UNIVERSIDAD DE INVESTIGACIÓN DE TECNOLOGÍA EXPERIMENTAL YACHAY

Escuela de Ciencias Matemáticas y Computacionales

TÍTULO: C_0-Semigroups in the Controllability of Semilinear Systems of Differential Equations with Fractional Perturbations, Impulses and Delay: Reaction-Diffusion and Wave Equations

Trabajo de integración curricular presentado como requisito para la obtención del título de Matemático

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Urcuquí, diciembre de 2021

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To my beloved sister Valentina,
for you to enjoy the beautiful
world of mathematics
and the infinity of dreams
where they appear

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Abstract

Control theory of differential equations has been extensively studied during the last decades. This important field in mathematics has applications which range from dynamical systems in engineering to economic models. In this work, the mathematical theory needed for developing controllability results of two specific systems of partial differential equations is exhibited. Approximate controllability of a semilinear reaction-diffusion equation and exact controllability of a semilinear perturbed wave equation are obtained. The novelty of developing these results relies on the inclusion of fractional perturbation terms, instantaneous impulses and delay. A functional analysis approach dealing with C_0-semigroups is used for deriving the abstract formulations of the problems. This is the key stone for developing these results when considering the solution to belong to an infinite-dimensional Banach function space. Fixed-point theorems, such as Rothe-Isac and Banach Contraction Mapping, are used for fulfilling the existence of the desired control variables. This project is an extension of results based on the works done by H. Leiva, O. Camacho and N. Merentes since 2003.

Keywords: Controllability · reaction-diffusion system · perturbed wave equation · PDE · fractional perturbation · C_0-semigroups · perturbation principle

Resumen

La teoría de control de ecuaciones diferenciales se ha estudiado ampliamente durante las últimas décadas. Este importante campo de las matemáticas tiene aplicaciones que van desde sistemas dinámicos en ingeniería hasta modelos económicos. En este reporte de trabajo de titulación, se presenta la teoría matemática necesaria para desarrollar resultados de controlabilidad de dos sistemas específicos de ecuaciones en derivadas parciales. Se obtienen la controlabilidad aproximada de una ecuación de reacción-difusión semilineal y la controlabilidad exacta de una ecuación de onda perturbada semilineal. La novedad en desarrollar estos resultados radica en la inclusión de términos de perturbación fraccionaria, impulsos instantáneos y retardo. Se utiliza un enfoque de análisis funcional que trabaja con C_0-semigrupos para construir las formulaciones abstractas de los problemas. Esta es la pieza clave para desarrollar estos resultados cuando se considera una solución perteneciente a un espacio funcional de Banach de dimensión infinita. Los teoremas de punto fijo, tales como los de Rothe-Isac y de la Aplicación Contractiva de Banach, se utilizan para satisfacer la existencia de las variables de control. Este proyecto es una extensión de resultados basados en los trabajos de H. Leiva, O. Camacho y N. Merentes desde 2003.

Palabras clave: Controlabilidad · sistema de reacción-difusión · ecuación de onda perturbada · EDP · perturbación fraccionaria · C_0-semigrupos · principio de perturbación

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Chapter 1

Introduction

1.1 Historical background and motivation

In mathematical analysis and engineering, control theory is at the core of the uttermost scientific research areas. Thanks to its applications in industry, economics, electronics, epidemiology and many other disciplines, it has caught the attention of applied mathematicians during the last decades. In fact, most models arising from phenomena in these areas involve dynamical systems with a great deal of mathematical objects. From this point of view, specific properties, such as existence of solutions, controllability, invertibility, observability, among others, are of high interest to researchers.

Obtaining controllability of systems of differential equations from a mathematical approach can be understood as proving the existence of a control variable for which the associated system has a solution such that it is steered from a fixed initial state to a desired final state.

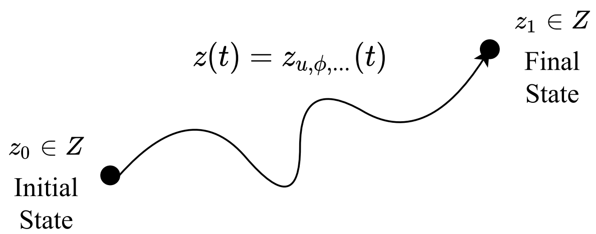

Fig. 1.1: Diagram of the solution $z(t)$ of a control system dependent on a control variable u and other conditions $\phi, \ldots$, which steers the system from an initial state z_0 to a final state z_1 in the state space Z.

In this work, we are interested in studying controllability results for two systems of partial differential equations (PDEs): a reaction-diffusion and a perturbed-wave equations. The main relevance in the methods used is the inclusion of several perturbation conditions over the systems: instantaneous impulses, delay, non-local conditions, fractional and semilinear perturbations. These conditions are specific to each phenomena and can have different interpretations. Nevertheless, to our interest, only the main mathematical framework needed to arrive at the controllability results is presented in detail.

The controllability of PDEs has been extensively developed during the last three decades. Most of the works in the area are conceived under a variational approach of the problem setting. Still, other methods can be used to arrive at the same results, and in some cases, even go further in this study. Therefore, in this work, a combination of results from (strongly continuous) C_0-semigroups and Operator Theory will help us to set the problems from a dynamical system focus, and arrive at the results by using adequate fixed-points theorems.

The theory of C_0-semigroups was mainly developed by Einar Hille, Ralph Saul Phillips and Kōsaku Yosida during the twentieth century [29]. One of the fundamental results in this theory is the generation theorem named after them (see Theorem 2.25). The most important notion of this theory is the fact that these operators represent a generalization of the exponential matrix dependent on a single time parameter, which is necessary to work in function spaces of infinite dimension such as $L^2(\Omega)$.

Regarding the variational approach, the introduction of these novel methods in control theory of PDEs was principally done by the renowned French mathematician Jacques-Louis Lions [45]. An important summary of the results obtained in the linear cases of the heat and wave equations is given by Enrique Zuazua [54]. Further, [17] and [47] studied the controllability of the heat equation in the semilinear case. In fact, this approach involved many rigorous tools typically used in the study of PDEs. Later, [4] and [53] gave important advances regarding the fractional perturbation to the reaction-diffusion equation.

In this project, the development of the first problem is mainly based on the recent results obtained in [8], [38] and [39]. In contrast to the aforementioned works, the authors use an Operator Theory approach for the approximate controllability of these systems. The main difference used in this particular case is the inclusion of the fractional perturbation in the Laplace operator. Thus, the problem setting requires the operator semigroup to be redefined in a different manner.

Our motivation is to prove that under all these simultaneous perturbations of different nature, the controllability of the systems is preserved. There are many works which have studied problems like this in other different kind of systems. For example, the well-posedness of the inclusion of impulsive terms in autonomous systems was studied in [22], [44], [25], and [14], and in non-autonomous systems by [48], [52], [42]. In addition, problems with non-local conditions that also include impulses can be found in [41] and [35]. In our project, we consider all the perturbations in a simultaneous way. Some recent works in this fashion are [36], [37] and [30]. Further, the one-dimensional heat equation involving a new kind of non-instantaneous impulses has been studied by [43].

The second problem is motivated by unbounded perturbation ideas presented in [31], [32], and exemplified in [33]. These works give an important result (see Theorem 2.28), which states the preservation of the exact controllability of a system of second-order differential equations with the referred unbounded perturbation. Other authors have also studied C_0-semigroups with perturbations, such as [5], [21] and [1].

Finally, it is worth to mention that under the Operator Theory approach, there exists many different kind of fixed-point theorems used to prove the existence of the desired solution, which depend on the context of the problem and the properties being studied. In some works like [35], [7] and [30], Karakostas' fixed-point theorem, an extension of the theorem presented by M. A. Krasnosel'skiĭ and developed in [26], is used for the well-posedness of the systems. Moreover, many articles on the controllability of dynamical systems use Rothe's fixed-point theorem for studying the approximate and exact controllability of this family of systems with delays, instantaneous impulses, and memory considerations [34], [20], [19], [46]. Lastly, recent works by [3] have introduced interesting ideas to obtain approximate controllability with methods avoiding fixed-point theorems.

1.2 Problem setting

Throughout this report, I_p shall denote the set of the first p natural numbers, $p \in \mathbb{N}$, and $I_0 = I_1$, by convention. Also, let $\Omega \subset \mathbb{R}^N$ be a bounded domain. Consider the following notation, for some $r, \tau \in \mathbb{R}$ with $0 < r < \tau$,

$$\Omega_\tau = (0, \tau] \times \Omega, \quad \Omega_{-r} = [-r, 0] \times \Omega, \quad (\partial\Omega)_\tau = [0, \tau] \times \partial\Omega,$$

where $\partial\Omega$ denotes the boundary of Ω.

1.2.1 A semilinear reaction-diffusion system

Let $p \in \mathbb{N}$ be fixed. For an open set $\omega \subset \Omega$, consider the following system:

$$\begin{cases} \dfrac{\partial}{\partial t} z(t,x) = \Delta_x z(t,x) + (-\Delta_x)^\eta z(t,x) & \\ \qquad\qquad + \mathbb{1}_\omega u(t,x) + f(t, z(t-r,x), u(t,x)), & (t,x) \in \Omega_\tau,\ t \neq t_k, \\ z(t,x) = \phi(t,x), & (t,x) \in \Omega_{-r}, \\ z(t,x) = 0, & (t,x) \in (\partial\Omega)_\tau, \\ z(t_k^+, x) = z(t_k^-, x) + \mathcal{I}_k\left(t_k, z\left(t_k, x\right), u(t_k, x)\right), & x \in \Omega,\ k \in I_p, \end{cases} \tag{1.1}$$

such that $0 < t_k < t_{k+1} < \tau$, $k \in I_{p-1}$ and $\phi \in \mathcal{PC}_{pr}$, a normed space of piece-wise continuous functions, as defined in detail later in Section 2.1.1. The mapping $\mathbb{1}_\omega$ is the characteristic function on ω, meaning that the control $u \in L^2([0,\tau], U)$, with $U = L^2(\Omega)$, is acting in the interior of Ω (see Figure 1.2). The non-linear function $f : [0,\tau] \times \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{R}$ and the jump functions $\mathcal{I}_k : [0,\tau] \times \mathbb{R} \longrightarrow \mathbb{R}$, $k \in I_p$, are smooth enough.

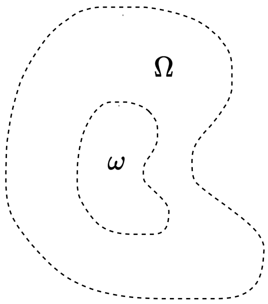

Fig. 1.2: Open set ω inside domain Ω.

Suppose that the following hypotheses for system (1.1) hold:

(I) $0 < \eta < 1$,

(II) These conditions are satisfied:

$$|f(t,z,u)| < a_0 \|z\|^{\alpha_0} + b_0 \|u\|^{\beta_0} + c_0, \tag{1.2}$$
$$|\mathcal{I}_k(t,z,u)| < a_k \|z\|^{\alpha_k} + b_k \|u\|^{\beta_k} + c_k, \quad k \in I_p, \tag{1.3}$$

where $a_k, b_k, c_k \geq 0$, and $\dfrac{1}{2} \leq \alpha_k, \beta_k < 1$, $k \in I_p \cup \{0\}$. The norms of the spaces for which z and u belong will be precised later.

Corresponding to the system (1.1), there exist the following two associated (linear) systems. For some z_0 in the respective state space,

$$\begin{cases} \dfrac{\partial}{\partial t} z(t,x) = \Delta_x z(t,x) + \mathbb{1}_\omega u(t,x), & (t,x) \in \Omega_\tau, \\ z = 0, & \text{on } (\partial\Omega)_\tau, \\ z(0,\cdot) = z_0, & \text{on } \Omega, \end{cases} \tag{1.4}$$

$$\begin{cases} \dfrac{\partial}{\partial t} z(t,x) = \Delta_x z(t,x) + (-\Delta_x)^\eta z(t,x) + \mathbb{1}_\omega u(t,x), & (t,x) \in \Omega_\tau, \\ z = 0, & \text{on } (\partial\Omega)_\tau, \\ z(0,\cdot) = z_0, & \text{on } \Omega. \end{cases} \tag{1.5}$$

It is well known that the problem (1.4) is approximate controllable on $[0,\tau]$, for every $\tau > 0$. This can be proven (at least) in two ways: using the aforementioned variational approach or by methods that require theory of C_0-semigroups. Both of these notions are explained in Chapter 3 of this report.

1.2.2 A semilinear perturbed wave equation system

Let $m, q \in \mathbb{N}$ be fixed. The following system of partial differential equations on Ω over an infinite-dimensional Banach space Z is considered:

$$\begin{cases} \dfrac{\partial^2}{\partial s^2} y(s,x) = \Delta_x y(s,x) + \epsilon(-\Delta_x)^{1/2} y(s,x) & \\ \qquad\qquad + \mathbb{1}_\omega u(s,x) + f\left(s, y(s-r,x), \dfrac{\partial}{\partial s} y(s-r,x)\right), & (s,x) \in \Omega_\tau, \\ y(s,x) + h_1\left(y(\theta_1 + s, x), \ldots, y(\theta_q + s, x)\right) = \rho_1(s,x), & (s,x) \in \Omega_{-r} \\ \dfrac{\partial}{\partial s} y(s,x) + h_2\left(\dfrac{\partial}{\partial s} y(\theta_1 + s, x), \ldots, \dfrac{\partial}{\partial s} y(\theta_q + s, x)\right) = \rho_2(s,x), & (s,x) \in \Omega_{-r}, \\ y(s,x) = 0, & (s,x) \in (\partial\Omega)_\tau, \\ \dfrac{\partial}{\partial s} y(s_k^+, x) = \dfrac{\partial}{\partial s} y(s_k^-, x) + J_k\left(s_k, y(s_k,x), \dfrac{\partial}{\partial s} y(s_k,x)\right), & x \in \Omega,\ k \in I_m, \end{cases} \tag{1.6}$$

where $0 < s_k < s_{k+1} < \tau$, $k \in I_{m-1}$, $0 < \theta_j < \theta_{j+1} < r < \tau$, $j \in I_{q-1}$ and $\epsilon > 0$ is small enough. The distributed control u belongs to $L^2([0,\tau], U)$, with $U = L^2(\Omega)$, and the initial functions or historical passes ρ_1, ρ_2 belong to the Banach space $\mathcal{PW}_{mr}$, antoher version of a piece-wise continuous function space, as defined in Section 2.1.1. In this case, the variational approach for the exact controllability of the linear case is presented in Chapter 3. Nonetheless, there exists an important condition that shall be imposed over $\tau > 0$. In contrast to the previous system, exact controllability can be attained in the semilinear case.

Structure of the report

The rest of the content in this document is organized in the following manner:

- **Chapter 2:** The preliminary definitions and mathematical framework in which this work has been developed is given. Sections 2.1 and 2.2 state well-known topics of Functional Analysis and Operator Theory and include a subsection devoted to state the specific function spaces required for the solutions of our systems. Important tools regarding dense-range operators and the existence of a spectral decomposition of the Laplace operator are included in sections 2.3 and 2.4. Finally, relevant advanced theory in C_0-semigroups, perturbation theory and the main definitions and notions of Control Theory are given in sections 2.5–2.7.

- **Chapter 3:** The abstract formulations of systems (1.1) and (1.6) are developed so that the notation for the next chapters is set. Then, the controllability of the linear heat equation is discussed in detail, and the notions of the variational approach for the controllability of the linear wave equation are presented. With all this theory, the full context is set to work on the semilinear cases.

- **Chapter 4:** This chapter includes the main results based on the Operator Theory approach and the abstract formulations from the previous chapter. It includes a section devoted to each PDE problem. The main tools used are the generalization of Rothe's and Banach's fixed-point theorems. A brief justification of the well-posedness of these problems is mentioned in the opening section.

- **Chapter 5:** In this last part, the main conclusions about the work done are given, including some guidelines for further open problems.

Chapter 2

Preliminary Definitions and Framework

This chapter is devoted to present some background in Functional Analysis and related theory required for the development of the results of this work. In mathematics, setting the problem properly is crucial, and accordingly, setting the framework must be correctly done.

The contents of this chapter concern basic definitions in Functional Analysis of normed and inner product spaces, some properties and classical results, introductory concepts in Control Theory and Perturbation Theory, and preliminary results in the analysis of PDEs.

Finally, another underlying goal in this chapter is to present enough details and references so that this work can be used itself as a short reference on the topics covered. This will allow a better understanding of the studied problems and techniques. Nevertheless, it is worth to mention that this work is not –in any way– aimed to give a complete review of the subsequent theory.

2.1 Some topics of Functional Analysis

Functional Analysis is a broad area of mathematical analysis dealing with functions as mathematical objects, the spaces were they belong, and the different properties that these spaces and objects exhibit according to their definitions in terms of topology, binary operations and dimensionality. The main references for this section are [6] and [10]. Consider $\mathbb{K} = \mathbb{R}$ in most of the results.

Let us start by recalling the definition of a topological vector space. The couple $(Z, \mathcal{T})$ is called a *topological space* if Z is a set and $\mathcal{T}$, a family of elements in the set of parts of Z, $\mathcal{P}(Z)$, is a topology, i.e., if the following three conditions are satisfied:

(i) The empty set $\emptyset$ and Z are in $\mathcal{T}$.

(ii) Every finite intersection of elements in $\mathcal{T}$ belongs to $\mathcal{T}$.

(iii) Any arbitrary union of elements in $\mathcal{T}$ belongs to $\mathcal{T}$.

The elements of $\mathcal{T}$ are called *open sets*. Moreover, $(Z, \mathcal{T})$ is called a *topological vector space* if Z is also a vector space and the sum and scalar-multiplication operations are continuous (with respect to $\mathcal{T}$).

In this work (and in many references), it is common to refer to the topological space $(Z, \mathcal{T})$ only as Z when the context generates no confusion. The same applies for the posterior spaces definitions.

In any topological space, there exist several separability properties that can be defined. Among them, the following is one of the most common in theoretical and applied mathematics. The topological vector space $(Z, \mathcal{T})$ is a *separated space* or *Hausdorff space* whenever for $x, y \in Z$, distinct, there exist $U, V \in \mathcal{T}$, disjoint and such that

$$x \in U \quad \text{and} \quad y \in V.$$

In this case, the topology $\mathcal{T}$ is also called a *T_2-topology*.

Furthermore, a set Z together with a binary operation $d : Z \times Z \longrightarrow [0, +\infty)$ is called a *metric space* (Z, d) if d is a metric, that is, if it satisfies the properties:

(i) **Symmetry.** $\forall x, y \in Z : \; d(x, y) = d(y, x)$.

(ii) **Separability.** $d(x, y) = 0 \iff x = y$.

(iii) **Triangular Inequality.** $\forall x, y, z \in Z : \; d(x, y) \leq d(x, z) + d(z, y)$.

So far, the next result can be easily verified.

Proposition 2.1. *Every metric space (Z, d) is a Hausdorff topological space. A classical topology is the topology of* open balls*, i.e, whose topological basis is comprised of elements of the form*

$$B_r(x_0) = \{x \in Z : \; d(x, x_0) < r\}$$

for $x_0 \in Z$ and $r > 0$.

Continuity and convergence notions can be defined on the frame of topological spaces only. Nonetheless, our focus need only convergence in normed spaces, as it will be presented later. However, an accurate generalization of such spaces in our context is given in the next concept.

A topological vector space is called a *locally convex space* if each one of its open sets to which the element 0 belongs comprises an open set $\mathcal{O}$ with the following properties:

(i) **Convex.** $\forall x, y \in \mathcal{O}, \; \forall t \in [0, 1] : \; tx + (1 - t)y \in \mathcal{O}$.

(ii) **Balanced.** $\forall x \in \mathcal{O}, \; \forall \alpha \in [-1, 1] : \; \alpha x \in \mathcal{O}$.

(iii) **Absorbing.** $\forall x \in \mathcal{O}, \; \exists \beta > 0 : \; \dfrac{1}{\beta} x \in \mathcal{O}$.

Finally, let Z be a vector space. A *semi-norm* is a mapping $p : Z \longrightarrow [0, +\infty)$ with the following properties

(i) **Absolute Homogeneity.** $\forall x \in Z, \; \forall \alpha \in \mathbb{K} : \; p(\alpha x) = |\alpha| \, p(x)$.

(ii) **Triangular Inequality.** $\forall x, y \in Z : \; p(x + y) \leq p(x) + p(y)$.

If moreover, the semi-norm satisfies:

(iii) **Separability.** $p(x) = 0 \iff x = 0$,

then, it is called a *norm*, usually denoted by $\|\cdot\|_Z$. The couples (Z, p) and $(Z, \|\cdot\|_Z)$ are called *semi-normed* and *normed* spaces, respectively.

As it can be noted, the main difference between semi-normed and normed spaces is the separability property of the operation(s). Intuitively, in order to somehow compensate this lack of separability in a semi-normed space, it is usually endowed with a set of semi-norms.

Proposition 2.2. *Any vector space can be endowed with a topology such that it becomes a topological locally convex separated vector space.*

A detailed proof of this result can be found in pp. 23–25 in [10, vol. 2].

Remark 2.1. In the light of this last proposition, any locally convex space, which also is a vector space, can be endowed with a topology of semi-norms. That is why in most references, locally convex separated spaces are endowed as such.

Now, it is time to pass onto the completeness of these spaces. We will denote[1] $(x_n)_{n\in\mathbb{N}}$ as a sequence of elements in a metric space Z, and will call it a *Cauchy sequence* whenever

$$\forall \varepsilon > 0,\ \exists N \in \mathbb{N}: \quad d(x_n, x_m) < \varepsilon, \qquad \text{for } n, m \geq N.$$

Then, a metric space Z is a *complete* space if every Cauchy sequence $(x_n)_{n\in\mathbb{N}} \subset Z$ is convergent in Z. A complete normed space is called a *Banach space*.

Since any normed space can be regarded as a metric space with the metric induced by the norm, it can also be considered as a topological space with the induced topology of open balls.

Moreover, we can consider a product of normed spaces and induce a topology on the basis of each of the norms.

Proposition 2.3 (Product space topology)**.** *Let $Z_1, Z_2, \dots, Z_n$ be normed spaces, each induced with its respective norm. Then, the norm*

$$\|x\|_Z = \|x_1\|_{Z_1} + \|x_2\|_{Z_2} + \dots + \|x_n\|_{Z_n} = \sum_{k=1}^{n} \|x_k\|_{Z_k}, \quad x = (x_1, x_2, \dots, x_n) \in Z \tag{2.1}$$

induces a normed space in the product space $Z = Z_1 \times Z_2 \times \dots \times Z_n$.

The proof of this fact is reduced to verifying the above mentioned conditions. Finally, let us introduce the concept of inner product endowed in a vector space. Let H be a vector space, an *inner product* is a bi-linear continuous form $\langle \cdot, \cdot \rangle_H : H \times H \longrightarrow \mathbb{R}$ satisfying the following properties:

(i) **Symmetry.** $\forall x, y \in H: \ \langle x, y\rangle_H = \langle y, x\rangle_H$.

(ii) **Positive semi-definiteness.** $\forall x \in H: \quad \langle x, x\rangle_H \geq 0$.

(iii) **Separability.** $\langle x, x\rangle = 0 \iff x = 0$.

Moreover, the couple $(H, \langle \cdot, \cdot \rangle_H)$ is an *inner product space*. A complete inner product space is called a *Hilbert space*.

Remark 2.2. Every inner product space H is a normed space with the norm given by

$$\|x\|_H = \langle x, x\rangle^{1/2}, \quad x \in H.$$

Now, let us present some important and well known properties for bounded linear operators in normed spaces. For a given operator T, denote $\mathcal{D}(T) = \mathrm{Dom}(T)$ as its given domain and $\mathrm{Im}(T)$ as its image or range. In most cases, the notation $\mathcal{D}(T) \subseteq Z$ refers to $\mathcal{D}(T)$ as a linear subspace of Z.

Theorem 2.1. *Let Z and U be normed spaces, and $T : \mathcal{D}(T) \subseteq Z \longrightarrow U$ a linear operator. Then, the following assertions are equivalent:*

(i) T is continuous on $\mathcal{D}(T)$.

(ii) T is bounded, i.e.,

$$\exists C > 0,\ \forall x \in \mathcal{D}(T): \quad \|T(x)\|_U \leq C\,\|x\|_Z.$$

(iii) T is continuous at $0 \in Z$.

(iv) T is uniformly continuous on $\mathcal{D}(T)$.

[1] To make a distinction when required, an arbitrary family (not necessarily countable) of elements will be denoted by $\{f_\alpha\}_{\alpha\in\mathcal{A}}$ for an index set $\mathcal{A}$.

The details for the proof of this result are based on the topological characterization of continuity in normed spaces. They can be checked in [10].

The set of linear and bounded operators of the form $T : Z \longrightarrow U$ is denoted by $\mathscr{L}(Z, U)$. Whenever there is no confusion, $\mathscr{L}(Z, Z)$ will be denoted only as $\mathscr{L}(Z)$. The space $\mathscr{L}(Z, U)$ can be equipped with the following norm

$$\|T\|_{\mathscr{L}(Z,U)} = \inf\left\{C > 0 \;/\; \forall x \in Z : \; \|T(x)\|_U \leq C \|x\|_Z\right\} = \sup_{\substack{x\in Z \\ \|x\|\leq 1}} \|T(x)\|_U \,,$$

which transforms it into a Banach space if and only if U is Banach, as well.

Following, the next concept is the basis of an extensively rich area of mathematics and even more important in the analysis of differential equations.

Definition 2.1 (Dual space and weak convergence). *Consider a normed space Z. The* dual *space Z' of Z is the set containing all the bounded linear functionals of the form $f : Z \longrightarrow \mathbb{R}$, i.e., $Z' = \mathscr{L}(Z, \mathbb{R})$. A sequence $(x_n)_{n\in\mathbb{N}} \subset Z$ is said to be* weakly convergent *to $x \in Z$ if and only if*

$$\forall f \in Z' : \quad \lim_{n\to+\infty} f(x_n) = f(x).$$

It is worth to mention that the notion of weak convergence leads to the definition of the *weak* topology over a normed space. Thus, this topology can be understood as the smallest topology (denoted as the initial topology $\sigma(Z, Z')$, see Section 1.4 in [10]) such that every linear functional in Z' is still continuous.

Moreover, this topology helps to analyse the associated operators in different ways. For example, strong convergence (in the norm sense) implies weak convergence, but the opposite is not necessarily true in general. Therefore, the solutions of differential equations can have different meanings according to the space where they live and the topology endowed in such space.

In some books, the following *dual product* notation is used. Considering a normed space Z, whenever $f \in Z', \; z \in Z$,

$$\langle f, z\rangle_{Z',Z} = f(z).$$

In this sense, the dual product is a generalization of an mathematical structure that puts in *duality* the two spaces Z' and Z. Whenever $Z = H$ is a Hilbert space, Theorem 2.9 gives an important result. In fact, this notation is used to make agreement with analogies regarding the definitions of the weak topology and the adjoint operator in a Hilbert space, for instance.

On the basis of Theorem 2.1, many other useful properties of bounded linear operators can be derived. These ideas are key when attempting to prove the controllability of multiple systems.

For Z a topological space and U a separated topological space, consider an operator $S : Z \longrightarrow U$, not necessarily linear. S is said to be a *compact* operator if for every bounded set $B \subset Z$, $S(B)$ is relatively compact in the topology of U.

In most classical theory (such as the presented by [6], [10] and [28]), this last definition requires the operator to be linear. Nevertheless, the fixed-point theorems that are used in this work extend this concept towards operators that are not necessarily linear.

Proposition 2.4 (Bounded and compact operators). *Let Z be a normed space. Consider $S \in \mathscr{L}(Z)$, and T a compact linear operator. Then, $T \circ S$ and $S \circ T$ are compact.*

Proof. Consider a bounded set $B \subset Z$, then, because S is continuous, $S(B)$ is also bounded. Thus, due to the compactness of T, we have that $T \circ S(B) = T(S(B))$ is relatively compact in Z. Therefore, by definition $T \circ S$ is compact (and linear).

The compactness of a linear operator can be characterized in the following way. Consider a bounded sequence $(z_n)_{n\in\mathbb{N}} \subset Z$, then, T is compact if and only if there exists $(Tz_{n_k})_{k\in\mathbb{N}} \subset (Tz_n)_{n\in\mathbb{N}}$ a convergent subsequence.

Therefore, consider an arbitrary bounded sequence $(z_n)_{n\in\mathbb{N}}$ of Z. Since T is compact, there exists $(Tz_{n_k})_{k\in\mathbb{N}}$ convergent subsequence, and in turn $(S\circ Tz_{n_k})_{k\in\mathbb{N}}$ is also a convergent subsequence of $(S\circ Tz_n)_{n\in\mathbb{N}}$ since S is continuous. This yields that $S\circ T$ is compact, as well. □

For more properties of compact linear operators, the reader can see Chapter 8 in [28].

Moreover, let $(Z,\mathcal{T}_Z)$ and $(U,\mathcal{T}_U)$ be two topological spaces. A mapping of the form $T:Z\longrightarrow U$ is said to be an *open mapping* if

$$\forall\ \mathcal{O}\in\mathcal{T}_Z:\quad T(\mathcal{O})\in\mathcal{T}_U,$$

that is, T maps every open set in Z to an open set in U.

Similarly, whenever Z and U are Banach spaces, an application $T:Z\longrightarrow U$ is said to be *closed* if the set

$$\operatorname{Gr}(T)=\{(x,y)\in Z\times U\ :\ T(x)=y\},$$

which is called the *graph* of T, is closed in $Z\times U$.

Finally, let us end this section with the concept of the Lebesgue measure function spaces. Given a Banach space X and a field $\mathbb{K}$ ($\mathbb{R}$ or $\mathbb{C}$), consider a subset $\Omega\subseteq X$, and the following function vector space

$$\widetilde{L^p}(\Omega,\mathbb{K})=\left\{f:\Omega\subseteq X\longrightarrow\mathbb{K}\ /\ f\text{ is measurable, and }\int_\Omega|f(x)|^p\,\mu(dx)<+\infty\right\}$$

where μ is the measure on X, $1\le p<+\infty$, and it is endowed with the norm

$$\|f\|_{L^p(\Omega,\mathbb{K})}=\left(\int_\Omega|f(x)|^p\,\mu(dx)\right)^{1/p},\quad f\in\widetilde{L^p}(\Omega,\mathbb{K}).$$

For the case $p=+\infty$, we define

$$\widetilde{L^\infty}(\Omega,\mathbb{K})=\left\{f:\Omega\subseteq X\longrightarrow\mathbb{K}\ /\ f\text{ is measurable, and }\sup_{x\in\Omega}|f(x)|<+\infty,\ \text{a.e.}\right\}$$

endowed with the essential supremum norm

$$\|f\|_\infty=\operatorname*{ess\,sup}_{x\in\Omega}|f(x)|,\quad f\in\widetilde{L^\infty}(\Omega,\mathbb{K}).$$

That is to say

$$\|f\|_\infty=\sup_{x\in\Omega}|f(x)|,\ \text{almost everywhere (a.e.)},\quad f\in\widetilde{L^\infty}(\Omega,\mathbb{K}).$$

Therefore, it is intuitive by definition that each element of a Lebesgue space will have the same properties of another when they take the same values up to a set of null measure. In other words, the Lebesgue spaces are often seen as equivalence classes of functions with the equivalence relation defined as

$$\forall f,g\in\widetilde{L^p}(\Omega,\mathbb{K}):\quad f\sim g\quad\Longleftrightarrow\quad f(x)=g(x),\quad\text{a.e. on }\Omega.$$

Indeed, for $1\le p\le+\infty$,

$$L^p(\Omega,\mathbb{K})=\left\{[f]_\sim\ :\ f\in\widetilde{L^p}(\Omega,\mathbb{K})\right\}.$$

Whenever $\mathbb{K}=\mathbb{R}$, the notation reduces to $L^p(\Omega)=L^p(\Omega,\mathbb{R})$.

In a more general manner, the Banach space X shall be replaced by a measure space (X,Σ,μ) preserving the well-posedness of these definitions. Nevertheless, for the purpose of this project, we will require only $X=\mathbb{R}^N$ induced by the Lebesgue measure.

