UNIVERSIDAD DE INVESTIGACIÓN DE
TECNOLOGÍA EXPERIMENTAL YACHAY

Escuela de Ciencias Matemáticas y Computacionales

TÍTULO: CONTROL THEORY OF DIFFERENTIAL SYSTEM

Trabajo de integración curricular presentado como
requisito para la obtención
del título de Matemática

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Resumen

Por muchos años, uno de los objetivos de teoría de control ha sido probar la siguiente conjetura: bajo la influencia de ciertos fenómenos intrínsecos, tales como impulsos, retrasos y condiciones no locales, que son fenómenos intrínsecos, la controlabilidad no cambia. Es decir, si consideramos estas tres características como perturbaciones del sistema, lo cual es muy natural en los problemas de la vida real, la controlabilidad mantiene su robustez. Tomando esto en cuenta, este trabajo está dedicado a estudiar la existencia, unicidad de las soluciones y la controlabilidad de un sistema semilineal impulsivo con retardo infinito y condiciones no locales. Para lograr este objetivo, primero seleccionamos adecuadamente el espacio de fase de tal manera que satisfaga la teoría axiomática formulada por Hale y Kato para estudiar ecuaciones diferenciales con retardo infinito. Después de definir el espacio en el que trabajaremos, desarrollamos las tres pruebas principales de nuestro estudio. La existencia de soluciones y la controlabilidad exacta se reducen al problema de encontrar los puntos fijos de operadores, para lo cual aplicamos el teorema del punto fijo de Karakosta, que es una extensión del teorema del punto fijo de Krasnosel'skii y el teorema del punto fijo de Rothe, respectivamente. La última prueba trata del uso de una técnica desarrollada por A. Bashirov et. at, que evaden el uso de teoremas de punto fijo y se aplicarán para demostrar la controlabilidad aproximada del sistema semilineal. Al final de la prueba de existencia mostramos un ejemplo que involucra impulsos, retardo infinito y condiciones no locales.

Palabras Clave: sistema semilineal, impulsos no instantáneous, retardo infinito, condiciones no locales, controlabilidad, existencia, teorema del punto fijo, bashirov, unicidad

Abstract

For many years, one of the goals of control theory has been to prove the following conjecture: under conditions such as impulses, delays, and non-local conditions, which are intrinsic phenomena, the controllability of a system does not change. That is, if we consider these three characteristics as disturbances of the system, which is very natural in real-life problems, the controllability of the system turns out to be robust. Taking into account this phenomena, this work is devoted to study the existence, uniqueness of solutions, and the controllability of an impulsive semilinear system with infinite delay and non-local conditions. To achieve this goal, we first select the phase space adequately in such a way that it satisfies the axiomatic theory formulated by Hale and Kato to study differential equations with infinite delay. After defining the space we will be working on, we develop the three main proofs of our study. The existence of solution, and the exact controllability are reduced to the problem of finding the fixed points of an operator, for doing so, we apply Karakosta's Fixed Point Theorem (an Extension of Krasnosel'skii's Fixed Point Theorem) and Rothe's Fixed Point Theorem, respectively. The third proof use a technique developed by A. Bashirov et. al, which evades the use of fixed point theorems and will be applied to prove the approximate controllability of the semi-linear system. At the end of the existence proof, we show an example that involves impulses, infinite delay, and non-local conditions.

Keywords: semilinear system, non-instantaneous impulses, infinite delay, non-local conditions, controllability, existence, fixed point theorem, bashirov, uniqueness

Contents

Chapter 1

Introduction

1.1 Background

Modeling real-life problems that help to predict future changes has become of great importance in the latest times, and control them even more. Nevertheless, it was not always considered as indispensable as it is now. During the development of society in ancient times and nowadays, control systems appear naturally, but not each improvement made has had a mathematical foundation on it. Control problems have been studied for a long time, but not from the mathematical point of view; the first work in control theory that is worth mentioning is a regulatory mechanism with a float in Greece, around 250 BC, [1]. However, the most relevant antecedent was developed in 1788: James Watt's centrifugal regulator that worked with automatic feedback, [2]. Approximately until the end of the 19^{th} century, control problems were considered merely intuitive, but as the necessity to improve the responses and the precision of the control system appears, control theory started its development. The 20^{th} century, in particular between 1960 and 1990, control theory was roughly investigated. The most important and remarkable were the works presented by Kalman et al, which represent the base of the control theory we have now.

To achieve a better model, and in the attempt of making it as precise as possible, there exist the need of setting conditions such as impulses, delay, and non-local conditions. These perturbations are intrinsic phenomena of a real-life problem. Lately, several studies in control theory have focused on proving the conjecture that controllability is preserved under the three perturbations mentioned above.

In fact, let's consider abrupt changes in the state which would imply the use of impulsive equations. These kinds of equations had caught the interest of many scientists in a variety of fields such as biology, economy, neural networks, and others. In [3] the authors study some mathematical models involving impulsive equations, i.e. Lasota-Wazewska model, Hematopoiesis models, and others regarding Biological models. In the rest of fields, to mention some we have impulsive models in Populations dynamics, impulsive Hopfield Neural Network, impulsive Price Fluctuations Models. The impulses could be instantaneous, where the changes on the states are short and the non-instantaneous ones, that remain for a finite interval of time. Considering the first case, we can see studies such as [4] and [5], where the authors prove the controllability of a system by using fixed point theorems. On the other hand, systems that take into consideration non-instantaneous impulses are considered in [6], where the existence of a solution of this type of impulsive system is studied. In [7], a study of controllability by using different approaches is carried out. In both cases, instantaneous and non-instantaneous, the conjecture restricted to impulses is proved.

The other two characteristics, non-local conditions and delayed differential equations are also of our interest. The first one was a concept introduced in [8]. It is a Cauchy problem that helps to get a more precise model of a real-life problem. To mention a few, we have [9] and [10] that are studies related to systems containing these constraints and where no problem arises in the controllability. Concerning to the second part, in [3] it can be found models in Neural Networks and Economy that also delays,i.e., differential equations where the time derivatives depend on the solution at previous times:

$$\frac{d}{dt}z(t) = f(t, z_t)$$

where, $f\colon \mathbb{R}^+ \times \mathfrak{H} \to \mathbb{R}$, $z_t(\theta) = z(\theta + t)$ and $\mathfrak{H}$ is the phase space to be specified later, defined on $(-\infty, 0]$. It is known that when $r < \infty$, the use of the space of continuous functions like the phase space of the solution, as it is defined on [11], [12] and [13], is not always an option to be considered in retarded equations. Especially, if we want to avoid problems on the existence, uniqueness, and stability of the solution of the system under study. This problem becomes even worst when we take $r = \infty$, that is, part of the initial functions is always contained by the state z_t. As a consequence, a deeper analysis must be taken into consideration for differential equations with infinite delay. Hale and Kato in [14] defined appropriated conditions for this kind of system, which will be detailed later. Notice that neither in a system with non-local conditions nor with delay, the controllability is destroyed. So, the conjecture with each of these two perturbations holds.

It can also be considered more than one perturbation simultaneously in the same system. That is the case of [15], [16] and [17], to mention a few, where impulses and non-local conditions are added to the known linear control system. We can see in [18] and [19] that the authors focus on bounded and unbounded delayed systems with impulses, respectively. It can even be possible to add more than two characteristics as it was done in [20], [21] and [22] where the controllability of systems is studied by using fixed point theorems, and in the case of the last one, techniques avoiding it, used by Bashirov in [23], [24] and [25]. It is worth emphasizing, that the controllability under the influence of one, two, or even three of the perturbations is not ruined.

1.2 Setting of the problem

The system under study contains these three mentioned perturbations simultaneously, that is, non-instantaneous impulses, infinite delay and non-local conditions at the same time, and it is defined as follows:

$$\begin{cases} z'(t) = A(t)z(t) + \mathcal{F}(t, z_t), \quad t \in I_k, \quad k = 0, 1, 2, \dots, \\ z(s) + h(z_{\pi_1}, \dots, z_{\pi_q})(s) = \phi(s), \quad s \in \mathbb{R}_- = (-\infty, 0], \\ z(t) = G_k(t, z(t_k^-)), \quad t \in J_k, \quad k = 1, 2, \dots, \end{cases} \tag{1.1}$$

where $I_0 = (0, t_1]$, $I_k = (s_k, t_{k+1}]$, $J_k = (t_k, s_k]$, $0 = s_0 < t_1 < s_1 < t_2 < s_2 < \cdots < s_{k-1} < t_k \longrightarrow \infty$, as $k \longrightarrow \infty$. There exists a fixed number $\zeta > 0$ such that $\pi_q \leq \min\{\zeta, \tau\}$, where $[0, \tau)$ is the maximal interval of local existence of solutions of problem (1.1); and $0 \leq \pi_1 < \pi_2 < \cdots < \pi_q$, $i = 1, \ldots, q$, selected under certain rules marked by the real life problem that the mathematical model could represent, such as: $\pi_i = i\pi_q/q, \quad i = 1, \ldots, q$. Some measurements at more places are incorporated with the use of non-local conditions, which is one the advantages of it in order to get better models. $h : \mathfrak{H}^q \to \mathfrak{H}$, $\phi : \mathbb{R}_- \longrightarrow \mathbb{R}^n, \quad \phi \in \mathfrak{H}$. $\mathcal{F} : \mathbb{R}_+ \times \mathfrak{H} \longrightarrow \mathbb{R}^n$ is an smooth enough function, $G_k : J_k \times \mathbb{R}^n \longrightarrow \mathbb{R}^n, \quad k = 1, 2, 3, \ldots,$ are continuous and represents the impulsive effect in the system (1.1), i.e., we are considering that the system can have abrupt changes that stay there for a finite interval of time. These alterations in state might be due to certain external factors, which cannot be well described by pure ordinary differential equations, (see, for instance, [26] and reference therein). Here, $A(t) \in \mathbb{R}^{n\times n}$ is a continuous matrix function and the function $z_t(\theta) = z(t+\theta)$ for $\theta \in (-\infty, 0]$ illustrate the history of the state up to the time t, and also remembers much of the historical past of ϕ, carrying part of the present to the past. It is important to remark that this system will be used to prove the existence of solutions of (1.1).

In order to study the controllability of (1.1), the function $\mathcal{F}$ in (1.1) will be taken as $\mathcal{F}(t, z_t) = \mathcal{B}(t)u(t) + f(t, z_t, u(t))$ for every fixed $u \in L^2(0, \tau; \mathbb{R}^m)$. Thus, the system becomes:

$$\begin{cases} z'(t) = \mathcal{A}(t)z(t) + \mathcal{B}(t)u(t) + f(t, z_t, u(t)), \quad t \in I_k, \quad k = 0, 1, 2, \ldots, p \\ z(s) + h(z_{\pi_1}, \ldots, z_{\pi_q}) = \varphi(s), \quad s \in \mathbb{R}_- = (-\infty, 0], \\ z(t) = G_k(t, z(t_k^-)), \quad t \in J_k, \quad k = 1, 2, \cdots, p \end{cases} \tag{1.2}$$

where $0 = s_0 < t_1 < s_1 < t_2 < s_2 < \cdots < s_p < t_{p+1} = \tau$, $\mathcal{B} \in \mathbb{R}^{n\times m}$, the control u belongs to $L^2(0, \tau; \mathbb{R}^m)$. $f : \mathbb{R}_+ \times \mathfrak{H} \times \mathbb{R}^m \longrightarrow \mathbb{R}^n$ is an smooth enough function in (1.1) and $\varphi \in \mathfrak{H}$. The rest of terms are defined as in (1.1). Here, notice that we have a finite number of impulses, since our main objective is to prove the controllability of the system on a finite interval $[0, \tau]$.

Something that it is worth to mention, as we will use it, is that some authors have considered the differential system with non-instantaneous impulses of the following form

$$\begin{cases} y'(t) = Ay(t) + \mathcal{F}(t, y(t)), & t \in (s_i, t_{i+1}], i = 0, 1, \ldots, m, \\ y(t) = G_i(t, y(t)), & t \in (t_i, s_i], i = 1, \ldots, m, \\ y(0) = y_0, \end{cases} \tag{1.3}$$

who was firstly introduced by Hernandez in [27]. Fečkan et. al. consider in [28] a special remark over the impulsive condition of (1.3). It specifies that, there are positive constants $d_k, \ k = 1, 2, \ldots, p$ such that

$$\|G_k(t, z^1) - G_k(t, z^2)\| \leq d_k \|z^1 - z^2\|, \quad \forall\, z^1, z^2 \in, \mathbb{R}^n, \ t \in [t_k, s_k],$$

where, $G_k \in \mathcal{C}\,([t_k, s_k]; \mathbb{R})$ and $\max\{d_k \,:\, k = 1, 2, \ldots, p\} < 1$ is a necessary condition. Then the Banach fixed point theorem gives a unique $y_k \in C([t_k, s_k], \mathbb{R})$ such that $z = G_k(t, z)$ iff $z = y_k(t)$. So (1.3) is equivalent to

$$z(t) = y_k(t), \quad t \in (t_k, s_k], \quad k = 1, 2, 3, \ldots, p,$$

which does not depend on the state variable $z(\cdot)$. Thus, it was necessary to modify the impulses of the system (1.3) and consider the condition:

$$z(t) = G_k(t, z(t_k^-)), \ t \in (t_k, s_k], \ k = 1, 2, 3, \ldots, p.$$

This work shall prove that under some conditions on $\mathcal{F}$, G_k and h, the problem (1.1) admits a solution on $(-\infty, \tau]$, for some $\tau > 0$. Then, under some additional conditions, we shall prove that this solution can be extended on the whole real line $\mathbb{R}$. Finally, after the existence, uniqueness, and the prolongation of solutions are proved, we consider (1.2) for the controllability proofs under some other different conditions on f, h and G_k, which will be stated later on the Chapter 3

The next Chapter, that is Chapter 2, is focused on state some important definitions, theorems, and lemmas, that are a clue key in the development of the main results of this work. It begins with some necessary linear control theory, then it continues with the spaces where the operators of the proofs are defined and finally it mentions some of the two fixed point theorems that we used, Arzelà Ascoli theorem and other results.

Chapter 3 aim is to show the main results of this work. It starts with the proofs, uniqueness, and prolongation of solutions of (1.1), where Karakosta's fixed point theorem is used, followed by a mathematical example. Then, it continues with the proofs of approximate controllability, where Bashirov techniques that avoid the use of fixed-point is used, and it concludes with the proof of exact controllability by applying Rothe's fixed point theorem.

Finally, Chapter 4 presents a conclusion of the work and at the end, a final remark is given.

Chapter 2

Theoretical Framework

This chapter is devoted to state, without proofs, some of the necessary definitions, theorems and lemmas that are fundamental to achieve the main objective of this work.

