mathbf{x}_0, \bar{\mu})) := F\left(t, \mathbf{z}, \nu, \mu, \bar{\mu}\right). \tag{3.4}$$

We will now focus on the equation

$$\dot{\mathbf{z}} = F\left(t, \mathbf{z}, \nu, \mu, \bar{\mu}\right), \tag{3.5}$$

where $F : [0, \infty) \times \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^m \times \mathbb{R}^m \to \mathbb{R}^n$ is a continuous function. It is assumed that F can be decomposed as the sum of three functions

$$F\left(t, \mathbf{z}, \nu, \mu, \bar{\mu}\right) = B\mathbf{z} + G\left(t, \mu, \bar{\mu}\right) + H\left(t, \mathbf{z}, \mu\right), \tag{3.6}$$

and for this decomposition we impose the following hypothesis:

H1) B is a constant real $n \times n$ matrix for which all the eigenvalues have negative real part.

H2) $G : [0, \infty) \times \mathbb{R}^m \times \mathbb{R}^m \to \mathbb{R}^n$ is continuous and satisfies: if given $\varepsilon > 0$, then $\delta > 0$ exists such that
$$\|G(t, \mu, \bar{\mu})\| < \varepsilon,$$
for any $t \geq 0$ and $\mu, \bar{\mu}$ with $\|\mu - \bar{\mu}\| < \delta$.

H3) $H : [0, \infty) \times \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ is a continuous function such that $H(t, 0, \mu) = 0$ for any $\mu \in \mathbb{R}^m$ and $t \geq 0$. Also it satisfies the following type of Lipschitz condition: for any $\mu \in \mathbb{R}^m$ there is a positive constant $L = L(\mu)$ such that
$$\|H(t, \mathbf{z}_1, \mu) - H(t, \mathbf{z}_2, \mu)\| \leq L\|\mathbf{z}_1 - \mathbf{z}_2\| \ , t \geq 0.$$

About H1), H2) and H3:

Corresponding to hypothesis H1), let us propose an interesting result that will be useful in the next section.

Proposition 5. *If B is a constant real $n \times n$ matrix for which all the eigenvalues have negative real part, then there are positive constants K, γ such that*
$$\left\|e^{Bt}\mathbf{z}\right\| \leq Ke^{-\gamma t}\|\mathbf{z}\|, \quad t \geq 0, \quad \mathbf{z} \in \mathbb{R}^n. \tag{3.7}$$

Proof. The proof of this result is strongly based on the Jordan canonical form of the matrix B and can be seen in [11] *(Theorem 4.2., (ii))*. □

Hypothesis H2) tells us that the norm of G, depending on how close the parameters μ and $\bar{\mu}$ are, can be made sufficiently small. H3) is a type of condition that is often considered when looking for existence and uniqueness of solutions for differential equations.

Now, in our problem we study the system (3.5) with the decomposition given in (3.6) and under the hypotheses H1), H2) and H3). We pursuit to find solutions of (3.5) that, when associated with the transformation (3.3), guarantee synchronization of the master−slave system (3.1) − (3.2).

3.2 Theoretical Results

We start with a lemma relating the equation (3.5) with an integral equation

Lemma 2. *The initial value problem*
$$\begin{cases} \dot{\mathbf{z}} = F(t, \mathbf{z}, \mu, \bar{\mu}) \\ \mathbf{z}(0) = \mathbf{z}_0, \end{cases} \tag{3.8}$$
where F is decomposed as in (3.6), is equivalent to
$$\mathbf{z}(t) = e^{Bt}\mathbf{z}_0 + \int_0^t e^{B(t-s)} \left(G(s, \mu, \bar{\mu}) + H(s, \mathbf{z}(s), \mu)\right) ds. \tag{3.9}$$

Proof. It is a direct consequence of the main Theorem of Calculus. □

Suppose K, γ are the constants appearing in (3.7). Let $\rho > 0$ and $\mu, \bar{\mu} \in \mathbb{R}^m$ such that

$$\|G(t, \mu, \bar{\mu})\| < \frac{\rho\gamma}{4K}, \quad t \geq 0. \tag{3.10}$$

With this choice of ρ, μ and $\bar{\mu}$, and for $\mathbf{z}_0 \in \mathbb{R}^n$ satisfying $\|\mathbf{z}_0\| < \dfrac{\rho}{2K}$ we define

$$\mathcal{G}(\mathbf{z}_0, \rho, \mu, \bar{\mu}) := \left\{ \mathbf{z} \in \mathcal{C}_b\left([0,\infty), \mathbb{R}^{\mathbf{n}}\right) : \|\mathbf{z}\|_\infty := \sup_{\mathbf{t} \geq 0} \|\mathbf{z}(t)\| \leq \rho \text{ and } \mathbf{z}(0) = \mathbf{z}_0 \right\}.$$

Proposition 6. *$\mathcal{G}(\mathbf{z}_0, \rho, \mu, \bar{\mu})$ is a closed subset of the Banach space $X = \mathcal{C}_b([0,\infty), \mathbb{R}^{\mathbf{n}})$ with the supremum norm.*

Proof. We are going to show that $X - \mathcal{G}(\mathbf{z}_0, \rho, \mu, \bar{\mu})$ is open i.e. given $\bar{\mathbf{z}} \in X - \mathcal{G}(\mathbf{z}_0, \rho, \mu, \bar{\mu})$ there exists an $\varepsilon-$neighborhood of $\bar{\mathbf{z}}$ which is contained in $X - \mathcal{G}(\mathbf{z}_0, \rho, \mu, \bar{\mu})$. If $\bar{\mathbf{z}} \in X - \mathcal{G}(\mathbf{z}_0, \rho, \mu, \bar{\mu})$ then $\bar{\mathbf{z}}(0) \neq \mathbf{z}_0$ or for some $\bar{t} \geq 0$ we have that $\left\|\bar{\mathbf{z}}(\bar{t})\right\| > \rho$.

If $\bar{\mathbf{z}}(0) \neq \mathbf{z_0}$, then $\|\bar{\mathbf{z}}(0) - \mathbf{z_0}\| := \bar{r} > 0$. Choose $\varepsilon = \frac{\bar{r}}{2}$ and consider the set

$$\left\{ \mathbf{z} \in C_b([0,\infty), \mathbb{R}^n) : \|\mathbf{z} - \bar{\mathbf{z}}\|_\infty < \varepsilon \right\}.$$

We have that this ε -neighborhood is contained in $X - \mathcal{G}(\mathbf{z}_0, \rho, \mu, \bar{\mu})$. In fact: If $\mathbf{z} \in \left\{ \mathbf{z} \in C_b([0,\infty), \mathbb{R}^n) : \|\mathbf{z} - \bar{\mathbf{z}}\|_\infty < \varepsilon \right\}$, then

$$\begin{aligned}
\|\mathbf{z}(0) - \mathbf{z}_0\| &= \|\mathbf{z}(0) - \bar{\mathbf{z}}(0) + \bar{\mathbf{z}}(0) - \mathbf{z}_0\| \\
&\geqslant \|\bar{\mathbf{z}}(0) - \mathbf{z}_0\| - \|\mathbf{z}(0) - \bar{\mathbf{z}}(0)\| \\
&= \bar{r} - \frac{\bar{r}}{2} > 0.
\end{aligned}$$

Thus, $\mathbf{z}(0) \neq \mathbf{z_0}$.