For some $1<p<+\infty$, its *conjugate* is defined as the number $1<p'<+\infty$ such that

$$\frac{1}{p}+\frac{1}{p'}=1.$$

With this in mind, the following useful results are set.

Lemma 2.1 (Young's Inequality)**.** *Given some a, b, non-negative real numbers, and $1 < p < +\infty$, the next inequality holds:*

$$ab \leq \frac{1}{p}a^p + \frac{1}{p'}b^{p'}.$$

Proof. In fact, recall that the natural logarithm ln is a concave function. For a and b, it follows that:

$$\begin{aligned} \ln\left(\frac{1}{p}a^p + \left(1 - \frac{1}{p}\right)b^{p'}\right) &\geq \frac{1}{p}\ln\left(a^p\right) + \left(1 - \frac{1}{p}\right)\ln\left(b^{p'}\right) \\ &= \frac{1}{p}p\ln a + \frac{1}{p'}p'\ln b \\ &= \ln(ab). \end{aligned}$$

Thus, we get the desired inequality after taking the exponential on both sides. □

Theorem 2.2 (Hölder's Inequality)**.** *Let $\Omega \subset \mathbb{R}^N$, bounded, and consider the Lebesgue measure μ. Let also $f \in L^p(\Omega)$ and $g \in L^{p'}(\Omega)$. Then, $fg \in L^1(\Omega)$ and*

$$\|fg\|_{L^1(\Omega)} \leq \|f\|_{L^p(\Omega)} \|g\|_{L^{p'}(\Omega)},$$

or in other words,

$$\int_\Omega |f(x)g(x)| \ \mu(dx) \leq \left(\int_\Omega |f(x)|^p \ \mu(dx)\right)^{1/p} \left(\int_\Omega |g(x)|^{p'} \ \mu(dx)\right)^{1/p'}.$$

For a proof, see Theorem 4.6 in [6]. Furthermore, a generalized result follows from induction using this theorem.

Theorem 2.3 (Generalized Hölder's Inequality)**.** *Let $\Omega \subset \mathbb{R}^N$, bounded, and consider the Lebesgue measure μ. Let also $f_1, f_2, \ldots, f_k : \Omega \longrightarrow \mathbb{R}$, with $f_i \in L^{p_i}(\Omega)$, $i \in I_k$ and such that*

$$\sum_{i=1}^{k} \frac{1}{p_i} = \frac{1}{p_1} + \frac{1}{p_2} + \cdots + \frac{1}{p_k} = 1.$$

Then, $f = f_1 f_2 \cdots f_k \in L^1(\Omega)$ and

$$\|f\|_{L^1(\Omega)} \leq \|f_1\|_{L^{p_1}(\Omega)} \|f_2\|_{L^{p_2}(\Omega)} \cdots \|f_k\|_{L^{p_k}(\Omega)}.$$

In a similar way, the next holds.

Proposition 2.5. *Let $\Omega \subset \mathbb{R}^N$, bounded and with finite Lebesgue measure $\mu(\Omega)$. Consider $1 \leq q \leq p \leq +\infty$. Then, $L^p(\Omega) \subset L^q(\Omega)$ and the following relation holds*

$$\|f\|_{L^q(\Omega)} \leq \mu(\Omega)^{\frac{p-q}{pq}} \|f\|_{L^p(\Omega)}, \quad \textit{for all } f \in L^p(\Omega).$$

Proof. This result is a consequence of Hölder's Inequality. Indeed, consider the case $1 < q < p < +\infty$ and an arbitrary $f \in L^p(\Omega)$. We have that

$$\frac{q}{p} + \frac{p-q}{p} = 1,$$

so

$$\begin{aligned} \int_\Omega |f(x)|^q \ \mu(dx) &\leq \left(\int_\Omega |f(x)|^{q\frac{p}{q}} \ \mu(dx)\right)^{q/p} \left(\int_\Omega \mathbb{1}_\Omega(x)^{\frac{p}{p-q}} \ \mu(dx)\right)^{(p-q)/p} \\ &= \left[\left(\int_\Omega |f(x)|^p \ \mu(dx)\right)^{1/p}\right]^q \cdot \mu(\Omega)^{\frac{p-q}{p}} < +\infty. \end{aligned}$$

Then, $f \in L^q(\Omega)$ and the desired relation follows from

$$\|f\|^q_{L^q(\Omega)} \leq \mu(\Omega)^{\frac{p-q}{p}} \|f\|^q_{L^p(\Omega)} .$$

□

Furthermore, let us present the definition of a Sobolev space, essential for the study of PDEs.

Definition 2.2 (Sobolev spaces)**.** *Let $\Omega \subseteq \mathbb{R}^N$. For $1 \leq p < +\infty$, the general* Sobolev space *is defined by*

$$W^{k,p}(\Omega) = \{f \in L^p(\Omega) \;:\; D^\gamma f \in L^p(\Omega), \quad |\gamma| \leq k\},$$

considering $D^\gamma f$ as the derivative in the distribution sense for each multiindex γ with order $|\gamma| \leq k$ (see Section 5.2 in [16]), and induced by the norm

$$\|f\|_{W^{k,p}(\Omega)} = \left(\sum_{|\gamma|\leq k} \|D^\gamma f\|^p_{L^p(\Omega)} \right)^{1/p} .$$

In particular, the following notation is used $H^k(\Omega) = W^{k,2}(\Omega)$.

Also, the space of smooth and compact support functions is denoted as

$$C_0^\infty(\Omega) = \left\{ \varphi : \Omega \subseteq \mathbb{R}^N \longrightarrow \mathbb{R} \;/\; \varphi \text{ is infinitely (strongly) differentiable and } \operatorname{supp}(\varphi) \text{ is compact} \right\}$$

where the support $\operatorname{supp}(\varphi)$ of φ is the set

$$\operatorname{supp}(\varphi) = \overline{\{x \in \Omega \;:\; \varphi(x) \neq 0\}} \subseteq \Omega.$$

The elements of $C_0^\infty(\Omega)$ are usually called *test functions*. Finally, the following space is defined

$$H_0^1(\Omega) = \overline{C_0^\infty(\Omega)}^{\|\cdot\|_{H^1(\Omega)}},$$

that is, the closure of $C_0^\infty(\Omega)$ with respect to the norm of $H^1(\Omega)$. Note that this norm can be expressed as

$$\|f\|_{H^1(\Omega)} = \|f\|_{W^{1,2}(\Omega)} = \left(\|f\|^2_{L^2(\Omega)} + \sum_{j=1}^N \left\| \frac{\partial f}{\partial x_j} \right\|^2_{L^2(\Omega)} \right)^{1/2} = \left(\|f\|^2_{L^2(\Omega)} + \|\nabla f\|^2_{L^2(\Omega)} \right)^{1/2}.$$

Remark 2.3. Sometimes the norm topology of the Sobolev spaces $H^k(\Omega)$, $k \in \mathbb{N}$, can be characterized with an equivalent norm in terms of the Fourier transform of the function. This idea is highly relevant for defining fractional Sobolev spaces. Nevertheless, the main tools in this project do not require to consider those spaces, even though the equations deal with a perturbed fractional Laplacian.

Now, it seems interesting to study the relations among these function spaces. For instance, if we consider $\Omega = \mathbb{R}^N$, it can be checked that $H_0^1(\Omega)$ coincides with $H^1(\Omega)$. However, in our context, Ω is set to be a bounded domain in $\mathbb{R}^N$. In this case, the result which holds is

$$\overline{C_0^\infty\left(\overline{\Omega}\right)} = H^1(\Omega).$$

Some other properties deal with continuous and compact embeddings. The next theorem provides an interesting and useful result which relates functions in Sobolev spaces and their continuity (See Theorem 1 in Section 5.7 from [16]).

Theorem 2.4 (Rellich-Kondrachov). *Consider the open and bounded subset Ω of $\mathbb{R}^N$. Assume that $\partial\Omega$ is of class C^1, and $1 \le p < N$, Then, the identity operator*

$$\mathbf{I} : W^{1,p}(\Omega) \longrightarrow L^q(\Omega)$$

is compact for $1 \le q < p'$. This shall also be denoted as

$$W^{1,p}(\Omega) \subset\subset L^q(\Omega) \quad \text{or} \quad W^{1,p}(\Omega) \overset{c}{\hookrightarrow} L^q(\Omega).$$

Remark 2.4. Similarly, whenever Ω is only bounded (not necessarily of class C^1) the next result holds on the test functions space:

$$H_0^1(\Omega) \overset{c}{\hookrightarrow} L^2(\Omega).$$

Indeed, this idea is similar to the result of Arzelà-Ascoli Theorem, but extended to multiple dimensions.

2.1.1 Special function spaces

In order to set properly the problems associated to (1.1) and (1.6), we will require some specific function spaces of piece-wise continuous functions. These definitions take already into account the nature of the delay and impulses that perturb the equations. All the spaces defined here are Banach spaces when they are equipped with the respective uniform norm. The proof of this fact follows in a similar fashion to the proof of the completeness of the continuous functions spaces under the sup norm.

For each of the two considered equations in this report, we shall define a space of functions whose codomain (or say, the state space of the problem) has an specific topology relying on the way the abstract formulation is set. However, the purpose of this section is to address an specific notation for these spaces so as to give the details of the abstract problem in the latter chapters.

Firstly, consider Z to be a Banach space. For some function $z : \mathbb{R} \longrightarrow Z$ and $t_0 \in \mathbb{R}$, we denote

$$z(t_0^+) = \lim_{t \to t_0^+} z(t) = \lim_{t \,\downarrow\, t_0} z(t), \qquad \text{and} \qquad z(t_0^-) = \lim_{t \to t_0^-} z(t) = \lim_{t \,\uparrow\, t_0} z(t).$$

Then, having fixed $p \in \mathbb{N}$, the number of impulses appearing in (1.1), and relying on the value $r > 0$, which defines the delay of the equations, consider the function space:

$$\mathcal{PC}_{pr} = \Big\{ \phi : [-r, 0] \longrightarrow Z \ / \ \exists\ \theta_k, k \in I_p : \ \phi\big|_{[-r,0]\setminus\{\theta_k\}_{k=1}^p} \in C\left([-r,0]\setminus\{\theta_k\}_{k=1}^p, Z\right), \\ \phi(\theta_k^+), \phi(\theta_k^-) \text{ exist and } \phi(\theta_k) = \phi(\theta_k^+),\ \forall k \in I_p \Big\}$$

equipped with the norm

$$\|\phi\|_{\mathcal{PC}_{pr}} := \sup_{t \in [-r,0]} \|\phi(t)\|_Z, \quad \phi \in \mathcal{PC}_{pr},$$

that is, the space of right-continuous functions but up to a finite number p of points on the interval $[-r, 0]$. Now, considering the whole interval $[-r, \tau]$, define the following space

$$\mathcal{PC}_p^\tau = \mathcal{PC}_{t_1,\dots,t_p}([-r,\tau], Z) = \Big\{ z : [-r,\tau] \longrightarrow Z \ / \ z\big|_{[-r,0]} \in \mathcal{PC}_{pr},\ z\big|_{[0,\tau]'} \in C([0,\tau]', Z), \\ z(t_k^+), z(t_k^-) \text{ exist and } z(t_k) = z(t_k^+),\ \forall k \in I_p \Big\}$$

where $[0,\tau]' = [0,\tau]\setminus\{t_1, t_2, \dots, t_p\}$. In other words, the elements of $\mathcal{PC}_p^\tau$ are the piece-wise continuous functions on $[-r, \tau]$, right-continuous on the points of impulse, whose state space is Z, and when restricted to $[-r, 0]$ satisfy a similar piece-wise condition. As it was mentioned, these spaces are complete with the topology induced by the uniform norm. In this case,

$$\|z\|_{\mathcal{PC}_p^\tau} := \sup_{t \in [-r,\tau]} \|z(t)\|_Z, \quad z \in \mathcal{PC}_p^\tau.$$

If there is no confusion, $\|z\|_{\mathcal{PC}_p^\tau}$ will be denoted only as $\|z\|_\infty$.

Note that we must take into account the whole interval $[-r,\tau]$ since otherwise, the space would not be complete. The solutions of the abstract equation associated to our semilinear reaction-diffusion equation will belong to the space $\mathcal{PC}_p^\tau$, with $Z = L^2(\Omega)$.

Now, let us present the spaces that allow us to deal with the solutions of the second problem, the perturbed wave equation (1.6). For this purpose, by the nature of the abstract formulation, the state space will have a product form. In this sense, the solutions will belong to spaces similar to those stated before, but with the next characteristics.

For $0 < s < 1$, the state space denoted by $\mathcal{Z}^s$ is the product of the two spaces

$$\mathcal{Z}^s = Z^s \times Z := \mathcal{D}(A^s) \times L^2(\Omega),$$

with $A = -\Delta : \mathcal{D}(A) \subseteq Z \longrightarrow Z$, and whose domain is defined as

$$\mathcal{D}(A) = H^2(\Omega) \cap H_0^1(\Omega).$$

In this sense, Z^s is defined as

$$Z^s = \mathcal{D}(A^s) = \left\{ z \in L^2(\Omega) \ : \ \sum_{n=1}^{+\infty} \lambda_n^{2s} \|S_n z\|^2_{L^2(\Omega)} < +\infty \right\},$$

with $\{S_n\}_{n=1}^{+\infty}$, the set of projection operators of the Laplacian spectral decomposition (as it will be presented in Section 2.4). In this Hilbert space, the inner product can be given as

$$\langle z, y \rangle_{Z^s} =: \sum_{n=1}^{+\infty} \lambda_n^{2s} \langle S_n z, S_n y \rangle_{L^2(\Omega)}, \quad z, y \in Z^s,$$

and thus, the space $\mathcal{Z}^s$ is also a Hilbert space when endowed by the inner product

$$\langle z, y \rangle_{\mathcal{Z}^s} = \langle z_1, y_1 \rangle_{Z^s} + \langle z_2, y_2 \rangle_{L^2(\Omega)}, \quad z, y \in \mathcal{Z}^s.$$

Then, having fixed $m \in \mathbb{N}$, the number of impulses in the second equation (1.6), consider the following Banach spaces:

$$\begin{aligned}\mathcal{PW}_{mr} = \Big\{ \rho : [-r,0] \longrightarrow \mathcal{Z}^{1/2} \ / \ \exists\, \theta_k, k \in I_m : \ \rho\big|_{[-r,0]\setminus\{\theta_k\}_{k=1}^m} \in C\left([-r,0]\setminus\{\theta_k\}_{k=1}^m, \mathcal{Z}^{1/2}\right), \\ \rho(\theta_k^+), \rho(\theta_k^-) \text{ exist, and } \rho(\theta_k) = \rho(\theta_k^+), \ \forall k \in I_m \Big\},\end{aligned}$$

$$\begin{aligned}\mathcal{PW}_m^\tau &= \mathcal{PW}_{s_1,\dots,s_m}\left([-r,\tau], \mathcal{Z}^{1/2}\right) \\ &= \Big\{ y : [-r,\tau] \longrightarrow \mathcal{Z}^{1/2} \ / \ y\big|_{[-r,0]} \in \mathcal{PW}_{mr}, \text{ and } y\big|_{[0,\tau]'} \in C([0,\tau]', \mathcal{Z}^{1/2}), \\ &\qquad y(s_k^+), y(s_k^-) \text{ and } y(s_k) = y(s_k^+), \ \forall k \in I_m \Big\},\end{aligned}$$

where $[0,\tau]' = [0,\tau]\setminus\{s_1, s_2, \dots, s_m\}$, and endowed with the respective norms

$$\begin{aligned}\|\rho\|_{\mathcal{PW}_{mr}} &= \sup_{s\in[-r,0]} \|\rho(s)\|_{\mathcal{Z}^{1/2}}, \quad \rho \in \mathcal{PW}_{mr}, \\ \|y\|_{\mathcal{PW}_m^\tau} &= \sup_{s\in[-r,\tau]} \|y(s)\|_{\mathcal{Z}^{1/2}}, \quad y \in \mathcal{PW}_m^\tau.\end{aligned}$$

As before, $\|y\|_\infty$ will be used for denoting $\|y\|_{\mathcal{PW}_m^\tau}$ when there is no confusion. Analogously, we define the Banach space:

$$\mathcal{PW}_{mr}^q = \Big\{ \varsigma : [-r,0] \longrightarrow \left(\mathcal{Z}^{1/2}\right)^q \ / \\ \exists\, \theta_k, k \in I_m : \ \varsigma\big|_{[-r,0]\setminus\{\theta_1,\dots,\theta_m\}} \in C\left([-r,0]\setminus\{\theta_1,\dots,\theta_m\}, \left(\mathcal{Z}^{1/2}\right)^q\right), \\ \varsigma(\theta_k^+), \varsigma(\theta_k^-) \text{ exist and } \varsigma(\theta_k) = \varsigma(\theta_k^+),\ \forall k \in I_m \Big\},$$

equipped with the norm

$$\|\varsigma\|_{\infty,q} = \sup_{s\in[-r,0]} \|\varsigma(s)\|_{\mathcal{Z}^q} = \sum_{i=1}^q \sup_{s\in[-r,0]} \|\varsigma_i(s)\|_{\mathcal{Z}}, \quad \varsigma \in \mathcal{PW}_{mr}^q.$$

2.1.2 Some important theorems

Hereby, a small collection of important theorems in Functional Analysis and Operator Theory is presented. These results, which represent classical and some well-known properties in mathematical analysis, have been developed in different contexts and generalizations. These will be crucial in the coming chapters.

In particular, one of the most important theorems in analysis is the one developed by Italian mathematicians Cesare Arzelà and Giulio Ascoli, which yields a compactness result. The classical version states the following.

Theorem 2.5 (Arzelà–Ascoli)**.** *Let $\{f_\alpha\}_{\alpha\in\mathcal{A}}$ a uniformly bounded and equicontinuous infinite family of functions defined on a bounded set B of $\mathbb{R}^N$ onto $\mathbb{R}^d$. Then, there exists a sequence $(f_n)_{n\in\mathbb{N}} \subset \{f_\alpha\}_{\alpha\in\mathcal{A}}$ uniformly convergent in B.*

A proof of this result can found in pp. 62–64 from [13]. However, since we intend to work on infinite-dimensional spaces, a generalized version of the previous theorem can help to give more detail to this compactness notion (see p. 3 from [29] and pp. 167–169 in [49]).

Theorem 2.6 (Generalized Arzelà–Ascoli)**.** *Let E be a Banach space and $\{f_\alpha\}_{\alpha\in\mathcal{A}}$ be an equicontinuous family of functions from $[a,b] \subset \mathbb{R}$ into E. Let $(f_n)_{n\in\mathbb{N}}$ be a sequence belonging to this family such that for each $t \in [a,b]$ the set $\{f_n(t)\}_{n\in\mathbb{N}}$ is relatively compact in E. Then, there exists a subsequence $(f_{n_k})_{k\in\mathbb{N}}$ of $(f_n)_{n\in\mathbb{N}}$ which is uniformly convergent on $[a,b]$.*

Furthermore, other important results in functional analysis which are of our interest are presented along the following lines.

Theorem 2.7 (Open Mapping)**.** *Consider two Banach spaces E and F, and $T \in \mathscr{L}(E,F)$, a bounded linear and surjective operator. Then, there exists some $c > 0$ such that*

$$B_F(0,c) \subseteq T\left(B_E(0,1)\right),$$

and T is an open mapping.

Theorem 2.8 (Uniform Boundedness Principle)**.** *Let E and F be two Banach spaces. Consider a family $\{T_i\}_{i\in\mathcal{I}}$ (need not be numerable) of bounded linear operators from E to F and such that*

$$\forall x \in E: \quad \sup_{i\in\mathcal{I}} \|T_i(x)\|_F < +\infty.$$

Then, there exists a constant c such that

$$\forall x \in E,\ \forall i \in \mathcal{I}: \quad \|T_i(x)\|_F \le c\,\|x\|_E,$$

that is

$$\sup_{i\in\mathcal{I}} \|T_i\|_{\mathscr{L}(E,F)} < +\infty.$$

Theorem 2.9 (Riesz-Fréchet representation). *Let $(H, \langle\cdot,\cdot\rangle_H)$ be a Hilbert space. Then, for each $f \in H'$, the dual of H, there exists a unique element $u \in H$ such that:*

$$\forall h \in H : \quad \langle f, h\rangle_{H',H} = \langle u, h\rangle_H .$$

Moreover, the following relation holds:

$$\|f\|_{H'} = \|f\|_{\mathscr{L}(H)} = \|u\|_H .$$

For more detail of the derivation of this result, the reader can see Section 5.2 from [6].

Theorem 2.10. *Let E be a reflexive Banach space, $\mathcal{O} \subseteq E$ a closed and convex subset, and a convex and continuous functional $\varphi : \mathcal{O} \subseteq E \longrightarrow \mathbb{R}$ satisfying*

$$\|x\|_E \longrightarrow +\infty \quad \Longrightarrow \quad |\varphi(x)| \longrightarrow +\infty. \tag{2.2}$$

Then, there exists $\hat{x} \in \mathcal{O}$ such that

$$\varphi(\hat{x}) = \min_{x\in\mathcal{O}} \varphi(x).$$

Property (2.2) is referred to as the coercivity *of φ.*

For further detail, check Section 3.5 presented by [6]. Now, let us present the main tools for obtaining our results in the Operator Theory approach.

Definition 2.3. *Consider a vector space X. Let T be an operator defined from X onto itself. Then, it is said that T has a* fixed point *if there exists $x^* \in X$ such that*

$$T(x^*) = x^*.$$

Theorems concerning fixed points have been used in many different fields, mostly to provide existence results for certain operator equations. To our aim, some fixed-point theorems shall be used for attaining the existence of the control variable in our system.

The first theorem presented is a classical result in mathematical analysis, one of the first fixed-point theorems formulated, and perhaps the most-known. It is also called *Banach's Contraction Principle* or *Contraction Mapping Theorem*.

Theorem 2.11 (Banach's fixed-point). *Let $T : X \longrightarrow X$ be a contraction over a complete non-empty metric space X, i.e., there exists a number $0 < k < 1$ such that*

$$\forall x, y \in X : \quad d(Tx, Ty) \le k d(x, y).$$

Then, T has a unique fixed point.

Proof. The next proof is taken from pp. 2–3 of [51]. Consider some point $x \in X$. Then, for any $n \in \mathbb{N}$, we have the following

$$d(T^{n+1}x, T^n x) \le k d(T^n x, T^{n-1}x),$$

since T is a contraction. Inductively,

$$d(T^{n+1}x, T^n x) \le k^n d(Tx, x).$$

Then, the next statement holds, for arbitrary $n, m \in \mathbb{N}$, $n < m$,

$$\begin{aligned} d(T^m x, T^n x) &\le d(T^m x, T^{m-1}x) + d(T^{m-1}x, T^{m-2}x) + \cdots + d(T^{n+1}x, T^n x) \\ &\le \left(k^{m-1} + k^{m-2} + \cdots + k^n\right) d(Tx, x) \\ &\le k^n \frac{1}{1-k} d(Tx, x). \end{aligned}$$

Because $0 < k < 1$, for any $\varepsilon > 0$, there exists $N \in \mathbb{N}$ big enough such that:

$$\forall n, m \in \mathbb{N} : \quad N < n < m \implies d(T^m x, T^n x) < \varepsilon.$$

Therefore, $(T^n x)_{n\in\mathbb{N}}$ is a Cauchy sequence in X, and thus, there exists $z \in X$ such that

$$\lim_{n\to+\infty} T^n x = z.$$

Finally, we see that since T is continuous,

$$T(z) = T\left(\lim_{n\to+\infty} T^n x\right) = \lim_{n\to+\infty} TT^n x = \lim_{n\to+\infty} T^{n+1} x = z,$$

and z is a fixed point of T. Suppose there exists $y \in X$, a different fixed point of T. We would have that

$$d(y, z) = d(Ty, Tz) \leq kd(y, z).$$

Then,

$$(1 - k)d(y, z) \leq 0,$$

which implies that $d(y, z) = 0$, and this gives a contradiction. Thus, we conclude such fixed point is unique. $\square$

In 1930, Julius Schauder made the proof of the following fixed-point theorem:

Theorem 2.12 (Schauder's fixed-point)**.** *Let M be a non-empty convex subset of a complete normed space X. Let T be a continuous mapping of M into a compact set $K \subset M$. Then T has a fixed point.*

Successively, some generalizations of this theorem have appeared [51]. Among them, the classical Rothe's Theorem was proved in 1937. The original version is given as follows:

Theorem 2.13 (Rothe's fixed-point)**.** *Let X be a normed space, M the closed unit ball and ∂M the unit sphere in X. Let T be a continuous compact mapping of M into X such that $T(\partial M) \subset M$. Then, T has a fixed point.*

Furthermore, in this work, a more useful version will be used according to our formulation of the problem. This version is a trivially-valid modification of Rothe's Theorem as presented by [24] in a generalized topological vector space.

Theorem 2.14 (Rothe-Isac)**.** *Let X be a Hausdorff topological vector space. Consider $B \subset X$, a closed and convex subset such that the zero of X is contained in the interior of B. Let $\Phi : B \longrightarrow X$ be a continuous compact mapping with $\Phi(\partial B) \subset B$. Then, Φ has a fixed-point in B.*

2.2 Spectral Theory

The spectral theory of continuous linear operators have been a topic of high interest with numerous applications in diverse areas. Most of these results help to deal with different kinds of PDEs. The results of this section give the main foundation for presenting the eigenvalue problem, and thus, the existence of a strongly continuous semigroup for the specific operators in our problems. Our specific attention will be given to compact and self-adjoint operators in Banach spaces. The main references for this theory are Chapters 7 to 9 from [28].

Consider a non-trivial Banach space Z and a linear operator $T : \mathcal{D}(T) \subset Z \longrightarrow Z$. For $\lambda \in \mathbb{C}$, define

$$T_\lambda = T - \lambda \mathbf{I}, \tag{2.3}$$

where $\mathbf{I} : Z \longrightarrow Z$ is the identity operator in Z.

Whenever T_λ is injective, denote $R_\lambda(T)$ as the inverse of T_λ with $\mathcal{D}(R_\lambda(T)) = \mathrm{Im}(T_\lambda)$. Moreover, let us define the next conditions:

(R1) $R_\lambda(T)$ exists (only in the injectivity sense).

(R2) $R_\lambda(T)$ is bounded.

(R3) $R_\lambda(T)$ can be defined on a dense subset of Z.

Then, consider the following concept.

Definition 2.4 (Resolvent set and spectrum)**.** *The* resolvent set $\rho(T)$ *of the operator* T *is the subset of* $\mathbb{C}$ *for which conditions (R1)–(R3) are satisfied.*

The spectrum $\sigma(T)$ *of* T *is the complement of the resolvent set. By construction, it is partioned as follows:*

- ***Point spectrum*** $\sigma_p(T)$***:*** *All values of* $\mathbb{C}$ *for which* $R_\lambda(T)$ *does not exists, i.e., (R1) is not satisfied. Its elements are called* eigenvalues*.*
- ***Continuous spectrum*** $\sigma_c(T)$***:*** *All values of* $\mathbb{C}$ *for which* $R_\lambda(T)$ *exists and is densely defined on* Z*, but is not bounded, i.e., (R1) and (R3) are satisfied but (R2) is not.*
- ***Residual spectrum*** $\sigma_r(T)$***:*** *All values of* $\mathbb{C}$ *for which* $R_\lambda(T)$ *exists but is not densely defined on* Z*, i.e., (R1) satisfied but (R3) is not.*

Whenever Z is a finite-dimensional space, the spectrum only consists of eigenvalues. Therefore, the formula (2.3) resembles the generalization of the eigenvalue of a matrix associated to the finite-dimensional linear operator T. In the infinite-dimensional case, the spectrum can have other special properties as we shall note throughout this theory. This is a key consideration to bear in mind when comparing results in finite and infinite dimension.

An immediate consequence is that if T is a linear bounded operator on the Banach space Z, then, its spectrum is compact. Moreover, the complex norm of the spectral values are bounded by $\|T\|_{\mathscr{L}(Z)}$. Indeed, the spectral radius is given by the formula

$$r_\sigma(T) = \lim_{n\to+\infty} \sqrt[n]{\|T^n\|_{\mathscr{L}(Z)}}.$$

Theorem 2.15. *Consider a Banach space* Z*. Let* $T \in \mathscr{L}(Z)$*. Then,* $\sigma(T)$ *is compact and*

$$\forall \lambda \in \sigma(T): \quad |\lambda| \le \|T\|_{Z'}.$$

See Chapter 6 from [6]. Forthwith, recall the definition of a compact operator. In this section, the results given are valid for the linear case of a compact operator T. An important theorem is presented as follows.

Theorem 2.16 (Compact operator spectral properties)**.** *Let* Z *be a normed space, and* $T : Z \longrightarrow Z$*, a linear and compact operator. Then, the following are satisfied:*

(i) The set $\sigma_p(T)$ *is countable, and* $\lambda = 0$ *is the only possible accumulation point.*

(ii) If $\lambda \neq 0$*, the null space associated to* T_λ *is of finite dimension. Moreover Im*(T_λ) *is a closed set.*

(iii) For all $\lambda \in \sigma(T)\backslash\{0\}: \quad \lambda \in \sigma_p(T)$*.*

Detailed proofs can be found in Theorems 8.3-1, 8.3-3 and 8.4-4 from [28] or checked in [6] and [16].

Remark 2.5. If $\dim(Z) = +\infty$, we have that $0 \in \rho(T)$. And moreover, if Z is not necessarily complete, consequence (iii) from the previous theorem is still valid (c.f. Theorem 8.6-4 from [28]).

Furthermore, some of the operators that will appear later will have an specific property when considering its adjoint.

Definition 2.5. *Consider the Banach spaces U and Z, and let $T : U \longrightarrow Z$ be a linear bounded operator. Then, the* adjoint *operator of T, $T^* : U' \longrightarrow Z'$ is defined such that:*

$$\forall u \in U', \ \forall x \in U : \quad \langle u, Tx \rangle_{U',Z} = \langle T^* u, x \rangle_{Z',U} \, . \tag{2.4}$$

If $Z = U = H$ is a Hilbert space endowed with the inner product $\langle \cdot, \cdot \rangle_H$, by Riesz-Fréchet representation Theorem 2.9, the adjoint operator can be deemed as a *Hilbert*-adjoint operator $T^* : H \longrightarrow H$ in the sense that (2.4) gets transformed onto

$$\forall u, v \in H : \quad \langle u, Tv \rangle_H = \langle Tu, v \rangle_H \, .$$

In particular, the spaces based on $L^2(\Omega)$, such as $H^1(\Omega)$, can help to define Hilbert-adjoint operators. Throughout this work, any Hilbert-adjoint operator shall be mentioned only as adjoint, depending on the context.

The spectral theory of adjoint operators is highly relevant when considering *self-adjoint* operators, i.e., whenever $T^* = T$. In finite dimension, an operator associated with a matrix is self-adjoint when the matrix is a Hermitian (or symmetric in the real case), that is, if it coincides with its conjugate transpose.

In this last case, there are some well-known properties of the eigenvalues of this matrix that can be generalized to infinite-dimensional spaces (See Section 6.4 in [6]).

Theorem 2.17 (Bound for spectrum of self-adjoint operator)**.** *Consider a Hilbert space H, and $T \in \mathscr{L}(H)$, a self-adjoint operator. Define*

$$m = \inf_{\|u\|=1} \{ \langle Tu, u \rangle_H \} \qquad \textit{and} \qquad M = \sup_{\|u\|=1} \{ \langle Tu, u \rangle_H \} \, .$$

Then,

$$\sigma(T) \subset [m, M], \qquad \textit{with} \qquad m, M \in \sigma(T),$$

and

$$\|T\|_{H'} = \max\{|m|\,, |M|\}.$$

We have seen that the spectrum of a bounded self-adjoint operator is thus real. In particular, this last result gives a stronger restriction than Theorem 2.15 for bounding the spectrum of T when it is self-adjoint.

The following theorem constitutes an crucial result for the formulation of the abstract problem to solve the controllability of the proposed PDEs (See Chapter 6 from [6] or [16]).

Theorem 2.18 (Eigenvectors basis)**.** *Consider a separable Hilbert space H, and $T \in \mathscr{L}(H)$ a compact and self-adjoint operator. Then, eigenvectors corresponding to distinct eigenvalues of T are orthogonal. Moreover, a Hilbert basis of H can be constructed from eigenvectors of T.*

Finally, let us note that the eigenvalues of a positive operator are thus positive as well. Its definition is as follows.