2.1 Linear control systems in finite dimensional spaces

Since we are focusing on non-autonomous linear systems, we shall consider, in the first place, the following linear system:

$$z'(t) = \mathcal{A}(t)z(t) \tag{2.1}$$

where, $\mathcal{A}(\cdot)$ is a $n \times n$ continuous matrix and $z(t) \in \mathbb{R}^n$. Its fundamental matrix is denoted by Φ and it is the solution of the Cauchy Problem

$$\begin{cases} \dfrac{d\Phi(t)}{dt} = \mathcal{A}(t)\Phi(t), \\ \Phi(0) = I. \end{cases}$$

The evolution operator is defined by $\mathcal{U}(t,s) = \Phi(t)\Phi^{-1}(s),\ s,t \in \mathbb{R}$, also we will consider the following bound

$$M = \sup_{t,s\in[0,\tau]} \|\mathcal{U}(t,s)\|$$

Let's now take into consideration the following linear control system with initial condition

$$\begin{cases} z'(t) = \mathcal{A}(t)z(t) + \mathcal{B}(t)u(t), \qquad z(t) \in \mathbb{R}^n, t \in [t_0, \tau] \\ z(t_0) = z^0 \end{cases} \tag{2.2}$$

where $\mathcal{B}(\cdot)$ is a continuous matrix of dimension $n \times m$. The rest of terms are defined in the same way as in (2.1). The previous system admits only one solution, which is given by

$$z(t) = \mathcal{U}(t,t_0)z^0 + \int_{t_0}^{t} \mathcal{U}(t,\varrho)\mathcal{B}(\varrho)u(\varrho)d\varrho, \quad t \in [t_0, \tau] \tag{2.3}$$

Definition 2.1 *The system in* (2.2) *is controllable on* $[t_0, \tau]$ *if given two points* z^0 *and* z^1*, there exist a control* $u \in L^2(t_0, \tau; \mathbb{R}^m)$ *such that the corresponding solution of* (2.2)*,* $z(\cdot)$*, satisfies the boundary conditions*

$$z(t_0) = z^0 \qquad \textit{and} \qquad z(\tau) = z^1.$$

It is known that the controllability of (2.2), with $t_0 = 0$, is obtained by the surjectivity of the operator $\mathcal{G} : L^2(0, \tau; \mathbb{R}^m) \to \mathbb{R}^n$, which is defined by

$$\mathcal{G}u = \int_0^\tau \mathcal{U}(\tau, \varrho)\mathcal{B}(\varrho)u(\varrho)d\varrho, \tag{2.4}$$

and, a control $u \in L^2([0, \tau]; \mathbb{R}^m)$, that steers the system (2.2) from the initial state z^0 to a final state z^1 on $[0, \tau]$, is given as follows:

$$u(\varrho) = \mathcal{B}^*(t)\mathcal{U}^*(\tau, \varrho)(\mathcal{W}_{[0,\tau]})^{-1}(z^1 - \mathcal{U}(\tau, 0)z^0), \quad \varrho \in [0, \tau], \tag{2.5}$$

where, $\mathcal{W}_{[0,\tau]} : \mathbb{R}^n \to \mathbb{R}^n$ is the Controllability Gramian Operator in the interval $[0, \tau]$, defined as

$$\mathcal{W}_{[0,\tau]}z = \mathcal{G}\mathcal{G}^*z = \int_0^\tau \mathcal{U}(\tau, \varrho)\mathcal{B}(\varrho)\mathcal{B}^*(\varrho)\mathcal{U}^*(\tau, \varrho)z \, d\varrho. \tag{2.6}$$

In fact, the next theorem is a characterization of the controllability of system (2.2). The proof can be seen in [29].

Theorem 2.1 *The following statements are equivalent:*

i) *The system* (2.2) *is controllable on the interval* $[0, \tau]$.

ii) $\mathrm{Rg}(\mathcal{G}) = \mathbb{R}^n$.

iii) *There exist* $\lambda > 0$ *such that*

$$\lambda \left\| \mathcal{B}^*(\cdot)\Phi^{-1*}(\cdot)z \right\|_{L^2} \geq \|z\|_{\mathbb{R}^n}, \quad z \in \mathbb{R}^n.$$

iv) *If* $\mathcal{B}^*(t)\Phi^{-1*}(t)z = 0$ *with* $0 \leq t \leq \tau$*, then* $z = 0$.

v) *The matrix*

$$\mathcal{W}_{[0,\tau]} = \int_0^\tau \Phi^{-1}(\varrho)\mathcal{B}(\varrho)\mathcal{B}^*(\varrho)\Phi^{-1*}d\varrho$$

is positive definite, i.e., there exist $\beta > 0$ *such that*

$$\langle \mathcal{W}z, z \rangle \geq \beta \|z\|^2.$$

Moreover, given $z^1, z^0 \in [0, \tau]$*, the control a* (2.5) *transfer the system from the initial state* z^0 *to the final state* z^1.

The adjoint operator $\mathcal{G}^* : \mathbb{R}^n \to L^2([0,\tau];\mathbb{R}^m)$, which is actually used in the proof of last theorem, is given by

$$(\mathcal{G}^* z)(\varrho) = \mathcal{B}^*(\varrho)\mathcal{U}^*(\tau,\varrho)z \qquad \varrho \in [0,\tau]. \tag{2.7}$$

In the same way, the linear system (2.2) is controllable on $[\alpha,\beta] \subseteq [0,\tau]$, if and only if, the controllability operator given by

$$\mathcal{G}_{\alpha\beta}u = \int_{\alpha}^{\beta} \mathcal{U}(\beta,\varrho)\mathcal{B}(\varrho)u(\varrho)d\varrho, \qquad u \in L^2([\alpha,\beta];\mathbb{R}^m), \tag{2.8}$$

is surjective. i.e., The Gramian Operator $\mathcal{W}_{[\alpha,\beta]}$ given by

$$\mathcal{G}_{\alpha\beta}\mathcal{G}^*_{\alpha\beta}z = \mathcal{W}_{[\alpha,\beta]}z = \int_{\alpha}^{\beta} \mathcal{U}(\beta,\varrho)\mathcal{B}(\varrho)\mathcal{B}^*(\varrho)\mathcal{U}^*(\beta,\varrho)zd\varrho, \tag{2.9}$$

is invertible. For the foregoing matrix, there exist $\delta_\alpha > 0$ such that $\left\|\mathcal{W}^{-1}_{[\alpha,\beta]}\right\| < \dfrac{1}{\delta_\alpha}$, and a control u steering the linear system (2.2) from z^α to z^β on $[\alpha,\beta]$ is given by

$$u(\varrho) = \mathcal{B}^*(\varrho)\mathcal{U}^*(\beta,\varrho)(\mathcal{W}_{[\alpha,\beta]})^{-1}(z^\beta - \mathcal{U}(\beta,\alpha)z^\alpha), \quad \varrho \in [\alpha,\beta]. \tag{2.10}$$

In particular, for $\tau > 0$ and $0 < \delta < \tau$, we consider the following system

$$\begin{cases} y' = \mathcal{A}(t)y(t) + \mathcal{B}(t)u(t), \quad y \in \mathbb{R}^n, \quad t \in [\tau-\delta,\tau], \\ y(\tau-\delta) = z^0, \end{cases} \tag{2.11}$$

which admits only one solution given by

$$y(t) = \mathcal{U}(t,\tau-\delta)z^0 + \int_{\tau-\delta}^{t} \mathcal{U}(t,\varrho)\mathcal{B}(\varrho)u(\varrho)d\varrho, \;\; t \in [\tau-\delta,\tau], \tag{2.12}$$

Corresponding with (2.11), we shall denote the Gramian controllability matrix by:

$$\mathcal{W}_{\tau\delta} = \int_{\tau-\delta}^{\tau} \mathcal{U}(\tau,\varrho)\mathcal{B}(\varrho)\mathcal{B}^*(\varrho)\mathcal{U}^*(\tau,\varrho)d\varrho. \tag{2.13}$$

As it can be seen in [20], the system (2.11) is controllable on $[\tau-\delta,\tau]$ if, and only if, the matrix $\mathcal{W}_{\tau\delta}$ is invertible. Moreover, a control that steers the system (2.11) from the initial state z^0 to a final state z^1 on the interval $[\tau-\delta,\tau]$ is given by

$$v^\delta(\varrho) = \mathcal{B}^*(\varrho)\mathcal{U}^*(\tau,\varrho)\mathcal{W}^{-1}_{\tau\delta}(z_1 - \mathcal{U}(\tau,\tau-\delta)z^0), \quad \varrho \in [\tau-\delta,\tau]. \tag{2.14}$$

i.e., the corresponding solution $y^\delta(t)$ of the linear system (2.11) satisfies the boundary condition:

$$y^\delta(\tau-\delta) = z^0 \quad \text{and} \quad y^\delta(\tau) = z^1. \tag{2.15}$$

Remark 2.1 *When we study the exact controllability in L^2-spaces and we are dealing with finite-dimensional linear control systems, it is important to keep in mind that the system under study is controllable iff, it is controllable with controls in any dense subspace of L^2. Thus, the system* (2.2) *is controllable with controls on L^2 iff, it is controllable with controls on $\mathcal{PW}_u$(see [30]).*

2.2 Spaces, definitions, lemmas and theorems

In this section, we shall define the spaces where our problem will be studied, and will review some definitions, theorems and lemmas that are used to prove the main results of this work.

First of all, let us define the control function space $\mathcal{PW}_u = \mathcal{PW}_u(((0,\tau];\mathbb{R}^m)$, by

$$\mathcal{PW}_u = \{u : (0,\tau] \to \mathbb{R}^m : u \text{ is bounded and } u \in \mathcal{C}(I;\mathbb{R}^m)\}.$$

where $I = \bigcup_{i=0}^{N}(s_i, t_{i+1}]$, endowed with the norm

$$\|u\|_0 = \sup_{t\in[0,\tau]} \|u(t)\|_{\mathbb{R}^m}$$

Also, let us define $\mathcal{PW} = \mathcal{PW}((-\infty,0];\mathbb{R}^n)$ as the normalized piecewise continuous functions, as follows:

$$\mathcal{PW} = \left\{\varphi : \ (-\infty,0] \longrightarrow \mathbb{R}^n : \varphi\Big|_{[a,0]} \text{is a piecewise continuous function}, \quad \forall a < 0\right\}$$

Using ideas from [31], we consider a function $g : \mathbb{R} \to \mathbb{R}_+$ satisfying the following conditions.

a) $g(0) = 1$,

b) $g(-\infty) = +\infty$,

c) g is decreasing.

Remark 2.2 *A particular function, that holds the conditions above-mentioned, is* $g(s) = \exp(-as)$*, with* $a > 0$.

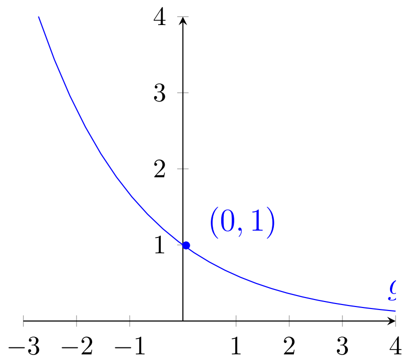

Figure 2.1: Example of g function described above.

Now, we define the following functions space

$$C_g = \left\{z \in \mathcal{PW} : \sup_{s\leq 0} \frac{\|z(s)\|}{g(s)} < \infty\right\}.$$

C_g is a Banach space, and a sketch of the proof is given in [32]:

Lemma 2.1 *The space C_g equipped with the norm*

$$\|z\|_{C_g} = \sup_{s\leq 0} \frac{\|z(s)\|}{g(s)}, \quad z \in C_g,$$

is a Banach space.

Our phase space will be

$$\mathfrak{H} := C_g,$$

equipped with the norm

$$\|z\|_{C_g} = \|z\|_{\mathfrak{H}}.$$

We shall take into consideration the following space $\mathcal{PW}_{g\tau} := \mathcal{PW}_{g\tau}((-\infty,\tau];\mathbb{R}^n)$ defined by

$$\mathcal{PW}_{g\tau} = \Bigg\{ z : (-\infty,\tau] \to \mathbb{R}^n : z\Big|_{\mathbb{R}_-} \in \mathfrak{H} \text{ and } z\Big|_{(0,\tau]} \text{ is a continuous except at } t_k,$$
$$k = 1,2,...,p \text{ ,with } s_{p-1} < \tau \text{ where side limits } z(t_k^+), z(t_k^-) \text{ exist and } z(t_k^+) = z(t_k)\},$$

which is larger, and where $z(t_k^+) = \lim_{t\to t_k^+} z(t)$, $z(t_k^-) = \lim_{t\to t_k^-} z(t)$. From Lemma (2.1), we have the following lemma:

Lemma 2.2 *$\mathcal{PW}_{g\tau}$ is a Banach space endowed with the norm*

$$\|z\|_{\mathcal{PW}_{g\tau}} = \left\| z|_{\mathbb{R}_-} \right\|_{C_g} + \|z|_I\|_\infty$$

where $\|z|_I\|_\infty = \sup\limits_{t\in I=(0,\tau]} \|z(t)\|.$

It is not hard to verify that the axiomatic theory proposed by Hale and Kato for the phase space of retarded equations with infinite delay is satisfied:

$A1)$ If z belongs to the whole space where the differential equation is defined, then for every $t \in [0,\tau]$ the following conditions hold:

  i) z_t is in $\mathfrak{H}$;

  ii) $\|z(t)\|_{\mathbb{R}^n} \leq H\|z_t\|_{\mathfrak{H}}$;

  iii) $\|z_t\|_{\mathfrak{H}} \leq K(t)\sup\{\|z(s)\| : 0 \leq s \leq t\} + M(t)\|z_0\|_{\mathfrak{H}}$, where $H \geq 0$ is a constant, $K,\ M : [0,\infty) \to [0,\infty)$, K is continuous and M is locally bounded, and $H,\ K,\ M$ are independent of $z(t)$.

$A2)$ For the function $z(\cdot)$ in A1), z_t is a $\mathfrak{H}-$valued continuous function on $[0,\tau]$.

$A3)$ The space $\mathfrak{H}$ is complete.

More detail about this theory can be found in[14, 31, 33, 34].

Now, let us denote by

$$\mathfrak{H}^q = \mathfrak{H} \times \mathfrak{H} \times ... \times \mathfrak{H} = \prod_{i=1}^{q} \mathfrak{H},$$

i.e.,

$$z = (z_1, ..., z_q)^T \in \mathfrak{H}^q,$$

and the norm in the space $\mathfrak{H}^q$ is given by

$$\|y\|_{\mathfrak{H}^q} = \sum_{i=1}^{q} \|y_i\|_{\mathfrak{H}}\,.$$

The following Lemma is a key to obtain our results. In fact, it is stronger than the axiom A1)-iii) from Hale and Kato axiomatic theory previously mentioned. Its proof, which can be found in [32], is due to the fact that the function g is defined on the whole real line.

Lemma 2.3 *For all function $z \in \mathcal{PW}_{g\tau}$ the following estimate holds for all $\varrho \in [0, \tau]$:*

$$\|z_\varrho\|_{\mathfrak{H}} \le \|z\|_{\mathcal{PW}_{g\tau}}.$$

Definition 2.2 (Exact Controllability) *The system* (1.2) *is said to be exactly controllable on* $[0, \tau]$ *if for every* $\phi \in \mathfrak{H}$, $z^1 \in I\!R^n$, *there exists* $u \in L^2(0, \tau; \mathbb{R}^m)$ *such that the solution* $z(t)$ *of* (1.2) *corresponding to* u *verifies:*

$$z(0) + h(z_{\pi_1}, \dots, z_{\pi_q})(0) = \phi(0) \quad \textit{and} \quad z(\tau) = z^1.$$

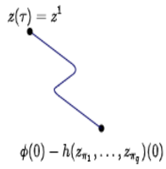

Figure 2.2: Exact Controllability

Definition 2.3 (Approximate Controllability) *The system* (1.2) *is said to be approximately controllable on* $[0, \tau]$ *if for every* $\phi \in \mathfrak{H}$, $z_1 \in I\!R^n$ *and* $\epsilon > 0$, *there exists* $u \in L_2([0, \tau]; \mathbb{R}^m)$ *such that the solution* $z(t)$ *of* (1.2) *corresponding to* u *verifies:*

$$z(0) + h(z_{\pi_1}, \dots, z_{\pi_q})(0) = \phi(0), \quad \textit{and} \quad \left\| z(\tau) - z^1 \right\|_{\mathbb{R}^n} < \epsilon.$$

In addition to the definitions of the spaces and some related lemmas, it is also necessary to state some extra definitions, lemmas and theorems that will take part in the development of the proofs of the main results.

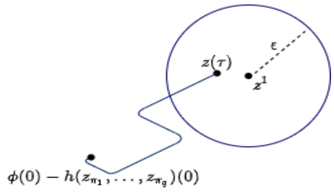

Figure 2.3: Approximate Controllability

Definition 2.4 (***Equicontractivity***) *Let Z be a Banach space and $\{T_n\}_{n\in I}$ be a family of operators $T_n\colon Z \to Z$. The family $\{T_n\}_{n\in\mathbb{N}}$ is said to be equicontractive, if there exists $0 < L < 1$ such that:*

$$\|T_n z_1 - T_n z_2\| \leq L\|z_1 - z_2\|\,, \qquad z_1, z_2 \in Z, \qquad n \in I = \mathbb{N}$$

Lemma 2.4 (***Gronwall inequality***) *Let $v\colon [a,b] \to \mathbb{R}$ and $\eta\colon [a,b] \to \mathbb{R}^+$ be continuous functions. Consider the continuous function $y\colon [a,b] \to \mathbb{R}$ such that*

$$y(t) \leq v(t) + \int_a^t \eta(s)y(s)ds, \qquad\qquad t \in [a,b],$$

Then, for all $t \in [a,b]$, we have

$$y(t)t \leq v(t)\int_a^t v(s)\eta(s)exp\left(\int_s^t v(u)du\right)ds,$$

In particular, if $f(t) \equiv k$

$$y(t) \leq kexp\left(\int_a^t v(s)ds\right)$$

,

Theorem 2.2 (*G.L. Karakostas Fixed Point Theorem, [35]) Let Z and Y be Banach spaces and D be a closed convex subset of Z. Also, let $\mathcal{C} : D \to Y$ be a continuous operator such that $\mathcal{C}(D)$ is a relatively compact subset of Y, and*

$$\mathcal{T} : D \times \overline{\mathcal{C}(D)} \to D$$

is a continuous operator such that the family $\{\mathcal{T}(\cdot, y) : y \in \overline{\mathcal{C}(D)}\}$ is an equicontractive family. Then, the operator equation

$$\mathcal{T}(z, \mathcal{C}(z)) = z$$

admits a solution on D.

