UNIVERSIDAD DE INVESTIGACIÓN DE TECNOLOGÍA EXPERIMENTAL YACHAY

Escuela de Ciencias Computacionales y Matemáticas

TÍTULO: Concentration and multiplicity of solutions for a non-linear Schrödinger equation with critical frequency: Infinite Case

Trabajo de integración curricular presentado como requisito para la obtención del título de Matemático

Autor:

Aguas Barreno Ariel Alejandro

Tutor:

Dr. Mayorga Zambrano, Juan Ricardo, Ph.D.

Urcuquí, septiembre del 2021

Urcuquí, 28 de julio de 2021

SECRETARÍA GENERAL
(Vicerrectorado Académico/Cancillería)
ESCUELA DE CIENCIAS MATEMÁTICAS Y COMPUTACIONALES
CARRERA DE MATEMÁTICA
ACTA DE DEFENSA No. UITEY-ITE-2021-00024-AD

A los 28 días del mes de julio de 2021, a las 15:30 horas, de manera virtual mediante videoconferencia, y ante el Tribunal Calificador, integrado por los docentes:

Presidente Tribunal de Defensa	Dr. LOPEZ RIOS, JUAN CARLOS , Ph.D.
Miembro No Tutor	Dr. FERNANDES CAMPOS, HUGO MIGUEL , Ph.D.
Tutor	Dr. MAYORGA ZAMBRANO, JUAN RICARDO , Ph.D.

El(la) señor(ita) estudiante **AGUAS BARRENO, ARIEL ALEJANDRO**, con cédula de identidad No. **1004288591**, de la **ESCUELA DE CIENCIAS MATEMÁTICAS Y COMPUTACIONALES**, de la Carrera de **MATEMÁTICA**, aprobada por el Consejo de Educación Superior (CES), mediante Resolución **RPC-SO-15-No.174-2015**, realiza a través de videoconferencia, la sustentación de su trabajo de titulación denominado: **Concentration and multiplicity of solutions for a non-linear Schrödinger equation with critical frequency: Infinite Case.** , previa a la obtención del título de **MATEMÁTICO/A.**

El citado trabajo de titulación, fue debidamente aprobado por el(los) docente(s):

Tutor	Dr. MAYORGA ZAMBRANO, JUAN RICARDO , Ph.D.

Y recibió las observaciones de los otros miembros del Tribunal Calificador, las mismas que han sido incorporadas por el(la) estudiante.

Previamente cumplidos los requisitos legales y reglamentarios, el trabajo de titulación fue sustentado por el(la) estudiante y examinado por los miembros del Tribunal Calificador. Escuchada la sustentación del trabajo de titulación a través de videoconferencia, que integró la exposición de el(la) estudiante sobre el contenido de la misma y las preguntas formuladas por los miembros del Tribunal, se califica la sustentación del trabajo de titulación con las siguientes calificaciones:

Tipo	Docente	Calificación
Presidente Tribunal De Defensa	Dr. LOPEZ RIOS, JUAN CARLOS , Ph.D.	9,5
Miembro Tribunal De Defensa	Dr. FERNANDES CAMPOS, HUGO MIGUEL , Ph.D.	10,0
Tutor	Dr. MAYORGA ZAMBRANO, JUAN RICARDO , Ph.D.	10,0

Lo que da un promedio de: **9.8 (Nueve punto Ocho)**, sobre 10 (diez), equivalente a: **APROBADO**

Para constancia de lo actuado, firman los miembros del Tribunal Calificador, el/la estudiante y el/la secretario ad-hoc.

Certifico que *en cumplimiento del Decreto Ejecutivo 1017 de 16 de marzo de 2020, la defensa de trabajo de titulación (o examen de grado modalidad teórico práctica) se realizó vía virtual, por lo que las firmas de los miembros del Tribunal de Defensa de Grado, constan en forma digital.*

AGUAS BARRENO, ARIEL ALEJANDRO
Estudiante

ARIEL ALEJANDRO AGUAS BARRENO
Firmado digitalmente por ARIEL ALEJANDRO AGUAS BARRENO
Fecha: 2021.08.02 16:12:46 -05'00'

Dr. LOPEZ RIOS, JUAN CARLOS , Ph.D.
Presidente Tribunal de Defensa

Firmado electrónicamente por:
JUAN CARLOS LOPEZ RIOS

Dr. MAYORGA ZAMBRANO, JUAN RICARDO , Ph.D.
Tutor

Firmado electrónicamente por:
JUAN RICARDO MAYORGA ZAMBRANO

HUGO MIGUEL FERNANDES CAMPOS Assinado de forma digital por HUGO MIGUEL FERNANDES CAMPOS Dados: 2021.08.02 14:45:18 -05'00'

Dr. FERNANDES CAMPOS, HUGO MIGUEL , Ph.D.
Miembro No Tutor

TATIANA BEATRIZ TORRES MONTALVAN Firmado digitalmente por TATIANA BEATRIZ TORRES MONTALVAN Fecha: 2021.07.29 09:14:28 -05'00'

TORRES MONTALVÁN, TATIANA BEATRIZ
Secretario Ad-hoc

Autoría

Yo, **Ariel Alejandro Aguas Barreno**, con cédula de identidad **100428859-1**, declaro que las ideas, juicios, valoraciones, interpretaciones, consultas bibliográficas, definiciones y conceptualizaciones expuestas en el presente trabajo; así cómo, los procedimientos y herramientas utilizadas en la investigación, son de absoluta responsabilidad del autor del trabajo de integración curricular. Así mismo, me acojo a los reglamentos internos de la Universidad de Investigación de Tecnología Experimental Yachay.

Urcuquí, Septiembre del 2021

ARIEL ALEJANDRO AGUAS BARRENO

Digitally signed by ARIEL ALEJANDRO AGUAS BARRENO
Date: 2021.09.23 13:07:39 -05'00'

Ariel Aguas
CI: 100428859-1

Autorización de publicación

Yo, **Ariel Alejandro Aguas Barreno**, con cédula de identidad **100428859-1**, cedo a la Universidad de Tecnología Experimental Yachay, los derechos de publicación de la presente obra, sin que deba haber un reconocimiento económico por este concepto. Declaro además que el texto del presente trabajo de titulación no podrá ser cedido a ninguna empresa editorial para su publicación u otros fines, sin contar previamente con la autorización escrita de la Universidad.

Asimismo, autorizo a la Universidad que realice la digitalización y publicación de este trabajo de integración curricular en el repositorio virtual, de conformidad a lo dispuesto en el Art. 144 de la Ley Orgánica de Educación Superior.

Urcuquí, Septiembre del 2021

ARIEL ALEJANDRO AGUAS BARRENO

Digitally signed by ARIEL ALEJANDRO AGUAS BARRENO
Date: 2021.09.23 13:08:07 -05'00'

Ariel Aguas
CI: 100428859-1

Dedication

"Dedicated to my family, closest friends and my pets.
Without you I wouldn't be here, neither be the person I am now.
You will forever hold a piece of my soul."

Acknowledgments

I would like to start by expressing my most sincere gratitude to all the professors that compose the Yachay Tech family, and to those who left for different reasons. You have shown us and the rest of the world that having a little bit of hope can go a long way. Your unique way of teaching, your friendships, your warmth, your love for science and your dedication to teaching us, even through the most dire of situations, have changed my life and the lives of a thousand Ecuadorian students.

The professor who stands out the most is my advisor and mentor Dr. Juan Mayorga-Zambrano. Your talent, knowledge, passion and an unyielding commitment to teach me and all of your students has truly been an inspiration. Thank you for your support, pressure, unending patience and understanding as I kept losing myself. Throughout all these years I couldn't have asked for a better master to pull me to the light side, nor YT asked for a better faculty member. Special thanks to Juan Carlos López, Cédric M. Campos, Raúl Manzanilla and Antonio Acosta. Without all of your teachings I would have never been in the place I am now or studied mathematics. You will forever hold a special place in my heart.

I would also like to thank my parents, Lilian Barreno and Jaime Aguas, for teaching me to be a man with principles, criteria, for giving me the tools to be here and for your eternal support.

My life would've not been the same had I not chose to come to Yachay Tech, a decision that I will never regret. I have met the most wonderful people here, I truly believe that the future of science in Ecuador rests in the hands of students of Yachay Tech University. It harbors the best and brightest of all Ecuador, people to enact true change for our country. My time here has been a blessing and the best years of my life.

Finally, I would like to thank my friends in the group *Yachaysitos*, I cannot fathom my time in University without any of you. I love you all. My gratitude to the fourth generation of mathematicians, it was a pleasure to watch you all grow, to my roommates from H1-5 and many others with whom I shared unique, beautiful moments. A special mention goes to Juan Sebastián Burbano, for tolerating, trying to understand and putting up with me since the very beginning of our time here. It has been a shared journey, full of ups, downs and distinct experiences, but one I would gladly take again. We have learned much from each other and it is a privilege to be able to call someone like you my friend.

Abstract

In this work we prove existence, multiplicity, concentration phenomena and decay of solutions for the nonlinear Schrödinger equation

$$\begin{cases} \varepsilon^2 \Delta v(x) - V(x)\, v(x) + |v(x)|^{p-1} v(x) = 0, & x \in \mathbb{R}^N, \\ v(x) \to 0, & \text{as } |x| \to \infty, \end{cases} \tag{P_ε}$$

where $\varepsilon > 0$.

We consider the *Infinite Case* as presented by Byeon & Wang, under the restrictions:

(V1) $V \in \mathrm{C}(\mathbb{R}^N)$ is non-negative;

(V2) $V(x) \to \infty$, as $|x| \to \infty$;

(V3) $\overline{\mathcal{Z}} = \{x \in \mathbb{R}^N \ / \ V(x) = \inf(V)\} = \{0\}$;

($\mathrm{V_{inf}}$) $\forall |x| \leq 1 : V(x) = \exp\left(-\frac{1}{a(x)}\right)$.

where $b \in C(\mathbb{R}^N)$ is an Ω-*quasi homogeneous function* and $a \in C(\mathbb{R}^N \setminus \{0\})$ is an *asymptotically* (Ω, b)-*quasi homogeneous* function.

Under conditions (V1), (V2), (V3) and ($\mathrm{V_{inf}}$) the corresponding limit problem of (P_ε) as $\varepsilon \to 0$ is:

$$\begin{cases} \Delta w(x) + |w(x)|^{p-1}\, w(x) = 0, & x \in \Omega, \\ w(x) = 0, & x \in \partial\Omega, \end{cases} \tag{P}$$

where $\Omega \subseteq \mathbb{R}^N$ is a strictly star-shaped domain.

Using the properties of the Krasnoselskii genus and by a Ljusternik-Schnirelman scheme we prove the existence of an infinite number of solutions $v_{k,\varepsilon}, w_k$ for (P_ε) and (P) while presenting concentration results about the solutions of (P_ε). We prove the subconvergence, up to scaling of $v_{k,\varepsilon}$ to a solution of (P) and exponential decay of solutions away from Ω. Our results are congruent with the ones obtained by Byeon & Wang (2002), Felmer & Mayorga (2007) and Mayorga, Medina & Muñoz (2020) in each of their respective studies of the *Critical Frequency* cases.

Keywords: Nonlinear Schrödinger equation, infinite case, critical frequency, multiplicity, concentration.

Resumen

En este trabajo demostramos la existencia, multiplicidad, concentración y decaimiento de soluciones del problema relacionado a la ecuación no-lineal de Schrödinger:

$$\begin{cases} \varepsilon^2 \Delta v(x) - V(x)\, v(x) + |v(x)|^{p-1} v(x) = 0, & x \in \mathbb{R}^N, \\ v(x) \to 0, & \text{mientras } |x| \to \infty, \end{cases} \tag{P_ε}$$

con $\varepsilon > 0$.

Consideramos el *Caso Infinito* presentado por Byeon & Wang, bajo las restricciones:

(V1) $V \in \mathrm{C}(\mathbb{R}^N)$ es no-negativo;
(V2) $V(x) \to \infty$, si $|x| \to \infty$;
(V3) $\overline{\mathcal{Z}} = \{x \in \mathbb{R}^N \ / \ V(x) = \inf(V)\} = \{0\}$;
($\mathrm{V_{inf}}$) $\forall |x| \leq 1 : V(x) = \exp\left(-\frac{1}{a(x)}\right)$,

donde $b \in C(\mathbb{R}^N)$ es una función Ω-*cuasi homogénea* y $a \in C(\mathbb{R}^N \setminus \{0\})$ es una función *asintóticamente* (Ω, b) *cuasi-homogénea.*

El problema límite correspondiente a (P_ε) bajo las restricciones $(\mathrm{V1}), (\mathrm{V2}), (\mathrm{V3})$ y $(\mathrm{V_{inf}})$ es:

$$\begin{cases} \Delta w(x) + |w(x)|^{p-1}\, w(x) = 0, & x \in \Omega, \\ w(x) = 0, & x \in \partial\Omega, \end{cases} \tag{P}$$

donde $\Omega \subseteq \mathbb{R}^N$ es un dominio estríctamente estrellado.

Usamos un esquema de Ljusternik-Schnirelman y las propiedades del género de Krasnoselskii para demostrar la existencia de un número infinito de soluciones $v_{k,\varepsilon}, w_k$ de (P_ε) y (P), respectivamente. También presentamos resultados de concentración referentes a la solución de (P_ε). Dado un escalamiento, demostramos la subconvergencia de $v_{k,\varepsilon}$ a una solución de (P) y el decaimiento exponencial de soluciones por fuera de Ω. Nuestros resultados son congruentes con los obtenidos por Byeon & Wang (2002), Felmer & Mayorga (2007) y Mayorga, Medina & Muñoz (2020) en cada uno de sus respectivos estudios referentes a los problemas con *Frecuencia Crítica.*

Plabras Clave: Ecuación de Schrödinger No-linear, caso infinito, frecuencia crítica, multiplicidad, concentración.

Contents

Chapter 1

Introduction

As humanity advances in it's understanding of the Universe, ever more complex, bold and precise theories to describe it arise. As these theories emerge, so does the need for a more accurate and sophisticated way in which they are written. Naturally, this requires significant improvements in the ever so intertwined fields of Mathematics and Physics given their ability to precisely describe physical phenomena.

Such is the case of Quantum Mechanics. It arose due to Classical Mechanics' inability to accurately describe small scaled phenomena. By obtaining a solid mathematical foundation in the mid 1920's, it became the standard way in which we understand the Universe at a microscopic scale. The famous *Schrödinger equation* has been the most commonly used tool to study the state of a quantum system. In this work we are interested in its nonlinear version

$$i\hbar\psi_t(x,t) + \frac{\hbar^2}{2}\Delta\psi(x,t) - V_0(x)\psi(x,t) + |\psi(x,t)|^{p-1}\psi(x,t) = 0, \quad \forall x \in \mathbb{R}^N, \forall t \geq 0, \tag{1.1}$$

where i denotes the imaginary unit, $N \in \mathbb{N}$ and $p > 1$. ψ is called the state function. It contains information about the system and

$$\hbar = 6.62607015 \times 10^{-34} Kg \cdot m^2 \cdot s^{-1},$$

is known as the reduced Planck constant. V is a real valued function known as the potential of the system and the Laplacian operator Δ is given by:

$$\Delta := \sum_{i=1}^{N} \frac{\partial^2}{\partial x_i^2}.$$

This equation arises in the study of the evolution of Bose-Einstein condensates, [23], and it is relevant to model the propagation of light in some nonlinear optical materials, [24].

When searching for solutions of (1.1), it is natural to search for *standing waves.* i.e., semi-classical states of the form

$$\psi(x,t) = v(x)e^{-iEt/\hbar}, \quad x \in \mathbb{R}^N, t \geq 0,$$

with

$$E = \inf(V_0).$$

Here the time-independent component, v, should verify

$$\varepsilon^2 \Delta v(x) - V(x)v(x) + |v(x)|^{p-1}v(x) = 0, \tag{1.2}$$

with $\varepsilon^2 = \hbar^2/2$ and $V(x) = V_0(x) - E$. The term semi-classical is justified as it is an asymptotic method which stops considering $\hbar$ as a constant but instead considers it as a parameter that decreases to zero, that is, passing to the limit as $\varepsilon \to 0$.

We consider the situation where

$$\overline{\mathcal{Z}} = \{x \in \mathbb{R}^N \ / \ V(x) = \inf(V)\} \neq \emptyset.$$

The case when $\inf(V) > 0$ is referred to as the non-critical frequency case. The critical frequency case arises when $\inf(V) = 0$, this term is justified since the behavior of the solutions notably changes.

There exist a large number of works regarding the non-critical case such as [2], [8], [12], [13], [24], [25], [28] and [32], based on the Lyapunov-Schmidt reduction, the variational method or a combination of both. Some common results are:

(I) v_ε^*, a solution of (1.2), is bounded away from zero, i.e.,

$$\liminf_{\varepsilon \to 0} \max_x |v_\varepsilon(x)| > 0; \tag{1.3}$$

(II) v_ε^* concentrates around some critical points of V;

(III) v_ε^* exponentially decays to zero away from such critical points, as $\varepsilon \to 0$; and,

(IV) there is a unique limit problem and, therefore, a unique profile, as $\varepsilon \to 0$.

In our work, we continue the study of the case when $\inf(V) = 0$ presented in the pioneer work [6] for the critical-frequency problem

$$\begin{cases} \varepsilon^2 \Delta v(x) - V(x)v(x) + |v(x)|^{p-1}v(x) = 0, & x \in \mathbb{R}^N, \\ v(x) \to 0, & \text{as } |x| \to \infty, \end{cases} \tag{P_ε}$$

with

$$\begin{cases} 2 < 1 + p < 2^* = 2N/(N-2), & \text{if } N \geq 3; \\ 2 < 1 + p, & \text{if } N = 1, 2. \end{cases} \tag{1.4}$$

where it is proved the existence of v_ε, a positive standing wave, least energy solution, (see [6]), for which

(C1) (1.3) no longer holds. Instead, the following behavior is verified

$$\lim_{\varepsilon \to 0} \|v_\varepsilon\|_{\mathrm{L}^\infty(\mathbb{R}^N)} = 0, \tag{1.5}$$

$$\liminf_{\varepsilon \to 0} \frac{\|v_\varepsilon\|_{\mathrm{L}^\infty(\mathbb{R}^N)}}{\varepsilon^{2/(p-1)}} > 0; \tag{1.6}$$

(C2) v_ε concentrates around an isolated component of $\overline{\mathcal{Z}} = \{V = 0\}$;

(C3) v_ε exponentially decays outside the region $\overline{\mathcal{Z}}$;

(C4) There no longer exists a unique limit problem and, therefore, neither a unique profile. They depend on the behavior of V nearby $\overline{\mathcal{Z}}$.

Three cases were considered.

Flat: $\mathrm{int}\overline{\mathcal{Z}} = \mathcal{Z} \neq \emptyset$ is bounded;

Finite: $\overline{\mathcal{Z}}$ is finite and V vanishes polinomially around it;

Infinite: $\overline{\mathcal{Z}}$ is finite and V vanishes exponentially around it.

For these cases it was also proved that

(C5) a scaling of the positive least-energy solution v_ε converges to u, a positive least-energy solution of a corresponding limit problem;

(C6) the energy of v_ε converges to the energy of u.

The papers [11] and [21] focus on the flat case, assuming the potential satisfies the following conditions:

(V1) $V \in \mathrm{C}(\mathbb{R}^N)$ is non-negative;

(V2) $V(x) \to \infty$, as $|x| \to \infty$;

(V_{flat}) $\mathrm{int}(\mathcal{Z}) \neq \emptyset$ is connected and smooth.

In this context, the corresponding limit problem is

$$\begin{cases} \Delta u(x) + |u(x)|^{p-1}u(x) = 0, & x \in \mathcal{Z}, \\ u(x) = 0, & x \in \partial\mathcal{Z}. \end{cases} \tag{P_{flat}}$$

Authors Felmer and Mayorga-Zambrano applied the Ljusternik-Schnirelman theory to the even energy functionals I_ε and I associated with (P_ε) and (P_{flat}), respectively. They proved that:

i) There exist sequences of solutions, $(v_{k,\varepsilon})_{k\in\mathbb{N}}$ and $(u_{k,\varepsilon})_{k\in\mathbb{N}}$, for (P_ε) and (P_{flat}) respectively.

ii) For a fixed k and as $\varepsilon \to 0$ the solution, not necessarily positive, $v_{k,\varepsilon}$ behaves like those found in [6]. That is, conditions (C1), (C2) and (C3) hold.

iii) Point (C6) also holds, that is:

$$\lim_{\varepsilon\to 0} I_\varepsilon(v_{k,\varepsilon}) = I(u_k). \tag{1.7}$$

iv) Point (C5) holds in the sense that a scaling of $v_{k,\varepsilon}$ converges, up to subsequences, to w_k a solution of (P_{flat}) with the same energy of u_k:

$$I(w_k) = I(u_k).$$

Additionally, further asymptotic estimates on the boundary of $\mathcal{Z}$ were obtained.

Remark 1.0.1. In the context of the Ljusternik-Schnirelman theory for even functionals, the indices k of the critical values are representative of the topological characteristic of the level sets, given by the Krasnoselskii genus.

Remark 1.0.2. Condition (V2) is more restrictive than the one in [6]. Where it was assumed that for some $\gamma > 0$ we have $\liminf_{|x|\to\infty} V(x) > 2\gamma$.

In [21] the results of [11] were experimentally shown via a numerical approach for $N = 1$. For a fixed ε, a variation of the shooting scheme developed in [17] was applied. Here, instead of the commonly-used Newton method to adjust the initial slope, a secant method was applied because the manipulation of two initial slopes provided control on k, which corresponds to the number of changes of sign that u_k and $v_{k,\varepsilon}$ have, in the one dimensional case. For several values of k, the following concentration property was numerically shown

$$\|u_k - \varepsilon^{-2} v_{k,\varepsilon}\|_{L^2(\Omega)} \to 0, \quad \text{as} \quad \varepsilon \to 0.$$

This is difficult because of the nonlinearity of the equations and, by the properties of the Krasnoselskii genus, there are at least two solutions for each topological level $k \in \mathbb{N}$.

In [22], the finite case was studied. The same kind of results as in [11] were obtained. In this project, similarly, we show that the same types of results of [11] hold for the N-dimensional *infinite case*. Grossly speaking, that is when $V(x)$ decays at an exponential rate as x gets close to $\overline{\mathcal{Z}} = \{0\}$. This document is organized as follows:

- In Section 2 we present the basic mathematical tools required for our project. We review some definitions, theorems and results from normed spaces, Functional Analysis, Partial Differential Equations, Sobolev spaces, Variational Calculus and Nonlinear Analysis.

- In Section 3 we start with a historical overview of Quantum Mechanics and proceed with some brief but essential concepts from its foundation, such as wave functions, the Schrödinger equation and the Heisenberg uncertainty principle.

- In Section 4 we formally state our problem and the main results. Then we set up a Ljusternik-Schnirelman scheme to prove that our problem has infinite solutions. Next we prove the convergence of critical values and make an asymptotical analysis of our solutions. We conclude with the decay results of our problem.

Chapter 2

Mathematical Framework

In this chapter we present topics that are needed to handle our problem. We will provide proofs of some of the most relevant results while referring the reader to our main sources, such as [4], [10], [18], [19], [20], [31]. We will use standard notation across this entire chapter.

2.1 Basic definitions

We start with some fundamental results from linear algebra and topology.

Definition 2.1.1 (Linear Space)**.** Let V be a non-void set. Let $(V,+)$ be an Abelian group with an external operation $\cdot : \mathbb{R} \times V \to V$. We say that $(V,+,\cdot)$ is a linear space over $\mathbb{R}$ iff:

i) (Harmlessness of 1) $\forall u \in V : 1 \cdot u = u$;

ii) (Mixed associativity) $\forall u \in V, \forall \alpha, \beta \in \mathbb{R} : (\alpha\beta) \cdot u = \alpha \cdot (\beta \cdot u)$;

iii) (Vector Associativity) $\forall u \in V, \forall \alpha, \beta \in \mathbb{R} : (\alpha + \beta) \cdot u = \alpha \cdot u + \beta \cdot u$;

iv) (Scalar Distributivity) $\forall \alpha \in \mathbb{R}, \forall u, v \in V : \alpha \cdot (u + v) = \alpha \cdot u + \alpha \cdot v$.

The pair $(V, \mathcal{T})$ is called a topological space iff V is a non-void set and $\mathcal{T} \subseteq \mathcal{P}(V)$ verifies

i) $\emptyset \in \mathcal{T}$ and $V \in \mathcal{T}$;

ii) $\forall A, B \in \mathcal{T} : A \cap B \in \mathcal{T}$;

iii) $(A_\lambda)_{\lambda \in \Lambda} \subseteq \mathcal{T} \implies \bigcup\limits_{\lambda \in \Lambda} A_\lambda \in \mathcal{T}$.

Elements of the topology $\mathcal{T}$ are called *open sets*, we will later give a characterization using the notion of norm.

Now let us define the concept of a metric. Let X be a set. We say that $d : X \times X \longrightarrow \mathbb{R}$ is a metric over $\mathbb{R}$ iff for every $x, y, z \in X$ the following properties hold:

i) $d(x, y) = 0 \iff x = y$;

ii) $d(x, y) = d(y, x)$;

iii) $d(x, z) \leq d(x, y) + d(y, z)$.

and from points *i*) and *iii*) it follows that

$$0 \leq d(x, y).$$

The ordered pair (X, d) is called a metric space.

Definition 2.1.2 (Norm, normed space)**.** Let E be a linear space and $||\cdot|| : E \longrightarrow \mathbb{R}$. We say that $(E, ||\cdot||)$ is a normed space iff for every $x, y \in E$ and $\lambda \in \mathbb{R}$ the following hold:

i) $0 \leq ||x||$;

ii) $||x|| = 0 \iff x = 0$;

iii) $||\lambda x|| = |\lambda| ||x||$;

iv) $||x + y|| \leq ||x|| + ||y||$.

Remark 2.1.1 (Every metric space is a topological space)**.** Every metric space X is a topological space when, by definition, $A \subseteq X$ is open iff A can be expressed as a union of sets of the form

$$B(x_0, r) = \{x \in X : \quad d(x_0, x) < r\}, \quad x_0 \in X, r > 0.$$

Remark 2.1.2 (Notation)**.** If it generates no confusion, we shall use the following notation for the rest of the chapter $||\cdot||_E := ||\cdot||$ and we will denote $(E, ||\cdot||_E) = E$, $(F, ||\cdot||_F) = F$ as normed spaces unless stated otherwise.

It is important to know that every norm induces a metric in the sense of

$$d(x, y) := ||x - y||. \tag{2.1}$$

Hence, every normed space is also a metric space.

We now introduce the concept of balls. Given $x_0 \in E$ and $r > 0$, we denote:

$$\text{(Open ball)} \;\; B(x_0, r) = \{x \in E : ||x_0 - x|| < r\},$$

$$\text{(Closed ball)} \;\; \overline{B}(x_0, r) = \{x \in E : ||x_0 - x|| \leq r\},$$

$$\text{(Sphere)} \;\; S(x_0, r) = \{x \in E : ||x_0 - x|| = r\}.$$

We say that all of these objects are centered in x_0 with radius r. Using this, let us give the characterization of open sets.
We have that $U \subseteq E$ is open iff

$$\forall x \in U, \exists r > 0 : \quad B(x, r) \subseteq U.$$

We say that $C \subseteq E$ is closed iff $C^c := E - C$ is open.

Definition 2.1.3 (Interior, closure and boundary of a set)**.** Let E be a linear space, $U \subseteq E$. We denote:

i) *(Interior of U)* $\mathrm{int}(U) = \{x \in E \;/\; \exists r > 0 : \quad B(x, r) \subseteq U\}$,

ii) *(Closure of U)* $\overline{U} = \{x \in E \;/\; \forall r > 0 : \quad B(x, r) \cap U \neq \emptyset\}$,

iii) *(Boundary of U)* $\partial U = \overline{U} \cap \overline{U}^c$.