If $\|\bar{\mathbf{z}}(\bar{t})\| > \rho$ for some $\bar{t} \geqslant 0$, then the ε -neighburhood of $\bar{\mathbf{z}}$, with $\varepsilon = \|\bar{\mathbf{z}}(\bar{t})\| - \rho$, is contained in $X - \mathcal{G}(\mathbf{z}_0, \rho, \mu, \bar{\mu})$. In fact: If $\mathbf{z} \in \left\{ \mathbf{z} \in C_b([0,\infty), \mathbb{R}^n) : \|\mathbf{z} - \bar{\mathbf{z}}\|_\infty < \varepsilon \right\}$, then

$$\begin{aligned}
\left\|\mathbf{z}(\bar{t})\right\| &= \left\|\mathbf{z}(\bar{t}) - \bar{\mathbf{z}}(\bar{t}) + \bar{\mathbf{z}}(\bar{t})\right\| \\
&\geqslant \left\|\bar{\mathbf{z}}(\bar{t})\right\| - \left\|\mathbf{z}(\bar{t}) - \bar{\mathbf{z}}(\bar{t})\right\| \\
&= \varepsilon + \rho - \left\|\mathbf{z}(\bar{t}) - \bar{\mathbf{z}}(\bar{t})\right\| \\
&> \rho.
\end{aligned}$$

Thus, $\left\|\mathbf{z}(\bar{t})\right\| > \rho$. □

Inspired in the last lemma, for any $\mathbf{z} \in \mathcal{G}(\mathbf{z}_\prime, \rho, \mu, \bar{\mu})$ we define an operator $T\mathbf{z}$ by

$$(T\mathbf{z})(t) = e^{Bt}\mathbf{z}_0 + \int_0^t e^{B(t-s)} \left(G(s, \mu, \bar{\mu}) + H(s, \mathbf{z}(s), \mu)\right) ds, \ t \geq 0.$$

We have, due to the fact that $e^{B(\cdot)}\mathbf{z}_0$ is continuous and the hypothesis H1), H2), that $T\mathbf{z}$ is a continuous function for $t \geq 0$.

Now, we state our main theoretical result.

Teorema 1. *If*

$$\frac{KL}{\gamma} \leq \frac{1}{4}, \tag{3.11}$$

where $L = L(\bar{\mu})$ is the constant given in H3), then T acts from $\mathcal{G}(\mathbf{z}_{\prime}, \rho, \mu, \bar{\mu})$ into itself and also has a unique fixed point in $\mathcal{G}(\mathbf{z_0}, \rho, \mu, \bar{\mu})$.

Proof. Let $\mathbf{z} \in \mathcal{G}(\mathbf{z_0}, \rho, \mu, \bar{\mu})$. From H1), (3.10), (3.11), we obtain

$$\begin{aligned}
\|(T\mathbf{z})(t)\| \leq& \left\|e^{Bt}\mathbf{z}_0\right\| + \int_0^t \left\|e^{B(t-s)}\left(G(s,\mu,\bar{\mu}) + H(s,\mathbf{z}(s),\mu)\right)\right\| ds \\
\leq& Ke^{-\gamma t}\|\mathbf{z_0}\| + K\int_0^t e^{-\gamma(t-s)}\left\|(G(s,\mu,\bar{\mu}) + H(s,\mathbf{z}(s),\mu))\right\| ds \\
\leq& Ke^{-\gamma t}\|\mathbf{z_0}\| + K\int_0^t e^{-\gamma(t-s)}\left\|(G(s,\mu,\bar{\mu}))\right\| ds \\
&+ K\int_0^t e^{-\gamma(t-s)}\left\|(H(s,\mathbf{z}(s),\mu))\right\| ds \\
\leq& \frac{\rho}{2} + \frac{\rho\gamma}{4}\int_0^t e^{-\gamma(t-s)} ds + KL\int_0^t e^{-\gamma(t-s)}\|\mathbf{z}(s)\| ds \\
\leq& \frac{\rho}{2} + \frac{\rho\gamma}{4}\int_0^t e^{-\gamma(t-s)} ds + KL\|\mathbf{z}\|_\infty\int_0^t e^{-\gamma(t-s)} ds \\
=& \frac{\rho}{2} + \left(\frac{\rho\gamma}{4} + KL\|\mathbf{z}\|_\infty\right)\frac{1}{\gamma}(1 - e^{-\gamma t}) \\
\leq& \frac{\rho}{2} + \left(\frac{\rho\gamma}{4} + KL\|\mathbf{z}\|_\infty\right)\frac{1}{\gamma} \\
=& \frac{\rho}{2} + \frac{\rho}{4} + \frac{KL}{\gamma}\|\mathbf{z}\|_\infty \\
\leq& \rho.
\end{aligned}$$

Thus, $\|T\mathbf{z}\| \leq \rho$ and $T\mathbf{z} \in \mathcal{G}(\mathbf{z}_{\prime}, \rho, \mu, \bar{\mu})$.

Now, let us take $\mathbf{z_1}, \mathbf{z_2} \in \mathcal{G}(\mathbf{z}_{\prime}, \rho, \mu, \bar{\mu})$, the same type of estimates yields

$$\begin{aligned}
\|(T\mathbf{z_2})(t) - (T\mathbf{z_1})(t)\| &\leq \int_0^t \left\|e^{B(t-s)}\left(H(s,\mathbf{z_2}(s),\mu) - H(s,\mathbf{z_1}(s),\mu)\right)\right\| ds \\
&\leq \int_0^t Ke^{-\gamma(t-s)}\left\|(H(s,\mathbf{z_2}(s),\mu) - H(s,\mathbf{z_1}(s),\mu))\right\| ds \\
&\leq \int_0^t KLe^{-\gamma(t-s)}\|\mathbf{z_2}(s) - \mathbf{z_1}(s)\| ds \\
&\leq \left(\int_0^t KLe^{-\gamma(t-s)} ds\right)\|\mathbf{z_2} - \mathbf{z_1}\|_\infty \\
&= \frac{KL}{\gamma}\left(1 - e^{-\gamma t}\right)\|\mathbf{z_2} - \mathbf{z_1}\|_\infty \\
&\leq \frac{KL}{\gamma}\|\mathbf{z_2} - \mathbf{z_1}\|_\infty \\
&\leq \frac{1}{4}\|\mathbf{z_2} - \mathbf{z_1}\|_\infty.
\end{aligned}$$

Thus, T is a contraction on $\mathcal{G}(\mathbf{z}_{\prime}, \rho, \mu, \bar{\mu})$ and there is a unique fixed in $\mathcal{G}(\mathbf{z}_{\prime}, \rho, \mu, \bar{\mu})$. $\square$

We close this chapter with an interesting application of the famous Gronwall's inequality which, of course, here is related to the *Theorem 1.* First, let us to state and give a proof of the Gronwall's inequality

Lema 3. *Let M be a non-negative constant and let f and g be continuous non-negative functions, for $a \leq t \leq b$, satisfying*

$$f(t) \leq M + \int_a^t f(s)g(s)ds, \;\; a \leqslant t \leqslant b,$$

then

$$f(t) \leq Me^{\int_a^b g(s)ds} \quad a \leqslant t \leqslant b \; .$$

Proof. Define $h(t) = M + \int_a^t f(s)g(s)ds$. We have that $h(a) = M$ and $\dot{h}(t) = f(t)g(t)$. Now, since $f(t) \leqslant h(t)$, $f(t) \geqslant 0$ and $g(t) \geqslant 0$ for $a \leqslant t \leqslant b$ it is obtained

$$\dot{h}(t) = f(t)g(t) \leqslant h(t)g(t) \; .$$

By multiplying both members of this inequality by $e^{-\int_a^t g(s)ds}$, we obtain

$$e^{-\int_a^t g(s)ds}\dot{h}(t) \leqslant e^{-\int_a^t g(s)ds}h(t)g(t) \; .$$

Thus,

$$e^{-\int_a^t g(s)ds}\left(\dot{h}(t) - h(t)g(t)\right) \leqslant 0 \;\; \text{and} \;\; \frac{d}{dt}\left(e^{-\int_a^t g(s)ds}h(t)\right) \leqslant 0 \; .$$