Theorem 2.3 *(Rothe's Fixed Theorem, [36]) Let Z be a Banach space. Let $\mathbb{B} \subset Z$ be a closed convex subset such that the zero of Z is contained in the interior of $\mathbb{B}$. Let $\Phi : \mathbb{B} \to Z$ be a continuous mapping with $\Phi(\mathbb{B})$ relatively compact in Z and $\Phi(\partial\mathbb{B}) \subset \mathbb{B}$. Then there is a point $z^* \in \mathbb{B}$ such that $\Phi(z^*) = z^*$.*

Chapter 3

Main Results

As it was already mentioned, the aim of this chapter is to prove the results of this work.

3.1 Existence and uniqueness of a solution

This first section is devoted to prove the existence and uniqueness of solution with some other important results and not less important to find a solution for the semilinear system with non-instantaneous impulses, infinite delay and non-local conditions (1.1).

3.1.1 Integral formula of the solution

Proposition 3.1 *Let $\mathcal{F}$, G_k and h be smooth functions. Then, problem (1.1) admits a solution $z(\cdot)$ on $(-\infty, \tau]$, if and only if, $z(\cdot)$ satisfies the following integral equation for $k = 1, 2, \ldots$*

$$z(t) = \begin{cases} \mathcal{U}(t,0)[\phi(0) - h(z_{\pi_1}, \ldots, z_{\pi_q})(0)] + \int_0^t \mathcal{U}(t,s)\mathcal{F}(s, z_s)ds, & t \in I_0 = [0, t_1] \\ \mathcal{U}(t, s_k)G_k(s_k, z(t_k^-)) + \int_{s_k}^t \mathcal{U}(t,s)\mathcal{F}(s, z_s)ds, & t \in I_k \\ G_k(t, z(t_k^-))), & t \in J_k, \\ \phi(t) - h(z_{\pi_1}, \ldots, z_{\pi_q})(t) & t \in (-\infty, 0] \end{cases} \tag{3.1}$$

Proof $\Rightarrow$) Suppose that z is a solution of the problem (1.1).

- By the variation of constant formula, for $t \in [0, t_1]$, we obtain

$$z(t) = \mathcal{U}(t,0)[\phi(0) - h(z_{\pi_1}, \ldots, z_{\pi_q})(0)] + \int_0^t \mathcal{U}(t,s)\mathcal{F}(s, z_s)ds$$

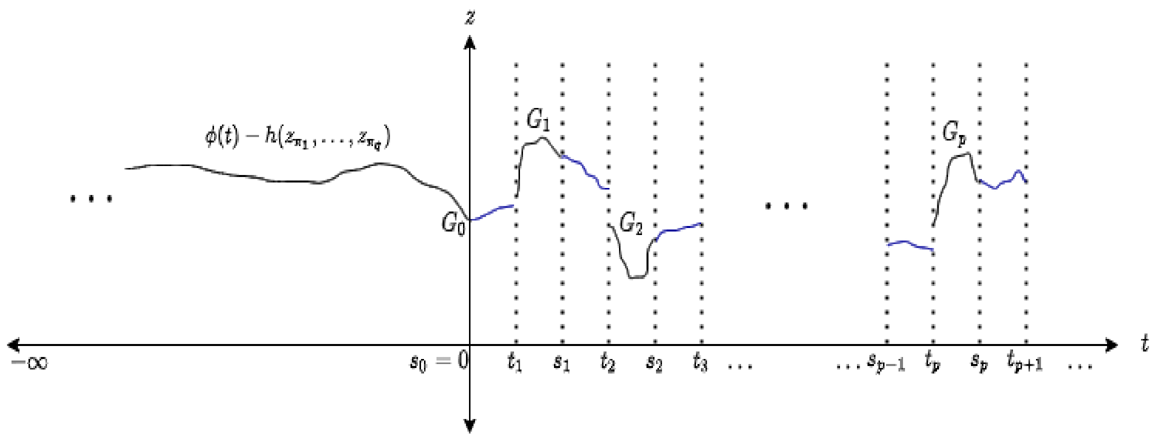

Figure 3.1: Scheme of the behaviour of the solution

- For $t \in I_k$, we use the variation constant formula again

$$\begin{aligned} z(t) &= \mathcal{U}(t, s_k)z(s_k) + \int_{s_k}^{t} \mathcal{U}(t,s)\mathcal{F}(s, z_s)ds \\ &= \mathcal{U}(t, s_k)G_k(s_k, z(t_k^-)) + \int_{s_k}^{t} \mathcal{U}(t,s)\mathcal{F}(s, z_s)ds \end{aligned}$$

The other two cases are explicitly defined, then it is not necessary to prove them.
By continuity, the solution z shall be defined on each s_k, $k = 0, 1, 2, ...$ by

$$z(s_k) = z(s_k^-) = z(s_k^+) = G(s_k, z(t_k^-))$$

and for $k = 0$

$$z(s_0) = z(0)) = \phi(0) - h(z_{\pi_1}, \ldots, z_{\pi_q})(0)$$

$\Leftarrow$)

- Let us consider $t \in (0, t_1]$. Then, applying Leibniz's rule, we get

$$\begin{aligned} z'(t) &= \frac{d}{dt}\left[\mathcal{U}(t,0)[\phi(0) - h(z_{\pi_1}, \ldots, z_{\pi_q})(0)] + \int_0^t \mathcal{U}(t,s)\mathcal{F}(s, z_s)ds\right] \\ &= A(t)\mathcal{U}(t,0)[\phi(0) - h(z_{\pi_1}, \ldots, z_{\pi_q})(0)] + \int_0^t \frac{\partial}{\partial t}\mathcal{U}(t,s)\mathcal{F}(s, z_s)ds \\ &+ \mathcal{U}(t,t)\mathcal{F}(t, z_t) \\ &= A(t)\mathcal{U}(t,0)[\phi(0) - h(z_{\pi_1}, \ldots, z_{\pi_q})(0)] + A(t)\int_0^t \mathcal{U}(t,s)\mathcal{F}(s, z_s)ds + \mathcal{F}(t, z_t) \\ &= A(t)\left\{\mathcal{U}(t,0)[\phi(0) - h(z_{\pi_1}, \ldots, z_{\pi_q})(0)] + \int_0^t \mathcal{U}(t,s)\mathcal{F}(s, z_s)ds\right\} + \mathcal{F}(t, z_t) \\ &= A(t)z(t) + \mathcal{F}(t, z_t). \end{aligned}$$

- Consider now $t \in I_k$, $k = 1, 2, 3, \cdots$

$$\begin{aligned} z'(t) &= \frac{d}{dt}\left[\mathcal{U}(t, s_k)G_k(s_k, z(t_k^-)) + \int_{s_k}^{t} \mathcal{U}(t, s)\mathcal{F}(s, z_s)ds\right] \\ &= A(t)\mathcal{U}(t, s_k)G_k(s_k, z(t_k^-)) + \int_{s_k}^{t} \frac{\partial}{\partial t}\mathcal{U}(t, s)\mathcal{F}(s, z_s)ds + \mathcal{U}(t, t)\mathcal{F}(t, z_t) \\ &= A(t)\left[\mathcal{U}(t, s_k)G_k(s_k, z(t_k^-)) + \int_{s_k}^{t} \mathcal{U}(t, s)\mathcal{F}(s, z_s)ds\right] + \mathcal{F}(t, z_t) \\ &= A(t)z(t) + \mathcal{F}(t, z_t). \end{aligned}$$

Since the other two cases are explicitly defined, it is not necessary to prove them. □

3.1.2 Hypotheses

It is necessary, in order to use Karakosta's fixed point theorem, to state some conditions on the functions and operators. The hypotheses that we shall consider are the following:

(H1) The function $\mathcal{F} : \mathbb{R}_+ \times \mathfrak{H} \longrightarrow \mathbb{R}^n$ satisfies the following conditions:

i) $\|\mathcal{F}(t, z) - \mathcal{F}(t, x)\|_{\mathbb{R}^n} \leq \mathcal{K}(\|z\|_{\mathfrak{H}}, \|x\|_{\mathfrak{H}}) \|z - x\|_{\mathfrak{H}}, \quad \forall z, x \in \mathfrak{H}, \quad \forall t \in I_k$

ii) $\|\mathcal{F}(t, z)\|_{\mathbb{R}^n} \leq \tilde{\psi}(\|z\|_{\mathfrak{H}}), \quad \forall z \in \mathfrak{H},$

where $\mathcal{K} : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ and $\tilde{\psi} : \mathbb{R}_+ \to \mathbb{R}_+$ are continuous and increasing functions.

(H2) There exist constants $d_q, L > 0$ such that, for all $k = 1, 2, ...$, and $y, z \in \mathbb{R}^n, \ell, t \in J_k$ we have that:

i) $\|G_k(t, y) - G_k(\ell, z)\|_{\mathbb{R}^n} \leq L\{|t - \ell| + \|y - z\|_{\mathbb{R}^n}\}$.

ii) There exists $\Theta \geq 0$ such that $\|G_k(t, 0)\| \leq \Theta, \quad k = 1, 2, \cdots, t \in J_k$, and

$$\|h(x) - h(y)\|_{\mathfrak{H}} \leq d_q \|x - y\|_{\mathfrak{H}^q}, \quad \forall x, y \in \mathfrak{H}^q,$$

with $h(0) = 0$, where,

$$M(L + d_q q) < \frac{1}{2}$$

(H3) There exist $\tau, \rho > 0$ such that

$$M\left(\left(d_q q + L\right)(\|\tilde{\phi}\| + \rho) + \tau\tilde{\psi}(\|\tilde{\phi}\| + \rho) + \Theta\right) < \frac{\rho}{2},$$

where the function $\tilde{\phi} \in \mathcal{PW}_{g\tau}$ is defined by

$$\tilde{\phi} = \begin{cases} \mathcal{U}(t, 0)\phi(0), & t \in I_0, \\ \phi(t), & t \in \mathbb{R}_-, \\ 0, & t \in I_k, \\ 0, & t \in J_k. \end{cases} \tag{3.2}$$

Theorem 3.1 *Suppose that the hypothesis (H1)-(H3) hold. Then, system (1.1) has at least one solution on* $(-\infty, \tau]$.

3.1.3 Existence of solutions

Since we want to apply Karakosta's fixed point theorem, we consider the following operators:

$$\mathcal{T} : \mathcal{PW}_{g\tau} \times \mathcal{PW}_{g\tau} \longrightarrow \mathcal{PW}_{g\tau},$$
$$\mathcal{C} : \mathcal{PW}_{g\tau} \longrightarrow \mathcal{PW}_{g\tau},$$

where

$$\mathcal{T}(z,y)(t) = \begin{cases} \phi(t) - h(z_{\pi_1}, \dots, z_{\pi_q})(t), & t \in (-\infty, 0], \\ y(t), & t \in I_0, \\ y(t) + \mathcal{U}(t, s_k) G_k(s_k, z(t_k^-)), & t \in I_k, \\ G_k(t, z(t_k^-))), & t \in J_k \end{cases}$$

and

$$\mathcal{C}(z)(t) = \begin{cases} \mathcal{U}(t,0)[\phi(0) - h(z_{\pi_1}, \dots, z_{\pi_q})(0)] + \int_0^t \mathcal{U}(t,s)\mathcal{F}(s, z_s) ds, & t \in I_0, \\ \int_{s_k}^t \mathcal{U}(t,s)\mathcal{F}(s, z_s) ds, & t \in I_k, \\ \phi(t), & t \in \mathbb{R}_-, \\ 0 & t \in J_k \end{cases}$$

Also, the following closed and convex set

$$D = D(\rho, \tau, \phi) = \left\{ y \in \mathcal{PW}_{g\tau} : \|y - \tilde{\phi}\| \leq \rho \right\}, \tag{3.3}$$

where the function $\tilde{\phi}$ is defined in (3.2). Therefore, the problem of solving system (1.1) is reduced to find solutions of the operator equation

$$\mathcal{T}(z, \mathcal{C}(z)) = z.$$

To find solutions of such equation, we shall apply Karakostas Fixed Point Theorem as it was mentioned. We are going to verify that the operators $\mathcal{C}$ and $\mathcal{T}$ satisfy the assumptions presented in Theorem 2.2. First, we will prove that the operator $\mathcal{C}$ is continuous and that $\mathcal{C}(D)$ is a relatively compact set. After that, we shall prove that $\{\mathcal{T}(\cdot, y) : y \in \overline{\mathcal{C}(D)}\}$ is equicontractive and that $\mathcal{T}(\cdot, \mathcal{C}(\cdot))(D) \subseteq D$. Therefore, we divide the proof in the following steps:

Step 1: $\mathcal{C}$ is a continuous operator.
In order to prove this, we shall use the hypotheses (H1)-i),(H2)-ii) and Lemma 2.3. We have the following equalities for $z, y \in \mathcal{PW}_{g\tau}$.

- Consider $t \in (-\infty, 0]$. Then,

$$\|\mathcal{C}(z)(t) - \mathcal{C}(y)(t)\|_{\mathbb{R}^n} = \|\phi(t) - \phi(t)\|_{\mathbb{R}^n} = 0, \tag{3.4}$$

  that is, $\| \left(\mathcal{C}(z) - \mathcal{C}(y)\right)\Big|_{\mathbb{R}_-} \|_{\mathfrak{H}} = 0.$

- Now, let's consider $t \in (0, t_1]$; then we have that

$$\begin{aligned}\|\mathcal{C}(z)(t)-\mathcal{C}(y)(t)\|_{\mathbb{R}^n} &\leq \Bigg\|\mathcal{U}(t,0)h(z_{\pi_1},\dots,z_{\pi_q})(0)+\int_0^t \mathcal{U}(t,s)\mathcal{F}(s,z_s)ds \\ &-\mathcal{U}(t,0)h(y_{\pi_1},\dots,y_{\pi_q})(0)-\int_0^t \mathcal{U}(t,s)\mathcal{F}(s,y_s)ds\Bigg\|_{\mathbb{R}^n} \\ &\leq M\|h(y_{\pi_1},\dots,y_{\pi_q})(0)-h(z_{\pi_1},\dots,z_{\pi_q})(0)\|_{\mathbb{R}^n} \\ &+M\int_0^t \|\mathcal{F}(s,z_s)-\mathcal{F}(s,y_s)\|_{\mathbb{R}^n}ds \\ &\leq Md_q\|\tilde{z}-\tilde{y}\|_{\mathfrak{H}^q}+M\int_0^t \mathcal{K}(\|z_s\|_{\mathfrak{H}},\|y_s\|_{\mathfrak{H}})\|z_s-y_s\|_{\mathfrak{H}}ds \\ &\leq Md_q q\|z-y\|_{\mathfrak{H}}+M\int_0^t \mathcal{K}(\|z\|,\|y\|)\|z-y\|ds \\ &\leq Md_q q\|z-y\|+Mt_1\mathcal{K}(\|z\|,\|y\|)\|z-y\|\end{aligned}$$

Hence, on the interval $(0, t_1]$, we get that

$$\|\mathcal{C}(z)(t)-\mathcal{C}(y)(t)\|_{\mathbb{R}^n} \leq \Big(Md_q q+M\tau\mathcal{K}(\|z\|,\|y\|)\Big)\|z-y\|. \tag{3.5}$$

- Consider $t \in I_k$, for $k = 1, 2, \dots,$. Then

$$\begin{aligned}\|\mathcal{C}(z)(t)-\mathcal{C}(y)(t)\|_{\mathbb{R}^n} &= \left\|\int_{s_k}^t \mathcal{U}(t,s)\mathcal{F}(s,z_s)ds-\int_{s_k}^t \mathcal{U}(t,s)\mathcal{F}(s,y_s)ds\right\|_{\mathbb{R}^n} \\ &\leq \int_{s_k}^t \|\mathcal{U}(t,s)\|\|\mathcal{F}(s,z_s)-\mathcal{F}(s,y_s)\|_{\mathbb{R}^n}ds \\ &\leq M\int_{s_k}^t \mathcal{K}(\|z_s\|_{\mathfrak{H}},\|y_s\|_{\mathfrak{H}})\|z_s-y_s\|_{\mathfrak{H}}ds \\ &\leq M\mathcal{K}(\|z\|,\|y\|)\|z-y\|\tau.\end{aligned}$$

Therefore, on I_k we get that

$$\|\mathcal{C}(z)(t)-\mathcal{C}(y)(t)\|_{\mathbb{R}^n} \leq \tau M\mathcal{K}(\|z\|,\|y\|)\|z-y\|. \tag{3.6}$$

Since $\|\mathcal{C}(z)(t)-\mathcal{C}(y)(t)\|_{\mathbb{R}^n} = 0$ for $t \in J_k$, $k = 1, 2, \dots,$ we get in combination with (3.4), (3.5) and (3.6) that the operator $\mathcal{C}$ is locally Lipschitz, which implies the continuity of the operator $\mathcal{C}$.

