UNIVERSIDAD DE INVESTIGACIÓN DE TECNOLOGÍA EXPERIMENTAL YACHAY

Escuela de Ciencias Físicas y Nanotecnología

TÍTULO: STUDY OF THE TENSOR POWER SPECTRUM FOR THE STAROBINSKY INFLATIONARY MODEL USING THE UNIFORM APPROXIMATION METHOD

Trabajo de integración curricular presentado como requisito para la obtención del título de Físico

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Dedication

To the infinity love of my family.

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David Alejandro Altamirano Coello

Resumen

El origen del universo ha sido una gran pregunta a lo largo de la existencia humana. El modelo del Big Bang ha sido el primer avance científico que ha tratado de responder esta pregunta. Luego, vino lo que se conoce como Cosmología Inflacionaria para resolver 'the fine-tuning problems' que emergían de la teoría clásica del BB. A la inflación se entiende como una expansión acelerada que ocurre en el universo temprano, y a pesar de su gran ayuda para entender su comportamiento, todavía no es un model que explique la inflación satisfactoriamente. La manera de darnos cuenta la validez del modelo es comparando los valores calculados con los valores observados experimentalmente. Así, para este presente trabajo se utilizará el modelo inflacionario propuesto por Starobinsky el cual es el más aceptado por los resultados que arroja. Con este modelo calculamos el espectro de potencia para perturbaciones tensoriales usando tres métodos diferentes, integración numérica, aproximación slow-roll y aproximación uniforme, haciendo más énfasis en el tercero. El espectro de potencia nos permitirá estimar cantidades observables como el índice tensorial espectral n_T y la relación tensor-escalar r. Demostraremos que la aproximación slow-roll da un mejor resultado cuando evaluamos el espectro de potencia. Pero, cuando calculamos el parámetro r, la precisión de la aproximación uniforme se compara favorablemente con los resultados numéricos.

Palabras claves: Cosmología Inflacionaria, Perturbaciones Cosmológicas, Modelo de Starobinsky, Espectro de Potencia, Integración Numérica, Aproximación Slow-Roll, Método de Aproximación Uniforme.

Abstract

The origin of the universe has been a big question along human existence. The Big Bang model was the first scientific step in order to answer this question. Then, Inflationary Cosmology comes to solve the fine-tuning problems emerging from the classical theory of the BB. Inflation is understood as an exponential expansion that occurred in the early universe, and despite its great help in order to understand its behavior there is still no model that explains it satisfactorily. The way to realize the validity of the model is comparing the calculated values with the experimentally observed values. Then, this work will make use of the Starobinsky model which is the most accepted model for the results it yields, in order to calculate the power spectrum of tensor perturbations by three different methods, numerical integration, slow-roll approximation and uniform approximation, making more emphasis in the third one. By consequence, we can also calculate the tensor spectral index n_T and the tensor-to-scalar ratio r. We will show that the slow-roll method gives a better result when we evaluate the power spectrum. But, when we calculate r the accuracy of the uniform approximation compares favorably with the numerical results.

Keywords: Inflationary Cosmology, Cosmological Perturbations, Starobinsky Model, Power Spectrum, Numerical Integration, Slow-Roll Approximation, Uniform Approximation Method.

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Chapter 1

Introduction

The need to explain what the universe is has moved humanity since its inception, and thanks to the technological development of the last century we have been able to glimpse certain doubts. The modern point of view of the universe is evidenced by the cosmological principle, which states that, at large scales the universe is homogeneous and isotropic[1], it means that if we compare two different space regions we realize that have very similar composition. On the other hand, theoretical developments have also allowed a better understanding, as General Relativity by Albert Einstein[2 3]. The Big Bang idea which originates from GR also broke in strongly, forging the beginning of Cosmology as such. The Big Bang states that the universe was created from a hot, dense and infinitesimally small point in the vacuum space. After that, the universe began to expand over the space and with it also cooled, taking place to a process where the very light elements were created, known as nucleosynthesis[4]. The first observational step for consolidating this theory is the expansion of the universe described by Hubble[5] while measuring the redshift of distant galaxies. It is established that galaxies are moving away from each other and, in general, that the universe was expanding. Other observational fact predicted by the Big Bang is the existence of a remnant that evidences this process known as the Cosmic Microwave Background (CMB)[6] discovered in 1964. By these facts, the Big Bang model assures a correct description for the universe creation. However, this is not a complete model, in fact it has certain drawbacks related to the initial conditions[7] that gave rise to the universe and are summarized in the "Fine-tuning problems". In this context Alan Guth proposed a way to solve these problems, known as Inflation[8] that consists in a short period of time in the early universe where there was a huge and astonishing expansion. This idea not only solved these problems but also helped explain the CMB anisotropies or the large-scale structure of the Universe. It also generates a spectrum of density perturbations and predicts primordial gravitational waves. For inflation take place, the energy density of the Universe was dominated by the potential of the scalar field. To evaluate the potential useful, we should calculate the power spectrum and compare with the experimental measurements of CMB anisotropies[9]. The present work makes use of the Starobinsky potential to calculate the tensor power spectrum of cosmological perturbations.

1.1 The Big Bang theory

Nowadays, if we think about the Big Bang theory we intrinsically assume the inflation era. But now we will present a summary about the initial idea of this theory, called the Hot Big Bang. For this, it is important start with the Cosmological Principle which had been established thanks to observations. Then, the number of galaxies per unit volume appear to be uniform throughout large space regions (homogeneity), and the number of galaxies per unit solid angle appears to be the same in all directions (isotropy)[2]. To reproduce this features, Friedman[10] proposed a simple model as a solution of the Einstein field equations, and then Robertson and Walker introduce the part of the cosmological principle. Then, an expanding (or contracting) homogeneous and isotropic Universe is described by the Friedmann-Lamaitre-Robertson-Walker (FLRW) metric,

$$ds^2 \equiv g_{\mu\nu}\, dx^\mu dx^\nu = -dt^2 + a^2(t) \left[\frac{dr^2}{1 - \kappa r^2} + r^2\, (d\theta^2 + sin^2\theta\, d\phi^2) \right], \tag{1.1}$$

expressed in polar coordinates where t, r, θ and ϕ are the comoving coordinates. The constant κ describes the curvature of the Universe. The variable a is the scale factor and explain the expansion of the Universe. As is known, the expansion is due to the deformation of the space-time itself, but not by the motion of its constituents, making the distinction between physical distance, x, and comoving distance, r, necessary. The relationship between both is given by,

$$x(t) = a(t)\, r. \tag{1.2}$$

However, the properties of the constituents, like the energy density $\rho(t)$ or the pressure $p(t)$, also play an important role in order to explain the expansion. They are related by a equation of state and the classic example are,

$$p = \frac{\rho}{3}, \tag{1.3}$$

for radiation, and

$$p = 0, \tag{1.4}$$

for non-relativistic matter. However, the behaviour of the constituents could be more complex like that.

The dynamics (expansion or contraction) of the Universe is described by solving the Einstein Field Equations,

$$G_{\mu\nu} + \Lambda\, g_{\mu\nu} = 8\, \pi\, G\, T_{\mu\nu}, \tag{1.5}$$

where G is the universal gravitational constant, $g_{\mu\nu}$ is the metric tensor and $G_{\mu\nu}$ is the Einstein tensor. The cosmological constant Λ is related to the dark energy. $T_{\mu\nu}$ is the energy-momentum tensor which describes the properties of the constituents of the Universe. As is known, this equation relates the geometry (left hand side) of the space with the matter (right hand side) of the Universe. To solve the left hand side of the equation the line element described in Eq.(1.1) is used. While to solve the right hand side is necessary to assume that the Universe is a collection of N perfect fluids, means that is uniform along all the space. Then, the energy-momentum tensor is,

$$T_{\mu\nu} = \sum_{i=1}^{i=N} T_{\mu\nu}^{(i)} = \sum_{i=1}^{i=N} \left\{ \left[\rho_i(t) + p_i(t)\right] u_\mu u_\nu + p_i(t)\; g_{\mu\nu} \right\}, \tag{1.6}$$

where p_i is the pressure and ρ_i is the density of the i perfect fluid, both depending on time. The vector u_μ is a four velocity in comoving coordinates given by $u = (1, 0, 0, 0)$ and satisfies the relation $u_\mu u^\mu = -1$. In order to determine the cosmological evolution and to solve the Einstein equations, let to assume the properties of the fluid represented in the equation of state $p_i = \omega_i \rho_i$, where ω is a constant which depends on the kind of matter. Then, the obtained equation after solving the Einstein field equations are two coupled nonlinear differential equations,

$$\frac{\dot{a}^2}{a^2} + \frac{\kappa}{a^2} = \frac{8\pi G}{3M_{pl}} \sum_{i=1}^{N} \rho_i + \frac{\Lambda}{3}, \tag{1.7}$$

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3M_{pl}} \sum_{i=1}^{N} (\rho_i + 3p_i) + \frac{\Lambda}{3}, \tag{1.8}$$

where overdots are time derivatives and M_{pl} is the reduced Planck mass. In order to make these equations more manageable we will assume that the Universe is mainly filled with just one fluid, we vanish the cosmological constant and we say that the curvature of the Universe is $\kappa = 0$ as observations suggest. Thus, we get the Friedmann equation,

$$\left(\frac{\dot{a}}{a}\right)^2 \equiv H^2 = \frac{8\pi G}{3}\rho, \tag{1.9}$$

and the acceleration equation,

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}(\rho + 3p), \tag{1.10}$$

where H is the Hubble parameter. Combining both equations we can prove that the continuity equation is fulfilled which assures the energy conservation,

$$\dot{\rho} + 3H(\rho + p) = 0. \tag{1.11}$$

This equation can be also obtained from the conservation of the energy-momentum tensor $T^{\mu\nu}{}_{;\nu} = 0$. In order to solve it we integrate the conservation equation (1.11) which show the dynamics of the density,

$$p(t) = p_f \left(\frac{a_f}{a}\right)^{3(1+\omega)}, \tag{1.12}$$

where $p_f = p(t_f)$ and $a_f = a(t_f)$. Substituting in the Friedmann equation (1.7), we get

$$\frac{\dot{a}^2}{a^2} = \frac{8\,\pi\,G\,\rho_f}{3}\left(\frac{a_f}{a}\right)^{3(1+\omega)}, \tag{1.13}$$

and solving we get the scale factor,

$$a(t) = a_f \left[\frac{3}{2}(1+\omega)\,H_f\,(t - t_f) + 1\right]^{\frac{2}{3(1+\omega)}}. \tag{1.14}$$

From this equation we realize that the scale factor vanishes when $t = t_{BB} = t_f - 2/[3(1+\omega)H_f]$. According to the current knowledge, it is unimaginable that a time existed prior to the Big Bang, therefore we say that time begins at t_{BB}. For convenience, the scale factor is rewritten in terms of t_{BB} and fix $t_{BB} = 0$, then the scale factor has the form

$$a(t) = a_f \left(\frac{t}{t_f}\right)^{\frac{2}{3(1+\omega)}}. \tag{1.15}$$

This equation shows that the scale factor depends on the constant ω which is determined by the constituents of the Universe. In order to analyze this constituents we should establish the epoch to treat. There are three predominant epochs where a different constituent dominates each one. To cross from one epoch to another, the process must be smooth, it means that a and H must be continuous along time. The first epoch was dominated by radiation and the corresponding equation of state is $\omega = 1/3$, giving that the scale factor is $a \propto t^{1/2}$. The second epoch (current one) is dominated by matter, described by the equation of state $\omega = 0$, then $a \propto t^{2/3}$. Finally, dark energy dominates the third epoch, described by $\omega = -1$ and the scale factor is ill-defined because we have a exponential solution, $a(t) = a_f \, exp[H_f \, (t - t_f)]$, known as the Sitter solution.