Now, integrating from a to t we get

$$e^{-\int_a^t g(s)ds}h(t) - e^{-\int_a^t g(s)ds}h(a) \leqslant 0 \; ,$$

which implies $e^{-\int_a^t g(s)ds}h(t) \leqslant h(a) = M$ and $h(t) \leqslant Me^{\int_a^t g(s)ds}$. Finally, the fact that $f(t) \leqslant h(t)$ produces the result

$$f(t) \leqslant Me^{\int_a^t g(s)ds}.$$

□

Consider the sets $\mathcal{G}(\mathbf{z_0}, \rho, \mu, \bar{\mu})$ and $\mathcal{G}(\tilde{\mathbf{z}}_\mathbf{0}, \rho, \mu, \bar{\mu})$. Let us denote by $\mathbf{z}^\star := \mathbf{z}^\star(\cdot, \mathbf{z_0})$ and $\tilde{\mathbf{z}}^\star := \tilde{\mathbf{z}}^\star(\cdot, \tilde{\mathbf{z}}_\mathbf{0})$ the fixed points of the operator T on $\mathcal{G}(\mathbf{z_0}, \rho, \mu, \bar{\mu})$ and $\mathcal{G}(\tilde{\mathbf{z}}_\mathbf{0}, \rho, \mu, \bar{\mu})$, respectively, i.e. $T\mathbf{z}^\star = \mathbf{z}^\star$ and $T\tilde{\mathbf{z}}^\star = \tilde{\mathbf{z}}^\star$. We have

$$\mathbf{z}^\star(t, \mathbf{z_0}) - \tilde{\mathbf{z}}^\star(t, \tilde{\mathbf{z}}_\mathbf{0}) = e^{Bt}(\mathbf{z_0} - \tilde{\mathbf{z}}_\mathbf{0}) + \int_0^t e^{B(t-s)}\left(H(s, \mathbf{z}^\star(s, \mathbf{z_0}), \mu) - H(s, \tilde{\mathbf{z}}^\star(s, \tilde{\mathbf{z}}_\mathbf{0}), \mu)\right) ds.$$

Now,

$$\|\mathbf{z}^\star(t, \mathbf{z_0}) - \tilde{\mathbf{z}}^\star(t, \tilde{\mathbf{z}}_\mathbf{0})\| \leq Ke^{-\gamma t}\|\mathbf{z_0} - \tilde{\mathbf{z}}_\mathbf{0}\| + \int_0^t KLe^{-\gamma(t-s)}\|\mathbf{z}^\star(s, \mathbf{z_0}) - \tilde{\mathbf{z}}^\star(s, \tilde{\mathbf{z}}_\mathbf{0})\|ds.$$

By multiplying both members of the previous inequality by $e^{\gamma t}$, we obtain

$$e^{\gamma t}\|\mathbf{z}^{\star}(t,\mathbf{z_0})-\tilde{\mathbf{z}}^{\star}(t,\tilde{\mathbf{z}}_\mathbf{0})\| \leq K\|\mathbf{z_0}-\tilde{\mathbf{z}}_\mathbf{0}\| + \int_0^t KLe^{\gamma s}\|\mathbf{z}^{\star}(s,\mathbf{z_0})-\tilde{\mathbf{z}}^{\star}(s,\tilde{\mathbf{z}}_\mathbf{0})\|ds.$$

For this inequality we have, with $M = K\|\mathbf{z_0}-\tilde{\mathbf{z}}_\mathbf{0}\|$, $f(t) = e^{\gamma t}\|\mathbf{z}^{\star}(t,\mathbf{z_0})-\tilde{\mathbf{z}}^{\star}(t,\tilde{\mathbf{z}}_\mathbf{0})\|$ and $g(t) = KL$, the hypotheses of Gronwall's lemma. Therefore, we can conclude that

$$e^{\gamma t}\|\mathbf{z}^{\star}(t,\mathbf{z_0})-\tilde{\mathbf{z}}^{\star}(t,\tilde{\mathbf{z}}_\mathbf{0})\| \leqslant K\|\mathbf{z_0}-\tilde{\mathbf{z}}_\mathbf{0}\|e^{\int_0^t KLds}, \quad t \geqslant 0.$$

Thus,

$$\|\mathbf{z}^{\star}(t,\mathbf{z_0})-\tilde{\mathbf{z}}^{\star}(t,\tilde{\mathbf{z}}_\mathbf{0})\| \leqslant Ke^{(KL-\gamma)t}\|\mathbf{z_0}-\tilde{\mathbf{z}}_\mathbf{0}\|, \quad t \geqslant 0, \tag{3.12}$$

because in the *Theorem 1*, $\dfrac{KL}{\gamma} \leqslant \dfrac{1}{4}$. Relation (3.12) implies $\|\mathbf{z}^{\star}(t,\mathbf{z_0})-\tilde{\mathbf{z}}^{\star}(t,\tilde{\mathbf{z}}_\mathbf{0})\|$ approaches zero exponentially as $t \to \infty$ and $\|\mathbf{z}^{\star}(\cdot,\mathbf{z_0})-\tilde{\mathbf{z}}^{\star}(\cdot,\tilde{\mathbf{z}}_\mathbf{0})\|_\infty \leqslant K\|\mathbf{z_0}-\tilde{\mathbf{z}}_\mathbf{0}\|$.

Chapter 4

Application in Chua's equation

In this chapter, we present the application of the theoretical results obtained in the last chapter to Chua's equations. We begin with a brief description of Chua's circuit. Then, we prove that Chua's equations satisfy the conditions described above and, using a Computer Algebra System, we show graphically the result of the system to be synchronized.

4.1 A brief aspects about Chua's circuit

The absence of a reproducible functioning chaotic system would imply that chaos could be a phenomenon that will solely exist in mathematical abstraction and computer simulations [15]. In 1983, intending to build an autonomous electronic circuit that exhibits a chaotic electronic natural behavior, Professor Leon O. Chua proposes an electronic circuit which models non-linear dynamics and present convoluted bifurcations and chaos.

Chua's circuit is the bridge to understand the characteristic associated to dynamics of nonlinear phenomena as stable orbits, bifurcations, and attractors, and to study experimentally the chaos control. This circuit, in its classic configuration, is one of the simplest chaotic systems containing an inductor L, two capacitors C_1, C_2 which are the linear energy-storage elements, a linear resistor i_L and one 2-terminal nonlinear resistor N_R characterized by a current-voltage $v - i$ characteristic which has a negative slope [16].

All circuit elements are passive except for the nonlinear resistor N_R; this element must be active in order for the circuit to become chaotic, and hence the instability condition implies that each equilibrium point must lie on a segment of piecewise linear $v - i$ [15].

Reference [17] shows that Chua's circuit can be analyzed using Kirchoff's Laws. Thus, the following equations are obtained describing the circuit behavior

$$C_1 \frac{dv_{C_1}}{dt} = \frac{1}{R}(v_{C_2} - v_{C_1}) - f(v_{C_1}) \tag{4.1}$$

$$C_2 \frac{dv_{C_2}}{dt} = \frac{1}{R}(v_{C_1} - v_{C_2}) + i_L \tag{4.2}$$

$$L \frac{di_l}{dt} = -v_{C_2}, \tag{4.3}$$

where

$$f(v_R) = G_b v_R + \frac{1}{2}(G_a - G_b)\left[|v_R + B_p| - |v_R - B_p|\right],$$

is the response function of non-linear element $(v_R - i_R)$. It consists of five linear segments, as shown in Figure (4.1).