Definition 2.6 (Positive operator)**.** *Let H be a Hilbert space. Consider $T \in \mathscr{L}(H)$, a self-adjoint operator. T is said to be* positive*, denoted as $T \geq 0$, if and only if*

$$\forall h \in H : \quad \langle T(h), h \rangle_H \geq 0.$$

It is clear to obtain the next result.

Proposition 2.6 (Positive operator eigenvalues)**.** *Let H be a Hilbert space. Consider $T \in \mathscr{L}(H)$, a self-adjoint positive operator. Then, all eigenvalues of T are non-negative real numbers.*

Proof. By Theorem 2.17, the eigenvalues of T are real numbers. Then, take an arbitrary $\lambda \in \sigma_p(T)$. For any $\omega_\lambda \in H$, eigenvector associated to λ, we have that

$$0 \leq \langle T\omega_\lambda, \omega_\lambda \rangle_H = \lambda \langle \omega_\lambda, \omega_\lambda \rangle_H = \lambda \|\omega_\lambda\|_H^2 .$$

Since $\omega_\lambda \neq 0$ because it is an eigenvector, $\|\omega_\lambda\| > 0$, so $\lambda \geq 0$. By the arbitrariness of λ, we conclude the result.

□

Definition 2.7 (Projection operator)**.** *Let H be a Hilbert space. A* projection *is a bounded linear operator $P : H \longrightarrow H$ which is self-adjoint and idempotent.*

2.3 Dense-range operators

The topics presented in this section will be essential for the proofs built in the next chapters. In fact, these characterizations of surjective and dense-range operators are intuitively analogous to the definitions of exact and approximate controllability that will be given in the next section. The main references for these results are [12] and [40].

In this section, consider two Hilbert spaces U and Z, each one endowed with their respective inner products $\langle \cdot, \cdot \rangle_U$ and $\langle \cdot, \cdot \rangle_Z$, and $\mathcal{G} \in \mathscr{L}(U, Z)$. U and Z need not be of finite dimension. As defined in Section 2.1, the range of $\mathcal{G}$, $\mathrm{Im}(\mathcal{G})$, is said to be dense in its codomain if

$$\overline{\mathrm{Im}(\mathcal{G})} = Z.$$

Moreover, recall that if $\mathrm{Im}(\mathcal{G}) = Z$, then $\mathcal{G}$ is said to be surjective. In Lemmas 2.2 and 2.3, some properties of surjective and dense-range operators are used to characterize them, respectively.

Lemma 2.2 (Characterization of surjective operators)**.** *Let U and Z be Hilbert spaces, and $\mathcal{G} \in \mathscr{L}(U, Z)$. Then, the following statements are equivalent:*

(i) $Im(\mathcal{G}) = Z$.

(ii) *There exists $\gamma > 0$ such that:* $\quad \|\mathcal{G}^* z\|_U \geq \gamma \|z\|_Z , \quad \forall z \in Z.$

(iii) *The inverse of $\mathcal{G}\mathcal{G}^*$ exists.*

(iv) $\sup\limits_{\varepsilon > 0} \left\| (\varepsilon \mathbf{I}_Z + \mathcal{G}\mathcal{G}^*)^{-1} \right\|_{\mathscr{L}(Z)} < +\infty.$

(v) $\sup\limits_{\varepsilon > 0} \left\{ \varepsilon \left\| (\varepsilon \mathbf{I}_Z + \mathcal{G}\mathcal{G}^*)^{-1} \right\|_{\mathscr{L}(Z)} \right\} < 1.$

Remark 2.6. Consider $z \in Z$. Note that if the operator $\mathcal{G}$ is surjective, the equation

$$\mathcal{G}u = z, \quad u \in U$$

has a unique solution (with minimum norm) $u_z = \mathcal{G}^* \left(\mathcal{G}\mathcal{G}^* \right)^{-1} z$, via the adjoint operator. Further, the operator

$$\begin{aligned} \Gamma : Z &\longrightarrow \quad U \\ z &\longmapsto \Gamma z = \mathcal{G}^*(\mathcal{G}\mathcal{G}^*)^{-1} z \end{aligned}$$

is a right inverse of $\mathcal{G}$.

Lemma 2.3 (Characterization of dense-range operators). *Let U and Z be Hilbert spaces, and $\mathcal{G} \in \mathscr{L}(U, Z)$. Then, the following statements are equivalent:*

(i) $\overline{Im(\mathcal{G})} = Z$.

(ii) $\overline{Im\left(\mathcal{G}\mathcal{G}^*\right)} = Z$.

(iii) $\mathcal{G}^*$ *is injective.*

(iv) For all $z \in Z$, with $z \neq 0$, it is satisfied that: $\langle \mathcal{G}\mathcal{G}^* z, z \rangle_Z > 0$.

(v) For all $z \in Z$, it is satisfied that: $\lim\limits_{\varepsilon \to 0^+} \varepsilon \left(\varepsilon \mathbf{I}_Z + \mathcal{G}\mathcal{G}^*\right)^{-1} z = 0$.

The proof of Lemmas 2.2 and 2.3 can be found in [40] and [7], which in turn are based on [12].

Remark 2.7. For each $z \in Z$, set $u_\varepsilon = \mathcal{G}^* \left(\varepsilon \mathbf{I}_Z + \mathcal{G}\mathcal{G}^*\right)^{-1} z$. Note that

$$\mathcal{G} u_\varepsilon = z - \varepsilon \left(\varepsilon \mathbf{I}_Z + \mathcal{G}\mathcal{G}^*\right)^{-1} z, \quad \text{and} \quad \lim_{\varepsilon \to 0^+} \mathcal{G} u_\varepsilon = z.$$

Thus, the operator

$$\begin{aligned} \Gamma_\varepsilon : Z &\longrightarrow \quad U \\ z &\longmapsto \Gamma_\varepsilon z = \mathcal{G}^* \left(\varepsilon \mathbf{I}_Z + \mathcal{G}\mathcal{G}^*\right)^{-1} z \end{aligned}$$

is an *approximate* right inverse of $\mathcal{G}$, in the sense that, under the strong operator topology,

$$\lim_{\varepsilon \to 0^+} \mathcal{G}\Gamma_\varepsilon = \mathbf{I}_Z.$$

Remark 2.8. In the case of finite dimension of the Hilbert space Z, the injectivity of $\mathcal{G}^*$ yields a stronger result since every finite-dimensional subspace of a vector space is closed. Thus, $\mathrm{Im}(\mathcal{G})$ would be the whole space Z.

Remark 2.9. It is worth to mention that statement (ii) in Lemma 2.2 implies the injectivity of $\mathcal{G}^*$, but the opposite does not hold in general. Indeed, an additional condition for reaching the equivalence is that $\mathrm{Im}(\mathcal{G})$ would be closed.

2.4 Eigenvalue problem for elliptic equations

So far, we have reviewed some main well-known foundation needed for setting the mathematical context in our problems. Nevertheless, there are some other important tools required for applying certain properties of the operators that appear in the abstract formulations.

As it was mentioned in the introduction section, the main purpose of this thesis is to extend some controllability results by considering particular terms of perturbation in the heat and wave equations. A notorious characteristic of these equations is the Laplace operator or Laplacian. This section is devoted to exhibit the eigenvalue problem of the Laplacian, which will be related to the central hypothesis in the generation of the semigroup associated to the PDEs.

Consider the space $\mathbb{R}^N$, the Laplacian operator defined over a functional normed space Z, such as $L^2\left(\mathbb{R}^N\right)$ or the others defined previously, is given by

$$\begin{aligned} \Delta : Z &\longrightarrow \quad Z \\ z &\longmapsto \Delta z = \sum_{k=1}^{N} z_{x_k}. \end{aligned} \tag{2.5}$$

where

$$\Delta z(x) = \sum_{k=1}^{N} z_{x_k}(x) = \sum_{k=1}^{N} \frac{\partial z}{\partial x_k}(x), \quad x \in \mathbb{R}^N.$$

Clearly the dimension of $\mathbb{R}^N$ is finite, and in fact, in most applications, the space considered will be of dimension no greater than those of $\mathbb{R}^2$ or $\mathbb{R}^3$. However, the space $L^2\left(\mathbb{R}^N\right)$ and the spaces of piece-wise continuous functions work as infinite-dimensional spaces where many properties of finite-dimensional spaces are not obtained straightforward or in a compact manner, and even some others do not hold.

In [16], the theory for the generalized case of a second order differential operator in elliptic equations is developed. Nonetheless, hereby these results will be presented considering the specific case of the Laplacian for a better understanding of the properties involved in our particular PDEs. For detailed information about the eigenvalue problem, one can also review [27].

On the bounded domain $\Omega \subset \mathbb{R}^N$, consider the eigenvalue problem (sometimes also known as *Helmholtz* equation)

$$\begin{cases} Az = \lambda z & \text{in } \ \Omega \\ z = 0 & \text{on } \ \partial\Omega \end{cases} \tag{2.6}$$

where $A = -\Delta$, and $z \in C^{\infty}(\Omega)$, $\lambda \in \mathbb{R}$ are an associated eigenvector and eigenvalue, respectively.

We will consider the concept of weak solution to problem (2.6) in order to proof the following theorem retrieved from [27].

Theorem 2.19. *There exists an orthonormal basis $\{\phi_k\}_{k\in\mathbb{N}}$ of $L^2(\Omega)$ and a sequence of positive real numbers $\{\lambda_k\}_{k\in\mathbb{N}}$ with*

$$0 < \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_k \leq \cdots$$

and

$$\lim_{k\to+\infty} \lambda_k = +\infty, \tag{2.7}$$

such that each $\phi_k \in H_0^1(\Omega) \cap C^{\infty}(\Omega)$ is an eigenfunction of A associated to the eigenvalue λ_k, i.e., satisfying (2.6).

Proof. Define the operator $S : L^2(\Omega) \longrightarrow L^2(\Omega)$ such that for each $f \in L^2(\Omega)$, $S[f] \in H_0^1(\Omega)$ is the weak solution of the problem

$$\begin{cases} Az = f & \text{in } \ \Omega \\ z = 0 & \text{on } \ \partial\Omega. \end{cases}$$

Note that

$$\forall f, g \in L^2(\Omega), \ \forall \alpha \in \mathbb{R}: \quad S[\alpha f + g] = \alpha S[f] + S[g].$$

As a matter of fact, for $f, g \in L^2(\Omega), \ \alpha \in \mathbb{R}$,

$$\Delta(\alpha S[f] + S[g]) = \alpha \Delta S[f] + \Delta S[g] = \alpha f + g.$$

Clearly, $\alpha S[f] + S[g] = 0$ on $\partial\Omega$. So, because α, f, g were generic, S is a linear operator. Then, we have that for any $f \in L^2(\Omega)$ and $v \in H_0^1(\Omega)$, by integrating by parts,

$$-\int_{\Omega} v(x)\Delta u(x)\ dx = \int_{\Omega} \nabla S[f](x) \cdot \nabla v(x)\ dx + \int_{\partial\Omega} v(x) \frac{\partial S[f]}{\partial \eta}(x)\ ds(x)$$

so

$$\int_{\Omega} f(x)v(x)\ dx = \int_{\Omega} \nabla S[f](x) \cdot \nabla v(x)\ dx \tag{2.8}$$

Consider $f, g \in L^2(\Omega)$, arbitrary elements. Then, by the last result,

$$\begin{aligned}\langle g, S[f]\rangle_{L^2(\Omega)} &= \int_\Omega g(x)S[f](x)\ dx \\ &= \int_\Omega \nabla S[g](x)\cdot\nabla S[f](x)\ dx \\ &= \int_\Omega S[g](x)f(x)\ dx \\ &= \langle S[g], f\rangle_{L^2(\Omega)}\,.\end{aligned}$$

Hence, since f and g were chosen arbitrarily, it yields that $S^* = S$, that is, S is self-adjoint. Further, since Ω is a bounded set, by a remark to Rellich-Kondrachov Compactness Theorem 2.4 the identity operator $\mathbf{I} : H_0^1(\Omega) \longrightarrow L^2(\Omega)$ is compact. Thus, by Proposition 2.4, the operator $S = S\circ\mathbf{I}$ is compact. This last relation is well-defined since, indeed by definition, $\mathrm{Im}(S) \subseteq H_0^1(\Omega)$.

In addition, we have that for any $f \in L^2(\Omega)$, (2.8) yields

$$\langle S[f], f\rangle_{L^2(\Omega)} = \int_\Omega S[f](x)f(x)\ dx = \int_\Omega \nabla S[f](x)\cdot\nabla S[f](x)\ dx = \|S[f]\|^2_{L^2(\Omega)} \geq 0.$$

Therefore, S is also a positive operator. In fact, S is an *strict* positive operator since for any $f \in L^2(\Omega)$ with $f \neq 0$,

$$\langle S[f], f\rangle_{L^2(\Omega)} > 0.$$

Thus, recalling Proposition 2.6, the eigenvalues of S are no just non-negative, but strictly positive.

By Theorem 2.18, there exists a countable orthonormal basis $\{\phi_k\}_{k\in\mathbb{N}}$ of eigenvectors of S associated to eigenvalues β_k, $k \in \mathbb{N}$. If the constructed basis is finite, then $\dim(L^2(\Omega)) < +\infty$, which would be a contradiction. Whence, by Theorem 2.16, the sequence $(\beta_k)_{k\in\mathbb{N}} \subset \mathbb{R}_+$ is such that

$$\lim_{k\to+\infty} \beta_k = 0. \tag{2.9}$$

Now, for each $k \in \mathbb{N}$, denoting $\lambda_k = \beta_k^{-1}$, we have that

$$S[\phi_k] = \beta_k\phi_k \quad\Longrightarrow\quad \phi_k = S[\lambda_k\phi_k].$$

And in turn, this relation together with (2.8) give

$$\forall v \in H_0^1(\Omega): \quad \int_\Omega \nabla\phi_k(x)\cdot\nabla v(x)\ dx = \lambda_k\int_\Omega \phi_k(x)v(x)\ dx,$$

which implies that ϕ_k is a weak solution of (2.6) and eigenvector of A associated to the eigenvalue λ_k.

Thanks to (2.9), we have (2.7) and these eigenvalues can be ordered such that

$$0 < \lambda_1 \leq \lambda_2 \leq \lambda_3 \leq \cdots$$

Finally, it is left to prove that each $\phi_k \in C^\infty(\Omega)$. Indeed, take $k \in \mathbb{N}$. Without loss of generality, we can assume the boundary of Ω to be smooth enough. We have that

$$\Delta\phi_k = -\lambda_k\phi_k \in L^2(\Omega) \quad\Longrightarrow\quad \phi_k \in H^2(\Omega),$$

and thus,

$$\Delta\phi_k = -\lambda_k\phi_k \in H^2(\Omega) \quad\Longrightarrow\quad \phi_k \in H^4(\Omega),$$

so, we obtain that for any $m \in \mathbb{N}$, $\phi_k \in H^m(\Omega)$ by regularity of the weak solution. This, in turn, yields by Sobolev Embedding Theorem that $\phi_k \in C^\infty(\Omega)$.

□

Lastly, let us write the previous result in a useful manner of our interest for the coming chapters.

Corollary 2.1. *Let $A = -\Delta$ on the Hilbert space $L^2(\Omega)$. Then, there exists a formula for A in terms of an orthonormal expansion of projection operators:*

$$Az = \sum_{n=1}^{+\infty} \lambda_n S_n z, \quad z \in \mathcal{D}(A), \tag{2.10}$$

where $(\lambda_n)_{n\in\mathbb{N}}$ is a strictly increasing sequence of positive distinct eigenvalues of A.

Proof. By the previous theorem, it is evident that there is an orthonormal basis $\{\phi_n\}_{n\in\mathbb{N}}$ of $L^2(\Omega)$ and a sequence $\left(\tilde{\lambda}_n\right)_{n\in\mathbb{N}}$ of positive eigenvalues of A such that

$$\lim_{n\to+\infty} \tilde{\lambda}_n = +\infty.$$

Then, this sequence can be ordered without repetition so that the new sequence $(\lambda_n)_{n\in\mathbb{N}}$ fulfills

$$0 < \lambda_1 < \lambda_2 < \lambda_3 < \cdots$$

Take an arbitrary $n \in \mathbb{N}$. Denote γ_n as the finite geometric multiplicity of λ_n. Now, there exists γ_n linearly independent normal eigenvectors $\phi_n^1, \phi_n^2, \ldots, \phi_n^{\gamma_n}$ in $\{\phi_n\}_{n\in\mathbb{N}}$ associated to this eigenvalue. Further, define

$$\begin{aligned} S_n : L^2(\Omega) &\longrightarrow \quad L^2(\Omega) \\ z &\longmapsto S_n z = \sum_{k=1}^{\gamma_n} \phi_n^k \left\langle z, \phi_n^k \right\rangle_{L^2(\Omega)} \end{aligned}$$

as the orthogonal (or orthonormal) projection associated to λ_n. In fact, it is clear that S_n is linear and continuous. Let us show it is also idempotent. For $z \in L^2(\Omega)$, we have that

$$\begin{aligned} S_n S_n z &= \sum_{k=1}^{\gamma_n} \left\langle \sum_{j=1}^{\gamma_n} \langle z, \phi_n^j \rangle_{L^2(\Omega)} \phi_n^j, \phi_n^k \right\rangle_{L^2(\Omega)} \phi_n^k \\ &= \sum_{k=1}^{\gamma_n} \sum_{j=1}^{\gamma_n} \langle z, \phi_n^j \rangle_{L^2(\Omega)} \left\langle \phi_n^j, \phi_n^k \right\rangle_{L^2(\Omega)} \phi_n^k \\ &= \sum_{k=1}^{\gamma_n} \left\langle z, \phi_n^k \right\rangle_{L^2(\Omega)} \phi_n^k \\ &= S_n z, \end{aligned}$$

and note that S_n is self-adjoint, as well. Taking $y, z \in L^2(\Omega)$,

$$\begin{aligned} \langle y, S_n z \rangle_{L^2(\Omega)} &= \left\langle y, \sum_{k=1}^{\gamma_n} \left\langle z, \phi_n^k \right\rangle_{L^2(\Omega)} \phi_n^k \right\rangle_{L^2(\Omega)} \\ &= \sum_{k=1}^{\gamma_n} \left\langle z, \phi_n^k \right\rangle_{L^2(\Omega)} \left\langle y, \phi_n^k \right\rangle_{L^2(\Omega)} \\ &= \left\langle \sum_{k=1}^{\gamma_n} \left\langle y, \phi_n^k \right\rangle_{L^2(\Omega)} \phi_n^k, z \right\rangle_{L^2(\Omega)} \\ &= \langle S_n y, z \rangle_{L^2(\Omega)} . \end{aligned}$$

Now, denoting the orthonormal basis as $\{\phi_n^k\}_{n\in\mathbb{N}}$, we have for each $z \in L^2(\Omega)$, the orthonormal expansion is given as

$$z = \sum_{j=1}^{+\infty} \langle z, \phi_j \rangle_{L^2(\Omega)} \phi_j = \sum_{n=1}^{+\infty} \sum_{k=1}^{\gamma_n} \phi_n^k \left\langle z, \phi_n^k \right\rangle_{L^2(\Omega)}$$

and since the basis is composed of eigenvalues of A, whenever $z \in \mathcal{D}(A)$

$$Az = \sum_{j=1}^{+\infty} \tilde{\lambda}_j \langle z, \phi_j \rangle_{L^2(\Omega)} \phi_j = \sum_{n=1}^{+\infty} \lambda_n \sum_{k=1}^{\gamma_n} \phi_n^k \left\langle z, \phi_n^k \right\rangle_{L^2(\Omega)} = \sum_{n=1}^{+\infty} \lambda_n S_n z.$$

□

2.5 C_0-semigroups of operators

The main framework for setting the problem formulations is presented in this section. An introduction to the concept of strongly continuous (or C_0) semigroups is given, including the primary required properties of these mathematical elements. The main reference used in this section is [29].

Before arriving at the formal definitions, let us review a motivation for this theory. Consider the abstract Cauchy problem (or Initial Value Problem) on a Banach space Z:

$$\begin{cases} \dfrac{dz}{dt}(t) = Az(t), \quad t > 0, \\ z(0) = z_0, \end{cases} \tag{2.11}$$

for $A : \mathcal{D}(A) \subseteq Z \longrightarrow Z$, a linear operator on a Banach space Z, $z : \mathbb{R}_+ \longrightarrow Z$ and $z_0 \in \mathcal{D}(A)$. The problem is *well-set* or *well-posed* if

(i) there is a unique solution to the problem for any given initial datum $z_0 \in \mathcal{D}(A)$;

(ii) the solution changes continuously along with the initial data.

For the purpose of illustrating the main notion of a C_0-semigroup, define for each $t \geq 0$ an operator $T(t) : Z \longrightarrow Z$ which maps the solution $z(s)$ of (2.11) to $z(t+s)$. For instance,

$$T(t) \circ T(s) z_0 = T(t)T(s)z(0) = T(t)z(s) = z(t+s) = T(t+s)z_0, \quad t, s \geq 0$$

and similarly, the following associativity property holds

$$T(t) \circ T(s) = T(t+s), \quad t, s \geq 0.$$

Therefore, the set $\{T(t)\}_{t\geq 0}$ is an algebraic semigroup when endowed with the composition operation $\circ$.

In order to satisfy equation (2.11), we must have

$$\lim_{t\to 0^+} T(t)z_0 = z_0, \qquad \text{and so} \qquad \lim_{t\to 0^+} T(t) = \mathbf{I}_Z,$$

in the strong topology. We have then, that the set $\{T(t)\}_{t\geq 0}$ constitutes a semigroup of operators in the algebraic sense, that is, it is endowed with the associative property.

In the finite-dimensional case, whenever $A \in \mathcal{M}_n(\mathbb{R})$, the solution to (2.11) is given as

$$z(t) = T(t)z_0 = \exp(tA)z_0, \quad t \geq 0.$$

In this case, tA is called the *generator* of the exponential matrix $\exp(tA)$ at each $t \geq 0$.

Consider A to be a bounded operator, as well. For any $t \in \mathbb{R}$, define

$$E_n = \sum_{k=0}^{n} \frac{1}{k!} t^k A^k,$$

then, if $m < n$,

$$\|E_n - E_m\|_{\mathscr{L}(Z)} = \left\| \sum_{k=m+1}^{n} \frac{1}{k!} t^k A^k \right\|_{\mathscr{L}(Z)} \leq \sum_{k=m+1}^{n} \frac{1}{k!} |t|^k \|A\|_{\mathscr{L}(Z)}^k$$

which converges to 0 as $m, n \to +\infty$. Henceforth, $(E_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in $\mathscr{L}(Z)$, and thus, has a limit, denoted by $\exp(tA)$. In this sense, $\{\exp(tA)\}_{t\geq 0}$ is a uniformly continuous semigroup in Z, and moreover, it satisfies

$$\frac{d}{dt}\exp(tA) = A\exp(tA).$$

In fact,

$$\begin{aligned}
\frac{d}{dt}\exp(tA) &= \frac{d}{dt} \lim_{n\to+\infty} \sum_{k=0}^{n} \frac{1}{k!} t^k A^k \\
&= \lim_{n\to+\infty} \sum_{k=1}^{n} \frac{1}{(k-1)!} t^{k-1} A^k \\
&= \lim_{n\to+\infty} \sum_{k=0}^{n} \frac{1}{k!} t^k A^{k+1} \\
&= A\exp(tA).
\end{aligned}$$

The concept of the semigroup of operators thereupon generalizes these ideas towards the infinite-dimensional case.

Definition 2.8. *A family of bounded linear operators $\{T(t)\}_{t\geq 0}$ which map a Banach space Z into itself is called a* strongly continuous semigroup *or* C_0-semigroup *of operators if the following conditions are satisfied:*

(i) $T(t+s) = T(t)T(s), \quad t, s \geq 0,$

(ii) $T(0) = \mathbf{I}_Z$, *where* $\mathbf{I}_Z$ *is the identity operator in* Z,

(iii) for each $x \in Z$, $T(t)x$ is strongly continuous on t in the set $\mathbb{R}_+ \cup \{0\}$, i.e.,

$$\lim_{\Delta t\to 0} \|T(t+\Delta t)x - T(t)x\| = 0, \quad t,\ t+\Delta t \geq 0.$$

Moreover, if the map T given by $t \longmapsto T(t)$ is continuous in the uniform operator topology, then, the family $\{T(t)\}_{t\geq 0}$ is called a uniform continuous semigroup *in Z. Also, if the property $\|T(t)\| \leq 1,\ t \geq 0$ is satisfied by the semigroup, it is called an* contraction semigroup.

The notation C_0 refers to the one parameter t of the continuous semigroup. Note that property (i) of Definition 2.8 yields the commutative property under the composition operation.

An important property of these structures is the *exponential decay.* For the purpose of illustrating this concept, consider the following lemma.

Lemma 2.4. *Let $\omega : \mathbb{R}_+ \cup \{0\} \longrightarrow \mathbb{R}$ be a sub-additive and bounded above function on each finite subinterval. Then,*

$$\omega_0 := \inf_{t>0} \frac{\omega(t)}{t}$$

is finite or $-\infty$ and

$$\omega_0 = \lim_{t\to\infty} \frac{\omega(t)}{t}.$$

Proof. It is clear that the quantity $\omega_0 = \inf_{t>0} \dfrac{\omega(t)}{t}$ is finite or tends to $-\infty$ by the boundedness above condition. Then, for any $\delta > \omega_0$, there exists $t_0 > 0$ such that:

$$\frac{\omega(t_0)}{t_0} < \delta.$$

Moreover, whenever $t \geq 0$, we can decompose it as

$$t = n(t)t_0 + r$$

for some non-negative integer $n(t)$ and $r \in [0, t_0)$. Then, by the sub-additivity of ω,

$$\frac{\omega(t)}{t} = \frac{\omega\left(n(t)t_0 + r\right)}{t} \leq \frac{n(t)\omega(t_0) + \omega(r)}{t} = \frac{\omega(t_0)}{t_0 + r/n(t)} + \frac{\omega(r)}{t}.$$

And so, by taking the lim sup,

$$\limsup_{t\to+\infty} \frac{\omega(t)}{t} \leq \frac{\omega(t_0)}{t_0+0} + 0 = \frac{\omega(t_0)}{t_0} < \delta.$$

Note that since $\omega_0 = \inf_{t>0} \dfrac{\omega(t)}{t}$, and by definition of lim inf and lim sup,

$$\omega_0 \leq \liminf_{t\to+\infty} \frac{\omega(t)}{t} \leq \limsup_{t\to+\infty} \frac{\omega(t)}{t} < \delta.$$

Therefore, because $\delta > \omega_0$ was arbitrary, passing to the limit as $\delta \longrightarrow \omega_0^+$,

$$\limsup_{t\to+\infty} \frac{\omega(t)}{t} = \liminf_{t\to+\infty} \frac{\omega(t)}{t} = \omega_0,$$

and the proof is complete.

$\square$

Next, let us prove the exponential decay condition.

Theorem 2.20. *Consider a C_0-semigroup $\{T(t)\}_{t\geq 0}$ on a Banach space Z. The limit*

$$\omega_0 = \lim_{t\to\infty} \frac{\log \|T(t)\|_{\mathscr{L}(Z)}}{t} \tag{2.12}$$

exists, and for each $\delta > \omega_0$, there exists a constant M_δ such that

$$\|T(t)\|_{\mathscr{L}(Z)} \leq M_\delta \exp(\delta t), \quad t \geq 0.$$

The number ω_0 is called the type *of the C_0-semigroup.*

Proof. Define a real function from the formula

$$\omega(t) = \log \|T(t)\|_{\mathscr{L}(Z)}, \quad t \geq 0.$$

Note that ω is sub-additive. Suppose, by contradiction, that there exists a sequence $(t_n)_{n\in\mathbb{N}} \subset [0, t_0]$ such that

$$\lim_{n\to+\infty} t_n = t^* \in [0, t_0], \quad \text{and} \quad \lim_{n\to+\infty} \|T(t_n)\|_{\mathscr{L}(Z)} = +\infty, \tag{2.13}$$

For each $x \in Z$, we have that

$$T(t_n)x \longrightarrow T(t^*)x \quad \text{as} \quad n \longrightarrow +\infty,$$

and that $\|T(t^*)\|_{\mathscr{L}(Z)} < +\infty$. Then,

$$\sup_{n\in\mathbb{N}} \|T(t_n)x\|_Z < +\infty, \qquad \text{for } x \in Z.$$

Thanks to Theorem 2.8,

$$\sup_{n\in\mathbb{N}} \|T(t_n)\|_{\mathscr{L}(Z)} < +\infty.$$

Thus, it yields a contradiction, so there is no sequence satisfying (2.13), and therefore there exists $C_{t_0} > 0$ such that:

$$\|T(t)\|_{\mathscr{L}(Z)} < C_{t_0} \qquad \text{with } t \text{ in any finite interval } [0, t_0]. \tag{2.14}$$

Due to Lemma 2.4, ω_0 as in (2.12) exists as a finite number or $-\infty$. Considering $\delta > \omega_0$, there is $\widehat{t_0} = \widehat{t_0}(\delta)$ such that

$$\frac{\log \|T(t)\|_{\mathscr{L}(Z)}}{t} < \delta \qquad \Longrightarrow \qquad \|T(t)\|_{\mathscr{L}(Z)} < \exp(\delta t), \quad t \geq \widehat{t_0}.$$

Then, the results follows from (2.14). In fact, for $M_\delta = \max\left\{1, C_{\widehat{t_0}}\right\}$,

$$\|T(t)\|_{\mathscr{L}(Z)} \leq C_{\widehat{t_0}} \leq C_{\widehat{t_0}} \exp(\delta t) \leq M_\delta \exp(\delta t), \quad 0 \leq t < \widehat{t_0}.$$

□

Following with the approach of generalization of the exponential matrix, we arrive at the concept of infinitesimal generator, which is the analogue to the generator operator of the exponential matrix. In this case, an operator with certain features is defined so that the properties of the semigroup are satisfied, specifically, the strongly continuity.

Consider a C_0-semigroup $\{T(t)\}_{t\geq 0}$ on a Banach space Z and let $h > 0$. Denote

$$A_h x := \frac{1}{h}\left(T(h)x - x\right), \quad x \in Z.$$

Also, define

$$\mathcal{D}(A) = \left\{ x \in Z \ : \ \lim_{h\to 0^+} A_h x \text{ is finite.} \right\},$$

and $A : \mathcal{D}(A) \subseteq Z \longrightarrow Z$ by

$$A(x) = \lim_{h\to 0^+} A_h x, \quad x \in \mathcal{D}(A). \tag{2.15}$$

Definition 2.9 (Infinitesimal generator)**.** *The operator A defined by* (2.15) *is called the* infinitesimal generator *of the semigroup $\{T(t)\}_{t\geq 0}$.*

Now, let us review some interesting properties of this new operator. Although it is a linear and closed operator as we shall see forthwith, it need not be bounded.

Theorem 2.21 (Infinitesimal generator properties)**.** *Let $\{T(t)\}_{t\geq 0}$ be a C_0-semigroup on a Banach space Z and $A : \mathcal{D}(A) \subseteq Z \longrightarrow Z$ its infinitesimal generator. Then, the following are satisfied:*

(i) $\mathcal{D}(A)$ is a vector space on Z and A is a linear operator.

(ii) If $x \in \mathcal{D}(A)$, then $T(t)x \in \mathcal{D}(A)$ for $t \geq 0$ and it is differentiable on $t > 0$ such that

$$\frac{d}{dt} T(t)x = AT(t)x = T(t)Ax. \tag{2.16}$$

(iii) If $x \in \mathcal{D}(A)$, then

$$T(t)x - T(s)x = \int_s^t T(\tau)Ax \, d\tau, \quad t, s \geq 0.$$

(iv) If f is a continuous real-valued function on $\mathbb{R}_+ \cup \{0\}$, then

$$\lim_{h \to 0^+} \frac{1}{h} \int_t^{t+h} f(s)T(s)x \, ds = f(t)T(t)x, \quad x \in Z, \; t \geq 0.$$

(v) The integral $\int_0^t T(s)x \, ds$ belongs to $\mathcal{D}(A)$ and

$$T(t)x = x + A \int_0^t T(s)xds, \quad x \in Z$$

(vi) $\mathcal{D}(A)$ is dense in Z and A is a closed operator.