From this, it follows that $\mathrm{int}(U) \subseteq U$ and $U \subseteq \overline{U}$. Moreover, U is open iff $U = \mathrm{int}(U)$.

Lemma 2.1.4. *In the definition above, we note that the set $\overline{U}$ is closed.*

Proof. Let's prove that

$$\overline{A}^c = \mathrm{int}(A^c),$$

which implies that $\overline{A}^c$ is open, so that $\overline{A}$ is closed.

i) First, we will prove that $\overline{A}^c \subseteq \mathrm{int}(A^c)$. Let $x \in \overline{A}^c$, generic. Then $x \notin \overline{A}$, and therefore, there exists $r > 0$ such that

$$B(x, r) \cap U = \emptyset.$$

So that $B(x, r) \subseteq U^c$ and $x \in \mathrm{int}(U)$. By the arbitrariness of x, we have $\overline{A}^c \subseteq \mathrm{int}(A^c)$.

ii) Let's prove that $\mathrm{int}(A^c) \subseteq \overline{A}^c$. Let $x \in \mathrm{int}(A^c)$, then there exists $r > 0$ such that

$$B(x, r) \subseteq U^c.$$

Hence, $x \notin \overline{U}$. That is $x \in \overline{U}^c$. We conclude by the arbitrariness of x.

□

Theorem 2.1.5 (Characterization of a closed set)**.** *Let $(X, \mathcal{T})$ be a topological space, $A \subseteq X$. Then*

$$A \text{ is closed} \iff A = \overline{A}.$$

Proof. $\Leftarrow$) Assume that $A = \overline{A}$. By the previous lemma, A is closed.

$\Rightarrow$) Assume that A is closed. Therefore A^c is open and, by the previous lemma, $\overline{U}^c = U^c$. This implies that

$$\overline{U} = \left(\overline{U}^c\right)^c = (U^c)^c = U.$$

We conclude. □

Definition 2.1.6 (Convergent sequence)**.** Let E be a normed space, $(x_n)_{n\in\mathbb{N}} \subset E$. We say that x_n converges to $x \in E$ iff:

$$\forall \varepsilon > 0, \exists N \in \mathbb{N} : \quad n > N \implies ||x_n - x|| < \varepsilon,$$

and it is denoted by

$$\lim_{x\to\infty} ||x_n - x|| = 0,$$

$$x_n \to x, \quad \text{as} \quad n \to \infty.$$

Using the notion of a convergent sequence we can define compact sets. Namely, $K \subseteq E$ is compact iff every sequence in K has a convergent subsequence, that is

$$\forall (x_n)_{n\in\mathbb{N}} \subseteq K, \exists (x_{n_k})_{k\in\mathbb{N}} \subseteq (x_n)_{n\in\mathbb{N}} \text{ - convergent }.$$

We say that $A \subseteq E$ is relatively compact iff $\overline{A}$ is compact.

Let's now define the concept of continuity of a function. Let $(E, d), (F, \rho)$ be metric spaces. We say that a function $f : E \longrightarrow F$ is continuous at $x_0 \in E$ iff:

$$\forall \varepsilon > 0, \exists \delta > 0 : \quad d(x, x_0) < \delta \implies \rho(f(x), f(x_0)) < \varepsilon.$$

Moreover, if f is continuous at every $x_0 \in E$, we say that f is continuous in E.

Proposition 2.1.1 (Continuity by inverse image)**.** *Let, E, F be topological spaces, $f : E \to F$. Then, f is continuous iff the inverse image of an open subset of F belongs to E. That is*

$$\forall A \subseteq F \text{ - } open \ : f^{-1}(A) \in E.$$

A proof of this proposition can be found in [18].

Let's now define a particularly useful type of continuity. For E, F normed spaces we say that $f : E \to F$ is Lipschitz continuous iff there exists $C > 0$ such that

$$\forall x, y \in E : \quad ||f(x) - f(y)|| < C||x - y||.$$

In particular, if $C < 1$ we say that f is a *contraction*.

Remark 2.1.3. Note that any function f that is Lipschitz continuous is continuous, that is, Lipschitz continuity implies continuity.

Definition 2.1.7 (Comparable norms)**.** Let V be a linear space and $||\cdot||_1$,$||\cdot||_2$ be two norms on V. We say that $||\cdot||_1$ dominates $||\cdot||_2$ iff

$$\exists c > 0, \forall u \in V : \quad ||u||_2 \leq c||u||_1.$$

And say that the norms are **comparable** iff one of the norms dominates the other. Additionally, $||\cdot||_1$ and $||\cdot||_2$ are **equivalent** iff they dominate each other, that is:

$$\exists c_1, c_2 > 0, \forall u \in V : \quad c_1||u||_1 \leq ||u||_2 \leq c_2||u||_1.$$

We say that a sequence $(x_n)_{n\in\mathbb{N}} \subset E$ is *Cauchy* iff

$$\forall \varepsilon > 0, \exists N \in \mathbb{N} : \quad m, n > N \implies ||x_m - x_n|| < \varepsilon.$$

Cauchy sequences are also known as fundamental sequences. The last equation tells us that from a certain point, the *tail* of the sequence gets infinitely close or stays within a certain ball. Cauchy sequences are used in numerical approximation when ε is taken as the approximation value.

Proposition 2.1.2 (Convergent implies Cauchy)**.** *Every convergent sequence in a normed space is a Cauchy sequence.*

Proof. Let $(x_n)_{n\in\mathbb{N}} \subseteq E$ be a convergent sequence, let's denote as as its limit. Then

$$\forall \varepsilon > 0, \exists N = N(\varepsilon) : n > N \implies ||x_n - x|| < \frac{\varepsilon}{2}.$$

Now, the triangle inequality implies that, for $m, n < N$

$$||x_m - x_n|| \leq ||x_m - x|| + ||x - x_n|| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

That is, $(x_n)_{n\in\mathbb{N}}$ is Cauchy. We conclude. $\square$

Definition 2.1.8 (Complete space)**.** Let (X, δ) be a metric space. Then, X is complete iff every Cauchy sequence converges in X.

From the previous definition we say that any normed space $(E, \|\cdot\|)$ that is complete with the metric induced by the norm $\|\cdot\|$ is a **Banach space**.

Remark 2.1.4. It is very important to note that completeness is a property that depends on the metric. A space E may be complete with the norm $||\cdot||_1$ but not with the norm $||\cdot||_2$.

Definition 2.1.9 (Uniformly convex space)**.** We say that the normed space E is *uniformly convex* iff:

$$\forall \varepsilon > 0, \exists \delta > 0, \forall u, v \in E : (||u|| \leq 1 \ \wedge \ ||v|| \leq 1 \ \wedge \ ||x - y|| \geq \varepsilon) \implies \left\| \frac{x+y}{2} \right\| < 1 - \delta.$$

Uniform convexity is a geometric property related to the unit sphere $S(0,1)$. Namely, the unit sphere must be *round* and cannot admit any line segment.

Moreover, the *Milman-Pettis* theorem [4, Th. 3.31] states that every uniformly convex Banach space is reflexive.

Theorem 2.1.10 (Closed set in a complete space)**.** *Let (X,δ) be a complete metric space and $A \subseteq X$. Then, A is closed iff A is complete.*

A proof of this theorem can be found in [20].

2.2 Some topics on Functional Analysis

In this section we shall introduce some of the most important spaces in Analysis and provide some important concepts that are useful for our work.

2.2.1 Linear operators and functionals

Definition 2.2.1 (Linear operator and functionals)**.** Let E, F be normed spaces, $T : E \to F$ be an operator. We say that T is linear iff

$$\forall \lambda \in \mathbb{R}, \forall x, y \in E : T(\lambda x + y) = \lambda T(x) + T(y).$$

In particular, if $F = \mathbb{R}$, we say that T is a *functional*.

Remark 2.2.1. Note that, in the definition above, the term operator is used to define a function. These terms are equivalent and change depending on the context.

We say that T is bounded iff:

$$\exists c > 0, \forall x \in E : ||T(x)|| \leq c||x||,$$

and denote, the space of bounded linear operators by

$$\mathcal{L}(E,F) := \{T; E \to F \ / \ T \text{ is linear and bounded.}\}.$$

This space is equipped with the norm $||\cdot||_{\mathcal{L}(E,F)}$ given by:

$$||T||_{\mathcal{L}(E,F)} = \inf\{c > 0 \ / \ \forall x \in E : \quad ||T(x)|| \leq c||x||\}.$$

As a consequence,

$$\forall x \in E : ||T(x)|| \leq ||T||_{\mathcal{L}(E,F)}||x||.$$

A proof that $||\cdot||_{\mathcal{L}(E,F)}$ is a norm can be found in [20].

Remark 2.2.2. Let's note that, by the linearity of the operators, all bounded operators are Lipschitz, that is

$$\forall x \in E, \forall T \in \mathcal{L}(E, F) : ||T(x)|| \leq ||T|| ||x||,$$

implies that

$$\forall x, y \in E : ||T(x) - T(v)|| \leq ||T|| \cdot ||u - v||.$$

Definition 2.2.2 (Dual space)**.** The topological dual or, simply, dual space of E is

$$E^* := \mathcal{L}(E, \mathbb{R}).$$

This space is complete since $\mathbb{R}$ is complete.

Theorem 2.2.3 (Continuity and boundedness)**.** *Let E, F be normed spaces, $T \in \mathcal{L}(E, F)$. Then, T is bounded iff T is continuous.*

Proof. $\Rightarrow$) By Remark 2.2.2 we have that T is Lipschitz continuous. Therefore, T is continuous.
$\Leftarrow$) Now assume that T is continuous, we have to prove the boundedness of T, i.e.,

$$\exists c > 0, \forall u \in E : \quad \|T(u)\| \leq c\|u\|. \tag{2.2}$$

By the purpose of contradiction, assume that (2.2) is false, that is

$$\forall c > 0, \exists u \in E : \quad \|T(u)\| > c\|u\|.$$

Therefore we can choose a sequence $(u_n)_{n\in\mathbb{N}} \subseteq E$ such that

$$\forall n \in \mathbb{N} : \quad T(u_n) > n\|u_n\|.$$

For each $n \in \mathbb{N}$ we set

$$v_n = \frac{1}{n\|u_n\|} u_n,$$

so that $\|v_n\| = 1/n$. Then,

$$\lim_{n\to\infty} v_n = 0, \tag{2.3}$$

and

$$\|T(u_n)\| = \frac{1}{n\|u_n\|}\|T(u_n)\| \geq \frac{1}{n\|u_n\|} n\|u_n\| = 1. \tag{2.4}$$

Points (2.3) and (2.4) contradict the continuity of T so we conclude that T is bounded. □

Corollary 2.2.1 (Composition of bounded operators is bounded)**.** *Let V, W, U be normed spaces. If $T \in \mathcal{L}(V, W)$ and $S \in \mathcal{L}(W, U)$, then $ST \in \mathcal{L}(V, U)$. Moreover,*

$$||ST|| \leq ||S|| ||T||.$$

A proof of this corollary can be found in [18].

In the context of proposition 2.1.7. Let's assume that $E \subseteq F$ and that in the normed spaces $(E, ||\cdot||_1), (F, ||\cdot||_2)$, $||\cdot|||_2$ dominates $||\cdot||_1$. In this case, we have that

$$(E, ||\cdot||_2) \subseteq (E, ||\cdot||_1),$$

and the embedding operator $I : E \to F$ given by

$$I(u) = u.$$

is continuous. We say that $(E, ||\cdot||_1)$ is continuously embedded in $(F, ||\cdot||_2)$.

2.2.2 Weak and weak * convergence.

We start by noting that any sequence that converges in the context of definition 2.1.6 is said to be strongly convergent or converge strongly to x.

Definition 2.2.4 (Weak convergence)**.** Let E be a normed space. A sequence $(x_n)_{n\in\mathbb{N}} \subseteq E$ is said to be weakly convergent to $x \in E$ iff

$$\forall T \in E^* : \lim_{n\to\infty} T(x_n) = T(x),$$

which is written as

$$x_n \xrightarrow{w} x \quad \text{or} \quad x_n \rightharpoonup x.$$

The element x is called the weak limit of $(x_n)_{n\in\mathbb{N}}$.

This type of convergence is incredibly important in analysis, it is widely used e.g. in the calculus of variations and differential equations.

Lemma 2.2.5 (Weak convergence)**.** *Let E be a normed space. Let $(x_n)_{n\in\mathbb{N}} \subseteq E$ be a weakly convergent sequence. Then,*

i) the weak limit x is unique;

ii) every subsequence of $(x_n)_{n\in\mathbb{N}}$ converges weakly to x;

iii) the sequence $(||x||_n)_{n\in\mathbb{N}}$ is bounded.

A proof of this lemma can be found in [18].

Theorem 2.2.6 (Strong and weak convergence)**.** *Let E be a normed space, $(x_n)_{n\in\mathbb{N}} \subseteq E$ be a sequence that converges strongly. Then, $(x_n)_{n\in\mathbb{N}}$ converges weakly with the same limit. That is, strong convergence implies weak convergence.*

Proof. Assume that $(x_n)_{n\in\mathbb{N}}$ is strongly convergent, that is

$$x_n \longrightarrow x,$$

which means that

$$||x_n - x|| \longrightarrow 0.$$

This implies, by the linearity and boundedness of $T \in E^*$, generic, that

$$|T(x_n) - T(x)| = |T(x_n - x)| \leq \|f\|\|x_n - x\| \longrightarrow 0.$$

We conclude by the arbitrariness of T. $\square$

Definition 2.2.7 (Weak * convergence)**.** Let E be a normed space, $(T_n)_{n\in\mathbb{N}} \subseteq E^*$ be a sequence of functionals. We say that $(T_n)_{n\in\mathbb{N}}$ converges *weakly to $T \in E^*$ iff

$$\forall x \in E : \lim_{x\to\infty} T_n(x) = T(x).$$

T is called the weak* limit of $(T_n)_{n\in\mathbb{N}}$.

Remark 2.2.3. Note that weak convergence implies *weak convergence. A proof of this fact can be found in [20]

Now let X be a topological space. We say that a family of functions $(f_\varepsilon)_{\varepsilon>0} \subseteq X$ *subconverges* in X as $\varepsilon \to 0$ iff from every sequence $(\varepsilon_n)_{n\in\mathbb{N}}$ converging to zero it is possible to extract a subsequence $(\varepsilon_{n_i})_{i\in\mathbb{N}}$ such that $(f_{\varepsilon_i})_{i\in\mathbb{N}}$ converges in X as $i \to \infty$.

A linear operator $T : E \to F$ is called a compact linear operator, or completely continuous linear operator iff for any $U \subseteq E$ bounded, $T(U)$ is relatively compact, that is, $\overline{T(U)}$ is compact.

Lemma 2.2.8 (Continuity of compact operators)**.** *Let $T : E \to F$ be any compact operator. Then, T is bounded, hence, continuous.*

Proof. Since T is compact, we have that for any bounded $U \subseteq E$, then $\overline{T(U)}$ is compact. Now consider the unit ball in E, that is $B(0,1) \subseteq E$, we have that

$$\overline{T(B(0,1))} \subseteq F,$$

is compact and, therefore, bounded. So we have that,

$$||T|| \leq \sup_{||x||=1} ||Tx|| < \infty,$$

which implies that T is bounded. We conclude. $\square$

Theorem 2.2.9 (Compactness criterion)**.** *Let E, F be normed spaces, $T : E \to F$ be a linear operator. Then T is compact iff T maps every bounded sequence $(x_n)_{n\in\mathbb{N}} \subseteq E$ onto a sequence $(Tx_n)_{n\in\mathbb{N}} \subseteq F$ which has a convergent subsequence.*

Proof. If T is compact and $(u_n)_{n\in\mathbb{N}}$ is bounded, then $\overline{T(x_n)}$ is compact in F and by the definition of a compact sequence, it has a convergent subsequence.

Conversely, we assume that every generic bounded sequence $(x_n)_{n\in\mathbb{N}}$ contains a subsequence $(x_{n_k})_{k\in\mathbb{N}}$ such that $T(x_{n_k})_{k\in\mathbb{N}}$ converges in F. We consider any bounded subset of X, $B \subset X$, and let $(y_n)_{n\in\mathbb{N}}$ be any sequence in $T(B)$.
Then, $T(x_n) = y_n$ or some $x_n \in B$, we have that $(x_n)_{n\in\mathbb{N}}$ is bounded since B is bounded. Then, as $T(x_n)$ contains a convergent subsequence $\overline{T(B)}$ is compact by the arbitrariness of $(y_n)_{n\in\mathbb{N}}$ in $T(B)$.

□

Theorem 2.2.10 (Weak convergence and linear compact operators)**.** *Let $T : E \to F$ be a linear compact operator and $(x_n)_{n\in\mathbb{N}} \subseteq E$ be a sequence that converges weakly to $x \in E$. Then $(Tx_n)_{n\in\mathbb{N}} \subseteq F$ converges strongly to Tx.*

A proof of this theorem can be found in [18].

Theorem 2.2.11. *Let E be a uniformly convex Banach space, $(x_n)_{n\in\mathbb{N}} \subseteq E$ and $x \in E$ such that*

$$x_n \rightharpoonup x,$$

and

$$\limsup_{x\to\infty} \|x_n\|_E \leq \|x\|_E.$$

Then, x_n converges strongly to x.

A proof of this theorem can be found in [4, Prop.3.32].

2.2.3 Lebesgue spaces

In order to ease the contents of this document, in this section we assume the reader has some notions about measure theory such as *Lebesgue measure, measurable sets, measurable spaces, measurable functions and integrable functions.* We shall state some commonly known results in $L^p(\Omega)$ spaces that are important to our work.

Definitions, norm and properties

Let m denote the Lebesgue measure. As it's usually done, we will adopt the abuse of notation $u = v$ for equality of functions if they coincide almost everywhere (a.e.), that is

$$m\left(\{x \in \mathbb{R}^N : u(x) \neq v(x)\}\right) = 0.$$

Let $\Omega \subseteq \mathbb{R}^N$ open, we set

$$L^1(\Omega) := \left\{u : \Omega \to \mathbb{R} \ / \int_\Omega |u(x)|dx < \infty\right\},$$

with the functional $||\cdot||_{L^1(\Omega)}$ given by

$$||u||_{L^1(\Omega)} = \int_\Omega |u(x)|dx,$$

as the space of integrable functions.

Definition 2.2.12 ($L^p(\Omega)$ spaces)**.** Let $1 \leq p < \infty$, for $\Omega \subseteq \mathbb{R}^N$ open we define

$$L^p(\Omega) := \left\{u : \Omega \to \mathbb{R} \ / \quad |u|^p \in L^1(\Omega)\right\},$$

with the functional $||\cdot||_{L^p(\Omega)} : L^p(\Omega) \to \mathbb{R}$ given by

$$||u||_{\mathrm{L}^p(\Omega)} = \left(\int_\Omega |u(x)|^p dx\right)^{\frac{1}{p}}.$$

Remark 2.2.4. For $p = \infty$ we set

$$L^\infty(\Omega) := \{u : \Omega \to \mathbb{R} \ / \ \exists C > 0 : |u(x)| \leq C, \quad \text{for a.e. } x \in \Omega\}.$$

Remark 2.2.5. We define the space $L^1_{loc}(\Omega)$ as

$$L^1_{loc}(\Omega) := \left\{u : \Omega \to \mathbb{R} \ / \ \forall K \subseteq \Omega \text{ - compact: } \quad \int_K |u(x)|dx < \infty\right\},$$

the space of locally integrable functions.

$L^p(\Omega)$ spaces are normed spaces for any p. But before we state and prove the theorem, we shall state some useful results.

To simplify notation, whenever there is no confusion, we set $||\cdot||_{L^p(\Omega)} = ||\cdot||_p$. For $1 \leq p \leq \infty$ we denote the conjugate exponent of p, p', by

$$\frac{1}{p} + \frac{1}{p'} = 1. \tag{2.5}$$

Lemma 2.2.13 (Young's Inequality)**.** *Let $a, b \geq 0$ and $p > 1$. Then*

$$ab \leq \frac{a^p}{p} + \frac{b^{p'}}{p'}.$$

Theorem 2.2.14. *[Hölder's inequality] Let $1 \leq p \leq \infty$. Assume that $u \in L^p(\Omega)$ and $v \in L^{p'}(\Omega)$. Then, $uv \in L^1(\Omega)$ and*

$$\int_\Omega |uv| \leq ||u||_p ||v||_{p'}.$$

A proof of this theorem can be found in [4].

Remark 2.2.6. A particular case of Hölder's inequality, known as the *interpolation inequality*, holds for $1 \leq p \leq q \leq \infty$, $u \in L^p(\Omega) \cap L^q(\Omega)$ and $p \leq r \leq q$. For $\beta \in [0,1]$ such that

$$\frac{1}{r} = \frac{\beta}{p} + \frac{1-\beta}{q},$$

we have

$$||u||_r \leq ||u||_p^{\beta} ||u||_q^{1-\beta}.$$

We now state the theorem and prove it.

Theorem 2.2.15 ($L^p(\Omega)$ is a normed space)**.** *Let* $1 \leq p \leq \infty$. *Then,* $\left(\mathrm{L}^p(\Omega), ||\cdot||_p\right)$ *is a normed space.*

Proof. The cases $p = 1$ and $p = \infty$ are trivial. We assume that $1 < p < \infty$ and let $u, v \in L^p(\Omega)$, generic. For any $x \in \Omega$ and since $|\cdot|$ is a norm on $\mathbb{R}$, by the triangle inequality for $|\cdot|$, we have that

$$|u(x) + v(x)|^p \leq (|u(x)| + |v(x)|)^p \leq 2^p(|(u(x)|^p + |v(x)|^p),$$

which implies that $u + v \in L^p(\Omega)$ and proves that $L^p(\Omega)$ is a vector space since $L^p(\Omega)$ is a subset of the space of linear functions and $f \in L^p(\Omega) \implies \lambda f \in L^p(\Omega)$.

Let's now prove that $||\cdot||_p$ is a norm. Conditions $i) - iii)$ are trivial, so we shall only prove the triangle inequality. We have that

$$\begin{aligned}
||u+v||_p^p &= \int_\Omega |u(x) + v(x)|^{p-1} |u(x) + v(x)| dx \\
&\leq \int_\Omega |u(x) + v(x)|^{p-1} (|u(x)| + |v(x)|) dx \\
&= \int_\Omega |u(x) + v(x)|^{p-1} |u(x)| dx + \int_\Omega |u(x) + v(x)|^{p-1} |v(x)| dx.
\end{aligned}$$

and by (2.5), noting that $p'(p-1) = p$,

$$|f + g|^{p-1} \in L^{p'}(\Omega).$$

Therefore, by Hölder's inequality we have that

$$||u+v||_p^p \leq ||u+v||_p^{p-1}(||u||_p + ||v||_p),$$

that is

$$||u+v||_p \leq ||u||_p + ||v||_p.$$

We conclude by the arbitrariness of u and v. $\square$

Theorem 2.2.16 (Fischer-Riesz)**.** $L^p(\Omega)$ *is a Banach space for any* p *such that* $1 \leq p \leq \infty$.

A proof of this theorem can be found in [4].

The following theorems are useful when doing computations on $L^p(\Omega)$ spaces.

Theorem 2.2.17 (Monotone convergence theorem). *Let* $(f_n)_{n\in\mathbb{N}} \subset \mathrm{L}^1(\Omega)$ *satisfying*

i) $f_n \leq f_{n+1}$ *a.e. on* Ω *for all* $n \in \mathbb{N}$*;*

ii) $\sup\limits_{n\in\mathbb{N}} \int f_n < \infty$.

Then, there exists $f \in L^1(\Omega)$ *such that* $f_n(x) \to f(x)$ *a.e. and* $||f_n - f||_1 \to 0$ *as* $n \to \infty$.

Theorem 2.2.18 (Dominated convergence theorem). *Let* $(f_n)_{n\in\mathbb{N}} \subset \mathrm{L}^1(\Omega)$ *satisfying*

i) $f_n(x) \to f(x)$ *a.e. on* Ω,

ii) $\exists g \in L^1(\Omega) : \forall n \in \mathbb{N}$*:* $|f_n(x)| \leq |g(x)|$ *a.e. on* Ω.

Then $f \in L^1$ *and* $||f_n - f||_1 \to 0$, *as* $n \to \infty$.

We denote by $C_c(\mathbb{R}^N)$ the space of all continuous functions on $\mathbb{R}^N$ with compact support, that is:

$$C_c(\mathbb{R}^N) := \{f \in C(\mathbb{R}^N) \ / \ \exists K \subseteq \mathbb{R}^N \text{ compact}, \forall x \in \mathbb{R}^N \setminus K : f(x) = 0\}.$$

This space is dense on $L^1(\Omega)$, a proof of this fact can be found in [4]. The notation $C_0(\mathbb{R}^N)$ is also used.

Theorem 2.2.19 (Kolmogorov-Riesz-Fréchet). *Let* $\mathcal{F} \subseteq L^p(\Omega)$ *be a bounded set,* $1 \leq p < \infty$. *Assume that*

$$\forall \varepsilon > 0, \exists \delta > 0, \forall f \in \mathcal{F} : |h| < \delta \implies ||\tau_h f - f||_p < \varepsilon.$$

Then, $\mathcal{F}|_\Omega$ *is relatively compact in* $\mathrm{L}^p(\Omega)$ *for any* $\Omega \subseteq \mathbb{R}^N$ *measurable. Here we denote* $\tau_h f := f(x + h)$ *as the shift of* f *for* $x, h \in \mathbb{R}^N$.

A proof of this theorem can be found in [4].

Theorem 2.2.20. *Let* $(f_n)_{n\in\mathbb{N}} \subseteq L^p(\Omega)$, $f \in L^p(\Omega)$ *such that*

$$\|f_n - f\|_{L^p(\Omega)} \to 0.$$

Then, there exists a subsequence $(f_{n_k})_k \in \mathbb{N} \subseteq L^p(\Omega)$ *and a function* $h \in L^p(\Omega)$ *such that*

a) $f_{n_k}(x) \to f(x)$ *almost everywhere on* Ω*;*

b) $\forall k \in \mathbb{N} : \quad |f_{n_k}| \leq h(x)$ *almost everywhere on* Ω.

A proof of this Theorem can be found in [4].

2.2.4 Hilbert spaces

We say that $(V, (\cdot,\cdot))$ is an inner-product space iff the functional $(\cdot,\cdot) : V \times V \to \mathbb{R}$ verifies:

i) $\forall u, v, w \in V : (u + v, w) = (u, w) + (v, w)$;

ii) $\forall \lambda \in \mathbb{R}, \forall u, v \in V : (\lambda u, v) = \lambda(u, v)$;

iii) $\forall u, v \in V : (u, v) = (v, u)$;

iv) $\forall u \in V : (u, u) \geq 0$.

Thus, the functional $(\cdot,\cdot)$ is known as an inner product.

Remark 2.2.7. Note that from the conditions above, it follows that

$$\forall u \in V : (u, u) = 0 \iff u = 0.$$

Proposition 2.2.1 (Inner product induces a norm)**.** *The inner product* $(\cdot,\cdot)$ *induces a norm* $||\cdot||$ *given by*

$$||u|| = \sqrt{(u, u)}.$$

Before proving proposition 2.2.1 we shall state and prove the Cauchy-Bunyakovsky-Schwarz (CBS) inequality for inner product spaces.