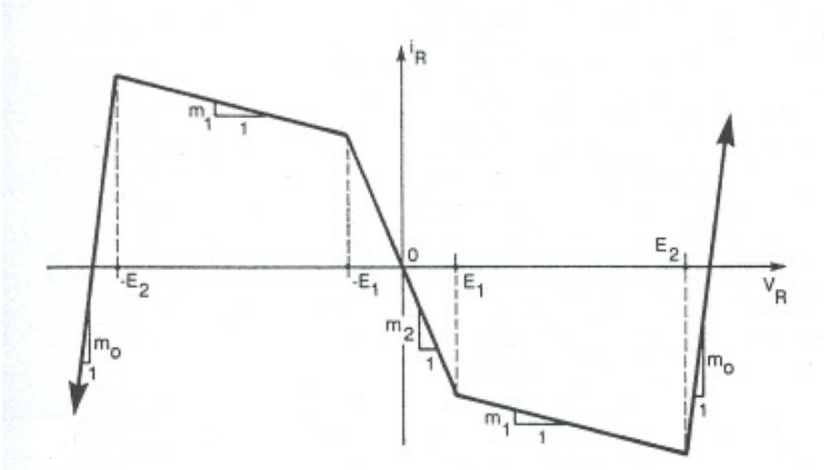

Figure 4.1: Illustration of the 5-segment $v_R - i_R$ characteristic for the nonlinear resistor corresponding to Chua's circuit (4.1)−(4.3) used in the first research of the chaotic behavior of this circuit [15]

In this work, we will use the adimensional form of equation system which is obtained by rescaling the parameters of the system

$$
\begin{gathered}
x = \frac{v_{C_1}}{B_p}, \quad y = \frac{v_{C_2}}{B_p}, \quad z = R\frac{i_L}{B_p}, \\
t = tRC_2, \quad \alpha = \frac{C_2}{C_1}, \quad \beta = R^2\frac{C_2}{L}, \\
a = -RG_a, \quad b = -RG_b.
\end{gathered}
$$

For (4.1) the following procedure is carry out. Taking, $x = \frac{v_{C_1}}{B_p}$ then $\dfrac{dx}{dt} = \dfrac{1}{B_p}\dfrac{dv_{C_1}}{dt}$ and taking $y = \frac{v_{C_2}}{B_p}$, so

$$
\begin{aligned}
C_1\frac{dv_{C_1}}{dt} &= \frac{1}{R}(v_{C_2} - v_{C_1}) - f(v_{C_1}) \\
C_1B_p\frac{dx}{dt} &= \frac{B_p}{R}(y - x) - f(xB_p) \\
C_1B_p\frac{dx}{dt} &= \frac{B_p}{R}(y - x) - \left(G_bxB_p + \frac{1}{2}\left(\frac{b}{R} - \frac{a}{R}\right)\left[|B_p(x+1)| - |B_p(x-1)|\right]\right) \\
\frac{dx}{dt} &= \frac{1}{RC_1}(y - x) - \left(x\frac{b}{RC_1} + \frac{1}{2RC_1}(b-a)\left(|x+1| - |x-1|\right)\right),
\end{aligned}
$$

since $t = tRC_2$ then $C_2 = \frac{1}{R}$, it follows that

$$\begin{aligned}\frac{dx}{dt} &= \frac{C_2}{C_1}(y-x) - \left[bx\frac{C_2}{C_1} + \frac{C_2}{2C_1}(b-a)\left(|x+1| - |x-1|\right)\right] \\ \frac{dx}{dt} &= \alpha(y-x) - \alpha\left[bx + \frac{1}{2}(b-a)\left(|x+1| - |x-1|\right)\right],\end{aligned}$$

by taking $f(x) = bx + \frac{1}{2}(b-a)\left(|x+1| - |x-1|\right)$, then (4.1) becomes

$$\dot{x} = \alpha(y-x) - \alpha f(x).$$

For (4.2) the following procedure is carry out. Consider $x = \dfrac{v_{C_1}}{B_p}$ then $\dfrac{dy}{dt} = \dfrac{1}{B_p}\dfrac{dv_{C_2}}{dt}$ and taking $x = \frac{v_{C_1}}{B_p}$ and $i_L = \frac{zB_p}{R}$, so

$$\begin{aligned}C_2\frac{dv_{C_2}}{dt} &= \frac{1}{R}(v_{C_1} - v_{C_2}) + i_L \\ C_2B_p\frac{dy}{dt} &= \frac{B_p}{R}(x-y) + \frac{zB_p}{R} \\ \frac{dy}{dt} &= \frac{1}{RC_2}(x-y) + \frac{z}{RC_2},\end{aligned}$$

as before, since $t = tRC_2$ then $\frac{1}{R} = C_2$. Thus

$$\dot{y} = x - y + z.$$

For the last equation, (4.3), Let us consider $v_{C_2} = yB_p$, and $z = R\frac{i_L}{B_p}$ then $\frac{dz}{dt} = \frac{R}{B_p}\frac{di_L}{dt}$, so

$$\begin{aligned}\frac{LB_p}{R}\frac{dz}{dt} &= -yB_p \\ \frac{dz}{dt} &= -\frac{R}{L}y,\end{aligned}$$

since $t = tRC_2$ then $\dfrac{R}{L} = \dfrac{R^2C_2}{L} = \beta$, so that

$$\dot{z} = -\beta y.$$

Thus, the resulting system is a set of interdependent equations in the form of a 3-Dimensional autonomous piece-wise linear ordinary differential equation (flow) described by

$$\begin{cases}\dot{x} &= \alpha(y - x - f(x)), \\ \dot{y} &= x - y + z, \\ \dot{z} &= -\beta y,\end{cases} \tag{4.4}$$

where

$$f(x) = bx + \frac{1}{2}(b-a)\left[|x+1| - |x-1|\right],$$

and a, b, α and β are real parameter, $a < b < 0$, $\alpha > 0$ and $\beta > 0$. The system (4.4) is known as Chua's equation.

This piece-wise function can be written as follows

$$f(x) = \begin{cases} bx - (a-b), & \text{if } x \leq -1, \\ ax, & \text{if } -1 \leq x \geq 1, \\ bx + (a-b), & \text{if } x \geq 1. \end{cases} \tag{4.5}$$

Unlike the Figure (4.1), the non-linear function $f(x)$ has three negative slopes as shown in the following figure

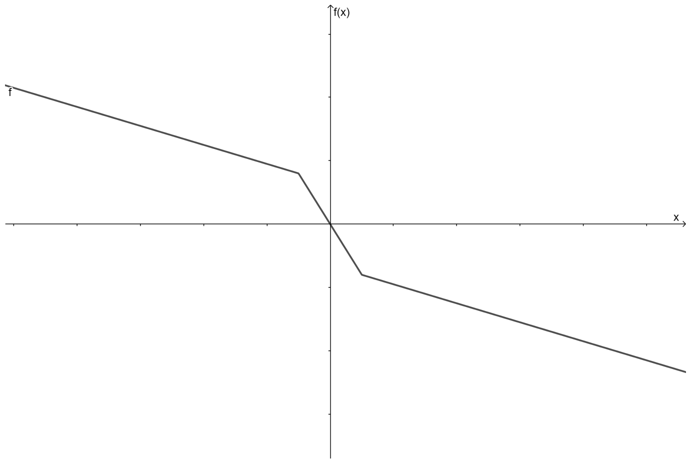

Figure 4.2: Illustration of the three negative slopes corresponding to the non- linear continuous function (4.5) of the Chua's Equation (4.4).

Each coordinate of (4.4) corresponds to physical quantities describing by the circuit

- x is the voltage drop on the first capacitor,
- y is the voltage drop on the second capacitor, and
- z is the current through the coil.