Proof. Let us prove each numeral.

(i) It is clear by definition. Indeed, for $x, y \in \mathcal{D}(A)$ and $\alpha \in \mathbb{R}$, generic, since for each $t \geq 0$, $T(t)$ is linear, the limit

$$\begin{aligned}
\lim_{h \to 0^+} Ah(\alpha x + y) &= \lim_{h \to 0^+} \frac{1}{h} \left[T(h)(\alpha x + y) - (\alpha x + y) \right] \\
&= \alpha \lim_{h \to 0^+} \frac{1}{h}(T(h)x - x) + \lim_{h \to 0^+} \frac{1}{h}(T(h)y - y)
\end{aligned}$$

exists so $\alpha x + y \in \mathcal{D}(A)$. Because these elements were taken arbitrarily, $\mathcal{D}(A)$ is a sub-vector space of Z, and A is linear.

(ii) Consider $x \in \mathcal{D}(A)$ and $t \geq 0$. As a result of the continuity of the semigroup,

$$\begin{aligned}
A_h \left[T(t)x \right] &= \frac{1}{h} \left[T(h)T(t)x - T(t)x \right] = T(t) \frac{1}{h} \left(T(h)x - x \right) \\
&= T(t)A_h x \longrightarrow T(t)Ax
\end{aligned} \tag{2.17}$$

as $h \longrightarrow 0^+$. So, $T(t)x \in D(A)$ by definition. Now, let us consider $t > 0$,

$$\lim_{h \to 0^+} \left[\frac{1}{h} \left(T(t+h)x - T(t)x \right) - T(t)Ax \right] = \lim_{h \to 0^+} \left(T(t)A_h x - T(t)Ax \right) = 0, \tag{2.18}$$

and similarly,

$$\begin{aligned}
&\lim_{h \to 0^-} \left[\frac{1}{h} \left(T(t+h)x - T(t)x \right) - T(t)Ax \right] \\
&= \lim_{h \to 0^-} \left\{ T(t+h) \left[\frac{1}{-h} \left(T(-h)x - T(0)x \right) - Ax \right] + \left(T(t+h) - T(t) \right) Ax \right\} \\
&= T(t) \left[\lim_{h \to 0^+} \frac{1}{h} \left(T(h)x - x \right) - Ax \right] + \lim_{h \to 0^-} \left(T(t+h) - T(t) \right) Ax \\
&= 0.
\end{aligned} \tag{2.19}$$

Hence, thanks to (2.17)–(2.19), (2.16) is satisfied.

(iii) This result is trivial, following from (ii) and the Fundamental Theorem of Calculus on normed spaces.

(iv) Let $x \in Z$. The operator defined by the formula $S(t) = f(t)T(t)x$ is continuous in t for all $t \geq 0$. Now, consider $\xi \geq 0$ and define

$$F(\xi) = \int_t^{t+\xi} S(s)\ ds.$$

It follows that, by definition,

$$F'(0) = \lim_{h \to 0} \frac{1}{h} \int_t^{t+h} S(s)\ ds,$$

and by the rule of differentiation under the integral sign,

$$F'(\xi) = S(t+\xi) \quad \Longrightarrow \quad F'(0) = S(t) = f(t)T(t)x.$$

Thus, the desired result holds.

(v) For $x \in Z$ and $t, h > 0$,

$$\begin{aligned} A_h \int_0^t T(s)x\ ds &= \frac{1}{h} \int_0^t (T(h+s)x - T(s)x)\ ds \\ &= \frac{1}{h} \left[\int_0^t T(h+s)x\ ds - \int_0^t T(s)x\ ds \right] \\ &= \frac{1}{h} \left[\int_h^{t+h} T(s)x\ ds - \int_0^t T(s)x\ ds \right] \\ &= \frac{1}{h} \left[\int_t^{t+h} T(s)x\ ds - \int_0^h T(s)x\ ds \right]. \end{aligned}$$

Then, by (iv) with $f = \mathbb{1}_{\mathbb{R}_+ \cup \{0\}}$,

$$\begin{aligned} \lim_{h \to 0^+} A_h \int_0^t T(s)x\ ds &= \lim_{h \to 0^+} \frac{1}{h} \left[\int_t^{t+h} T(s)x\ ds - \int_0^h T(s)x\ ds \right] \\ &= T(t)x - T(0)x \\ &= T(t)x - x. \end{aligned}$$

Therefore,

$$T(t)x = x + A \int_0^t T(s)x\ ds.$$

(vi) Take an arbitrary $x \in Z$. Likewise, from (iv),

$$x = \lim_{t \to 0} \frac{1}{t} \int_0^t T(s)x\ ds,$$

and from (v),

$$\int_0^t T(s)x\ ds \in \mathcal{D}(A), \quad t \geq 0.$$

So, the sequence $(x_n)_{n \in \mathbb{N}} \subset \mathcal{D}(A)$ defined by

$$x_n = n \int_0^{1/n} T(s)x\ ds, \quad n \in \mathbb{N}$$

approaches to x as n goes to $+\infty$. By the arbitrariness of x, $\mathcal{D}(A)$ is dense in Z.

Subsequently, consider $(x_n)_{n \in \mathbb{N}} \subset \mathcal{D}(A)$ with

$$\lim_{n \to +\infty} x_n = x \quad \text{and} \quad \lim_{n \to +\infty} Ax_n = y.$$

Take any $t \geq 0$. Since the operator $T(t)$ is continuous,

$$T(t)Ax_n \longrightarrow T(t)y \quad \text{as} \quad n \to \infty,$$

and together with (iii),

$$T(t)x - x = \lim_{n\to+\infty} (T(t)x_n - x_n) = \lim_{n\to+\infty} \int_0^t T(s)Ax_n\, ds = \int_0^t T(s)y\, ds.$$

Furthermore, due to result in (iv),

$$\lim_{h\to 0^+} A_h x = \lim_{h\to 0^+} \frac{1}{h}(T(h)x - x) = \lim_{h\to 0^+} \frac{1}{h}\int_0^h T(s)y\, ds = T(0)y = y$$

Then, we have that $Ax = y$ and that $x \in \mathcal{D}(A)$. By definition, the operator A is closed. □

Theorem 2.22. *Let A on $\mathcal{D}(A)$ be the infinitesimal generator of a C_0-semigroup $\{T(t)\}_{t\geq 0}$ over a Banach space Z. Then, the abstract Cauchy problem given by (2.11) has a unique solution given by*

$$x(t) = T(t)x_0, \quad t \geq 0. \tag{2.20}$$

Proof. Let us note that, by result (ii) of the previous theorem, finding the derivative of (2.20) yields

$$\frac{d}{dt}x(t) = \frac{d}{dt}T(t)x_0 = AT(t)x_0 = Ax(t),$$

and it is clear that

$$x(0) = T(0)x_0 = x_0.$$

So, such a solution exists. Now, consider y as an arbitrary solution of the Cauchy problem. Then, for $F(s) = T(t-s)y(s),\ s \in [0,t]$, by Theorem 2.21, and since $y(s) \in \mathcal{D}(A)$, the strong derivative of F in s can be found. Certainly, whenever $s \in [0,t]$

$$\begin{aligned}\frac{d}{ds}F(s) &= \frac{d}{ds}T(t-s)\cdot y(s) + T(t-s)\cdot y'(s)\\ &= -AT(t-s)y(s) + T(t-s)Ay(s)\\ &= 0.\end{aligned}$$

Consequently, $F(s) = c$ on $[0,t]$, for some $c \in Z$. Then,

$$x(t) = T(t)x_0 = T(t)y(0) = F(0) = c$$

and

$$y(t) = T(0)y(t) = F(t) = c.$$

By the arbitrariness of t, this implies that the solution is, further, unique. □

Now, let us consider the non-homogeneous Cauchy problem

$$\begin{cases}\dfrac{dz}{dt}(t) - Az(t) = f(t), & t > 0,\\ z(0) = z_0, & z_0 \in \mathcal{D}(A).\end{cases} \tag{2.21}$$

with f continuous and differentiable on $\mathbb{R}_+ \cup \{0\}$. Then, the following theorem holds.

Theorem 2.23. *Let A be the infinitesimal generator of a C_0-semigroup $\{T(t)\}_{t\geq 0}$. Let $f : [0,\infty) \longrightarrow Z$ be a (strongly) continuous and differentiable function. Then, the non-homogeneous Cauchy problem (2.21) has a unique solution given by*

$$z(t) = T(t)z_0 + \int_0^t T(t-s)f(s)\, ds, \quad t \geq 0. \tag{2.22}$$

Proof. Similarly as in the previous theorem, for $t \geq 0$, generic, assuming formula in the right side of (2.22) and that the derivative of the integral term can be found, we must have that

$$\begin{aligned}\frac{dz}{dt}(t) &= AT(t)z_0 + \frac{d}{dt}\int_0^t T(t-s)f(s)\ ds \\ &= AT(t)z_0 + A\int_0^t T(t-s)f(s)\ ds + T(0)f(t) \\ &= Az(t) + f(t).\end{aligned}$$

Also, under this scenario, we must get that

$$z(0) = T(0)z_0 + \int_0^0 T(t-s)f(s)ds = z_0,$$

and thus, this formula indeed characterizes a solution. Nevertheless, let us check the assumption made. Define g such that

$$g(t) = \int_0^t T(t-s)f(s)\ ds, \quad t > 0.$$

For $h > 0$, it follows that

$$\begin{aligned}\frac{1}{h}(g(t+h) - g(t)) &= \frac{1}{h}\int_0^{t+h} T(t+h-s)f(s)ds - \frac{1}{h}\int_0^t T(t-s)f(s)ds \\ &= \frac{1}{h}\left(T(h) - \mathbf{I}_Z\right)\int_0^t T(t-s)f(s)\ ds + \frac{1}{h}\int_t^{t+h} T(t+h-s)f(s)\ ds.\end{aligned}$$

Then, thanks to Theorem 2.21,

$$g'(t) = \lim_{h\to 0}\frac{1}{h}(g(t+h) - g(t)) = A\int_0^t T(t-s)f(s)ds + f(t).$$

Moreover, the well-posedness of the derivative of g is given by noting that

$$g'(t) = \int_0^t T(s)f'(t-s)ds + T(t)f(0).$$

The uniqueness is trivial to see with aid of the previous theorem. For two solutions x and y of (2.21), denote $w = x - y$ we have that

$$\frac{dw}{dt}(t) = \frac{d}{dt}[x(t) - y(t)] = Ax(t) - f(t) - Ay(t) + f(t) = A[x(t) - y(t)] = Aw(t),$$

and

$$w(0) = x(0) - y(0) = x_0 - x_0 = 0$$

But, by Theorem 2.22, the unique solution to

$$\begin{cases}\dfrac{dw}{dt}(t) = Aw(t), & t > 0 \\ w(0) = 0.\end{cases}$$

is $w = 0$, so $x = y$. By arbitrariness of x and y, the solution to (2.21) is unique. □

Remark 2.10. By construction of the proof, it can be noted that a weaker condition on f for Theorem 2.23 to be valid is that f and $Af(\cdot)$ are strongly continuous on t and $f(t) \in \mathcal{D}(A)$, for all $t \geq 0$.

Finally, let us see that the infinitesimal generator only generates one sole semigroup of operators, and that every dense-domain operator can be a infinitesimal generator of at most one C_0-semigroup, as well.

Theorem 2.24. *An operator A with domain $\mathcal{D}(A)$ dense in the Banach space Z can be the infinitesimal generator of at most one strongly continuous semigroup $\{T(t)\}_{t\geq 0}$.*

Proof. Consider two arbitrary C_0-semigroups generated by A, $\{T(t)\}_{t\geq 0}$ and $\{S(t)\}_{t\geq 0}$. Then, we have that, for the abstract Cauchy problem established by A, with some generic initial data $z_0 \in \mathcal{D}(A)$,

$$x(t) = T(t)z_0 \quad \text{and} \quad y(t) = S(t)z_0$$

are both its solutions. Then, by the arbitrariness of z_0, the uniqueness of this solution, and the density of $\mathcal{D}(A)$ on Z,

$$T(t)z_0 = S(t)z_0, \quad z_0 \in D(A) \quad \Longrightarrow \quad T(t)z = S(t)z, \quad z \in Z,$$

for all $t \geq 0$. Hence, the desired result is obtained. □

Finally, there are some relevant results concerning other operators which generate a C_0-semigroup. The following theorem is a classical result in this theory which has plenty application to different kind of systems. Later on, a result concerning perturbation theory of particular interest to our PDE problems will be presented.

Theorem 2.25 (Hille-Yosida-Phillips)**.** *A necessary and sufficient condition for a closed operator A with dense domain $\mathcal{D}(A)$ in the Banach space Z to be the infinitesimal generator of a C_0-semigroup $\{T(t)\}_{t\geq 0}$ is that there exist positive real numbers M and ω such that for every $\lambda > w$,*

$$\lambda \in \rho(A) \quad \textit{and} \quad \|R_\lambda(A)^n\|_{\mathscr{L}(Z)} \leq \frac{M}{(\lambda-\omega)^n}, \quad \textit{for each } n \in \mathbb{N}.$$

A proof in detail of this result can be found in pp. 33–39 of [29].

2.6 Control Theory

Throughout the last decades, many control problems have been studied from analytical perspectives. There have been proposed different definitions for the concept of controllability. For PDEs, the main difference lies on whether control variable acts on the interior of the considered domain Ω, from the exterior, on the boundary of it, or whether the specific type of control is exact, approximate, null, among others.

The main fixed-point techniques applied in this work are built upon the analogy that the operator equation equivalent to solving to the control problem is defined from a surjective operator resembles the exact controllability of it. Likewise, if such equation is established solely from a range-dense operator (and no more than that), then the approximate controllability of the system is applied.

Each author presents in a different fashion their control theory definitions, on the basis of the problem in question. These definitions will be presented here so that the reader takes into account our aims.

Consider a classical arbitrary system of differential equations with delay

$$\begin{cases} z' = F(t, z(t-r), u(t)) & \text{on} \ (0, \tau] \\ z = \phi & \text{on} \ [-r, 0] \end{cases} \tag{2.23}$$

where $z : [0, \tau] \longrightarrow Z$ takes values on a normed space Z, $u : [0, \tau] \longrightarrow U$ takes values on a normed space U, usually $u \in L^2([0, \tau], U)$, and $F : [0, \tau] \times \mathbb{R} \times \mathbb{R} \longrightarrow Z$ is a generic (possibly non-linear) function. For $r \geq 0$, $[-r, 0]$ is a subset of the real numbers on which the control does not act.

We say that $\phi : [-r, 0] \longrightarrow Z$ is the *initial condition function* and r is the *length* of the delay. Whenever, there is no delay, the initial condition function can be regarded as a *initial state* $\phi = z_0 \in Z$, and $r = 0$. In either case, we will refer to these elements as *initial data* or *initial conditions*.

For the purpose of analyzing the behavior in the time and space variables of systems (1.1) and (1.6), we consider $Z = U = L^2(\Omega)$ in the following chapters. Mostly in theses cases, the control variable is considered to act only on a specified set $\omega \subset \mathbb{R}^N$. The positive value τ is the *final time* of the problem. The set Z is usually appointed as the *state space*, and its elements as *states*, of the control system.

Now, the well-posedness of system (2.23) indicates that given some arbitrary initial condition ϕ and control function u, there exists a unique solution $z = z(\phi, u)$ depending on these elements. Depending on the nature of the problem, the solution will be of certain regularity, and the involved terms of the main equation in (2.23) must work under specific conditions.

Finally, observe that for a controllability problem, there must be a *final state* or *final condition* $z_f \in Z$ given to which the system must be steered as needed.

Definition 2.10 (Exact controllability). *A system of the form* (2.23) *is said to be* exactly controllable *in time* $\tau > 0$ *(or in the interval* $[0, \tau]$ *), if for any initial and final conditions* $\phi : [-r, 0] \longrightarrow Z$*,* $z_f \in Z$*, there is a control* $u \in L^2([0, \tau], U)$ *such that the associated solution* $z = z(\phi, u, z_f)$ *satisfies*

$$z(\tau) = z_f. \tag{2.24}$$

However, not every system of the form (2.23) shall have an exact solution that reaches the desired final state. In this case, the condition (2.24) is relaxed in the following sense.

Definition 2.11 (Approximate controllability). *A system of the form* (2.23) *is said to be* approximately controllable *in time* $\tau > 0$ *(or in the interval* $[0, \tau]$ *), if for any initial and final conditions* $\phi : [-r, 0] \longrightarrow Z$*,* $z_f \in Z$*, there is a sequence of controls* $(u_n)_{n \in \mathbb{N}}$*, with* $u_n \in L^2([0, \tau], U)$*,* $n \in \mathbb{N}$*, such that the associated solutions* $z_n = z(\phi, u_n, z_f)$ *satisfy*

$$\lim_{n \to +\infty} z_n(\tau) = z_f, \tag{2.25}$$

or equivalently, considering a precision $\varepsilon > 0$*, there exists a control* u_ε *such that the associated solution* $z_\varepsilon = z(\phi, u_\varepsilon, z_f)$ *fulfills*

$$\|z_f - z_\varepsilon(\tau)\|_Z < \varepsilon.$$

Evidently, exact controllability implies approximate controllability, nevertheless, the relevance on the research on these type of control systems roots in the assumptions made for obtaining the desired result and the techniques used in the way. That is why in this work, the main purpose will be to obtain specific controllability results on the semilinear systems, but by setting the problem from different perspectives.

In the analysis of PDEs, there exists a particular fashion of presenting the controllability definitions since the set $\omega \subset \mathbb{R}^N$ can be imposed distinct conditions. In fact, one can regard the control variable to work solely inside a strict subset of the bounded domain, at the exterior of it, or on the its boundary.

Consider the following formulation of a PDE control system dependent on $t \in [-r, \tau]$ and $x \in \Omega$

$$\begin{cases} P(z) = F(t, z(t-r, x), u(t, x)) & \text{on} \quad (0, \tau] \times \Omega \\ z = \phi & \text{on} \quad [-r, 0] \times \Omega \end{cases} \tag{2.26}$$

where P is a linear differential operator and $F : [0, \tau] \times \mathbb{R} \times \mathbb{R} \longrightarrow Z$ is a generic function. The other variables are similarly defined as in (2.23).

Now, let us review the definitions established for these kind of systems. From these concepts, one can study the specific type of controllability based on the previous definitions.

Definition 2.12 (Controllability with respect to ω). *Considering a system of the form* (2.26)*, the following control problems can be studied:*

(i) ***Interior controllability:*** *whenever* $\omega \subset \Omega \subset \mathbb{R}^N$*, and the indicator function* $\mathbb{1}_\omega$ *is accompanying the control variable such that*

$$F(t, z(t-r, x), u(t, x)) = F\left(t, z(t-r), \mathbb{1}_\omega u(t, x)\right).$$

(ii) ***Boundary controllability:*** *whenever* $\omega = \partial\Omega \subset \mathbb{R}^N$*, the function* F *does not depend on* u

$$F(t, z(t-r, x), u(t, x)) = F(t, z(t-r, x)),$$

and the following additional condition is attached to (2.26)

$$z(t, x) = u(t, x), \quad (t, x) \in (0, \tau] \times \partial\Omega.$$

(iii) ***Exterior controllability:*** *whenever* $\omega \subset \mathbb{R}^N \backslash \Omega$*,* F *does not depend on* u*, and moreover the following is added to the system* (2.26)

$$z(t, x) = \mathbb{1}_\omega u(t, x), \quad (t, x) \in (0, \tau] \times \omega.$$

Noticeably, in each one of these definitions, the specific case of exact or approximate controllability can be studied. In fact, there exists variants such that the localization of the control towards just a subset of the boundary or a restriction towards the half-space. As it can be observed, throughout this work, we will exclusively focus on the interior controllability of systems (1.1) and (1.6). However, the form of the abstract systems that will be formulated shortly will be as in (2.23) for the purpose of considering the Operator Theory approach.

Finally, in the controllability of PDEs, it is a common practice to deliver these definitions considering the next concepts, specially in the filed of optimal control theory.

Definition 2.13 (Admissible controls)**.** *Fixing an initial condition* $\phi : [-r, 0] \longrightarrow Z$ *and final state* z_f*, a control function* $u \in L^2([0, \tau], U)$ *is called an* admissible control *for* (2.23) *if the associated solution* $z = z(\phi, u, z_f)$ *of the control system verifies*

$$z(\tau) = z_f.$$

As a matter of fact, the search for a desired optimal control is based on its size or norm, roughly speaking.

Definition 2.14 (Reachable states)**.** *Given the initial condition* $\phi : [-r, 0] \longrightarrow Z$*, the set of* reachable *states of a system of the form* (2.23) *for an specific final time* τ *is the subset of elements of the state space for which there exists an admissible control that steers the systems towards it. In other words,*

$$\mathcal{R}_\phi(\tau) = \left\{z_f \in Z \;/\quad \exists u \in L^2([0, \tau], U) : \; z_u(\tau) = z_f, \; \textit{with } z_u = z(\phi, u) \textit{ a solution of } (2.23)\right\}.$$

Now, definitions 2.10 and 2.11 will be equivalent to

Definition 2.15 (Exact controllability)**.** *A system of the form* (2.23) *is said to be* exactly controllable in time $\tau > 0$ *(or in the interval* $[0, \tau]$ *), if for all initial conditions* $\phi : [-r, 0] \longrightarrow Z$*, the set of reachable states coincides with the entire state space, i.e.,*

$$\mathcal{R}_\phi(\tau) = Z.$$

Definition 2.16 (Approximate controllability)**.** *A system of the form* (2.23) *is said to be* approximately controllable in time $\tau > 0$ *(or in the interval* $[0, \tau]$ *), if for all initial conditions* $\phi : [-r, 0] \longrightarrow Z$*, the set of reachable states is dense in the state space, i.e.,*

$$\overline{\mathcal{R}_\phi(\tau)} = Z.$$

Lastly, it is worth to mention an important definition commonly studied in this type of systems.

Definition 2.17 (Null controllability). *A system of the form* (2.23) *is said to be* null-controllable in time $\tau > 0$ *(or in the interval* $[0, \tau]$ *), if for all initial conditions* $\phi : [-r, 0] \longrightarrow Z$*, the zero* $0 \in Z$ *of the state space is a reachable state, i.e.,*

$$0 \in \mathcal{R}_\phi(\tau).$$

Thus, the question arising in the controllability of systems of PDEs regard the existing relation between null controllability and approximate controllability. Some of these studies have been developed in [11] and [50].

2.7 Perturbation Theory

The result presented in this section follows from a combination of Theorem 19 in [15] and Chapter XIII of [23]. It is well know that, if A is the infinitesimal generator of a C_0-semigroup $\{T_A(t)\}_{t\geq 0}$ in the Banach space Z and P is a bounded linear operator in Z ($P \in \mathscr{L}(Z)$), then $A + P$ is the infinitesimal generator of a C_0-semigroup $\{T_{A+P}(t)\}_{t\geq 0}$ which is given the following formula

$$T_{A+P}(t)z = T_A(t)z + \int_0^t T_A(t-s)PT_{A+P}(s)zds, \quad z \in Z. \tag{2.27}$$

Now, we shall see that if P is an unbounded linear operator which is not too irregular relative to A, then $A + P$ is the infinitesimal generator of a C_0-semigroup $\{T_{A+P}(t)\}_{t\geq 0}$, but the formula (2.27) is not true in general.

Consider the following class of unbounded operators.

Definition 2.18 ($\mathscr{P}(A)$ class). *Let A be the infinitesimal generator of a C_0-semigroup $\{T_A(t)\}_{t\geq 0}$, we denote by $\mathscr{P}(A)$ the class of closed linear operators P satisfying the following conditions:*

(i) $\mathcal{D}(A) \subset \mathcal{D}(P)$,

(ii) for each $t > 0$*, there exist a constant* $h(t) \geq 0$ *such that:*

$$\|PT_A(t)z\|_Z \leq h(t)\,\|z\|_Z\,, \quad \forall z \in \mathcal{D}(A),$$

(iii) the integral $\int_0^1 h(t)dt$ *exists.*

Remark 2.11. Note that A is bounded if and only if $A \in \mathscr{P}(A)$.

Let us study some properties regarding this specific class of unbounded operators. The following results can be found in books like [15] and [23].

Lemma 2.5. *Let $P \in \mathscr{P}(A)$, for A an infinitesimal generator of a C_0-semigroup $\{T(t)\}_{t\geq 0}$. Then,*

(i) $\bigcup_{t\geq 0} T_A(t)z \subset \mathcal{D}(P)$,

(ii) the mapping $z \mapsto PT_A(t)z, z \in \mathcal{D}(A)$*, has a unique extension to a bounded operator defined on* Z*. In order to simplify the notation, we will call this extension* $PT(t)$*,*

(iii) $PT(t)$ *is a continuous in* $t > 0$ *at each* $z \in Z$*. If* $\omega_0 = \lim_{t\to+\infty} \frac{\log \|T(t)\|_{\mathscr{L}(Z)}}{t}$*, then*

$$\limsup_{t\to+\infty} \frac{\log \|PT(t)\|_{\mathscr{L}(Z)}}{t} \leq \omega_0,$$

(iv) if $\mathcal{R}(\lambda) > \omega_0$*, then*

$$PR_\lambda(A)z = \int_0^\infty e^{-\lambda t} PT(t)z dt, \quad z \in Z,$$

where $R_\lambda(A) = (A - \lambda I)^{-1}$*.*

Lemma 2.6. *Under the same scenario as in the previous lemma, we have that:*

(e) If $\omega > \omega_0$*, then there exist* $M_\omega < +\infty$ *such that*

$$\|T(t)\|_{\mathscr{L}(Z)} \leq M_\omega e^{\omega t} \quad \text{and} \quad \|PT(t)\|_{\mathscr{L}(Z)} \leq M_\omega e^{\omega t} \quad t \geq 0,$$

(f) for all $\beta > 0$

$$\int_0^\beta \|PT(t)\|_{\mathscr{L}(Z)} dt < +\infty.$$

The following theorem will be extremely useful for defining the C_0-semigroup which will be used throughout the controllability proofs.

Theorem 2.26. *Let* A *be the infinitesimal generator of a* C_0*-semigroup* $\{T_A(t)\}_{t\geq 0}$ *in* Z*. If* $P \in \mathscr{P}(A)$*, then* $A + P$ *defined on* $\mathcal{D}(A+P) = \mathcal{D}(A)$ *is the infinitesimal generator of a* C_0*-semigroup* $\{T_{A+P}(t)\}_{t\geq 0}$*. Furthermore,*

$$T_{A+P}(t)z = \sum_{n=0}^{+\infty} E_n(t)z, \quad t \geq 0, \tag{2.28}$$

where

$$E_0(t) = T_A(t) \quad \text{and} \quad E_n(t)z = \int_0^t T_A(t-s)PE_{n-1}(s)z ds, \quad n \geq 1, \; z \in Z.$$

and the series (2.28) is absolutely convergent in the uniform norm of $\mathscr{L}(Z)$*, uniformly with respect to the variable* t *in each finite interval. For each* n *and* z *the function* $E_n(t)z$ *is continuous for* $t \geq 0$*.*

The proof of this theorem is based on the previous Lemmas 2.5 and 2.6, and includes the convolution of the semigroup. It is included in detail in the first section of Chapter VIII from [15].

Moreover, the following proposition is needed for establishing an equivalent metric with which the controllability results will be obtained.

Proposition 2.7. *Let* A *be the infinitesimal generator of a* C_0*-semigroup* $\{T_A(t)\}_{t\geq 0}$ *of type* ω_0*. Define the function from* $\mathscr{P}(A) \times \mathscr{P}(A)$ *to* $\mathbb{R}_+ \cup \{0\}$*,*

$$d_A(P_1, P_2) = \int_0^1 \|(P_1 - P_2)T_A(t)\|_{\mathscr{L}(Z)} dt, \quad P_1, P_2 \in \mathscr{P}(A), \tag{2.29}$$

and for a fixed $\omega > \omega_0$ *the function*

$$\delta_A(P_1, P_2) = \int_0^\infty e^{-\omega t} \|(P_1 - P_2)T_A(t)\|_{\mathscr{L}(Z)} dt, \quad P_1, P_2 \in \mathscr{P}(A). \tag{2.30}$$

Then δ_A *and* d_A *are equivalent metrics on* $\mathscr{P}(A)$*, i.e., there exist constants* $M_A, m_A > 0$ *such that*

$$m_A \delta(P_1, P_2) \leq d_A(P_1, P_2) \leq M_A \delta_A(P_1, P_2), \quad \forall P_1, P_2 \in \mathscr{P}(A).$$

Remark 2.12. If $P_1 - P_2$ is bounded, then

$$d_A(P_1, P_2) \leq \|P_1 - P_2\|_{\mathscr{L}(Z)} \int_0^1 \|T_A(t)\|_{\mathscr{L}(Z)} dt.$$

Theorem 2.27. *The mapping $P \longmapsto T_{A+P}(t) \in \mathscr{L}(Z)$, for $P \in \mathscr{P}(A)$, is continuous, i.e.,*

$$\lim_{d_A(P,P_0)\to 0} \|T_{A+P}(t) - T_{A+P_0}(t)\|_{\mathscr{L}(Z)} = 0,$$

uniformly with respect to t in each interval of the form $[0,\beta]$, $\beta > 0$.

Furthermore, if $\delta_A(P,P_0) < 1$, then there exist a constant $M = M_{P_0}$ such that

$$\|T_{A+P}(t) - T_{A+P_0}(t)\|_{\mathscr{L}(Z)} \leq \frac{\delta_A(P,P_0)}{1-\delta_A(P,P_0)} M e^{\omega t}, \quad t \geq 0. \tag{2.31}$$

Remark 2.13. The couple $(\mathscr{P}(A), d_A)$ is a metric space endowed with the metric d_A of formula (2.29).

Now, we are ready to present a result from [32] about unbounded perturbation of the exact controllability of linear systems in infinite-dimensional Banach spaces. This result will be applied in the proof of the controllability of the fractional perturbed wave equation system associated with the semilinear equations (1.6).

Theorem 2.28 (Main result from [32])**.** *If for $P_0 \in (\mathscr{P}(A), d_A)$ the perturbed linear system*

$$z' = (A + P_0)z + B(t)u(t), \quad t > 0, \tag{2.32}$$

is exactly controllable on $[0,\tau]$, then there exists a neighborhood $\mathcal{N}(P_0)$ such that for each $P \in \mathcal{N}(P_0)$ the system

$$z' = (A + P)z + B(t)u(t), \quad t > 0, \tag{2.33}$$

is also exactly controllable on $[0,\tau]$.

Chapter 3

Problem Formulation and Controllability of the Linear Systems

3.1 Abstract formulation

Once all the required theory for proving our main results has been reviewed, in this section the abstract formulation for each problem is presented. At this point, the notation that shall be used throughout the development of the controllability in the semilinear cases in Chapter 4 must be established. Moreover, this setting will be useful for presenting the controllability results of the linear cases in the next section.

By constructing the abstract formulation of problems (1.1) and (1.6), the second order equations can be transformed into dynamical systems of the form:

$$z' = Az + Bu + F(z, u, \ldots),$$

where the notation $'$ regards the derivative with respect to time. Intuitively, this consists of dragging the dependence of the variables of the equation in space towards the co-domain of each functional variable in the abstract counterpart. That is, instead of defining, for example,

$$z : [-r, \tau] \times \Omega \longrightarrow \mathbb{R}, \quad z \in L^2([-r, \tau] \times \Omega),$$

one can consider

$$z : [-r, \tau] \longrightarrow L^2(\Omega), \quad z \in L^2([-r, \tau]).$$

Clearly, these two viewpoints are equivalent. The difference is that the second allows to state the problem as a first order dynamical system in which the derivative is only considered explicitly in time, while the space differentials are "contained" in versions of the Laplace operator.

All definitions presented here will be mainly useful for the approach dealing with Operator Theory. Establishing an abstract formulation in this way is a common practice with this approach for studying controllability properties, and it is mostly used when working with dynamical systems.