Lemma 2.2.21. *[Cauchy-Bunyakovsky-Schwarz inequality] Let* $(V, ||\cdot||)$ *be an inner-product space. Then,*

$$\forall x, y \in V : |(x, y)| = \sqrt{(x, x)}\sqrt{(y, y)}. \tag{2.6}$$

Proof. Let $x, y \in V$, generic. Let's denote

$$\|x\| = (x, x)^{1/2}.$$

We have that

$$0 \leq ||x - y||^2 = ||x||^2 - 2(x, y) + ||y||^2 \implies (x, y) \leq \frac{1}{2}(||x||^2 + ||y||^2),$$
$$0 \leq ||x + y||^2 = ||x||^2 + 2(x, y) + ||y||^2 \implies (x, y) \leq -\frac{1}{2}(||x||^2 + ||y||^2).$$

Then

$$|(x, y)| \leq \frac{1}{2}(||x||^2 + ||y||^2).$$

Therefore,

$$\begin{aligned} |(x, y)| &= |(\lambda x, \lambda^{-1} y)| \\ &\leq \frac{1}{2}\lambda^2||x||^2 + \frac{1}{2\lambda^2}||y||^2. \end{aligned}$$

By taking, in particular, $\lambda = ||y||||x||^{-1}$ we get our result. We conclude by the arbitrariness of x and y. $\square$

Proof of proposition 2.2.1. Let $x, y \in V$, generic. We shall only prove the triangle inequality for the induced norm since points i) and ii) are trivial. By the CBS inequality, we have that

$$\begin{aligned} ||x+y||^2 &:= (x+y, x+y) \\ &= ||x||^2 + 2(x,y) + ||y||^2 \\ &\leq ||x||^2 + ||x||||y|| + ||y||^2 \\ &= (||x|| + ||y||)^2. \end{aligned}$$

We conclude by the arbitrariness of x and y. □

Corollary 2.2.2 (Parallelogram equality)**.** *From the proof of Proposition 2.2.1, the following equality, known as the parallelogram equality, holds:*

$$\forall x, y \in V : \left\| \frac{x+y}{2} \right\|^2 + \left\| \frac{x-y}{2} \right\|^2 = \frac{1}{2}(||x||^2 + ||y||^2).$$

Definition 2.2.22 (Hilbert space)**.** An inner-product space $(V, (\cdot, \cdot))$ is said to be a *Hilbert space* iff it's complete with the norm induced by the inner product.

From now on, we shall refer to any arbitrary Hilbert space as H. We will now state some properties of Hilbert spaces.

Proposition 2.2.2. *Let H be a Hilbert space. Then, H is uniformly convex.*

Proof. Let $\varepsilon > 0$ and $u, v \in H$ generic be such that

$$|u| \leq 1, |v| \leq 1 \text{ and } |u - v| \geq \varepsilon.$$

By the parallelogram equality, we have

$$\left| \frac{u+v}{2} \right|^2 \leq 1 - \frac{\varepsilon^2}{4},$$

so that

$$\left| \frac{u+v}{2} \right| \leq 1 - \delta,$$

with

$$\delta = 1 - \left(1 - \frac{\varepsilon^2}{4}\right)^{\frac{1}{2}} > 0.$$

Since δ does not depend on u or v, we conclude our proof. □

Lemma 2.2.23 (Equality by using the inner product)**.** *Let V be an inner-product space and $u, v \in V$, then*

$$[\forall w \in V : (w, u) = (w, v)] \implies u = v.$$

Proof. Assume that $(w, u) = (w, v)$. Let $w \in V$, generic. Then, we have that

$$(w, u) = (w, v) \implies (w, u) - (w, v) = 0.$$

Now, by the linearity of the inner product and taking $w = u - v$ we obtain

$$\begin{aligned} 0 &= (w, u - v) \\ &= (u - v, u - v) \\ &= ||u - v|| \\ &\implies u - v = 0 \\ &\implies u = v. \end{aligned}$$

We conclude by the arbitrariness of w. □

Theorem 2.2.24 (Riesz-Fréchet representation theorem)**.** *Let H be a Hilbert space and $\psi \in H^*$. Then*

$$\exists! v \in H \ \ s.t \ \ \forall u \in H : \psi(u) = (u, v),$$

and

$$||\psi||_{H^*} = ||v||_H.$$

A proof of this theorem can be found in [18].

Definition 2.2.25 (Self-adjoint operator)**.** Let H be a real Hilbert space, we say that a linear operator $T : H \to H$ is self-adjoint iff

$$\forall u, v \in H : (Tu, v) = (u, Tv).$$

Theorem 2.2.26 (Banach-Alaoglu)**.** *Let H be a Hilbert space and $(x_n)_{n \in \mathbb{N}} \subseteq$ H be bounded. Then $(x_n)_{n \in \mathbb{N}}$ has a weakly convergent subsequence.*

A proof of this theorem can be found in [33, p.126].

2.3 Some topics on Sobolev spaces and partial differential equations

In this chapter we introduce some concepts about partial differential equations and Sobolev spaces, the main setting of our work, and establish some of their most important properties. Our sources for this section are [1], [4] and [10].

2.3.1 Partial Differential Equations

We call multiindex to a vector $\alpha = (\alpha_1, ..., \alpha_N) \in \mathbb{N}_*^N$ of order

$$|\alpha| = \alpha_1 + \cdots + \alpha_N.$$

Now let $U : \Omega \subseteq \mathbb{R}^N \longrightarrow \mathbb{R}$. Given a multiindex α, we write:

$$D^\alpha U(x) := \frac{\partial^{|\alpha|} U(x)}{\partial_{x_1}^{\alpha_1} \cdots \partial_{x_N}^{\alpha_N}} \equiv \partial_{x_1}^{\alpha_1} \cdots \partial_{x_N}^{\alpha_N} U.$$

Definition 2.3.1 (Partial differential equation)**.** Let's fix an integer $k \geq 1$. For $\Omega \subseteq \mathbb{R}^N$ open and a given $F : \mathbb{R}^{N^k} \times \mathbb{R}^{N^{k-1}} \times ... \times \mathbb{R}^N \times \mathbb{R} \times \Omega \to \mathbb{R}$ we define a partial differential equation (PDE) as an expression of the form

$$F(D^k u(x), D^{k-1} u(x), Du(x), u(x), x) = 0, x \in \Omega, \tag{2.7}$$

where we want to find, if it exists, a function $u : \Omega \to \mathbb{R}$ that satisfies (2.7). In this case, u is known as the solution of (2.7). The order of the PDE is the order of the highest derivative appearing in it.

Ideally, we want to find an explicit solution (or family of solutions) of (2.7) by adding some constraints such as initial or boundary conditions that reflect characteristics of a physical phenomena. It is not always possible to immediately find an explicit solution, so we are limited to proving that it exists, finding some of it's properties or approximating it using numerical methods.

We say that (2.7) is:

i) *Linear:* if F is linear with respect to u and it's derivatives, that is:

$$\sum_{|\alpha| \leq k} a_\alpha(x) D^\alpha u = f(x),$$

for given functions a, f. This PDE is homogeneous if $f \equiv 0$.

ii) *Semilinear:* if F is nonlinear with respect to u, but linear for its derivatives. That is, it has the form

$$\sum_{|\alpha| = k} a_\alpha(x) D^\alpha u + a_0(D^{k-1} u, ..., Du, u, x) = 0,$$

iii) *Quasilinear:* if F is linear for the highest derivatives of u. That is, it has the form

$$\sum_{|\alpha| = k} a_\alpha(D^{k-1} u, ..., Du, u, x) + a_0(D^{k-1} u, ..., Du, u, x) = 0,$$

iv) *Fully nonlinear:* if F is nonlinear for the highest order of derivatives.

Let $L \in C^k(\Omega)$ be a linear operator given by

$$Lu = \sum_{|\beta| \leq k} a_\beta(x) D^\beta u,$$

where β is a multiindex. Then, we say that L is elliptic iff

$$\forall x \in \Omega, \forall \zeta \in \mathbb{R}^N \setminus 0 : \quad \sum_{|\beta|=k} a_\beta(x)\zeta^\beta \neq 0.$$

Let $u \in C^2(\mathbb{R}^N)$, we say that u is harmonic if:

$$\Delta u = 0.$$

Additionally, if $\Delta u \leq 0$ and $\Delta u \geq 0$ we say that u is superharmonic and subharmonic, respectively.

We will now state the mean value theorem for harmonic functions, which says that any harmonic function u must be equal to the *average* of its values in a neighborhood of any point. For $\Omega \subseteq \mathbb{R}^N$ we use the following notation

$$\fint_\Omega u(x) \equiv \frac{1}{|\Omega|} \int_\Omega u(x)dx,$$

where $|\Omega|$ is the measure of Ω.

Theorem 2.3.2 (Mean value theorem for harmonic functions)**.** *Let $\Omega \subseteq \mathbb{R}^N$ be open and $u \in C^2(\Omega)$. If u is harmonic in Ω, then for any $x \in \Omega, r > 0$ such that $B_r(x) \subset\subset \Omega$, we have that*

$$u(x) = \fint_{B_r(x)} u(y)dx = \fint_{\partial B_r(x)} u(y)dS(y).$$

Conversely, if u verifies

$$u(x) = \fint_{\partial B_r(x)} u(y)dS(y),$$

then u is harmonic.

A proof of this theorem can be found in [10, Th.2 & 3, p.26], since its quite lengthy, it is not included. Additionally, we use the notation $B_r(x) \subset\subset \Omega$ to denote a compactly contained set, that is:

$$\overline{B_r(x)} \subset \Omega.$$

Theorem 2.3.3 (Strong maximum principle.)**.** *Let $\Omega \subseteq \mathbb{R}^N$ be bounded, $u \in C^2(\Omega)$ be a harmonic function. Then, if Ω is connected and u attains its maximum M on* $\mathrm{int}(\Omega)$ *then u is constant in Ω.*

Proof. Assume that

$$u(x_0) = \max_{\overline{\Omega}} u = M.$$

Let's set

$$A = \{x \in \Omega \ / \ u(x) = M\}.$$

Since u is continuous, A is closed. We shall prove that A is open. Let $B_r(x) \subset\subset \Omega$ be such that $x \in A$. Recall that

$$\forall y \in \Omega : \quad u(y) \leq M. \tag{2.8}$$

Now, by (2.8) and since u is harmonic, by Theorem 2.3.2, we have that

$$\begin{aligned} 0 &= u(x) - M \\ &= \fint_{B_r(x)} u(y)dy - M \\ &= \fint_{B_r(x)} [u(y) - M]dy \\ &\leq 0. \end{aligned}$$

And since $M - u(y) \geq 0$ we obtain

$$\forall y \in B_r(x) : u(y) = M.$$

That is, $B_r(x) \subset A$ and A is open. Since Ω is connected, $A = \Omega$. We conclude. □

2.3.2 $\mathbf{W^{1,p}}$ spaces

Let $\Omega \subseteq \mathbb{R}^N$, $1 \leq p \leq \infty$. We say $u \in L^p(\Omega)$ belongs to the Sobolev space $W^{1,p}(\Omega)$ if there exist $f_1, f_2, ..., f_N \in L^p(\Omega)$ such that

$$\forall \psi \in C_c^\infty(\Omega), \forall i = 1, 2, ..., N : \quad \int_\Omega u \frac{\partial \psi}{\partial x_i} = -\int_\Omega f_i \psi. \tag{2.9}$$

For the case when $p = 2$ we set

$$H^1(\Omega) = W^{1,2}(\Omega).$$

For $u \in W^{1,p}(\Omega)$ we define $\dfrac{\partial u}{\partial x_i} = f_i$ and we write

$$\nabla u = \left(\frac{\partial u}{\partial x_1}, \frac{\partial u}{\partial x_2}, ..., \frac{\partial u}{\partial x_N} \right) = \text{grad}(u),$$

which makes sense since f_i is unique a.e. The derivatives in the $W^{1,p}$ space are called *weak derivatives*. Note that when $N = 1$ the definition in (2.9) above coincides with the integration by parts formula.

The space $W^{1,p}(\Omega)$ is equipped with the norm:

$$||u||_{W^{1,p}(\Omega)} = ||u||_p + \sum_{i=1}^N \left\| \frac{\partial u}{\partial x_i} \right\|_p.$$

The space $H^1(\Omega)$ is equipped with the scalar product

$$(u,v)_{H^1(\Omega)} = (u,v)_{L^2(\Omega)} + \sum_{i=1}^{N} \left(\frac{\partial u}{\partial x_i}, \frac{\partial v}{\partial x_i} \right) = \int_\Omega uv + \sum_{i=1}^{N} \frac{\partial u}{\partial x_i}\frac{\partial v}{\partial x_i},$$

and the associated norm

$$||u||_{H^1(\Omega)} = \left(||u||_2^2 + \sum_{i=1}^{N} \left\| \frac{\partial u}{\partial x_i} \right\|_2^2 \right)^{1/2},$$

which is equivalent to the $W^{1,2}$ norm.

Remark 2.3.1. It is important to note that if $u \in C^1(\Omega) \cap L^p(\Omega)$ such that its partial derivatives belong to $L^p(\Omega)$, then they coincide with (2.9).

Some properties of $W^{1,p}(\Omega)$ are

i) $\forall 1 \le p \le \infty$: $\mathrm{W}^{1,p}(\Omega)$ is a Banach space.

ii) $\forall 1 < p < \infty$: $\mathrm{W}^{1,p}(\Omega)$ is reflexive and separable.

iii) $\mathrm{H}^1(\Omega)$ is a separable Hilbert space.

A proof of these facts can be found in [4, Prop. 9.1].

Remark 2.3.2. It is important to note that we could have also defined the $\mathrm{W}^{1,p}(\Omega)$ spaces as the completion of

$$\{u \in C^\infty(\Omega) / \quad \|u\|_{W^{1,p}(\Omega)} < \infty\},$$

with the $|| \cdot ||_{W^{1,p}(\Omega)}$ norm.

Similarly, for $1 \le p < \infty$ we can define the spaces $\mathrm{W}_0^{1,p}(\Omega)$ as the closure of $\mathrm{C}_0^\infty(\Omega)$ in $\mathrm{W}^{1,p}(\Omega)$. Therefore, we set $\mathrm{W}_0^{1,2}(\Omega) = \mathrm{H}_0^1(\Omega)$ as the closure of $\mathrm{C}_0^\infty(\Omega)$ respect to the norm $|| \cdot ||$ given by

$$||u||^2 := \int_\Omega |\nabla u(x)|^2 dx,$$

which is equivalent to the usual $H^1(\Omega)$ norm by Poincaré's inequality (2.13).

The space $\mathrm{W}_0^{1,p}(\Omega)$ is a separable Banach space, reflexive when $1 < p < \infty$. $\mathrm{H}_0^1(\Omega)$ is a Hilbert space.

Remark 2.3.3. In general, $\mathrm{W}^{1,p}(\Omega) \neq \mathrm{W}_0^{1,p}(\Omega)$ on an arbitrary subset of $\mathbb{R}^N$. These spaces are equal whenever $\Omega = \mathbb{R}^N$. Functions of $\mathrm{W}_0^{1,p}(\Omega)$ are "roughly" those of $\mathrm{W}^{1,p}(\Omega)$ that vanish on $\partial\Omega$. This is important since a function in $\mathrm{W}^{1,p}(\Omega)$ is only defined a.e. and the measure of $\partial\Omega$ is zero.

Remark 2.3.4. Given $U \subseteq \mathbb{R}^N$ we usually identify any $u \in \mathrm{H}_0^1(U)$ with its extension by zero, $\overline{u}$, of u in $\mathbb{R}^N \setminus U$ as an element of $\mathrm{H}^1(\mathbb{R}^N)$ in the sense

$$\overline{u} = \begin{cases} u(x), & \text{if } x \in U; \\ 0, & \text{if } x \in \mathbb{R}^N \setminus U. \end{cases} \tag{2.10}$$

We now consider $\Omega \subseteq \mathbb{R}^N$ to be open and of class C^1. Geometrically, this means that $\partial\Omega$ is smooth and, locally, is the image of a continuous function whose image is contained within Ω. Namely, $\partial\Omega$ is similar, locally, to $\partial B_1(0) \subseteq \mathbb{R}^N$, for more information about domains of class C^1 we refer the reader to [4] or [10].

Proposition 2.3.1. *Assume that $\Omega \subseteq \mathbb{R}^N$ is of class C^1. Let $1 < p < \infty$, $u \in L^p(\Omega)$. The following statements are equivalent:*

i) $u \in W_0^{1,p}(\Omega)$;

ii) there exists a constant $C > 0$ such that

$$\forall \psi \in \mathrm{C}_0^1(\mathbb{R}^N): \quad \left| \int_\Omega u \frac{\partial \psi}{\partial x_i} \right| \leq C \|\psi\|_{L^{p'}(\Omega)}, \quad \forall i = 1, 2, \ldots, N.$$

iii) the function

$$\overline{u} = \begin{cases} u(x), & \text{if } x \in \Omega; \\ 0, & \text{if } x \in \mathbb{R}^N \setminus \Omega. \end{cases}$$

Belongs to $W^{1,p}(\mathbb{R}^N)$ and, in this case $\dfrac{\partial \overline{u}}{\partial x_i} = \overline{\dfrac{\partial u}{\partial x_i}}$.

A proof of this proposition can be found in [4, p. 304].

Theorem 2.3.4 (A weak maximum principle)**.** *Let $\Omega \subseteq \mathbb{R}^N$, $u \in \mathrm{H}^1(\Omega)$. Let L be an elliptic operator satisfying $Lu \geq 0$ ($Lu \leq 0$) on Ω, then*

$$\sup_\Omega u \leq \sup_{\partial\Omega} u^+ \quad (\inf_\Omega u \geq \inf_{\partial\Omega} u^-),$$

where $u^+ := \{x \in \Omega : \quad u(x) > 0\}$, $u^- := \{x \in \Omega : \quad u(x) < 0\}$.

A proof of this theorem can be found in [15, p. 179].

2.3.3 Sobolev inequalities and immersion results

In this section we shall state, for dimension $N \geq 2$, the *Sobolev embedding theorem*. We begin by considering the following when $\Omega = \mathbb{R}^N$.

Theorem 2.3.5. *[Sobolev, Gagliardo, Niremberg] Let $1 \leq p < N$. Then*

$$W^{1,p}(\mathbb{R}^N) \subset L^{p^*}(\mathbb{R}^N), \tag{2.11}$$

and there exists a constant $C = C(p, N)$ such that

$$\forall u \in W^{1,p}(\mathbb{R}^N) : ||u||_{p^*} \leq C||\nabla u||_p. \tag{2.12}$$

Where, p^* is given by $p^* = \dfrac{pN}{N-p}$.

A proof of this theorem can be found in [4].

Corollary 2.3.1. *Let $1 \leq p < N$. Then*

$$\forall q \in [p, p^*] : \quad W^{1,p}(\mathbb{R}^N) \subset L^q(\mathbb{R}^N).$$

Proof. Let $u \in L^q(\mathbb{R}^N)$, $q \in [p, p^*]$ generic. Then, for any $\lambda \in [0, 1]$ we have that:

$$\frac{1}{q} = \frac{\lambda}{p} + \frac{1-\lambda}{p^*},$$

then the interpolation inequality implies that

$$||u||_q \leq ||u||_p^\lambda ||u||_{p^*}^{1-\lambda},$$

so that

$$||u||_q \leq ||u||_p^\lambda ||u||_{p^*}^{1-\lambda} \leq C||u||_{W^{1,p}}.$$

Since u was taken arbitrarily, we conclude. □

Corollary 2.3.2. *Let $1 \leq p < \infty$. We have the following continuous injections*

$$\begin{cases} \mathrm{W}^{1,p}(\Omega) \subset \mathrm{L}^{p^*}(\Omega) & \text{if} \quad p < N, \\ \forall q \in [p.\infty) : \mathrm{W}^{1,p}(\Omega) \subset \mathrm{L}^q(\Omega) & \text{if} \quad p = N, \\ \mathrm{W}^{1,p}(\Omega) \subset \mathrm{L}^\infty(\Omega) & \text{if} \quad p > N. \end{cases}$$

Moreover, if $p > N$, for any $u \in \mathrm{W}^{1,p}(\Omega)$ and for almost all $x, y \in \Omega$ we have that

$$|u(x) - u(y)| \leq C||u||_{W^{1,p}}|x - y|^\alpha,$$

with $\alpha = 1 - (N/p)$, $C = C(\Omega, p, N)$. In particular, we have

$$\mathrm{W}^{1,p}(\Omega) \subset C(\overline{\Omega}).$$

A proof of this result can be found in [4].

Theorem 2.3.6 (Rellich-Kondrachov). *Assume that Ω is bounded and of class C^1. Then, we have the following* ***compact*** *injection for $p < N$.*

$$\forall q \in [1, p^*) : \mathrm{W}^{1,p}(\Omega) \subset \mathrm{L}^q(\Omega).$$

A proof of this result can be found in [4].

Remark 2.3.5. Theorem 2.3.6 tells us that we can transform bounded sequences in $\mathrm{W}^{1,p}(\Omega)$ into sequences that have convergent subsequences converging in $\mathrm{L}^q(\Omega)$.

We shall finish by stating Poincaré's inequality.

Corollary 2.3.3. *Assume that $1 \leq p < \infty$ and that Ω is open and bounded. Then, there exists a constant $C = C(\Omega, p)$ such that:*

$$\forall u \in \mathrm{W}_0^{1,p}(\Omega) : ||u||_{\mathrm{L}^p(\Omega)} \leq C||\nabla u||_{\mathrm{L}^p(\Omega)}. \tag{2.13}$$

2.4 Some topics on nonlinear analysis and variational calculus

On this chapter we will present some results of nonlinear analysis that are relevant to our work. The main references for this section are [7],[10], [19] and [31].

2.4.1 Differentiability in normed spaces

We will start with the definitions of small o and directional derivative in order to lead us to weak and strong derivatives and their properties.

Let E, F be normed spaces, $\mathcal{O} \subseteq E$ open such that $0 \in \mathcal{O}$, and $g : \mathcal{O} \to F$ such that $g(0) = 0$. Let

$$\varepsilon : B(0, r) \subseteq E \to F,$$

be a mapping such that

$$\lim_{h \to 0} \varepsilon(h) = 0,$$

then, we say that g is a *small o of h*, denoted

$$g(h) = o(h)$$

iff

$$g(h) = ||h||\varepsilon(h)$$

which is equivalent to

$$\lim_{h\to 0}\frac{1}{||h||}g(h)=0=\lim_{h\to 0}\frac{||g(h)||}{||h||}.$$

Let $u\in\mathcal{O}$ be a point, $h\in E$ a direction and $f:\mathcal{O}\to F$. We call the ***directional derivative*** *of f at u in the direction h* to the limit

$$\partial_h f(u)=\lim_{t\to 0}\frac{1}{t}[f(u+th)-f(u)], \tag{2.14}$$

if it exists. Note that in (2.14) we have that $\partial_{\lambda h}f(u)=\lambda\partial_h f(u)$, for some $\lambda\in\mathbb{R}$. Indeed, by taking $\alpha\lambda=t$

$$\begin{aligned}\lambda\partial_h f(u)&=\lambda\lim_{t\to 0}\frac{1}{t}[f(u+th)-f(u)]\\&=\frac{t}{\alpha}\lim_{t\to 0}\frac{1}{t}[f(u+\lambda\alpha h)-f(u)]\\&=\lim_{t\to 0}\frac{1}{\alpha}[f(u+\lambda\alpha h)-f(u)]\\&=\partial_{\lambda h}f(u).\end{aligned}$$

This allows us to define the following concepts:

Definition 2.4.1 (Weak derivative)**.** We say that f is *Gateaux (or weakly) differentiable* iff for every direction h the directional derivative exists and

$$\exists f'_G(u)\in\mathcal{L}(E,F),\forall h\in E:\quad \partial_h f(u)=f'_G(u)h. \tag{2.15}$$

Since the operator that satisfies (2.15) is unique, it is referred to as the *Gateaux (or weak) differential* of f at u.

Definition 2.4.2 (Fréchet differentiability)**.** Let $u\in\mathcal{O}$ be a point and $f:\mathcal{O}\to F$. If $\exists\phi\in\mathcal{L}(E,F)$ such that

$$\forall h\in E: u+h\in\mathcal{O}\implies f(u+h)-f(u)=\phi(h)+o(h), \tag{2.16}$$

then we say that f is **Fréchet, or strongly, differentiable at** u.

If f is differentiable in every point of $\mathcal{O}_1\subseteq\mathcal{O}$ then we say that f is *differentiable on* $\mathcal{O}_1$. If f is differentiable at all the points in its domain, we simply say that f is **differentiable**.

As in weak differentiation, we also have uniqueness of the bounded operator shown in (2.16). This is stated in the following proposition.

Proposition 2.4.1 (Uniqueness of the Fréchet differential)**.** *Let E,F be normed spaces, $\mathcal{O}\subseteq E$, $u\in\mathcal{O}$ and $f:\mathcal{O}\to F$. If f is Fréchet differentiable, then its differential is unique.*

Proof. Let $\varphi \in \mathcal{L}(E,F)$ be such that for every $h \in E$, with $u+h \in \mathcal{O}$, it implies that

$$f(u+h) - f(u) = \varphi(h) + o(h). \tag{2.17}$$

We have to prove that $\phi = \varphi$, that is,

$$\forall v \in E : \phi(v) = \varphi(v).$$

So let $v \in E$, generic. Since $\mathcal{O}$ is open, there exists $r > 0$ such that

$$B(u,r) = u + B(0,r) \subseteq \mathcal{O}.$$

Then, from (2.16) and (2.17) we get

$$\forall h \in B(0,r) : \quad \phi(h) + ||h||\varepsilon_1(h) = \varphi(h) + ||h||\varepsilon_2(h), \tag{2.18}$$

with functions $\varepsilon_1, \varepsilon_2 : B(0,r) \to F$ vanishing to 0 as h goes to 0. We distinguish two cases:

1. If $v = 0$, then, by linearity, we have that $0 = \phi(v) = \varphi(v)$.
2. If $v \neq 0$ we choose $N \in \mathbb{N}$ such that for every $n \in \mathbb{N}$ with $n > N$

$$h_n = \frac{1}{n} \cdot \frac{1}{||v||} v \in B(0,r).$$

So that (2.18) provides

$$\phi(h_n) - \varphi(h_n) = ||h_n||[\varepsilon_2(h_n) - \varepsilon_1(h_n)]$$

and

$$\phi(v) - \varphi(v) = ||v||[\varepsilon_2(h_n) - \varepsilon_1(h_n)].$$

Now, since ε_1 and ε_2 vanish at 0, we let $n \to \infty$ and get our desired result. We conclude by the arbitrariness of v. $\square$

The previous allows us to rewrite (2.16) as:

$$f(u+h) - f(u) = f'(u)h + o(h).$$

And refer to the operator

$$f'(u) \in \mathcal{L}(E,F),$$

as the *differential* of f at u. This operator is sometimes referred as the Fréchet differential of f at u. Moreover, we shall denote $f'(u) = \phi(u)$ evaluated at $v \in E$ by $\phi(u)v$ and $Df(u) = f'(u)$.

Remark 2.4.1. It is clear that Fréchet (strong) differentiation implies Gateaux (weak) differentiation, and whenever this happens we write

$$Df(u) = f'(u) = f'_G.$$

Whenever f is a functional, then $f'(u) \in E^*$ and it is called the (first) *variation* of f at u, written as:

$$f(u+h) - f(u) = \langle f'(u), h \rangle + o(h).$$

Let us state some useful proposition and properties of the differential.

Proposition 2.4.2 (Differentiability implies continuity). *Let $f : \mathcal{O} \subseteq E \to F$. If f is differentiable at $u \in \mathcal{O}$, then f is continuous at u.*

Proof. We have that

$$\forall h \in E : u + h \in \mathcal{O} \implies f(u+h) = f(u) + f'(u)h + o(h).$$

Now, since $f'(u)$ is a continuous linear operator and since $\lim_{h\to 0} o(h) = 0$, we have that

$$\lim_{h\to 0} f(u+h) = f(u),$$

that is

$$\lim_{x\to u} f(x) = f(u),$$

so that, f is continuous at u. □

The next two results involve the operator $f : \mathcal{O} \subseteq E \to F$. We have that,

1. If f is a constant operator, then $\forall u \in \mathcal{O} : f'(u) = 0$.
2. If $f \in \mathcal{L}(E, F)$, then $\forall u \in E : f'(u) = f$.