When some types of matter coexist, to quantify the contribution of each component to the total energy density of the universe, the density parameter of component i is defined as

$$\Omega_{(i)} = \frac{\rho_0^{(i)}}{\rho_{cr}}, \tag{1.16}$$

where $\rho_0^{(i)}$ is the density of the components i (baryons, photons, ...) and ρ_{cr} is the critical energy density that corresponds to the value of a spatially flat universe ($k = 0$) and has the form

$$\rho_{cr}(t) = \frac{3M_{pl}^2 H_0^2}{8\pi G}, \tag{1.17}$$

where H_0 is the current Hubble constant. By consequence the total density parameter is

$$\Omega_T = \sum_i \Omega_i. \tag{1.18}$$

This equation, (1.18) shows the fraction that contributes each material to the total energy density of the Universe.

1.2 Fine-Tuning Problems

Despite the fact that the cosmological model explained above, the BB model, describes the dynamics of the universe on a large scale with an acceptable precision, it raises some problems in explaining various observational facts. We will explore three of them. For this, we will follow the textbooks[1 7].

1.2.1 Flatness Problem

In order to analyze this problem, we are going to rewrite the Friedmann equation (1.9) in terms of the critical energy density $\rho_{cr}(t)$ and the density parameter $\Omega(t)$, then

$$|\Omega_T - 1| = \frac{|\kappa|}{a^2 H^2}. \tag{1.19}$$

During the Big Bang model described above, a^2H^2 is decreasing, and so Ω moves away from one, for example for an Universe dominated by matter $|\Omega - 1| \propto t^{2/3}$, and in an Universe dominated by radiation $|\Omega - 1| \propto t$, where the solutions apply provided Ω is far to one.

In order to understand more mathematically, we review this problem using the notes of Martin[7]. So, we replace the equation (1.12) in (1.19),

$$\Omega_T = \frac{H_o^2}{H^2} \sum_i \Omega_i^o \left(\frac{a_o}{a}\right)^{3(1+\omega_i)}, \tag{1.20}$$

where the subscript or superscript o means that is about the present time. Now, we use equation (1.19) assuming an approximately flat Universe,

$$\Omega_T - 1 = \left(\frac{H_o}{H}\right)^2 (\Omega_t^o - 1) \left(\frac{a_o}{a}\right)^2. \tag{1.21}$$

Combining (1.20) and (1.21) we find that

$$\Omega_T(t) = \frac{\sum_i \Omega_i^o \left(\frac{a_o}{a}\right)^{3(1+\omega_i)}}{\sum_i \Omega_i^o \left(\frac{a_o}{a}\right)^{3(1+\omega_i)} - (\Omega_T^o - 1)\left(\frac{a_o}{a}\right)^2}. \tag{1.22}$$

Considering the two known epoch, matter and radiation the equation turns in,

$$\Omega_T(t) = \left[1 - \left(\frac{a^2}{a_o}\right) \frac{\Omega_T^o - 1}{a\Omega_m^o \; a_o\Omega_{rad}^o}\right]^{-1}, \tag{1.23}$$

and doing a Taylor expansion and rearranging we get,

$$\Omega_T^o - 1 = (\Omega_T(z) - 1)\left[(z+1)\,\Omega_m^o + (z+1)^2\,\Omega_{rad}^o\right]. \tag{1.24}$$

We know from the data obtained by Planck mission published in 2018[11] that $|\Omega_T^o - 1| \leq 0.003$. To know the scope of this results we analyze for different scenarios. For $\Omega = 1$, the solution is an unstable critical point. This leads us to estimate upper limits that this quantity should have throughout the life of the Universe. For a larger redshift z (go backwards in time), $\Omega_T^o - 1$ must be less and less to satisfy the observational condition. For example, for Big Bang Nucleosynthesis where $t \sim 1s$ we have $|\Omega - 1| < O(10^{-16})$. For the Planck time $t \sim 10^{-43}s$ we have $|\Omega - 1| < O(10^{-60})$. Then, the question is why the initial energy density of the universe was so finely tuned to its critical value. In order to answer this question, a new paradigm which has this features is proposed.

1.2.2 Horizon Problem

This problem arises as a consequence of the finite age of the Universe, t_0. It means that we can only see the Universe up to such a distance that its light has had time to reach us. For further distant regions, the light would take a time $t > t_0$ to reach us, therefore the information from these regions would not be received yet, that is, these regions are outside our light cone. Then, in the Universe there are regions which are not causally connected between them, that means that no event that happens in one region will affect the other. This behaviour also affects to the temperature which looks almost the same along all the space. The better explanation is that the Universe has indeed reached a thermal equilibrium state, but unfortunately in the Big Bang theory this is not possible.

As we saw in the previous subsection, the expansion of an Universe dominated by matter like the current one has the form $a(t) \propto t^{2/3}$. This implies that the *particle horizon* (horizon that delimits the causally connected regions after a given time) along the time has been increasing, so that now there are causally connected regions that in the past were not. Specifically, no pair of points separated by more than 1.7 degrees would have been in causal contact at the moment of their formation [12].

We can evidence this behaviour in the CMB measurement. According to the Big Bang theory, the CMB is formed by causally disconnected regions, but experimentally it has been seen that the CMB has the same characteristics in all space. To explain this property it should be consider as an initial condition of the Universe, but it is a trivial answer. In order to analyze the problem, consider the size of a causally connected sphere on the CMB which was established when matter decoupled from radiation at the last scattering surface (lss). Then, the angular size of the horizon at lss is given by

$$\delta\theta = \left[\int_{t_{lss}}^{t_0} \frac{\mathrm{d}\tau}{a(\tau)}\right]^{-1} \int_0^{t_{lss}} \frac{\mathrm{d}\tau}{a(\tau)}. \tag{1.25}$$

To solve this equation we should know the behaviour of the scale factor $a(t)$, but according to the Big Bang model is not possible. However, an approximate solution is given by equation (1.14). Then, it is important to explore the behaviour of the scale factor in each phase of the Universe evolution, using the following time scale $0 < t_i < t_{end} < t_{eq} < t_{de} < t_0$. Phase I correspond to $0 < t < t_i$ when radiation dominates the Universe and the scale factor is $a(t) = a_i\ (2H_i\ t)^{1/2}$, where a_i and H_i are free parameters. At $t = 0$ (Big Bang) the scale factor vanishes and the scalar curvature blows up. In phase II, the scale factor is given by equation (1.14) for $t_i < t < t_{end}$, where the behaviour of $a(t)$ changes but staying continuous during the transition. The quantity ω is a free parameter related to an equation of state of matter. Phase III has similar features with Phase I where dominates the radiation and the scale factor is $a(t) = a_{end}\ [2H_{end}\ (t - t_{end}) + 1]^{1/2}$ for times $t_{end} < t < t_{eq}$. In Phase IV where matter dominates the Universe composition the scale factor is $a(t) = a_{eq}\ [(3/2)\ H_{eq}\ (t - t_{eq}) + 1]^{2/3}$ for $t_{eq} < t < t_{de}$. And the last Phase V is dominated by the cosmological constant with a scale factor of the form $a(t) = a_{de}\ e^{H_0\ (t - t_{de})}$ valid until the present time $t_{de} < t < t_0$.Then, we start solving the integrals by the denominator of equation (1.25), then the integral reads

$$\begin{aligned}
\int_{t_{lss}}^{t_0} \frac{\mathrm{d}\tau}{a(\tau)} &= \int_{t_{lss}}^{t_{de}} \frac{\mathrm{d}\tau}{a(\tau)} + \int_{t_{de}}^{t_0} \frac{\mathrm{d}\tau}{a(\tau)}, \\
&= \frac{1}{a_{eq}} \int_{t_{lss}}^{t_{de}} \mathrm{d}\tau \left[\frac{3}{2}\ H_{eq}(\tau - \tau_{eq} + 1\right]^{-2/3} + \frac{1}{a_{de}} \int_{t_{de}}^{t_0} \mathrm{d}\tau \left[e^{-H_0(\tau - \tau_{de})}\right], \\
&= \frac{2}{a_{eq}H_{eq}} \left(\frac{a_0}{a_{eq}}\right)^{1/2} \left[\left(\frac{a_{de}}{a_0}\right)^{1/2} - \left(\frac{a_{lss}}{a_0}\right)^{1/2}\right] + \frac{1}{a_0 H_0} \left(\frac{a_0}{a_{de}} - 1\right).
\end{aligned} \tag{1.26}$$

Applying the chain rule to the Hubble parameter at this epoch, we have

$$\frac{2}{a_{eq}H_{eq}} = \frac{2}{a_0 H_0} \frac{a_0 H_0}{a_{de} H_{de}} \frac{a_{de} H_{de}}{a_{eq} H_{eq}} = \frac{2}{a_0 H_0} \frac{a_0}{a_{de}} \left(\frac{a_{de}}{a_0}\right)^{-1/2} \left(\frac{a_0}{a_{eq}}\right)^{-1/2}. \tag{1.27}$$

Replacing in (1.26) we get the integral in terms of scale factor ratios at different times,

$$\int_{t_{lss}}^{t_0} \frac{\mathrm{d}\tau}{a(\tau)} = \frac{2}{a_0 H_0} \left(\frac{a_0}{a_{de}}\right)^{3/2} \left[\left(\frac{a_{de}}{a_0}\right)^{1/2} - \left(\frac{a_{lss}}{a_0}\right)^{1/2}\right] + \frac{1}{a_0 H_0} \left(\frac{a_0}{a_{de}} - 1\right). \tag{1.28}$$

In order to solve the second part of equation (1.25), the numerator, we apply the same procedure as before,

$$\int_0^{t_{lss}} \frac{d\tau}{a(\tau)} = \int_0^{t_i} \frac{d\tau}{a(\tau)} + \int_{t_i}^{t_{end}} \frac{d\tau}{a(\tau)} + \int_{t_{end}}^{t_{eq}} \frac{d\tau}{a(\tau)} + \int_{t_{eq}}^{t_{lss}} \frac{d\tau}{a(\tau)}, \tag{1.29}$$

and, using the piece-wise solution described before and replacing the respective scale factors, we have

$$\int_0^{t_{lss}} \frac{d\tau}{a(\tau)} = \frac{1}{a_i H_i} + \frac{1}{a_i H_i}\frac{2}{1+3\omega}\left[\left(\frac{a_{end}}{a_i}\right)^{\frac{1+3\omega}{2}} - 1\right] + \frac{1}{a_{end}H_{end}}\left(\frac{a_{eq}}{a_{end}} - 1\right) + \frac{2}{a_{eq}H_{eq}}\left[\left(\frac{a_{lss}}{a_{eq}}\right)^{1/2} - 1\right]. \tag{1.30}$$