Step 2: $\mathcal{C}$ maps bounded sets of $\mathcal{PW}_{g\tau}$ into bounded sets of $\mathcal{PW}_{g\tau}$.
It is enough to prove that for any $R > 0$ there exists $r > 0$ such that for each $y \in B_R = \Big\{z \in \mathcal{PW}_{g\tau} : \|z\| \leq R\Big\}$, we have that $\|\mathcal{C}(y)\| \leq r$.
Indeed, let's consider an arbitrary $R > 0$ and $z \in B_R$. Then, due to Lemma 2.3 and hypotheses (H1)-ii)-(H2)-ii), we get the following:

- For $t \in (-\infty, 0]$, we obtain that

$$\|\mathcal{C}(z)(t)\|_{\mathbb{R}^n} = \|\phi(t)\|_{\mathbb{R}^n},$$

from which follows that,

$$\Big\|(\mathcal{C}(z))|_{\mathbb{R}_-}\Big\|_{\mathfrak{H}} = \sup_{t\leq 0}\frac{\|\mathcal{C}(z)(t)\|_{\mathbb{R}^n}}{g(t)} = \sup_{t\leq 0}\frac{\|\phi(t)\|_{\mathbb{R}^n}}{g(t)} = \|\phi\|_{\mathfrak{H}} := R_1 \tag{3.7}$$

- For $t \in (0, t_1]$, we have instead that,

$$\begin{aligned}
\|\mathcal{C}(z)(t)\|_{\mathbb{R}^n} &\leq \left\|\mathcal{U}(t,0)\left\{\phi(0) - h(z_{\pi_1}, \ldots, z_{\pi_q})(0)\right\}\right\|_{\mathbb{R}^n} \\
&+ \int_0^t \|\mathcal{U}(t,s)\mathcal{F}(s,z_s)\|_{\mathbb{R}^n} ds \\
&\leq M\|\phi(0) - h(z_{\pi_1}, \ldots, z_{\pi_q})(0)\|_{\mathbb{R}^n} + M\int_0^t \psi(\|z_s\|_{\mathfrak{H}})ds \\
&\leq M\|\phi(0)\|_{\mathbb{R}^n} + M\|h(z_{\pi_1}, \ldots, z_{\pi_q})(0)\|_{\mathbb{R}^n} + Mt_1\psi(\|z\|) \\
&\leq M\|\phi(0)\|_{\mathbb{R}^n} + Md_q\|\tilde{z}\|_{\mathfrak{H}^q} + Mt_1\psi(\|z\|) \\
&\leq M\|\phi(0)\|_{\mathbb{R}^n} + Md_q q\|z\|_{\mathfrak{H}} + Mt_1\psi(\|z\|) \\
&\leq M\|\phi(0)\|_{\mathbb{R}^n} + Md_q q\|z\| + Mt_1\psi(\|z\|) \\
&\leq M\|\phi(0)\|_{\mathbb{R}^n} + Md_q qR + Mt_1\psi(R) := R_2
\end{aligned}$$

- For $t \in I_k$, we have

$$\begin{aligned}
\|\mathcal{C}(z)(t)\|_{\mathbb{R}^n} &= \int_{s_k}^t \|\mathcal{U}(t,s)\mathcal{F}(s,z_s)\|_{\mathbb{R}^n} ds \\
&\leq M\int_{s_k}^t \psi(\|z_s\|_{\mathfrak{H}})ds \\
&\leq M\psi(\|z\|)\tau \leq \tau M\psi(R) := R_3.
\end{aligned}$$

Hence, letting $r = R_1 + R_2 + R_3$, we get that $\|\mathcal{C}(z)\| \leq r$.

Step 3: $\mathcal{C}$ maps bounded sets of $\mathcal{PW}_{g\tau}$ into equicontinuous sets of $PW_{g\tau}$.
Let's consider B_R as it was previously defined in the foregoing step. We shall prove that $\mathcal{C}(B_R)$ is an equicontinuous family.
Since the equicontinuity on $(-\infty, 0]$ is trivial, we only need to prove the equicontinuity in the remain part.
Let's take $y \in B_R$, and consider Lemma 2.3 and hypotheses (H1)-ii), (H2)-ii). Then, we get that

- For $t_1, t_2 \in I_0$ such that $0 < t_1 < t_2$, it turns out that

$$\begin{aligned}
\|\mathcal{C}(y)(t_2) - \mathcal{C}(y)(t_1)\|_{\mathbb{R}^n} = &\Big\|\mathcal{U}(t_2,0)\left\{\phi(0) - h(y_{\pi_1},, \ldots, y_{\pi_q})(0)\right\} \\
&+ \int_0^{t_2} \mathcal{U}(t_2,s)\mathcal{F}(s,y_s)ds \\
&- \mathcal{U}(t_1,0)\left\{\phi(0) - h(y_{\pi_1},, \ldots, y_{\pi_q})(0)\right\} \\
&- \int_0^{t_1} \mathcal{U}(t_1,s)\mathcal{F}(s,y_s)ds\Big\|_{\mathbb{R}^n} \\
\leq &\|(\mathcal{U}(t_2,0) - \mathcal{U}(t_1,0))\left\{\phi(0) - h(y_{\pi_1}, \ldots, y_{\pi_q})(0)\right\}\|_{\mathbb{R}^n} \\
&+ \Big\|\int_0^{t_1} \mathcal{U}(t_2,s)\mathcal{F}(s,y_s)ds + \int_{t_1}^{t_2} \mathcal{U}(t_2,s)\mathcal{F}(s,y_s)ds \\
&- \int_0^{t_1} \mathcal{U}(t_1,s)\mathcal{F}(s,y_s)ds\Big\|_{\mathbb{R}^n}
\end{aligned}$$

$$
\begin{aligned}
&\leq \|\mathcal{U}(t_2,0)-\mathcal{U}(t_1,0)\|\|\phi(0)-h(y_{\pi_1},\ldots,y_{\pi_q})(0)\|_{\mathbb{R}^n}\\
&+\int_0^{t_1}\|\mathcal{U}(t_2,s)-\mathcal{U}(t_1,s)\|\|\mathcal{F}(s,y_s)\|ds\\
&+\left\|\int_{t_1}^{t_2}\mathcal{U}(t_2,s)\mathcal{F}(s,y_s)ds\right\|_{\mathbb{R}^n}\\
&\leq \|\mathcal{U}(t_2,0)-\mathcal{U}(t_1,0)\|\left(\|\phi(0)\|+d_q\|y\|_{\mathfrak{H}^q}\right)\\
&+M\int_{t_1}^{t_2}\tilde{\psi}(\|y_s\|_{\mathfrak{H}})ds+\int_0^{t_1}\|\mathcal{U}(t_2,s)-\mathcal{U}(t_1,s)\|\tilde{\psi}(\|y_s\|_{\mathfrak{H}})ds\\
&\leq \|\mathcal{U}(t_2,0)-\mathcal{U}(t_1,0)\|\left(\|\phi(0)\|+d_q q\|y\|\right)\\
&+M\tilde{\psi}(\|y\|)(t_2-t_1)+\tilde{\psi}(\|y\|)\int_0^{t_1}\|\mathcal{U}(t_2,s)-\mathcal{U}(t_1,s)\|ds\\
&\leq \|\mathcal{U}(t_2,0)-\mathcal{U}(t_1,0)\|\left(\|\phi(0)\|+d_q qR\right)\\
&+M\tilde{\psi}(R)(t_2-t_1)+\tilde{\psi}(R)\int_0^{t_1}\|\mathcal{U}(t_2,s)-\mathcal{U}(t_1,s)\|ds.
\end{aligned}
$$

By the continuity of the evolution operator, we have that

$$\|\mathcal{C}(y)(t_2)-\mathcal{C}(y)(t_1)\|_{\mathbb{R}^n}\to 0 \quad as \quad t_2\to t_1 \tag{3.8}$$

independently on $y\in B_R$.

- for $t_1,t_2\in I_k$ such that $0<t_1<t_2$, we have that

$$
\begin{aligned}
\|\mathcal{C}(y)(t_2)-\mathcal{C}(y)(t_1)\|_{\mathbb{R}^n} &= \left\|\int_{s_k}^{t_2}\mathcal{U}(t_2,s)\mathcal{F}(s,y_s)ds-\int_{s_k}^{t_1}\mathcal{U}(t_1,s)\mathcal{F}(s,y_s)ds\right\|_{\mathbb{R}^n}\\
&= \left\|\int_{s_k}^{t_1}(\mathcal{U}(t_2,s)-\mathcal{U}(t_1,s))\mathcal{F}(s,y_s)ds\right.\\
&\left.+\int_{t_1}^{t_2}\mathcal{U}(t_2,s)\mathcal{F}(s,y_s)ds\right\|_{\mathbb{R}^n}\\
&\leq \int_{s_k}^{t_1}\|\mathcal{U}(t_2,s)-\mathcal{U}(t_1,s)\|\|\mathcal{F}(s,y_s)\|ds\\
&+\int_{t_1}^{t_2}\|\mathcal{U}(t_2,s)\|\|\mathcal{F}(s,y_s)\|ds\\
&\leq \int_{s_k}^{t_1}\|\mathcal{U}(t_2,s)-\mathcal{U}(t_1,s)\|\tilde{\psi}(\|y_s\|_{\mathfrak{H}})ds+M\int_{t_1}^{t_2}\tilde{\psi}(\|y_s\|_{\mathfrak{H}})\\
&\leq \tilde{\psi}(\|y\|)\int_{s_k}^{t_1}\|\mathcal{U}(t_2,s)-\mathcal{U}(t_1,s)\|ds+M\tilde{\psi}(\|y\|)(t_2-t_1)\\
&\leq \tilde{\psi}(R)\int_{s_k}^{t_1}\|\mathcal{U}(t_2,s)-\mathcal{U}(t_1,s)\|ds+M\tilde{\psi}(R)(t_2-t_1)
\end{aligned}
$$

The continuity of $\mathcal{U}(t,s)$ implies that

$$\|\mathcal{C}(y)(t_2)-\mathcal{C}(y)(t_1)\|_{\mathbb{R}^n}\to 0 \quad as \quad t_2\to t_1 \tag{3.9}$$

independently on the chosen y.

Therefore, from the definition of the operator $\mathcal{C}$ and (3.8) and (3.9), we can conclude that B_R is an equicontinuous family.

Step 4: The subset $\mathcal{C}(D)$ is relatively compact in $\mathcal{PW}_{g\tau}$. Without loss of generality we can assume that $t_p \leq \tau$. Let $D \subset \mathcal{PW}_{g\tau}$ be the bounded set defined in (3.3) and let us take a sequence $\{y_n\}_{n\in\mathbb{N}} \subseteq \mathcal{C}(D)$. By steps 2 and 3, it is bounded and equicontinuous in $\mathcal{PW}_{g\tau}$. Note that $y_n|_{(-\infty,0]} = \phi$, then by Arzelá-Ascoli theorem applied to $\left\{y_n|_{(0,t_1]}\right\}_{n\in\mathbb{N}} \subset \mathcal{C}((0,t_1])$, there exist an uniformly convergent subsequence $\{y_n^1\}_{n\in\mathbb{N}}$ on $(-\infty, t_1]$. Let's consider now the sequence $\{y_n^1\}_{n\in\mathbb{N}}$ on the interval $(t_1,t_2]$, which is also bounded and equicontinuous. Then, applying Arzelá-Ascoli theorem, it has a convergent subsequence $\{y_n^2\}_{n\in\mathbb{N}}$ over $(t_1,t_2]$. This sequence is actually an uniformly convergent subsequence of $\{y_n\}_{n\in\mathbb{N}}$ over $(-\infty,t_2]$. We continue this process iteratively over each interval $(t_2,t_3],\cdots,(t_p,\tau]$ and finally arrive to the conclusion that the subsequence $\{y_n^p\}_{n\in\mathbb{N}} \subseteq \{y_n\}_{n\in\mathbb{N}}$ is uniformly convergent on the whole interval $(-\infty,\tau]$. This implies that $\overline{\mathcal{C}(D)}$ is compact, and so the operator $\mathcal{C}$.

Step 5: The family $\{\mathcal{T}(\cdot,y) : y \in \overline{\mathcal{C}(D)}\}$ is equicontractive.

Let us take $z,x \in PW_{g\tau}$ and $y \in \overline{\mathcal{C}(D)}$. Also, consider Lemma 2.3 and (H2), then

- Let us chose $t \in (-\infty,0]$. Then

$$\begin{aligned}
\frac{\|\mathcal{T}(z,\mathcal{C}(y))(t)-\mathcal{T}(x,\mathcal{C}(y))(t)\|_{\mathbb{R}^n}}{g(t)} &= \frac{\left\|h(z_{\pi_1},\ldots,z_{\pi_q})(t)-h(x_{\pi_1},\cdots,x_{\pi_q})(t)\right\|_{\mathbb{R}^n}}{g(t)} \\
&\leq \left\|h(z_{\pi_1},\ldots,z_{\pi_q})-h(x_{\pi_1},\cdots,x_{\pi_q})\right\|_{\mathfrak{H}} \\
&\leq d_q \left\|\tilde{z}-\tilde{x}\right\|_{\mathfrak{H}^q} \\
&\leq d_q q \left\|z-x\right\|_{\mathfrak{H}} \\
&\leq d_q q \|z-x\|.
\end{aligned}$$

  By taking the supremum on $t \in \mathbb{R}_-$, we have that,

$$\left\|(\mathcal{T}(z,\mathcal{C}(y))-\mathcal{T}(x,\mathcal{C}(y)))|_{\mathbb{R}_-}\right\|_{\mathfrak{H}} \leq d_q q\|z-x\|. \tag{3.10}$$

- Let $t \in I_0$. Then, we have that

$$\|\mathcal{T}(z,\mathcal{C}(y))(t)-\mathcal{T}(x,\mathcal{C}(y))(t)\|_{\mathbb{R}^n} = \|y(t)-y(t)\|_{\mathbb{R}^n} = 0.$$

- Let $t \in I_k$. Then,we have

$$\begin{aligned}
\|\mathcal{T}(z,\mathcal{C}(y))(t)-\mathcal{T}(x,\mathcal{C}(y))(t)\|_{\mathbb{R}^n} &= \Big\|\mathcal{U}(t,s_k)G_k(s_k,z(t_k^-)) \\
&\quad -\mathcal{U}(t,s_k)G_k(s_k,x(t_k^-)))\Big\|_{\mathbb{R}^n} \\
&\leq M\left\|G_k(s_k,z(t_k^-))-G_k(s_k,x(t_k^-)))\right\|_{\mathbb{R}^n} \\
&\leq ML\|z-x\|.
\end{aligned}$$

  Thus,

$$\|\mathcal{T}(z,\mathcal{C}(y))(t)-\mathcal{T}(x,\mathcal{C}(y))(t)\|_{\mathbb{R}^n} \leq ML\|z-x\|, \quad t \in (0,\tau]. \tag{3.11}$$

- Consider $t \in J_k$. Then, we get

$$\begin{aligned}\|\mathcal{T}(z,\mathcal{C}(y))(t)-\mathcal{T}(x,\mathcal{C}(y))(t)\|_{\mathbb{R}^n} &\leq \left\|G_k(t,z(t_k^-)))-G_k(t,x(t_k^-)))\right\|_{\mathbb{R}^n}\\ &\leq L\|z(t_k^-))-x(t_k^-))\|_{\mathbb{R}^n}\\ &\leq L\|z-x\|.\end{aligned}$$

  Hence,

$$\|\mathcal{T}(z,\mathcal{C}(y))(t)-\mathcal{T}(x,\mathcal{C}(y))(t)\|_{\mathbb{R}^n} \leq L\|z-x\|. \tag{3.12}$$

Therefore, from the foregoing inequalities and equation (3.12), we get that

$$\|\mathcal{T}(z,\mathcal{C}(y))-\mathcal{T}(x,\mathcal{C}(y))\| < \frac{1}{2}\|z-x\|.$$

which is a contraction independently of $y \in \overline{\mathcal{C}(D)}$. So , the family $\{\mathcal{T}(\cdot,y) : y \in \overline{\mathcal{C}(D)}\}$ is equicontractive.