Additionally, differentiability is linear, that is:

Proposition 2.4.3 (Linearity). *Let $\lambda \in \mathbb{R}$ and $f, g : \mathcal{O} \subseteq E \to F$ be differentiable at $u \in \mathcal{O}$. Then $\lambda f + g$ is differentiable at u and:*

$$(\lambda f + g)'(u) = \lambda f'(u) + g'(u).$$

Proof. Since f and g are differentiable, we have that:

$$\forall h \in E : \; u + h \in \mathcal{O} \implies f(u+h) - f(u) = f'(u)(h) + ||h|| \cdot \varepsilon_1(h), \tag{2.19}$$

$$\forall h \in E : \; u + h \in \mathcal{O} \implies g(u+h) - g(u) = g'(u)(h) + ||h|| \cdot \varepsilon_2(h), \tag{2.20}$$

where

$$\varepsilon_1 \to 0 \quad \text{and} \quad \varepsilon_2 \to 0, \quad \text{as} \quad h \to 0.$$

We have to prove that:

$$\forall h \in E : u + h \in \mathcal{O} \implies (\lambda f + g)(u + h) - (\lambda f + g)(u) = \varphi(h) + ||h|| \cdot \varepsilon(h), \tag{2.21}$$

where $\varphi = \lambda f'(u) + g'(u) \in \mathcal{L}(E, F)$ and

$$\varepsilon(h) \to 0 \quad \text{as} \quad h \to 0.$$

So let $h \in E$ be such that $u + h \in \mathcal{O}$. By (2.19) and (2.20), we have that:

$$\begin{aligned}(\lambda f + g)(u + h) - (\lambda f + g)(u) &= \lambda[f(u + h) - f(u)] + [g(u + h) - g(u)] \\ &= \lambda[f'(u)(h) + ||h|| \cdot \varepsilon_1(h)] + [g'(u)(h) + ||h|| \cdot \varepsilon_2(h)] \\ &= \varphi(h) + ||h||\varepsilon(h),\end{aligned}$$

where

$$\varepsilon(h) = \varepsilon_1(h) + \varepsilon_2(h) \to 0 \quad \text{as} \quad h \to 0.$$

We conclude by the arbitrariness of h. □

Proposition 2.4.4 (Differential of a product)**.** *Let F be a commutative algebra. Let $f, g : \mathcal{O} \to F$ be differentiable at $u \in \mathcal{O}$, then, $f \cdot g$ is differentiable at u and*

$$(f \cdot g)'(u) = g(u)f'(u) + g'(u)f(u).$$

For the proof of this proposition we refer our reader to [7] or [19].

Now, assume that G is a normed space, $\mathcal{U} \subseteq F$ is open and that the operators

$$f : \mathcal{O} \to F, \qquad g : \mathcal{U} \to G,$$

verify that $f(\mathcal{O}) \subseteq \mathcal{U}$. So that the mapping $g \circ f$ is defined on $\mathcal{O}$, i.e.,

$$\begin{aligned} g \circ f : \mathcal{O} &\longrightarrow G \\ X &\longmapsto (g \circ f)(u) = g(f(u)). \end{aligned}$$

and we have:

Theorem 2.4.3 (Chain rule)**.** *Assume that $f(\mathcal{O}) \subseteq \mathcal{U}$. If f is differentiable at $u \in \mathcal{O}$ and g is differentiable at $f(u)$, then $g \circ f$ is differentiable at u and*

$$(g \circ f)' = g(f(u)) \circ f'(u).$$

The proof of this theorem is based on (2.16) and and it can be reviewed in [7] and [19].

2.4.2 The C^1 class

Now we will define an important concept such as the class of continuous functions with one continuous derivative, or C^1 class. As before, consider E, F as normed spaces and $\mathcal{O} \subseteq E$ as an open set.
We say that $f : \mathcal{O} \to F$ belongs to the class $C^1(\mathcal{O}, F)$ iff

1. f is differentiable.

2. the function

$$\begin{aligned} f' : \mathcal{O} \subseteq E &\longrightarrow \mathcal{L}(E, F) \\ x &\longmapsto f'(x), \end{aligned}$$

   is continuous.

Whenever these two conditions hold, and there is no confusion, we say that f is of class C^1.

Remark 2.4.2. We have the following result on the C^1 class. Let $\Omega \subseteq \mathbb{R}^N$ open and $f : \Omega \longrightarrow \mathbb{R}$ be such that all its partial derivatives exist and are continuous at Ω. Then

$$f \in C^1. \tag{2.22}$$

2.4.3 Palais-Smale condition and the Krasnoselskii genus

Before explaining what the Palais-Smale condition for sequences is, we shall explain a little bit about critical points and extremum.

Consider X as a non-void set and $f : X \longrightarrow \mathbb{R}$, we have the following definitions:

i) We say that $a \in X$ is *a* point of (global) minimum iff

$$\forall a \in X : \quad f(a) \leq f(x).$$

   If this holds, we say that the value $f(a)$ is *the* minimum of f.

ii) We say that $a \in X$ is *a* point of (global) maximum iff

$$\forall a \in X : \quad f(a) \geq f(x).$$

   If this holds, we say that the value $f(a)$ is *the* maximum of f.

iii) A point $a \in X$ of either minimum or maximum is called *a point of (global) extremum.* In this case, we say the the corresponding value $f(a)$ is *an extremum of* f.

Additionally, we have that a continuous function on a compact set always achieves its extremums.

Theorem 2.4.4 (Extremum on a compact)**.** *Let $(X, \mathcal{T})$ be a topological space, $A \subseteq X$ compact and $f \in C(X)$. Then f achieves its extremums in A, that is*

$$\exists x_m, x_M \in A, \forall x \in A : \quad f(x_m) \leq f(x) \leq f(x_M).$$

Theorem 2.4.4 provides an existence result but provides no uniqueness, however, whenever $x_m \in A$ is the only point where f achieves its minimum, we say that x_m is *the point of strict minimum* and that the value $f(a)$ is *the strict minimum.* We can speak the same way of the *strict maximum.*

Consider now $Y \subseteq X$ and $g : Y \longrightarrow \mathbb{R}$. We say that $a \in Y$ is a point of local minimum of f iff

$$\exists G \in \mathcal{N}(a), \forall x \in G \cap Y : g(a) \leq g(x).$$

So whenever this holds, we say that the value $f(a)$ is a *local* minimum of f. We can say the same for *local maximum* and *local extremum.*

Definition 2.4.5 (Critical point)**.** Let E, F be normed spaces, $\mathcal{O} \subseteq E$ open and $f : \mathcal{O} \longrightarrow \mathbb{R}$. We say that $x \in \mathcal{O}$ is a critical point of f iff f is differentiable at x and

$$f'(x) = 0.$$

We shall denote the set of all critical points of f as:

$$\mathcal{K}(f) := \{x \in \mathcal{O} / f'(x) = 0\}.$$

The corresponding value $f(x) = c \in \mathbb{R}$ is called a *critical value* or a *critical level.* If c is not a critical value, it is called a *regular value.*

Corollary 2.4.1. *Let E be a normed space, $\mathcal{O} \subseteq$ and $f : \mathcal{O} \to \mathbb{R}$. Assume that:*

i) f has a local extremum at $x \in \mathcal{O}$;

ii) f is differentiable at x.

Then, x is a critical point of f, i.e.,

$$f'(x) = 0.$$

We are now ready to define the Palais-Smale condition for sequences.

Definition 2.4.6 (Palais-Smale)**.** Let E be a Banach space and $\Phi \in C^1(E)$. Then $(u_n)_{n\in\mathbb{N}} \subseteq E$ is called a Palais-Smale (PS) sequence iff

$$(\Phi(u_n)) \text{ is bounded} \quad \text{and} \quad \Phi'(u_n) \to 0.$$

If $\Phi(u_n) \to c \in \mathbb{R}$ and $\Phi'(u_n) \to 0$, then (u_n) is a $(PS)_c$-sequence. The functional Φ is said to satisfy the (PS) condition (or $(PS)_c$ condition) if each (PS) (or $(PS)_c$) sequence has a convergent subsequence.

Remark 2.4.3. It is clear that if a (PS) sequence, or a subsequence, converges to u, then u is a critical point of Φ.

In the following subsections we will use the concept of manifold, so let us define it.

Definition 2.4.7 (C^k differentiable manifold)**.** Let H be a Hilbert space, $O \subseteq H$ open. $\mathcal{M} \subseteq H$ closed is said to be a C^k differentiable manifold iff there exists $\Phi \in C^k(U)$, $c \in \mathbb{R}$ a regular value of Φ and

$$\mathcal{M} = \Phi^{-1}(c).$$

Moreover, for $u \in \mathcal{M}$ we say that

$$T_u\mathcal{M} := \text{Ker}(D\Phi(u)),$$

is the tangent space of $\mathcal{M}$ at u.

Krasnoselskii's genus

Let E be a Banach space. We define the class of all closed, symmetric subsets of $A \subseteq E$ that do not contain 0 as

$$\Sigma_E = \{A \in E / \quad A = \overline{A}, A = -A, 0 \notin A\}.$$

The *genus* of A, denoted $\gamma(A)$, is the smallest integer k such that there exists and odd mapping $h \in C(A, \mathbb{R}^k - \{0\})$. That is, for a given $A \in \Sigma_E$ we set

$$K = \{k \in \mathbb{N} / \quad \exists f \in C(A, \mathbb{R}^k \setminus \{0\}) \text{ odd }\}.$$

and the Krasnoselskii genus as:

$$\gamma(A) := \inf(K),$$

We set $\gamma(\emptyset) = 0$ and $\gamma(A) = \infty$ if $K = \emptyset$.
The concept of Krasnoselskii's genus is a generalization of the concept of dimension and it is possible to use it to explain the previously described critical point theory.

The following lemma states some properties of the genus.

Lemma 2.4.8. *Let $A_1, A_2 \in \Sigma_E$. Then, we have that:*

i) *If $A_1 \subseteq A_2$ then $\gamma(A_1) \leq \gamma(A_2)$,*

ii) $\gamma(A_1 \cup A_2) \leq \gamma(A_1) + \gamma(A_2)$,

iii) *If $\eta \in C(A, E)$ is odd, then $\gamma(A) \leq \gamma(\eta(A))$.*

iv) *If A is compact then $\gamma(A) < \infty$ and there exists a symmetric neighborhood U_A of A such that*

$$\gamma(\overline{U_A}) = \gamma(A).$$

A proof of this lemma can be found in [3].

Let S denote the unit sphere in E, that is:

$$S_1(0) := \{u \in E : \quad ||u|| = 1\},$$

the following theorem regarding critical points and the genus holds.

Theorem 2.4.9. *Let E be an infinite dimensional Banach space. Assume that $\Phi \in C^1(S, \mathbb{R})$ is bounded from below satisfies the PS condition, then*

$$\Phi \text{ has infinitely many pairs of critical points.}$$

A proof of this theorem can be found in [27]. While we do not include the proof here, it is important to show that for any $j \in \mathbb{N}$ it defines

$$\Gamma_j = \{A \subset S : \ A = -A, A \text{ compact and } \gamma(A) > j\},$$

and point out that it shows that every

$$c_j = \inf_{A \in \Gamma_j} \sup_{u \in A} \Phi(u),$$

is a critical level. Moreover if $c_j = ... = c_{j+p}$ for some $p \geq 0$, then:

$$\gamma(K_{c_j}) \geq p + 1,$$

where

$$K_{c_j} := \{u \in S : \quad \Phi(u) = c_j \ \text{ and } \ \Phi'(u) = 0\}.$$

Hence, as in [31], the number of critical points is infinite regardless of whether the number of distinct c_j's is finite or not.

2.4.4 Nehari Manifolds

In order to describe Nehari manifolds in an abstract setting we shall assume that E is a real Banach space and $\Phi \in C^1(E)$. Recall that the Fréchet derivative of Φ at u belongs to the dual space, that is:

$$\Phi'(u) \in E^*.$$

Furthermore, assume that $u \neq 0$ is a critical point of Φ. Then, necessarily, u is contained in the set

$$\mathcal{N} := \{u \in E \setminus \{0\} : \quad \Phi'(u)u = 0\}. \tag{2.23}$$

So that $\mathcal{N}$ is a natural constraint for the problem of finding nontrivial critical points of Φ and it is called the *Nehari manifold* although, in general, it may not be a manifold. By setting

$$c := \inf_{u \in \mathcal{N}} \Phi(u), \tag{2.24}$$

we hope that c is attained at some $u_0 \in \mathcal{N}$ under appropriate conditions.

Now, without loss of generality, we assume that E is uniformly convex, $\Phi(0) = 0$. We say that $\varphi \in C(\mathbb{R})$ is a *normalization function* iff

i) $\varphi(0) = 0$;

ii) φ is strictly increasing;

iii) $\varphi(t) \to \infty$ as $t \to \infty$.

We shall need the following assumptions:

(A1) There exists a normalization function φ such that

$$u \mapsto \varphi(u) := \int_0^{||u||} \varphi(t)dt \quad \in C^1(E \setminus \{0\}, \mathbb{R}).$$

So that $J := \varphi'$ is bounded on bounded sets and for every $w \in S : J(w)w = 1$.

(A2) For any $w \in E \setminus \{0\}$ there exists $s_w \in (0, \infty)$ such that if we set the function $\alpha_w(s) := \Phi(sw)$ we have that:

$$\begin{aligned} 0 < s < s_w &\implies \alpha'_w(s) > 0, \\ s > s_w &\implies \alpha'_w(s) < 0. \end{aligned}$$

(A3.a) There exists $\delta > 0$ such that

$$s_w \geq \delta;$$

(A3.b) for any $\mathcal{W} \subset S$ compact, there exists $C_W \in \mathbb{R}$ such that

$$\forall w \in \mathcal{W} : \quad s_w \leq C_W.$$

The functional J in (A1) is called the duality mapping corresponding to φ. For (A1) to hold we need $||\cdot|| \in C^1(E \setminus \{0\}, \mathbb{R})$. The case that will interest us the most is $E = \mathrm{W}_0^{1,p}(\Omega)$ with $\Omega \subset \mathbb{R}^N$ bounded, $p > 1$ and setting $\varphi(t) := t^{p-1}$. The associated functional ψ is given by

$$\psi(u) = \frac{1}{p}||u||^p,$$

and the duality mapping

$$J = \psi' : E \to E^*,$$

given by

$$\langle J(w), v\rangle = \int_\Omega |\nabla w(x)|^{p-2} \nabla w(x) \cdot \nabla v(x) dx,$$

is continuous and bounded on bounded sets.

From (A2) we have that for any $w \in S_1(0)$ the function α_w attains a unique maximum $s_w \in (0, \infty)$ such that

$$\begin{aligned} 0 < s < s_w &\implies \alpha'_w(s) > 0, \\ s > s_w &\implies \alpha'_w(s) < 0. \end{aligned}$$

And, for some $\delta > 0$ independent of w

$$s_w > \delta.$$

Then

$$\alpha'_w(s_w) = \Phi'(s_w w) = 0.$$

So that $s_w w$ is the unique point on the ray $s \mapsto sw, s > 0$, which intersects $\mathcal{N}$.

Moreover $\mathcal{N}$ is closed and bounded away from 0 by the first part of (A3), let's use the mappings defined in [31]

$$\hat{m} := E \setminus \{0\} \longrightarrow \mathcal{N} \quad \text{and} \quad m : S \longrightarrow \mathcal{N},$$

by setting

$$\hat{m}(w) = s_w w \quad \text{and} \quad m := \hat{m}|_S.$$

So that the following proposition holds.

Proposition 2.4.5. *Assume that Φ satisfies (A2), (A3.a) and (A3.b). Then*

a) The mapping $\hat{m}$ is continuous.

b) The mapping m is a homeomorphism (bijective and bicontinuous) between S and $\mathcal{N}$, and the inverse of m is given by $m^{-1}(u) : u/||u||$.

Proof. (a) Assume that $w_n \to w \neq 0$. Since

$$\forall t > 0 : \hat{m}(tw) = \hat{m}(w),$$

we assume that $w_n \in S$ for any $n \in \mathbb{N}$, so we need to prove that

$$\hat{m}(w_n) = \hat{m}(w), \tag{2.25}$$

after passing to a subsequence. Let's denote

$$\hat{m}(w_n) = s_n w_n.$$

By (A2), (A3.a) and (A3.b), we have that the sequence $(s_n)_{n\in\mathbb{N}}$ is bounded and bounded away from 0, so that we can take a subsequence

$$s_n \to \overline{s} > 0. \tag{2.26}$$

Since $\mathcal{N}$ is closed and by (2.26) we have that

$$\overline{s}w \in \mathcal{N}.$$

hence

$$\overline{s}w = s_w w = \hat{m}(w).$$

Which proves (2.25).

(b) Is a direct consequence of (a).

We conclude our proof.

□

Additionally, it is clear that c in (2.24) is positive if attained. We also have that $u_0 \in \mathcal{N}$ is a critical point whenever $\Phi(u_0) = c$. Note that since $s \mapsto \alpha_w(s)$ is increasing for any $w \in S$, 0 is a local minimum and a critical point of Φ.
Since u_0 is a solution of the equation

$$\Phi'(u) = 0,$$

that has minimal energy Φ in the set of all nontrivial solutions, we shall call it a *ground state*.

Let's remark that a point $u \in E$ is a nonzero critical point of Φ if and only if $u \in \mathcal{N}$ and u is critical for the restriction of Φ to $\mathcal{N}$. So that we can apply critical point theory on the Nehari manifold in order to find critical points of Φ.

Chapter 3

A short introduction to Quantum Mechanics

In this chapter we provide some elementary ideas about Quantum Mechanics (QM). We begin by presenting a historical overview. Then, in Sections 3.3 and 3.4 we consider the mathematical approaches followed by Erwin Schrödinger and Werner Heisenberg, which helped to give birth to the field. The main sources for this chapter are [14], [16], [18] and [26].

3.1 Historical background

Throughout its years of existence, quantum theory has proven to be exceptionally fruitful and interesting. The perception that our physical world, that had been considered to be an area of clear, determined and consistent problems, was shattered in the early years of the 20th century because of various discoveries that showed that, at a subatomic level, our world was erratic and cloudy in it's behavior, [26]. This was the biggest change in the way we understood our world since the days of Sir Isaac Newton, so much that the progenitor of relativity theory, Albert Einstein, resolutely opposed it up onto the very end of his life.

The starting point of QM was Planck's idea that electromagnetic radiation is emitted and absorbed in discrete amounts of energy, called quanta. This was further strengthened by Einstein since quanta proved useful in fixing an inconsistency in the framework of the *photoelectric effect*, [26]. Thus, the first quantum particles to be named emerged: photons. These provided a glimpse into the wave-particle duality of light. Then came the atomic model proposed by Bohr, [26], which stated that electrons orbited around fixed orbits, jumping from orbit to orbit without going through intermediate states.

All of this culminated in Heisenberg and Schrödinger's work: QM. Also came the incorporation of quantum electro-magnetic radiation which was accomplished by Jordan, Pauli, Heisenberg, Born, Dirac and Fermi. We will briefly describe the Geiger-Marsden experiments, [26], and present, as shown in [14], a version of the double-slit experiment that confirms the particle-wave duality of light.

In 1911 Rutherford conducted the Geiger-Marsden experiments in which the planetary model of an atom was established by studying how some small, positively charged projectiles called α-*particles* behaved when they hit a thin gold film. Many of these α-*particles* passed trough unaffected but some were substantially deflected, making Rutherford theorize that it was because the positive charge of gold atoms could not be spread around and instead must be concentrated around the centre of the atom. This model proposed that almost all the weight of the atom would be concentrated on a tiny nucleus of $10^{-13} - 10^{-12}$ cm at the center, with electrons orbiting around it, [14]. With the electrons being repelled or attracted to the nucleus via Coulomb forces. However, in classical physics this model is unstable since the size of an atom is about 10^{-8} cm.

The double-slit experiment

This experiment is a modern version equivalent to the scattering of electrons conducted by Young in 1805. As seen in [14], it can be abstracted as the double-slit experiment, wherein an interference pattern for electrons is displayed, similar to that of waves.

We assume that a current of electrons is fired at a wall, acting as a shield, in which two slits have been cut. On the other side of the wall there is a detector screen.

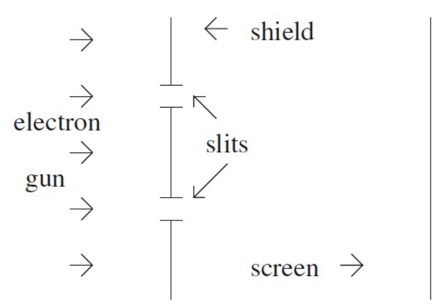

Figure 3.1: Shield set up and electron firing. Source [14].

The electrons that pass trough the slits will hit the sensor barrier, in whose case their impact positions will be recorded. If either one of the slits is closed, after sufficient impacts there will be enough data for a intensity distribution, pictured in Figures 3.2 and 3.3

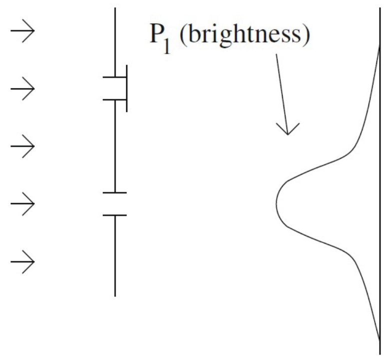

Figure 3.2: Intensity distribution when slit 1 is blocked. Source [14].

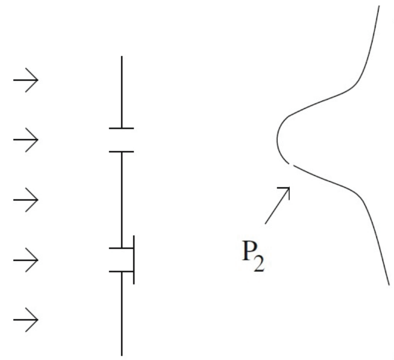

Figure 3.3: Intensity distribution when slit 2 is blocked. Source [14].

A striking result is shown in Figure 3.4 when both slits are open.

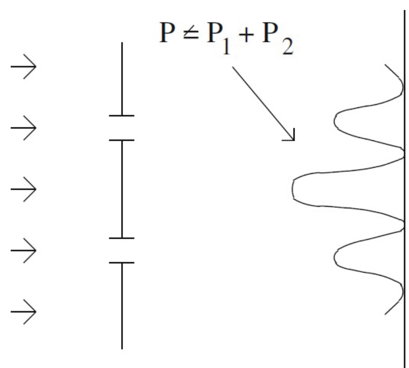

Figure 3.4: Intensity distribution when both slits are open. Source [14].

Contrary to what we would expect, the intensity distribution is not the sum of the previous, that is; $P \neq P1+P2$. Based on this observations, we claim that matter behaves in a random way since we cannot exactly predict where a given electron will hit the sensor, we can only determine the distribution of impacts, [14].

Remark 3.1.1. It is important to observe that the intensity pattern we observe when both slits are open is similar to the one seen when a wave propagates trough the slits, as seen in Figure 3.5 below.

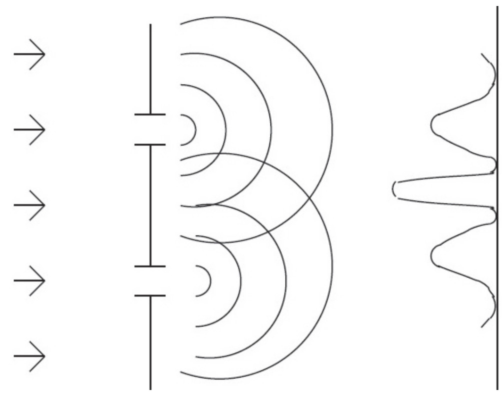

Figure 3.5: Pattern generated when waves go through the slits. Source [14]

A wave enters through each slit, splitting into new waves E_1 and E_2, represented by complex numbers that encode the information amplitude and phase, that crash into each other, the split waves generate a combined pattern that is proportional to:

$$|E_1 + E_2|^2 \neq |E_1|^2 + |E_2|^2.$$

This allows us to conclude that matter also exhibits wave-like properties. These observations form a central part of the impact introduced by QM.

3.2 Wave functions and the state space

In QM, the state of a particle is described by a complex-valued function

$$\begin{aligned} \psi : \mathbb{R}^3 \times \mathbb{R} &\longrightarrow \mathbb{C} \\ (x,t) &\longmapsto \psi(x,t), \end{aligned}$$

where x and t represent position and time, respectively. We refer to this function as a state function or *wave function*, [14]. Recall that the state of a particle at a time t is related to the probability distribution of its position. Thus, the following conditions seems natural for our wave function:

1. The probability distribution of the position of a particle at time t will be given by $|\psi(\cdot,t)|^2$.

2. As a consequence, it's required the normalization

$$\int_{\mathbb{R}^3} |\psi(x,t)|^2 dx = 1.$$

3. The probability that a particle is in a region $\Omega \subseteq \mathbb{R}^3$ at time t is given by

$$|\psi_x|^1 = \int_{\Omega} |\psi(x,t)|^2 dx.$$

An immediate consequence of points 1 and 2 is that for any time t, the wave function has to be square integrable, that is;

$$\forall t \in \mathbb{R} : \psi(\cdot, t) \in L^2(\mathbb{R}^3),$$

Recall that $L^2(\mathbb{R}^3)$ is a Hilbert space with its own inner product, see Section 2.2.3.

3.3 The Schrödinger equation

Based on what we have presented, let's give a motivation to the equation that determines the evolution of a particle's wave function and total energy, the Schrödinger equation. First, by [14] we shall state three conditions that any wave function should satisfy.

1. *Causality:* If we know $\psi(t_0)$ at a time $t = t_0$, then we should be able to determine the state at all the following times $t > t_0$. Therefore ψ must satisfy the following equation, for some operator $T \in L^2(\mathbb{R}^3)$,

$$\frac{\partial}{\partial t}\psi = T\psi. \tag{3.1}$$

2. *Superposition principle:* If ψ and ϕ are any state functions, then their sum, $\psi + \phi$ is also a state function. This suggests that the operator T must be linear.

3. *Correspondence principle:* Quantum theory results must be in accordance to those obtained from classical methods, when dealing within the same setting.

Applying the third principle to the Hamilton-Jacobi equation (HJ), and using an analogy with the *eikonal* equation in the transition from wave optics to geometrical optics, seen in [14], leads us to an explicit expression for (3.1), indeed:

$$\frac{\partial S}{\partial t} = -h(x, \Delta S), \tag{HJ}$$

where $h(\cdot, \cdot)$ is the classical Hamiltonian, where for a particle of mass m moving in a *potential* V is given by

$$h(x, k) = \frac{1}{2m}|k|^2 + V(x),$$

and $S(x, t)$ is the classical *action.* We look for solutions of (3.1) in the form

$$\psi(x, t) = a(x, t)e^{i\frac{S(x,t)}{\hbar}},$$

where

$$\hbar = 6.62607015 \times 10^{-34} Kg \cdot m^2 \cdot s^{-1}$$

is the reduced Planck constant and $S(\cdot,\cdot)$ satisfies (HJ). Now, by assuming that a is independent of $\hbar$, it is shown that, to leading order, ψ then satisfies the equation

$$i\hbar\frac{\partial}{\partial t}\psi(x,t) = -\frac{\hbar^2}{2m}\Delta_x\psi(x,t) + V(x,t)\psi(x,t). \tag{3.2}$$

$$= H\psi(x,t), \tag{3.3}$$

Equation (3.2) is also known as the *time-dependant Schrödinger equation.* Let's also note that in equation (3.3) the linear operator H is called the Schrödinger operator and it is given by:

$$H\psi := -\frac{\hbar^2}{2m}\Delta\psi + V\psi. \tag{3.4}$$

The Laplacian operator in (3.4) describes the kinetic energy of the system while the potential V is related to the potential energy. If the reader wants to know more about the process used to obtain (3.2), we refer to [14] and [16].