Now, we apply again a chain rule where $1/(a_i H_i) = 1/[(a_{end}H_{end})(a_i/a_{end})^{(1+3\omega)/2}]$ and $1/(a_{end}H_{end}) = 1/[(a_{eq}H_{eq})(a_{eq}/a_{end})^{-1}]$ given,

$$\int_0^{t_{lss}} \frac{d\tau}{a(\tau)} = \frac{1}{a_{eq}H_{eq}}\left[1 + \frac{1-3\omega}{1+3\omega}\frac{a_{end}}{a_{eq}} - \frac{1-3\omega}{1+3\omega}\frac{a_{end}}{a_{eq}}\left(\frac{a_i}{a_{end}}\right)^{\frac{1+3\omega}{2}}\right] + \frac{2}{a_{eq}H_{eq}}\left[\left(\frac{a_{lss}}{a_{eq}}\right)^{1/2} - 1\right]. \tag{1.31}$$

Finally, we should do a clarification about the phases. The Phase II was introduced to explain the dominance of the fluid in the Universe with the equation of state ω, therefore in the MCE model this phase is absent, being Phase I, II and III almost the same. Then, we say that $a_i = a_{end}$, and approximating $a_0 \simeq a_{de}$ and $a_{lss} \simeq a_{eq}$, so we get

$$\delta\theta \simeq \frac{1}{2}(1 + z_{lss})^{-1/2} \simeq 0.0138. \tag{1.32}$$

This means that the sky should be formed by many patches of different temperatures each one. But if we see this property in the CMB we can identify because everything looks similar at large scales, the anisotropies appears just in order to 10^{-5}. The solution that arises from this problem is the inflation which supposes a very early Universe in causal contact and after an accelerated expansion period this contact is break down. Then, the homogeneity that existed previously is printed when the CMB was formed.

1.2.3 Monopoles Problem

The BB model with help of particle physics assumes the spontaneous symmetry breaking of the vacuum that took place when the temperature of the universe was of the order of $T \sim 10^{16} GeV$. From this process, magnetic monopoles must have been produced which should be easily observed in the current universe and even more should govern it due to its great density that would dominate the total density. However, these relics have not been observed to date.

Inflation could dilute the monopoles by its accelerated expansion period, since the energy density during inflation falls more slowly than the relic density. Then, relic density quickly becomes negligible and does not dominate.

1.3 General and Specific Objectives

The principal objective of the present work is to evaluate a specific model for inflation proposed by Starobinsky [13] and to calculate, using the uniform approximation method up to the second order, the tensor power spectrum of cosmological perturbations. This result will be compared with the numerical method (exact result) and the slow-roll method (standard in inflationary cosmology).

The Chapter 2 contains the theoretical background necessary to our discussion. Beginning with the inflationary model proposed by Guth[8] and how this idea can solve the fine-tuning problems. Also, the slow-roll approximation is revised which is the most useful technique in Inflationary Cosmology. Then, both the Starobinsky model that currently best explains the observations and the tensor cosmological perturbations are also introduced. At the end of this chapter we explore a semi-classical method to solve the equation of tensor perturbations since it is a second-order differential equation that has no real solution. The method is known as uniform approximation and is the main contribution of this work to the scientific community, how an alternative method to solve the cosmological perturbations equation. In Chapter 3 the results are presented. Both the results of background dynamics, as the solutions to the Mukhanov-Sasaki equation and the full power spectrum of cosmological perturbations are presented with its respective spectral index. The Chapter 4 consists of a summary of the work and the possibilities which can take to continue researching.

For the present work we will use a notation that is specified below and that will remain constant throughout it. We use the Einstein summation convention where a greek indices run from 0 to 3, and latin indexes go from 1 to 3. The signature of the metric as usual is (-, +, +, +). The value of some constants like $\hbar$ or c is setting at 1. While the reduced mass Planck is defined as $M_{pl} = (8\pi G)^{-1/2}$. The dot over some variable denotes differentiation with respect to the cosmic time t, and $'$ is the differentiation with respect to the conformal time η. The cosmic time and the conformal time are related by $dt = a\, d\eta$

Chapter 2

Methodology

2.1 Inflation

First at all, inflation was proposed in 1981 by Alan Guth[8] as a solution to the fine-tuning problems of Big Bang theory but along the time these theories have become complementary to each other. By consequence the inflationary theory has been widely accepted not only for responding to these problems, but also because it is supported by the observational facts.

Inflationary theory is a period of accelerated expansion of the Universe, strictly speaking the scale factor was accelerating

$$\ddot{a}(t) > 0, \tag{2.1}$$

or rewriting in terms of the Hubble length or Hubble horizon, $(Ha)^{-1}$, which are synonymous, we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{H^{-1}}{a} < 0. \tag{2.2}$$

It says that the Hubble length decreases during inflation. Reminding the acceleration equation (1.10), we see how a positive acceleration involves that

$$\rho + 3p < 0, \tag{2.3}$$

showing that either pressure or density should be negative. Since a negative density is unimaginable, the "exotic matter" that drives inflation must have a negative pressure. The equations (2.1), (2.2) and (2.3) are different conditions to express the inflationary process.

Continuing the analysis of the dynamic behaviour of the Universe with this assumption, where we have a perfect fluid with a negative pressure, the solution of the Friedmann equation (1.9) is

$$a(t) = a_0\, e^{Ht}. \tag{2.4}$$

Also, it is necessary to determine the amount of inflation, it means how much the Universe expands during this epoch. In order to quantify it we use the number of e-foldings N from a initial time t_i to a time t, and is given by

$$N(t) = ln\left[\frac{a(t)}{a(t_i)}\right]. \tag{2.5}$$

Until this point, we will see that inflation can solve the cosmological problems, but is not a clear scenario which can describes how inflation played out and why it has only been around for a short period of time. In order to answer this, the idea of a potential that governed inflation arises. For simplicity this potential is in the form of a single massless scalar field, known as inflaton $\phi = \phi(\vec{x}, t)$, which is defined in all space-time presenting a potential $V(\phi)$. It is used to calculate the CMB observables of inflation, like spectral indices. The behaviour of the field is described by the Klein-Gordon equation

$$\nabla_\mu \nabla^\mu \phi = V'(\phi). \tag{2.6}$$

The dynamics of the scalar field in the case minimally coupled to gravity is governed by the action,

$$S_\phi = \int d^4x \ \sqrt{-g} \left[-\frac{1}{2}\partial^\mu \phi \ \partial_\mu \phi - V(\phi)\right], \tag{2.7}$$

where $g = det(g_{\mu\nu})$. The corresponding energy-momentum tensor is derived using (1.6), then

$$T^\mu_\nu = \partial^\mu \phi \ \partial_\nu \phi - \left[\frac{1}{2}\partial^\sigma \phi \ \partial_\sigma \phi + V(\phi)\right] \delta^\mu_\nu. \tag{2.8}$$

Taking account the homogeneity of the Universe, the scalar field takes the form $\phi(\vec{x}, t) = \phi(t)$. Then, the energy density and pressure for a Universe containing a single scalar field is defined by

$$\rho = \frac{1}{2}\dot{\phi}^2 + V(\phi), \tag{2.9}$$

$$p = \frac{1}{2}\dot{\phi}^2 - V(\phi), \tag{2.10}$$

where $\dot{\phi}$ is the kinetic term and $V(\phi)$ is the potential term. In order to fulfills the inflationary condition of a negative pressure, from equation (2.10) we assume that potential dominates over the kinetic energy. Then, inflation will exist as long as this condition is met. Replacing these equation in equations (1.9), (1.10) and (1.11) gives the equations of motion of a FLRW Universe during inflation governed by a scalar field,

$$H^2 = \frac{8\pi G}{3}\left[V(\phi) + \frac{1}{2}\dot{\phi}^2\right], \tag{2.11}$$

$$\frac{\ddot{a}}{a} = -\frac{8\pi G}{3}\left[\dot{\phi}^2 - V(\phi)\right], \tag{2.12}$$

$$\ddot{\phi} + 3H\dot{\phi} + \frac{\partial V}{\partial \phi} = 0. \tag{2.13}$$

The equation (2.11) is the Friedmann equation of the field. The equation (2.13) has a similar form with the Klein-Gordon equation (2.6), but with an extra term, $3H\dot{\phi}$, which is known as the friction term.

2.1.1 Inflation to solve the flatness problem

From the above explanation about the phases of the Universe evolution, it is necessary remarks the importance of the introduced Phase II corresponding to $t_i < t < t_{end}$, because as explained, in the Cosmological Standard Model this phase can be omitted. But, it is important when we introduce inflation, then in this phase we consider a new fluid, with an unknown equation of state ω which dominates the energy density of the Universe at this epoch. The point where both phases are connecting is t_{end}, then the possible redshift that can be reached is z_{end}. So, we need to rewrite z_{end} in terms of z_i and from equation (1.22) we get

$$\Omega_T(z_{end}) \simeq \frac{\Omega_X^{ini}}{\Omega_X^{ini} - (\Omega_T^{ini} - 1)\left(\frac{a_{ini}}{a_{end}}\right)^{-1-3\omega}}, \tag{2.14}$$

where X represent the matter that governs the inflation at this epoch. Or in terms of the inflationary condition, we get

$$\Omega_T(z_{end}) \simeq 1 - \frac{\Omega_T(z_{ini}) - 1}{\Omega_X^{ini}} e^{-N_T|1+3\omega|}. \tag{2.15}$$

Using the result of equation (1.24) we get the condition to solve the flatness problem,

$$\Omega_T^0 - 1 = \frac{\Omega_T(z_{ini}) - 1}{\Omega_X^{ini}}\; e^{-N_T|1+3\omega|}\; \left[(z_{end}+1)\;\Omega_m^o + (z_{end}+1)^2\;\Omega_{rad}^o\right] \lesssim 0.003. \tag{2.16}$$

Assuming $\Omega_T^i \approx \Omega_T^0$ which implies no fine-tuning, then the amount of inflation or e-foldings necessary to the flatness problem is,

$$N \gtrsim \frac{-1}{1+3\omega}\; ln\left[(z_{end}+1)\;\Omega_m^0 + (z_{end}+1)^2\;\Omega_{rad}^0\right]. \tag{2.17}$$

Considering $\omega \approx -1$ and $z_{end} \approx 10 \times 10^{27}$ given by the Grand Unified Theory scale[7], we get $N \gtrsim 60$. Then, if inflation provides at least 60 e-foldings, we do not assume any fine-tuning.