Step 6: Finally, we shall prove that

$$\mathcal{T}(\cdot,\mathcal{C}(\cdot))(D(\rho,\tau,\phi)) \subseteq D(\rho,\tau,\phi)$$

Let us consider $z \in D(\rho,\tau,\phi)$. In order to prove Step 6, we shall take into consideration Lemma 2.3, the hypotheses (H2)-ii), (H1)-ii) and (H3).

- Let $t \in (-\infty,0]$. Then, we have the following estimate

$$\begin{aligned}\frac{1}{g(t)}\|\mathcal{T}(z,\mathcal{C}(z))(t)-\tilde{\phi}(t)\|_{\mathbb{R}^n} &= \frac{1}{g(t)}\|h(z_{\pi_1},\dots,z_{\pi_q})(t)\|_{\mathbb{R}^n}\\ &\leq d_q\|\tilde{z}\|_{\mathfrak{H}^q}\\ &\leq d_q q\|z\|_{\mathfrak{H}}\\ &\leq d_q q\|z\|\\ &\leq d_q q(\|\tilde{\phi}\|+\rho) < \rho/2.\end{aligned}$$

- Next, for $t \in I_0$, we get that

$$\begin{aligned}\|\mathcal{T}(z,\mathcal{C}(z))(t)-\tilde{\phi}(t)\|_{\mathbb{R}^n} &\leq Md_q\left\|\tilde{z}\right\|_{\mathfrak{H}^q}+\int_0^t\|\mathcal{U}(t,s)\mathcal{F}\left(s,z_s\right)\|_{\mathbb{R}^n}\,ds\\ &\leq Md_q q\left\|z\right\|_{\mathfrak{H}}+M\int_0^t\tilde{\psi}(\|z_s\|_{\mathfrak{H}})ds\\ &\leq Md_q q\left\|z\right\|+t_1M\tilde{\psi}(\|z\|)\\ &\leq Md_q q(\|\tilde{\phi}\|+\rho)+M\tau\tilde{\psi}(\|\tilde{\phi}\|+\rho)\\ &< \rho/2.\end{aligned}$$

- Considering $t \in I_k$, we get that

$$\begin{aligned}\|\mathcal{T}(z,\mathcal{C}(z))(t)-\tilde{\phi}(t)\|_{\mathbb{R}^n} \leq& \|\mathcal{U}(t,s_k)G_k(s_k,z(t_k^-))\|+\int_{s_k}^t\|\mathcal{U}(t,s)\mathcal{F}(s,z_s)\|\,ds\\ \leq& \|\mathcal{U}(t,s_k)\|\left\|G_k(s_k,z(t_k^-))-G_k(s_k,0)+G_k(s_k,0)]\right\|\\ &+\int_{s_k}^t\|\mathcal{U}(t,s)\|\|\mathcal{F}(s,z_s)\|\,ds\\ \leq& M[L(\|\tilde{\phi}\|+\rho)+\tau\tilde{\psi}(\|\tilde{\phi}\|+\rho)+\Theta] < \rho/2\end{aligned}$$

- Finally, to complete this part, if $t \in J_k$, we get that

$$\begin{aligned}\|\mathcal{T}(z,\mathcal{C}(z))(t)-\tilde{\phi}(t)\|_{\mathbb{R}^n} &= \|G_k(t,z(t_k^-)))\| \leq L\|z\|+\Theta\\ &\leq L(\|\tilde{\phi}\|+\rho)+\Theta<\rho/2\end{aligned}$$

Hence, $\mathcal{T}(\cdot,\mathcal{C}(\cdot))D(\rho,\tau,\phi)\subseteq D(\rho,\tau,\phi)$.
Since Step 1, Step 4 and Step 5 hold, the conditions of Karakostas Fixed Point Theorem are satisfied for the closed and convex set given in (3.3), and the proof of Theorem 3.1 immediately follows by applying Theorem 2.2. □

3.1.4 Uniqueness and prolongation of solutions

Theorem 3.2 *In addition to the conditions of Theorem 3.1, we suppose that for* $\rho,\tau>0$ *the following inequality holds*

$$\tau M\mathcal{K}(\|\tilde{\phi}\|+\rho,\|\tilde{\phi}\|+\rho)+M[d_q q+L]<\frac{1}{2}.$$

Then, the problem (1.1) has only one solution on $(-\infty,\tau]$.

In order to prove the uniqueness of the solution, let z^1 and z^2 be two solutions for problem (1.1). Then, we have that:

- Consider the following estimate for $t\in(-\infty,0]$:

$$\begin{aligned}\frac{1}{g(t)}\|z^1(t)-z^2(t)\|_{\mathbb{R}^n} &= \frac{1}{g(t)}\|h(z^2_{\pi_1},\dots,z^2_{\pi_q})(t)-h(z^1_{\pi_1},\dots,z^1_{\pi_q})(t)\|\\ &\leq d_q\|\tilde{z^2}-\tilde{z^1}\|_{\mathfrak{H}^q}\\ &\leq d_q q\|z^2-z^1\|_{\mathfrak{H}}\\ &<\frac{1}{2}\|z^2-z^1\|.\end{aligned}$$

- Now, let $t\in(0,t_1]$,

$$\begin{aligned}\|z^2(t)-z^1(t)\| &\leq \|\mathcal{U}(t,0)[\phi(0)-h(z^2_{\pi_1},\dots,z^2_{\pi_q})(0)]+\int_0^t \mathcal{U}(t,s)\mathcal{F}(s,z^2_s)ds\\ &-\mathcal{U}(t,0)[\phi(0)-h(z^1_{\pi_1},\cdots,z^1_{\pi_q})(0)]-\int_0^t\mathcal{U}(t,s)\mathcal{F}(s,z^1_s)ds\|\\ &\leq M\|h(z^1_{\pi_1},\dots,z^1_{\pi_q})(0)-h(z^2_{\pi_1},\dots,z^2_{\pi_q})(0)\|_{\mathbb{R}^n}\\ &+\int_0^t\|\mathcal{U}(t,s)\|\|\mathcal{F}(s,z^2_s)-\mathcal{F}(s,z^1_s)\|_{\mathbb{R}^n}ds\\ &\leq Md_q\|z^1-z^2\|_{\mathfrak{H}^q}+M\int_0^t\mathcal{K}(\|z^2_s\|_{\mathfrak{H}},\|z^1_s\|_{\mathfrak{H}})\|z^1_s-z^2_s\|_{\mathfrak{H}}ds\\ &\leq\left\{Md_q q+Mt_1\mathcal{K}(\|z^2\|,\|z^1\|)\right\}\|z^1-z^2\|\\ &\leq\left\{Md_q q+Mt_1\mathcal{K}(\|\tilde{\phi}\|+\rho,\|\tilde{\phi}\|+\rho)\right\}\|z^1-z^2\|\\ &<\frac{1}{2}\|z^1-z^2\|.\end{aligned}$$

- Now, we consider $t \in I_k$. Then

$$\begin{aligned}
\|z^2(t) - z^1(t)\| = & \bigg\| \mathcal{U}(t, s_k) G_k(s_k, z^2(t_k^-)) + \int_{s_k}^t \mathcal{U}(t,s)\mathcal{F}(s, z_s^2) \\
& - \mathcal{U}(t, s_k) G_k(s_k, z^1(t_k^-)) - \int_{s_k}^t \mathcal{U}(t,s)\mathcal{F}(s, z_s^2) ds \bigg\| \\
& \leq M \|G_k(s_k, z^2(t_k^-)) - G_k(s_k, z^1(t_k^-))\| \\
& + M \int_{s_k}^t \|\mathcal{F}(s, z_s^2) - \mathcal{F}(s, z_s^1)\|_{\mathbb{R}^n} ds \\
& \leq ML\|z^2 - z^1\| + M \int_{s_k}^t \mathcal{K}(\|z_s^2\|_{\mathfrak{H}}, \|z_s^1\|_{\mathfrak{H}}) \|z_s^2 - z_s^1\|_{\mathfrak{H}} ds \\
& \leq ML\|z^2 - z^1\| + M(t_{k+1} - s_k)\mathcal{K}(\|z^2\|, \|z^1\|)\|z^2 - z^1\| \\
& \leq ML\|z^2 - z^1\| + M(t_{k+1} - s_k)\mathcal{K}(\|\tilde{\phi}\| + \rho, \|\tilde{\phi}\| + \rho)\|z^2 - z^1\| \\
& \leq \left[ML + M\tau\mathcal{K}(\|\tilde{\phi}\| + \rho, \|\tilde{\phi}\| + \rho) \right] \|z^2 - z^1\| \\
& < \frac{1}{2}\|z^2 - z^1\|.
\end{aligned}$$

- Consider $t \in J_k$. Then we have that

$$\begin{aligned}
\|z^2(t) - z^1(t)\| &= \|G_k(t, z^2(t_k^-)) - G_k(t, z^1(t_k^-))\| \\
&\leq L\|z^2(t_k^-) - z^1(t_k^-)\| \\
&< \frac{1}{2}\|z^2 - z^1\|.
\end{aligned}$$

Hence, from the foregoing inequalities and the last expression, we get that

$$\|z^2 - z^1\| < \|z^2 - z^1\|,$$

which implies that $z^1 = z^2$. □

In remaining part of this subsection we shall study the prolongation of the solutions of problem (1.1). To this end, we shall consider the following subset $\tilde{D}$ of $\mathbb{R}^n$:

$$\tilde{D} = \{y \in \mathbb{R}^n : \|y\|_{\mathbb{R}^n} \leq \rho\}. \tag{3.13}$$

Therefore, for all $z \in D$, we have that $z(t) - \tilde{\phi}(t) \in \tilde{D}$ for $-\infty < t \leq \tau$.

Definition 3.1 *We shall say that $(-\infty, \tau_1)$ is a maximal interval of existence for the solution $z(\cdot)$ of problem (1.1) if there is not solution of the (1.1) on $(-\infty, \tau_2)$ with $\tau_2 > \tau_1$.*

Theorem 3.3 *Suppose that the conditions of Theorem 3.2 hold. If z is a solution of problem (1.1) on $(-\infty, \tau_1)$ and τ_1 is maximal, then either $\tau_1 = +\infty$ or there exists a sequence $\tau_n \to \tau_1$ as $n \to \infty$ such that $z(\tau_n) - \tilde{\phi}(\tau_n) \to \partial\tilde{D}$.*

Proof Suppose, for the purpose of contradiction, that $\tau_1 < \infty$ and there exist a neighborhood N of $\partial\tilde{D}$ such that $z(t) - \tilde{\phi}(t)$ does not enter in it, for $0 < s_2 \le t < \tau_1$. We can take $N = \tilde{D} \backslash B$, where B is a closed subset of $\tilde{D}$, then $z(t) - \tilde{\phi}(t) \in B$ for $0 < s_{p-1} < t < \tau_1$. We need to prove that $\lim_{t \to \tau_1^-} \{z(t) - \tilde{\phi}(t)\} = z_1 - \tilde{\phi}(\tau_1) \in B$. For that purpose, it enough to prove that $\lim_{t \to \tau_1^-} z(t) = z_1$, and we will divide the proof in two cases:

- Suppose that $0 \le s_{p-1} < t_p \le t < \tau_1$. Then consider $t, \ell > 0$ such that
$$0 < t_p < \ell < t < \tau_1 \le s_p.$$
Hence $t, l \in J_p$ and
$$\|z(t) - z(\ell)\| = \|G_p(t, z(t_p^-) - G_p(\ell, z(t_p^-))\| \le L\{|t - \ell|\}.$$
Then
$$\|z(t) - z(\ell)\| \le L|t - \ell| \to 0 \quad \text{as} \quad t, \ell \to 0.$$
Therefore, $\lim_{t \to \tau_1} z(t) = z_1$ exists in $\mathbb{R}^n$, and since B is closed, $z_1 - \tilde{\phi}(\tau_1)$ belongs to B.

- Suppose that $0 \le s_{p-1} < \tau_1 \le t_p$. Indeed, if we consider $0 \le s_{p-1} < \ell < t < \tau_1 \le t_p$, then $t, \ell \in I_p$ and
$$\begin{aligned}
\|z(t) - z(\ell)\|_{\mathbb{R}^n} &\le \|\mathcal{U}(t, s_{p-1}) - \mathcal{U}(\ell, s_{p-1})\|\|G_p(s_{p-1}, z(t_{p-1}^-))\|_{\mathbb{R}^n} \\
&+ \int_{s_{p-1}}^{\ell} \|\mathcal{U}(t, s) - U(\ell, s)\|\|f(s, z_s)\|ds \\
&+ \int_{\ell}^{t} \|(\mathcal{U}(t, s)\|\|f(s, z_s)\|ds \\
&\le \|\mathcal{U}(t, s_{p-1}) - \mathcal{U}(\ell, s_{p-1})\|\|G_p(s_{p-1}, z(t_{p-1}^-))\|_{\mathbb{R}^n} \\
&+ \left(\int_{s_{p-1}}^{\ell} \|\mathcal{U}(t, s) - \mathcal{U}(\ell, s)\|ds + \int_{\ell}^{t} \|(\mathcal{U}(t, s)\|ds\right)\tilde{\psi}(\|z\|) \\
&\le \|\mathcal{U}(t, s_{p-1}) - \mathcal{U}(\ell, s_{p-1})(\|L\|z\| + \Theta) \\
&+ \left(\int_{s_{p-1}}^{\ell} \|\mathcal{U}(t, s) - \mathcal{U}(\ell, s)\|ds + \int_{\ell}^{t} \|(\mathcal{U}(t, s)\|ds\right)\tilde{\psi}(\|z\|) \\
&\le \|\mathcal{U}(t, s_{p-1}) - \mathcal{U}(\ell, s_{p-1})(\|L\|z\| + \Theta) \\
&+ \left(\int_{s_{p-1}}^{\ell} \|\mathcal{U}(t, s) - \mathcal{U}(\ell, s)\|ds + \int_{\ell}^{t} \|(\mathcal{U}(t, s)\|ds\right)\tilde{\psi}(R)
\end{aligned}$$

Since $\mathcal{U}(t, s)$ is uniformly continuous on $t, s \ge 0$, then $\|z(t) - z(l)\|_{\mathbb{R}^n}$ goes to zero as $\ell \to \tau_1$. Therefore, $\lim_{t \to \tau_1} z(t) = z_1$ exists in $\mathbb{R}^n$, and since B is closed, $z_1 - \tilde{\phi}(\tau_1)$ belongs to B. This will contradict the maximality of τ_1. In fact, we have that $z_1 \in B + \tilde{\phi}(\tau_1)$,which is contained in the interior of the ball $\tilde{D} + \tilde{\phi}(\tau_1)$. Hence, $z(\cdot)$ can be extended to $(-\infty, \tau_1]$. In this regard, for ϵ small enough, the following initial value problem admit only one solutions on $(-\infty, \tau_1 + \epsilon)$

$$\begin{cases} v'(t) = A(t)v(t) + f(t, v_t), & t \in (\tau_1, \tau_1 + \epsilon) \\ v(s) + h(v_{\pi_1}, v_{\pi_2}, v_{\pi_3}, \ldots, v_{\pi_q})(s) = \phi(s), & s \in (-\infty, \tau_1], \end{cases} \tag{3.14}$$

This is a contradiction with the maximality of τ_1. So, the proof is completed.