The time-independent Schrödinger equation

To describe a particle we need non-trivial solutions of equation (3.2), [16]. We do this by applying the separation of variables method, that is, we look for solutions of the form:

$$\psi(x,t) = v(x)\phi(t), \qquad \forall x \in \mathbb{R}^3, t \in \mathbb{R}. \tag{3.5}$$

So, by assuming that V is independent of time, $V(t,x) = V(x)$, and replacing (3.5) into (3.2), as seen in [16], we obtain the following:

$$i\hbar v(x)\frac{\partial\phi(t)}{\partial t} = -\frac{\hbar^2}{2m}\Delta v(x)\phi(t) + V(x)v(x)\phi(t),$$

$$i\hbar\frac{1}{\phi(t)}\frac{\partial\phi(t)}{\partial t} = -\frac{\hbar^2}{2m}\frac{1}{v(x)}\Delta v(x) + V(x). \tag{3.6}$$

We note that the left side of (3.6) does not depend on position and the right side does not depend on time, hence both must be equal to a separation constant. We will call this constant E. We obtain:

$$i\hbar\frac{1}{\phi(t)}\frac{\partial\phi(t)}{\partial t} = E, \tag{3.7}$$

and

$$-\frac{\hbar^2}{2m}\Delta v(x) + V(x)v(x) = Ev(x). \tag{3.8}$$

Moreover, (3.7) is a well known ordinary differential equation whose general solution is $\phi(t) = e^{\frac{-iEt}{\hbar}}$ and equation (3.8) is known as the *time-independent Schrödinger equation*, [14]. No further can be done without specifying the potential V. However, it is immediate that a solution of (3.8), does not need to be complex-valued, so that the solution has the form:

$$\psi(x,t) = v(x)e^{\frac{-iEt}{\hbar}}, \tag{3.9}$$

which is also known as a *standing wave.*

3.4 The Heisenberg uncertainty principle

Before stating the Heisenberg uncertainty principle we must recall that, in QM, the state of a particle at a time t is described by its state function ψ. We will briefly describe one of the tools used to extract information about state functions, the kind of operators referred to as observables, [14].

3.4.1 Observables

An *observable* is an unbounded self-adjoint operator, [18], defined on $L^2(\mathbb{R}^3)$. It represents, roughly speaking, any measurable property of a physical system, namely; position, spin, energy and momentum, among others. We have already stated and studied some parts of the behavior of the Schrödinger operator, this operator describes the total energy of a particle described by its wave function. We refer the reader to [14] for the specific proof of the self-adjointedness of H and for further information about other observables. However, at a formal level, for any $u, v \in H_0^1(\mathbb{R}^3) \cap H^2(\mathbb{R}^3)$, applying integration by parts twice, we obtain

$$\begin{aligned}
(-\Delta u, v) &= \int_{\mathbb{R}^3} -\Delta u(x) v(x) dx \\
&= -\int_{\mathbb{R}^3} \Delta u(x) v(x) dx \\
&= \int_{\mathbb{R}^3} \nabla u(x) \nabla v(x) dx \\
&= -\int_{\mathbb{R}^3} u(x) \Delta v(x) dx \\
&= \int_{\mathbb{R}^3} u(x)(-\Delta v(x)) dx \\
&= (u, -\Delta v).
\end{aligned}$$

The border integrals disappear since u has compact support.

We define the mean value of an operator $T \in L^2(\mathbb{R}^3)$ at a state ψ by

$$\langle \psi, T\psi \rangle = \langle T \rangle_\psi. \tag{3.10}$$

Moreover, for any ψ satisfying (3.2), we can compute:

$$\begin{aligned}
\frac{d}{dt}\langle T \rangle_\psi &= \frac{d}{dt}\langle \psi, T\psi \rangle \\
&= \langle \dot{\psi}, T\psi \rangle + \langle \psi, T\dot{\psi} \rangle \\
&= \left\langle \frac{1}{i\hbar} H\psi, T\psi \right\rangle + \left\langle \psi, T\frac{1}{i\hbar} H\psi \right\rangle \\
&= \left\langle \psi, \frac{i}{\hbar} HT\psi \right\rangle - \left\langle \psi, T\frac{i}{\hbar} H\psi \right\rangle \\
&= \left\langle \psi, \frac{i}{\hbar} [H, T]\psi \right\rangle,
\end{aligned} \tag{3.11}$$

to analyze how the mean value of an observable in a state ψ evolves in time. Note that in (3.11)

$$[H, T] := HT - TH$$

is the *commutator* of H and T. We define $\hat{x}_j : \psi(x) \mapsto x_j\psi(x)$ as the coordinate multiplication operator, we will explain more about $\hat{x}_j$ in Section (3.4).

Recall that

$$\begin{aligned} H =& -\frac{\hbar^2}{2m}\Delta + V, \\ \Delta(x_j\psi) =& x_j\Delta\psi + 2\frac{\partial}{\partial x_j}\psi. \end{aligned} \tag{3.12}$$

Now, using the commutator operator, for $\hat{x}_j$ we obtain

$$\frac{i}{\hbar}[H, \hat{x}_j] = -\frac{i\hbar}{m}\nabla_j,$$

leading us to the equation

$$\frac{d}{dt}\langle\psi, x_j\psi\rangle = \frac{1}{m}\langle\psi, -i\hbar\nabla_j\psi\rangle. \tag{3.13}$$

So that, by denoting the operator $-i\hbar\nabla_j$ as p_j and using (3.10), equation (3.13) becomes

$$m\frac{d}{dt}\langle\hat{x}_j\rangle_\psi = \langle p_j\rangle_\psi, \tag{3.14}$$

which is reminiscent of the definition of classical momentum. The operator p is known as the *momentum operator*. Using the Fourier transform, [14, 18], we can compute the mean value of p as

$$\begin{aligned} \langle\psi, p_j\psi\rangle &= \langle\hat{\psi}, \widehat{p_j\psi}\rangle \\ &= \langle\hat{\psi}, k_j\hat{\psi}\rangle \\ &= \int_{\mathbb{R}^3} k_j|\hat{\psi}(k)|^2 dk. \end{aligned}$$

This and similar computations, show that $|\hat{\psi}(k)|^2$ is the probability distribution for the particle momentum.

3.4.2 The Heisenberg representation

The framework outlined until this point is known as the *Schrödinger representation* of QM. However, chronologically, QM was first formulated in the *Heisenberg representation*, which we will describe using the former. We consider two observables known as *position* and *momentum*, the latter being briefly described above.

For $j \in \{1, ..., N\}$ and ψ a state function, we have:

1. The position observable for the jth coordinate x_j:

$$
\begin{aligned}
\hat{x}_j : D_j \subseteq L^2(\mathbb{R}^N) &\longrightarrow L^2(\mathbb{R}^N) \\
\psi(x) &\longmapsto \hat{x}_j[\psi](x) = x_j \psi(x),
\end{aligned}
$$

   where

$$
D_j := \left\{ \psi \in L^2(\mathbb{R}^N) \ / \ \int_{\mathbb{R}^N} |x_j \psi(x)|^2 < +\infty \right\}
$$

2. The momentum observable for the jth coordinate p_j:

$$
\begin{aligned}
p_j : U_j \subseteq L^2(\mathbb{R}^N) &\longrightarrow L^2(\mathbb{R}^N) \\
\psi(x) &\longmapsto p_j[\psi](x) = \frac{\partial}{\partial x_j} \psi(x),
\end{aligned}
$$

   where

$$
U_j := \left\{ \psi \in L^2(\mathbb{R}^N) \ / \ \frac{\partial}{\partial x_j} \psi \text{ exists and belongs to } L^2(\mathbb{R}^N) \right\}
$$

We want to compute $\frac{d}{dt}\langle \hat{x}_j \rangle$ using (3.12) to obtain what we will refer to as the *Heisenberg equations*. Since $\Delta(x_j \psi) = x_j \Delta \psi + 2 \frac{\partial}{\partial x_j} \psi$, it follows that:

$$
\begin{aligned}
\frac{i}{\hbar}[H, \hat{x}_j] &= -\frac{i\hbar}{m} \frac{\partial}{\partial x_j} \\
&= -\frac{i\hbar}{m} p_j.
\end{aligned}
$$

Now, (3.11) implies that

$$
\frac{d}{dt} \langle \hat{x}_j \rangle_\psi = \frac{1}{m} \langle p_j \rangle_\psi. \tag{3.15}
$$

Notice that the previous expression is similar to (3.14). Similarly, we can compute $\dfrac{d}{dt}\langle p_j \rangle_\psi$ using the facts that:

1. $[\Delta, p_j] = 0$ and,
2. $[V, p_j] = i\hbar \frac{\partial}{\partial x_j} V$.

We obtain

$$
\frac{i}{\hbar}[H, p_j] = -\frac{\partial}{\partial x_j} V,
$$

thus,

$$
\frac{d}{dt} \langle p_j \rangle_\psi = \left(\frac{\partial}{\partial x_j} V \right)_\psi. \tag{3.16}
$$

We refer to equations (3.15) and (3.16) as the Heisenberg equations.

3.4.3 Uncertainty principle

Consider a particle in a state ψ. We can think of the observables $\hat{x}_j$ and p as random variables with their respective probability distributions, as shown in Sections 3.2 and 3.4.1. Then, we can describe their standard deviations, or dispersion in the state ψ, by use of their mean values $\langle \hat{x}_j \rangle_\psi$ and $\langle p_j \rangle_\psi$, as

$$\sigma(\hat{x}_j)^2 := \langle (\hat{x}_j - \langle \hat{x}_j \rangle_\psi)^2 \rangle_\psi,$$

and

$$\sigma(p_j)^2 := \langle (p_j - \langle p_j \rangle_\psi)^2 \rangle_\psi,$$

respectively. This allows us to state the Heisenberg Uncertainty Principle as follows:

Theorem 3.4.1 (Heisenberg Uncertainty Principle)**.** *For any state function ψ, we have that:*

$$(\sigma_{\hat{x}_j})^2(\sigma_{p_j})^2 \geq \frac{\hbar}{2}. \tag{HUP}$$

Proof. In order to prove (HUP) we will consider the commutation relation

$$\frac{i}{\hbar}[p_j, \hat{x}_j] = 1 = \delta_{jk}.$$

As in [14], we will assume that

$$\langle \hat{x}_j \rangle_\psi = \langle p \rangle_\psi = 0,$$

for notational simplicity. Let's now recall that for two self-adjoint operators T and S and $\psi \in D(T) \cap D(S)$ we have that

$$\langle i[T, S] \rangle_\psi = -2\mathrm{Im}\langle T\psi, B\psi \rangle. \tag{3.17}$$

Now, by assuming that $||\psi||_{L^2} = 1$ and that $\psi \in D_j \cap U_j$, by (3.17) we obtain

$$\begin{aligned} 1 = \langle \psi, \psi \rangle &= \langle \psi, \frac{i}{\hbar}[p_j, \hat{x}_j]\psi \rangle \\ &= -\frac{2}{\hbar}\mathrm{Im}\langle p_j\psi, \hat{x}_j\psi \rangle \\ &\leq \frac{2}{\hbar}|\langle p_j\psi, \hat{x}_j\psi \rangle| \\ &\leq \frac{2}{\hbar}||p_j\psi||||\hat{x}_j\psi|| = \frac{2}{\hbar}(\sigma_{\hat{x}_j})^2(\sigma_{p_j})^2. \end{aligned}$$

This concludes our proof. □

This famous theorem says that a small dispersion of the momentum implies greater dispersion of the position and vice-versa. Physically, it means that we cannot simultaneously measure momentum and position of a particle with unlimited accuracy, [18].

3.5 The nonlinear Schrödinger equation

From the description provided by thermodynamics, within a gas it can be assumed as if all the particles behave in the same manner and, in principle, they can occupy certain energy states, or rather, quantum states, [29]. If the particles are *fermions*, they cannot occupy the same quantum state, this is given by the Pauli exclusion principle. This is not the case with *bosons* as any number of them can occupy the same quantum space and moreover, they will increase occupation of the states of minimum energy as the temperature tends to zero.

For collections of bosons, as the temperature goes to zero, all the particles are going to occupy the same energy state, the ground state of the system. This implies that when the temperature is low enough, the majority of the particles will have the same velocity in the same quantum state. Thus, in this setting the collection of bosons acts like a macroscopic fluid with new properties, such as superfluidity, [29]. This is known as Bose-Einstein condensation.

In order to study these properties, we only concentrate on the ground state. We must mention that interactions between bosons are not necessary for condensation to take place, and yet, they play a very important role in the properties of the condensate. In this sense, the usual Schrödinger equation is not enough to provide a good description, instead we need the *Gross-Pitaevskii equation* which is also known as the (main) nonlinear Schrödinger Equation, [29],

$$i\hbar\frac{\partial}{\partial t}\Phi(x,t) = \left[-\frac{\hbar^2}{2m}\Delta + V(x) + g|\Phi(x,t)|^2\right]\Phi(x,t), \quad g > 0. \tag{3.18}$$

This equation contains another energy term $g|\Phi(x,t)|^2$, proportional to the local density $|\Phi(x,t)|^2$ of the condensate, which is referred to as twice the mean-field energy of the condensate. It plays a fundamental role in its dynamics.

Remark 3.5.1. It is important to note that comparing the atom-atom interactions to *Kerr nonlinearity* in optics suggest that they play a role similar to that of a non linear medium for light, [23]. For more information about the derivation of (3.18) and the relation with nonlinear optics, we refer to [16], [23] and [29].

Chapter 4

Results

4.1 Problem statement

Let's consider the following nonlinear Schrödinger equation:

$$i\hbar\psi_t(x,t) + \frac{\hbar^2}{2}\Delta\psi(x,t) - V_0(x)\psi(x,t) + |\psi(x,t)|^{p-1}\psi(x,t) = 0, \tag{SchEq}$$

for $x \in \mathbb{R}^N, t \in \mathbb{R}$ and $N \geq 3, p+1 \in (2, 2^*)$ with

$$2^* = \frac{2N}{N-2}.$$

We look for solutions in the form of standing waves of the form

$$\psi(x,t) = v(x)e^{-iEt/\hbar}, \quad x \in \mathbb{R}^N, t \in \mathbb{R}. \tag{4.1}$$

By replacing (4.1) in (SchEq) we obtain

$$\begin{aligned}
&i\hbar v(x)e^{-iEt/\hbar} + \frac{\hbar^2}{2}\Delta v(x)e^{-iEt/\hbar} - V_0(x)v(x)e^{-iEt/\hbar} + |v(x)e^{-iEt/\hbar}|^{p-1}v(x)e^{-iEt/\hbar} = 0 \\
&Ee^{-iEt/\hbar}v(x) + \frac{\hbar^2}{2}e^{-iEt/\hbar}\Delta v(x) - V_0(x)v(x)e^{-iEt/\hbar} + |v(x)|^{p-1}|e^{-iEt/\hbar}|^{p-1}v(x)e^{-iEt/\hbar} = 0 \\
&Ev(x) + \frac{\hbar^2}{2}\Delta v(x) - V_0(x)v(x) + |v(x)|^{p-1}v(x) = 0. \qquad (4.2)
\end{aligned}$$

By setting $\varepsilon^2 = \hbar^2/2$ and $V(x) = V_0(x) - E$ in (4.2) we obtain

$$\begin{aligned}
\frac{\hbar^2}{2}\Delta v(x) - (V_0(x) - E)v(x) + |v(x)|^{p-1}v(x) = 0, \\
\varepsilon^2\Delta v(x) - V(x)v(x) + |v(x)|^{p-1}v(x) = 0
\end{aligned}$$

and thus, our problem becomes

$$\begin{cases} \varepsilon^2\Delta v(x) - V(x)\,v(x) + |v(x)|^{p-1}v(x) = 0, & x \in \mathbb{R}^N, \\ v(x) \to 0, & \text{as } |x| \to \infty. \end{cases} \tag{P_ε}$$

Let's consider the situation where

$$\overline{\mathcal{Z}} = \{x \in \mathbb{R}^N \,/\, V(x) = \inf(V)\} \neq \emptyset.$$

The case when inf(V) >0 is referred to as non-critical frequency, critical frequency or energy corresponds to the case inf(V) $=0$. In this work we shall focus on the *infinite case* as presented in the pioneer work [6]. Our objective is to make an asymptotical analysis of (P_ε) through a semiclassical approach, that is, study the behavior of (P_ε) as $\varepsilon \to 0$ and obtain similar results to the ones presented in [11]. We consider the following conditions:

(V1) $V \in \mathrm{C}(\mathbb{R}^N)$ is non-negative;
(V2) $V(x) \to \infty$, as $|x| \to \infty$;
(V3) $\overline{\mathcal{Z}} = \{0\}$

Remark 4.1.1. In the papers by Byeon & Wang, [6], and Felmer & Mayorga, [11], three cases were considered.

Flat: $\mathrm{int}\overline{\mathcal{Z}} = \mathcal{Z} \neq \emptyset$ is bounded;

Finite: $\overline{\mathcal{Z}}$ is finite and V vanishes polinomially around it;

Infinite: $\overline{\mathcal{Z}}$ is finite and V vanishes exponentially around it.

The condition $(V_{\inf})$, below, differentiates our situation with that of the finite case. It corresponds, grossly speaking, to the decay of V as we get close to $\overline{\mathcal{Z}}$. However before stating it we need to define a couple of concepts.
Let $\Omega \subseteq \mathbb{R}^N$ be a smooth bounded strictly star-shaped domain, i.e., there exists a ball $B \subseteq \Omega$ such that

$$\forall x \in B, \forall y \in \Omega : \quad [x,y] \subseteq \Omega.$$

Ω is a q-Poincaré domain for all $q \geq 1$, [30], i.e., there exists a constant $M_{q,\Omega} > 0$ such that

$$\forall u \in \mathrm{C}^1(\Omega) : \quad \|u - u_\Omega\|_{\mathrm{L}^q(\Omega)} \leq M_{q,\Omega} \left(\int_\Omega |\nabla u(x)|^q \, dx \right)^{1/q}.$$

Moreover, we assume that Ω is generated by a positive capturer function $r \in \mathrm{C}(\mathbb{R}^N \backslash \{0\})$ such that

$$\begin{cases} t = r(x) \implies \frac{1}{t}x \in \partial\Omega, \\ t > r(x) \implies \frac{1}{t}x \in \Omega, \\ t < r(x) \implies \frac{1}{t}x \in \mathbb{R}^N \setminus \Omega. \end{cases} \tag{4.3}$$

Point (4.3) implies that every non-zero point is well determined by a point in the boundary of Ω:

$$\forall x \in \mathbb{R}^N \setminus \{0\}, \exists!(r(x), s(x)) \in (0,+\infty) \times \partial\Omega : \quad x = r(x)s(x).$$

Now, as in [6], we assume that $b \in \mathrm{C}(\mathbb{R}^N)$ is an Ω-*quasi homogeneous function* and $a \in \mathrm{C}(\mathbb{R}^N \setminus \{0\})$, is an *asymptotically* (Ω, b)-*quasi homogeneous function.* That is, there exists $\beta : [0,+\infty[\to \mathbb{R}$ such that

b.1) $\forall x \in \mathbb{R}^N: \quad b(x) = b(r(x)\,s(x)) = \beta(r(x))$;
b.2) β is non-negative and strictly-increasing;
b.3) for $L = \lim_{r\to 0} \dfrac{\beta(cr)}{\beta(r)}$ it holds

$$\begin{cases} c < 1 \ \Rightarrow\ L < 1, \\ c > 1 \ \Rightarrow\ L > 1; \end{cases} \tag{4.4}$$

a) a is positive and

$$\lim_{|x|\to 0} \frac{a(x)}{b(x)} = 1.$$

Now we can write the condition that characterizes our case:

($\mathrm{V}_{\inf}$) $\forall |x| \leq 1 : V(x) = \exp\left(-\dfrac{1}{a(x)}\right)$.

Under conditions (V1), (V2), (V3) and ($\mathrm{V}_{\inf}$) the limit problem of (P_ε) is

$$\begin{cases} \Delta w(x) + |w(x)|^{p-1}\, w(x) = 0, & x \in \Omega, \\ w(x) = 0, & x \in \partial\Omega. \end{cases} \tag{$P_{\inf}$}$$

As it's done in [6], for any $\varepsilon > 0$ we set

$$V_\varepsilon(x) = \frac{1}{[\varepsilon g(\varepsilon)]^2} V\left(\frac{x}{g(\varepsilon)}\right), \tag{4.5}$$

with $g : (0, +\infty) \to \mathbb{R}$ given by

$$g(\varepsilon) = \frac{1}{b^{-1}\left(\frac{-1}{\ln(\varepsilon)^2}\right)}.$$

We consider the space

$$\mathrm{H}_\varepsilon := \left\{ u \in \mathrm{H}^1(\mathbb{R}^N) : \int_{\mathbb{R}^N} \left[|\nabla u(x)|^2 + V_\varepsilon(x) u^2(x) \right] dx < \infty \right\},$$

with the norm

$$\|\cdot\|_\varepsilon = \int_{\mathbb{R}^N} \left[|\nabla u(x)|^2 + V_\varepsilon(x) u^2(x) \right] dx.$$

Proposition 4.1.1. *The functional* $(\cdot,\cdot)_\varepsilon : \mathrm{H}_\varepsilon \times \mathrm{H}_\varepsilon \to \mathbb{R}$ *given by*

$$(u, v)_\varepsilon = \int_{\mathbb{R}^N} \left[\nabla u \cdot \nabla v + V_\varepsilon(x) u(x) v(x) \right] dx,$$

defines an inner product on H_ε *and induces* $\|\cdot\|_\varepsilon$

Proof. Let $u, v, w \in \mathrm{H}_\varepsilon, \lambda \in \mathbb{R}$. We shall prove conditions i) - iv) described at the start of Section 2.2.4.

i) Let's prove that

$$(\lambda u + v, w)_\varepsilon = \lambda(u, w)_\varepsilon + (v, w)_\varepsilon.$$

We have that

$$\begin{aligned}
(u + v, w)_\varepsilon &= \int_{\mathbb{R}^N} \left[\nabla(\lambda u + v)(x)\nabla w(x) + V_\varepsilon(x)(\lambda u + v)(x)w(x)\right] dx \\
&= \int_{\mathbb{R}^N} \left[\lambda \nabla u(x)\nabla w(x) + V_\varepsilon(x)\lambda u(x)w(x)\right] dx + \int_{\mathbb{R}^N} \left[\nabla v(x)\nabla w(x) + V_\varepsilon(x)v(x)w(x)\right] dx \\
&= \lambda \int_{\mathbb{R}^N} \left[\nabla u(x)\nabla w(x) + V_\varepsilon(x)u(x)w(x)\right] dx + \int_{\mathbb{R}^N} \left[\nabla v(x)\nabla w(x) + V_\varepsilon(x)v(x)w(x)\right] dx \\
&= \lambda(u, w)_\varepsilon + (v, w)_\varepsilon
\end{aligned}$$

So that $(\cdot, \cdot)_\varepsilon$ is linear in the first argument.

ii) Let's prove that

$$(u, v)_\varepsilon = (v, u)_\varepsilon.$$

We have that

$$\begin{aligned}
(u, v)_\varepsilon &= \int_{\mathbb{R}^N} [\nabla u(x)\nabla v(x) + V_\varepsilon(x)u(x)v(x)]dx \\
&= \int_{\mathbb{R}^N} \left[\nabla v(x)\nabla u(x) + V_\varepsilon(x)v(x)u(x)\right] dx \\
&= (v, u)_\varepsilon.
\end{aligned}$$

So $(\cdot . \cdot)_\varepsilon$ is symmetric. As a consequence, $(\cdot, \cdot)_\varepsilon$ is bilinear.

iii) Let's prove that $(\cdot, \cdot)_\varepsilon \geq 0$. By (4.5) and since $\mathrm{V} \geq 0$ we have that

$$(u, u)_\varepsilon = \int_{\mathbb{R}^N} \left[|\nabla u(x)|^2 + V_\varepsilon(x)u^2(x)\right] dx \geq 0.$$

Clearly, $(u, u)_\varepsilon = 0 \iff u = 0$.

Hence, we have proved that $(u, v)_\varepsilon$ defines an inner product, and thus, induces a norm $\|\cdot\|_\varepsilon$ in H_ε. We conclude by the arbitrariness of u, v, w and λ. □

Furthermore, as is usually done, the functional $(\cdot, \cdot) : \mathrm{H}_0^1(\mathbb{R}^N) \times \mathrm{H}_0^1(\mathbb{R}^N) \to \mathbb{R}$ given by

$$(u, v)_{\mathrm{H}_0^1(\mathbb{R}^N)} = \int_{\mathbb{R}^N} \nabla u(x) \cdot \nabla v(x) dx,$$

defines an inner product on $\mathrm{H}_0^1(\mathbb{R}^N)$ and induces $\|\cdot\|_{\mathrm{H}_0^1(\mathbb{R}^N)}$.

The proof that $(\cdot, \cdot)$ is an inner product is analogous to that of Proposition 4.1.1.

Remark 4.1.2. We can also consider the Hilbert spaces $(\mathrm{H}_\varepsilon, \|\cdot\|_\varepsilon)$ and $(\mathrm{H}^1_0(\Omega), \|\cdot\|_{\mathrm{H}^1_0(\Omega)})$ as the completion of $\mathrm{C}^\infty_0(\mathbb{R}^N)$ with the norms induced by the inner products $(u,v)_\varepsilon$ and $(u,v)_{\mathrm{H}^1_0(\mathbb{R}^N)}$, respectively.

Remark 4.1.3. The following problems are closely related to (P_ε).

$$\begin{cases} \Delta w(x) - V_\varepsilon(x)\, w(x) + |w(x)|^{p-1} w(x) = 0, & x \in \mathbb{R}^N, \\ w(x) \to 0, & \text{as } |x| \to \infty, \end{cases} \tag{P'_ε}$$

$$\begin{cases} \Delta \hat{w}(x) - V_\varepsilon(x)\, \hat{w}(x) + 2\Theta\, |\hat{w}(x)|^{p-1} \hat{w}(x) = 0, & x \in \mathbb{R}^N, \\ \hat{w}(x) \to 0, & \text{as } |x| \to \infty, \end{cases} \tag{$\hat{P}_\varepsilon$}$$

where

$$\Theta = \frac{1}{2}(\hat{w}, \hat{w})_\varepsilon. \tag{4.6}$$

Indeed, if $\hat{w}$ is a solution of $(\hat{P}_\varepsilon)$, then

$$w(x) = (2\Theta)^{1/(p-1)} \hat{w}(x), \quad x \in \mathbb{R}^N,$$

is a solution of (P_ε).

Remark 4.1.4. Related to $(P_{\inf})$ is

$$\begin{cases} \Delta \hat{w}(x) + 2\Upsilon\, |\hat{w}(x)|^{p-1}\, \hat{w}(x) = 0, & x \in \Omega, \\ \hat{w}(x) = 0, & x \in \partial\Omega, \end{cases} \tag{$\hat{P}$}$$

where

$$\Upsilon = \frac{1}{2}(\hat{w}, \hat{w})_{\mathrm{H}^1_0(\Omega)}.$$

Indeed, if $\hat{w}$ is a solution of $(\hat{P})$, then

$$w(x) = (2\Upsilon)^{1/(p-1)} \hat{w}(x), \quad x \in \Omega,$$

is a solution of $(P_{\inf})$.

Main results

We define the functional $J_\varepsilon : \mathcal{M}_\varepsilon \subseteq \mathrm{H}_\varepsilon \to \mathbb{R}$ given by

$$\begin{aligned} J_\varepsilon(u) &= \frac{1}{2}\, \|u\|^2_\varepsilon \\ &= \frac{1}{2} \int_{\mathbb{R}^N} [|\nabla u(x)|^2 + V_\varepsilon |u(x)|^2] dx, \end{aligned} \tag{4.7}$$

where

$$\mathcal{M}_\varepsilon := \{w \in \mathrm{H}_\varepsilon / \, \|w\|_{\mathrm{L}^{p+1}(\mathbb{R}^N)} = 1\} \tag{4.8}$$

is a Nehari manifold, see [31].