4.2 Application of Theoretical Results

In order to apply the theoretical results obtained to the system (3.1) – (3.2) in the Chua's sytems (4.4), we consider

$$\mathbf{z} = \begin{pmatrix} \bar{x} \\ \bar{y} \\ \bar{z} \end{pmatrix}, \quad \mathbf{x}(t, \mathbf{x}_0, \bar{\mu}) = \begin{pmatrix} x \\ y \\ z \end{pmatrix}, \quad \mu = \begin{pmatrix} \alpha \\ \beta \end{pmatrix}, \quad \bar{\mu} = \begin{pmatrix} \bar{\alpha} \\ \bar{\beta} \end{pmatrix}.$$

We have to find the representation of (3.4) for the case of (4.4). For that, first let us compute $\mathbf{f}(\mathbf{z}+\mathbf{x}(t,\mathbf{x}_0,\bar{\mu}),\mu) - \mathbf{f}\left(\mathbf{x}(t,\mathbf{x}_0,\bar{\mu}),\bar{\mu}\right)$. Then,

$$\mathbf{f}(\mathbf{z}+\mathbf{x}(t,\mathbf{x}_0,\bar{\mu}),\mu) = \begin{pmatrix} \alpha((\bar{y}+y)-(\bar{x}+x)-f(\bar{x}+x)) \\ (\bar{x}+x)-(\bar{y}+y)+(\bar{z}+z) \\ -\beta(\bar{y}+y) \end{pmatrix},$$

so that,

$$\begin{aligned}\mathbf{f}(\mathbf{z}+\mathbf{x}(t,\mathbf{x}_0,\bar{\mu}),\mu) - \mathbf{f}\left(\mathbf{x}(t,\mathbf{x}_0,\bar{\mu}),\bar{\mu}\right) &= \begin{pmatrix} \alpha(\bar{y}-\bar{x}) \\ \bar{x}-\bar{y}+\bar{z} \\ -\beta\bar{y} \end{pmatrix} + \begin{pmatrix} (\alpha-\bar{\alpha})(y-x-f(x)) \\ 0 \\ (\bar{\beta}-\beta)y \end{pmatrix} \\ &+ \begin{pmatrix} \alpha(f(x)-f(\bar{x}+x)) \\ 0 \\ 0 \end{pmatrix}.\end{aligned}$$

Rewriting the last expression we have

$$\begin{aligned}\mathbf{f}(\mathbf{z}+\mathbf{x}(t,\mathbf{x}_0,\bar{\mu}),\mu) - \mathbf{f}\left(\mathbf{x}(t,\mathbf{x}_0,\bar{\mu}),\bar{\mu}\right) &= \begin{pmatrix} -\alpha & \alpha & 0 \\ 1 & -1 & 1 \\ 0 & -\beta & 0 \end{pmatrix}\begin{pmatrix} \bar{x} \\ \bar{y} \\ \bar{z} \end{pmatrix} + \begin{pmatrix} (\alpha-\bar{\alpha})(y-x-f(x)) \\ 0 \\ (\bar{\beta}-\beta)y \end{pmatrix} \\ &+ \begin{pmatrix} \alpha(f(x)-f(\bar{x}+x)) \\ 0 \\ 0 \end{pmatrix}.\end{aligned}$$

Now, setting

$$A = A(\mu) := \begin{pmatrix} -\alpha & \alpha & 0 \\ 1 & -1 & 1 \\ 0 & -\beta & 0 \end{pmatrix}, \tag{4.6}$$

$$G(t,\mu,\bar{\mu}) := \begin{pmatrix} (\alpha-\bar{\alpha})(y-x-f(x)) \\ 0 \\ (\bar{\beta}-\beta)y \end{pmatrix}, \tag{4.7}$$

and

$$H(t,\mu,\mathbf{z}) = \begin{pmatrix} \alpha(f(x)-f(\bar{x}+x)) \\ 0 \\ 0 \end{pmatrix}, \tag{4.8}$$

the system (3.1) – (3.2), in the context of Chua's equations (4.4), becomes

$$\begin{aligned}\dot{\mathbf{z}} &= \nu\mathbf{z} + A\mathbf{z} + G(t,\mu,\bar{\mu}) + H(t,\mu,\mathbf{z}) \\ &= (A+\nu I)\mathbf{z} + G(t,\mu,\bar{\mu}) + H(t,\mu,\mathbf{z}).\end{aligned} \tag{4.9}$$

Notice that, at this point, we get a mathematical formula written as in (3.6). Our next goal is to establish that for the expression given in (4.9) the hypothesis H1), H2) and H3) are satisfied. To prove H1), we consider $B = A + \nu I$ where $B \in \mathcal{M}_{3\times 3}$ so that, it is important to know some properties about A matrix. Using a Computer Algebraic System, Maxima [18], we can obtain the eigenvalues and eigenvectors of A. Then, for any α, $\beta \in \mathbb{R}$, the eigenvalues of A are given by

$$\begin{aligned}
\lambda_{1_A} =& \left(-\frac{1}{2}-\frac{\sqrt{3}i}{2}\right) h_2(\alpha,\beta) - \frac{\left(\frac{\sqrt{3}i}{2}-\frac{1}{2}\right)\left(\frac{\beta}{3}+\frac{(-1)\,(\alpha+1)^2}{9}\right)}{h_2(\alpha,\beta)} + \frac{(-1)\;(\alpha+1)}{3},\\
\lambda_{2A} =& \left(-\frac{1}{2}+\frac{\sqrt{3}i}{2}\right) h_2(\alpha,\beta) - \frac{\left(-\frac{1}{2}-\frac{\sqrt{3}i}{2}\right)\left(\frac{\beta}{3}+\frac{(-1)\,(\alpha+1)^2}{9}\right)}{h_2(\alpha,\beta)} + \frac{(-1)\;(\alpha+1)}{3},\\
\lambda_{3A} =& h_2(\alpha,\beta) - \frac{\frac{\beta}{3}+\frac{(-1)\,(\alpha+1)^2}{9}}{h_2(\alpha,\beta)} + \frac{(-1)\;(\alpha+1)}{3},
\end{aligned}$$

where

$$h_2(\alpha,\beta) = \left(h_1(\alpha,\beta) + \frac{(\alpha+1)\,\beta - 3\alpha\beta}{6} + \frac{(-1)\;(\alpha+1)^3}{27}\right)^{\frac{1}{3}},$$

and

$$h_1(\alpha,\beta) = \frac{\sqrt{\beta\;(4\beta^2+(8\alpha^2-20\alpha-1)\,\beta+4\alpha^4+12\alpha^3+12\alpha^2+4\alpha)}}{23^{\frac{3}{2}}}.$$

From here, we consider $\alpha = 9$ and $\beta = \frac{100}{7}$ so, the eigevalues of A are

$$\begin{aligned}
\lambda_{1A} &= -0.0639 - 3.608i,\\
\lambda_{2A} &= -0.0639 + 3.608i,\\
\lambda_{3A} &= -9.8721,
\end{aligned}$$

and the eigenvectors asociated to each λ_i for $i = 1, 2, 3$ are given by

$$v_{1A} = \begin{pmatrix} 1 \\ 0.9929 \\ -1.5172 \end{pmatrix}, \quad v_{2A} = \begin{pmatrix} 0 \\ -0.4009 \\ -3.9579 \end{pmatrix}, \quad v_{3A} = \begin{pmatrix} 1 \\ -0.0969 \\ -0.1402 \end{pmatrix},$$

respectively.

By *Proposition 2*, for each $i = 1, 2, 3$, v_{iA} is an eigenvector of B. In order to prove H1) in the Chua's equation (4.4) we consider the following proposition.