2.1.2 Inflation to solve the horizon problem

In order to explain how inflation solve the horizon problem we are going to use again the Phase II where the behaviour of the Universe was dominated by a fluid with a equation of state ω. Then, the angular size of the angular connected sphere (1.25) is,

$$\delta\theta = \left[\int_0^{t_i} \frac{d\tau}{a(\tau)} + \int_{t_i}^{t_{end}} \frac{d\tau}{a(\tau)} + \int_{t_{end}}^{t_{eq}} \frac{d\tau}{a(\tau)}\right]\left[\int_{t_{eq}}^{t_0} \frac{d\tau}{a(\tau)}\right]^{-1}, \tag{2.18}$$

but we can vanish the first and third integrals of the numerator due to they give small contribution to the angular size, then

$$\delta\theta \simeq \left[\int_{t_i}^{t_{end}} \frac{d\tau}{a(\tau)}\right]\left[\int_{t_{eq}}^{t_0} \frac{d\tau}{a(\tau)}\right]^{-1}. \tag{2.19}$$

For calculating the integrals we make the same procedure than in subsection above, then for solve the numerator we use the scale factor defined by (1.14) changing f by i, then

$$\begin{aligned}\int_{t_i}^{t_{end}} \frac{\mathrm{d}\tau}{a(\tau)} &= \frac{1}{a_i}\int_{t_i}^{t_{end}} \left[\frac{3}{2}(1+\omega)\, H_i\, (\tau - t_i) + 1\right]^{-\frac{2}{3(1+\omega)}}, \\ &= \left[\frac{2}{(1+3\omega)\, H(\tau)\, a(\tau)}\right]_{t_i}^{t_{end}}, \\ &= \frac{2}{1+3\omega}\left(\frac{1}{a_{end}H_{end}} - \frac{1}{a_i H_i}\right), \\ &= \frac{2}{a_i H_i\,(1+3\omega)}\left[\left(\frac{a_{end}}{a_i}\right)^{\frac{1+3\omega}{2}} - 1\right].\end{aligned} \tag{2.20}$$

And using the relation,

$$\frac{1}{a_i H_i} = \frac{1}{a_0 H_0}\, \frac{a_i^{(1+3\omega)/2}}{a_0^{1/2}\, a_{eq}^{1/2}\, a_{end}^{(3\omega-1)/2}}, \tag{2.21}$$

we get,

$$\int_{t_i}^{t_{end}} \frac{\mathrm{d}\tau}{a(\tau)} = \frac{2}{a_0 H_0\,(1+3\omega)}\, \frac{a_{end}}{a_0^{1/2}\, a_{eq}^{1/2}}\left[1 - \left(\frac{a_i}{a_{end}}\right)^{\frac{1+3\omega}{2}}\right]. \tag{2.22}$$

Then, we get an expression for equation (2.19), imposing $\delta\theta \geqslant 2\pi$ to solve the horizon problem, then

$$\delta\theta = \frac{a_{end}}{a_0^{1/2}\, a_{eq}^{1/2} - a_{eq}}\, \frac{1}{(1+3\omega)}\left[1 - e^{-\frac{N_T}{2}(1+3\omega)}\right] \geqslant 2\pi, \tag{2.23}$$

where N_T is the total number of e-foldings during inflation given by,

$$N_T \geqslant \frac{-2}{1+3\omega}\, ln\left[\frac{-2\pi\,(1+3\omega)}{z_{eq}+1}\, \left(\sqrt{z_{eq}+1} - 1\right)\, (z_{end}+1)\right]. \tag{2.24}$$

In the same way as in the above subsection, we consider $z_{end} \approx 10^{27}$, $z_{eq} \approx 1089$ and $\omega \approx -1$, giving $N_T \gtrsim 60$. Thus, if inflation provides at least 60 e-foldings the horizon problem vanishes.

2.2 Slow-Roll Approximation

This section should be part of the inflation itself because the slow-roll approximation is the standard strategy for analyzing this process, however we present as a separated section due to the importance for this work. The slow-roll approximation states that the scale factor $\phi(t)$ decays slowly along the time, by consequence the potential energy would be governs over the kinetic energy, given the slow-roll condition,

$$V(\phi) >> \dot{\phi}^2. \tag{2.25}$$

This condition is easily seen if we define an effective field ω as the quotient between pressure and density, thus

$$\omega = \frac{p}{\rho} = \frac{\frac{1}{2}\dot{\phi}^2 - V(\phi)}{\frac{1}{2}\dot{\phi}^2 + V(\phi)}. \tag{2.26}$$

Thus, for inflation occurs $\omega \approx -1$, then the condition (2.25) is satisfied. By consequence from this condition, we add that the potential must be flat enough to dominate the time required to solve the fine-tuning problems,

$$\ddot{\phi} << V_{\phi}. \tag{2.27}$$

Physically, it is understood as a slow slide through the potential, which will reach a minimum where inflation will end through a process of overheating, giving place to the epoch dominated by radiation. Applying the conditions evidenced in equations (2.25) and (2.27) to the equations of motion (2.11) and (2.13), we have,

$$H^2 \simeq \frac{8\pi G}{3} V(\phi), \tag{2.28}$$

$$3H\dot{\phi} \simeq -\frac{\partial V}{\partial \phi}. \tag{2.29}$$

This kind of approximation is quantified by the slow-roll parameters, ϵ and η[4], given by

$$\epsilon \equiv \frac{1}{16\pi G}\left(\frac{V'}{V}\right)^2, \tag{2.30}$$

$$\eta \equiv \frac{1}{8\pi G}\frac{V''}{V}, \tag{2.31}$$

where ϵ measures the slope of the potential and η the curvature. Both are dimensionless quantities. The inflation takes place when,

$$\epsilon << 1, \qquad \textit{and} \qquad |\eta| << 1. \tag{2.32}$$

Thus, the inflation ends when $\epsilon \to 1$ and $\eta \to 1$. In the same way, applying the slow-roll approximation the amount of inflation (2.5), can be rewritten as

$$N\left[\phi(t)\right] \simeq \int_{\phi_{end}}^{\phi} \frac{V}{V'}\, d\phi. \tag{2.33}$$

The behaviour of the Universe after the inflationary process is well described by Kofman, Linde and Starobinsky [14]. This process is known as reheating, and is a transition period between inflation and radiation epochs. Technically, reheating occurs when the potential loses energy (until be less than kinetic energy) and reaches the minimum energy, then the scalar field rapidly decays into 4 particles, or into other bosons which in turn will decay into other particles [14]. This soup of particles will interact between them to get a thermal equilibrium state at T_r, known as reheating temperature.

2.3 Inflationary Models

Now we explore some inflationary models used to try to explain inflation in order to highlight the importance of the Starobinsky model which is the used in the present work.

The first idea about inflationary mechanisms was proposed by Starobinsky [13] in 1980, although it went unnoticed. His model was based on quantum corrections to the gravity. One year after, Guth proposed the first model [8] based specifically to solve the fine-tuning problems. The model is known as old-inflation and consists in a first-order phase transition of the field, from a false vacuum with non zero energy to a true vacuum with zero energy by a quantum tunneling effect. This model has a problem because the inflation does not end properly. To solve it, the new-inflation model is proposed both by Linde [15] and Albrecht and Steinhardt [16] in separated works. They rely on a second order phase transition assuming that the scalar field is in thermal equilibrium. Two years after, in 1983 Linde proposed another model called the chaotic inflation [17] which is a extremely simple potential. There are also hybrid models with more than one scalar field

2.3.1 Starobinsky model

The Starobinsky model [13] was first proposed to resolve the singularity that gives rise to the Big Bang, through quantum corrections to gravity at an early age. Thus, the Hilbert-Einstein action is modified by adding a quadratic term on the curvature scalar, given the action

$$S = \frac{1}{2}\int d^4x \, \sqrt{-g}\, \left(M_{pl}^2\, R + \frac{1}{6M^2}\, R^2\right), \tag{2.34}$$

where M is a parameter with mass dimension and R is the curvature scalar. The quadratic term, under specific conditions, can acts as a cosmological constant giving rise to an accelerated unstable expansion which will decay giving way to the standard evolution of the universe. This case does not present a scalar field, the effective inflaton is given purely by geometry. The importance of this model is that for large field values, some other models take the Starobinsky form [18].

After to write the above equation in terms of the Einstein frame by means of the conformal transformation [18],

$$g_{\mu\nu} \rightarrow \overline{g}_{\mu\nu} = e^{-\sqrt{\frac{2}{3}}\frac{\phi}{M_{pl}}}\, g_{\mu\nu}, \tag{2.35}$$

we arrive to an action of the form

$$S = \int d^4x \, \sqrt{-g}\, \left[\frac{M_{pl}^2}{2}\, R - \frac{1}{2}\, \partial_\mu\phi\, \partial^\mu\phi - \frac{3}{4}\, M_{pl}^4\, M^2 \left(1 - e^{-\sqrt{2/3}\,\phi/M_{pl}}\right)^2\right], \tag{2.36}$$

which is equivalent to the effective potential

$$V(\phi) = \frac{3}{4}M^2\, M_{pl}^2\left(1 - e^{-\sqrt{\frac{2}{3}}\,\frac{\phi}{M_{pl}}}\right)^2. \tag{2.37}$$

Figure 2.1 represents the graphical behaviour of this expression. In the limit, when $\frac{\phi}{M_{pl}} >> 1$ the potential has the form $V(\phi) = \frac{3}{4}M^2 M_{pl}^2$ where we realize that it is flat enough to have slow-roll. On the other hand, when $\frac{|\phi|}{M_{pl}} << 1$ the potential has the form of the harmonic potential $V(\phi) = \frac{1}{2}M^2\phi^2$, indicating that at the origin the potential oscillates giving place to the reheating.

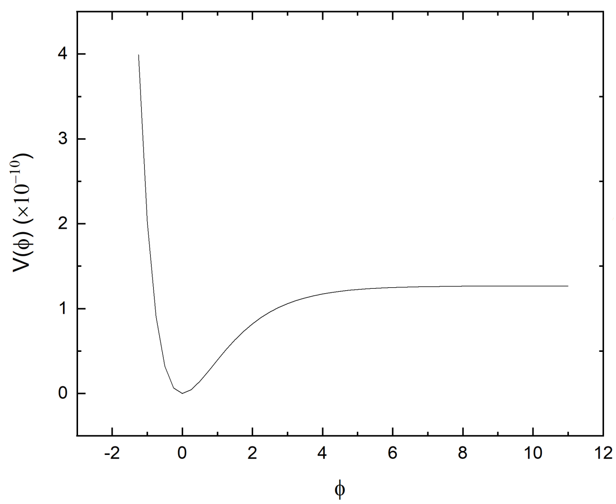

Figure 2.1: Starobinsky potential

2.4 Cosmological Perturbations Theory

The cornerstone of the Cosmology is to understand how the large-scale structure of the Universe was formed. Inflation can explain it because generates primordial perturbations, besides solving the fine-tuning problems and respond to the existence of small temperature fluctuations in the CMB.

The idea of an homogeneous and isotropic Universe is valid only to large scales. For small scales, there are deviations that tend to grow due to the attractive nature of gravity. In the moment when the CMB was formed these inhomogeneities were of the order of 10^{-5}[4]. Then, due to their small amplitude we can treat these inhomogeneities as linear perturbations of the FLRW universe. By consequence, every variable which depends of the space-time could be separated in a homogeneous part which depends only on the cosmic time, and a perturbation which contains the variations with respect to the background, where the perturbations would be much less than the homogeneous part.