Remark 4.1.5. Since every norm is continuous, J_ε is continuous.

Proposition 4.1.2 (J_ε is strongly differentiable)**.** *The functional J_ε defined above is of class C^1. Moreover, its Fréchet differential is given by*

$$\langle DJ_\varepsilon(u), h\rangle = \int_{\mathbb{R}^N} [\nabla u(x)\nabla h(x) + V_\varepsilon(x)u(x)h(x)]\, dx = (u,h)_\varepsilon.$$

Proof. 1. First we will now prove that J_ε is Gateaux differentiable. Let $u, h \in \mathrm{H}_\varepsilon$ and $\lambda \in \mathbb{R}$ be generic. By (4.7) we have that

$$\begin{aligned} J(u+\lambda h) &= \frac{1}{2}\int_{\mathbb{R}^N} |\nabla(u+\lambda h)(x)|^2 + V_\varepsilon|(u+\lambda h)|^2 dx \\ &= \frac{1}{2}\int_{\mathbb{R}^N} [|\nabla u(x)|^2 + 2\lambda \nabla u(x)\nabla h(x) + \lambda^2|h(x)|^2 + V_\varepsilon(x)|u(x)|^2 + \\ &+ 2\lambda V_\varepsilon(x)u(x)h(x) + \lambda^2 V_\varepsilon|h(x)|^2]dx, \end{aligned}$$

so that

$$\frac{d}{d\lambda}J(u+\lambda h) = \frac{1}{2}\int_{\mathbb{R}^N}\left[2\nabla u(x)\nabla h(x) + 2\lambda|h(x)|^2 + 2V_\varepsilon(x)u(x)h(x) + 2\lambda V_\varepsilon|h(x)|^2\right] dx,$$

we obtain

$$\frac{d}{d\lambda}J(u+\lambda h)\Big|_{\lambda=0} = \frac{1}{2}\int_{\mathbb{R}^N}[2\nabla u(x)\nabla h(x) + 2V_\varepsilon(x)u(x)h(x)]\, dx.$$

Therefore

$$\partial_h J_\varepsilon(u) = \int_{\mathbb{R}^N}[\nabla u(x)\nabla h(x) + V_\varepsilon(x)u(x)h(x)]\, dx.$$

We define the functional $\Phi : \mathrm{H}_\varepsilon \to \mathbb{R}$ given by

$$\Phi(y) = \int_{\mathbb{R}^N}[\nabla u(x)\nabla y(x) + V_\varepsilon(x)u(x)y(x)]\, dx. \tag{4.9}$$

Clearly Φ is linear, we shall prove that it's bounded, that is

$$\exists c > 0, \forall y \in \mathrm{H}_\varepsilon : |\Phi(y)| \le c\,\|y\|_\varepsilon$$

We choose

$$c > 2\,\|u\|_\varepsilon\,.$$

Let $y \in \mathrm{H}_\varepsilon$, by the Hölder, (2.2.14), and CBS, (2.2.21), inequalities we have that

$$\begin{aligned}
|\Phi(y)| &= \left| \int_{\mathbb{R}^N} \nabla u(x) \nabla y(x) + V_\varepsilon(x) u(x) y(x) dx \right| \\
&\leq \int_{\mathbb{R}^N} |\nabla u(x) \nabla y(x)| dx + \int_{\mathbb{R}^N} |V_\varepsilon(x) u(x) y(x)| dx \\
&\leq \int_{\mathbb{R}^N} |\nabla u(x)||\nabla y(x)| dx + \int_{\mathbb{R}^N} |[V_\varepsilon(x)]^{1/2} u(x) [V_\varepsilon]^{1/2} y(x)| dx \\
&\leq \left(\int_{\mathbb{R}^N} |\nabla u(x)|^2 \right)^{1/2} \left(\int_{\mathbb{R}^N} |\nabla y(x)|^2 dx \right)^{1/2} + \\
&\left(\int_{\mathbb{R}^N} V_\varepsilon(x) |u(x)|^2 dx \right)^{1/2} \left(\int_{\mathbb{R}^N} V_\varepsilon(x) |y(x)|^2 dx \right)^{1/2} \\
&= \|\nabla u\|_{L^2} \|\nabla y\|_{L^2} + \left\| V_\varepsilon^{1/2} u \right\|_{L^2} + \left\| V_\varepsilon^{1/2} y \right\|_{L^2} \\
&\leq 2 \|u\|_\varepsilon \|y\|_\varepsilon .
\end{aligned}$$

Hence the functional Φ is bounded and therefore $\Phi \in \mathrm{H}_\varepsilon^*$ is Gateaux differentiable, namely

$$J'_{\varepsilon G}(u) h = \Phi(h). \tag{4.10}$$

2. Let's prove that J_ε is Fréchet differentiable. By (4.7) we have that

$$J_\varepsilon(u+h) = \frac{1}{2} \int_{\mathbb{R}^N} \left[|\nabla u(x) + \nabla h(x)|^2 + V_\varepsilon(x) |u(x) + h(x)|^2 \right] dx,$$

so that

$$\begin{aligned}
J_\varepsilon(u+h) - J_\varepsilon(u) &= \frac{1}{2} \int_{\mathbb{R}^N} [|\nabla u(x) + \nabla h(x)|^2 + V_\varepsilon(x) |u(x) + h(x)|^2 - \\
&\qquad - |\nabla u(x)|^2 - V_\varepsilon(x) |u(x)|^2] dx \\
&= \int_{\mathbb{R}^N} \left[\nabla u(x) \cdot \nabla h(x) + V_\varepsilon(x) u(x) h(x) \right] dx + \qquad (4.11) \\
&\qquad + \frac{1}{2} \int_{\mathbb{R}^N} \left[|\nabla h(x)|^2 + V_\varepsilon(x) |h(x)|^2 \right] dx. \qquad (4.12) \\
&= (u, h)_\varepsilon + \frac{1}{2} \|h\|_\varepsilon^2
\end{aligned}$$

Now, since $(u, \cdot)_\varepsilon \in \mathrm{H}_e^*$ and clearly $\|h\|_\varepsilon^2$ is $o(h)$ we have that J is differentiable and its Fréchet differential is given by (4.9) and thus

$$\langle DJ_\varepsilon(u), h \rangle = J'_\varepsilon(u) h = \int_{\mathbb{R}^N} \left[\nabla u(x) \nabla h(x) + V_\varepsilon(x) u(x) h(x) \right] dx.$$

We conclude that J_ε is Fréchet differentiable by the arbitrariness of u, h, y and λ.

Finally, by the CBS inequality, Lemma (2.2.21), we have that for any $u, v \in \mathrm{H}_\varepsilon$

$$\langle DJ_e(u), v\rangle = (u, v)_\varepsilon \leq \|u\|_\varepsilon \|v\|_\varepsilon,$$

which implies that

$$\forall u \in \mathrm{H}_\varepsilon : \quad \|DJ_\varepsilon(u)\|_{H_\varepsilon^*} \leq \|u\|_\varepsilon.$$

so that DJ_ε is continuous and linear. Therefore J_ε is of class C^1. We conclude our proof. □

Additionally we consider the functional $J : \mathcal{M} \subseteq \mathrm{H}_0^1(\Omega) \to \mathbb{R}$ given by

$$\begin{aligned} J(u) &= \frac{1}{2}\|u\|_{\mathrm{H}_0^1(\Omega)}^2 \\ &= \frac{1}{2}\int_\Omega |\nabla u(x)|^2 dx, \end{aligned} \tag{4.13}$$

where

$$\mathcal{M} := \{w \in \mathrm{H}_0^1(\Omega) / \|w\|_{\mathrm{L}^{p+1}\left(\mathbb{R}^N\right)} = 1\} \tag{4.14}$$

is a Nehari manifold, see [31].

Remark 4.1.6 (J is of class C^1)**.** Notice that J is also continuous and it can be easily proved that it is strongly differentiable in the same way as J_ε. Namely, it's Fréchet differential is given by

$$\langle DJ(u), h\rangle = \int_\Omega \nabla u(x)\nabla h(x) = (u, h)_{\mathrm{H}_0^1(\Omega)}. \tag{4.15}$$

Indeed, we have that

$$\begin{aligned} J(u+h) - J(u) &= \frac{1}{2}\int_\Omega |\nabla(u+h)(x)|^2 dx - \frac{1}{2}\int_\Omega |\nabla u(x)|^2 dx \\ &= \frac{1}{2}\int_\Omega |\nabla u(x) + \nabla h(x)|^2 dx - \frac{1}{2}\int_\Omega |\nabla u(x)|^2 dx \\ &= \frac{1}{2}\int_\Omega \left[|\nabla u(x)|^2 + 2\nabla u(x)\nabla h(x) + |\nabla h(x)|^2\right] dx - \frac{1}{2}\int_\Omega |\nabla u(x)|^2 dx \\ &= \int_\Omega \nabla u(x)\nabla h(x) dx + \frac{1}{2}\int_\Omega |\nabla h(x)|^2 dx \\ &= (u, h)_{\mathrm{H}_0^1(\Omega)} + \frac{1}{2}\|h\|_{\mathrm{H}_0^1(\Omega)}. \end{aligned}$$

And since $(u, \cdot) \in (\mathrm{H}_0^1(\Omega))^* = \mathrm{H}^{-1}$ and $\|h\|_{\mathrm{H}_0^1(\Omega)}$ is clearly $o(h)$ we conclude that J is differentiable, and its differential, which we shall denote as DJ, is given by (4.15). Moreover, by the CBS inequality, for any $u, v \in \mathrm{H}_0^1(\Omega)$, we have that

$$\langle DJ(u), v\rangle = (u, v)_{\mathrm{H}_0^1(\Omega)} \leq \|u\|_{\mathrm{H}_0^1(\Omega)} \|v\|_{\mathrm{H}_0^1(\Omega)}. \tag{4.16}$$

And thus, (4.16) implies that

$$\forall u \in \mathrm{H}_0^1(\Omega) : \quad \|DJ(u)\|_{\mathrm{H}^{-1}} \leq \|u\|_{\mathrm{H}_0^1(\Omega)}.$$

So that the differential of J is bounded, thus, continuous. We conclude that J is also of class C^1.

We assume that (V1), (V2), (V3), (V_{inf}) and that

$$\begin{cases} 2 < 1+p < 2^* = 2N/(N-2), & \text{if } N \geq 3; \\ 2 < 1+p, & \text{if } N = 1,2, \end{cases} \tag{4.17}$$

always hold.

We start by stating the main results and in the following sections we will provide the proofs of the theorems. We start with the multiplicity result.

Theorem 4.1.1. *The following points hold.*

i) Given $\varepsilon > 0$, the functional J_ε has a sequence of different critical points $(\hat{w}_{k,\varepsilon})_{k\in\mathbb{N}} \subseteq \mathcal{M}_\varepsilon$.

ii) The functional J has a sequence of different critical points $(\hat{w}_k)_{k\in\mathbb{N}} \subseteq \mathcal{M}$.

Remark 4.1.7. By Remarks 4.1.3 and 4.1.4, for $\varepsilon > 0$ and $k \in \mathbb{N}$ we have that the function given by

$$v_{k,\varepsilon}(x) = \left[2c_{k,\varepsilon}\left(\varepsilon\, g(\varepsilon)\right)^2\right]^{1/(p-1)} \hat{w}_{k,\varepsilon}\left(\frac{x}{g(\varepsilon)}\right), \tag{4.18}$$

where

$$c_{k,\varepsilon} = J_\varepsilon(\hat{w}_{k,\varepsilon}), \tag{4.19}$$

is a solution of (P_ε) and the function given by

$$w_k(x) = (2c_k)^{1/(p-1)}\, \hat{w}_k(x), \tag{4.20}$$

is a solution of (P_{inf}).

What follows is the convergence of energies result.

Theorem 4.1.2. *Let $k \in \mathbb{N}$. Then*

$$\lim_{\varepsilon\to 0} c_{k,\varepsilon} = c_k. \tag{4.21}$$

We also have the subconvergence of critical points and the result about exponential decay.

Theorem 4.1.3. *Let $k \in \mathbb{N}$. As $\varepsilon \to 0, (w_{k,\varepsilon})_{\varepsilon>0}$ subconverges in $\mathrm{H}^1(\mathbb{R}^N)$ to some $u_k \in \mathrm{H}^1(\mathbb{R}^N)$ such that its restriction to Ω is another solution of (P_{inf}), verifying*

$$J(\hat{u}_k|_\Omega) = c_k,$$

where

$$\hat{u}_k = (2c_k)^{1/p-1} u_k.$$

Theorem 4.1.4. *Let $k \in \mathbb{N}$ and $\delta > 0$, there exists $\varepsilon_\delta > 0$ and $C = C(N, k, p, \delta)$ such that*

$$\forall x \in \mathbb{R}^N, \forall \varepsilon \in (0, \varepsilon_\delta) : \quad |\hat{w}_{k,\varepsilon}(x)| < \frac{C}{(2c_k)^{1/p-1}} \cdot \exp\left(\gamma_{\delta,\varepsilon} \cdot \operatorname{dist}(x, \Omega^\delta)\right), \tag{4.22}$$

where

$$\Omega^\delta = \{x \in \mathbb{R}^N : \quad \operatorname{dist}(x, \Omega) < \delta\},$$

and

$$\gamma_{\delta,\varepsilon} = \gamma_{\delta,\varepsilon}(N, k, p) \to -\infty, \quad as \quad \varepsilon \to 0.$$

Preliminary results

The following results are stated in [6] and are rewritten in our setting. They are helpful in proving our results and involve the potential V, the scaling V_ε and the functions g, b, a and β. We have that:

$$\lim_{\varepsilon \to 0} g(\varepsilon) = \infty; \tag{4.23}$$

$$\exists \alpha > 0 : \quad \lim_{r \to 0} \frac{\beta(r)}{r^\alpha} = 0 \ \wedge \ \lim_{\varepsilon \to 0} \frac{g(\varepsilon)}{|\ln(\varepsilon)|^{1/\alpha}} = 0; \tag{4.24}$$

$$\forall c > 0 : \quad \lim_{\varepsilon \to 0} \frac{1}{[\varepsilon^c g(\varepsilon)]^2} = \lim_{\varepsilon \to 0} \frac{1}{g^2(\varepsilon)} \exp\left(\frac{c}{b\left(\frac{1}{g(\varepsilon)}\right)}\right) = \infty. \tag{4.25}$$

By ($V_{\inf}$), for every $\varepsilon > 0$ and $|x| \leq g(\varepsilon)$,

$$\begin{aligned} V_\varepsilon(x) &= \frac{1}{[\varepsilon\, g(\varepsilon)]^2} \exp\left(-\frac{1}{a\left(\frac{x}{g(\varepsilon)}\right)}\right) & (4.26) \\ &= \frac{1}{g^2(\varepsilon)} \cdot \exp\left(\frac{1}{\beta\left(\frac{1}{g(\varepsilon)}\right)}\left[1 - \frac{\beta\left(\frac{1}{g(\varepsilon)}\right)\beta\left(\frac{r(x)}{g(\varepsilon)}\right)}{\beta\left(\frac{r(x)}{g(\varepsilon)}\right) a\left(\frac{x}{g(\varepsilon)}\right)}\right]\right). & (4.27) \end{aligned}$$

Additionally we have the following propositions.

Proposition 4.1.3. *For every $B \subseteq \Omega$ measurable,*

$$\lim_{\varepsilon \to 0} \|V_\varepsilon\|_{L^\infty(B)} = \lim_{\varepsilon \to 0} \operatorname*{esssup}_{x \in B} |V_\varepsilon(x)| = 0. \tag{4.28}$$

Proposition 4.1.4. *There exists $D \in]0, 1[$ such that for all $d > 1$,*

$$\lim_{\varepsilon \to 0} \min_{x \in R_{\varepsilon,D,d}} V_\varepsilon(x) = \infty, \tag{4.29}$$

where

$$R_{\varepsilon,D,d} = \{x \in \mathbb{R}^N \,/\, |x| \leq D\, g(\varepsilon) \ \wedge \ r(x) \geq d\}.$$

The following result is similar to the embedding theorems stated in Section 2.3.3 involving the weighted Sobolev spaces such that the weighted functions verify (V1) and (V2).

Theorem 4.1.5. *Assume that* $U \in C(\mathbb{R}^N)$ *is non-negative and such that*

$$U(x) \to \infty \quad \text{as} \quad |x| \to \infty.$$

Let H_U *be the Hilbert space resulting of the completion of* $\mathrm{C}_0^\infty(\mathbb{R}^N)$ *whenever it is equipped with the norm induced by the interior product given by*

$$(v, w)_U = \int_{\mathbb{R}^N} [\nabla v(x) \nabla w(x) + U(x) v(x) w(x)] dx.$$

Then, for every $q \in [2, Q)$*, the embedding*

$$\mathrm{H}_U \subseteq \mathrm{L}^q(\mathbb{R}^N),$$

is compact. Where,

$$Q = \begin{cases} 2^*, & \text{if } N \geq 3; \\ \infty, & \text{if } N = 1, 2. \end{cases}$$

As mentioned in [11], this Theorem is obtained by an application of the Kolmogorov-Riesz-Fréchet Theorem, 2.2.19, and [4, Cor 4.27]. Compensating the non-boundedness of the domain by letting U explode at infinity.

The following holds by Remark 2.3.4.

Proposition 4.1.5. *Let* $\varepsilon > 0$*, then:*

1. *The embedding* $\mathrm{H}_0^1(\Omega) \subseteq \mathrm{H}_\varepsilon$ *is continuous.*

2. *The norms* $\|\cdot\|_\varepsilon$ *and* $\|\cdot\|_{\mathrm{H}_0^1(\Omega)}$ *are equivalent in* $\mathrm{H}_0^1(\Omega)$.

Proof. By direct computation we have

$$\forall u \in \mathrm{H}_0^1(\Omega) : \|u\|_{\mathrm{H}_0^1(\Omega)} \leq \|u\|_\varepsilon \leq C_{\Omega,\varepsilon} \|u\|_{\mathrm{H}_0^1(\Omega)}, \tag{4.30}$$

where

$$C_{\Omega,\varepsilon}^2 = 1 + C_\Omega^2 \|V_\varepsilon\|_{\mathrm{L}^\infty(\Omega)} > 0, \tag{4.31}$$

with C_Ω being the constant appearing in (2.13).

Remark 4.1.8. Recall that for any measurable set $\Lambda \subseteq \mathbb{R}^N$ such that $|\Lambda| < \infty$ it holds that

$$\forall w \in L^{p+1}(\Lambda) : \quad \|w\|_{L^2(\Lambda)} \leq |\Lambda|^{(p-1)/2(p+1)} \|w\|_{L^{p+1}(\Lambda)},$$

by applying Hölder's inequality, (2.2.14).

4.2 Multiplicity by a Ljusternik-Schnirelman scheme

This section is dedicated to the proof of Theorem 4.1.1, showing how a Ljusternik-Schnirelman scheme, [3], provides the desired result in a very direct way. The tools used in this section are the Krasnoselskii genus and Palais-Smale sequences, briefly described in Section 2, alongside the tools provided by nonlinear analysis and variational calculus. The main tool for this section is the following theorem.

Theorem 4.2.1. *Let $\mathcal{M} \in \Sigma_E$ be a C^1 manifold of E and let $f \in C^1(E)$ be even. Assume that $(\mathcal{M}, f)$ satisfy the Palais-Smale condition and let*

$$C_k(f) = \inf_{A \in A_k(\mathcal{M})} \max_{u \in A} f(u), \tag{4.32}$$

where $A_k(\mathcal{M}) = \{A \in \Sigma_E \cap \mathcal{M} : \gamma(A) \geq k\}$. Denote K_c as the set of critical points of f corresponding to the value c. Then

a) f has at least $\gamma(\mathcal{M})$ pairs of critical points on $\mathcal{M}$:

$$\gamma(\mathcal{M}) \leq \sum_{c \in \mathbb{R}} \gamma(K_c).$$

b) If $C_k(f) \in \mathbb{R}$, then $C_k(f)$ is a critical value of f. Moreover, if

$$c = C_k(f) = \cdots = C_{k+m}(f),$$

then $\gamma(K_c) \geq m + 1$. Particularly, if $m \geq 1$, then K_c contains infinitely many elements.

Further information about this theorem and its proof can be found in [27].

Lemma 4.2.2. $(\mathcal{M}, J)$ *verifies the Palais-Smale condition.*

Proof. Let's consider $(u_n)_{n\in\mathbb{N}} \subseteq \mathcal{M}$ a Palais-Smale sequence, that is,

1) $(J(u_n))_{n\in\mathbb{N}} \subseteq \mathbb{R}$ is bounded.

2) $J'(u_n) \to 0$ as $n \to \infty$.

We have to prove that there exists a convergent subsequence of $(u_n)_{n\in\mathbb{N}}$.

From assumption 2) we have that

$$\lim_{n\to 0} \left\| J'(u_n) \right\|_{\mathrm{H}_0^1(\Omega)} = 0$$

so that J' converges to 0 strongly, so by Theorem 2.2.6 we have that J' converges weakly to 0. In particular, we have

$$\forall n \in \mathbb{N} : \quad \lim_{n\to\infty} \langle J'(u_n), u_n \rangle = 0.$$

Since
$$J(u) = \frac{1}{2} \|u\|^2_{\mathrm{H}^1_0(\Omega)},$$
it is bounded from below. Moreover, by this and since $(J(u_n))_{n\in\mathbb{N}} \subseteq \mathbb{R}$ is bounded, we have that
$$\exists k_1 \in \mathbb{R}, \exists k_2 \in (\max\{0, k_1\}, \infty), \forall n \in \mathbb{N} : \quad k_1 \leq J(u_n) \leq k_2.$$
By the compactness of the Sobolev embedding shown in Remark 4.1.5 and the equivalence of the $\|\cdot\|_\varepsilon$ and $\|\cdot\|_{\mathrm{H}^1_0(\Omega)}$ norms, we have that
$$\exists C_1 > 0 : \|u_n\|_{\mathrm{L}^p(\Omega)} \leq C_1 \|u_n\|_\varepsilon \leq C_{\Omega,\varepsilon} \|u_n\|_{\mathrm{H}^1_0(\Omega)}, \tag{4.33}$$
with $C_{\Omega,\varepsilon}$ being the constant appearing in (2.13). So that, by (4.33), up to a subsequence, for some $u \in \mathcal{M}$, it follows that
$$u_n \rightharpoonup u \quad \text{in } \mathcal{M}.$$
Now, viewing it as weak convergence, we have that
$$\forall \nu \in \mathrm{H}^{-1} : \langle \nu, u_n - u \rangle \to 0 \quad \text{as} \quad n \to \infty,$$
which implies that
$$\langle J'(u) - J'(u_n), u_n - u \rangle \to 0 \quad \text{as} \quad n \to 0.$$
Now, by (4.15), the triangle and Cauchy-Schwarz inequalities we compute
$$\begin{aligned}
\Big|\langle J'(u_n) - J'(u), u_n - u \rangle\Big| &= \Big|\langle J'(u_n), u_n - u\rangle - \langle J'(u), u_n - u\rangle\Big| \\
&= \Big|\langle J'(u_n), u_n - u\rangle + \langle -J'(u), u_n - u\rangle\Big| \\
&\leq \Big|\langle J'(u_n), u_n - u\rangle\Big| + \Big|\langle -J'(u), u_n - u\rangle\Big| \\
&= \Big|\langle J'(u_n), u_n - u\rangle\Big| + \Big|\langle J'(u), u_n - u\rangle\Big| \\
&\leq \|J'(u_n)\| \|u_n - u\|_{\mathrm{H}^1_0(\Omega)} + \|J'(u)\| \|u_n - u\|_{\mathrm{H}^1_0(\Omega)}.
\end{aligned} \tag{4.34}$$
Finally, by (4.34), we have that
$$\Big|\langle J'(u_n) - J'(u), u_n - u \rangle\Big| \to 0 \quad \text{as} \quad n \to \infty.$$
So that
$$u_n \to u, \quad \text{as} \quad n \to \infty,$$
whence, the functional J satisfies the (PS) condition. □

Since the functional J is of class C^1 and even, it satisfies the hypotheses of Theorem 4.2.1. So, for $k \in \mathbb{N}$ we write
$$\begin{cases} \Sigma = \Sigma_{\mathrm{H}^1_0(\Omega)}, & \mathcal{A}_k = \mathcal{A}_k(\mathcal{M}), \\ c_k = C_k(J) = J(\hat{w}_k) \in]0, \infty[. \end{cases} \tag{4.35}$$

Related to (P_{inf}) we have the following intermediate problem that will be of use

$$\begin{cases} \Delta u(x) + |u(x)|^{p-1}u(x) = 0, & x \in \Omega^\delta \\ u(x) = 0, & x \in \partial\Omega^\delta, \end{cases} \tag{P^δ}$$

where

$$\Omega^\delta = \{x \in \mathbb{R}^N \ / \ \text{dist}(x, \Omega) < \delta\}.$$

We set

$$\mathcal{M}^\delta = \{u \in \text{H}_0^1(\Omega^\delta) \ / \ \|u\|_{L^p(\Omega^\delta)} = 1\} \tag{4.36}$$

in order to define the functional $J^\delta : \mathcal{M}^\delta \to \mathbb{R}$ which is given by

$$J^\delta = \int_{\Omega^\delta} |\nabla u(x)|^2 dx,$$

which also satisfies the conditions of Theorem 4.2.1. Hence, for $k \in \mathbb{N}$ we write

$$\begin{cases} \Sigma^\delta = \Sigma_{\text{H}_0^1(\Omega^\delta)}, & \mathcal{A}_k^\delta = \mathcal{A}_k(\mathcal{M}^\delta), \\ c_k^\delta = C_k(J^\delta) = J^\delta(\hat{w}_k) \in (0, \infty). \end{cases}$$

And by the scaling shown in Remark 4.1.4, it follows that the function

$$w_k^\delta = (2c_k^\delta)^{1/(p-1)} \hat{w}_k^\delta(x), \quad x \in \Omega^\delta, \tag{4.37}$$

is a solution of (P^δ).

Now, since V and V_ε satisfy the conditions of the embedding Theorem 4.1.5 we have that, in particular, this result holds for $\text{H}_\varepsilon = \text{H}_{V_\varepsilon}$. Therefore, it is proved that the functional J_ε satisfies the Palais-Smale condition on $\mathcal{M}_\varepsilon$. Hence, we write for $\varepsilon > 0$ and for $k \in \mathbb{N}$,

$$\begin{cases} \Sigma_\varepsilon = \Sigma_{\text{H}_\varepsilon}, & \mathcal{A}_{k,\varepsilon} = \mathcal{A}_k(\mathcal{M}_\varepsilon), \\ c_{k,\varepsilon} = C_k(J_\varepsilon) = J(\hat{w}_{k,\varepsilon}) \in (0, \infty). \end{cases} \tag{4.38}$$

Remark 4.2.1. We can strengthen assumption (V1) so that for some $\eta > 0, V \in C^\eta(\mathbb{R}^N)$. Then, by using standard regularity arguments, it can be proved that $v_{k,\varepsilon}, w_k$ and w_k^δ are of class $C^{2,\eta}$, becoming classical solutions of (P_ε), (P_{inf}) and (P^δ), respectively.

4.3 Limits for the critical values

In this section we shall prove Theorem 4.1.2, that is, for any $k \in \mathbb{N}$

$$\lim_{\varepsilon \to 0} c_{k,\varepsilon} = c_k. \tag{4.39}$$

which means that the k-th level sets of the functionals J_ε and J are topologically equivalent by the Ljusternik-Schnirelman theory for even functionals used before. We begin by clarifying some notation and establishing three propositions that are used in the proof of Theorem 4.1.2.