Proposition 7. *The matrix B satisfies H1). Moreover, there are positive contants K, γ such that*

$$\left\| e^{Bt} \right\| \leq K e^{-\gamma t}, \quad t \geq 0. \tag{4.10}$$

Proof. Let B be a 3×3 real matrix given by

$$B = A + \nu I = \begin{pmatrix} -\alpha+\nu & \alpha & 0 \\ 1 & -1+\nu & 1 \\ 0 & -\beta & \nu \end{pmatrix}. \tag{4.11}$$

By *Proposition 1*, for any value of α, β, the eigenvalues of B are given by

$$\lambda_{1B} = \lambda_{1A} + \nu, \quad i = 1, 2, 3,$$

then for values of $\alpha = 9$ and $\beta = \frac{100}{7}$, the eigenvalues become

$$\begin{aligned}
\lambda_{1B} &= -0.0639 + \nu - 3.6082i, \\
\lambda_{2B} &= -0.0639 + \nu + 3.6082i, \\
\lambda_{3B} &= -9.8721 + \nu.
\end{aligned}$$

By choosen $\nu < 0.064$, we have estabished that all the eigenvalues have negative real part. By *Proposition 3*, B has the descomposition

$$B = PJP^{-1},$$

where J is given by

$$J = \begin{pmatrix} -0.0639 + \nu & -3.6082 & 0 \\ 3.6082 & -0.0639 + \nu & 0 \\ 0 & 0 & -9.8721 + \nu \end{pmatrix}.$$

By *Proposition 2*, the eigenvalues of A are eigenvalues for B so P becomes

$$P = \begin{pmatrix} 1 & 0 & 1 \\ 0.9929 & -0.4009 & -0.0969 \\ -1.5172 & -3.9579 & -0.1402 \end{pmatrix},$$

the inverse corresponding to this matrix is

$$P^{-1} = \begin{pmatrix} 0.06728 & 0.8135 & -0.0824 \\ -0.0588 & -0.2830 & -0.2240 \\ 0.9327 & -0.8135 & 0.0824 \end{pmatrix}.$$

We know that

$$e^{Bt} = Pe^{Jt}P^{-1}. \tag{4.12}$$

By taking the Euclidean norm in both sides, we get

$$\begin{aligned}
\left\|e^{Bt}\right\| &= \left\|Pe^{Jt}P^{-1}\right\| \\
&\leq \|P\|\left\|P^{-1}\right\|\left\|e^{Jt}\right\| \\
&= \|P\|\left\|P^{-1}\right\|\left\|\begin{pmatrix} e^{(-0.0639+\nu)t}\cos(3.6082t) & -e^{-0.0639t}\sin(3.6082t) & 0 \\ e^{-0.0639t}\sin(3.6082t) & e^{(-0.0639+\nu)t}\cos(3.6082t) & 0 \\ 0 & 0 & e^{(-9.8721+\nu)t} \end{pmatrix}\right\| \\
&\leq 5.8470\left(e^{(-9.936+2\nu)t}\right) \\
&= Ke^{-\gamma t},
\end{aligned}$$

where $K = 5.8470$ and $\gamma = 9.936 - 2\nu$. □

Now, let us prove that H2) is satisfied. Let $G(t, \mu, \bar{\mu})$ be a continuous function given by (4.7).

Proposition 8. *Given $\varepsilon > 0$, there exists $\delta > 0$ such that*

$$\|G(t, \mu, \bar{\mu}\| < \varepsilon \tag{4.13}$$

for $t \geq 0$ and $\|\mu - \bar{\mu}\| < \delta$.

Proof. Let $\varepsilon > 0$ be arbitrary. For any $x, y \in \mathbb{R}$ and $t \geq 0$ choose

$$0 < \delta < \frac{\varepsilon}{\mid x \mid + \mid y \mid}, \tag{4.14}$$

and assume that $\|\mu - \bar{\mu}\| < \delta$, i.e.,

$$\|\mu - \bar{\mu}\| = \left\| \begin{pmatrix} \alpha - \bar{\alpha} \\ \beta - \bar{\beta} \end{pmatrix} \right\| < \delta. \tag{4.15}$$

Then, by (4.14) and (4.15)

$$\begin{aligned}
\|G(t, \mu, \bar{\mu})\| &= \left\| \begin{pmatrix} (\alpha - \bar{\alpha})(y - x - f(x)) \\ 0 \\ (\bar{\beta} - \beta)y \end{pmatrix} \right\| \\
&\leq \left\| \begin{pmatrix} (\alpha - \bar{\alpha})y \\ 0 \\ (\bar{\beta} - \beta)y \end{pmatrix} \right\| + \left\| \begin{pmatrix} (\alpha - \bar{\alpha})(-x - f(x)) \\ 0 \\ 0 \end{pmatrix} \right\| \\
&\leq \left\| \begin{pmatrix} \alpha - \bar{\alpha} \\ \beta - \bar{\beta} \end{pmatrix} \right\| \|y\| + \left\| \begin{pmatrix} \alpha - \bar{\alpha} \\ \beta - \bar{\beta} \end{pmatrix} \right\| \|x + f(x)\| \\
&< \delta \left(\mid y \mid + \mid x + f(x) \mid\right) \\
&\leq \delta(\mid x \mid + \mid y \mid) < \varepsilon.
\end{aligned}$$

Since ε was arbitrarily chosen, we have proved (4.13). □

Now, to prove H3) we consider $H(t, \mu, z)$ given by (4.8) and the following proposition.

Proposition 9. *$H(t, \mu, z)$ is Globally Lipschitz in $\mathbf{z}$.*

Proof. We have to prove that

$$\exists\, c > 0, \forall\, \mathbf{z_1},\ \mathbf{z_2} \in \mathbb{R}^3 :\ \|H(t, \mu, \mathbf{z_2}) - H(t, \mu, \mathbf{z_1})\| \leq\ c\|\mathbf{z_2} - \mathbf{z_1}\|. \tag{4.16}$$

Let $\mathbf{z_1},\ \mathbf{z_2} \in \mathbb{R}^3$, be arbitrary elements. Then,

$$H(t,\mu,\mathbf{z_2}) - H(t,\mu,\mathbf{z_1}) = \begin{pmatrix} \alpha(f(\bar{x}_2+x) - f(\bar{x}_1+x)) \\ 0 \\ 0 \end{pmatrix},$$

where $\alpha > 0$ and $f(x)$ is given by (4.5). It reminds to find a real constant $M > 0$ such that

$$\|\alpha(f(\bar{x}_2+x) - f(\bar{x}_1+x))\| \leq M\|\bar{x}_2 - \bar{x}_1\|. \tag{4.17}$$

By the formula of f, we have to study 9 cases according the values that $\bar{x}_2 + x$ and $\bar{x}_1 + x$ take.

The cases where $\bar{x}_2 + x$ and $\bar{x}_1 + x$ have the same conditions are immediately from the definition of f. In the other cases, we have to work a little more. So,

i) if $\bar{x}_1 + x \geq 1$ and $\bar{x}_2 + x \geq 1$, then

$$\|\alpha(f(\bar{x}_2+x) - f(\bar{x}_1+x))\| = \mid \alpha \mathrm{b} \mid \|\bar{x}_2 - \bar{x}_1\|,$$

ii) if $\bar{x}_1 + x \leq -1$ and $\bar{x}_2 + x \leq -1$, then

$$\|\alpha(f(\bar{x}_2+x) - f(\bar{x}_1+x))\| = \mid \alpha \mathrm{b} \mid \|\bar{x}_2 - \bar{x}_1\|,$$

iii) if $\mid \bar{x}_1 + x \mid \leq 1$ and $\mid \bar{x}_2 + x \mid \leq 1$, then

$$\|\alpha(f(\bar{x}_2+x) - f(\bar{x}_1+x))\| = \mid \alpha \mathrm{a} \mid \|\bar{x}_2 - \bar{x}_1\|.$$

iv) If $\bar{x}_1 + x \geq 1$ and $\bar{x}_2 + x \leq -1$, we deduce that $\bar{x}_1 - \bar{x}_2 \geq 2$. Then,

$$\left\|\alpha\left(f\left(\bar{x}_2+x\right) - f\left(\bar{x}_1+x\right)\right)\right\| = \left\|\alpha\left(b\left(\bar{x}_2 - \bar{x}_1\right) + 2(b-a)\right)\right\|. \tag{4.18}$$

Let us concentrate in $b\left(\bar{x}_2 - \bar{x}_1\right) + 2(b-a)$. By one hand we have

$$b\left(\bar{x}_2 - \bar{x}_1\right) + 2(b-a) \leq b\left(\bar{x}_2 - \bar{x}_1\right) + (b-a)(\bar{x}_1 - \bar{x}_2) = a(x_2 - \bar{x}_1). \tag{4.19}$$

Now, since $b\left(\bar{x}_2 - \bar{x}_1\right) + 2(b-a) > 0$, it follows that

$$b\left(\bar{x}_2 - \bar{x}_1\right) + 2(b-a) \geq -a(x_2 - \bar{x}_1). \tag{4.20}$$