3.1.1 Semilinear reaction-diffusion system

To begin with, let us recall some important results from sections 2.2 – 2.4, and establish our notation. From now on, let the negative Laplace operator on space be $A = -\Delta_x = -\Delta$. Consider the expansion given by Corollary 2.1

$$Az = \sum_{n=1}^{+\infty} \lambda_n S_n z = \sum_{n=1}^{+\infty} \lambda_n \sum_{k=1}^{\gamma_n} \left\langle z, \phi_n^k \right\rangle \phi_n^k, \quad z \in \mathcal{D}(A) \tag{3.1}$$

where the sequence $(\lambda_n)_{n\in\mathbb{N}}$ is composed of its eigenvalues, such that

$$0 < \lambda_1 < \lambda_2 < \dots, \qquad \text{and} \qquad \lim_{n\to+\infty} \lambda_n = \infty, \tag{3.2}$$

with geometric multiplicities γ_n, $n \in \mathbb{N}$, respectively. The orthonormal set $\left\{\phi_n^k \;:\; k \in I_{\gamma_n},\; n \in \mathbb{N}\right\}$ comprises the associated eigenvectors.

Hence, $-A$ is the infinitesimal generator of a strongly continuous semigroup $\{T(t)\}_{t\geq 0}$ such that for $t \geq 0$,

$$T(t)z = \sum_{n=1}^{+\infty} e^{-\lambda_n t} S_n z, \quad z \in Z. \tag{3.3}$$

It is not hard to see that the elements of this semigroup are self-adjoint operators. Now, considering the perturbed system (1.1) for $0 < \eta < 1$, and assuming without loss of generality $\lambda_n > 1$, $n \in \mathbb{N}$, it is obtained that

$$0 < \lambda_n - \lambda_n^\eta = \lambda_n^\eta \left(\lambda_n^{1-\eta} - 1\right) =: \nu_n, \quad n \in \mathbb{N}.$$

So, the set $\{\nu_n\}_{n=1}^{+\infty}$ corresponds to the eigenvalues of the perturbed operator

$$A_\eta = A - A^\eta = -(\Delta + (-\Delta)^\eta).$$

In fact, let $z \in Z = \mathcal{D}(A^\eta) \subset L^2(\Omega)$ be an eigenvector of A associated to $\hat{\lambda}_n$, it yields

$$A^\eta z = \hat{\lambda}_n^\eta z,$$

and then

$$A_\eta z = (A - A^\eta)z = Az - A^\eta z = \hat{\lambda}_n z - \hat{\lambda}_n^\eta z = \left(\hat{\lambda}_n - \hat{\lambda}_n^\eta\right) z.$$

In this sense, a new semigroup of strongly continuous operators $\{T_\eta(t)\}_{t\geq 0}$ is generated by $-A_\eta$, with

$$T_\eta(t)z = \sum_{n=1}^{\infty} e^{-\nu_n t} S_n z, \quad z \in Z. \tag{3.4}$$

Moreover, note that since each ν_n increases,

$$\|T_\eta(t)\|_{\mathscr{L}(Z)} \leq e^{-\nu_1 t}, \quad t \geq 0. \tag{3.5}$$

Now, let $U = L^2(\Omega)$. Define the following functions

$$B_\omega : U \longrightarrow Z, \quad f^e,\ \mathcal{I}_k^e : [0,\tau] \times Z \times U \longrightarrow Z$$

for each $k \in I_p$, with formulas

$$\begin{aligned} B_\omega u\ (x) &= \mathbb{1}_\omega(x) u(x), \quad u \in L^2(\Omega), \\ f^e\,(t,\varphi,u)\,(x) &= f\,(t,\varphi(x),u(x))\,, \quad \varphi, u \in L^2(\Omega), \\ \mathcal{I}_k^e\,(t,z,u)\,(x) &= \mathcal{I}_k\,(t,z(x),u(x))\,, \quad z, u \in L^2(\Omega), \end{aligned}$$

whenever $x \in \Omega$. Finally, consider $u \in L^2\left([0,\tau], L^2(\Omega)\right)$, $z \in \mathcal{PC}_p^\tau$. Then, given $t \in [0,\tau]$, denote

$$z[t](s) = z(t+s), \qquad \text{for } s \in [-r, 0],$$

and thus, for $(t,x) \in \Omega_\tau$,

$$\begin{aligned} B_\omega u(t)\ (x) &= \mathbb{1}_\omega(x) u(t,x), \\ f^e\,(t, z[t](-r), u(t))\,(x) &= f\,(t, z(t-r,x), u(t,x))\,, \\ \mathcal{I}_k^e\,(t, z(t), u(t))\,(x) &= \mathcal{I}_k\,(t, z(t,x), u(t,x))\,. \end{aligned}$$

Therefore, the problem given by equation (1.1) can be formulated as an abstract system dependent on time as follows:

$$\begin{cases} z' = -A_\eta z + B_\omega u + f^e\left(\cdot, z[\cdot](-r), u(\cdot)\right), & \text{on } I' = (0,\tau] \setminus \{t_k\}_{k=1}^p, \\ z = \phi, & \text{on } [-r,0], \\ z\left(t_k^+\right) = z\left(t_k^-\right) + \mathcal{I}_k^e\left(t_k, z(t_k), u(t_k)\right), & k \in I_p, \end{cases} \tag{3.6}$$

with $z \in \mathcal{PC}_p^\tau$, $u \in L^2\left([0,\tau], L^2(\Omega)\right)$ and $\phi \in \mathcal{PC}_{pr}$.

Similarly as in [38] and [8], the next proposition holds for the bounding conditions presented in Section 1.2.1 over the functions defined above.

Proposition 3.1. *Let statements of hypothesis (II) be satisfied. Then, for any $z, u \in Z = L^2(\Omega)$ and $\varphi \in \mathcal{PC}_{pr} \subset L^2(\Omega)$, the following conditions are satisfied for functions f^e, $\mathcal{I}_k^e$ stated before:*

$$\|f^e(t,\varphi(t),u)\|_{L^2(\Omega)} \leq \tilde{a}_0 \|\varphi(-r)\|_{L^2(\Omega)}^{\alpha_0} + \tilde{b}_0 \|u\|_{L^2(\Omega)}^{\beta_0} + \tilde{c}_0, \tag{3.7}$$

$$\|\mathcal{I}_k^e(t,z(t),u)\|_{L^2(\Omega)} \leq \tilde{a}_k \|z\|_{L^2(\Omega)}^{\alpha_k} + \tilde{b}_k \|u\|_{L^2(\Omega)}^{\beta_k} + \tilde{c}_k, \quad k \in I_p. \tag{3.8}$$

Proof. Take some arbitrary elements $t \in (0,\tau]$, $\varphi \in PC([-r,0],Z)$, $u \in L^2([0,\tau],Z)$. Then, it yields that

$$\begin{aligned} \|f^e(t,\varphi(t),u)\|_{L^2(\Omega)}^2 &= \int_\Omega |f(t,\varphi(t,x),u(x))|^2\, dx \\ &\leq \int_\Omega \Big(a_0 |\varphi(-r,x)|^{\alpha_0} + b_0 |u(x)|^{\beta_0} + c_0\Big)^2 dx \\ &\leq \int_\Omega 4 \left(a_0^2 |\varphi(-r,x)|^{2\alpha_0} + \Big(b_0 |u(x)|^{\beta_0} + c_0\Big)^2\right) dx \\ &\leq 4a_0^2 \int_\Omega |\varphi(-r,x)|^{2\alpha_0}\, dx + 8b_0^2 \int_\Omega |u(x)|^{2\beta_0}\, dx + 8c_0^2 \int_\Omega 1 dx \end{aligned}$$

So,

$$\begin{aligned} \|f^e(t,\varphi(t),u)\|_{L^2(\Omega)} &\leq 2a_0 \left(\int_\Omega |\varphi(-r,x)|^{2\alpha_0}\, dx\right)^{\frac{1}{2\alpha_0}\alpha_0} \\ &\quad + \sqrt{8} b_0 \left(\int_\Omega |u(x)|^{2\beta_0}\, dx\right)^{\frac{1}{2\beta_0}\beta_0} + \sqrt{8} c_0 \sqrt{\mu(\Omega)} \\ &= 2a_0 \|\varphi(-r)\|_{L^{2\alpha_0}(\Omega)}^{\alpha_0} + \sqrt{8} b_0 \|u\|_{L^{2\beta_0}(\Omega)}^{\beta_0} + \sqrt{8} c_0 \sqrt{\mu(\Omega)} \end{aligned}$$

Finally, since

$$\frac{1}{2} \leq \alpha_0 < 1 \implies 1 \leq 2\alpha_0 < 2 \quad \text{and} \quad \frac{1}{2} \leq \beta_0 < 1 \implies 1 \leq 2\beta_0 < 2,$$

Proposition 2.5 yields that

$$\begin{aligned} \|f^e(t,\varphi(t),u)\|_{L^2(\Omega)} &\leq 2a_0 \mu(\Omega)^{\frac{1-\alpha_0}{\alpha_0}} \|\varphi(-r)\|_{L^2(\Omega)}^{\alpha_0} \\ &\quad + \sqrt{8} b_0 \mu(\Omega)^{\frac{1-\beta_0}{\beta_0}} \|u\|_{L^2(\Omega)}^{\beta_0} + \sqrt{8} c_0 \sqrt{\mu(\Omega)} \\ &= \tilde{a}_0 \|\varphi(-r)\|_{L^2(\Omega)}^{\alpha_0} + \tilde{b}_0 \|u\|_{L^2(\Omega)}^{\beta_0} + \tilde{c}_0, \end{aligned}$$

Likewise, the next estimation is obtained:

$$\begin{aligned} &\|\mathcal{I}_k^e(t,z,u)\|_{L^2(\Omega)} \\ &\quad \leq 2a_k \mu(\Omega)^{\frac{1-\alpha_k}{\alpha_k}} \|z\|_{L^2(\Omega)}^{\alpha_k} + \sqrt{8} b_k \mu(\Omega)^{\frac{1-\beta_k}{\beta_k}} \|u\|_{L^2(\Omega)}^{\beta_k} + \sqrt{8} c_k \sqrt{\mu(\Omega)} \\ &\quad = \tilde{a}_k \|z\|_{L^2(\Omega)}^{\alpha_k} + \tilde{b}_k \|u\|_{L^2(\Omega)}^{\beta_k} + \tilde{c}_k, \quad k \in I_p. \end{aligned}$$

□

Remark 3.1. Note that $\|B_\omega\|_{\mathscr{L}(U,Z)} = 1$.

In fact, consider a generic $u \in U = L^2(\Omega)$ with $\|u\|_{L^2(\Omega)} = 1$. It follows that

$$\|B_\omega u\|^2_{L^2(\Omega)} = \int_\Omega |\mathbb{1}_\omega(x)u(x)|^2 \, dx = \int_\omega |u(x)|^2 \, dx \le \int_\Omega |u(x)|^2 \, dx = 1 \cdot \|u\|^2_{L^2(\Omega)} .$$

So, by definition, $\|B_\omega\|_{\mathscr{L}(U,Z)} \le 1$.

Now, since Ω is bounded, setting $l = \mu(\omega) < +\infty$, for $\tilde{u} = \dfrac{1}{\sqrt{l}}\mathbb{1}_\Omega \in L^2(\Omega)$, one gets that

$$\|B_\omega\|_{\mathscr{L}(U,Z)} = \sup_{\|u\|=1} \|B_\omega u\|_{L^2(\Omega)} \ge \|B_\omega \tilde{u}\|_{L^2(\Omega)} = \int_\Omega |\mathbb{1}_\omega(x)\tilde{u}(x)|^2 \, dx = \frac{1}{l}\int_\omega dx = \frac{1}{l} l = 1,$$

and the remark follows.

3.1.2 Semilinear perturbed wave equation system

Similarly as in the previous section, we shall define the notation on the specific operators to obtain a suitable formulation that allows us to reach the controllability result by means of fixed point theorems in Banach spaces.

Using the change of variable $v = y'$, we transform the second order equation (1.6) into the following first order system of ordinary differential equations with impulses, delays, and non-local conditions.

Recalling the spaces definitions from Section 2.1.1, denote in matrix notation

$$\mathbf{z} = \begin{bmatrix} y \\ v \end{bmatrix}, \quad \mathcal{A} = \begin{bmatrix} 0 & \mathbf{I}_Z \\ -A & 0 \end{bmatrix}, \quad \mathcal{P} = \begin{bmatrix} 0 & 0 \\ A^{1/2} & 0 \end{bmatrix}, \quad \mathcal{B} = \begin{bmatrix} 0 \\ \mathbb{1}_\omega \end{bmatrix}, \quad \boldsymbol{\rho} = \begin{bmatrix} \rho_1 \\ \rho_2 \end{bmatrix}. \tag{3.9}$$

for $A = -\Delta : \mathcal{D}(A) \subset L^2(\Omega) \longrightarrow L^2(\Omega)$, the negative of the Laplace operator. Recall the definition of the following spaces:

$$\mathcal{Z}^{1/2} = Z^{1/2} \times Z := \mathcal{D}(A^{1/2}) \times L^2(\Omega),$$

with

$$\mathcal{D}(A^{1/2}) = \left\{ z \in L^2(\Omega) \ : \ \sum_{n=1}^{+\infty} \lambda_n \|S_n z\|^2_{L^2(\Omega)} < +\infty \right\}.$$

Also, define the functions, for $U = L^2(\Omega)$,

$$\begin{aligned}
\mathcal{J}_k : [0,\tau] \times \mathcal{Z}^{1/2} &\longrightarrow \quad \mathcal{Z}^{1/2} \\
(t,\varphi) &\longmapsto \begin{bmatrix} 0 \\ J_k\left(t, \varphi_1(t), \varphi_2(t)\right) \end{bmatrix}, \\
\mathcal{F} : [0,\tau] \times \mathcal{PW}_{mr} &\longrightarrow \quad \mathcal{Z}^{1/2} \\
(t,\varphi) &\longmapsto \begin{bmatrix} 0 \\ f\left(t, \varphi_1(t), \varphi_2(t)\right) \end{bmatrix}, \\
\boldsymbol{G} : \quad \mathcal{PW}^q_{mr} &\longrightarrow \quad \mathcal{PW}_{mr} \\
\left(\varphi^k\right)_{k=1}^q &\longmapsto \begin{bmatrix} h_1\left(\varphi_1^1(s,\cdot), \ldots, \varphi_1^q(s,\cdot)\right) \\ h_2\left(\varphi_2^1(s,\cdot), \ldots, \varphi_2^q(s,\cdot)\right) \end{bmatrix},
\end{aligned}$$

Thus, system (1.6) can be understood as the following first order system:

$$\begin{cases} \mathbf{z}' = (\mathcal{A} + \epsilon\mathcal{P})\,\mathbf{z} + \mathcal{B}u + \mathcal{F}\left(\cdot, \mathbf{z}[\cdot](-r)\right), & \text{on } (0,\tau] \backslash \{s_k\}_{k=1}^m , \\ \mathbf{z}(s) + \boldsymbol{G}\left(\mathbf{z}_{\theta_1}, \ldots, \mathbf{z}_{\theta_q}\right)(s) = \boldsymbol{\rho}(s), & s \in [-r, 0], \\ \mathbf{z}(s_k^+) = \mathbf{z}(s_k^-) + \mathcal{J}_k\left(s_k, \mathbf{z}(s_k)\right), & k \in I_m \end{cases} \tag{3.10}$$

Proposition 3.2. *The operator $P_j : \mathcal{Z}^{1/2} \longrightarrow \mathcal{Z}^{1/2}$, for each $n \in \mathbb{N}$, defined in matrix notation as*

$$P_j = \begin{bmatrix} S_j & 0 \\ 0 & S_j \end{bmatrix} \tag{3.11}$$

is a continuous and bounded orthogonal projections in the Hilbert space $\mathcal{Z}^{1/2}$.

Proof. Firstly, it will be showed that $P_j\left(\mathcal{Z}^{1/2}\right) \subset \mathcal{Z}^{1/2}$, which is equivalent to show that $S_j\left(Z^{1/2}\right) \subset Z^{1/2}$. In fact, let z be an arbitrary element in $Z^{1/2}$ and consider the eigenvalues of the Laplace operator $\{\lambda_n\}_{n=1}^{+\infty}$

$$\|S_j z\|_{Z^{1/2}}^2 = \sum_{n=1}^{+\infty} \lambda_n \|S_n S_j z\|_{L^2(\Omega)}^2 = \lambda_j \|S_j z\|_{L^2(\Omega)}^2 < +\infty.$$

Therefore, $S_j z \in Z^{1/2}$, for all $z \in Z^{1/2}$. Now, let us verify that this projection is bounded. In fact, from the continuous inclusion $D\left(A^{1/2}\right) = Z^{1/2} \subset Z = L^2(\Omega)$, there exists a constant $c > 0$ such that

$$\|z\|_{L^2(\Omega)} \leq C\|z\|_{Z^{1/2}}, \quad z \in Z^{1/2}.$$

Then, for all $z \in Z^{1/2}$, we have the following estimate, thanks to the boundedness of each S_j,

$$\|S_j z\|_{Z^{1/2}}^2 = \sum_{n=1}^{+\infty} \lambda_n \|S_n S_j z\|_{L^2(\Omega)}^2 = \lambda_j \|S_j z\|_{L^2(\Omega)}^2 \leq \lambda_j \|z\|_{L^2(\Omega)}^2 \leq \lambda_j C^2 \|z\|_{Z^{1/2}}^2.$$

Thus, this implies the continuity of $S_j : Z^{1/2} \longrightarrow Z^{1/2}$. So, P_j is a continuous operator on $\mathcal{Z}^{1/2}$. Finally, it is not hard to see that $P_j P_j = P_j$ and $P_i P_j = \mathbf{0}$ if $i \neq j$. So each P_j is a projection. □

The following theorem is direct result from the work done by [31]. This is the basis for developing our results in exact controllability.

Theorem 3.1. *The operator $\mathcal{A}$ given in (3.9) is the infinitesimal generator of a strongly continuous semigroup $\{T_{\mathcal{A}}(t)\}_{t\geq 0}$ given by*

$$T_{\mathcal{A}}(t)\mathbf{z} = \sum_{j=1}^{+\infty} e^{A_j t} P_j \mathbf{z}, \quad t \geq 0, \ \mathbf{z} \in \mathcal{Z}^{1/2}, \tag{3.12}$$

where $\{P_j\}_{j\in\mathbb{N}}$ is a complete family of orthogonal projections in the Hilbert space $\mathcal{Z}^{1/2}$ given by

$$P_j = \begin{bmatrix} S_j & 0 \\ 0 & S_j \end{bmatrix}, \quad j \in \mathbb{N}, \tag{3.13}$$

and

$$A_j = R_j P_j, \quad R_j = \begin{bmatrix} 0 & 1 \\ -\lambda_j & 0 \end{bmatrix}, \quad j \in \mathbb{N}. \tag{3.14}$$

3.2 Controllability of the linear systems

3.2.1 Heat equation

Now, it is appropriate to outline the results on the controllability of the linear reaction-difussion equation. As schemed previously, most of the proofs of the controllability of semilinear systems lie in the fact that the linear case can also be controllable and that the non-linear terms play the role of *bounded* perturbations under which controllability is preserved.

Given some $z_0 \in Z$, it is important to state that the following linear system

$$\begin{cases} \dfrac{\partial z}{\partial t}(t,x) = \Delta_x z(t,x) + \mathbb{1}_\omega(x)u(t,x), & (t,x) \in \Omega_\tau, \\ z(t,\cdot) = z_0, & t = 0, \\ z(t,x) = 0, & (t,x) \in (\partial\Omega)_\tau, \end{cases} \tag{3.15}$$

has been proven to be non-exactly controllable in the sense of Definition 2.10 [2]. Therefore, the controllability proofs constructed upon can not be of this specific type of control.

Hereby, the details for the approximate controllability of the linear system with delay

$$\begin{cases} \dfrac{\partial z}{\partial t}(t,x) = \Delta_x z(t,x) + \mathbb{1}_\omega(x)u(t,x) & (t,x) \in \Omega_\tau, \\ z(t,x) = \phi(t,x), & (t,x) \in \Omega_{-r}, \\ z(t,x) = 0, & (t,x) \in (\partial\Omega)_\tau, \end{cases} \tag{3.16}$$

will be given. The main objective is to make a contrast between these two approaches.

Approach from Operator Theory

By using the same notation as in the abstract formulation given by (3.6), a linear system of an abstract differential equation on $[-r,\tau]$ is obtained:

$$\begin{cases} z' = -Az + B_\omega u, & \text{on } (0,\tau], \\ z = \phi, & \text{on } [-r,0] \end{cases} \tag{3.17}$$

where $z \in C([0,\tau],Z)$, and $A = -\Delta$ is regarded as the abstraction of the unperturbed Laplace operator, generator of the C_0-semigroup $\{T(t)\}_{t\geq 0}$.

To begin with, consider the following *controllability* operator:

$$\begin{aligned} \mathcal{G} : L^2(\Omega)([0,\tau],U) &\longrightarrow \quad Z \\ u \quad &\longmapsto \mathcal{G}u = \int_0^\tau T(\tau - s)B_\omega u(s)ds \end{aligned} \tag{3.18}$$

Hence, the *Grammian* associated to the control heat equation (3.17) will be defined by means of the adjoint of the controllability operator:

$$\begin{aligned} \mathcal{W} : Z &\longrightarrow \quad Z \\ z &\longmapsto \mathcal{W}z = \mathcal{G}\mathcal{G}^* z, \end{aligned} \tag{3.19}$$

i.e.,

$$\mathcal{W}z = \int_0^\tau T(\tau - s)B_\omega B_\omega^* T^*(t-s)zds, \quad z \in Z.$$

Forthwith, the approximate controllability of system (3.16) is characterized in the following theorem. The main strategy presented in [38] is to prove the controllability of system (3.17) by using Lemma 2.2.

Theorem 3.2. *Consider $\phi \in C([-r,0],Z)$ and $z_f \in Z$, initial and final states. Then, system (3.16) is approximately controllable on $[0,\tau]$.*

Proof. Firstly, let us note that the adjoint of $\mathcal{G}$ is injective. In fact, assume

$$\mathcal{G}^* z = 0, \quad z \in Z.$$

Note that

$$B_\omega^* T^*(\tau - s)z = 0, \quad s \in [0,\tau]$$

implies, for $\nu = \tau - s$,

$$B_\omega^* T^*(\nu) = 0, \quad \nu \in [0,\tau].$$

Then, for $\nu \in [0,\tau]$, and since each $T(t)$, $t \geq 0$, is self-adjoint,

$$0 = B_\omega^* T^*(\nu) = B_\omega^* \sum_{n=1}^{+\infty} e^{-\lambda_n \nu} S_n z = \sum_{n=1}^{+\infty} e^{-\lambda_n \nu} \sum_{k=1}^{\gamma_n} \left\langle z, \phi_n^k \right\rangle \mathbb{1}_\omega \phi_n^k.$$

Thus,

$$\sum_{n=1}^{+\infty} e^{-\lambda_n \nu} \sum_{k=1}^{\gamma_n} \left\langle z, \phi_n^k \right\rangle \phi_n^k(x) = 0, \quad x \in \omega \subset \Omega, \ \nu \in [0,\tau]. \tag{3.20}$$

Now, for each $n \in \mathbb{N}$, denote $E_n(x) = \sum_{k=1}^{\gamma_n} \left\langle z, \phi_n^k \right\rangle \phi_n^k(x)$. By analyticity, equation (3.20) can be expanded to all $\nu \geq 0$. Then, since $\lambda_1 > 0$, and setting $\lambda_n = \lambda_1 + \sigma_n$ for some $\sigma_n > 0$,

$$e^{-\lambda_1 \nu} \left[E_1(x) + \sum_{n=2}^{+\infty} e^{-\sigma_n \nu} E_n(x) \right] = 0, \quad x \in \omega \subset \Omega, \ \nu \geq 0.$$

So,

$$E_1(x) + \sum_{n=2}^{+\infty} e^{-\sigma_n \nu} E_n(x) = 0, \quad x \in \omega \subset \Omega, \ \nu \geq 0,$$

and passing to the limit,

$$E_1(x) + \lim_{\nu \to +\infty} \sum_{n=2}^{+\infty} e^{-\sigma_n \nu} E_n(x) = E_1(x) = 0, \quad x \in \omega \subset \Omega.$$

In the same fashion for each $n \in \mathbb{N}$, it can be obtained that,

$$E_n(x) = \sum_{k=1}^{\gamma_n} \left\langle z, \phi_n^k \right\rangle \phi_n^k(x) = 0, \quad x \in \omega \subset \Omega. \tag{3.21}$$

Define

$$\begin{aligned} f_n : \Omega &\longrightarrow \quad \mathbb{R} \\ x &\longmapsto f_n(x) = E_n(x) = \sum_{k=1}^{\gamma_n} \left\langle z, \phi_n^k \right\rangle \phi_n^k(x). \end{aligned}$$

Since f_n can be understood linear combination of eigenvectors from the eigenspace associated to λ_n, with (3.21), the following system can be formulated at each $n \in \mathbb{N}$,

$$\begin{cases} (\Delta_x + \lambda_n \mathbf{I}) f_n \equiv 0, & \text{in } \Omega, \\ f_n = 0, & \text{in } \omega. \end{cases} \tag{3.22}$$

Then, from the classical unique continuation principle for elliptic equations, (3.21) is valid for all points in Ω, i.e.,

$$\sum_{k=1}^{\gamma_n} \left\langle z, \phi_n^k \right\rangle \phi_n^k(x) = 0, \quad x \in \Omega,$$

and since each ϕ_n^k is orthonormal,

$$\forall \in n \in \mathbb{N}, \ \forall k \in I_{\gamma_n} : \quad \left\langle z, \phi_n^k \right\rangle = 0.$$

Hence,

$$z = \sum_{n=1}^{+\infty} \lambda_n \sum_{k=1}^{\gamma_n} \left\langle z, \phi_n^k \right\rangle \phi_n^k = 0.$$

Therefore, because z was chosen arbitrarily, it has been shown that $\mathcal{G}^*$ is injective. By Lemma 2.6,

$$\overline{\mathrm{Im}(\mathcal{G})} = Z. \tag{3.23}$$

Now, note that, for some control $u \in U$, the mild solution of (3.17) on $[0, \tau]$ is given by the formula

$$z(t) = z_{u,\phi}(t) = T(t)\phi(t) + \int_0^t T(t-s)B_\omega u(s)ds.$$

In fact, whenever $t \in [0, \tau]$,

$$\begin{aligned} z'(t) &= \frac{d}{dt}T(t)\phi(t) + \frac{d}{dt}\int_0^t T(t-s)B_\omega u(s)ds \\ &= AT(t)\phi(t) + A\int_0^t T(t-s)B_\omega u(s)ds + T(t-t)B_\omega u(t) \\ &= Az(t) + B_\omega u(t). \end{aligned}$$

In this sense, consider $\varepsilon > 0$. For $z_f - T(\tau)\phi(T) \in Z$, because of (3.23), there exists a control $u_\varepsilon \in U$ such that

$$\|z_f - T(\tau)\phi(T) - \mathcal{G}u_\varepsilon\|_Z < \varepsilon. \tag{3.24}$$

Forthwith, consider the mild solution z_ε dependent on u_ε and ϕ. It follows that

$$\begin{aligned} \|z_f - z_\varepsilon(\tau)\|_Z &= \left\| z_f - T(\tau)\phi(\tau) - \int_0^\tau T(\tau-s)B_\omega u(s)ds \right\|_Z \\ &= \|z_f - T(\tau)\phi(\tau) - \mathcal{G}u_\varepsilon\|_Z \\ &< \varepsilon. \end{aligned}$$

In this way, a sequence of controls $(u_{\varepsilon_n})_{n\in\mathbb{N}}$, with $\varepsilon_n = \dfrac{1}{n}$, is obtained such that the set of mild solutions of (3.17) $\{z_{\varepsilon_n}\}_{n\in\mathbb{N}}$ satisfy

$$\lim_{n\to+\infty} z_{\varepsilon_n}(\tau) = z_f.$$

It is immediate to see that the approximate controllability obtained for system (3.17) is equivalent to the approximate controllability of (3.16), and the theorem follows.

□

Variational Approach

The purpose of providing another technique for proving the controllability of the linear systems relies on contrasting the different approaches which depend on the problem formulation. In [54], there exist two approaches given for obtaining the approximate controllability of the linear heat equation (3.16) from a variational perspective.

With this mind, in this section, an alternative proof of the approximate controllability of (3.16) is given. Let us consider an arbitrary desired final state $z_f \in Z$. Define the following associated differential equation system

$$\begin{cases} \dfrac{\partial y}{\partial t}(t,x) + \Delta_x y(t,x) = 0, & (t,x) \in \Omega_\tau, \\ y(\tau,x) = y_f(x), & x \in \Omega, \\ y(t,x) = 0, & (t,x) \in (\partial\Omega)_\tau. \end{cases} \tag{3.25}$$

In contrast to the previous approach, an abstract formulation is not taken into account since the formulation is done in the original equation itself.

For each $\varepsilon > 0$, consider the following functional

$$\begin{aligned} J_\varepsilon : L^2(\Omega) &\longrightarrow \mathbb{R} \\ y_f &\longmapsto \frac{1}{2}\int_{\omega\times[0,\tau]} y^2(t,x)d(t,x) + \varepsilon \|y_f\|_{L^2(\Omega)} - \int_\Omega z_f(x)y_f(x)dx \end{aligned} \tag{3.26}$$

where y is the solution to associated equation (3.25) for the given $y_f \in L^2(\Omega)$. Let us review some of its properties.

Proposition 3.3. *Given $\varepsilon > 0$, the functional J_ε given by (3.26) is convex, continuous and has a point of minimum on $L^2(\Omega)$.*

Proof. Take an arbitrary sequence $\left(y_{f\,n}\right)_{n\in\mathbb{N}} \subset L^2(\Omega)$ of arbitrary final states of (3.16) such that

$$\lim_{n\to+\infty} \|y_{f\,n}\|_{L^2(\Omega)} = +\infty.$$

Note that these correspond to the initial conditions of system (3.25). For each $n \in \mathbb{N}$, denote the normalized element as

$$\eta_{f\,n} = \frac{1}{\|y_{f\,n}\|_{L^2(\Omega)}} y_{f\,n}.$$

Likewise, denote η_n and y_n as the solutions of (3.25) associated to $\eta_{f\,n}$ and $y_{f\,n}$, respectively. Thus, for $n \in \mathbb{N}$,

$$\begin{aligned} &\frac{1}{\|y_{f\,n}\|_{L^2(\Omega)}} J_\varepsilon\left(y_{f\,n}\right) \\ &= \frac{1}{\|y_{f\,n}\|_{L^2(\Omega)}}\left(\frac{1}{2}\int_{\omega\times[0,\tau]} y_n^2(t,x)d(t,x) + \varepsilon\|y_{f\,n}\|_{L^2(\Omega)} - \int_\Omega z_f(x)y_{f\,n}(x)dx\right) \\ &= \frac{\|y_{f\,n}\|_{L^2(\Omega)}}{2}\int_{\omega\times[0,\tau]} \eta_n^2(t,x)d(t,x) - \int_\Omega z_f(x)\eta_{f\,n}(x)dx + \varepsilon \\ &= I_1 - I_2 + \varepsilon \end{aligned} \tag{3.27}$$

Now, assume

$$\liminf_{n\to+\infty} I_1 = \liminf_{n\to+\infty} \frac{\|y_{f\,n}\|_{L^2(\Omega)}}{2}\int_{\omega\times[0,\tau]} \eta_n^2(t,x)d(t,x) = 0. \tag{3.28}$$

Thus, there is some $h_0 \in L^2(\Omega)$ such that

$$\eta_{f_n} \rightharpoonup h_0 \quad \left(\text{ in } L^2(\Omega)\right) \quad \text{as} \quad n \longrightarrow +\infty.$$

Moreover, for the associated solution h of (3.25),

$$\eta_n \rightharpoonup h \quad \left(\text{ in } L^2\left([0,\tau], H_0^1(\Omega)\right)\right) \quad \text{as} \quad n \longrightarrow +\infty,$$

and we have that (3.28) yields

$$\int_{\omega\times(0,\tau)} h^2(t,x)\ d(t,x) \leq \liminf_{n\to+\infty} \int_{\omega\times(0,\tau)} \eta_n^2(t,x)\ d(t,x) = 0.$$

Then, it is clear that $h \equiv 0$ on $\omega \times (0,\tau)$. Further, by the Uniqueness Continuation Principle, we have that $\omega \equiv 0$ on Ω_τ, and so, $h_0 \equiv 0$. Thus,

$$\eta_{f_n} \rightharpoonup 0 \quad \left(\text{ in } L^2(\Omega)\right) \quad \text{as} \quad n \longrightarrow +\infty,$$

and

$$\lim_{n\to+\infty} I_2 = \lim_{n\to+\infty} \int_\Omega z_{f_n}(x)\eta_{f_n}(x)dx = 0.$$

Finally, it follows from this last result, together with (3.27) and (3.28), that

$$\liminf_{n\to+\infty} \frac{1}{\left\|y_{f_n}\right\|_{L^2(\Omega)}} J_\varepsilon\left(y_{f_n}\right) \geq \varepsilon - \liminf_{n\to+\infty} I_2 = \varepsilon.$$

So, it is clear that there exists $m \in \mathbb{N}$ big enough such that, whenever $n \geq m$,

$$\frac{1}{\left\|y_{f_n}\right\|_{L^2(\Omega)}} J_\varepsilon\left(y_{f_n}\right) \geq \varepsilon \quad \Longrightarrow \quad \left|J_\varepsilon\left(y_{f_n}\right)\right| \geq \varepsilon \left\|y_{f_n}\right\|_{L^2(\Omega)}.$$

Therefore, by passing to the limit as $n \longrightarrow +\infty$ it can be concluded that

$$\left|J_\varepsilon\left(y_{f_n}\right)\right| \longrightarrow +\infty \quad \text{as} \quad \left\|y_{f_n}\right\|_{L^2(\Omega)} \longrightarrow +\infty.$$

Finally, if (3.28) does not hold, the result follows similarly since by (3.27),

$$\liminf_{n\to+\infty} \frac{1}{\left\|y_{f_n}\right\|_{L^2(\Omega)}} J_\varepsilon\left(y_{f_n}\right) = +\infty,$$

and then, for some $K > 0$, there is $m \in \mathbb{N}$ big enough such that

$$\frac{1}{\left\|y_{f_n}\right\|_{L^2(\Omega)}} J_\varepsilon\left(y_{f_n}\right) \geq K.$$

At this point, the hypotheses of Theorem 2.10 have been satisfied, and we conclude the existence of a point of minimum $\widehat{y_f} \in L^2(\Omega)$ of J_ε:

$$J_\varepsilon\left(\widehat{y_f}\right) = \min_{y_f \in L^2(\Omega)} J_\varepsilon\left(y_f\right).$$

□

Now, let us establish the analogous version of Theorem 3.2 under this approach.