We have that, for any $k \in \mathbb{N}$ and $\varepsilon, \delta > 0$

$$c_{k,\varepsilon} = \inf_{A \in \mathcal{A}_{k,\varepsilon}} \max_{u \in A} J_\varepsilon(u), \tag{4.40}$$

$$c_k = \inf_{A \in \mathcal{A}_k} \max_{u \in A} J(u), \tag{4.41}$$

$$c_k^\delta = \inf_{A \in \mathcal{A}_k^\delta} \max_{u \in A} J^\delta(u). \tag{4.42}$$

Proposition 4.3.1. *Let $k \in \mathbb{N}$. Then the following points hold*

$$\forall \varepsilon > 0 : \quad \mathcal{A}_k \subseteq \mathcal{A}_{k,\varepsilon}, \tag{4.43}$$

$$\forall \varepsilon > 0 : \quad c_{k,\varepsilon} \leq c_k \cdot C_{\Omega,\varepsilon}, \tag{4.44}$$

$$\limsup_{\varepsilon \to 0} c_{k,\varepsilon} \leq c_k, \tag{4.45}$$

where $C_{\Omega,\varepsilon}$ is as given in (4.31).

Proof. By Proposition 4.1.5, we have that the norms $\|\cdot\|_{\mathrm{H}_0^1(\Omega)}$ and $\|\cdot\|_\varepsilon$ induce the same topology. Therefore

$$\mathcal{A}_k \subseteq \mathcal{A}_{k,\varepsilon}.$$

Moreover, by (4.30), (4.40) and (4.41) we obtain

$$\begin{aligned} c_{k,\varepsilon} &= \inf_{A \in \mathcal{A}_{k,\varepsilon}} \max_{u \in A} J_\varepsilon(u) \\ &= \inf_{A \in \mathcal{A}_k} \max_{u \in A} J_\varepsilon(u) \\ &\leq C_{\Omega,\varepsilon} \cdot \inf_{A \in \mathcal{A}_k} \max_{u \in A} J(u) \\ &= C_{\Omega,\varepsilon} \cdot c_k. \end{aligned} \tag{4.46}$$

Now, by Proposition 4.1.3 we have that

$$\lim_{\varepsilon \to 0} \|V_\varepsilon\|_{L^\infty(\Omega)} = 0,$$

which, alongside (4.30) and (4.46), implies (4.45). □

Proposition 4.3.2. *Let $k \in \mathbb{N}$ and $\sigma > 0$. Then, $\exists \delta_0, \varepsilon_2 > 0$ such that*

$$\forall \delta \in (0, \delta_0), \forall \varepsilon \in (0, \varepsilon_2) : c_k^\delta \leq c_{k,\varepsilon} + \sigma. \tag{4.47}$$

Proof. 1.- Assume that $\varepsilon > 0$. By point (4.40) we have that

$$\exists A_\sigma(\varepsilon) \in \mathcal{A}_{k,\varepsilon}$$

such that

$$\max_{u \in \mathcal{A}_\sigma(\varepsilon)} J_\varepsilon(u) \leq c_{k,\varepsilon} + \frac{\sigma}{4}. \tag{4.48}$$

2.- Proposition 4.1.3 also implies that

$$\forall \mu > 0, \exists \hat{\varepsilon} = \hat{\varepsilon}(\mu) > 0 : \varepsilon \in (0, \hat{\varepsilon}) \implies \|V_\varepsilon\|_{L^\infty(\Omega)} < \mu. \tag{4.49}$$

Let's choose

$$\mu = \frac{8\sigma c_k + \sigma^2}{16 C_\Omega^2 c_k^2}, \tag{4.50}$$

and

$$\varepsilon_0 = \varepsilon_0(\sigma, k) = \hat{\varepsilon}(\mu).$$

From here on we assume that $0 < \varepsilon < \varepsilon_0$. Then, by points (4.1.3),(4.44), (4.48) and (4.50) we get

$$\begin{aligned} c_{k,\varepsilon}^2 &\leq c_k^2 + C_\Omega^2 \|V_\varepsilon\|_{L^\infty(\Omega)} c_k^2 \\ &\leq c_k^2 + \frac{\sigma}{2} c_k + \frac{\sigma^2}{16} \\ &= \left[c_k + \frac{\sigma}{4} \right]^2 . \end{aligned} \tag{4.51}$$

Therefore

$$c_{k,\varepsilon} \leq c_k + \frac{\sigma}{4}. \tag{4.52}$$

3. We choose

$$b_{k,\sigma} = c_k + \frac{\sigma}{2},$$

so that, points (4.49) and (4.52) imply that

$$\forall v \in A_\sigma(\varepsilon) : J_\varepsilon \leq b_{k,\sigma}, \tag{4.53}$$

thus obtaining, by (4.7),

$$\forall v \in A_\sigma(\varepsilon) : \int_{\mathbb{R}^N} |\nabla v(x)|^2 dx \leq 2 b_{k,\sigma}, \tag{4.54}$$

$$\forall v \in A_\sigma(\varepsilon) : \int_{\mathbb{R}^N} V_\varepsilon(x) \cdot |v(x)|^2 dx \leq 2 b_{k,\sigma}. \tag{4.55}$$

4. For $\rho > 0$ we denote

$$V_{\rho,\varepsilon} = \min_{x \in \mathbb{R}^N \setminus \Omega^\rho} V_\varepsilon(x). \tag{4.56}$$

Assume that $\delta > 0$, let $D \in (0, 1)$ provided by Proposition 4.1.4 and let's choose $\delta_* > 1$ such that

$$R_{\varepsilon,D,d_*} = \overline{B}(0, Dg(\varepsilon)) \setminus \Omega^\delta \subseteq \mathbb{R}^N \setminus \Omega^\delta.$$

Then

$$\min_{x \in R_{\varepsilon,D,d}} V_\varepsilon(x) \leq V_{\delta,\varepsilon},$$

by (4.29) we get

$$\lim_{\varepsilon\to 0} V_{\delta,\varepsilon} = \infty. \tag{4.57}$$

From point (4.55) we obtain that

$$\forall v \in A_\sigma(\varepsilon): \quad \|v\|^2_{\mathrm{L}^2\left(\mathbb{R}^N\setminus\Omega^\delta\right)} = \int_{\mathbb{R}^N\setminus\Omega^\delta} |v(x)|^2 dx \le \frac{2b_{k,\sigma}}{V_{\delta,\varepsilon}}. \tag{4.58}$$

On the other hand, by Theorem 2.3.5 and point (4.54) it holds that

$$\forall v \in A_\sigma(\varepsilon): \|v\|_{L^{2^*}(\mathbb{R}^N)} \le \theta\|\nabla v\|_{L^2(\mathbb{R}^N)} \le \theta(2b_{k,\sigma})^{1/2},$$

where $\theta = \theta_N > 0$ depends on the dimension. Now

$$\forall v \in A_\sigma(\varepsilon): \|v\|_{L^{2^*}(\mathbb{R}^N\setminus\Omega^\delta)} \le \theta(2b_{k,\varepsilon})^{1/2}. \tag{4.59}$$

We let $0 < \alpha < 1$ be such that

$$\frac{1}{p+1} = \frac{1-\alpha}{2} + \frac{\alpha}{2^*}.$$

So that by points (4.58), (4.59) and the Interpolation Inequality shown in Remark 2.2.6, it follows that, for any $v \in A_\sigma(\varepsilon)$

$$\begin{aligned}
\|v\|_{L^{p+1}(\mathbb{R}^N\setminus\Omega^\delta)} &\le \|v\|^{1-\alpha}_{L^2(\mathbb{R}^N\setminus\Omega^\delta)} \cdot \|v\|^{\alpha}_{L^{2^*}(\mathbb{R}^N\setminus\Omega^\delta)} \\
&\le \left(\frac{2b_{k,\sigma}}{V_{\delta,\varepsilon}}\right)^{(1-\alpha)/2} \cdot \theta^\alpha (2b_{k,\sigma})^{\alpha/2} \\
&= \frac{\theta^\alpha (2b_{k,\sigma})^{1/(p+1)}}{V_{\delta,\varepsilon}^{(1-\alpha)/2}},
\end{aligned}$$

which, by point (4.57), implies that

$$\lim_{\varepsilon\to 0} \max_{v\in A_\sigma(\varepsilon)} \|v\|_{L^{p+1}(\mathbb{R}^N\setminus\Omega^\delta)} = 0. \tag{4.60}$$

6. Now, by (4.60) and for any $s > 0$ there exists $\varepsilon_1 = \varepsilon_1(\delta, s; \sigma, k) \in (0, \varepsilon_0)$ such that

$$\forall \varepsilon \in (0, \varepsilon_1): \quad \max_{v\in A_\sigma(\varepsilon)} \|v\|_{L^{p+1}(\mathbb{R}^N\setminus\Omega^\delta)} < \delta^s. \tag{4.61}$$

We choose now $s = 1$ and $\hat{\varepsilon_1} = \varepsilon_1(\delta, 1; \sigma, k) \in (0, \varepsilon_0)$. Hence, we get

$$\forall \varepsilon \in (0, \hat{\varepsilon}_1), \forall v \in A_\sigma(\varepsilon): \quad \|v\|_{L^{p+1}(\mathbb{R}^N\setminus\Omega^\delta)} < \delta. \tag{4.62}$$

If we assume that $0 < \delta < 1$, point (4.62) implies that

$$\forall \varepsilon \in (0, \hat{\varepsilon}_1), \forall v \in A_\sigma(\varepsilon): \quad \|v\|_{L^{p+1}(\Omega^\delta)} < 1 - \delta. \tag{4.63}$$

7. We pick a cut-off function $\phi_\delta \in C_0^\infty(\mathbb{R}^N)$ such that, for some $r > 1$

$$\begin{cases} \forall x \in \Omega^{\delta/2} : \quad \phi_\delta(x) = 1; \\ \forall x \in \mathbb{R}^N \setminus \Omega^\delta : \quad \phi_\delta(x) = 0; \\ \forall x \in G^\delta : \quad 0 < \phi_\delta(x) < 1; \\ \forall x \in G^\delta : \quad |\nabla \phi_\delta(x)| \leq \dfrac{1}{\delta^r}, \end{cases} \tag{4.64}$$

where

$$G^\delta = \Omega^\delta \setminus \overline{\Omega^{\delta/2}}.$$

We define $\Phi_\delta : A_\sigma(\varepsilon) \subseteq \mathcal{M}_\varepsilon \to \mathcal{M}^\delta$ given by

$$\Phi_\delta[u] := \frac{\phi_\delta \cdot u}{\|\phi_\delta \cdot u\|_{L^{p+1}(\Omega^\delta)}}.$$

Since ϕ_δ is odd, then Φ_δ is odd, now we need to prove that Φ_δ is bounded to apply point iii) of Lemma 2.4.8 and get that

$$\Phi_\delta[A_\sigma(\varepsilon)] \in \mathcal{A}_k^\delta. \tag{4.65}$$

Let's prove that Φ is bounded and therefore continuous. Assume that $\varepsilon \in (0, \tilde{\varepsilon}_1)$, where

$$\tilde{\varepsilon}_1 = \min\{\hat{\varepsilon}_1, \varepsilon_1(\delta/2, 1; \sigma, k)\}.$$

a) By (4.63), for any $v \in A_\sigma(\varepsilon)$, that

$$\begin{aligned} 1 \geq & \|\phi_\delta v\|_{L^{p+1}(\Omega^\delta)}^{p+1} \\ = & \int_{\Omega^{\delta/2}} |v(x)|^{p+1} dx + \int_{G^\delta} |\phi_\delta(x) v(x)|^{p+1} dx \\ \geq & \int_{\Omega^{\delta/2}} |v(x)|^{p+1} dx \\ \geq & \left(1 - \frac{\delta}{2}\right)^{p+1} \\ \geq & (1-\delta)^{p+1} \end{aligned} \tag{4.66}$$

so that ϕ_δ is well defined.

b) For any $u, v \in A_\sigma(\varepsilon) \subseteq \mathcal{M}_\varepsilon$ we have that

$$\begin{aligned} \|\Phi_\delta[u] - \Phi_\delta[v]\|_{\mathrm{H}_0^1(\Omega^\delta)} = & \|\nabla(\phi_\delta(u-v))\|_{L^2(\Omega^\delta)} \\ \leq & \|\phi_\delta \nabla(u-v)\|_{L^2(\Omega^\delta)} + \|(u-v)\nabla\phi_\delta\|_{L^2(\Omega^\delta)}. \end{aligned} \tag{4.67}$$

Now, since the norms $\|\cdot\|_{\mathrm{H}_0^1(\Omega)}$ and $\|\cdot\|_\varepsilon$ are still equivalent if we replace Ω by any $U \in \mathbb{R}^N$ open and bounded, we get by (4.64) that

$$\begin{aligned}
\|\phi_\delta \nabla(u-v)\|_{L^2(\Omega^\delta)} &= \left(\int_{\Omega^\delta} \phi_\delta |\nabla(u-v)(x)|^2 dx\right)^{1/2} \\
&\leq \|u-v\|_{\mathrm{H}_0^1(\Omega^\delta)} \\
&\leq \|u-v\|_\varepsilon .
\end{aligned} \tag{4.68}$$

On the other hand, we have by (4.56) and (4.64) that

$$\begin{aligned}
\|(u-v)\nabla\phi_\delta\|_{L^2(\Omega^\delta)} &= \left(\int_{\Omega^\delta} |u(x)-v(x)|^2 \cdot |\nabla\phi_\delta(x)|^2\right)^{1/2} \\
&= \left(\int_{G^\delta} |u(x)-v(x)|^2 \cdot |\nabla\phi_\delta(x)|^2\right)^{1/2} \\
&\leq \frac{1}{\delta^r}\left(\int_{G^\delta} \frac{V_\varepsilon(x)}{V_\varepsilon(x)} \cdot |u(x)-v(x)|^2 dx\right)^{1/2} \\
&\leq \frac{1}{\delta^r \cdot \min\limits_{y\in G^\delta} V_\varepsilon(y)} \left(V_\varepsilon(x) \cdot |u(x)-v(x)|^2 dx\right)^{1/2} \\
&\leq \frac{1}{\delta^r \cdot V_{\delta/2,\varepsilon}} \|u-v\|_\varepsilon .
\end{aligned} \tag{4.69}$$

Moreover, from points (4.67),(4.68) and (4.69) it follows that

$$\|\Phi_\delta[u]-\Phi_\delta[v]\|_{\mathrm{H}_0^1(\Omega^\delta)} \leq \left(1+\frac{1}{\delta^r \cdot V_{\delta/2,\varepsilon}}\right)\|u-v\|_\varepsilon ,$$

showing that Φ_δ is Lipschitz continuous.

8. By (4.65) it follows that

$$c_k^\delta \leq \max_{v\in\Phi_\delta[A_\sigma(\varepsilon)]} J^\delta(v),$$

so that, we pick

$$u \in A_\sigma(\varepsilon), \quad \overline{v} = \Phi_\delta[u] \in \Phi_\delta[A_\sigma(\varepsilon)],$$

such that

$$c_k^\delta \leq \max_{v\in\Phi_\delta[A_\sigma(\varepsilon)]} J^\delta(\overline{v}) + \frac{\sigma}{4}. \tag{4.70}$$

And we claim that

$$\exists w \in A_\sigma(\varepsilon) : J^\delta(\overline{v}) \leq J_\varepsilon(w) + \frac{\sigma}{2}. \tag{4.71}$$

So that by (4.48), (4.70) and (4.71), it follows that

$$\begin{aligned}
c_k^\delta &\leq J^\delta(\overline{v}) + \frac{\sigma}{4} \leq J_\varepsilon(w) + \frac{3\sigma}{4} \\
&\leq \max_{u\in A_\sigma(\varepsilon)} J_\varepsilon(u) + \frac{3\sigma}{4} \leq c_{k,\varepsilon} + \sigma.
\end{aligned}$$

9. Recall our claim, (4.71), we need to prove it to conclude. Notice that taking $u = w$ is enough. Indeed, by (4.66) we have that

$$\begin{aligned}
2(1-\delta)^2\, J^{\delta}(\overline{v}) &\leq 2\,\|\phi_\delta u\|^2_{\mathrm{L}^{p+1}(\Omega^\delta)} \cdot J^{\delta}(\overline{v}) \\
&= \|\phi_\delta u\|^2_{\mathrm{L}^{p+1}(\Omega^\delta)} \cdot \left\| \frac{\phi_\delta\, u}{\|\phi_\delta\, u\|_{\mathrm{L}^{p+1}(\Omega^\delta)}} \right\|^2_{\mathrm{H}^1_0(\Omega^\delta)} \\
&= \|\nabla(\phi_\delta u)\|^2_{\mathrm{L}^2(\Omega^\delta)} \\
&= \int_{\Omega^\delta} [u^2|\nabla\phi_\delta|^2 + 2u\theta_\delta \nabla u \nabla \phi_\delta + \phi_\delta^2 |\nabla u|^2] dx. \qquad (4.72)
\end{aligned}$$

We have that

$$\int_{\Omega^\delta} \phi_\delta^2(x)|\nabla u(x)|^2 \leq \int_{\Omega^\delta} |\nabla u(x)|^2 dx \leq 2J_\varepsilon(u). \qquad (4.73)$$

Therefore, by remark (4.1.8) we obtain:

$$\begin{aligned}
\int_{\Omega^\delta} u^2(x)|\nabla\phi_\delta(x)|^2 dx &\leq \int_{G^\delta} u^2(x)\nabla\phi_\delta(x)|^2 dx \\
&\leq \frac{1}{\delta^{2r}} \int_{G^d} u^2(x) dx \\
&\leq \frac{1}{\delta^{2r}} |G^d|^{(p-1)/(p+1)}\, \|u\|^2_{\mathrm{L}^{p+1}(G^d)}. \qquad (4.74)
\end{aligned}$$

Finally by applying (4.1.8) again alongside (4.54) and the CBS inequality on $\mathbb{R}^N$ and $L^2(\mathbb{R}^N)$ we obtain

$$\begin{aligned}
\int_{\Omega^\delta} 2u\phi_\delta \nabla u \nabla\phi_\delta dx &\leq 2\int_{\Omega^\delta} |u|\,|\phi_\delta|\,|\nabla u|\,|\nabla \phi_\delta|\, dx \\
&\leq \frac{2}{\delta^r} \int_{G^\delta} |u|\,|\nabla u|\ dx \\
&\leq \frac{2}{\delta^r} \left(\int_{G^\delta} |\nabla u|^2 dx \right)^{1/2} \left(\int_{G^\delta} |u|^2 dx \right)^{1/2} \\
&\leq \frac{2}{\delta^r} (2b_{k,\sigma})^{1/2} |G^\delta|^{(p-1)/2(p+1)}\, \|u\|_{\mathrm{L}^{p+1}(G^\delta)}. \qquad (4.75)
\end{aligned}$$

We now assume that $\varepsilon \in (0, \varepsilon_2)$ where

$$\varepsilon_2 = \min\{\tilde{\varepsilon}_1, \varepsilon_1(\delta, s_*; \sigma, k)\},$$

for some $s_* > 2r$. Now, observe that $\delta^{2s_*} < \delta^{s_*}$ and by using (4.61) we get from points (4.72) to (4.75) that,

$$(1-\delta)^2 J^\delta(\overline{v}) \leq J_\varepsilon(u) + \frac{\zeta}{2}\delta^{s_*-2r}, \qquad (4.76)$$

where

$$\zeta = \max\left\{ |G^\delta|^{(p-1)/(p+1)} + 2(2b_{k,\sigma})^{1/2} |G^\delta|^{(p-1)/2(p+1)} \right\}.$$

Now let

$$0 < \delta < 1 - \frac{\sqrt{2}}{2}.$$

So that, by (4.53) and (4.76) we get

$$\begin{aligned}\frac{1}{2}J^{\delta}(\overline{v}) &\leq J_{\varepsilon}(u) + \frac{\zeta}{2}\delta^{s_*-2r} \\ &\leq b_{k,\sigma}\frac{\zeta}{2}\delta^{s_*-2r},\end{aligned} \tag{4.77}$$

so that by combining points (4.76) and (4.77) we have

$$\begin{aligned}J^{\delta}(\overline{v}) &\leq J_{\varepsilon}(u) + \frac{\zeta}{2}\delta^{s_*-2r} + 2\delta\, J^{\delta}(\overline{v}) - \delta^2\, J^{\delta}(\overline{v}) \\ &\leq J_{\varepsilon}(u) + \frac{\zeta}{2}\delta^{s_*-2r} + 2\delta\left(2b_{k,\sigma}) + \zeta\delta^{s_*-2r}\right),\end{aligned}$$

so that it's clear that we can find $\delta_0 \in (0, 1 - \frac{\sqrt{2}}{2})$ such that (4.71) holds for any $\delta \in (0, \delta_0)$ and $\varepsilon \in (0, \varepsilon_2)$. We conclude our proof. □

The following result collects the results of Lemmas 3.3 and 3.4 in [11].

Proposition 4.3.3. *Let $k \in \mathbb{N}$. Then*

$$\forall \delta > 0: \quad c_k^{\delta} \leq c_k; \tag{4.78}$$

$$\forall \sigma > 0, \exists \delta_{\sigma} > 0, \forall \delta \in]0, \delta_{\sigma}[: \quad c_k \leq c_k^{\delta} + \sigma. \tag{4.79}$$

Proof of Theorem 4.1.2. Let $\sigma > 0$ be small. Let's choose δ_{σ} as in (4.79), take $\delta_0 = \delta_0(\sigma) > 0$ and $\varepsilon_2 = \varepsilon_2(\sigma)$ from Proposition 4.3.2. We set

$$\hat{\delta_{\sigma}} = \min\{\delta_{\sigma}, \delta_0\}.$$

Then, by (4.44), (4.47) and (4.79) we have, for any $\delta \in (0, \hat{\delta_{\sigma}})$ and any $\varepsilon \in (0, \varepsilon_2)$, that

$$c_k \leq c_k^{\delta} + \sigma \leq c_{k,\varepsilon} + 2\sigma \leq c_k \cdot C_{\Omega,\varepsilon} + 2\sigma.$$

We conclude that

$$\lim_{\varepsilon \to 0} c_{k,\varepsilon} = c_k,$$

by the arbitrariness of σ. □

4.4 Asymptotic profiles and concentration phenomena

In this section want to prove Theorems 4.1.3 and 4.1.4.

Recall that Theorem 4.1.3 states that for some $k \in \mathbb{N}$, as $\varepsilon \to 0, (w_{k,\varepsilon})_{\varepsilon>0}$ subconverges in $\mathrm{H}^1(\mathbb{R}^N)$ to some $u_k \in \mathrm{H}^1(\mathbb{R}^N)$ such that its restriction to Ω is another solution of (P_{inf}), verifies

$$J(\hat{u}_k|_\Omega) = c_k,$$

where

$$\hat{u}_k = (2c_k)^{1/p-1}u_k.$$

Lemma 4.4.1. *Let $k \in \mathbb{N}$. Then $(\hat{w}_{k,\varepsilon})_{\varepsilon>0}$ weakly and pointwise subconverges to some $\hat{u}_k \in \mathrm{H}^1(\mathbb{R}^N)$, as $\varepsilon \to 0$.*

Proof. By Theorem 4.1.2, for a given $\sigma > 0$, there exists $\varepsilon_{\sigma,1} > 0$ such that, for any $\varepsilon \in (0, \varepsilon_{\sigma,1})$,

$$\int_{\mathbb{R}^N} [|\nabla \hat{w}_{k,\varepsilon}(x)|^2 + V_\varepsilon |\hat{w}_{k,\varepsilon}(x)|^2]dx = 2c_{k,\varepsilon} \leq 2c_k + \sigma \equiv B_{k,\sigma}. \tag{4.80}$$

Then, by Theorem 2.3.5, there exists $C_N > 0$ such that for any $\varepsilon \in (0, \varepsilon_{\sigma,1})$

$$\left\|\hat{w}_{k,\varepsilon}\right\|^2_{\mathrm{L}^{2^*}(\mathbb{R}^N)} \leq C_N^2 \int_{\mathbb{R}^N} |\hat{w}_{k,\varepsilon}(x)|^2 dx \leq C_N^2 B_{k,\sigma}. \tag{4.81}$$

Let $0 < \delta < 1$. By Hölders inequality and (4.81), for any $\varepsilon \in (0, \varepsilon_{\sigma,1})$ we have that

$$\begin{aligned}
\|\hat{w}_{k,\varepsilon}\|^2_{L^2(\Omega^\delta)} &\leq |\Omega^\delta|^{2/N} \cdot \|\hat{w}_{k,\varepsilon}\|^2_{L^{2^*}(\Omega^\delta)} \\
&\leq |\Omega^\delta|^{2/N} \cdot \|\hat{w}_{k,\varepsilon}\|^2_{L^{2^*}(\mathbb{R}^N)} \\
&\leq C_N^2 |\Omega^\delta|^{2/N} \cdot B_{k,\sigma}.
\end{aligned} \tag{4.82}$$

On the other hand, by (4.57), there exists $\varepsilon_{\sigma,2} \in (0, \varepsilon_{\sigma,1})$ such that, for any $\varepsilon \in (0, \varepsilon_{\sigma,2})$, it verifies that $\dfrac{1}{V_{\delta,\varepsilon}} < 1$. Then, by (4.80) we obtain,

$$\|\hat{w}_{k,\varepsilon}\|^2_{L^2(\mathbb{R}^N \setminus \Omega^\delta)} \leq \int_{\mathbb{R}^N \setminus \Omega^\delta} \frac{V_\varepsilon(x)}{V_{\delta,\varepsilon}} |\hat{w}_{k,\varepsilon}(x)|^2 dx \leq B_{k,\sigma}. \tag{4.83}$$

Then, (4.81), (4.82) and (4.83), for $\varepsilon \in (0, \varepsilon_{\sigma,2})$ it follows that

$$\begin{aligned}
\|\hat{w}_{k,\varepsilon}\|^2_{\mathrm{H}^1(\mathbb{R}^N)} &= \int_{\mathbb{R}^N} [|\nabla \hat{w}_{k,\varepsilon}(x)|^2 + |\hat{w}_{k,\varepsilon}(x)|^2]dx \\
&\leq B_{k,\sigma} + C_N^2 |\Omega^\delta|^{2/N} B_{k,\sigma} + B_{k,\sigma} \\
&= B_{k,\sigma} \cdot (2 + C_N^2 |\Omega^1|^{2/N}).
\end{aligned} \tag{4.84}$$

Hence, from (4.84) and Theorems 2.2.20 and [4, Th.3.18], there exists $\hat{u_k} \in \mathrm{H}^1(\mathbb{R}^N)$ such that $(\hat{w}_{k,\varepsilon})_{\varepsilon>0}$ subconverges in $\mathrm{H}^1(\mathbb{R}^N)$ weakly and pointwise. $\square$

Lemma 4.4.2. *Let $k \in \mathbb{N}$. The function $\hat{u}_k$ is a weak solution of $(P_{\inf})$ and verifies that $J(\hat{u}_k|_\Omega) = c_k$.*

Proof. Let $\varepsilon > 0$. Since $\hat{w}_{k,\varepsilon} \in \mathcal{M}_\varepsilon$ is a critical point of J_ε, by Remark 4.1.3, we have that for any $\varphi \in \mathrm{C}_0^\infty(\mathbb{R}^N)$

$$\int_{\mathbb{R}^N} [\nabla \hat{w}_{k,\varepsilon} \nabla \varphi + V_\varepsilon(x) \hat{w}_{k,\varepsilon} \varphi] dx = 2c_{k,\varepsilon} \int_{\mathbb{R}^N} |\hat{w}_{k,\varepsilon}|^{p-1} \hat{w}_{k,\varepsilon} dx. \tag{4.85}$$

Let $\varphi \in \mathrm{C}_0^\infty(\Omega)$. By points (4.82) and (4.83) we have, for any $\varepsilon \in (0, \varepsilon_{\sigma,2})$ that

$$\begin{aligned} \left| V_\varepsilon(x) \hat{w}_{k,\varepsilon}(x) \varphi(x) \right| &\leq \left\| \hat{w}_{k,\varepsilon} \right\|_{\mathrm{L}^2(\Omega)} \|\varphi\|_{\mathrm{L}^2(\Omega)} \|V_\varepsilon\|_{\mathrm{L}^\infty(\Omega)} \\ &\leq \left[1 + C_N^2 |\Omega^1|^{2/N} \right] \|\varphi\|_{\mathrm{L}^2(\Omega)} \|V_\varepsilon\|_{\mathrm{L}^\infty(\Omega)}. \end{aligned}$$

And, by Proposition 4.3.1, we have

$$\int_{\mathbb{R}^N} V_\varepsilon(x) \hat{w}_{k,\varepsilon} \varphi(x) \to 0, \quad \text{as} \quad \varepsilon \to 0. \tag{4.86}$$

Moreover, by the compactness of the embedding in Theorem 4.1.5, we have that $(\hat{w}_{k,\varepsilon})_{\varepsilon>0}$ subconverges in $L^{p+1}(\mathbb{R}^N)$ to $\hat{u}_k$. Whence, by (4.85), (4.86), Theorem 4.1.2 and since φ is arbitrary, we obtain

$$\forall \varphi \in \mathrm{C}_0^\infty(\Omega): \quad \int_\Omega \nabla \hat{u}_k \nabla \varphi dx = 2c_k \int_\Omega |\hat{u}_k|^{p-1} \hat{u}_k \varphi dx. \tag{4.87}$$

Now, let's take $(\varphi_n)_{n \in \mathbb{N}} \subseteq \mathrm{C}_0^\infty(\Omega)$ converging in $L^{p+1}(\Omega)$ to $\hat{u}_k|_\Omega$. Therefore, by replacing $\varphi = \varphi_n$ in (4.87) and letting $n \to \infty$, we get, by Lemma 4.4.2, $c_k = J(\hat{u}_k|_\Omega)$.