Thus, by (4.19) and (4.20)

$$\|b\left(\bar{x}_2 - \bar{x}_1\right) + 2(b-a)\| \leq \|a(x_2 - \bar{x}_1)\|,$$

therefore,

$$\left\|\alpha\left(b\left(\bar{x}_2 - \bar{x}_1\right) + 2(b-a)\right)\right\| \leq \ \mid \alpha a \mid \|(x_2 - \bar{x}_1)\|.$$

The procedure for the remaining cases is similar to iv).

v) If $| \bar{x}_1 + x | \leq -1$ and $\bar{x}_2 + x \geq 1$, then

$$\left\| \alpha \left(f\left(\bar{x}_2 + x\right) - f\left(\bar{x}_1 + x\right)\right) \right\| \leq | \alpha a | \, \|\bar{x}_2 - \bar{x}_1\|,$$

vi) if $| \bar{x}_1 + x | \leq 1$ and $\bar{x}_2 + x \geq 1$, then

$$\left\| \alpha \left(f\left(\bar{x}_2 + x\right) - f\left(\bar{x}_1 + x\right)\right) \right\| \leq \quad | \alpha b | \, \|\bar{x}_2 - \bar{x}_1\|,$$

vii) if $| \bar{x}_1 + x | \leq 1$ and $\bar{x}_2 + x \leq -1$, then

$$\left\| \alpha \left(f\left(\bar{x}_2 + x\right) - f\left(\bar{x}_1 + x\right)\right) \right\| \leq \quad | \alpha a | \, \|\bar{x}_2 - \bar{x}_1\|,$$

viii) if $\bar{x}_1 + x \leq -1$ and $| \bar{x}_2 + x | \leq 1$, then

$$\left\| \alpha \left(f\left(\bar{x}_2 + x\right) - f\left(\bar{x}_1 + x\right)\right) \right\| \leq \quad | \alpha a | \, \|\bar{x}_2 - \bar{x}_1\|,$$

ix) if $\bar{x}_1 + x \geq 1$ and $| \bar{x}_2 + x | \leq 1$, then

$$\left\| \alpha \left(f\left(\bar{x}_2 + x\right) - f\left(\bar{x}_1 + x\right)\right) \right\| \leq \quad | \alpha a | \, \|\bar{x}_2 - \bar{x}_1\|.$$

From i) - ix), Let us pick a constant $M = | \alpha a |$. Thus, we have proved (4.17), consequently we have proved,

$$\|H(t, \mu, \mathbf{z_2}) - H(t, \mu, \mathbf{z_1})\| \leq M\|\mathbf{z_2} - \mathbf{z_1}\|.$$

□

Thus, we say that the Chua's System fulfills the conditions H1), H2) and H3). Summarizing for $\alpha = 9, \beta = \frac{100}{7}, a = -\frac{8}{7}$ and $b = -\frac{5}{7}$, we found constants K, γ, and L such that

$$K = 5.8472, \quad \gamma = 9.936 - 2\nu \quad \text{and} \quad L = 9 \mid a \mid = \frac{72}{7}.$$

Thus, for values of $\nu < -1.79$, the condition $\frac{KL}{\gamma} < \frac{1}{4}$ is satisfied. Therefore, we can apply the *Theorem 1* so that

$$\|\mathbf{y} - \mathbf{x}(t, x_0, \bar{\mu})\| \leq \rho.$$

4.3 Numerical Results

The technique exposed in the system (3.1) − (3.2) considers two similar copies of the system to be synchronized with different initial conditions on them. Let $\mathbf{x} = (x, y, z) \in \mathbb{R}^3$ be the generalized coordinates corresponding to the master system, and $\mathbf{y} = (x_s, y_s, z_s) \in \mathbb{R}^3$ those of the slave system. Thus, the whole system is

$$\begin{cases} \dot{x} = \bar{\alpha}(y - x - f(x)), \\ \dot{y} = x - y + z, \\ \dot{z} = -\bar{\beta}y, \\ \dot{x}_s = \alpha(y_s - x_s - f(x_s)) + \nu(x_s - x), \\ \dot{y}_s = x_s - y_s + z_s + \nu(y_s - y), \\ \dot{z}_s = -\beta y_s + \nu(z_s - z). \end{cases}$$

As we mentioned before, we concentrate our attention on the master−slave system with the usual parameters $\bar{\mu} = (\bar{\alpha}, \bar{\beta}) = \left(9, \frac{100}{7}\right)$, and $(a, b) = \left(-\frac{8}{7}, -\frac{5}{7}\right)$. Let us choose $\nu = -10$ since it corresponds to the minor number respect to the negative real part of the eigenvalues mentioned in *Proposition 7*. Also, let us consider a vector $\mu = (\alpha, \beta) = \left(8.9, \frac{99}{7}\right)$, closer to $\bar{\mu}$.

To represent the master−slave synchronization we use a Computer Algebraic System, Maxima [18] and its graphical interface wxMaxima. To make the following figures, an important guide is found in [2]. To explore this synchronization technique, master−slave syncronization, we starting from two slightly initial conditions $(0, 0.5, 0.6)$ and $(-0.5, 2, 1)$ as shown in Figure 4.3

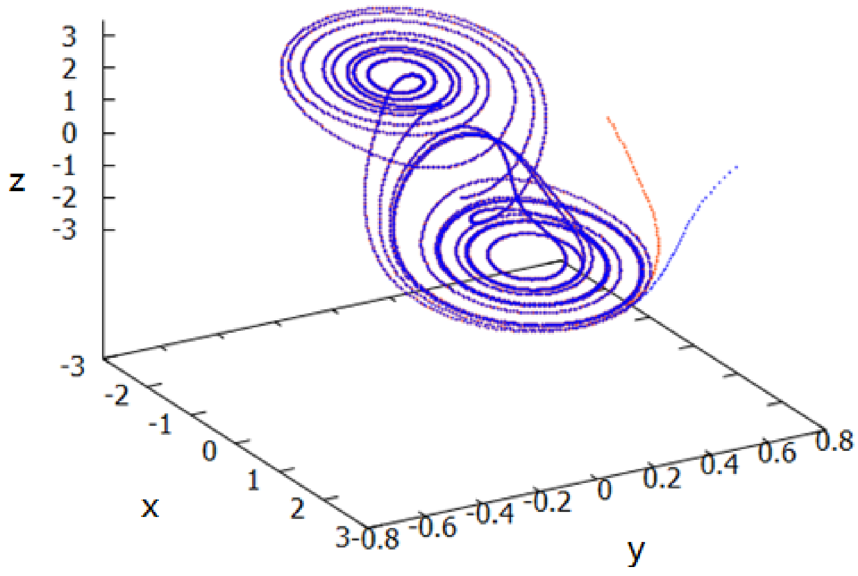

Figure 4.3: Illustration of two systems (master and slave), corresponding to Chua'a equation (4.4), after synchronization.

4.3.1 Synchronization by coordinates

In order to verify more explicitly the synchronization found in the Chua's system (4.4), we illustrate the synchronization through coordinates. In other words, we represent the synchronization in the coordinates $[x(t),\ x_s(t)]$ over time t, as the same way, we represent the synchroization in the coordinates $[y(t),\ y_s(t)]$ over time and $[z(t),\ z_s(t)]$ over time.

Let us begin with the coordinates $x(t)$ and $x_s(t)$. In Figure 4.4, it is represented the trajectories followed by $x(t)$ (coordinate corresponding to master system) and $x_s(t)$ (coordinate corresponding to slave system).