Corollary 3.1 *Under the conditions of Theorem 3.1, and assuming the following condition*

$$\|\mathcal{F}(t,\phi)\| \leq \mu(t)(1+\|\phi(0)\|_{\mathbb{R}^n}), \quad \phi \in \mathfrak{H}, \quad t \in \mathbb{R}, \tag{3.15}$$

where $\mu(\cdot)$ *is a continuous function on* $(-\infty,\infty)$*, the unique solution of problem (1.1) exists on* $(-\infty,\infty)$*.*

Proof We will divide the proof in two cases:

- suppose that $0 \leq s_{p-1} < t_p < \tau_1$. Then $0 \leq s_{p-1} < t_p < t < \tau_1 \leq s_p$. Therefore,

$$\|z(t)\|_{\mathbb{R}^n} = \|G_p(t, z(t_p^-))\|_{\mathbb{R}^n} \leq L\|z(t_p^-)\| + \Theta.$$

- Suppose that $0 \leq s_{p-1} < \tau_1 \leq t_p$. Then, for $t \in [s_{p-1}, \tau_1] \subset I_p$ we have that

$$\begin{aligned}
\|z(t)\|_{\mathbb{R}^n} &\leq \|\mathcal{U}(t,s_{p-1})\|\|G_k(s_{p-1}, z(t_p^-))\|_{\mathbb{R}^n} + \int_{s_{p-1}}^{t} \|\mathcal{U}(t,s)\|\|\mathcal{F}(s,z_s)\|ds \\
&\leq M(L\|z(t_p^-)\|_{\mathbb{R}^n} + \Theta) + \int_{s_{p-1}}^{t} M\mu(s)(1+\|z(s)\|_{\mathbb{R}^n})ds \\
&\leq M(L\|z(t_p^-)\|_{\mathbb{R}^n} + \Theta) + \int_{s_{p-1}}^{\tau} M\mu(s)ds + \int_{s_{p-1}}^{t} M\mu(s)\|z(s)\|_{\mathbb{R}^n}ds \\
&\leq M\left(L\|z(t_p^-)\|_{\mathbb{R}^n} + \Theta + \int_{s_{p-1}}^{\tau} \mu(s)ds\right) + \int_{s_{p-1}}^{t} M\mu(s)\|z(s)\|_{\mathbb{R}^n}ds
\end{aligned}$$

Then, applying Gronwall Inequality, we obtain that

$$\|z(t)\|_{\mathbb{R}^n} \leq M\left(L\|z(t_p^-)\|_{\mathbb{R}^n} + \Theta + \int_{s_{p-1}}^{\tau} \mu(s)ds\right) e^{\int_{s_{p-1}}^{\tau} M\mu(s)ds},$$

The two cases imply that $\|z(t)\|_{\mathbb{R}^n}$ remains bounded as $t \to \tau_1$, then applying Theorem 3.3 we get the required result.

3.1.5 Application

In this section we shall consider an example of semi-linear system with infinite delay, non-instantaneous impulses and non-local conditions where Theorem 3.1 can be applied

$$\begin{cases}
z'(t) = -z(t) + e^{-\frac{z_t(-1)}{10(t+5)^3}}, \quad t \in I_k \\
z(s) = \left(1 + \frac{\sin z}{30^2}\right)(s) + \phi(s), \quad s \in (-\infty, 0] \\
z(t) = \frac{\sin(z(t_k^-))}{4(t_k+8)^4} \cdot \cos(t - t_k), \quad t \in J_k
\end{cases} \tag{3.16}$$

In this case we have that the terms involving system (1.1) are given by: $A(t) = -1$, $\mathcal{F}(t,z) = exp\left\{-\frac{z}{10(t+5)^3}\right\}$, $h(z) = 1 + \frac{\sin(z)}{30^2}$ and $G_k(t,z) = \frac{\sin(z)}{4(t_k+8^4)} \cdot \cos(t-t_k)$. Then. we have,

$$\begin{array}{rcl} |\mathcal{F}(t,z)-\mathcal{F}(t,x)| & = & \left|e^{-\frac{z}{10(t+5)^3}}-e^{-\frac{x}{10(t+5)^3}}\right| \leq \frac{1}{10\cdot 5^3}|z-\tilde{z}|\,, \\ |G_k(t,z)-G_k(t,x)| & \leq & \frac{1}{4(t+8)^4}|\sin(z)-\sin(x)| \leq \frac{1}{4\cdot 8^4}|z-x|\,, \\ |h(z)-h(x)| & = & \frac{1}{30^2}|\sin(z)-\sin(x)| \leq \frac{1}{30^2}|z-x|\,, \end{array} \tag{3.17}$$

In this case, we have that

$$q=1, \qquad \mathcal{U}(t,s)=e^{-(t-s)}, \qquad M=1, \qquad \mathcal{K}=\frac{1}{10\cdot 5^3}$$

and

$$|\mathcal{F}(t,z)| \leq |\mathcal{F}(t,z)-\mathcal{F}(t,0)| + |\mathcal{F}(t,0)| \leq \frac{1}{10\cdot 5^3}|z|+1. \tag{3.18}$$

Therefore, if we put $\tilde{\Psi}(\xi)=\frac{1}{10\cdot 5^3}\xi+1$, with $\xi \geq 0$, then

$$|\mathcal{F}(t,z)| \leq \tilde{\Psi}(|z|)$$

Now, for $\varepsilon > 0$ small enough, let's take as initial function

$$\phi(s)=\varepsilon cos(s), \qquad s\in\mathbb{R}$$

and define

$$\tilde{\phi}(t)=\begin{cases} e^{-t}\phi(0) & ,t\geq 0 \\ \phi(t) & t\in(-\infty,0] \end{cases}$$

Then, we have

$$\begin{aligned} \tilde{\psi}\left(\left\|\tilde{\phi}\right\|+\rho\right) &= \frac{\left\|\tilde{\phi}\right\|+\rho}{10\cdot 5^3}+1 \\ &\leq \frac{\varepsilon+\rho}{10\cdot 5^3}+1 \end{aligned}$$

Therefore, the last condition of the hypothesis H2-ii) is satisfied. In fact,

$$\begin{aligned} M(L+d_q q) &= (L+d_q) \\ &= \left(\frac{1}{4\cdot 8^4}+\frac{1}{30^2}\right) \\ &= \frac{4321}{3686400} \\ &\leq \frac{1}{2} \end{aligned} \tag{3.19}$$

Since $G_k(t,0)=0$, then, $\Theta=0$. Hence, the condition of the hypothesis H3) is satisfied. In fact, the following inequality

$$\begin{aligned} M(d_q q+L)\left(\left\|\tilde{\phi}+\rho\right\|\right)+\tau\tilde{\psi}\left(\left\|\tilde{\phi}\right\|+\rho\right) &= (d_q+L)\left(\left\|\tilde{\phi}+\rho\right\|\right)+\tau\tilde{\psi}\left(\left\|\tilde{\phi}\right\|+\rho\right) \\ &= \left(\frac{1}{4\cdot 8^4}+\frac{1}{30^2}\right)(\varepsilon+\rho)+\tau\left(\frac{\varepsilon+\rho}{10\cdot 5^3}+1\right) \\ &\leq \frac{\rho}{2} \end{aligned}$$

holds for infinitely many values of τ, ρ and ε. In particular, we can take, for example, $\tau = \frac{1}{4}$, $\rho = 1$ and $\varepsilon = 1$. So, we get that

$$\begin{aligned}
\left(\frac{1}{4\cdot 8^4}+\frac{1}{30^2}\right)(\varepsilon+\rho)+\tau\left(\frac{\varepsilon+\rho}{10\cdot 5^3}+1\right) &= \left(\frac{1}{4\cdot 8^4}+\frac{1}{30^2}\right)(1+1)+\frac{1}{4}\left(\frac{1+1}{10\cdot 5^3}+1\right)\\
&= \frac{8642}{3686400}+\frac{1}{4}\left(\frac{1252}{1250}\right)\\
&= \frac{23292389}{92160000}\\
&\le \frac{1}{2}.
\end{aligned} \tag{3.20}$$

Thus, by (3.18),(3.17),(3.19),(3.20) we have that H1)-H3) holds. So, Theorem 3.1 ensures the existence of solutions for problem (3.16).

3.2 Approximate controllability

In order to study the controllability of system (1.2), with techniques that evade the use of fixed point Theorems, We will assume the following conditions on the nonlinear term f.

$$|f(t,\varphi,u)| \le \zeta(\|\varphi(-t_p)\|) \quad u\in\mathbb{R}^m, \quad \varphi\in\mathfrak{H}, \quad t\in[0,\tau] \tag{3.21}$$

where $\zeta : \mathbb{R}_+ \to [0,\infty)$ is a continuous function. In particular, $\zeta(\xi) = a(\xi)^\beta + b$, with $\beta \ge 1$.

Also, we shall assume the following hypothesis:

A1) The linear control system (2.11) is exactly controllable on any interval $[\tau-\delta,\tau]$, for all δ with $0<\delta<\tau$.

Since, the system (1.2) was slightly changed, the solution was also altered. Therefore, once the existence and uniqueness of solution is proved, for the controllability part, the solution for $k = 1,\dots,p$ is given by

$$z(t) = \begin{cases}
\quad \mathcal{U}(t,0)[\varphi(0)-h(z_{\pi_1},\dots,z_{\pi_q})(0)+\int_0^t \mathcal{U}(t,s)f(s,z_s,u(t))ds & \\
+\int_0^t \mathcal{U}(t,s)\mathcal{B}(s)u(s)ds, & t\in I_0\\
\quad \mathcal{U}(t,s_k)G_k(t,z(t_k^-))+\int_{s_k}^t \mathcal{U}(t,s)f(s,z_s,u(t))ds & \\
+\int_{s_k}^t \mathcal{U}(t,s)\mathcal{B}(s)u(s)ds, & t\in I_k\\
G_k(t,z(t_k^-)), & t\in J_k\\
\quad \varphi(t)-h(z_{\pi_1},\dots,z_{\pi_q})(t), & t\in(-\infty,0]
\end{cases} \tag{3.22}$$

Theorem 3.4 *If the functions f, G_k, h are smooth enough, condition* (3.21) *holds and the linear system* (2.11) *is exact controllable on any interval $[\tau-\delta,\tau]$, $0<\delta<\tau$, then system* (1.2) *is approximately controllable on $[0,\tau]$.*

Proof Consider $\phi \in \mathfrak{H}$, a final state z^1 and $\epsilon > 0$, we want to find a control $u^\epsilon \in L^2(0,\tau;\mathbb{R}^m)$ steering the system to a ball of center z^1 and radius $\epsilon > 0$ on $[0,\tau]$. In indeed, we consider any fixed control $u \in L^2(0,\tau;\mathbb{R}^m)$ and the corresponding solution $z(t) = z(t,0,\phi,u)$ of the problem (1.2).
For $0 < \delta < min\{\tau - s_p, s_p, \frac{\epsilon}{MK}\}$, we define the control $u^\epsilon \in L^2(0,\tau;\mathbb{R}^m)$ as follows

$$u^\epsilon(t) = \begin{cases} u(t), & \text{if } 0 \leq t \leq \tau - \delta, \\ v^\delta(t), & \text{if } \tau - \delta < t \leq \tau. \end{cases}$$

where $K = \sup_{s\in[0,\tau]}\{\zeta(\|z(s)\|\}$ and

$$v^\delta(t) = B^*(t)\mathcal{U}^*(\tau,t)(\mathcal{W}_{\tau\delta})^{-1}(z^1 - \mathcal{U}(\tau,\tau-\delta)z(\tau-\delta)), \quad \tau - \delta < t \leq \tau.$$

Since $0 < \delta < \tau - s_p$, then $\tau - \delta > s_p$; and using the cocycle property $\mathcal{U}(t,l)\mathcal{U}(l,s) = \mathcal{U}(t,s)$, the associated solution $z^\delta(t) = z(t,0,\phi,u^\epsilon)$ of the time-dependent impulsive semilinear retarded differential equation with infinite delay and nonlocal (1.2), at time τ, can be expressed as follows:

$$\begin{aligned} z^\delta(\tau) &= \mathcal{U}(\tau,s_p)G_k(s_p,z^\delta(s_p^-)) + \int_{s_p}^{\tau}\mathcal{U}(\tau,s)f(s,z_s,u^\epsilon(s))ds + \int_{s_p}^{\tau}\mathcal{U}(\tau,s)\mathcal{B}(s)u^\epsilon(s)ds \\ &= \mathcal{U}(\tau,\tau-\delta)\Big\{\mathcal{U}(\tau-\delta,s_p)G_p(s_p,z^\delta(s_p^-)) + \int_{s_p}^{\tau-\delta}\mathcal{U}(\tau-\delta,s)[\mathcal{B}(s)u(s)ds \\ &\quad + f(s,z_s^\delta,u(s))]ds\Big\} + \int_{\tau-\delta}^{\tau}\mathcal{U}(\tau,s)[\mathcal{B}(s)v^\delta(s) + f(s,z_s^\delta,v^\delta(s))]ds \end{aligned}$$

Therefore,

$$z^\delta(\tau) = \mathcal{U}(\tau,\tau-\delta)z(\tau-\delta) + \int_{\tau-\delta}^{\tau}\mathcal{U}(\tau,s)[\mathcal{B}(s)v^\delta(s) + f(s,z_s^\delta,v^\delta(s))]ds.$$

The corresponding solution $y^\delta(t) = y(t,\tau-\delta,z(\tau-\delta),v^\delta)$ of the initial value problem (2.11) at time τ, for the control v^δ and the initial condition $z^0 = z(\tau-\delta)$, is given by:

$$y^\delta(\tau) = \mathcal{U}(\tau,\tau-\delta)z(\tau-\delta) + \int_{\tau-\delta}^{\tau}\mathcal{U}(\tau,s)\mathcal{B}(s)v^\delta(s)ds,$$

and because of our assumption, we get that the conditions in (2.15) holds, particularly:

$$y^\delta(\tau) = z^1.$$

Thus,

$$\left\|z^\delta(\tau) - z_1\right\| \leq \int_{\tau-\delta}^{\tau}\|\mathcal{U}(\tau,s)\|\left\|f(s,z_s^\delta,v^\delta(s))\right\|ds. \tag{3.23}$$

Now, since $0 < \delta < s_p$ and $\tau - \delta \leq s \leq \tau$, then $s - s_p \leq \tau - s_p < \tau - \delta$ and

$$z^\delta(s - s_p) = z(s - s_p).$$

Hence, since δ satisfies $0 < \delta < \min\{s_p, \tau - s_p, \frac{\epsilon}{MK}\}$, from (3.23) we get:

$$\begin{aligned}\left\|z^{\delta}(\tau)-z_1\right\| &\leq \int_{\tau-\delta}^{\tau}\|\mathcal{U}(\tau,s)\|\left\|f(s,z_s,v^{\delta}(s))\right\|ds\\ &\leq M\int_{\tau-\delta}^{\tau}\zeta(\left\|z(s-s_p)\right\|)ds\\ &\leq MK(\tau-\tau+\delta)\leq\epsilon\end{aligned}$$

which completes the proof. □

The geometric representation of this theorem can be found below:

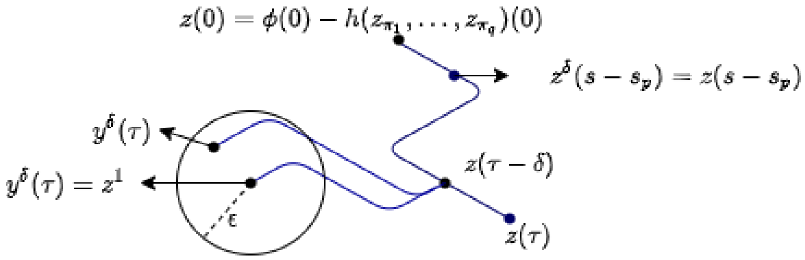

Figure 3.2: Geometric representation of Theorem 3.4

3.3 Exact controllability

The main objective of this section is to use Rothe's fixed point to prove that the system in (1.2) is exactly controllable. In order to do that, we shall consider the following hypotheses:

E1) The nonlinear function $f:\mathbb{R}_+\times\mathfrak{H}\times\mathbb{R}^m\longrightarrow\mathbb{R}^n$ satisfies

$$\|f(t,\nu_t,u)\|_{\mathbb{R}^n}\leq a_0\|\nu_t\|_{\mathfrak{H}}^{\alpha_0}+b_0\|u\|_{\mathbb{R}^m}^{\beta_0}+c_0,\quad t\in(0,\tau],\nu\in\mathfrak{H},u\in\mathbb{R}^m.$$

E2) The non instantaneous impulses function, $G_k\in\mathcal{C}\left((t_k,s_k]\times\mathbb{R}^n;\mathbb{R}^n\right)$ for all $k=1,2,3,\ldots,p$ and satisfies:

$$\|G_k(t,z)\|_{\mathbb{R}^n}\leq a_k\|z\|_{\mathbb{R}^n}^{\alpha_k}+c_k,$$

and

$$\|\,G_k(s,z)-G_k(t,w)\|\leq d_k\left(|s-t|+\|z-w\|\right).$$

E3) The function for the non local condition $h:\mathfrak{H}^q:\longrightarrow\mathfrak{H}$ satisfies, for $z,w\in\mathfrak{H}^q$, the following conditions:

$$\|h(z)\|_{\mathfrak{H}}\leq c\|z\|_{\mathfrak{H}^q}^{\eta},$$

and

$$\|h(z)-h(w)\|_{\mathfrak{H}}\leq d_q\|z-w\|_{\mathfrak{H}^q},$$

where $\eta,\alpha_k,\beta_0\in[0,1)$ and a_k,b_0,c_k,d_k,c,d_q are positive constants with $k=0,1,2,3,\cdots,p$.