For $\delta, \alpha > 0$ we write $\Gamma_{\delta,\alpha} = \{x \in \mathbb{R}^N \setminus \Omega^\delta \ / \ |\hat{u}_k(x)| \geq \alpha\}$. We have that

$$\forall \delta, \alpha: \quad |\Gamma_{\delta,\alpha}| = 0. \tag{4.88}$$

Indeed, looking for a contradiction, assume that

$$\exists \delta_*, \alpha_* > 0: \quad |\Gamma_{\delta_*,\alpha_*}| \neq 0, \tag{4.89}$$

which implies that there exists $\eta > 0$ such that

$$|\Gamma_{\delta_*,\alpha_*}| \geq \eta > 0.$$

Since for any $\delta \in (0, \delta_*)$ we have that $\Gamma_{\delta_*,\alpha_*} \subset \Gamma_{\delta,\alpha_*}$ it holds that

$$|\Gamma_{\delta,\alpha_*}| \geq \eta > 0.$$

Now, since Ω is bounded and connected, by condition (V3) we have that there is some $\delta' \in (0, \delta_*)$ such that for any $\varepsilon > 0$

$$\forall \delta \in (0, \delta'): \quad V_{\delta,\varepsilon} < \frac{\alpha_*^2 \eta}{2}. \tag{4.90}$$

Now let $\delta_0 \in (0, \delta')$ be fixed. We have that

$$\int_{\Gamma_{\delta_0,\alpha_*}} |\hat{u}_k|^2 \geq \alpha_*^2 \eta. \tag{4.91}$$

On the other hand, we associate to each $\delta > 0$

$$\varepsilon_\delta^* = \min\left\{\varepsilon_{\sigma,2}, \frac{V_{\delta,\varepsilon}}{(2c_k)^{1/2}}\right\}. \tag{4.92}$$

And for any $\sigma^* > 0$ there exists $\varepsilon_{\sigma^*} \in (0, \varepsilon_{\sigma,2})$ such that

$$\forall \varepsilon \in (0, \varepsilon_{\sigma^*}): \quad \|\hat{u}_k\|^2 \leq \|\hat{w}_{k,\varepsilon}\|^2 + \sigma^*.$$

Thus, for any $\varepsilon \in (0, \varepsilon_{\sigma^*})$ we set $\sigma^* = \dfrac{\alpha_*^2 \eta}{6}$. By points (4.83), (4.90) and (4.92) we have that

$$\int_{\Gamma_{\delta_0,\alpha_*}} |\hat{u}_k|^2 \leq \sigma^* + \int_{\Gamma_{\delta_0,\alpha_*}} |w_{k,\varepsilon}|^2 \leq \frac{\alpha_*^2\eta}{6} + B_{k,\sigma} < \frac{\alpha_*^2\eta}{6} + V_{\delta,\varepsilon} < \frac{2\alpha^2\eta}{3},$$

which contradicts (4.91). Hence $|\Gamma_{\delta,\alpha}| = 0$.

So that

$$\hat{u}_k = 0, \quad \text{for a.e.} \quad x \in \mathbb{R}^N \setminus \Omega, \tag{4.93}$$

and by Remark 2.3.4 and Theorem 2.3.1 it follows that

$$\hat{u}_k|_\Omega \in \mathrm{H}_0^1(\Omega).$$

We conclude by this and point (4.87). □

Proof of Theorem 4.1.3. By the compactness of the injection $\mathrm{H}_\varepsilon \subseteq L^2(\mathbb{R}^N)$, Lemma 4.4.1 and point (4.93) imply that

$$\lim_{\varepsilon \to 0} \|\hat{w}_{k,\varepsilon}\|_{L^2(\mathbb{R}^N)}^2 = \|\hat{u}_h\|_{L^2(\mathbb{R}^N)}^2. \tag{4.94}$$

Then, by (4.45) and (4.93) we have that

$$\limsup_{\varepsilon \to 0} \int_{\mathbb{R}^N} |\nabla \hat{w}_{k,\varepsilon}|^2 dx \leq 2 \limsup_{\varepsilon \to 0} c_{k,\varepsilon} \leq 2c_k = \int_{\mathbb{R}^N} |\nabla \hat{u}_k|^2 dx. \tag{4.95}$$

Whence, by points (4.94) and (4.95) we obtain that

$$\limsup_{\varepsilon \to 0} \|\hat{w}_{k,\varepsilon}\|_{\mathrm{H}^1(\mathbb{R}^N)} \leq \|\hat{u}_k\|_{\mathrm{H}^1(\mathbb{R}^N)}.$$

Finally, since $\mathrm{H}^1(\mathbb{R}^N)$ is a Hilbert space, it's uniformly convex by Proposition 2.2.2. So that, by Theorem 2.2.11 we have that $(\hat{w}_{k,\varepsilon})_{\varepsilon > 0}$ subconverges in $\mathrm{H}^1(\mathbb{R}^N)$ to $\hat{u}_k$, as $\varepsilon \to 0$. We conclude by Lemma 4.4.2. □

Remark 4.4.1 (Important)**.** From [11] we know that we can strengthen our assumption (V1) by assuming that V is of class C^{α} and that it can be proved that each weak solution presented is a classical solution. From now on we will assume this, i.e., $v_{k,\varepsilon}$ and w_k belong to $C^{2,\alpha}(\mathbb{R}^N)$ and are classical solutions of (P_ε) and $(P_{\inf})$, respectively.

Before proving Theorem 4.1.4 we shall present a useful proposition proven in [5] related to our problem and elliptic inequalities. This result was proved for any $\Omega \subset \mathbb{R}^N$ smooth and bounded, however it can be extended to a not necessarily bounded Ω or regular $\partial\Omega$.

Proposition 4.4.1. *Let $\Omega \subset \mathbb{R}^N$ be open and connected. If $w \in \mathrm{H}_0^1(\Omega)$, is a classical solution of the elliptic inequality*

$$\begin{cases} \Delta w - f(w) \geq 0 & \text{in } \Omega, \\ w > 0 & \text{in } \Omega, \\ w = 0 & \text{in } \partial\Omega, \end{cases} \tag{4.96}$$

where $N \geq 3, p+1 \in (2, 2^*)$, and for some $c > 0$ f satisfies

$$t \in \mathbb{R}^+ : \quad tf(t) \leq ct^{p+1}, \tag{4.97}$$

then there exists a constant $C = C(c, p, N) > 0$ such that

$$\|w\|_{L^\infty(\Omega)} \leq C\|w\|_{L^{2^*}(\Omega)}^{4/[(N+2)-p(N-2)]}. \tag{4.98}$$

Lemma 4.4.3. *For any $k \in \mathbb{N}$ and a given $\sigma > 0$, there exists $\varepsilon_{\sigma,2} > 0$ and $K = K(\sigma, N, k, p) > 0$ such that*

$$\forall \varepsilon \in (0, \varepsilon_{\sigma,2}) : \quad \|w_{k,\varepsilon}\|_{L^\infty(\mathbb{R}^N)} \leq K. \tag{4.99}$$

Proof. Let $\sigma > 0$ and $\varepsilon_{\sigma,2}$ as in the proof of Theorem 4.1.3. For any $\varepsilon \in (0, \varepsilon_{\sigma,2})$ we consider the connected component A_ε^+ of

$$W_\varepsilon^+ = \{x \in \mathbb{R}^N : \quad w_{k,\varepsilon} > 0\}.$$

Now, as $w_{k,\varepsilon}$ solves (P'_ε), by the non-negativity of V_ε it follows that

$$\begin{cases} \Delta w_{k,\varepsilon} + w_{k,\varepsilon}^p \geq 0 & x \in A_\varepsilon^+, \\ w_{k,\varepsilon} > 0 & x \in A_\varepsilon^+, \\ w_{k,\varepsilon} = 0 & x \in \partial A_\varepsilon^+. \end{cases} \tag{4.100}$$

So that, by Proposition 4.4.1 and (4.81) it holds that

$$\begin{aligned} \|w_{k,\varepsilon}\|_{L^\infty(A_\varepsilon^+)} &\leq C\|w_{k,\varepsilon}\|_{L^{2^*}(A_\varepsilon^+)}^{4/[(N+2)-p(N-2)]} \\ &\leq C\left[C_N^2 B_{k,\sigma}\right]^{2/[(N+2)-p(N-2)]} = K. \end{aligned} \tag{4.101}$$

By the arbitrariness of A_ε^+ we have that (4.101) holds for W_ε^+. That is,

$$\|w_{k,\varepsilon}\|_{L^\infty(W_\varepsilon^+)} \leq K.$$

Analogously we can obtain the same result for the region $W_\varepsilon^- = \{x \in \mathbb{R}^N : w_{k,\varepsilon} < 0\}$. $\square$

Remark 4.4.2. Recall that

$$v_{k,\varepsilon}(x) = \left[2c_{k,\varepsilon}(\varepsilon g(\varepsilon))^2\right]^{1/(p-1)} \hat{w}_{k,\varepsilon}\left(\frac{x}{g(\varepsilon)}\right).$$

Therefore, we have that

$$\begin{aligned}
\|v_{k,\varepsilon}\|_{L^\infty(\mathbb{R}^N)} &= \sup_{x\in\mathbb{R}^N}\left|\left[2c_{k,\varepsilon}(\varepsilon g(\varepsilon))^2\right]^{1/(p-1)} \hat{w}_{k,\varepsilon}\left(\frac{x}{g(\varepsilon)}\right)\right| \\
&\leq \left[2c_{k,\varepsilon}(\varepsilon g(\varepsilon))^2\right]^{1/(p-1)} \sup_{x\in\mathbb{R}^N}\left|\hat{w}_{k,\varepsilon}\left(\frac{x}{g(\varepsilon)}\right)\right| \\
&= \left[2c_{k,\varepsilon}(\varepsilon g(\varepsilon))^2\right]^{1/(p-1)} \|\hat{w}_{k,\varepsilon}\|_{L^\infty(\mathbb{R}^N)} \\
&\leq \left[2c_{k,\varepsilon}(\varepsilon g(\varepsilon))^2\right]^{1/(p-1)} K.
\end{aligned}$$

Now, by (4.25) we obtain

$$\lim_{\varepsilon\to 0}\|v_{k,\varepsilon}\|_{L^\infty(\mathbb{R}^N)} \leq \lim_{\varepsilon\to 0}\left[2c_{k,\varepsilon}(\varepsilon g(\varepsilon))^2\right]^{1/(p-1)} K = 0.$$

And since for any $k \in \mathbb{N} : \|\hat{u}_k\|_{L^{p+1}(\mathbb{R}^N)} = \|\hat{w}_{k,\varepsilon}\|_{L^{p+1}(\mathbb{R}^N)} = 1$, for any $\varepsilon \in (0, \varepsilon_{\sigma,2})$ we have that

$$\liminf_{\varepsilon\to 0}\frac{\|v_{k,\varepsilon}\|_{L^\infty}}{\left[2c_{k,\varepsilon}(\varepsilon g(\varepsilon))^2\right]^{1/(p-1)}} > 0.$$

We will apply a comparison argument in order to prove Theorem 4.1.4, following the same process as in [11]. Hence, the following remark will be of use.

Remark 4.4.3. Let $a, b, \delta > 0$, $\Omega \subseteq \mathbb{R}^N$ bounded. Let U be a positive solution of the problem

$$\begin{cases}\Delta U(x) - 2bU(x) = 0, & x \in \mathbb{R}^N \setminus \Omega^\delta,\\ U(x) = 0, & x \in \partial\Omega^\delta,\\ U(x) = 0, & \text{as } |x| \to \infty.\end{cases} \tag{4.102}$$

Then, U verifies

$$U(x) \leq C\cdot\exp\{-b\cdot\operatorname{dist}(x,\Omega^\delta)\}, x \in \mathbb{R}^N \setminus \Omega^\delta,$$

with

$$C = C(a, \Omega^\delta) > 0.$$

Recall the statement of Theorem 4.1.4. Let $k \in \mathbb{N}$ and $\delta > 0$, there exists $\varepsilon_\delta > 0$ and $C = C(N, k, p, \delta)$ such that

$$\forall x \in \mathbb{R}^N, \forall \varepsilon \in (0, \varepsilon_\delta): \quad |\hat{w}_{k,\varepsilon}(x)| < \frac{C}{(2c_k)^{1/p-1}} \cdot \exp\left(\gamma_{\delta,\varepsilon} \cdot \operatorname{dist}(x, \Omega^\delta)\right), \tag{4.103}$$

where

$$\Omega^\delta = \{x \in \mathbb{R}^N : \quad \operatorname{dist}(x, \Omega) < \delta\},$$

and

$$\gamma_{\delta,\varepsilon} = \gamma_{\delta,\varepsilon}(N, k, p) \to -\infty, \quad \text{as} \quad \varepsilon \to 0.$$

Proof of Theorem 4.1.4. Let $\sigma > 0$, $\varepsilon_{\sigma,2}$ and K as in Lemma 4.4.3. By (4.57) and Lemma 4.99 we can pick $\varepsilon_{\sigma,3} \in (0, \varepsilon_{\sigma,2})$ such that

$$\forall \varepsilon \in (0, \varepsilon_{\sigma,3}): \quad V_{\delta,\varepsilon} > K. \tag{4.104}$$

Similarly as in (4.56), for any $p > 0$, we have that

$$V_{p,\varepsilon} = \inf\{V_\varepsilon(x) : \quad |x| > pg(\varepsilon)\}.$$

So, for $p = \delta$ and $\varepsilon \in (0, \varepsilon_{\sigma,3})$ we have that

$$\begin{aligned} V_\varepsilon(x) &= \frac{1}{[\varepsilon g(\varepsilon)]^2} V\left(\frac{x}{g(\varepsilon)}\right) \\ &\geq \inf\{V_\varepsilon(x) : \quad |x| > \delta g(\varepsilon)\} \\ &= V_{\delta,\varepsilon}. \end{aligned}$$

From the above, (4.104) and by Lemma 4.4.3 it follows that

$$\begin{aligned} F_{k,\varepsilon} &\equiv V_\varepsilon(x) - |w_{k,\varepsilon}|^{p-1} \\ &\geq V_{\delta,\varepsilon} - K > 0. \end{aligned} \tag{4.105}$$

Let U be a positive solution of problem (4.102). From the previous, we obtain

$$a = K, \quad b = \frac{V_{\delta,\varepsilon} - K}{2} \equiv -\gamma_{\delta,\varepsilon}.$$

Then, we have that

$$\begin{cases} \Delta U(x) - [(V_{\delta,\varepsilon}) - K]U(x) = 0, & x \in \mathbb{R}^N \setminus \Omega^\delta, \\ U(x) = K, & x \in \partial\Omega^\delta, \\ U(x) = 0, & \text{as } |x| \to \infty. \end{cases} \tag{4.106}$$

Therefore, by (4.105) we have

$$\begin{cases} \Delta U(x) - F_{k,\varepsilon}(x)U \leq 0, & x \in \mathbb{R}^N \setminus \Omega^\delta, \\ U(x) = K, & x \in \partial\Omega^\delta, \\ U(x) \to 0, & \text{as } |x| \to \infty. \end{cases} \tag{4.107}$$

Hence, since $w_{k,\varepsilon}$ solves (P'_ε), from (4.106) and (4.107) it holds that

$$\begin{cases} \Delta[U(x) - w_{k,\varepsilon}(x)] - F_{k,\varepsilon}(x)[U(x) - w_{k,\varepsilon}(x)] \leq 0, & x \in \mathbb{R}^N \setminus \Omega^\delta, \\ U(x) - w_{k,\varepsilon}(x) > 0, & x \in \partial\Omega^\delta, \\ U(x) - w_{k,\varepsilon}(x) \to 0, & \text{as } |x| \to \infty. \end{cases} \tag{4.108}$$

Now as a consequence of Theorem 2.3.4 and by (4.108)

$$\forall x \in \mathbb{R}^N \setminus \Omega^\delta : \quad w_{k,\varepsilon}(x) \leq U(x).$$

Analogously, we can obtain

$$\forall x \in \mathbb{R}^N \setminus \Omega^\delta : \quad -U(x) \leq w_{k,\varepsilon}(x).$$

Finally, by Remark 4.4.3, for every $x \in \mathbb{R}^N \setminus \Omega^\delta$ it follows that

$$|w_{k,\varepsilon}(x)| \leq U(x) \leq \frac{C}{2c_{k,\varepsilon}} \cdot \exp\left(\gamma_{\delta,\varepsilon} \cdot \operatorname{dist}(x, \Omega^\delta)\right), \tag{4.109}$$

where $C = C(K,\delta) = C(\sigma, N, k, p, \delta) > 0$. Whence, we can obtain (4.103) by fixing a small value for σ and enlarging C to make it independent of σ and so that (4.109) becomes valid for $x \in \Omega^\delta$. Indeed, we get

$$\forall \varepsilon \in (o, \varepsilon_{\sigma,3}), \forall x \in \mathbb{R}^N : \quad |\hat{w}_{k,\varepsilon}(x)| \leq \frac{C}{(2c_k)^{1/(p-1)}} \exp\left(\frac{K - V_{\delta,\varepsilon}}{2} \cdot \operatorname{dist}(x, \Omega^\delta)\right).$$

Note that, clearly,

$$\frac{K - V_{\delta,\varepsilon}}{2} \equiv \gamma_{\delta,\varepsilon} \to -\infty \quad \text{as} \quad \varepsilon \to 0.$$

We conclude by the arbitrariness of ε. □

Chapter 5

Conclusions and recommendations

5.1 Conclusions

In this project we proved the existence, multiplicity, concentration phenomena and decay of solutions for the nonlinear Schrödinger equation

$$\begin{cases} \varepsilon^2 \Delta v(x) - V(x)\, v(x) + |v(x)|^{p-1} v(x) = 0, & x \in \mathbb{R}^N, \\ v(x) \to 0, & \text{as } |x| \to \infty, \end{cases} \tag{P_ε}$$

where $\varepsilon > 0$.

We studied the *Infinite Case* as presented by Byeon & Wang, under the restrictions:

(V1) $V \in \mathrm{C}(\mathbb{R}^N)$ is non-negative;
(V2) $V(x) \to \infty$, as $|x| \to \infty$;
(V3) $\overline{\mathcal{Z}} = \{0\}$;
($\mathrm{V_{inf}}$) $\forall |x| \leq 1 : V(x) = \exp\left(-\dfrac{1}{a(x)}\right)$.

where $b \in C(\mathbb{R}^N)$ was an Ω-*quasi homogeneous function* and $a \in C(\mathbb{R}^N \setminus \{0\})$ was an *asymptotically* (Ω, b)-*quasi homogeneous* function.

The corresponding limit problem of (P_ε) as $\varepsilon \to 0$ was:

$$\begin{cases} \Delta w(x) + |w(x)|^{p-1}\, w(x) = 0, & x \in \Omega, \\ w(x) = 0, & x \in \partial\Omega, \end{cases} \tag{P}$$

where $\Omega \subseteq \mathbb{R}^N$ was considered as a strictly star-shaped domain.

We can summarize the results present in this project as follows:

i) Using a Ljusternik-Schnirelman scheme and the properties of Krasnoselskii's genus we were able to prove the existence of an infinite number of solutions for (P_ε) and (P). Furthermore, we proved that for each topological level k there is at least a pair of solutions for each problem.

ii) We proved

- The critical levels of (P_ε) converge to those of (P) as $\varepsilon \to 0$.
- The solutions of (P_ε) converge, up to scaling and subsequences, to a solution of (P), for each topological level k.
- The solutions of (P_ε) decay exponentially away from Ω.

iii) We needed concepts and results studied in several courses such as Functional Analysis, Measure Theory, Operator Theory, Partial Differential Equations and Calculus of Variations for the development of this project. Moreover, results from Nonlinear Analysis were needed, such as Krasnoselkii's Genus, the Palais Smale condition and the Ljusternik-Schnirelman scheme, which were not covered in courses offered in the Mathematics career at Yachay Tech.

iv) The results we obtained are congruent to those of the works [6], [11] and [22]. This work concludes the studies of the cases presented by Byeon and Wang that involve a Laplacian operator of order one.

5.2 Recommendations

1. This work can be further expanded via numerical experimentation. Namely, one could set a fixed value for σ and ε and approximate the solution via a shooting method.

3. We conjecture that, by changing the Laplacian in (P_ε) to a p-Laplacian with $p > 1$ it can be shown that similar results hold for the three cases.

2. The need for more efficient and reliable administrations in Yachay Tech cannot be posponed any longer. The university has suffered enough incompetent administrations. The immense potential of this University and its students is being greatly wasted and yet, despite this, students and professors have managed to go forward and achieve global scientific products.

Bibliography

[1] R. Adams and J. Fournier *Sobolev Spaces*, Academic Press, London, 2003.

[2] A. Ambrosetti, M. Badiale, and S. Cingolani, *Semiclassical States of Nonlinear Shcrödinger Equations*, Archive for Rational Mechanics and Analysis, 140 (1997), pp. 285-300.

[3] A. Ambrosetti and A. Malchiodi, *Nonlinear Analysis and Semilinear Elliptic Problems*, Cambridge Studies in Advanced Mathematics 104, Cambridge University Press, Cambridge, 2007.

[4] H. Brezis, *Functional Analysis, Sobolev Spaces and Partial Differential Equations*, Universitext, Springer, 2011.
https://doi.org/10.1007/978-0-387-70914-7

[5] J. Byeon, *Existence of large positive solutions of some nonlinear elliptic equations on singularly perturbed domains*, Communications in Partial Differential Equations, 22 (1997), pp.1731-1769.

[6] J. Byeon and Z. Wang, *Standing Waves with a Critical Frequency for Nonlinear Schödinger Equations*, Archive for Rational Mechanics and Analysis, 165(2002), pp. 295-316.

[7] R. Coleman, *Calculus on Normed Vector Spaces*, Universitext, Springer, 2012.

[8] M. Del Pino and P. L. Felmer *Local Mountain Passes for Semilinear Elliptic Problems in Unbounded Domains*, Calculus of Variations and Partial Differential Equations, 4 (1996), pp. 121-137.

[9] B. Dracogona, *Introductionn to the Calculus of Variations*, Imperial College Press, 2004.

[10] L. Evans, *Partial Differential Equations*, vol 19 of Graduate Studies in Mathematics, American Mathematical Society, Providence, RI, 1998.

[11] P. Felmer and J. Mayorga-Zambrano, Multiplicity and concentration for the nonlinear Schrödinger equation with critical frequency, *Nonlinear Anal.* **66**(2007), No.1, 151-169.
https://doi.org/10.1016/j.na.2005.11.017

[12] A. Floer and A. Weinstein, *Nonspreading wave packets for the cubic Schrödinger equation with a bounded potential*, J. Functional Analysis, No. 3, 397-408.

[13] C. Gui, *Existence of multi-bump solutions for nonlinear Schrödinger equations via variational method*, Comm. Partial Differential Equations **21**(1996), No.5–6, 787–820

[14] S. J. Gustafson and I. M. Sigal, *Mathematical Concepts of Quantum Mechanics*, Springer Science & Business Media, 2011.

[15] D. Gilbarg and N. S. Trudinger, *Elliptic Partial Differential Equations of Second Order*, Springer, 2015.

[16] D.J Griffiths, *Introduction to Quantum Mechanics*, 2nd edition, Pearson, Chapter 2. The Time-Independent Schrödinger Equation, (2005).

[17] S. Ha, *A Nonlinear Shooting Method for Two-Point Boundary Value Problems*, Computers and Mathematics with Applications, **42**(2001), 1411–1420.

[18] E. Kreyszig, *Introductory Functional Analysis with Applications*, vol. 1, Wiley & Sons, New York, 1978

[19] J. Mayorga-Zambrano, *Introductory Calculus of Variations* (Working version, second draft), Yachay Tech University, Ecuador, 2020.

[20] J. Mayorga-Zambrano, *A course of Functional Analysis* (Working version), AMARUN, 2021.

[21] J. Mayrorga-Zambrano and R. Carrasco-Betancourt, *Concentration of solutions for a one-dimensional nonlinear Schrödinger equation with critical frequency*, Preprint, (2018).

[22] J. Mayorga-Zambrano, L. Medina-Espinosa & C. Muñoz-Moncayo *Concentration of infinitely many solutions for the finite case of a nonlinear Schrödinger equation with critical-frequency potential*, Mathematical Analysis and Applications, 2021, (submitted).

[23] P. Meystre, *Atom Optics*, Springer, New York, 2001, p.176-180.

[24] D. L Mills, *Nonlinear Optics: Basic Concepts*, Springer Science & Business Media, 2012.

[25] Y. Oh, *On positive multi-lump bound states of nonlinear Schrödinger equations under multiple well potential*, Comm. Math. Phys. **131**(1990), No.2, 223–253.

[26] J. Polkinghorne, *Quantum Theory: A Very Short Introduction*, Oxford University Press ,Oxford, 2002.

[27] P. Rabinowitz, *Minimax methods in critical point theory with applications to differential equations*, CBMS Regional Conference Series in Mathematics 65, 1986.

[28] P. Rabinowitz, On a class of nonlinear Schrödinger equations, Z. Angew. Math. Phys. **43**(1992), No. 2, 270–291.

[29] J. Rogel-Salazar, *The Gross-Pitaevskii Equation and Bose-Einstein condensates*, European Journal of Physics, 2013, p.247.

[30] W. Smith and D. Stregenga, *Hölder Domains and Pincaré Domains*, Transactions of the American mathematical Society 319(1990), No. 1, 67-100.

[31] A. Szulkin and T. Weth, *The method of Nehari manifold*, Handbook of Nonconvex Analysis and Applications, 2010.

[32] X. Wang, *On concentration of positive bound states of nonlinear Schrödinger equations*, Communications in Mathematics and Physics, **153**(1993), No. 2, 229–244.

[33] K. Yosida, *Functional Analysis*, Springer Science & Business Media, 2012.