Theorem 3.3. *Consider $\phi \in C([-r,0],Z)$ and $z_f \in Z$, initial and final states. Then, system (3.16) is approximately controllable on $[0,\tau]$.*

Proof. Consider an arbitrary precision $\varepsilon > 0$. By the last proposition, the functional given by (3.26) in the light of equation (3.25) has a point of minimum $\widehat{y_f}$. For this point of minimum, a solution $\hat{y} \in L^2(\Omega_\tau)$ is associated, such that by definition of J_ε,

$$\forall y_f \in L^2(\Omega),\ \forall \gamma \in \mathbb{R}: \quad J_\varepsilon(\widehat{y_f}) \leq J_\varepsilon(\widehat{y_f} + \gamma y_f).$$

Then, it follows that for some $\gamma > 0$ and $y_f \in L^2(\Omega)$,

$$\begin{aligned}
0 \leq J_\varepsilon(\widehat{y_f} + \gamma y_f) - J_\varepsilon(\widehat{y_f}) &= \frac{1}{2}\int_{[0,\tau]\times\omega} [\widehat{y}(t,x) + \gamma y(t,x)]^2 \ d(t,x) + \varepsilon \|\widehat{y_f} + \gamma y_f\|_{L^2(\Omega)} \\
&- \int_\Omega z_f(x)(\widehat{y_f}(x) + \gamma y_f(x)) \ dx - \frac{1}{2}\int_{[0,\tau]\times\omega} \widehat{y}(t,x)^2 d(t,x) \\
&- \varepsilon \|\widehat{y_f}\|_{L^2(\Omega)} + \int_\Omega z_f(x)\widehat{y_f}(x) \ dx \\
&= \frac{1}{2}\int_{(0,\tau)\times\omega} \left[2\gamma\widehat{y}(t,x)y(t,x) + \gamma^2 y(t,x)^2\right] \ d(t,x) \\
&+ \varepsilon\left\{\|\widehat{y_f} + \gamma y_f\|_{L^2(\Omega)} - \|\widehat{y_f}\|_{L^2(\Omega)}\right\} - \int_\Omega \gamma z_f(x) y_f(x) \ dx.
\end{aligned}$$

Now, note that

$$\begin{aligned}
\|\widehat{y_f} + \gamma y_f\|_{L^2(\Omega)} - \|\widehat{y_f}\|_{L^2(\Omega)} &\leq \|\widehat{y_f}\|_{L^2(\Omega)} + \gamma \|y_f\|_{L^2(\Omega)} - \|\widehat{y_f}\|_{L^2(\Omega)} \\
&= |\gamma| \|y_f\|_{L^2(\Omega)}.
\end{aligned}$$

Then, the previous result implies that

$$\begin{aligned}
0 &\leq \frac{\gamma^2}{2}\int_{(0,\tau)\times\omega} y^2(t,x) \ d(t,x) \\
&+ \gamma\left\{\int_{(0,\tau)\times\omega} \widehat{y}(t,x)y(t,x) \ d(t,x) - \int_\Omega z_f(x)y_f(x) \ dx\right\} + \varepsilon|\gamma| \|y_f\|_{L^2(\Omega)}.
\end{aligned}$$

And thus, after dividing by h, and passing to the limit as $h \longrightarrow 0$,

$$0 \leq \int_{(0,\tau)\times\omega} \widehat{y}(t,x)y(t,x) \ d(t,x) - \int_\Omega z_f(x)y_f(x) \ dx + \varepsilon \|y_f\|_{L^2(\Omega)},$$

yielding

$$\int_\Omega z_f(x)y_f(x) \ dx - \int_{(0,\tau)\times\omega} \widehat{y}(t,x)y(t,x) \ d(t,x) \leq \varepsilon \|y_f\|_{L^2(\Omega)}.$$

In the same fashion, whenever $h < 0$, we have that

$$-\varepsilon \|y_f\|_{L^2(\Omega)} \leq \int_\Omega z_f(x)y_f(x) \ dx - \int_{(0,\tau)\times\omega} \widehat{y}(t,x)y(t,x) \ d(t,x).$$

So,

$$\left|\int_\Omega z_f(x)y_f(x) \ dx - \int_{(0,\tau)\times\omega} \widehat{y}(t,x)y(t,x) \ d(t,x)\right| \leq \varepsilon \|y_f\|_{L^2(\Omega)}. \tag{3.29}$$

Furthermore, taking $u = \widehat{y} \in L^2(\Omega)$ as control yields a solution $z = z_{y,\phi}$ to the system (3.16) such that

$$\frac{\partial z}{\partial t}(t,x) - \Delta_x z(t,x) = \mathbb{1}_\omega \widehat{y}(t,x) \quad (t,x) \in \Omega_\tau. \tag{3.30}$$

Recalling that y is the solution of (3.25), by Green's formula of integration by parts it is clear that

$$\int_{\Omega_\tau} \frac{\partial z}{\partial t}(t,x)y(t,x)\ d(t,x) = \int_\Omega [z(\tau,x)y_f(x) - z(0,x)y(0,x)]\,dx - \int_{\Omega_\tau} z(t,x)\frac{\partial y}{\partial t}(t,x)\ d(t,x),$$

and

$$\begin{aligned}\int_{\Omega_\tau} \Delta_x z(t,x)y(t,x)\ d(t,x) &= \int_{\Omega_\tau} z(t,x)\Delta_x y(t,x)\ d(t,x)\\ &\quad + \int_{(\partial\Omega)_\tau} \left[y(t,x)\frac{\partial z}{\partial \eta}(t,x) - z(t,x)\frac{\partial y}{\partial \eta}(t,x)\right]\ d(t,S(x))\\ &= \int_{\Omega_\tau} z(t,x)\Delta_x y(t,x)\ d(t,x).\end{aligned}$$

Multiplying (3.30) by y, together with these two last results, we have that

$$\begin{aligned}\int_{(0,\tau)\times\omega} \widehat{y}(t,x)y(t,x)\ d(t,x) &= \int_{\Omega_\tau} \mathbb{1}_\omega \widehat{y}(t,x)y(t,x)\ d(t,x)\\ &= \int_\Omega [z(\tau,x)y_f(x) - z(0,x)y(0,x)]\,dx\\ &\quad - \int_{\Omega_\tau} z(t,x)\left[\frac{\partial y}{\partial t}(t,x) + \Delta_x y(t,x)\right]\ d(t,x)\\ &= \int_\Omega [z(\tau,x)y_f(x) - z(0,x)y(0,x)]\,dx.\end{aligned}$$

Finally, replacing in (3.29),

$$\begin{aligned}\left|\int_\Omega [z_f(x) - z(\tau,x)]\,y_f(x)dx\right| &= \left|\int_\Omega z_f(x)y_f(x)\ dx - \int_\Omega z(\tau,x)y_f(x)dx\right|\\ &\leq \varepsilon\,\|y_f\|_{L^2(\Omega)}\,.\end{aligned}$$

So, by the arbitrariness of $y_f \in L^2(\Omega)$, when taking $y_f = z_f - z(\tau)$,

$$\|z_f - z(\tau)\|^2_{L^2(\Omega)} \leq \varepsilon\,\|z_f - z(\tau)\|_{L^2(\Omega)}\,,$$

and thus,

$$\|z_f - z(\tau)\|_{L^2(\Omega)} \leq \varepsilon.$$

Because $\varepsilon > 0$ was chosen arbitrarily, there exists a sequence of controls $(u_n)_{n\in\mathbb{N}}$ which drive system (3.16) to the desired generic final state z_f.

□

Remark 3.2. According to [54], the importance of this variational proof lies on the fact that the same techniques may be used to obtain other types of controllability to the linear heat equation.

3.2.2 Wave equation

In the same fashion as in the previous section, hereby we will present the main notions of the controllability for the linear (unperturbed) wave equation associated to (1.6). In fact, given the initial states $y_0 \in H_0^1(\Omega)$, $y_1 \in L^2(\Omega)$, consider the following system

$$\begin{cases} \frac{\partial^2}{\partial s^2} y(s,x) = \Delta_x y(s,x) + \mathbb{1}_\omega u(s,x), & (s,x) \in \Omega_\tau, \\ y(0,x) = y_0(x), & x \in \Omega \\ \frac{\partial}{\partial s} y(0,x) = y_1(x), & x \in \Omega, \\ y(s,x) = 0, & (s,x) \in (\partial\Omega)_\tau. \end{cases} \tag{3.31}$$

Our aim in this section is to prove the exact controllability of (4.23) by means of the General Method of Calculus of Variations in a constructive manner. Based on the ideas presented in [54], consider the next associated systems:

$$\begin{cases} \frac{\partial^2}{\partial s^2} \psi(s,x) = \Delta_x \psi(s,x), & (s,x) \in \Omega_\tau, \\ \psi(0,x) = \psi_0(x), & x \in \Omega \\ \frac{\partial}{\partial s} \psi(0,x) = \psi_1(x), & x \in \Omega, \\ \psi(s,x) = 0, & (s,x) \in (\partial\Omega)_\tau, \end{cases} \tag{3.32}$$

and

$$\begin{cases} \frac{\partial^2}{\partial s^2} \varphi(s,x) = \Delta_x \varphi(s,x), & (s,x) \in \Omega_\tau, \\ \varphi(\tau,x) = \varphi_0^\tau(x), & x \in \Omega \\ \frac{\partial}{\partial s} \varphi(\tau,x) = \varphi_1^\tau(x), & x \in \Omega, \\ \varphi(s,x) = 0, & (s,x) \in (\partial\Omega)_\tau, \end{cases} \tag{3.33}$$

for $\psi_0, \varphi_0^\tau \in L^2(\Omega)$ and $\psi_1, \varphi_1^\tau \in H^{-1}(\Omega)$ (the dual of $H_0^1(\Omega)$ in the sense of tempered distributions). System (3.32) is simply the homogeneous linear wave equation while system (3.33) is the associated *backwards* homogeneous equation. Now, define the dual product between spaces $X_1 = H_0^1(\Omega) \times L^2(\Omega)$ and $X_2 = L^2(\Omega) \times H^{-1}(\Omega)$ as

$$\begin{aligned} \langle \cdot, \cdot \rangle_{X_2, X_1} : \qquad & X_2 \times X_1 \longrightarrow \mathbb{R} \\ & ((\psi_0, \psi_1), (y_0, y_1)) \longmapsto \langle \psi_1, y_0 \rangle_{H^{-1}, H_0^1} - \int_\Omega \psi_0(x) y_1(x) \, dx \end{aligned}$$

Furthermore, consider the next functional

$$\begin{aligned} \Phi : \qquad & X_2 \longrightarrow \mathbb{R} \\ & (\psi_0, \psi_1) \longmapsto \frac{1}{2} \int_0^\tau \int_\omega |\psi(s,x)|^2 \, dx ds + \langle (\psi_0, \psi_1), (y_0, y_1) \rangle_{X_2, X_1} \end{aligned} \tag{3.34}$$

for ψ, the solution of (3.32) associated to the initial states $\psi_0 \in L^2(\Omega)$ and $\psi_1 \in H^{-1}(\Omega)$. Indeed, it is not hard to see that the critical points of Φ must satisfy the following optimality condition:

$$\begin{aligned} & \exists u \in L^2((0,\tau) \times \omega), \ \forall \psi_0 \in L^2(\Omega), \ \forall \psi_1 \in H^{-1}(\Omega) : \\ & \qquad \int_0^\tau \int_\omega \psi(s,x) u(s,x) \, dx ds = \langle (\psi_0, \psi_1), (y_0, y_1) \rangle_{X_2, X_1}, \end{aligned} \tag{3.35}$$

for ψ the associated solution to (3.32).

Then, according to Lemma 2.2.2 as presented by in [54], to get relation (3.35) for the given of initial data $y_0 \in H_0^1(\Omega)$ and $y_1 \in L^2(\Omega)$ is an equivalent statement to have null controllability of (3.31). And thus, since the equation is linear, the null controllability of the wave equation implies its exact controllability for any desired final state. These ideas yield the following existence theorem.

Theorem 3.4. *Consider initial values $y_0 \in H_0^1(\Omega)$ and $y_1 \in L^2(\Omega)$. Let $\left(\widehat{\psi}_0, \widehat{\psi}_1\right) \in X_2$ be a point of minimum of functional Φ. Then, system (3.31) is null (and exactly) controllable in time τ, and such a control is given by*

$$u = \widehat{\psi}\big|_{\omega}.$$

Finally, the problem of controllability of equation (3.31) is reduced to the minimization of functional Φ. For this purpose, the notion of *observability* of the wave equation gives an important sufficient condition. Indeed, we say that the equation (3.31) is *observable in time* τ if there is a constant $K > 0$ such that:

$$\forall (\psi_0, \psi_1) \in X_2 : \quad \|(\psi_0, \psi_1)\|_{X_2} \leq K \int_0^\tau \int_\omega |\psi(s,x)|^2 \; dx ds, \tag{3.36}$$

where ψ is the solution to (3.32) associated to each ψ_0 and ψ_1.

In a general sense, condition (3.36) is the missing condition for having (3.35), since the other inequality depends on the continuity of the solutions of the homogeneous problem with respect to its initial data. Therefore, the observability of the equation will be satisfied (see [45] and [54]) with a condition in the final time $\tau > 0$. In fact, we require

$$\tau \geq 2\mu(\Omega). \tag{3.37}$$

This last theorem helps us to verify that indeed these notions give the desired controllability result.

Theorem 3.5. *Consider $y_0 \in H_0^1(\Omega)$ and $y_1 \in L^2(\Omega)$. Assume $\tau > 0$ such that equation (3.31) is observable. Thus, it is exactly controllable in time τ.*

This theorem follows as a consequence of Theorem 3.4 from the coercitivity of the functional Φ with aid of Theorem 2.10 (see [54]).

Chapter 4

Controllability of the Perturbed Semilinear Systems

4.1 Semilinear equations

Firstly, let us state some important results about the existence of solutions in equations which depend on non-linear terms in the following sense. Consider the next initial value problem, for Z a Hilbert space:

$$\begin{cases} \frac{d}{dt} z(t) = Az(t) + F(t, z(t)), & t > t_0, \\ z(t_0) = z_0, \end{cases} \tag{4.1}$$

where t_0 is the initial time and $z_0 \in Z$ is the initial state. In this general case, A is the infinitesimal generator of a C_0-semigroup $\{T(t)\}_{t \geq 0}$, and $F : \mathcal{O} \subset \mathbb{R} \times Z \longrightarrow Z$ satisfies a Lipschitz continuity condition.

Now, let us make an important distinction between the next two definitions.

Definition 4.1 (Strong solution)**.** *A* strong *or* classical *solution of system* (4.1) *is a continuous function of the form* $z : [t_0, \tau) \longrightarrow Z$ *such that:*

(i) $z(t_0) = z_0$*,*

(ii) $(t, z(t)) \in \mathcal{O}$*,*

(iii) $z'(t) \in Z$ *and* $z(t) \in \mathcal{D}(A)$*,*

(iv) $z(t)$ *satisfies* (4.1) *for all* $t \in (t_0, \tau)$*.*

Definition 4.2 (Weak solution)**.** *A* weak *or* mild *solution of system* (4.1) *is a function* $z : [t_0, \tau) \longrightarrow Z$ *such that:*

(i) $(t, z(t)) \in \mathcal{O}$*,*

(ii) $f(\cdot, z(\cdot)) \in L^1((t_0, \tau), Z)$*,*

(iii) $\forall t \in (t_0, \tau) : \quad z(t) = T(t - t_0)z_0 + \int_{t_0}^{\tau} T(t - s)F(s, z(s))\, ds.$

The main difference in the notion of mild solution is the fact the when we search for solutions of a system of differential equations, these need not be differentiable and shall only satisfy an equivalent condition that verifies the intrinsic behavior of the system such as condition (*iii*) in Definition 4.2.

In this project, the existence of solutions for the semilinear systems is assumed and not concerned in detail. A development of the proof would be based on the extension of the following theorems.

Theorem 4.1 (Local existence). *Consider an open set Ω in a Hilbert space Z, $z_0 \in \Omega$, and $F : \mathbb{R}\times\Omega \longrightarrow Z$ a continuous function satisfying the following hypothesis, for each $\tau > 0$,*

$$\exists K = K(\tau),\ \forall t \in [0,\tau],\ \forall z, y \in \Omega : \quad \|F(t,z) - F(t,y)\|_Z \leq K\,\|z - y\|_Z\,.$$

Then, for all $\tau > 0$, small-enough, there is a unique mild solution of (4.1) *in* $[0,\tau)$.

Theorem 4.2 (Global existence). *Let $z_0 \in Z$ and $F : \mathbb{R}\times Z \longrightarrow Z$ a continuous function satisfying the following hypothesis, for each $\tau > 0$,*

$$\exists K = K(\tau),\ \forall t \in [0,\tau],\ \forall z, y \in Z : \quad \|F(t,z) - F(t,y)\|_Z \leq K\,\|z - y\|_Z\,.$$

Then, for all $\tau > 0$ there is a unique mild solution of (4.1) *in* $[0,+\infty)$.

Proof of Theorems 4.1 and 4.2. See Section 2 of Chapter 2 in [18] and [9]. □

4.2 Approximate controllability of the semilinear reaction-diffusion system

This section is devoted to obtaining the first part of the main research results of this project, which is the approximate (or interior) controllability for the semilinear reaction diffusion system (1.1). Only the analytical approach regarding fixed-point and Operator Theory is developed in detail.

For achieving this goal, once the abstract formulation of the problem has been given in Section 3.1, and then, some functional operators are defined. With these inputs, the generalization of Rothe's Theorem presented by Isac is applied to prove the existence of a sequence of solutions that approaches the arbitrary final state, thus, approximately controlling the system.

As it has been explained in the foregoing theory, the abstract formulation of the problem allows to consider a relatively simpler version of (1.1). Now, the results of approximate controllability of the reaction-diffusion system will be extended to the semilinear case. As it was mentioned before, the main technique used works via a fixed point problem. The diagram in the next figure illustrates our scenario.

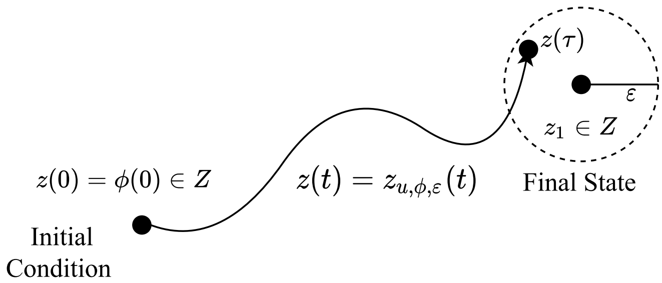

Fig. 4.1: Diagram of the solution $z(t)$ dependent on the control u, the historical pass ϕ and the precision ε, steering the system to an ε-neighborhood of the final state $z_1 \in Z$.

The main references for this results are [8] and [38]. The proof and corollaries of the approximately controllability are given in detail.

Similarly as in Section 3.2.1, for problem (3.6), let us define the non-linear operator

$$\mathcal{L} : \mathcal{D}(\mathcal{L}) = \mathcal{PC}_p^{\tau}([-r,\tau], Z) \times C([0,\tau], U) \ \longrightarrow\ Z,$$

where $Z = \mathcal{D}(A^\eta) \subset L^2(\Omega)$, $U = L^2(\Omega)$, and such that, for $(z,u) \in \mathcal{D}(\mathcal{L})$,

$$\begin{aligned}\mathcal{L}(z,u) = z_f - T_\eta(\tau)\phi - \int_0^\tau T_\eta(\tau - s) f^e(s, z[s](-r), u(s))ds \\ - \sum_{0<t_k<\tau} T_\eta(\tau - t_k)\mathcal{I}_k^e(t_k, z(t_k), u(t_k)).\end{aligned} \tag{4.2}$$

Let us see that this operator is continuous[1]. In fact, for $(z,u) \in \mathcal{D}(\mathcal{L})$, consider a neighborhood $\mathcal{V} \subset \mathcal{D}(\mathcal{L})$, then for $(h,u_h) \in \mathcal{D}(\mathcal{L})$ such that $(z,u)+(h,u_h) = (z+h, u+u_h) \in \mathcal{V}$,

$$
\begin{aligned}
&\|\mathcal{L}(z+h,u+u_h) - \mathcal{L}(z,u)\| \\
&\qquad \leq \int_0^\tau \|T_\eta(\tau-s)\left\{f^e(s,(z+h)[s](-r),(u+u_h)(s)) - f^e(s,z[s](-r),u(s))\right\}\| \, ds \\
&\qquad + \sum_{0<t_k<\tau} \|T_\eta(\tau-t_k)\left\{\mathcal{I}_k^e(t_k,(z+h)(t_k),(u+u_h)(t_k)) - \mathcal{I}_k^e(t_k,z(t_k),u(t_k))\right\}\| \\
&\qquad \leq \int_0^\tau e^{-\nu_1(t-s)} \|f^e(s,(z+h)[s](-r),(u+u_h)(s)) - f^e(s,z[s](-r),u(s))\| \, ds \\
&\qquad + \sum_{0<t_k<\tau} e^{-\nu_1(t-s)} \|\mathcal{I}_k^e(t_k,(z+h)(t_k),(u+u_h)(t_k)) - \mathcal{I}_k^e(t_k,z(t_k),u(t_k))\|
\end{aligned}
$$

So, by the continuity of functions f^e and $\mathcal{I}_k^e$, $k \in I_p$,

$$\lim_{(h,u_h)\to \mathbf{0}} \mathcal{L}(z+h,u+u_h) = \mathcal{L}(z,u).$$

Now, let us introduce the controllability operator associated to (3.6). With the same notions used in Section 3.1.1, consider

$$
\begin{aligned}
\mathcal{G}_\eta : C([0,\tau],U) &\longrightarrow Z \\
u &\longmapsto \mathcal{G}_\eta u = \int_0^\tau T_\eta(\tau-s)B_\omega u(s)ds,
\end{aligned}
$$

as well as the Grammian operator

$$
\begin{aligned}
\mathcal{W}_\eta : Z &\longrightarrow Z \\
z &\longmapsto \mathcal{W}_\eta z = \mathcal{G}_\eta \mathcal{G}_\eta{}^* z.
\end{aligned}
$$

Furthermore, for $\varepsilon > 0$, set

$$
\begin{aligned}
\Gamma_\varepsilon : Z &\longrightarrow C([0,\tau],U) \\
z &\longmapsto \Gamma_\varepsilon z = \mathcal{G}_\eta{}^*(\varepsilon \mathbf{I} + \mathcal{G}_\eta \mathcal{G}_\eta{}^*)^{-1} z.
\end{aligned}
\tag{4.3}
$$

Define the following operator

$$\mathcal{H}^{\eta,\varepsilon} : \mathcal{D}(\mathcal{H}^{\eta,\varepsilon}) = \mathcal{PC}_p^\tau([-r,\tau],Z) \times C([0,\tau],U) \longrightarrow \mathcal{PC}_p^\tau([-r,\tau],Z) \times C([0,\tau],U),$$

such that for each $(z,u) \in \mathcal{D}(\mathcal{H}^{\eta,\varepsilon})$,

$$\mathcal{H}^{\eta,\varepsilon}(z,u) = (\mathcal{H}_1^{\eta,\varepsilon}(z,u), \mathcal{H}_2^{\eta,\varepsilon}(z,u))$$

where $t \in [-r,\tau]$ yields

$$
\mathcal{H}_1^{\eta,\varepsilon}(z,u)(t) = \begin{cases}
T_\eta(t)\phi(t) + \displaystyle\int_0^t T_\eta(t-s)B_\omega\left(\Gamma_\varepsilon \mathcal{L}(z,u)\right)(s)ds & \\
+ \displaystyle\int_0^t T_\eta(t-s)f^e(s,z[s](-r),u(s))ds & \\
+ \displaystyle\sum_{0<t_k<t} T_\eta(t-t_k)\mathcal{I}_k^e(t_k,z(t_k),u(t_k)), & t \in (0,\tau] \\
\phi(t), & t \in [-r,0]
\end{cases}
\tag{4.4}
$$

[1]For the purpose of simplifying the notation, throughout this chapter and the rest of the capstone project, the notation on the norms will only be regarded as $\|\cdot\|$.

and, whenever $t \in [0, \tau]$,

$$\begin{aligned}\mathcal{H}_2^{\eta,\varepsilon}(z,u)(t) &= \left(\Gamma_\varepsilon \mathcal{L}(z,u)\right)(t) \\ &= B_\omega^* T_\eta{}^*(\tau - t)(\varepsilon \mathbf{I} + \mathcal{W}_\eta)^{-1}\mathcal{L}(z,u). \end{aligned} \tag{4.5}$$

Now, consider the following proposition.

Proposition 4.1. *The operator $\mathcal{H}^{\eta,\varepsilon}$ defined by (4.4) – (4.5) is a continuous and compact operator which satisfies that:*

$$\frac{\|\mathcal{H}^{\eta,\varepsilon}(z,u)\|}{\|z,u\|} \longrightarrow 0 \quad \textit{as} \quad \|(z,u)\| \longrightarrow +\infty. \tag{4.6}$$

Proof. Let us verify that $\mathcal{H}^{\eta,\varepsilon}$ is continuous. Consider some arbitrary elements $(z,u),(y,v) \in \mathcal{D}(\mathcal{H}^{\eta,\varepsilon})$. It is clear that if $t \in [-r, 0]$, then,

$$\mathcal{H}_1^{\eta,\varepsilon}(z,u)(t) - \mathcal{H}_1^{\eta,\varepsilon}(y,v) = 0. \tag{4.7}$$

Also, for any $t \in [0,\tau]$, it follows that

$$\begin{aligned}\|\mathcal{H}_1^{\eta,\varepsilon}(z,u)(t) - \mathcal{H}_1^{\eta,\varepsilon}(y,v)(t)\| &\leq \int_0^t \|T_\eta(t-s)\left\{B_\omega\left(\Gamma_\varepsilon\mathcal{L}(z,u)\right)(s) - B_\omega\left(\Gamma_\varepsilon\mathcal{L}(y,v)\right)(s)\right\}\| \, ds \\ &+ \int_0^t \|T_\eta(t-s)\left\{f^e(s,z[s](-r),u(s)) - f^e(s,y[s](-r),v(s))\right\}\| \, ds \\ &+ \sum_{0<t_k<t} \|T_\eta(t-t_k)\left\{\mathcal{I}_k^e(t_k,z(t_k),u(t_k)) - \mathcal{I}_k^e(t_k,y(t_k),v(t_k))\right\}\| \\ &\leq \int_0^t e^{-\nu_1(t-s)}\|B_w\|\,\|\Gamma_\varepsilon\|\,\|\mathcal{L}(z,u) - \mathcal{L}(y,v)\| \, ds \\ &+ \int_0^t e^{-\nu_1(t-s)}\|f^e(s,z[s](-r),u(s)) - f^e(s,y[s](-r),v(s))\| \, ds \\ &+ \sum_{0<t_k<t} e^{-\nu_1(t-s)}\|\{\mathcal{I}_k^e(t_k,z(t_k),u(t_k)) - \mathcal{I}_k^e(t_k,y(t_k),v(t_k))\}\| \\ &\leq \|\Gamma_\varepsilon\| \int_0^t \|\mathcal{L}(z,u) - \mathcal{L}(y,v)\| \, ds \\ &+ \int_0^t \|f^e(s,z[s](-r),u(s)) - f^e(s,y[s](-r),v(s))\| \, ds \\ &+ \sum_{0<t_k<t} \|\{\mathcal{I}_k^e(t_k,z(t_k),u(t_k)) - \mathcal{I}_k^e(t_k,y(t_k),v(t_k))\}\|\end{aligned}$$

Now, taking the uniform norm on $t \in [0,\tau]$, together with (4.7), it yields that

$$\begin{aligned}\|\mathcal{H}_1^{\eta,\varepsilon}(z,u) - \mathcal{H}_1^{\eta,\varepsilon}(y,v)\| &= \sup_{t\in[0,\tau]} \|\mathcal{H}_1^{\eta,\varepsilon}(z,u)(t) - \mathcal{H}_1^{\eta,\varepsilon}(y,v)(t)\| \\ &\leq \|\Gamma_\varepsilon\|\,\tau\,\|\mathcal{L}(z,u) - \mathcal{L}(y,v)\| \\ &+ \sup_{s\in[0,\tau]} \|f^e(s,z[s](-r),u(s)) - f^e(s,y[s](-r),v(s))\| \sup_{t\in[0,\tau]} \int_0^t ds \\ &+ \sup_{s\in[0,\tau]} \|\{\mathcal{I}_k^e(t_k,z(t_k),u(t_k)) - \mathcal{I}_k^e(t_k,y(t_k),v(t_k))\}\| \sup_{t\in[0,\tau]} \sum_{0<t_k<t} 1 \\ &\leq \|\Gamma_\varepsilon\|\,\tau\,\|\mathcal{L}(z,u) - \mathcal{L}(y,v)\| \\ &+ \tau \sup_{s\in[0,\tau]} \|f^e(s,z[s](-r),u(s)) - f^e(s,y[s](-r),v(s))\| \\ &+ p \sup_{s\in[0,\tau]} \|\{\mathcal{I}_k^e(t_k,z(t_k),u(t_k)) - \mathcal{I}_k^e(t_k,y(t_k),v(t_k))\}\|\end{aligned}$$

Then, since the integral terms are working on a compact interval, the continuity of $\mathcal{H}_1^{\eta,\varepsilon}$ follows from the continuity of $\mathcal{L}, f^e$ and $\mathcal{I}_k^e$, $k \in I_p$. In fact, passing to the limit in the last expression as $(y,v) \longrightarrow (z,u)$, it yields

$$\lim_{(y,v)\to(z,u)} \mathcal{H}_1^{\eta,\varepsilon}(y,v) = \mathcal{H}_1^{\eta,\varepsilon}(z,u). \tag{4.8}$$

Furthermore, it can be noted that, whenever $t \in (0,\tau]$,

$$\begin{aligned}\|\mathcal{H}_2^{\eta,\varepsilon}(z,u)(t) - \mathcal{H}_2^{\eta,\varepsilon}(y,v)(t)\| &= \|(\Gamma_\varepsilon \mathcal{L}(z,u))(t) - (\Gamma_\varepsilon \mathcal{L}(y,v))(t)\| \\ &\le \|\Gamma_\varepsilon\| \, \|\mathcal{L}(z,u)(t) - \mathcal{L}(y,v)(t)\| .\end{aligned}$$

Thus, by the continuity of $\mathcal{L}$, we get that $\mathcal{H}_2^{\eta,\varepsilon}$ is continuous at (z,u). Thanks to the arbitrariness of $(z,u) \in \mathcal{D}(\mathcal{H}^{\eta,\varepsilon})$, $\mathcal{H}_1^{\eta,\varepsilon}$ and $\mathcal{H}_2^{\eta,\varepsilon}$ are continuous, and so is the operator $\mathcal{H}^{\eta,\varepsilon}$.