In addition to the conditions imposed to the operators and functions involving the system, it is also necessary to define some operators that help us to prove the controllability of the system (1.2).

$$\begin{aligned} \mathcal{S}_1 : \mathcal{PW}_{g\tau} \times \mathcal{PW}_u &\longrightarrow \mathcal{PW}_{g\tau} \\ (z,u)(t) &\longmapsto y(t) := \mathcal{S}_1(z,u)(t) \end{aligned}$$

$$\begin{aligned} \mathcal{S}_2 : \mathcal{PW}_{g\tau} \times \mathcal{PW}_u &\longrightarrow \mathcal{PW}_u \\ (z,u)(t) &\longmapsto v(t) := \mathcal{S}_2(z,u)(t) \end{aligned}$$

where, arbitrary states $z^{t_{k+1}}, with \quad k = 0, 1, 2, \dots, p$, are given by:

$$y(t) = \begin{cases} \varphi(t) - h(z_{\pi_1}, \dots, z_{\pi_q})(t), & t \in (-\infty, 0] \\ \mathcal{U}(t,0)[\varphi(0) - h(z_{\pi_1}, \dots, z_{\pi_q})(0) + \int_0^{t_1} \mathcal{U}(t,s) f(s, z_s, u(t)) ds & \\ + \int_0^{t_1} \mathcal{U}(t,s) B(s)(\Upsilon_0 \mathfrak{L}_0(z,u))(s) ds, & t \in I_0 \\ \mathcal{U}(t, s_k) G_k(t, z(t_k^-)) + \int_{s_k}^t \mathcal{U}(t,s) f(s, z_s, u(t)) ds & \\ + \int_{s_k}^t \mathcal{U}(t,s) \mathcal{B}(s)(\Upsilon_k \mathfrak{L}_k(z,u))(s) ds, & t \in I_k \\ G_k(t, z(t_k^-)), & t \in J_k \end{cases} \tag{3.24}$$

and

$$v(t) = \begin{cases} \Upsilon_k \mathfrak{L}_k(z,u) := \mathcal{B}^*(t) \mathcal{U}^*(t_{k+1}, t)(\mathcal{W}_{[s_k, t_{k+1}]})^{-1} \mathfrak{L}_k(z,u)(t), & t \in (s_k, t_{k+1}] \\ 0, & t \in (t_k, s_k) \end{cases} \tag{3.25}$$

where,

$$\mathfrak{L}_k(z,u) = z^{t_{k+1}} - \mathcal{U}(t_{k+1}, s_k) G_k(s_k, z(t_k^-)) - \int_{s_k}^{t_{k+1}} \mathcal{U}(t_{k+1}, s) f(s, z_s, u(s)) ds, \tag{3.26}$$

and

$$\mathcal{W}_{[s_k, t_{k+1}]} z = \int_{s_k}^{t_{k+1}} \mathcal{U}(t_{k+1}, s) \mathcal{B}(s) \mathcal{B}^*(s) \mathcal{U}^*(t_{k+1}, s) z ds, \tag{3.27}$$

with $\delta_k > 0$, such that for each k, we have that the Grammian operator $\left\| (\mathcal{W}_{[s_k, t_{k+1}]})^{-1} \right\| < \frac{1}{\delta_k}$.

Now, using the foregoing operators, we shall define an operator $\mathcal{S}$ to transform the problem of controllability into a problem of finding fixed point of it.

$$\begin{aligned} \mathcal{S} : \mathcal{PW}_{g\tau} \times \mathcal{PW}_u &\longrightarrow \mathcal{PW}_{g\tau} \times \mathcal{PW}_u \\ (z(t), u(t)) &\longmapsto \mathcal{S}(z,u) = (\mathcal{S}_1(z,u)(t), \mathcal{S}_2(z,u)(t)) \end{aligned}$$

The following remark describes the properties of $\mathcal{S}$ and it can be trivially shown from the definition of it.

Remark 3.1 *The semi-linear system with non-instantaneous impulses, infinite delay, and nonlocal conditions* (1.2) *is controllable on* $[0,\tau]$, *iff, for all initial state* $\varphi \in \mathcal{PW}$ *and a final state* z^1 *the operator* $\mathcal{S}$ *has a fixed point. i.e., there exist* (z,u) *in the domain of* $\mathcal{S}$ *satisfying* $\mathcal{S}(z,u) = (z,u)$.

Theorem 3.5 *Under the conditions E1)-E3), the system* (1.2) *is controllable. This is equivalent to say that the operator* $\mathcal{S}$ *defined above has a fixed point. Moreover, given* $\varphi \in \mathfrak{H}$, $z^1 \in \mathbb{R}^n$ *and arbitrary points* $z^{t_{k+1}} \in \mathbb{R}^n, \quad k = 0,1,2,\dots,p$ *there exists a control* $u \in \mathcal{PW}_u$ *such that the corresponding solution* $z(\cdot)$ *of* (1.2) *satisfies:*

$$z(0) - h(z_{\pi_1},\dots,z_{\pi_q})(0) = \varphi(0), \qquad z(t_{k+1}) = z^{t_{k+1}}, \quad k = 0,1,2,\dots,p$$

where

$$z(t_{p+1}) = z^{t_{p+1}} = z^1.$$

In addition, for all $t \in (s_k, t_{k+1}]$ *and* $k = 0,1,2,\dots,p$

$$u(t) = \mathcal{B}^*(t)\mathcal{U}^*(t_{k+1},t)(\mathcal{W}_{(s_k,t_{k+1}]})^{-1}\mathfrak{L}_k(z,u),$$

with $\mathfrak{L}_k(z,u)$ *as showed in* (3.26)

The proof of this theorem will be given by steps.

Step 1 Operator $\mathcal{S}$ is continuous.
Consider hypotheses E2) and E3) and lemma (2.3). Since the solution depends on the inteval we have to consider the following cases:

i) $t \in (0, t_1]$

$$\|\mathcal{S}_1(z,u)(t) - \mathcal{S}_1(w,v)(t)\| \le \hat{C}_0\|z-w\| + \hat{D}_0 \sup_{s\in(0,t_1]} \|f(s,z_s,u(s)) - f(s,w_s,v(s))\|$$

ii) $t \in (t_k, s_k]$

$$\|\mathcal{S}_1(z,u)(t) - \mathcal{S}_1(w,v)(t)\| \le d_0\|z-w\|$$

iii) $t \in (s_k, t_{k+1}]$

$$\begin{aligned}\|\mathcal{S}_1(z,u)(t) - \mathcal{S}_1(w,v)(t)\| \le{}& \hat{C}_k\|z-w\| \\ &+ \hat{D}_k \sup_{s\in(s_k,t_{k+1}]} \|f(s,z_s,u(s)) - f(s,w_s,v(s))\|\end{aligned}$$

iv) $t \in (-\infty, 0]$, then

$$\|\mathcal{S}_1(z,u)(t) - \mathcal{S}_1(w,v)(t)\| \le d_q q\|w-z\|$$

where,

$$\hat{C}_k = C_k[1+\hat{K}_k], \qquad \hat{D}_k = D_k[1+\hat{D}], \qquad \hat{K}_k = \frac{(t_{k+1}-s_k)\|\mathcal{B}\|^2 M^2}{\delta_k},$$
$$C_0 = Md_q q, \qquad C_k = Md_k, \qquad D_k = M(t_{k+1}-s_k)$$

Then, because of the continuity of f, G_k, h, we get that $\mathcal{S}_1$ is continuous.
Aditionally, $\mathcal{S}_2$ is continuous since $\mathcal{B}$, $\mathcal{U}$, $\mathfrak{L}_k$, and $(\mathcal{W}_{[s_k,t_{k+1}]})^{-1}$ are also continuous. Using this two results we obtain as consequence, that the operator $\mathcal{S}$ is continuous. Note that in the interval $(-\infty, 0]$ we get right bound of $\mathcal{S}_1$ from the hypothesis E3), and the operator $\mathcal{S}_2$ is zero there.

Step 2 Operator $\mathcal{S}$ maps bounded sets into equicontinuous sets.

First, we notice that

$$\begin{aligned}\|\mathcal{S}(z,u)(t_2)-\mathcal{S}(z,u)(t_1)\| =& \|\mathcal{S}_1(z,u)(t_2)-\mathcal{S}_1(z,u)(t_1)\| \\ &+\|\mathcal{S}_2(z,u)(t_2)-\mathcal{S}_2(z,u)(t_1)\|\end{aligned} \tag{3.28}$$

Now, let $D \subset \mathcal{PW}_{g\tau}$ be a bounded set and recall that $\mathcal{S}(D) = (\mathcal{S}_1(D), \mathcal{S}_2(D))$. Then, for $\mathcal{S}_1$, we consider hypothesis E2) and the following cases:

i) Let $l_1, l_2 \in (0, t_1]$ such that $0 < l_1 < l_2 \leq t_1$

$$\begin{aligned}\|\mathcal{S}_1(z,u)(l_2)-\mathcal{S}_1(z,u)(l_1)\| \leq& \Big\|\mathcal{U}(l_2,0)\{\varphi(0)-h(z_{\pi_1},\dots,z_{\pi_q})(0)\} \\ &+ \int_0^{l_2}\mathcal{U}(l_2,s)\mathcal{B}(s)(\Upsilon\mathfrak{L}(z,u))(s)ds \\ &+ \int_0^{l_2}\mathcal{U}(l_2,s)f(s,z_s,u(s))ds \\ &- \mathcal{U}(l_1,0)\{\varphi(0)-h(z_{\pi_1},\dots,z_{\pi_q})(0)\} \\ &+ \int_0^{l_1}\mathcal{U}(l_1,s)\mathcal{B}(s)(\Upsilon\mathfrak{L}(z,u))(s)ds \\ &+ \int_0^{l_1}\mathcal{U}(l_1,s)f(s,z_s,u(s))ds\Big\| \\ \leq& \|\mathcal{U}(l_2,0)-\mathcal{U}(l_1,0)\|\Big\|\varphi(0)-h(z_{\pi_1},\dots,z_{\pi_q})(0)\Big\| \\ &+ \int_0^{l_1}\|\mathcal{U}(l_2,s)-\mathcal{U}(l_1,s)\|\|\mathcal{B}(s)(\Upsilon_0\mathfrak{L}_0(z,u))(s)\|\,ds \\ &+ \int_{l_1}^{l_2}\|\mathcal{U}(l_2,s)\|\|\mathcal{B}(s)(\Upsilon_0\mathfrak{L}_0(z,u))(s)\|\,ds \\ &+ \int_0^{l_1}\|\mathcal{U}(l_2,s)-\mathcal{U}(l_1,s)\|\|f(s,z_s,u(s))\|\,ds \\ &+ \int_{l_1}^{l_2}\|\mathcal{U}(l_2,s)\|\|f(s,z_s,u(s))\|\,ds\end{aligned}$$

ii) Let us take $l_1, l_2 \in (t_k, s_k]$ such that $t_k < l_1 < l_2 \leq s_k$

$$\begin{aligned}\|\mathcal{S}_1(z,u)(l_2) - \mathcal{S}_1(z,u)(l_1)\| &= \left\| G_k(l_2, z(t_k^-)) - G_k(l_1, z(t_k^-)\right\| \\ &\leq d_0|l_2 - l_1|\end{aligned}$$

iii) $l_1, l_2 \in (s_k, t_{k+1}]$ such that $s_k < l_1 < l_2 \leq t_{k+1}$

$$\begin{aligned}\|\mathcal{S}_1(z,u)(l_2) - \mathcal{S}_1(z,u)(l_1)\| \;\leq\; & \Big\| \mathcal{U}(l_2, s_k)G_k(s_k, z(t_k^-)) \\ + & \int_{s_k}^{l_2} \mathcal{U}(l_2, s)\mathcal{B}(s)(\Upsilon_k \mathfrak{L}_k(z,u))(s)ds \\ + & \int_{s_k}^{l_2} \mathcal{U}(l_2, s)f(s, z_s, u(s))ds - \mathcal{U}(l_1, s_k)G_k(s_k, z(t_k^-)) \\ + & \int_{s_k}^{l_1} \mathcal{U}(l_1, s)\mathcal{B}(s)(\Upsilon_k \mathfrak{L}_k(z,u))(s)ds \\ + & \int_{s_k}^{l_1} \mathcal{U}(l_1, s)f(s, z_s, u(s))ds \Big\| \\ \leq\; & \|\mathcal{U}(l_2, s_k) - \mathcal{U}(l_1, s_k)\| \left\| G_k(s_k, z(t_k^-)) \right\| \\ + & \int_{s_k}^{l_1} \|\mathcal{U}(l_2, s) - \mathcal{U}(l_1, s)\| \|\mathcal{B}(s)(\Upsilon_k \mathfrak{L}_k(z,u))(s)\| \, ds \\ + & \int_{l_1}^{l_2} \|\mathcal{U}(l_2, s)\| \|\mathcal{B}(s)(\Upsilon_k \mathfrak{L}_k(z,u))(s)\| \, ds \\ + & \int_{s_k}^{l_1} \|\mathcal{U}(l_2, s) - \mathcal{U}(l_1, s)\| \|f(s, z_s, u(s))\| \, ds \\ + & \int_{l_1}^{l_2} \|\mathcal{U}(l_2, s)\| \|f(s, z_s, u(s))\| \, ds\end{aligned}$$

and

$$\begin{aligned}\|\mathcal{S}_2(z,u)(l_2) - \mathcal{S}_2(z,u)(l_1)\| &\leq \|\mathcal{U}(t_{k+1}, l_2)\mathcal{B}(l_2) - \mathcal{U}(t_{k+1}, l_1)\mathcal{B}(l_1)\| \\ &\quad \times \left\| (\mathcal{W}_{[s_k, t_{k+1}]})^{-1} \mathfrak{L}_k(z,u) \right\|\end{aligned}$$

iv) Consider $l_1, l_2 \in (-\infty, 0]$ such that $-\infty \leq l_1 \leq l_2 \leq 0$, then we get

$$\begin{aligned}\|\mathcal{S}_1(z,u)(l_2) - \mathcal{S}_1(z,u)(l_1)\| &= \Big\| \varphi(l_2) - h(z_{\pi_1}, \ldots, z_{\pi_q})(l_2) - \varphi(l_1) \\ &\quad + h(z_{\pi_1}, \ldots, z_{\pi_q})(l_1) \Big\| \\ &\leq \|\varphi(l_2) - \varphi(l_1)\| \, \Big\| h(z_{\pi_1}, \ldots, z_{\pi_q})(l_2) \\ &\quad - h(z_{\pi_1}, \ldots, z_{\pi_q})(l_1) \Big\| .\end{aligned}$$

By the continuity of the evolution operator $\mathcal{U}$ and $\mathcal{W}_{[s_k, t_{k+1}]}$, the boundedness of h on D, with l_2 and l_1 close enough, and i), ii), iii), iv), the equicontinuity of the sets $\mathcal{S}_1(D)$ and $\mathcal{S}_2(D)$ is obtained, which at the same time implies the equicontinuity of $\mathcal{S}(D)$.