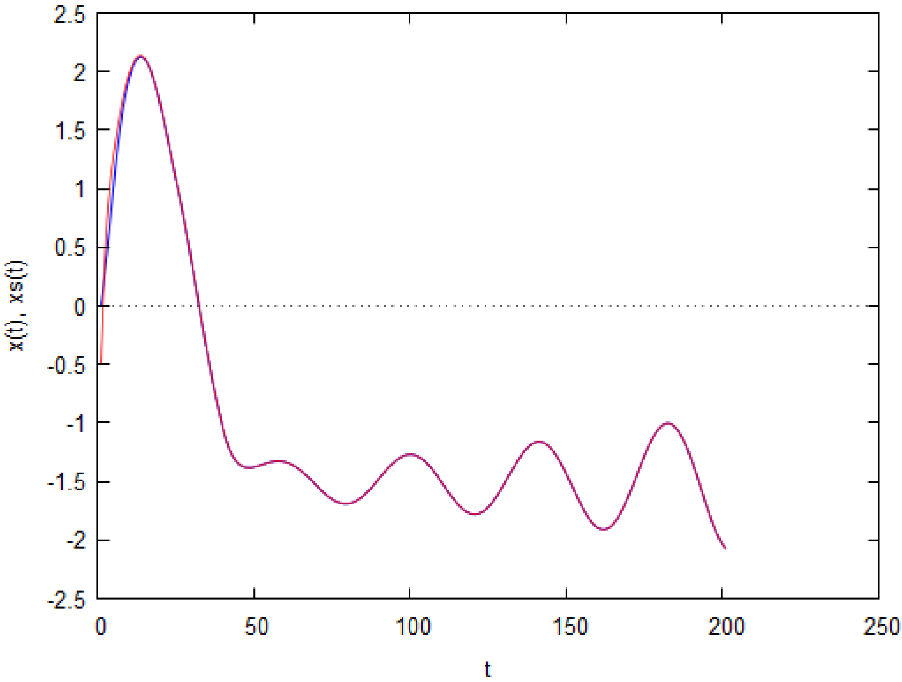

Figure 4.4: Simulation of trajectories of two initial conditions, $x(0) = 0$ and $x_s(0) = -0.5$, for two coupled Chua's systems. (4.4) .

Next, in Figure 4.5, it is represented the trajectories followed by $y(t)$ (coordinate corresponding to master system) and $y_s(t)$ (coordinate corresponding to slave system). Clearly, the initial values are far from each other, and over time, the synchronization achieved becomes more clearly evident.

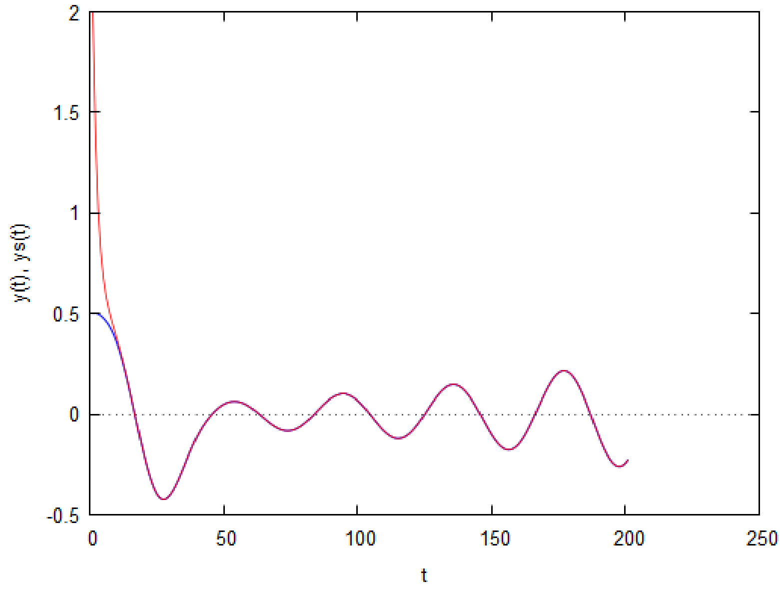

Figure 4.5: Simulation of trajectories of two initial conditions, $y(0) = 0.5$ and $y_s(0) = 2$, for two coupled Chua's systems. (4.4) .

Finally, we represent, in Figure 4.6, the trajectories followed by $z(t)$ (coordinate corresponding to master system) and $z_s(t)$ (coordinate corresponding to slave system). The trajectory of each coordinate has a different initial value which, over time, the synchronization achieved becomes more evident.

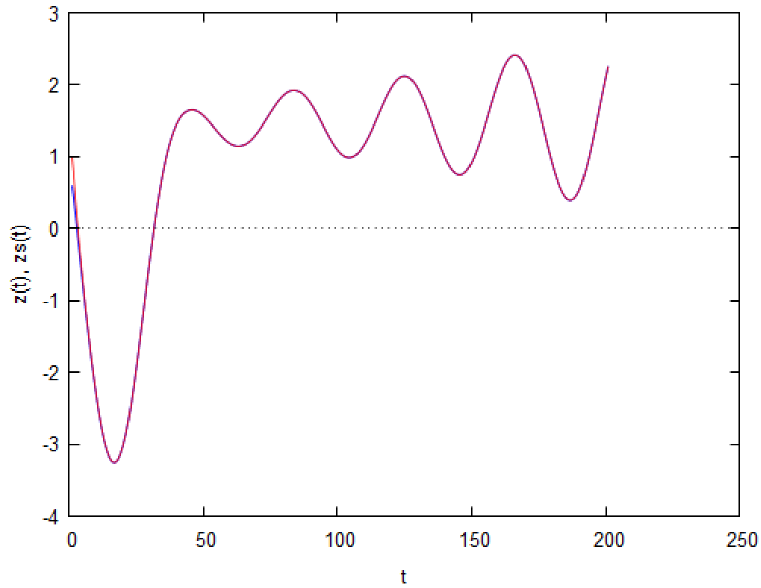

Figure 4.6: Simulation of trajectories of two initial conditions, $z(0) = 0.6$ and $z_s(0) = 1$, for two coupled Chua's systems. (4.4) .

Chapter 5

Conclusions

In this work, we have considered a finite-dimensional master-slave system, which is coupled through a linear term. The system has the form

$$\begin{aligned} \dot{\mathbf{x}} &= \mathbf{f}\left(\bar{\mu}, \mathbf{x}\right) && \text{(slave)} \\ \dot{\mathbf{y}} &= \mathbf{f}\left(\mu, \mathbf{y}\right) + \nu\left(\mathbf{y} - \mathbf{x}\right) && \text{(master)} \end{aligned}$$

Associated with this system and after the transformation $\mathbf{z} = \mathbf{y} - \mathbf{x}$, a non-autonomous system of ordinary differential equations has been considered

$$\dot{\mathbf{z}} = F\left(t, \mathbf{z}, \mu, \bar{\mu}\right) \ .$$

From this non-autonomous system, a class has been identified with the following characteristics:

- A linear part with a matrix possessing eigenvalues with negative real part.
- An expression which is under control in the sense of its size and which depends on the parameters involved in the problem as well as on a bounded solution of the master system.
- A Lipschitz condition, in the global sense. Lipschitz in the difference given by the coupling linear term of the master-slave system.

The class of conditions, on which synchronization is studied, is obtained without resorting to linearization. This, in principle, avoids the consideration of approximations and it can be an important point when considering numerical simulations.

As part of interest, it has been observed that Chua's equations, which correspond to a chaotic model, constitute a representation of the class identified here. We emphasize that in these equations the nonlinear part is given by a piecewise continuous function and this fact was fundamental for the global Lipschitz condition.

On the other hand, the theoretical results presented in this work come from very simple ideas and show a relationship between the parameters involved in the system. Furthermore, through the eigenvalues of the matrix mentioned above, we can give an estimate for the coupling parameter ν for which synchronization is guaranteed.

Finally, in order to provide more elements in the class of systems that have been identified, we have in mind, for future consideration, the Lorentz and Rossler equations. Although it is true that for these chaotic systems the non-linear terms are quadratic polynomials, there are works that have incorporated modifications that introduce piecewise linear functions, see for instance [19], [20]. In this sense, we believe that what has been developed in this work can be applied.

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