Classically, the perturbations theory arise after to apply an inflationary model to General Relativity, so we can think that the perturbed quantities are the matter and the geometry (metric) of the Universe studied. Then, the cosmological inflationary model which is a deep branch of GR gives two spectres both from scalar perturbations which is also known as density perturbations and from tensor perturbations which are also known as gravitational waves. Then, the Einstein equations takes the form,

$$G^{(0)}_{\mu\nu} + \delta G_{\mu\nu} = 8\pi G \left[T^{(0)}_{\mu\nu} + \delta T_{\mu\nu} \right], \tag{2.38}$$

where the superscript $^{(0)}$ corresponds to the background part which fulfill the Einstein equation. Then, we get only an equation for perturbations of the form

$$\delta G_{\mu\nu} = 8\pi G \delta T_{\mu\nu}. \tag{2.39}$$

Classical Perturbations

We start with a classical treatment of perturbations on General Relativity. First we go with the geometry and according to GR, the geometry of the Universe is described by the metric. In the same way as before the metric perturbations is given by $\delta g_{\mu\nu} = g^{(0)}_{\mu\nu} + \delta g_{\mu\nu}$. Cosmological perturbations can have different origin, they could be scalar, vector, and tensor perturbations. In a linear order these perturbations evolve independently, so that they can be studied separately. Vector perturbations have just vanishing solutions, for this reason its study is not relevant. Then, the most general lineal perturbation of the FLRW metric can be expressed as[19],

$$ds^2 = a^2 \left[-(1 + 2A)d\eta^2 + 2B_i dx^i d\eta + (\delta_{ij} + h_{ij})dx^i dx^j \right], \tag{2.40}$$

where η is the conformal time related to the cosmic time by $\eta = \int \frac{\mathrm{d}t}{a(t)}$. The latin indices are spatial indices. A is a scalar and B_i is a spatial vector. δ_{ij} is the spatial metric. And, h_{ij} is the symmetric tensor. If scalar perturbations are analyzed we realize that it presents instabilities that give rise to the current structures of the Universe. On the other hand, tensor perturbation does not present instabilities leading to gravitational waves that are not coupled to inhomogeneities. For this work, we will be mainly interested in **the metric** with only tensor perturbations which is characterized by $h_{00} = -1$, $h_{0i} = 0$ and spatial elements[20],

$$h_{ij} = a^2 \begin{pmatrix} 1 + h_+ & h_\times & 0 \\ h_\times & 1 - h_+ & 0 \\ 0 & 0 & 1 \end{pmatrix}, \tag{2.41}$$

where h_+ and $h_\times$ are two functions, assumed small. In order to be consistent we must specify that the perturbation occurs in the $x-y$ plane, advantageously obtaining the wavevector $\vec{k}$ in the $z-axis$. h_+ and $h_\times$ also are two components of a divergenless, traceless ans symmetric tensor. The pertrubation tensor of spatial elements also can be written as

$$\mathcal{H}_{ij} = \begin{pmatrix} h_+ & h_\times & 0 \\ h_\times & -h_+ & 0 \\ 0 & 0 & 0 \end{pmatrix}, \tag{2.42}$$

where divergenless means that $k^i \mathcal{H}_{ij} = k^j \mathcal{H}_{ij} = 0$.

Thus, we start to calculate the **Christoffel symbols** defined by

$$\Gamma^{\mu}_{\alpha\beta} = \frac{1}{2} g^{\mu\nu} \left(g_{\mu\nu,\beta} + g_{\mu\nu,\alpha} - g_{\alpha\beta,\mu} \right). \tag{2.43}$$

The explicit calculation is reported in Appendix A, then the obtained results from metric (2.41) are

$$\begin{aligned}
\Gamma^0_{00} &= 0, \\
\Gamma^0_{i0} &= \Gamma^0_{0i} = 0, \\
\Gamma^i_{00} &= 0, \\
\Gamma^0_{ij} &= Hg_{ij} + \frac{a^2\mathcal{H}_{ij,0}}{2}, \\
\Gamma^i_{0j} &= \Gamma^i_{j0} = H\delta_{ij} + \frac{1}{2}\mathcal{H}_{ij,0}, \\
\Gamma^i_{jk} &= \frac{i}{2}\left[k_k\mathcal{H}_{ij} + k_j\mathcal{H}_{ik} + k_i\mathcal{H}_{jk}\right].
\end{aligned} \tag{2.44}$$

Now, we calculate the **Ricci tensor** using

$$R_{\mu\nu} = \Gamma^\alpha_{\mu\nu,\alpha} - \Gamma^\alpha_{\mu\alpha,\nu} + \Gamma^\alpha_{\beta\alpha}\Gamma^\beta_{\mu\nu} - \Gamma^\alpha_{\beta\nu}\Gamma^\beta_{\mu\alpha}, \tag{2.45}$$

and the results are

$$\begin{aligned}
R_{00} &= -3\frac{\ddot{a}}{a}, \\
R_{0i} &= 0, \\
R_{ij} &= g_{ij}\left(\frac{\ddot{a}}{a} + 2H^2\right) + \frac{3}{2}a^2H\mathcal{H}_{ij,0} + a^2\frac{\mathcal{H}_{ij,00}}{2} + \frac{k^2}{2}\mathcal{H}_{ij}.
\end{aligned} \tag{2.46}$$

And the **Ricci scalar** defined by

$$\mathcal{R} = g^{00}R_{00} + g^{ij}R_{ij}. \tag{2.47}$$

Then, the obtained result is

$$\mathcal{R} = 0, \tag{2.48}$$

which means that tensor perturbations do no affect at first order the Ricci scalar.

With these results we can form the **Einstein tensor** which will be given by

$$G^\mu_\nu = R^\mu_\nu - \frac{1}{2}g^\mu_\nu\mathcal{R}, \tag{2.49}$$

due to the result (2.48). Then, the Einstein tensor perturbations are,

$$\begin{aligned}
\delta G^0_0 &= 0, \\
\delta G^i_0 &= 0, \\
\delta G^i_j &= \delta^{ik}\left[\frac{3}{2}H\mathcal{H}_{kj,0} + \frac{\mathcal{H}_{kj,00}}{2} + \frac{k^2}{2a^2}\mathcal{H}_{kj}\right],
\end{aligned} \tag{2.50}$$

Now, we proceed to analyze the perturbations in the matter. As we know from Einstein theory, the matter of the Universe is described by the energy-momentum tensor. Then, the matter perturbations is given by the right hand side of the equation (2.39). **The energy-momentum tensor** for a Universe with a single scalar field minimally coupled to gravity is given by equation (2.8). During inflation, due to the inflaton dominates over the other components, the perturbation are primarily limited to the scalar field, given

$$\phi(t, \vec{x}) = \phi_0(t) + \delta\phi(t, \vec{x}). \tag{2.51}$$

The potential is also expanded around the background using Taylor expansion,

$$V(\phi) = V(\phi_0) + \delta\phi\; V_{,\phi}. \tag{2.52}$$

Then, how we have been working the time-time and the time-space parts vanish, and also the space-space part vanishes. Thus, the tensor perturbation of the energy-momentum tensor is null because the anisotropic stress tensor has negligible linear order components in the perturbations.

Thus, the **equations of motion** for perturbations are obtained using the results of time-time, time-space and space-space components and replacing in (2.39). For this case we have only the space-space components $\delta G^i_j = 0$, then we will work on it in order to analyze the evolution of the tensor variables, h_+ and $h_\times$.

We start with h_+ component, so let us consider a difference as follow $\delta G^1_1 - \delta G^2_2$. We should realize that $\mathcal{H}_{11} = -\mathcal{H}_{22} = h_+$ due to the proportionality of $\mathcal{H}_{ij}$ and its derivatives in equation (2.50), therefore

$$\delta G^1_1 - \delta G^2_2 = 3Hh_{+,0} + h_{+,00} + \frac{k^2 h_+}{a^2}. \tag{2.53}$$

Then, the right-hand side of this components of the Einstein's equations is zero. And considering the same procedure for $h_\times$, we have

$$\ddot{h}_\alpha + 3H\dot{h}_\alpha + \frac{k^2}{a^2} h_\alpha = 0, \tag{2.54}$$

where $\alpha = +, \times$. If we change to the conformal time, we can rewrite the expressions $\dot{h}_\alpha = h'_\alpha/a$ and $\ddot{h}_\alpha = h''_\alpha/a^2 - (a'/a^3)h'_\alpha$. Then,

$$h''_\alpha + 2\mathcal{H}h'_\alpha + k^2 h_\alpha = 0, \tag{2.55}$$

where $\mathcal{H}$ is the conformal Hubble constant defined by $\mathcal{H} = a'/a$. Both equations (2.54) and (2.55) are wave equations and are called *gravity waves*. When the universe expands, the amplitude of a gravity wave described by euqation (2.55) falls off once the mode enters the horizon [20].

Quantum Perturbations

Until this point, we have reviewed the classical perturbations applied to GR and obtain an equation of motion which describes the tensor behaviour of the universe studied. But, during inflation the components that occupy the majority of the universe were a uniform scalar field and a uniform background metric. Then, we must also analyze quantum perturbations of the field against the background. In order to do it, we are going to follow the treatment present by Dodelson [20].

We proceed to quantize the equation of classical perturbations (2.55), so we must rewrite this equation in the form of a harmonic oscillator, then we define,

$$\tilde{h} \equiv \frac{ah}{\sqrt{16\pi G}}. \tag{2.56}$$

If we derive this equation one and two times with respect to the conformal time, we have, respectively,

$$h' = \sqrt{16\pi G}\left(\frac{\tilde{h}'}{a} - \frac{a'}{a^2}\,\tilde{h}\right), \tag{2.57}$$

$$h'' = \sqrt{16\pi G}\left(\frac{\tilde{h}''}{a} - 2\frac{a'}{a^2}\,\tilde{h}' - \frac{a''}{a^2}\,\tilde{h} + 2\frac{(a')^2}{a^3}\,\tilde{h}\right). \tag{2.58}$$

Inserting both results in equation (2.55) and multiplying by $1/\sqrt{16\pi G}$, gives

$$\frac{\tilde{h}''}{a} - 2\frac{a'}{a^2}\,\tilde{h}' - \frac{a''}{a^2}\,\tilde{h} + 2\frac{(a')^2}{a^3}\,\tilde{h} + 2\frac{a'}{a}\left(\frac{\tilde{h}'}{a} - \frac{a'}{a^2}\,\tilde{h}\right) + \frac{k^2}{a}\,\tilde{h} = 0, \tag{2.59}$$

simplifying and sorting out the equation, we have

$$\frac{1}{a}\left[\tilde{h}'' + \left(k^2 - \frac{a''}{a}\right)\tilde{h}\right] = 0. \tag{2.60}$$

This equation already has the form of a harmonic oscillator, it means that there are no damping terms ($\propto \dot{\tilde{h}}$), then h becomes a quantum operator described as follow,

$$\hat{\tilde{h}}(\vec{k},\eta) = v(k,\eta)\,\hat{a}_{\vec{k}} + v^*(k,\eta)\,a^\dagger_{\vec{k}}, \tag{2.61}$$

where the coefficients of the creation and annihilation operators satisfy this equation,

$$v_k'' + \left(k^2 - \frac{a''}{a}\right) v_k = 0. \tag{2.62}$$

In order to solve this equation, we consider the slow-roll regime and use the following transformation, $a' = a^2 H \simeq -a/\eta$, therefore

$$\frac{a''}{a} \simeq a\,\frac{1}{a}\frac{d}{d\eta}\left(\frac{a}{\eta}\right) \simeq \frac{2}{\eta^2}. \tag{2.63}$$

So the equation (2.62) is rewritten,

$$v_k'' + \left(k^2 - \frac{2}{\eta^2}\right) v_k = 0. \tag{2.64}$$

This is the Mukhanov-Sasaki equation.