Next, we shall see that $\mathcal{H}^{\eta,\varepsilon}$ is indeed compact. For this purpose, let us consider some bounded subset $B \subset \mathcal{D}(\mathcal{H}^{\eta,\varepsilon})$. Consider the following notation:

$$\|f^e(\cdot, z[\cdot](-r), u)\| = \sup_{t\in[0,\tau]} \|f^e(t, z[t](-r), u(t)\| .$$

Thanks to the conditions from the main hypotheses of the problem, there will exist positive constants l_3, l_4 and $p_j \in \mathbb{R}_+$, $j \in I_p$ such that for $(z,u) \in B$,

$$\begin{aligned}\|f^e(\cdot, z[\cdot](-r), u)\| &\le l_3, \\ \|\mathcal{I}_k^e(\cdot, z[\cdot](-r), u)\| &\le p_k, \quad k \in I_p, \\ \|\mathcal{L}(z,u)\|_Z &\le l_4.\end{aligned}$$

Consequently,

$$\sup_{t\in[0,\tau]} \|\mathcal{H}_1^{\eta,\varepsilon}(z,u)(t)\| \le \|\phi\|_\infty + \tau \|\Gamma_\varepsilon\| \, l_4 + \tau l_3 + \sum_{k=1}^{p} p_k =: C_1$$

and together with (4.7),

$$\|\mathcal{H}_1^{\eta,\varepsilon}(z,u)\| \le C_1. \tag{4.9}$$

Also,

$$\sup_{t\in[0,\tau]} \|\mathcal{H}_2^{\eta,\varepsilon}(z,u)(t)\| \le \|\Gamma_\varepsilon\| \, l_4 = C_2$$

and similarly,

$$\|\mathcal{H}_2^{\eta,\varepsilon}(z,u)\| \le C_2. \tag{4.10}$$

By (4.9) – (4.10), $\mathcal{H}^{\eta,\varepsilon}(B)$ is a uniformly bounded family of functions in $\mathcal{PC}_p^\tau([-r,\tau],Z) \times C([0,\tau],U)$.

At this point, it will be useful to show that $\mathcal{H}^{\eta,\varepsilon}(B)$ is a "piece-wise" equicontinuous family on $[-r,\tau]$, as well. For $(z,u) \in B$, let $t_A, t_B \in [0,\tau]\backslash\{t_k\}_{k=1}^p$ with $0 < t_A < t_B$. It follows that

$$\begin{aligned}&\|\mathcal{H}_1^{\eta,\varepsilon}(z,u)(t_B) - \mathcal{H}_1^{\eta,\varepsilon}(z,u)(t_A)\| \\ &\qquad \le \|T_\eta(t_B) - T_\eta(t_A)\| \, \|\phi\|_\infty + \int_{t_A}^{t_B} \|T_\eta(t_B - s)\| \, \|\Gamma_\varepsilon\| \, l_4 \, ds \\ &\qquad + \int_0^{t_A} \|T_\eta(t_B - s) - T_\eta(t_A - s)\| \, \|\Gamma_\varepsilon\| \, l_4 \, ds + \int_{t_A}^{t_B} \|T_\eta(t_B - s)\| \, l_3 \, ds \\ &\qquad + \int_0^{t_A} \|T_\eta(t_B - s) - T_\eta(t_A - s)\| \, l_3 \, ds + \sum_{\substack{k\in I_p \\ t_A \le t_k < t_B}} \|T_\eta(t_B - t_k)\| \, p_k \\ &\qquad + \sum_{\substack{k\in I_p \\ 0 < t_k < t_A}} \|T_\eta(t_B - t_k) - T_\eta(t_A - t_k)\| \, p_k\end{aligned}$$

$$
\begin{aligned}
&\leq \|T_\eta(t_B) - T_\eta(t_A)\| \, \|\phi\|_\infty + (\|\Gamma_\varepsilon\| \, l_4 + l_3) \, (t_B - t_A) \\
&+ (\|\Gamma_\varepsilon\| \, l_4 + l_3) \int_0^{t_A} \|T_\eta(t_B - s) - T_\eta(t_A - s)\| \, ds \\
&+ \sum_{t_A \leq t_k < t_B} \|T_\eta(t_B - t_k)\| \, \overline{p} + \sum_{0 < t_k < t_A} \|T_\eta(t_B - t_k) - T_\eta(t_A - t_k)\| \, \overline{p}
\end{aligned} \tag{4.11}
$$

where $\overline{p} = \max_{k \in I_p} \{p_k\}$. Furthermore, it is directly verifiable that

$$
\|\mathcal{H}_2^{\eta,\varepsilon}(z,u)(t_B) - \mathcal{H}_2^{\eta,\varepsilon}(z,u)(t_A)\| \leq \|B_\omega^*\| \, \|T_\eta{}^*(\tau - t_B) - T_\eta{}^*(\tau - t_A)\| \, \left\|(\varepsilon \mathbf{I} + \mathcal{W}_\eta)^{-1}\right\| \, l_4 \tag{4.12}
$$

Approaching by $t_B \longrightarrow t_A$, and because the mapping $t \longmapsto T(t),\ t \geq 0$ is uniform continuous away from zero, it yields together with (4.11) that the family $\mathcal{H}^{\eta,\varepsilon}(B)$ is equicontinuous up to a number of points p, i.e., when restricted to the interval $[0,\tau] \backslash \{t_k\}_{k=1}^p$. Moreover, by (4.8), $\mathcal{H}^{\eta,\varepsilon}(B)$ is also equicontinuous on $[-r,0]$.

In this sense, a sequence $(\omega_n)_{n\in\mathbb{N}}$ in $\mathcal{H}^{\eta,\varepsilon}(B)$ can be considered, for each $n \in \mathbb{N}$, such that there exists $(z_n, u_n) \in B$ with

$$
\mathcal{H}^{\eta,\varepsilon}(z_n, u_n) = \omega_n \in \mathcal{PC}_p^\tau([-r,\tau], Z) \times C([0,\tau], U).
$$

Note that $\left(\omega_n\big|_{[0,t_1]}\right)_{n\in\mathbb{N}}$ is a sequence in $C([-r,t_1], Z)$, and such that it is uniformly bounded and equicontinuous. Then, Arzelà-Ascoli Theorem yields that there exists a subsequence

$$
\left(\omega_n^1\right)_{n\in\mathbb{N}} \subseteq (\omega_n)_{n\in\mathbb{N}}
$$

convergent on the closed interval $[-r,t_1]$. Moreover, the restrictions of the elements of this sequence make another uniformly bounded and equicontinuous sequence on the interval $[t_1,t_2]$. Similarly, there exists a subsequence

$$
\left(\omega_n^2\right)_{n\in\mathbb{N}} \subseteq \left(\omega_n^1\right)_{n\in\mathbb{N}}
$$

convergent on $[t_1,t_2]$. Since $(\omega_n^1)_{n\in\mathbb{N}}$ was convergent on t_1, t_2 so is the last subsequence obtained. Following on this process for the closed intervals $[t_2,t_3], \ldots, [t_p,\tau]$, a final subsequence is obtained

$$
(\omega_n^p)_{n\in\mathbb{N}} \subseteq (\omega_n)_{n\in\mathbb{N}}
$$

which is convergent on the whole interval $[-r,\tau]$. Since the initial sequence was taken as a generic element, it can be stated that $\mathcal{H}^{\eta,\varepsilon}(B)$ is a relatively compact set. Thus, by definition, $\mathcal{H}^{\eta,\varepsilon}$ is a compact operator due to the arbitrariness of the bounded set B, as well.

Finally, let us prove (4.6). Consider an arbitrary $(z,u) \in \mathcal{D}(\mathcal{H}^{\eta,\varepsilon})$. It follows that, wherever $t \in [0,\tau]$

$$
\begin{aligned}
\|\mathcal{H}_1^{\eta,\varepsilon}(z,u)(t)\| &\leq e^{-\nu_1 t} \|\phi\|_\infty + \int_0^t e^{-\nu_1(t-s)} \|B_\omega\| \, \|\Gamma_\varepsilon\| \, \|\mathcal{L}(z,u)\| \, ds \\
&+ \int_0^t e^{-\nu_1(t-s)} \|f^e(s, z[s](-r), u(s))\| \, ds + \sum_{k\in I_p} e^{-\nu_1(t-t_k)} \|\mathcal{I}_k^e(t_k, z(t_k), u(t_k))\| \\
&\leq \|\phi\|_\infty + \tau \|\Gamma_\varepsilon\| \, \|\mathcal{L}(z,u)\| + \left(\tilde{a}_0 \|z(-r)\|^{\alpha_0} + \tilde{b}_0 \|u\|^{\beta_0} + \tilde{c}_0\right) \\
&+ \sum_{k\in I_p} \left(\tilde{a}_k \|z\|^{\alpha_k} + \tilde{b}_k \|u\|^{\beta_k} + \tilde{c}_k\right).
\end{aligned} \tag{4.13}
$$

Moreover,

$$\begin{aligned}\|\mathcal{L}(z,u)\| &\leq \|z_f\| + e^{-\nu_1\tau} + \int_0^\tau e^{-\nu_1(\tau-s)}\, \|f^e(s,z[s](-r),u(s))\|\, ds \\ &\quad + \sum_{k\in I_p} e^{-\nu_1(\tau-t_k)}\, \|\mathcal{I}_k^e(t_k,z(t_k),u(t_k))\| \\ &\leq \|z_f\| + e^{-\nu_1\tau} + \tau\left(\tilde{a}_0\,\|z(-r)\|^{\alpha_0} + \tilde{b}_0\,\|u\|^{\beta_0} + \tilde{c}_0\right) \\ &\quad + \sum_{k\in I_p}\left(\tilde{a}_k\,\|z\|^{\alpha_k} + \tilde{b}_k\,\|u\|^{\beta_k} + \tilde{c}_k\right).\end{aligned}$$

Hence, by (4.13),

$$\begin{aligned}\|\mathcal{H}_1^{\eta,\varepsilon}(z,u)(t)\| &\leq \|\phi\|_\infty + \tau\,\|\Gamma_\varepsilon\|\left(\|z_f\| + e^{-\nu_1\tau}\right) \\ &\quad + (\tau^2\,\|\Gamma_\varepsilon\| + 1)\left(\tilde{a}_0\,\|z(-r)\|^{\alpha_0} + \tilde{b}_0\,\|u\|^{\beta_0} + \tilde{c}_0\right) \\ &\quad + (\tau\,\|\Gamma_\varepsilon\| + 1)\sum_{k\in I_p}\left(\tilde{a}_k\,\|z\|^{\alpha_k} + \tilde{b}_k\,\|u\|^{\beta_k} + \tilde{c}_k\right).\end{aligned} \tag{4.14}$$

Similarly, for $t \in [0,\tau]$

$$\|\mathcal{H}_2^{\eta,\varepsilon}(z,u)(t)\| \leq \|\Gamma_\varepsilon\|\,\|\mathcal{L}(z,u)\|.$$

Thus, together with (4.14), it follows that

$$\begin{aligned}\|\mathcal{H}^{\eta,\varepsilon}(z,u)(t)\| &= \|\mathcal{H}_1^{\eta,\varepsilon}(z,u)(t)\| + \|\mathcal{H}_2^{\eta,\varepsilon}(z,u)(t)\| \\ &\leq \|\phi\|_\infty + \tau\,\|\Gamma_\varepsilon\|\left(\|z_f\|_Z + e^{-\nu_1\tau}\right) \\ &\quad + (\tau^2\,\|\Gamma_\varepsilon\| + 1)\left(\tilde{a}_0\,\|z(-r)\|^{\alpha_0} + \tilde{b}_0\,\|u\|^{\beta_0} + \tilde{c}_0\right) \\ &\quad + (\tau\,\|\Gamma_\varepsilon\| + 1)\sum_{k\in I_p}\left(\tilde{a}_k\,\|z\|^{\alpha_k} + \tilde{b}_k\,\|u\|^{\beta_k} + \tilde{c}_k\right) \\ &\quad + \|\Gamma_\varepsilon\|\,\|\mathcal{L}(z,u)\| \\ &\leq \|\phi\|_\infty + (\tau+1)\,\|\Gamma_\varepsilon\|\left(\|z_f\|_Z + e^{-\nu_1\tau}\right) \\ &\quad + (2\tau^2\,\|\Gamma_\varepsilon\| + 1)\left(\tilde{a}_0\,\|z(-r)\|^{\alpha_0} + \tilde{b}_0\,\|u\|^{\beta_0} + \tilde{c}_0\right) \\ &\quad + (2\tau\,\|\Gamma_\varepsilon\| + 1)\sum_{k\in I_p}\left(\tilde{a}_k\,\|z\|^{\alpha_k} + \tilde{b}_k\,\|u\|^{\beta_k} + \tilde{c}_k\right)\end{aligned}$$

for $t \in [0,\tau]$. Therefore, since

$$\|\mathcal{H}^{\eta,\varepsilon}(z,u)(t)\| = \|\mathcal{H}_1^{\eta,\varepsilon}(z,u)(t)\| + \|\mathcal{H}_2^{\eta,\varepsilon}(z,u)(t)\| = \|\phi\|_\infty + 0,$$

if $t \in [-r,\tau]$, and recalling

$$\|(z,u)\| = \|z\| + \|u\|,$$

we obtain that

$$
\begin{aligned}
\frac{\|\mathcal{H}^{\eta,\varepsilon}(z,u)\|}{\|(z,u)\|} \leq{}& \frac{1}{\|(z,u)\|}\left[\|\phi\|_\infty + (\tau+1)\|\Gamma_\varepsilon\|\left(\|z_f\|_Z + e^{-\nu_1\tau}\right)\right] \\
&+ \frac{2\tau^2\|\Gamma_\varepsilon\|+1}{\|(z,u)\|}\left[\tilde{a}_0(\|z\|+\|u\|)^{\alpha_0} + \tilde{b}_0(\|z\|+\|u\|)^{\beta_0} + \tilde{c}_0\right] \\
&+ \frac{2\tau\|\Gamma_\varepsilon\|+1}{\|(z,u)\|}\sum_{k\in I_p}\left[\tilde{a}_k(\|z\|+\|u\|)^{\alpha_k} + \tilde{b}_k(\|z\|+\|u\|)^{\beta_k} + \tilde{c}_k\right] \\
\leq{}& \frac{1}{\|(z,u)\|}\left[\|\phi\|_\infty + (\tau+1)\|\Gamma_\varepsilon\|\left(\|z_f\|_Z + e^{-\nu_1\tau}\right)\right] \\
&+ (2\tau^2\|\Gamma_\varepsilon\|+1)\left[\tilde{a}_0\left(\frac{1}{\|(z,u)\|^{1-\alpha_0}}\right) + \tilde{b}_0\left(\frac{1}{\|(z,u)\|^{1-\beta_0}}\right) + \frac{\tilde{c}_0}{\|(z,u)\|}\right] \\
&+ (2\tau\|\Gamma_\varepsilon\|+1)\sum_{k\in I_p}\left[\tilde{a}_k\left(\frac{1}{\|(z,u)\|^{1-\alpha_k}}\right) + \tilde{b}_k\left(\frac{1}{\|(z,u)\|^{1-\beta_k}}\right) + \frac{\tilde{c}_k}{\|(z,u)\|}\right]
\end{aligned}
$$

by Hypothesis (II). Then, by passing to the limit as $\|(z,u)\| \longrightarrow +\infty$,

$$\lim_{\|(z,u)\|\to+\infty} \frac{\|\mathcal{H}^{\eta,\varepsilon}(z,u)\|}{\|(z,u)\|} = 0.$$

This concludes the proof of the proposition.

□

In accordance with Definition 2.11, let us verify that the existence of a fixed point of operator $\mathcal{H}^{\eta,\varepsilon}$ gives an approximate solution to the control system (3.6).

Proposition 4.2. *If $\mathcal{H}^{\eta,\varepsilon}$ has a fixed-point on $\mathcal{D}(\mathcal{H}^{\eta,\varepsilon})$, then there exists a mild solution to problem (3.6).*

Proof. Assume there is $(z,u) \in \mathcal{D}(\mathcal{H}^{\eta,\varepsilon})$ such that

$$\mathcal{H}^{\eta,\varepsilon}(z,u) = (z,u).$$

Then, we have that whenever $t \in [0,\tau]$,

$$\left(\Gamma_\varepsilon \mathcal{L}(z,u)\right)(t) = \mathcal{H}_2^{\eta,\varepsilon}(t) = u(t)$$

and so,

$$
\begin{aligned}
T_\eta(t)\phi(t) &+ \int_0^t T_\eta(t-s)B_\omega u(s)ds + \int_0^t T_\eta(t-s)f^e(s,z[s](-r),u(s))ds \\
&+ \sum_{0<t_k<t} T_\eta(t-t_k)\mathcal{I}^e\left(t_k, z(t_k), u(t_k)\right) \\
&= T_\eta(t)\phi(t) + \int_0^t T_\eta(t-s)B_\omega\left(\Gamma_\varepsilon\mathcal{L}(z,u)\right)(s)\; ds \\
&+ \int_0^t T_\eta(t-s)f^e(s,z[s](-r),u(s))ds + \sum_{0<t_k<t} T_\eta(t-t_k)\mathcal{I}^e\left(t_k, z(t_k), u(t_k)\right) \\
&= \mathcal{H}_1^{\eta,\varepsilon}(z,u)(t) = z(t).
\end{aligned}
\tag{4.15}
$$

Moreover, if $t \in [-r,0]$,

$$(z(t),u(t)) = \mathcal{H}^{\eta,\varepsilon}(z,u)(t) = (\phi(t),0)\,.$$

Then, z is a mild solution for (3.6).

□

Henceforth, let us develop the main result of this chapter. So far, the control problem (1.1) has been set as an abstract differential equation problem with proper operators and function spaces definitions. Our technique used in the remaining of this chapter will be the application of Rothe's fixed-point theorem to operator $\mathcal{H}^{\eta,\varepsilon}$ defined above. In this way, we will obtain the existence of a sequence of controls that approximates the solution of (3.6) towards the desired final state. The properties shown in the last proposition will serve as the machinery to satisfy the fixed-point theorem's hypotheses.

Theorem 4.3. *Let $\phi \in \mathcal{PC}_{pr}$ be a initial-state delay function, and $z_f \in Z$ a final state. The semilinear system (3.6) is approximately controllable on $[0,\tau]$. Moreover, a generalized sequence of controls steering the system (3.6) from initial state to an ε-neighborhood of the final state at time $\tau > 0$ is given by*

$$u_{\varepsilon_n}(t) = B_\omega^* T^*(\tau - t)(\varepsilon_n \mathbf{I} + \mathcal{W})^{-1}\mathcal{L}(z,u), \quad t \in [0,\tau], \qquad n \in \mathbb{N}. \tag{4.16}$$

Proof. Consider an arbitrary $\varepsilon > 0$. Firstly, we have that the operator $\mathcal{H}^{\eta,\varepsilon}$ is continuous and compact thanks to Proposition 4.1. Now, for the purpose of using Rothe-Isac Theorem 2.14, it is left to find a subset $B \subset \mathcal{D}(\mathcal{H}^{\eta,\varepsilon})$ such that $\mathcal{H}^{\eta,\varepsilon}(\partial B) \subset B$.

In fact, by (4.6), for some $0 < K < 1$, there is $l > 0$, big enough such that

$$\frac{\|\mathcal{H}^{\eta,\varepsilon}(z,u)\|}{\|(z,u)\|} \leq K, \qquad \text{for} \quad (z,u) \in \partial B(0,l).$$

Thus, take $B = \overline{B}(0,l)$ and note that

$$\|\mathcal{H}^{\eta,\varepsilon}(z,u)\| \leq K\,\|(z,u)\| < \|(z,u)\| = l \qquad \text{for} \quad (z,u) \in \partial B(0,l).$$

Then,

$$\mathcal{H}^{\eta,\varepsilon}\Big|_B (\partial B) = \mathcal{H}^{\eta,\varepsilon}\left(\partial B(0,l)\right) \subset B(0,l) = B.$$

In this way, the hypotheses of Theorem 2.14 are satisfied and there exists a unique $\mathbf{z}^* = (z_\varepsilon, u_\varepsilon) \in B(0,l)$ such that

$$\mathcal{H}^{\eta,\varepsilon}(z_\varepsilon, u_\varepsilon) = (z_\varepsilon, u_\varepsilon). \tag{4.17}$$

By Proposition 4.2, this implies the existence of an approximate mild solution to the control system (3.6) in the following sense.

Consider the set $\{(z_\varepsilon, u_\varepsilon)\}_{0<\varepsilon\leq 1}$ of all the mild solutions to (3.6) for each $\varepsilon \in (0,1]$ and note that it is bounded. In fact, suppose by the purpose of contradiction that this is not the case, i.e., there exists a sequence $((z_{\varepsilon_n}, u_{\varepsilon_n}))_{n\in\mathbb{N}} \subset \{(z_\varepsilon, u_\varepsilon)\}_{0<\varepsilon\leq 1}$ with

$$\lim_{n\to+\infty} \|(z_{\varepsilon_n}, u_{\varepsilon_n})\| = +\infty.$$

Then, by Proposition 4.1, it yields, for each $0 < \varepsilon < 1$,

$$\lim_{n\to+\infty} \frac{\|\mathcal{H}^{\eta,\varepsilon}(z_{\varepsilon_n}, u_{\varepsilon_n})\|}{\|(z_{\varepsilon_n}, u_{\varepsilon_n})\|} = \lim_{\|(z_{\varepsilon_n}, u_{\varepsilon_n})\|\to+\infty} \frac{\|\mathcal{H}^{\eta,\varepsilon}(z_{\varepsilon_n}, u_{\varepsilon_n})\|}{\|(z_{\varepsilon_n}, u_{\varepsilon_n})\|} = 0. \tag{4.18}$$

In particular, by Cantor's Diagonalization Principle, we consider the sequence

$$\left(\frac{\|\mathcal{H}^{\eta,\varepsilon_n}(z_{\varepsilon_n}, u_{\varepsilon_n})\|}{\|(z_{\varepsilon_n}, u_{\varepsilon_n})\|}\right)_{n\in\mathbb{N}},$$

which fulfills

$$\lim_{n\to+\infty} \frac{\|\mathcal{H}^{\eta,\varepsilon_n}(z_{\varepsilon_n}, u_{\varepsilon_n})\|}{\|(z_{\varepsilon_n}, u_{\varepsilon_n})\|} = 1.$$

In fact, we can observe

$$
\begin{gathered}
\left(\frac{\|\mathcal{H}^{\eta,\varepsilon_1}(z_{\varepsilon_1},u_{\varepsilon_1})\|}{\|(z_{\varepsilon_1},u_{\varepsilon_1})\|},\frac{\|\mathcal{H}^{\eta,\varepsilon_1}(z_{\varepsilon_2},u_{\varepsilon_2})\|}{\|(z_{\varepsilon_2},u_{\varepsilon_2})\|},\ldots,\frac{\|\mathcal{H}^{\eta,\varepsilon_1}(z_{\varepsilon_n},u_{\varepsilon_n})\|}{\|(z_{\varepsilon_n},u_{\varepsilon_n})\|},\ldots\right)\longrightarrow 0\\
\left(\frac{\|\mathcal{H}^{\eta,\varepsilon_2}(z_{\varepsilon_1},u_{\varepsilon_1})\|}{\|(z_{\varepsilon_1},u_{\varepsilon_1})\|},\frac{\|\mathcal{H}^{\eta,\varepsilon_2}(z_{\varepsilon_2},u_{\varepsilon_2})\|}{\|(z_{\varepsilon_2},u_{\varepsilon_2})\|},\ldots,\frac{\|\mathcal{H}^{\eta,\varepsilon_2}(z_{\varepsilon_n},u_{\varepsilon_n})\|}{\|(z_{\varepsilon_n},u_{\varepsilon_n})\|},\ldots\right)\longrightarrow 0\\
\vdots\\
\left(\frac{\|\mathcal{H}^{\eta,\varepsilon_n}(z_{\varepsilon_1},u_{\varepsilon_1})\|}{\|(z_{\varepsilon_1},u_{\varepsilon_1})\|},\frac{\|\mathcal{H}^{\eta,\varepsilon_n}(z_{\varepsilon_2},u_{\varepsilon_2})\|}{\|(z_{\varepsilon_2},u_{\varepsilon_2})\|},\ldots,\frac{\|\mathcal{H}^{\eta,\varepsilon_n}(z_{\varepsilon_n},u_{\varepsilon_n})\|}{\|(z_{\varepsilon_n},u_{\varepsilon_n})\|},\ldots\right)\longrightarrow 0\\
\vdots
\end{gathered}
$$

Since (4.17) holds on this sequence,

$$
\lim_{n\to+\infty}\frac{\|\mathcal{H}^{\eta,\varepsilon_n}(z_{\varepsilon_n},u_{\varepsilon_n})\|}{\|(z_{\varepsilon_n},u_{\varepsilon_n})\|}=\lim_{n\to+\infty}\frac{\|(z_{\varepsilon_n},u_{\varepsilon_n})\|}{\|(z_{\varepsilon_n},u_{\varepsilon_n})\|}=1,
$$

which clearly is a contradiction with (4.18). Then, we can consider the specific subsequence $((z_{\varepsilon_n},u_{\varepsilon_n}))_{n\in\mathbb{N}}$ such that $\varepsilon_n=\dfrac{1}{n}$, for all $n\in\mathbb{N}$.

Now, it is true, by the form of the mild solution control variable obtained in Proposition 4.2, for each $\varepsilon>0$,

$$
u_\varepsilon=\Gamma_\varepsilon\mathcal{L}(z_\varepsilon,u_\varepsilon),
$$

so that

$$
\begin{aligned}
\mathcal{G}_\eta u_\varepsilon &= \mathcal{G}_\eta{\mathcal{G}_\eta}^*\left(\varepsilon\mathbf{I}+\mathcal{W}_\eta\right)^{-1}\mathcal{L}(z_\varepsilon,u_\varepsilon)\\
&=\left(\varepsilon\mathbf{I}+\mathcal{G}_\eta{\mathcal{G}_\eta}^*-\varepsilon\mathbf{I}\right)\left(\varepsilon\mathbf{I}+\mathcal{W}_\eta\right)^{-1}\mathcal{L}(z_\varepsilon,u_\varepsilon)\\
&=\left(\varepsilon\mathbf{I}+\mathcal{W}_\eta\right)\left(\varepsilon\mathbf{I}+\mathcal{W}_\eta\right)^{-1}\mathcal{L}(z_\varepsilon,u_\varepsilon)-\varepsilon\left(\varepsilon\mathbf{I}+\mathcal{W}_\eta\right)^{-1}\mathcal{L}(z_\varepsilon,u_\varepsilon)\\
&=\mathcal{L}(z_\varepsilon,u_\varepsilon)-\varepsilon\left(\varepsilon\mathbf{I}+\mathcal{W}_\eta\right)^{-1}\mathcal{L}(z_\varepsilon,u_\varepsilon)
\end{aligned}\tag{4.19}
$$

Moreover, it is not hard to verify that $\mathcal{L}$ is a compact operator. Then, without loss of generality, $\left(\mathcal{L}\left(z_{\varepsilon_n},u_{\varepsilon_n}\right)\right)_{n\in\mathbb{N}}$, with $\varepsilon_n=\dfrac{1}{n}$, for all $n\in\mathbb{N}$, can be assumed to converge to some $x\in Z$. Thus,

$$
\begin{aligned}
\lim_{n\to+\infty}\varepsilon_n\left(\varepsilon_n\mathbf{I}+\mathcal{W}_\eta\right)^{-1}\mathcal{L}(z_{\varepsilon_n},u_{\varepsilon_n}) &= \lim_{n\to+\infty}\varepsilon_n\left(\varepsilon_n\mathbf{I}+\mathcal{W}_\eta\right)^{-1}\left[x+\mathcal{L}(z_{\varepsilon_n},u_{\varepsilon_n})-x\right]\\
&=\lim_{n\to+\infty}\varepsilon_n\left(\varepsilon_n\mathbf{I}+\mathcal{W}_\eta\right)^{-1}x\\
&+\lim_{n\to+\infty}\varepsilon_n\left(\varepsilon_n\mathbf{I}+\mathcal{W}_\eta\right)^{-1}\left[\mathcal{L}(z_{\varepsilon_n},u_{\varepsilon_n})-x\right]\\
&=\lim_{n\to+\infty}\varepsilon_n\left(\varepsilon_n\mathbf{I}+\mathcal{W}_\eta\right)^{-1}\left[\mathcal{L}(z_{\varepsilon_n},u_{\varepsilon_n})-x\right].
\end{aligned}
$$

Therefore, by passing to the limit,

$$
\lim_{n\to+\infty}\left\|\varepsilon_n\left(\varepsilon_n\mathbf{I}+\mathcal{W}_\eta\right)^{-1}\left[\mathcal{L}(z_{\varepsilon_n},u_{\varepsilon_n})-x\right]\right\|\leq\lim_{n\to+\infty}\|\mathcal{L}(z_{\varepsilon_n},u_{\varepsilon_n})-x\|=0.
$$

And similarly,

$$
\lim_{n\to+\infty}\varepsilon_n\left(\varepsilon_n\mathbf{I}+\mathcal{W}_\eta\right)^{-1}\mathcal{L}(z_{\varepsilon_n},u_{\varepsilon_n})=\lim_{n\to+\infty}\varepsilon_n\left(\varepsilon_n\mathbf{I}+\mathcal{W}_\eta\right)^{-1}\left[\mathcal{L}(z_{\varepsilon_n},u_{\varepsilon_n})-x\right]=0.
$$

So, it follows from (4.19) that

$$
\lim_{n\to+\infty}\mathcal{G}_\eta u_{\varepsilon_n}=\lim_{n\to+\infty}\mathcal{L}\left(z_{\varepsilon_n},u_{\varepsilon_n}\right),
$$

that is,

$$\lim_{n\to+\infty}\left[\mathcal{G}_\eta u_{\varepsilon_n}-\mathcal{L}\left(z_{\varepsilon_n},u_{\varepsilon_n}\right)\right]=0. \tag{4.20}$$

Furthermore, note that the following holds, for all $n\in\mathbb{N}$,

$$\begin{aligned}
z_{\varepsilon_n}(\tau)-z_f &= T_\eta(\tau)\phi(\tau)+\int_0^\tau T_\eta(\tau-s)B_\omega u_{\varepsilon_n}ds \\
&\quad+\int_0^\tau T_\eta(\tau-s)f^e\left(s,z_{\varepsilon_n}[s](-r),u_{\varepsilon_n}(s)\right)ds \\
&\quad+\sum_{0<t_k<\tau}T_\eta(\tau-t_k)\mathcal{I}_k^e\left(t_k,z_{\varepsilon_n}(t_k),u_{\varepsilon_n}(t_k)\right)-z_f \\
&=\int_0^\tau T_\eta(\tau-s)B_\omega u_{\varepsilon_n}ds-\Bigg[z_f-T_\eta(\tau)\phi(\tau) \\
&\quad-\int_0^\tau T_\eta(\tau-s)f^e\left(s,z_{\varepsilon_n}[s](-r),u_{\varepsilon_n}(s)\right)ds \\
&\quad-\sum_{0<t_k<\tau}T_\eta(\tau-t_k)\mathcal{I}_k^e\left(t_k,z_{\varepsilon_n}(t_k),u_{\varepsilon_n}(t_k)\right)\Bigg] \\
&=\mathcal{G}_\eta u_{\varepsilon_n}-\mathcal{L}\left(z_{\varepsilon_n},u_{\varepsilon_n}\right).
\end{aligned} \tag{4.21}$$

Finally, this together with (4.20) yield the existence of $\tilde{n}\in\mathbb{N}$ big enough such that

$$\|z_{\varepsilon_n}(\tau)-z_f\|=\|\mathcal{G}_\eta u_{\varepsilon_n}-\mathcal{L}\left(z_{\varepsilon_n},u_{\varepsilon_n}\right)\|<\varepsilon,\qquad \text{for } n\geq\tilde{n}.$$

By the arbitrariness of $\varepsilon>0$, there exists such sequence of controls $(u_{\varepsilon_n})_{n\in\mathbb{N}}$ satisfying the formula, for $t\in[0,\tau]$,

$$\begin{aligned}
u_{\varepsilon_n}(t) &= \Gamma_\varepsilon\mathcal{L}\left(z_{\varepsilon_n},u_{\varepsilon_n}\right)\ (t) \\
&= \mathcal{G}^*\left(\varepsilon_n\mathbf{I}+\mathcal{W}_\eta\right)^{-1}\mathcal{L}\left(z_{\varepsilon_n},u_{\varepsilon_n}\right)\ (t) \\
&= B_\omega^*T^*(\tau-t)\left(\varepsilon_n\mathbf{I}+\mathcal{W}_\eta\right)^{-1}\mathcal{L}\left(z_{\varepsilon_n},u_{\varepsilon_n}\right).
\end{aligned}$$

and fulfilling that

$$\lim_{n\to+\infty}\|z_{\varepsilon_n}(\tau)-z_f\|=0.$$

Hence, Definition 2.11 is satisfied so that the system (3.6), and in turn (1.1), is approximately controllable on the interval $[0,\tau]$.