2.4.1 Power Spectrum of Perturbations

In order to get the power spectrum we must solve the Mukhanov-Sasaki equation (2.64). The solution obtained is

$$v_k = \frac{e^{-ik\eta}}{\sqrt{2k}}\left(1 - \frac{i}{k\eta}\right). \tag{2.65}$$

If we consider the initial condition when the perturbation v_k is at very early times before inflation occurs that is when $k|\eta| >> 1$, so the term k^2 will dominates and the equation behaves like a harmonic oscillator. The solutions would take the form,

$$v_k(k,\eta) \rightarrow \frac{e^{-ik\eta}}{\sqrt{2k}}. \tag{2.66}$$

On the other hand, in the super-horizon scales, $k|\eta| << 1$, which corresponds to time when inflation has worked for many e-folds. Then, the solution takes the form,

$$v_k(k,\eta) \rightarrow -\frac{e^{-ik\eta}}{\sqrt{2k}}\frac{i}{k\eta}. \tag{2.67}$$

With the before results in mind, we continue to calculate the power spectrum of the tensor perturbations. For this we return to the equation (2.64) and analogously to the harmonic oscillator, we state the variance of the perturbations in the $\tilde{h}$ field,

$$\begin{aligned} \langle \hat{\tilde{h}}^{\dagger}(\vec{k},\eta)\ \hat{\tilde{h}}(\vec{k}',\eta)\rangle &= |v(\vec{k},\eta)|^2\ (2\pi)^3\ \delta^3(\vec{k}-\vec{k}'), \\ &\equiv (2\pi)^3\ \mathcal{P}_h(k)\ \delta^3(\vec{k}-\vec{k}'), \end{aligned} \tag{2.68}$$

where $\mathcal{P}_h$ would be the *power spectrum* of the primordial perturbations to the metric which is defined by,

$$\mathcal{P}_h(k) = \frac{16\pi G}{a^2}\ |v(k,\eta)|^2. \tag{2.69}$$

Reminding the result (2.66) and considering that the primordial power spectrum which scales as $|v|^2/a^2$, must be constant in time after inflation stretched the mode beyond the horizon. Then,

$$\begin{aligned} \mathcal{P}_h(k) &= \frac{16\pi G}{a^2}\frac{1}{2k^3\eta^2}, \\ &= \frac{8\pi G H^2}{k^3}. \end{aligned} \tag{2.70}$$

This is the expression for primordial power spectrum for gravity waves. We can detect these waves measuring the Hubble rate during inflation. The power spectrum is both for h_+ and $h_\times$ separately, these are uncorrelated.

If we use the results of the comovil curvature perturbation $\mathcal{R}$ which is related to v by $\mathcal{R} = (v/a)(H/\dot{\phi})$, Then, the explicit power spectrum of the curvature perturbation for large scale is

$$\mathcal{P}_{\mathcal{R}}(k) = \mathcal{P}_v(k)\left(\frac{H}{a\dot{\phi}}\right)^2 \simeq \frac{1}{4\phi^2}\left(\frac{H^2}{\dot{\phi}}\right)^2\Bigg|_{k=aH}. \tag{2.71}$$

Because $\mathcal{R}$ remains constant at large scales in the slow-roll regime, when disturbances cross the Hubble horizon during inflation, so $\mathcal{R}$ determines the amplitude of the disturbances at the instant before the modes re-enter the determining horizon. Finally, we write the power spectrum of tensor perturbation, taking account the result (2.70),

$$\mathcal{P}_{\mathcal{T}}(k) = 2\times 4k^2 \times \left(\frac{H}{2\pi}\right)^2\Bigg|_{k=aH}, \tag{2.72}$$

where the factor 2 indicates the two polarizations associated with gravitational waves (h_+ and $h_\times$) and $2k^2$ is a normalization constant.

Finally, we define some values which characterize the spectrum. The dependence in the scale is characterized by the *tensor spectral index* defined by,

$$n_T(k) \equiv \frac{d\ ln\ \mathcal{P}_\mathcal{T}(k)}{d\ ln\ k}. \tag{2.73}$$

A spectrum independent of scale occurs when $n_s = 1$. And also, we define the ratio between the amplitude of the spectrum of tensor to scalar perturbations known as *the tensor-to-scalar ratio* given by,

$$r = 8\frac{\mathcal{P}_\mathcal{T}}{\mathcal{P}_\mathcal{R}}. \tag{2.74}$$

These three last results are measurable quantities by space satellites and serve to statistically characterize the anisotropies in the cosmic background radiation.

2.5 Uniform Approximation Method

Another way to solve the Mukhanov-Sasaki equation (2.64) is the uniform approximation method, from which we obtain the approximate solutions for linear second-order differential equations, such as MS equation. The best and most known example of an equation of this style is the Schrodinger equation. Herein lies the contribution of this work to the scientific community because a relatively unexplored method is applied to be compared with the results of the standard procedure in inflationary cosmology and with the observed results, with a specific form of the potential described by equation (2.37).

Coming back to the method, the theoretical development was carry out by Berry [21] and will be presented below based on it. The method consists of comparing equations, an unknown one which can have the following form,

$$\frac{d^2\Psi(x)}{dx^2} + \chi(x)\ \Psi(x) = 0, \tag{2.75}$$

with a known equation of the form,

$$\frac{d^2\Phi(\sigma)}{d\sigma^2} + \Gamma(\sigma)\ \Phi(\sigma) = 0, \tag{2.76}$$

where we will choose that $\Gamma(\sigma)$ is similar to $\chi(x)$, therefore the wavefunctions $\Phi(\sigma)$ and $\Psi(x)$ will be also similar between them, and are related by a mapping function described by,

$$\Psi(x) = f(x)\ \Phi(\sigma(x)). \tag{2.77}$$

If we replace equation (2.77) in (2.75), and we realize that $f = \left(\frac{d\sigma}{dx}\right)^{-1/2}$, we get that,

$$\frac{d\sigma}{dx} \simeq \left[\frac{\chi(x)}{\Gamma(\sigma)}\right]^{1/2}. \tag{2.78}$$

Then, the approximate solution to (2.75) is given by,

$$\Psi(x) \simeq \left[\frac{\Gamma(\sigma)}{\chi(x)}\right]^{1/4}\ \Phi(\sigma). \tag{2.79}$$

This solution is valid for the entire range of x including the turning points. In the semiclassical limit, all problems that have the same classical turning point structure are equivalent.

Chapter 3

Results & Discussion

In this section we are going to present all the results obtained using the Starobinsky potential (2.37), and the comparison of those results with the experimental data. The section will be separated in two parts, background dynamics of the universe and the power spectrum of all the methods used. The first part shows the background dynamics of the universe during inflation and how the variables a and ϕ which dominate the behavior during that time evolve. The second part presents the power spectra obtained from the numerical method, the uniform approximation method and the slow-roll approximation with their respective spectral index, also presents the comparison of the results. All the results presented below have been calculated using the software Mathematica 12.0.

3.1 Background dynamics

In order to illustrate the dynamic behaviour of the universe during inflation we must solve the Einstein Field Equations with the slow-roll procedure, then the equations that we have to solve are Eqs. (2.28) and (2.29). To do it, we must set an important initial condition, the scale factor at time $t = 0$ to be $a(0) = 1$ which means that at that time the universe is not expanding. Then, we can calculate the background behaviour of the scalar field ϕ and the dynamics of the scale factor a presented below, respectively,

$$\phi(t) = \sqrt{\frac{3}{2}}\, ln\left[e^{\sqrt{\frac{2}{3}}\phi_0} - \frac{2}{3}Mt\right], \tag{3.1}$$

$$a(t) = exp\left[\frac{1}{2}Mt + \frac{3}{4}\, ln\left(\frac{1}{3}e^{-\sqrt{\frac{2}{3}}\phi_0}\right) + \frac{3}{4}\, ln\left(3e^{\sqrt{\frac{2}{3}}\phi_0} - 2Mt\right)\right]. \tag{3.2}$$

Also, we can calculate the slow-roll parameter, ϵ and δ which takes the form,

$$\epsilon(t) = \frac{4}{3\left[e^{\sqrt{\frac{2}{3}}\phi_0} - 1\right]^2}, \tag{3.3}$$

$$\delta(t) = -\frac{4\left(e^{\sqrt{\frac{2}{3}}\phi_0} - 2\right)}{3\left(e^{\sqrt{\frac{2}{3}}\phi_0} - 1\right)^2}. \tag{3.4}$$

As we already studied before, the end of inflation occurs when $\epsilon = 1$, then, using Starobinsky model, the end of inflation occurs at

$$\phi_{end} = \sqrt{\frac{3}{2}}\,\log\left(\frac{2}{\sqrt{3}} + 1\right). \tag{3.5}$$

This results will be considered when we study the power spectrum of tensor perturbations in the last section of this chapter.

Another necessary initial condition in order to plot the obtained results is the value of M which appears in the Starobinsky potential and making use of the work of Mishra et al.[22], we define $M = 1.3 \times 10^{-5}$. This condition and the condition of the value of the scale factor at $t = 0$ defined before must be able to provide enough inflation for the horizon and flatness problems to be solved. The results of ϕ and a are presented in the Figure 3.1 and 3.2 respectively. Figure 3.1 shows the behaviour of the inflaton and how this field is rolling through the potential during inflation takes place. The end of inflation is described by (3.5) and using (3.1) we compute that the end of the Starobinsky inflationary model occurs at $t_{end} = 9.63049 \times 10^6$, after this time the field starts to oscillate. Figure 3.2 shows the evolution of the scale factor which behaves like an exponential function because during the inflation it is increasing rapidly which guarantees the accelerating universe expansion.

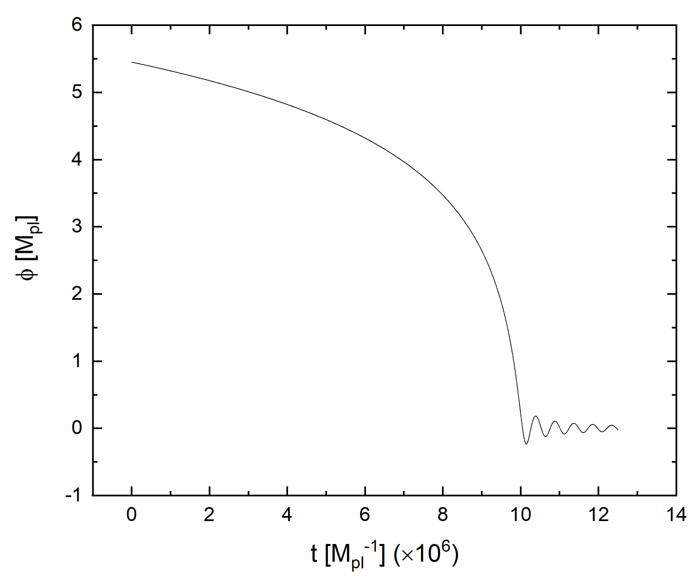

Figure 3.1: Behaviour of the background inflation field ϕ vs the cosmic time using the Starobinsky potential.

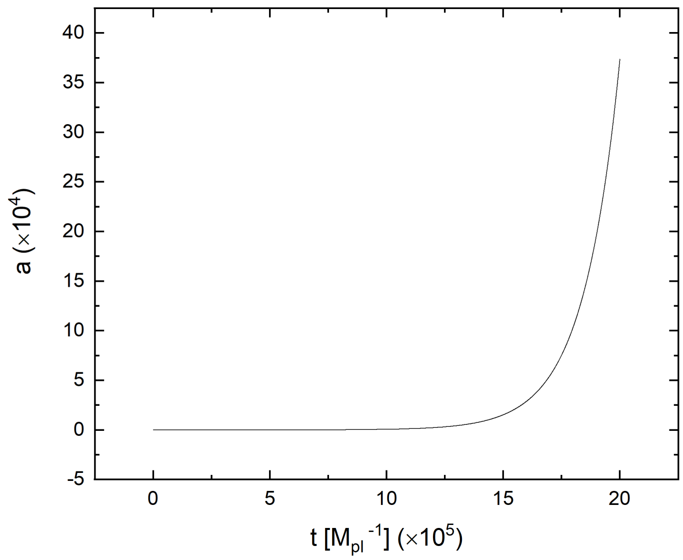

Figure 3.2: Behaviour of the scale factor a vs the cosmic time using the Starobinsky potential.

3.2 Perturbation dynamics

In order to know the dynamics of the perturbation we have to solve the Mukhanov-Sasaki equation (2.64) and thus we obtain the tensor power spectrum, the spectral index and the tensor-to-scalar ratio.

3.2.1 Slow-roll approximation

Also, we can explore the dynamics of the perturbations just knowing the power spectrum and, by the slow-roll approximation, the tensor power spectrum up-to first-order is given by,

$$P_T(k) \simeq \left[1 + (2b-2)\,\epsilon + \left(2b^2 - 2b - 3 + \frac{\pi^2}{2}\right)\epsilon^2\right]\left(\frac{H}{2\pi}\right)^2, \tag{3.6}$$

where $b = 0.729637$ is the Euler constant, and ϵ is described by equation (2.30). Also, the spectral index and the tensor-to-scalar are described, respectively, by[23]

$$n_T(k) \simeq -2\epsilon - 2\epsilon^2, \tag{3.7}$$

$$r \simeq 16\epsilon. \tag{3.8}$$

All the expression must be evaluated at the horizon crossing time which occurs when $k = aH$. We can obtain the k dependence of the tensor power spectrum using different values of k in the range 0.0001 $Mpc^1 \leq k \leq 10\ Mpc^1$. The results will presented in the next section.

3.2.2 Numerical integration

The first method used to solve the equation of tensor perturbations (2.64) is the numerical integration. To set this equation, we are going to use the results of solving equations (2.11) and (2.13). The resulting perturbation v_k is a complex function giving two equations to solve, a real part and an imaginary part. Both equations are solved basically by the same methodology.

The integration is divided in two parts. The first part corresponds when the perturbations are inside the horizon, it means that v_k presents an oscillatory behaviour. For this, we use the result (2.66) as initial condition. For the second part which go from before horizon crossing to roughly three times the horizon crossing, we use the final stage of the above solution as initial condition. Then, the tensor power spectrum is described by (2.69) while the spectral index and the tensor-to-scalar ratio are described by (2.73) and (2.74) respectively.

3.2.3 Uniform approximation

The second method in order to solve the equation (2.64) is the uniform approximation. For this, let's take the Habib's work[24] as a reference. So, before applying the method we rewrite the equation doing the change $a''/a \equiv C^2(\eta)/\eta^2$, then

$$v_k''(\eta) + \left[k^2 - \frac{C^2(\eta)}{\eta^2}\right] v_k(\eta) = 0. \tag{3.9}$$

If we consider that C is constant the solution is given in terms of Bessels functions, this would be the "easy" part. But, if we consider that C is not constant and varies slowly over time we have a problem. In order to solve this problem without forcing C to have any imposition on its behavior, we propose a generic differential equation similar to equation (3.9) represented by

$$\frac{d^2 v}{d\eta^2} = \left[b^2 g(\eta) + q(\eta)\right] v. \tag{3.10}$$

The solution will depend on the behaviour of $g(\eta)$ and $q(\eta)$. When $g(\tilde{\eta}) = 0$, where $\tilde{\eta}$ is a turning point, the solution is given in terms of Airy functions. If $g(\eta)$ has a pole of order $n \geqslant 2$ the Liouville-Green approximation is used. In any case, we must establish the convergence criteria described by

$$g(\eta) = \frac{1}{\eta^2}\left[C^2(\eta) + \frac{1}{4}\right] - k^2 = \frac{v_T^2(\eta)}{\eta^2} - k^2, \tag{3.11}$$

$$q(\eta) = -\frac{1}{4\eta^2}. \tag{3.12}$$

Then, the Mukhanov-Sasaki equation takes the form,

$$v_k'' = \left\{-k^2 + \frac{1}{\eta^2}\left[v_T^2(\eta) - \frac{1}{4}\right]\right\} v_k, \tag{3.13}$$

where $v_T = \left(\frac{a''}{a}\right)\eta^2 + \frac{1}{4}$ and the turning point is at $k^2 = \frac{v_T^2(\tilde{\eta}_T)}{\tilde{\eta}_T^2}$.

Now, we apply the uniform approximation method, starting by defining a new independent variable ξ and a new dependent variable U, given by

$$\xi\left(\frac{d\xi}{d\eta}\right)^2 = g(\eta), \tag{3.14}$$

$$u = \left(\frac{d\xi}{d\eta}\right)^{-1/2} U. \tag{3.15}$$

Then, we have a new equation from 3.10 described by

$$\frac{d^2U}{d^2\xi} = \left[b^2\xi + \psi(\xi)\right]U, \tag{3.16}$$

where, the mapping functions is

$$\psi(\xi) = \left[4g(\eta)g''(\eta) - 5g'^2(\eta)\right]\frac{\xi}{16g^3(\eta)} + \frac{\xi q(\eta)}{g(\eta)} + \frac{5}{16\xi^2}, \tag{3.17}$$

$$\frac{2}{3}\xi^{3/2} = -\int_\eta^{\tilde{\eta}} \sqrt{g(\eta)}\, d\eta \qquad \eta \geqslant \tilde{\eta}, \tag{3.18}$$

$$\frac{2}{3}(-\xi)^{3/2} = -\int_\eta^{\tilde{\eta}} \sqrt{-g(\eta)}\, d\eta \qquad \eta \leqslant \tilde{\eta}. \tag{3.19}$$

Then, we realize that the approximate solution for $v_k(\eta)$ is valid both to the left ($\eta \leqslant \tilde{\eta}_s$) and to the right ($\eta \geqslant \tilde{\eta}_s$) of the turning point. The solution can be written in terms of the Airy functions, where $Ai^{(1)} \equiv Ai$ and $Ai^{(2)} \equiv Bi$, then

$$v_{k\lessgtr}^{(1,2)}(\eta) = f_{\lessgtr}^{1/4}(\eta)\, g_s^{-1/4}(\eta)\, Ai^{(1,2)}\ \left[f_{\lessgtr}(\eta)\right], \tag{3.20}$$

$$f_{\lessgtr}(\eta) = \mp\left\{\pm\frac{3}{2}\int_\eta^{\tilde{\eta}_s} d\eta'\ \left[\mp g_s(\eta')\right]^{1/2}\right\}^{2/3}, \tag{3.21}$$

$$g_s(\eta) \equiv \frac{v_T^2(\eta)}{\eta^2} - k^2, \tag{3.22}$$

where the functions with the subscript $_<$ corresponds to the left side of the turning point, and those with the subscript $_>$ are the right side of the turning point. The solution for $v_k(\eta)$ is in turn a lineal combination of two solutions,

$$v_k(\eta) = A\ v_k^{(1)}(\eta) + B\ v_k^{(2)}(\eta). \tag{3.23}$$

We must fix the coefficients A and B in such a way that satisfy the results at the limit $k \to \infty$ where the solution is $v_k(\eta) = e^{-ik\eta}/\sqrt{2k}$. In this limit, $f_<(\eta)$ is large and negative, then asymptotic form is employed,

$$Ai(-x) = \frac{1}{\pi^{1/2}x^{1/4}}\ cos\left(\frac{2}{3}\ x^{3/2}\right) - \frac{\pi}{4}, \tag{3.24}$$

$$Bi(-x) = -\frac{1}{\pi^{1/2}x^{1/4}}\ sin\left(\frac{2}{3}\ x^{3/2}\right) - \frac{\pi}{4}, \tag{3.25}$$

choosing,

$$A = \sqrt{\frac{\pi}{2}}\, e^{i(\pi/4)}\ , \qquad B = -i\sqrt{\frac{\pi}{2}}\, e^{i(\pi/4)}. \tag{3.26}$$

Obtaining,

$$v_{k,1,<}\,(\eta) = \lim_{-k\eta\to\infty} \frac{C}{\sqrt{2k}}\, exp\left\{ i\, \frac{3}{2}\ [f_<(k,\eta)]^{3/2} \right\}. \tag{3.27}$$

On the other hand, when the limit is $\eta \to 0^-$, the $1/\eta^2$ pole dominates the behaviour of the solution. $f_>(k,\eta)$ becomes large, then we can use the asymtotic form,

$$Ai(x) = \frac{1}{2\sqrt{\pi}}\, x^{-1/4}\, exp\left(-\frac{2}{3}\, x^{2/3} \right), \tag{3.28}$$

$$Bi(x) = \frac{1}{\sqrt{\pi}}\, x^{-1/4}\, exp\left(\frac{2}{3}\, x^{2/3} \right), \tag{3.29}$$

which leads to

$$v_{k,1,>}(\eta) = \lim_{k\eta\to 0^-} -iC \sqrt{\frac{-\eta}{2v_T(\eta)}}\, exp\left\{ \frac{2}{3}\, [f_>(k,\eta)]^{3/2} \right\}, \tag{3.30}$$

where C is an irrelevant constant phase factor, then we only consider the growing to part of the solution in order to compute the power spectra which is given by,

$$P_T(k) = \lim_{-kt\to\infty} \frac{k^3}{2\pi^2} \left| \frac{v_k(t)}{a(t)} \right|^2 . \tag{3.31}$$

Finally, the spectral index and the tensor-to-scalar ratio, as before, are described by (2.73) and (2.74) respectively.