□

Remark 4.1. Note that the error of the approximation obtained from Theorem 4.3 can be regarded as a function given as:

$$\epsilon(\varepsilon)=E_\varepsilon\mathcal{L}(z_\varepsilon,u_\varepsilon)=-\varepsilon\left(\varepsilon\mathbf{I}+\mathcal{W}_\eta\right)^{-1}\mathcal{L}(z_\varepsilon,u_\varepsilon).$$

where z_ε is the solution associated to the control u_ε of approximation $\varepsilon>0$. Indeed, by (4.19) and (4.21), it follows that

$$z_\varepsilon-z_f=\mathcal{G}u_\varepsilon-\mathcal{L}(z_\varepsilon,u_\varepsilon)=\epsilon(\varepsilon).$$

4.3 Exact controllability of the semilinear perturbed wave equation system

Finally, in this section, the exact controllability of system (1.6) is obtained based on the previous results from Chapter 3. The complete form of abstract formulation (3.10) will be considered. However, before obtaining our main result, it is needed to analyze the controllability of the perturbed linear system. In this case, Theorem 2.28 (resulting from [32]) will be used in the particular case of system (4.22). The result obtained will be used later as an hypothesis for the exact controllability of the semilinear system.

4.3.1 Perturbed linear system

In this subsection we shall prove the exact controllability of the associated fractionally perturbed linear wave equation, for $\epsilon > 0$ small enough,

$$\mathbf{z}' = (\mathcal{A} + \epsilon\mathcal{P})\mathbf{z} + \mathcal{B}u. \tag{4.22}$$

Corresponding to this linear system, we have the controlled n-dimensional wave equation

$$z' = \mathcal{A}z + \mathcal{B}u. \tag{4.23}$$

By the analysis made in Chapter 3, it is well known that this wave equation is controllable for

$$\tau \geq 2\mu(\Omega). \tag{4.24}$$

In the case $\Omega = [0,1]$, we get that (4.23) is controllable for all $\tau \geq 2$. In [32], the following infinite-dimensional control system is studied

$$z'(t) = \boldsymbol{A}z(t) + \boldsymbol{B}(t)u(t), \quad t > 0,\ z \in Z,\ u \in U, \tag{4.25}$$

where Z and U are Banach spaces, and the mapping $t \longmapsto \boldsymbol{B}(t) : \mathbb{R} \longrightarrow \mathscr{L}(U, Z)$ is continuous in the strong operator topology of $\mathscr{L}(U, Z)$, $\boldsymbol{A}$ is the infinitesimal generator of a C_0-semigroup $\{T_{\boldsymbol{A}}(t)\}_{t\geq 0}$ in Z and the control function u belongs to the space $L^2([0,\tau], U)$. The following question was answered: if the control system (4.25) is exactly controllable, for which class of unbounded linear operators $\boldsymbol{P}$ on Z the perturbed system

$$z'(t) = (\boldsymbol{A} + \boldsymbol{P})z(t) + \boldsymbol{B}(t)u(t), \quad t > 0, \tag{4.26}$$

is also exactly controllable for $\boldsymbol{P}$ in a neighborhood $\mathcal{N}(\mathbf{0})$ of $\mathbf{0}$?. The background for these results in the finite-dimensional case was presented in [31].

As indicated by [32], if the system is of infinite dimension and $\boldsymbol{P}$ is unbounded, this result is not true in general. Nevertheless, for some particular family of evolution equations like the PDE considered in this section, we may obtain the controllability results even if $\boldsymbol{P}$ is unbounded.

From the perturbation theory developed in Section 2.7, as an application of Theorem 2.28, we shall prove the exact controllability of the system associated to the fractionally perturbed wave equation (4.22) knowing that the unperturbed wave equation is exactly controllable on $[0,\tau]$, with τ fulfilling the above mentioned conditions.

Let $\{T_{\mathcal{A}}(t)\}_{t\geq 0}$ be the C_0-semigroup generated by $\mathcal{A}$. We will need the next result:

Proposition 4.3. *Let $\epsilon \geq 0$, and define $\mathcal{P}_\epsilon = \epsilon\mathcal{P}$. Then, $\mathcal{P}_\epsilon \in \mathscr{P}(\mathcal{A})$, with*

$$\|\mathcal{P}_\epsilon T_{\mathcal{A}}(t)\mathbf{z}\| \leq 2\epsilon \|\mathbf{z}\|, \quad \forall \mathbf{z} \in \mathcal{Z}^{1/2},$$

and consequently, $\mathcal{A} + \mathcal{P}_\epsilon$ generates a strongly continuous semigroup $\{T_{\mathcal{A}+\mathcal{P}_\epsilon}(t)\}_{t\geq 0}$.

Proof. If $\epsilon = 0$, the result is trivial, so let us consider an arbitrary $\epsilon > 0$. Then, for a generic element $\mathbf{z} \in \mathcal{Z}^{1/2}$,

$$\|\mathbf{z}\|^2_{\mathcal{Z}^{1/2}} = \left\| \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} \right\|^2_{\mathcal{Z}^{1/2}} = \|z_1\|^2_{Z^{1/2}} + \|z_2\|^2_{L^2(\Omega)} = \sum_{n=1}^{+\infty} \lambda_n \|S_n z_1\|^2_{L^2(\Omega)} + \sum_{n=1}^{+\infty} \|S_n z_2\|^2_{L^2(\Omega)} < +\infty.$$

We have the following estimate

$$\begin{aligned}
\|P_\epsilon T_{\mathcal{A}}(t)\mathbf{z}\|^2_{\mathcal{Z}^{1/2}} &\leq \sum_{n=1}^{+\infty} \left\| \begin{bmatrix} 0 & 0 \\ \epsilon A^{1/2} & 0 \end{bmatrix} e^{A_n t} S_n \mathbf{z} \right\|^2_{\mathcal{Z}^{1/2}} \\
&= \sum_{n=1}^{+\infty} \left\| \begin{bmatrix} 0 & 0 \\ \epsilon A^{1/2} & 0 \end{bmatrix} \begin{bmatrix} \cos\sqrt{\lambda_n t} & \frac{1}{\sqrt{\lambda_n}} \sin\sqrt{\lambda_n t} \\ -\sqrt{\lambda_n} \sin\sqrt{\lambda_n t} & \cos\sqrt{\lambda_n t} \end{bmatrix} \begin{bmatrix} S_n z_1 \\ S_n z_2 \end{bmatrix} \right\|^2_{\mathcal{Z}^{1/2}} \\
&= \sum_{n=1}^{+\infty} \left\| \begin{bmatrix} 0 \\ \epsilon \cos\sqrt{\lambda_n t} A^{1/2} S_n z_1 + \epsilon \frac{1}{\sqrt{\lambda_n}} \sin\sqrt{\lambda_n t} A^{1/2} S_n z_2 \end{bmatrix} \right\|^2_{\mathcal{Z}^{1/2}} \\
&= \sum_{n=1}^{+\infty} \left\| A^{1/2} \left(\epsilon \cos\sqrt{\lambda_n t} S_n z_1 + \epsilon \frac{1}{\sqrt{\lambda_n}} \sin\sqrt{\lambda_n t} S_n z_2 \right) \right\|^2_{L^2(\Omega)} \\
&= \sum_{n=1}^{+\infty} \left\| \lambda_n^{1/2} \left(\epsilon \cos\sqrt{\lambda_n t} S_n z_1 + \epsilon \frac{1}{\sqrt{\lambda_n}} \sin\sqrt{\lambda_n t} S_n z_2 \right) \right\|^2_{L^2(\Omega)} \\
&\leq \sum_{n=1}^{+\infty} \left(2^2 \lambda_n \epsilon^2 \|S_n z_1\|^2_{L^2(\Omega)} + 2^2 \epsilon^2 \|S_n z_2\|^2_{L^2(\Omega)} \right) \\
&= \sum_{n=1}^{+\infty} 2^2 \epsilon^2 \left(\|S_n z_1\|^2_{Z^{1/2}} + \|S_n z_2\|^2_{L^2(\Omega)} \right) = 2^2 \epsilon^2 \|\mathbf{z}\|_{\mathcal{Z}^{1/2}}.
\end{aligned}$$

Therefore, since $\mathbf{z}$ was arbitrary,

$$\|\mathcal{P}_\epsilon T_{\mathcal{A}}(t)\mathbf{z}\|_{\mathcal{Z}^{1/2}} \leq 2\epsilon \|\mathbf{z}\|_{\mathcal{Z}^{1/2}}, \quad \forall \mathbf{z} \in \mathcal{Z}^{1/2}.$$

So, in this case $h(t) = 2\epsilon$ and trivially, by definition, conditions (i)-(iii) from Definition 2.18 are satisfied. So, $\mathcal{P}_\epsilon \in \mathscr{P}(\mathcal{A})$.

Consequently, applying Theorem 2.26, we obtain that $\mathcal{A} + \mathcal{P}_\epsilon$ generates a strongly continuous semigroup $\{T_{\mathcal{A}+\mathcal{P}_\epsilon}(t)\}_{t\geq 0}$ for all $\epsilon \geq 0$.

□

Hence, from all the previous development, we can get the first controllability result for the perturbed equation in the linear case. The proof is based on choosing a perturbation that is not too "irregular" from operator $\mathcal{A}$. In a mathematical sense, we have that to choose $\epsilon > 0$ small enough such that the perturbation is in a neighborhood of the zero operator. Then, Theorem 2.28 can be applied to this particular case.

Theorem 4.4. *Let $\epsilon > 0$ be small enough. Then, system (4.22) is exactly controllable on $[0, \tau]$ for τ satisfying condition (4.24).*

Proof. By the foregoing proposition, we know that $\mathcal{P}_\epsilon$ is an element of class $\mathscr{P}(\mathcal{A})$. Moreover, we obtain that

$$d_{\mathcal{A}}(\mathcal{P}_\epsilon, \mathbf{0}) = \int_0^1 \|(\mathcal{P}_\epsilon - \mathbf{0})\, T_{\mathcal{A}}(t)\|\, dt = \int_0^1 \|\mathcal{P}_\epsilon T_{\mathcal{A}}(t)\|\, dt \leq \int_0^1 2\epsilon dt = 2\epsilon. \tag{4.27}$$

On the other hand, it is well known that the wave equation system (4.23) is exactly controllable on $[0,\tau]$ for $\tau > 0$ satisying (4.24).

Thus, considering $A = \mathcal{A}$, $B(t) = \mathcal{B}$ and $P_0 = \mathbf{0}$, Theorem 2.28 yields that there exists a neighborhood $\mathcal{N}(\mathbf{0}) \subset (\mathscr{P}(\mathcal{A}), d_{\mathcal{A}})$, such that

$$\forall P \in \mathcal{N}(\mathbf{0}): \quad \text{Equation (4.22) is exactly controllable on } [0,\tau]. \tag{4.28}$$

Because $\mathscr{P}(\mathcal{A})$ is endowed with the topology induced by the metric, there exists a (non-empty) open ball

$$\mathcal{O}_\gamma = \{P \in \mathscr{P}(\mathcal{A}) : \ d_{\mathcal{A}}(P_\epsilon, \mathbf{0}) < \gamma\}$$

contained in $\mathcal{N}(\mathbf{0})$. Thus, it suffices to consider $\epsilon < \dfrac{\gamma}{2}$, so that by (4.27)

$$d_{\mathcal{A}}(\mathcal{P}_\epsilon, \mathbf{0}) \leq 2\epsilon < 2\frac{\gamma}{2} = \gamma$$

and in turn, $\mathcal{P} \in \mathcal{N}(\mathbf{0})$. Then, thanks to (4.28), system (4.22) is exactly controllable on $[0,\tau]$.

□

It is worth to mention that in contrast, this result is subject to some restrictions on the upper time limit τ. Nevertheless, the results from the previous chapter were on approximately controllability, whereas we prove here the existence of an specific control that drives the wave equation system (exactly) towards the desired final state.

4.3.2 Semilinear system

In this section, we use Theorem 2.11 to prove the exact controllability of system (1.6) in the case when the non-linear terms $\mathcal{F}$ and $\mathcal{J}_k$ do not depend on the control function u. To this end, we consider the system with non-linear terms independent of the control variable

$$\begin{cases} \mathbf{z}' = (\mathcal{A} + \epsilon\mathcal{P})\,\mathbf{z} + \mathcal{B}u + \mathcal{F}\left(t, \mathbf{z}[t]\right), & t \neq t_k, \\ \mathbf{z}(s) + \boldsymbol{G}\left(\mathbf{z}_{\theta_1}, \dots, \mathbf{z}_{\theta_q}\right)(s) = \rho(s), & s \in [-r, 0], \\ \mathbf{z}\left(t_k^+\right) = \mathbf{z}\left(t_k^-\right) + \mathcal{J}_k\left(t_k, \mathbf{z}\left(t_k\right)\right), & k \in I_m. \end{cases} \tag{4.29}$$

To prove the exact controllability of the previous system, it must be transformed into a fixed-point problem. Then, some bounding conditions must be imposed in the operators so that the existence of (at least) one fixed-point can be assured. In this sense, the required hypothesis is:

(A) There exists constants $l_1, l_2, m_k > 0$, $k \in I_m$, such that for each $t \in [0,\tau]$ and $s \in [-r, 0]$,

$$\|\mathcal{F}\left(t, \psi_1\right) - \mathcal{F}\left(t, \psi_2\right)\| \leq l_1 \left\|\psi_1 - \psi_2\right\|, \quad \psi_1, \psi_2 \in \mathcal{PW}_{mr},$$

$$\|\boldsymbol{G}(\mathbf{x})(s) - \boldsymbol{G}(\mathbf{v})(s)\| \leq l_2 \sum_{i=1}^{q} \left\|\mathbf{x}_i(s) - \mathbf{v}_i(s)\right\|, \quad \mathbf{x}, \mathbf{v} \in \mathcal{PW}_{mr}^q,$$

$$\|\mathcal{J}_k(t, \mathbf{x}) - \mathcal{J}_k(t, \mathbf{z})\| \leq m_k \|\mathbf{x} - \mathbf{z}\|, \quad \mathbf{x}, \mathbf{z} \in \mathcal{Z}^{1/2}.$$

In the next figure, the scenario of finding a exact solution to the control system is depicted.

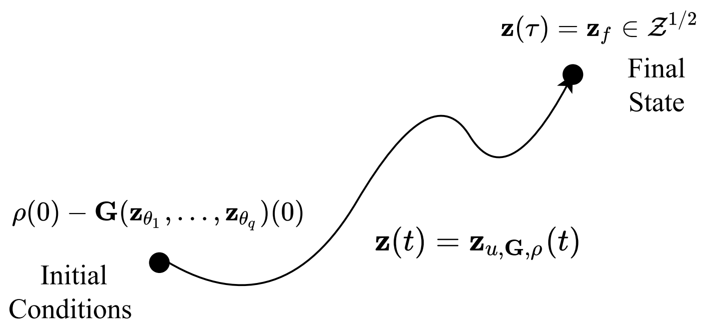

Fig. 4.2: Diagram of the solution $\mathbf{z}(t)$ dependent on the control u, the historical pass $\boldsymbol{\rho}$ and the non-local conditions function $\boldsymbol{G}$, steering the system exactly to the final state $z_1 \in Z$.

From Theorem 4.4, we know that system (4.22) is exactly controllable on $[0, \tau]$ for all τ satisfying (4.24). Therefore, the associated Grammian operator

$$\mathcal{W}(\cdot) = \int_0^\tau \mathcal{T}(\tau - s)\mathcal{B}\mathcal{B}^*\mathcal{T}(\tau - s)^* \cdot ds,$$

is invertible, where $\mathcal{T}(t)$ is used to represent each element of the semigroup $\{T_{\mathcal{A}+\mathcal{P}_\epsilon}(t)\}_{t\geq 0}$ generated by the perturbed operator $\mathcal{A} + \epsilon\mathcal{P}$. The steering operator $\Gamma : \mathcal{Z}^{1/2} \longrightarrow L^2\left([0, \tau], \mathcal{Z}^{1/2}\right)$, defined by

$$\Gamma\psi(s) = \mathcal{B}^*\mathcal{T}^*(\tau - s)\mathcal{W}^{-1}\psi, \quad s \in [0, \tau], \ \psi \in \mathcal{Z}^{1/2},$$

is a right inverse of the controllability operator $\mathcal{Q} : L^2([0, \tau], U) \longrightarrow \mathcal{Z}$ defined by

$$\mathcal{Q}(v) = \int_0^\tau \mathcal{T}(\tau - s)\mathcal{B}v(s)\mathrm{d}s, \quad v \in L^2([0, \tau], U)$$

Finally, consider the following notations:

$$\begin{aligned}
M &= \|\mathcal{T}\|_\infty = \sup_{s\in[0,\tau]} \|\mathcal{T}(s)\|, \\
\|\Gamma\| &= \|\Gamma\|_\infty = \sup_{s\in[0,\tau]} \left\|\mathcal{B}^*\mathcal{T}^*(\tau - s)\mathcal{W}^{-1}\right\|, \\
C &= Ml_2q + M\tau l_1 + MK, \\
K &= \sum_{k=1}^p m_k.
\end{aligned}$$

The following theorem is the final result in this chapter.

Theorem 4.5. *Assume that hypothesis (A) holds and that the following inequality is satisfied*

$$(M\tau\|\mathcal{B}\|\|\Gamma\| + 1)C < 1. \tag{4.30}$$

Then, the system (1.6) is exactly controllable on $[0, \tau]$.

Proof. Consider arbitrary elements for the initial and final elements of the control problem.It is not hard to see that a mild solution yielding the controllability of system (1.6) will be equivalent to the existence of a fixed-point for the following operator $\mathcal{K} : \mathcal{PW}_m^\tau \longrightarrow \mathcal{PW}_m^\tau$, defined by

$$\begin{aligned}
\mathcal{K}\mathbf{x}(t) = \mathcal{T}(t)\left[\boldsymbol{\rho}(0) - \boldsymbol{G}\left(\mathbf{x}_{\theta_1}, \ldots, \mathbf{x}_{\theta_q}\right)(0)\right] + \int_0^\tau \mathcal{T}(\tau - s)\mathcal{B}\Gamma\mathcal{L}(\mathbf{x})(s)ds \\
+ \int_0^\tau \mathcal{T}(\tau - s)\mathcal{F}\left(s, \mathbf{x}[s](-r)\right)ds + \sum_{0<t_k<t} \mathcal{T}(t - t_k)\mathcal{J}_k\left(t_k, \mathbf{x}\left(t_k\right)\right), \quad \mathbf{x} \in \mathcal{PW}_m^\tau
\end{aligned}$$

where the operator $\mathcal{L} : \mathcal{PW}_m \longrightarrow \mathcal{Z}$, is defined as, for each $\mathbf{x} \in \mathcal{PW}_m^\tau$

$$\mathcal{L}\mathbf{x} = \mathbf{z}_f - \mathcal{T}(\tau)\left[\boldsymbol{\rho}(0) - \boldsymbol{G}\left(\mathbf{x}_{\theta_1}, \ldots, \mathbf{x}_{\theta_q}\right)(0)\right] - \int_0^\tau \mathcal{T}(\tau - s)\mathcal{F}(s, \mathbf{x}[s])\, ds \\ - \sum_{0<t_k<\tau} \mathcal{T}(\tau - t_k)\mathcal{J}_k\left(t_k, \mathbf{x}(t_k)\right).$$

Now, we shall prove that $\mathcal{K}$ is a contraction mapping. In fact, let $\mathbf{x}, \mathbf{v} \in \mathcal{PW}_m^\tau$, then

$$\begin{aligned}\|(\mathcal{K}\,\mathbf{x})(t) - (\mathcal{K}\,\mathbf{v})(t)\| \leq{}& \|\mathcal{T}(t)\|\,\|\boldsymbol{G}\left(\mathbf{x}_{\theta_1}, \ldots, \mathbf{x}_{\theta_q}\right)(0) - \boldsymbol{G}\left(\mathbf{v}_{\theta_1}, \ldots, \mathbf{v}_{\theta_q}\right)(0)\| \\ &+ \int_0^\tau \|\mathcal{T}(\tau - s)\|\|\mathcal{B}\|\|\Gamma\|\|\mathcal{L}(\mathbf{x}) - \mathcal{L}(\mathbf{v})\|ds \\ &+ \int_0^\tau \|\mathcal{T}(\tau - s)\|\|\mathcal{F}(s, \mathbf{x}[s]) - \mathcal{F}(s, \mathbf{v}[s])\|ds \\ &+ \sum_{0<t_k<t} \|\mathcal{T}(t - t_k)\|\|\mathcal{J}_k(t_k, \mathbf{x}(t_k)) - \mathcal{J}_k(t_k, \mathbf{v}(t_k))\|.\end{aligned}$$

Then, using the above hypotheses, we get that

$$\begin{aligned}\|(\mathcal{K}\,\mathbf{x})(t) - (\mathcal{K}\,\mathbf{v})(t)\| \leq{}& Ml_2q\|\mathbf{x} - \mathbf{v}\| + M\tau\|\mathcal{B}\|\|\Gamma\|\|\mathcal{L}(\mathbf{x}) - \mathcal{L}(\mathbf{v})\| \\ &+ M\tau l_1\|\mathbf{x} - \mathbf{v}\| + MK\|\mathbf{x} - \mathbf{v}\|.\end{aligned}$$

Furthermore, we have the following estimate

$$\begin{aligned}\|\mathcal{L}(\mathbf{x}) - \mathcal{L}(\mathbf{v})\| &\leq Ml_2q\|\mathbf{x} - \mathbf{v}\| + M\tau l_1\|\mathbf{x} - \mathbf{v}\| + MK\|\mathbf{x} - \mathbf{v}\| \\ &= C\,\|\mathbf{x} - \mathbf{v}\|.\end{aligned}$$

Hence, by hypothesis,

$$\begin{aligned}\|(\mathcal{K}\,\mathbf{x})(t) - (\mathcal{K}\,\mathbf{v})(t)\| &\leq \left(Ml_2q + M\tau\|\mathcal{B}\|\|\Gamma\|\,C + M\tau l_1 + MK\right)\|\mathbf{x} - \mathbf{v}\| \\ &= \left(M\tau\|\mathcal{B}\|\|\Gamma\| + 1\right)C\|\mathbf{x} - \mathbf{v}\|\end{aligned}$$

Thus, by hypothesis (4.30), Definition 2.10 is satisfied and it yields that $\mathcal{K}$ is a contraction mapping, and consequently, by Banach's fixed-point theorem 2.11, $\mathcal{K}$ has a fixed point. That is to say, there exists $\tilde{\mathbf{z}} \in \mathcal{PW}_m^\tau$

$$\mathcal{K}(\tilde{\mathbf{z}}) = \tilde{\mathbf{z}}.$$

Since $u = \Gamma\mathcal{L}(\tilde{\mathbf{z}})$, we obtain

$$\begin{aligned}\mathcal{Q}u = \mathcal{L}(\tilde{\mathbf{z}}) \quad = \quad & \mathbf{z}_f - \mathcal{T}(t)\{\boldsymbol{\rho}(0) - \boldsymbol{G}\left(\tilde{\mathbf{z}}_{\theta_1}, \ldots, \tilde{\mathbf{z}}_{\theta_q}\right)(0)\} - \int_0^\tau \mathcal{T}(\tau - s)\mathcal{F}(s, \tilde{\mathbf{z}}[s])\, ds \\ & - \sum_{0<t_k<\tau} \mathcal{T}(\tau - t_k)\mathcal{J}_k\left(t_k, \tilde{\mathbf{z}}(t_k)\right).\end{aligned}$$

Then, the solution $\tilde{\mathbf{z}} = \tilde{\mathbf{z}}(\cdot, \boldsymbol{\rho}, u)$ of the system (1.6) verifies, from the definition of $\mathcal{K}$,

$$\tilde{\mathbf{z}}(\tau, \boldsymbol{\rho}, u) = \mathbf{z}_f.$$

□

Chapter 5

Conclusions and Final Remarks

In this report, the approximate controllability of the reaction-diffusion equation

$$\begin{cases} \dfrac{\partial}{\partial t}z(t,x) = \Delta_x z(t,x) + (-\Delta_x)^\eta z(t,x) & \\ \qquad\qquad + \mathbb{1}_\omega u(t,x) + f(t, z(t-r,x), u(t,x)), & (t,x) \in \Omega_\tau,\ t \neq t_k, \\ z(t,x) = \phi(t,x), & (t,x) \in \Omega_{-r}, \\ z(t,x) = 0, & (t,x) \in (\partial\Omega)_\tau, \\ z(t_k^+, x) = z(t_k^-, x) + \mathcal{I}_k\left(t_k, z\left(t_k, x\right), u(t_k, x)\right), & x \in \Omega,\ k \in I_p, \end{cases}$$

was proved for $\tau > 0$. To this end, this equation was formulated in the next form by aid of the C_0-semigroup $\{T_\eta(t)\}_{t\geq 0}$ generated by the Laplace operator and its fractional perturbation:

$$\begin{cases} z' = -A_\eta z + B_\omega u + f^e\left(\cdot, z[\cdot](-r), u(\cdot)\right), & \text{on } I' = (0,\tau]\setminus\{t_k\}_{k=1}^p, \\ z = \phi, & \text{on } [-r, 0], \\ z\left(t_k^+\right) = z\left(t_k^-\right) + \mathcal{I}_k^e\left(t_k, z(t_k), u(t_k)\right), & k \in I_p, \end{cases}$$

with $z \in \mathcal{PC}_p^\tau$, $u \in L^2\left([0,\tau], L^2(\Omega)\right)$ and $\phi \in \mathcal{PC}_{pr}$.

The infinite-dimensional function spaces for the solutions of this system were developed in detail, as well as the controllability in the linear cases from the two mentioned approaches. Then, the approximate controllability was reduced to satisfying the hypotheses of Theorem 2.14, for which some auxiliary operators, such as $\mathcal{H}^{\eta,\varepsilon}$, were defined.

The proof is mainly based on the properties proven in Proposition 4.1 for operator $\mathcal{H}^{\eta,\varepsilon}$, so that Rothe-Isac Theorem can be applied and a existence of a control variable for the system is attained. Then, a sequence of controls steering the system to the desired approximation of the final state is found.

In this sense, future projects can be done on the comparison between the different approaches towards the study of a PDE. In fact, there exist works with a variational focus to the controllability of the semilinear heat equation. Nevertheless, there are yet open problems where the inclusion of non-instantaneous impulses or different kinds of delay, such as infinite delay, can be taken into account. In this case, the observability of the problem shall be regarded as a bounding condition on the control operator of the equation. This is a topic that can motivate similar projects in this area.

Furthermore, the current project included the study of the exact controllability of the perturbed wave equation

$$\begin{cases} \dfrac{\partial^2}{\partial s^2}y(s,x) = \Delta_x y(s,x) + \epsilon(-\Delta_x)^{1/2}y(s,x) & \\ \qquad\qquad + \mathbb{1}_\omega u(s,x) + f\left(s, y(s-r,x), \dfrac{\partial}{\partial s}y(s-r,x)\right), & (s,x)\in\Omega_\tau, \\ y(s,x) + h_1\left(y\left(\theta_1+s,x\right),\ldots,\left(\theta_q+s,x\right)\right) = \rho_1(s,x), & (s,x)\in\Omega_{-r} \\ \dfrac{\partial}{\partial s}y(s,x) + h_2\left(\dfrac{\partial}{\partial s}y\left(\theta_1+s,x\right),\ldots,\dfrac{\partial}{\partial s}y\left(\theta_q+s,x\right)\right) = \rho_2(s,x), & (s,x)\in\Omega_{-r}, \\ y(s,x)=0, & (s,x)\in(\partial\Omega)_\tau, \\ \dfrac{\partial}{\partial s}y(s_k^+,x) = \dfrac{\partial}{\partial s}y(s_k^-,x) + J_k\left(s_k, y(s_k,x), \dfrac{\partial}{\partial s}y(s_k,x)\right), & x\in\Omega,\ k\in I_m, \end{cases}$$

for which the equivalent system in matrix notation was established:

$$\begin{cases} \mathbf{z}' = (\mathcal{A}+\epsilon\mathcal{P})\,\mathbf{z} + \mathcal{B}u + \mathcal{F}\left(\cdot, \mathbf{z}[\cdot](-r)\right), & \text{on } (0,\tau]\setminus\{s_k\}_{k=1}^m, \\ \mathbf{z}(s) + \boldsymbol{G}\left(\mathbf{z}_{\theta_1},\ldots,\mathbf{z}_{\theta_q}\right)(s) = \boldsymbol{\rho}(s), & s\in[-r,0], \\ \mathbf{z}(s_k^+) = \mathbf{z}(s_k^-) + \mathcal{J}_k\left(s_k, \mathbf{z}(s_k)\right), & k\in I_m. \end{cases}$$

Similarly, in this system, a bounding hypothesis was established on the Lipschitz continuity constants of the operators $\mathcal{F}, \boldsymbol{G}$ and $\mathcal{J}_k$:

$$(M\tau\|\mathcal{B}\|\|\Gamma\|+1)C<1.$$

Moreover, this result depended on the controllability of the linear system, which was shown, as noted in [54], to be valid for $\tau>0$, big enough, say satisfying (4.24). This important condition is restrictive when working also with other approaches. Thus, it is a open line which has not even been as studied as the previous problem. The important notion in this case is the application of the results obtained in [32], about preserving the controllability of these kind of systems under an unbounded perturbation.

The topics developed in this capstone project have required the study of advanced concepts in mathematical analysis, notably, theory of C_0-semigroups of operators, control theory in finite and infinite dimension, abstract formulation of Cauchy problems, generalizations of functional analysis theorems in infinite-dimensional spaces. Among the courses of the Mathematics curriculum at Yachay Tech, the most related with this capstone project are: Operators Theory, Functional Analysis, Partial Differential Equations (Introductory and Advanced), Calculus of Variations, Topology, Dynamical Systems and Measure Theory. Lastly, it is worth to mention that this research area has been the topic of study inside a research group of Yachay Tech students, alumni and colleagues from other parts of the world. This has enforced our learning and skills towards a competitive research career in mathematics and its applications. It is the will of the author that this project serves as a guide for those readers who wish to learn introductory notions about control theory in PDEs.

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