3.3 Power spectrum

We already know how to calculate the tensor power spectrum for any of the three presented methods. It is well known that the enough time for the spectrum to converge for all k modes is at $t = 1 \times 10^6 M_{Pl}^1$ which is the point until we evaluate the power spectrum. Figure 3.3 shows the main result of this work, the behaviour of the power spectrum of the tensor perturbations for the Starobinsky model for different k modes using the three different methods explained before. We can observe that the numerical solution and the second-order slow-roll solution are very close between both. However, the uniform approximation method neither gives a bad result, being acceptable. To corroborate this, we calculate the relative error of each approximation method with respect to the numerical result using the scale $k = 0.05 Mpc^{-1}$. The second-order slow roll approximation deviates in 0.0064% while the second-order uniform approximation deviates in 0.9394%.

Once the power spectrum was analyzed, we proceeded to calculate the tensor spectral index using the definitions described in (2.73). Table 3.1 presents the results obtained for the three different methods and their relative errors with respect to the numerical solution. The relative errors are very close in both cases. Then, we also calculate the tensor-to-scalar ratio described by (2.74) and the results are presented in Table 3.2. For this case, we have found that $r = 0.00271981$ is the value that is closest to the numerical result and corresponds to the second-order uniform

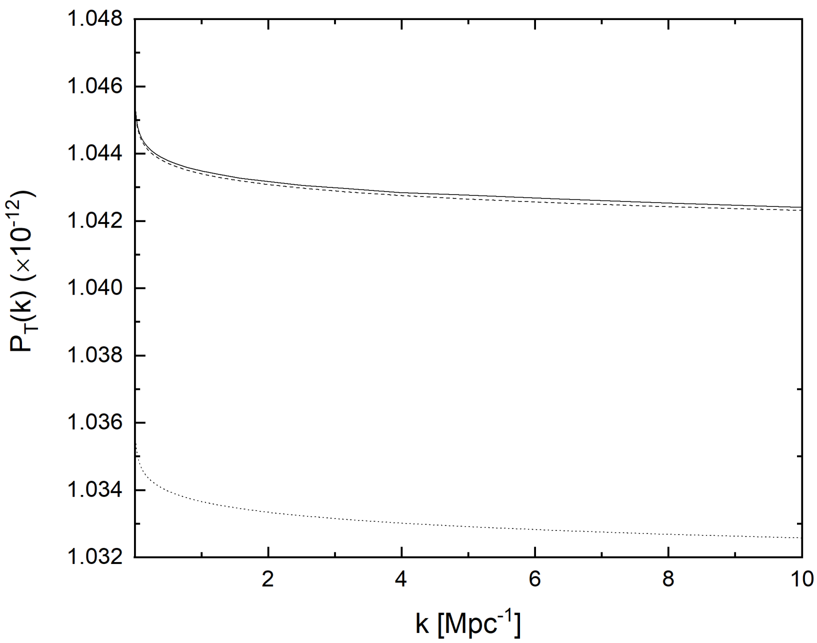

Figure 3.3: The power spectrum of the tensor cosmological perturbations for the Starobinsky inflationary model. Solid line is the numerical result, dashed line is the second-order slow-roll approximation result and the dotted line is the second-order uniform approximation.

approximation. Demonstrating that the use of a semiclassical method to solve the tensor perturbation equation is a good procedure.

Method	$n_T(k)$	rel. error (%)
Numerical integration	-0.000384929	
Second-order slow-roll	-0.000392819	2.049
Second-order uniform approximation	-0.000394894	2.589

Table 3.1: Values of $n_T(k)$ obtained with different methods for the Starobinsky inflationary model at the pivot scale $k = 0.05 Mpc^{-1}$

Method	r(k)	rel. error (%)
Numerical integration	0.00271453	
Second-order slow-roll	0.00281164	3.577
Second-order uniform approximation	0.00271981	0.195

Table 3.2: Values of the tensor-to-scalar ratio $r(k)$ obtained with different methods for the Starobinsky inflationary model at the pivot scale $k = 0.002 Mpc^{-1}$

Chapter 4

Conclusions & Outlook

This work consists of analyzing a model that describes the inflationary process in an almost satisfactory way known as the Starobinsky model, and by means of the specific application of the uniform approximation method to solve the perturbation equation we find the power spectra, and consequently the tensor spectral index for each method which are compared with the observed values by the Planck mission. To better understand what is being discussed we have started with the genesis of the universe, the Big Bang. We introduced this concept and the problems that brought. Also we review an inflationary period and how it solves the fine-tuning problems. The dynamics of the universe during inflation is characterized by the domination of the potential energy over the kinetic part also known as slow-roll approximation. Then, we studied the Starobinsky potential which is, currently, the best theoretical explanation to the observations. Cosmological perturbations were also discussed in their classical and quantum treatments, ending with the Mukhanov-Sasaki equation and its relation with the power spectrum. Also, we present a section to solve the Mukhanov-Sasaki equation applying a semi-classical method known as the uniform approximation, of great importance for this work. Then, the power spectra of tensor perturbation for Starobinsky inflationary model for the three different methods are presented in Figure 3.3 showing that the second-order slow-roll method is the one that best fits the numerical result with a deviation of 0.0064%. If we calculate the spectral index we realize that the deviation is imperceptible in any case. On the other hand, if we calculate the tensor-to-scalar ratio we realize that which fits better to the numerical solution is the second-order uniform approximation method which deviates just 0.195%. All the calculated values are in agreement with the results presented by Planck 2018 results. X[11].

For future developments, we can compute the power spectrum of tensor perturbations using the generalized Starobinsky model. Also, we can study the attractor behaviour of the system.

Appendix A

Explicit calculation of Christoffel symbols

To calculate the Christoffel symbols we are going to use the definition (2.43) and also the definition of the metric (2.41) or (2.42) which are related by,

$$g_{ij} = a^2 \left(\delta_{ij} + \mathcal{H}_{ij}\right). \tag{A.1}$$

We start setting $\alpha = 0$ and $\beta = 0$ giving Γ^{μ}_{00}. Then,

$$\begin{aligned}
\Gamma^{0}_{00} &= \frac{1}{2} g^{0\nu} \left(g_{\nu 0,0} + g_{\nu 0,0} - g_{00,\nu}\right), \\
&= \frac{1}{2} g^{00} \left(g_{00,0} + g_{00,0} - g_{00,0}\right), \\
&= \frac{1}{2} g^{00} \left(g_{00,0}\right), \\
&= 0,
\end{aligned} \tag{A.2}$$

and

$$\begin{aligned}
\Gamma^{i}_{00} &= \frac{1}{2} g^{i\nu} \left(g_{\nu 0,0} + g_{\nu 0,0} - g_{00,\nu}\right), \\
&= \frac{1}{2} g^{ij} \left(g_{j0,0} + g_{j0,0} - g_{00,j}\right), \\
&= \frac{1}{2} g^{ij} \left(-g_{00,j}\right), \\
&= 0.
\end{aligned} \tag{A.3}$$

Now we set $\alpha = i$ and $\beta = 0$ giving Γ^{μ}_{i0}. Then,

$$\begin{aligned}
\Gamma^{0}_{i0} &= \frac{1}{2} g^{0\nu} \left(g_{\nu i,0} + g_{\nu 0,i} - g_{i0,\nu} \right), \\
&= \frac{1}{2} g^{00} \left(g_{0i,0} + g_{00,i} - g_{i0,0} \right), \\
&= \frac{1}{2} g^{00} \left(g_{00,i} \right), \\
&= 0,
\end{aligned} \tag{A.4}$$

and

$$\begin{aligned}
\Gamma^{j}_{i0} &= \frac{1}{2} g^{j\nu} \left(g_{\nu i,0} + g_{\nu 0,i} - g_{i0,\nu} \right), \\
&= \frac{1}{2} g^{jk} \left(g_{ki,0} + g_{k0,i} - g_{i0,k} \right), \\
&= \frac{1}{2} g^{jk} \left(g_{ki,0} \right), \\
&= \frac{1}{2} g^{jk} \left\{ \frac{\partial}{\partial t} \left[a^2 \ (\delta_{ki} + \mathcal{H}_{ki}) \right] \right\}, \\
&= \frac{1}{2} g^{jk} \left[2a\dot{a} \left(\delta_{ki} + \mathcal{H}_{ki} \right) + a^2 \ \left(\mathcal{H}_{ki,0} \right) \right], \\
&= \frac{1}{2} g^{jk} \left[2H \left(g_{ki} \right) + a^2 \ \left(\mathcal{H}_{ki,0} \right) \right], \\
&= H \ \delta_{ij} + \frac{a^2}{2} \ g^{jk} \ \mathcal{H}_{jk,0}, \\
&= H \ \delta_{ij} + \frac{1}{2} \ \mathcal{H}_{ij,0},
\end{aligned} \tag{A.5}$$

where $g^{ij} = \frac{\delta_{ij}}{a^2}$.

Now, $\alpha = 0$ and $\beta = i$ giving Γ^{μ}_{0i}. Then,

$$\begin{aligned}
\Gamma^{0}_{0i} &= \frac{1}{2} g^{0\nu} \left(g_{\nu 0,i} + g_{\nu i,0} - g_{0i,\nu} \right), \\
&= \frac{1}{2} g^{00} \left(g_{00,i} + g_{0i,0} - g_{0i,0} \right), \\
&= \frac{1}{2} g^{00} \left(g_{00,i} \right), \\
&= 0,
\end{aligned} \tag{A.6}$$

and

$$\begin{aligned}
\Gamma^{j}_{0i} &= \frac{1}{2} g^{j\nu} \left(g_{\nu i,0} + g_{\nu 0,i} - g_{i0,\nu} \right), \\
&= \frac{1}{2} g^{jk} \left(g_{ki,0} + g_{k0,i} - g_{i0,k} \right), \\
&= \frac{1}{2} g^{jk} \left(g_{ki,0} \right), \\
&= H\ \delta_{ij} + \frac{1}{2}\ \mathcal{H}_{ij,0}.
\end{aligned} \tag{A.7}$$

Finally, we set $\alpha = i$ and $\beta = j$ giving Γ^{μ}_{ij}. Then,

$$\begin{aligned}
\Gamma^{0}_{ij} &= \frac{1}{2} g^{0\nu} \left(g_{\nu i,j} + g_{\nu j,i} - g_{ij,\nu} \right), \\
&= \frac{1}{2} g^{00} \left(g_{0i,j} + g_{0j,i} - g_{ij,0} \right), \\
&= \frac{1}{2} g^{00} \left(-g_{ij,0} \right), \\
&= \frac{1}{2}\ g_{ij,0}, \\
&= \frac{1}{2}\ (2H g_{ij} + a^2\ \mathcal{H}_{ij,0}), \\
&= H\ \delta_{ij} + \frac{a^2\ \mathcal{H}_{ij,0}}{2},
\end{aligned} \tag{A.8}$$

and

$$\begin{aligned}
\Gamma^{k}_{ij} &= \frac{1}{2} g^{k\nu} \left(g_{\nu i,j} + g_{\nu j,i} - g_{ij,\nu} \right), \\
&= \frac{1}{2} g^{kl} \left(g_{li,j} + g_{lj,i} - g_{ij,l} \right), \\
&= \frac{i}{2} \left[k_k\ \mathcal{H}_{ij} + k_j\ \mathcal{H}_i k - k_i\ \mathcal{H}_{jk} \right], \\
&= \frac{1}{2}\ g_{ij,0}.
\end{aligned} \tag{A.9}$$

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