Step 3 For any bounded subset $D \subset \mathcal{PW}_{g\tau} \times \mathcal{PW}_u$, $\mathcal{S}(D)$ is relatively compact. Let D be a bounded subset of $\mathcal{PW}_{g\tau} \times \mathcal{PW}_u$. By the continuity of f, $\mathfrak{L}$, and G_k, it follows that

$$\|f(\cdot, z, u)\|_0 \leq \sup_{s\in(0,\tau]} \|f(s, z_s, u(s))\|, \quad \|\mathcal{W}^{-1}_{[s_k,t_{k+1}]}\mathfrak{L}_k\| \leq T_k, \quad \|G_k\| \leq T_{k+p+1}$$

$$k = 1, 2, \ldots, p, \quad \forall (z, u) \in D,$$

where $\|G_k\| = \sup_{t\in(0,\tau]}\{\|G_k(t, z(t_k^-))\|_{\mathbb{R}^n}, T_1, \cdots, T_{2p+1} \in \mathbb{R}$. Therefore, $\mathcal{S}(D)$ is uniformly bounded.

Now, we consider a sequence $\{\psi_i = (y_i, v_i) : i = 1, 2, \ldots, \}$ in $\mathcal{S}(D)$. Since $\{v_i : i = 1, 2, \ldots, \}$ is contained in $\mathcal{S}_2(D) \subset \mathcal{PW}_u$ and $\mathcal{S}_2(D)$ is an uniformly bounded and equicontinuous family, by Arzelà-Ascoli Theorem we can assume, without loss of generality, that $\{v_i : i = 1, 2, \ldots, \}$ converges.

On the other hand, since $\{y_i : i = 1, 2, \ldots, \}$ is contained in $\mathcal{S}_1(D) \subset \mathcal{PW}_{g\tau}((-\infty, \tau]; \mathbb{R}^n)$, then $y_i\Big|_{(-\infty,-\tau_q]} = \phi - h(\phi_{\Pi_1}, \phi_{\Pi_2}, \ldots, \phi_{\Pi_q})$, $i = 1, 2, \ldots,$.

Taking into account that $y_i : i = 1, 2, \ldots, \}$ is bounded and equicontinuous in $[0, t_1]$, we can apply Arzelà-Ascoli Theorem to ensure the existence of a subsequence $\{y_i^1 : i = 1, 2, \ldots, \}$ of $\{y_i : i = 1, 2, \ldots, \}$, which is uniformly convergent on $[0, t_1]$. Now, consider the sequence $\{\phi_i^1 : i = 1, 2, \ldots, \}$ on the interval $[t_1, t_2]$. On this interval the sequence $\{y_i^1 : i = 1, 2, \ldots, \}$ is uniformly bounded and equicontinuous, and for the same reason, it has a subsequence $\{y_i^2 : i = 1, 2, \ldots, \}$ uniformly convergent on $[0, t_2]$. In this way, for the intervals $[t_2, t_3]$, $[t_3, t_4]$, $\ldots$, $[t_p, \tau]$, we see that the sequence $\{\phi_i^{p+1} : i = 1, 2, \ldots, \}$ converges uniformly on the interval $[0, \tau]$.

Besides, in the interval $[-\Pi_q, 0]$ the function y_i is piecewise continuous, then repeating the foregoing process we can assume that the subsequence $\{\psi_i^{p+1} = (y_i^{p+1}, v_i^{p+1}) : i = 1, 2, \ldots, p\}$ converges uniformly on $(-\infty, \tau]$. This means that $\overline{\mathcal{S}(D)}$ is compact, i.e., $\mathcal{S}(D)$ is relatively compact.

Step 4.

The following limit holds

$$\lim_{\|(z,u)\|\to\infty} \frac{\|\mathcal{S}(z, u)\|}{\|(z, u)\|} = 0,$$

where $\|\cdot\|$ is the norm in the space $\mathcal{PW}_{g\tau} \times \mathcal{PW}_u$. In fact, first we have to make a lot of computations:

i) For $t \in (0, t_1]$, we have that

$$\|\mathfrak{L}_0(z, u)\| \leq \left\|z^{t_1}\right\| + M\|\varphi\| + Mcq\|z\|^\eta + Mt_1[a_0\|z\|^{\alpha_0} + b_0\|u\|^{\beta_0}_{\mathbb{R}^m} + c_0]$$

Which implies that $\mathcal{S}_2$ and $\mathcal{S}_1$ as follows:

$$\|\mathcal{S}_2(z, u)(t)\| \leq \frac{M\|\mathcal{B}\|}{\delta_0}\left\|z^{t_1}\right\| + \frac{\|\mathcal{B}\|\, M^2}{\delta_0}\|\varphi\| + \frac{\|\mathcal{B}\|\, M^2}{\delta_0}cq\|z\|^\eta + \frac{\|\mathcal{B}\|\, M^2 t_1}{\delta_0}[a_0\|z\|^{\alpha_0} + b_0\|u\|^{\beta_0}_{\mathbb{R}^m} + c_0]$$

and,

$$\begin{aligned}\|\mathcal{S}_1(z,u)(t)\| \leq{}& M\|\varphi\| + Mcq\|z\|^\eta + Mt_1[a_0\|z\|^{\alpha_0} + b_0\|u\|^{\beta_0}_{\mathbb{R}^m} + c_0] + M\|\mathcal{B}\|\, t_1\\ &\times\left\{\frac{M\|\mathcal{B}\|}{\delta_0}\left\|z^{t_1}\right\| + \frac{\|\mathcal{B}\|\, M^2}{\delta_0}\|\varphi\| + \frac{\|\mathcal{B}\|\, M^2}{\delta_0}cq\|z\|^\eta\right.\\ &\left.+\frac{\|\mathcal{B}\|\, M^2 t_1}{\delta_0}[a_0\|z\|^{\alpha_0} + b_0\|u\|^{\beta_0}_{\mathbb{R}^m} + c_0]\right\}\end{aligned}$$

From the above inequality, we get the foregoing estimate $\mathcal{S}$

$$\begin{aligned}\|\mathcal{S}(z,u)\| ={}& \|\mathcal{S}_1(z,u)\| + \|\mathcal{S}_2(z,u)\|\\ \leq{}& E_0\|\varphi\| + H_0\left\|z^{t_1}\right\| + D_0\|z\|^\eta + F_0[a_0\|z\|^{\alpha_0} + b_0\|u\|^{\beta_0}_{\mathbb{R}^m} + c_0]\end{aligned}$$

and therefore,

$$\begin{aligned}\frac{\|\mathcal{S}(z,u)(t)\|}{\|(z,u)\|} \leq{}& \tfrac{E_0\|\varphi\| + H_0\left\|z^{t_1}\right\|}{\|z\|+\|u\|} + D_0\|z\|^{\eta-1} + F_0\Big[a_0\|z\|^{\alpha_0-1}\\ &+ b_0\|u\|^{\beta_0-1}_{\mathbb{R}^m} + \tfrac{c_0}{\|z\|+\|u\|}\Big]\end{aligned} \tag{3.29}$$

ii) For $t \in (s_k, t_{k+1}]$, we got the following bound

$$\|\mathfrak{L}_k(z,u)\| \leq \left\|z^{t_{k+1}}\right\| + M[a_k\|z\|^{\alpha_k} + c_k] + M(t_{k+1} - s_k)[a_0\|z\|^{\alpha_0} + b_0\|u\|^{\beta_0}_{\mathbb{R}^m} + c_0]$$

Which give us for operator $\mathcal{S}_2$ that

$$\begin{aligned}\|\mathcal{S}_2(z,u)(t)\| \leq{}& \frac{M\|\mathcal{B}\|}{\delta_k}\left\|z^{t_{k+1}}\right\| + \frac{\|\mathcal{B}\|\, M^2}{\delta_k}[a_k\|z\|^{\alpha_k} + c_k]\\ &+ \frac{\|\mathcal{B}\|\, M^2(t_{k+1} - s_k)}{\delta_k}[a_0\|z\|^{\alpha_0} + b_0\|u\|^{\beta_0}_{\mathbb{R}^m} + c_0]\end{aligned}$$

and, for operator $\mathcal{S}_1$ that

$$\begin{aligned}\|\mathcal{S}_1(z,u)(t)\| \leq{}& M[a_k\|z\|^{\alpha_k} + c_k] + M(t_{k+1} - s_k)\|\mathcal{B}\|\left[\frac{M\|\mathcal{B}\|}{\delta_k}\left\|z^{t_{k+1}}\right\|\right.\\ &\left.+ \frac{\|\mathcal{B}\|\, M^2}{\delta_k}[a_k\|z\|^{\alpha_k} + c_k] + \frac{\|\mathcal{B}\|\, M^2(t_{k+1} - s_k)}{\delta_k}[a_0\|z\|^{\alpha_0} + b_0\|u\|^{\beta_0}_{\mathbb{R}^m} + c_0]\right]\\ &+ M(t_{k+1} - s_k)[a_0\|z\|^{\alpha_0} + b_0\|u\|^{\beta_0}_{\mathbb{R}^m} + c_0]\end{aligned}$$

Hence, the operator $\mathcal{S}$, becomes:

$$\begin{aligned}\frac{\|\mathcal{S}(z,u)(t)\|}{\|(z,u)\|} \leq{}& E_k\left[a_k\|z\|^{\alpha_k-1} + \frac{c_k}{\|z\| + \|u\|}\right] + \frac{H_k\left\|z^{t_{k+1}}\right\|}{\|z\| + \|u\|}\\ &+ F_k\left[a_0\|z\|^{\alpha_0-1} + b_0\|u\|^{\beta_0-1}_{\mathbb{R}^m} + \frac{c_0}{\|z\| + \|u\|}\right]\end{aligned} \tag{3.30}$$

iii) For $t \in (t_k, s_k]$, we have that

$$\|\mathcal{S}_1(z,u)(t)\| \leq a_k \|z\|^{\alpha_k} + c_k$$

implying that

$$\frac{\|\mathcal{S}(z,u)(t)\|}{\|(z,u)\|} \leq a_k \|z\|^{\alpha_k - 1} + \frac{c_k}{\|z\| + \|u\|} \tag{3.31}$$

where,

$$D_0 = \left[Mcq + \frac{\|\mathcal{B}\|^2 M^3 t_1 cq}{\delta_0} + \frac{\|\mathcal{B}\| M^2 cq}{\delta_0} \right] \quad E_k = \left[M + \frac{(t_{k+1} - s_k)\|B\|^2 M^3}{\delta_k} + \frac{\|\mathcal{B}\| M^2}{\delta_k} \right]$$

$$H_k = \left[\frac{(t_{k+1} - s_k)\|B\|^2 M^2}{\delta_k} + \frac{\|\mathcal{B}\| M}{\delta_k} \right]$$

$$\text{and} \quad F_k = \left[\frac{(t_{k+1} - s_k)^2 \|\mathcal{B}\|^2 M^3}{\delta_k} + M(t_{k+1} - s_k) + \frac{(t_{k+1} - s_k)\|\mathcal{B}\| M^2}{\delta_k} \right]$$

Hence, considering the hypotheses (E1)-(E3) with the Lemma 2.3 and $0 < \alpha_k < 1$, $0 < \beta_0 < 1$, $k = 0, 1, \ldots, p$, $0 < \eta < 1$, it follows from (3.29), (3.30) and (3.31) that for any $t \in (0, \tau]$

$$\lim_{\|(z,u)\| \to \infty} \frac{\|\mathcal{S}(z,u)\|}{\|(z,u)\|} = 0,$$

Now, we are ready to prove that operator $\mathcal{S}$ has fixed point. In fact, for a fixed $0 < \rho < 1$, there exists $r > 0$ big enough, such that $\|\mathcal{S}(z,u)\| \leq \rho \|(z,u)\|$, for all $\|(z,u)\| \geq r$.
In particularly, if take $\|(z,u)\| = r$, then $\|\mathcal{S}(z,u)\| \leq \rho r < r$. Consequently,

$$\mathcal{S}(\partial B(0,r)) \subset B(0,r).$$

Hence, applying Rothe's Fixed Point Theorem 2.3, we conclude that the operator $\mathcal{S}$ has a fixed point $(z,u) \in \mathcal{PW}_{g\tau} \times \mathcal{PW}_u$. i.e., $\mathcal{S}(z,u) = (z,u)$, which prove the controllability of system (1.2).

Moreover, from the definition of the operator $\mathcal{S}$ and the prove of the above theorem, we got that letting $\varphi \in \mathfrak{H}$, $z^1 \in \mathbb{R}^n$ and arbitrary points $z^{t_{k+1}} \in \mathbb{R}^n$, $\quad k = 0, 1, 2, \ldots, p$, there exists a control $u \in \mathcal{PW}_u$ such that

$$u(t) = \mathcal{B}^*(t)\mathcal{U}^*(t_{k+1}, t)(\mathcal{W}_{(s_k, t_{k+1}]})^{-1}\mathfrak{L}_k(z,u)$$

for $t \in (s_k, t_{k+1}]$, $k = 1, 2, \cdots, p$. Replacing u into the solution (3.22), and evaluating it at $t = 0, t_1, t_{k+1}$, we obtain that:

$$z(0) + h(z_{\pi_1}, \ldots, z_{\pi_q})(0) = \varphi(0),$$

$$\begin{aligned}
z(t_1) &= \mathcal{U}(t_1,0)[\varphi(0)-h(z_{\pi_1},\dots,z_{\pi_q})(0)]+\int_0^{t_1}\mathcal{U}(t_1,s)f(s,z_s,u(s))ds\\
&\quad+\int_0^{t_1}\mathcal{U}(t_1,s)\mathcal{B}(s)\mathcal{B}^*(s)\mathcal{U}^*(t_1,s)(\mathcal{W}_{(0,t_1]})^{-1}\{z^{t_1}-\mathcal{U}(t_1,0)[\varphi(0)\\
&\quad-h(z_{\pi_1},\dots,z_{\pi_q})(0)]-\int_0^{t_1}\mathcal{U}(t_1,v)f(v,z_v,u(v))dv]\}ds\\
&= \mathcal{U}(t_1,0)[\varphi(0)-h(z_{\pi_1},\dots,z_{\pi_q})(0)]+\int_0^{t_1}\mathcal{U}(t_1,s)f(s,z_s,u(s))ds\\
&\quad+(\mathcal{W}_{(0,t_1]})(\mathcal{W}_{(0,t_1]})^{-1}\{z^{t_1}-\mathcal{U}(t_1,0)[\varphi(0)-h(z_{\pi_1},\dots,z_{\pi_q})(0)]\\
&\quad-\int_0^{t_1}\mathcal{U}(t_1,v)f(v,z_v,u(v))dv\}:=z^{t_1}
\end{aligned}$$

$$\begin{aligned}
z(t_{k+1}) &= \mathcal{U}(t_{k+1},s_k)G_k(s_k,z(t_k^-))+\int_{s_k}^{t_{k+1}}\mathcal{U}(t_{k+1},s)f(s,z_s,u(s))ds\\
&\quad+\int_{s_k}^{t_{k+1}}\mathcal{U}(t_{k+1},s)\mathcal{B}(s)\mathcal{B}^*(s)\mathcal{U}^*(t_{k+1},s)(\mathcal{W}_{(s_k,t_{k+1}]})^{-1}\{z_{t_{k+1}}\\
&\quad-\mathcal{U}(t_{k+1},s_i)G_k(s_k,z(t_k^-))-\int_{s_k}^{t_{k+1}}\mathcal{U}(t_1,v)f(v,z_v,u(v))dv\}ds\\
&= \mathcal{U}(t_{k+1},s_k)G_k(s_k,z(t_k^-))+\int_{s_k}^{t_{k+1}}\mathcal{U}(t_{k+1},s)f(s,z_s,u(s))ds\\
&\quad+(\mathcal{W}_{(s_k,t_{k+1}]})(\mathcal{W}_{(s_k,t_{k+1}]})^{-1}\{z^{t_{k+1}}-\mathcal{U}(t_{k+1},s_k)G_k(s_k,z(t_k^-))\\
&\quad-\int_{s_k}^{t_{k+1}}\mathcal{U}(t_{k+1},v)f(v,z_v,u(v))dv\}:=z^{t_{k+1}}.
\end{aligned}$$

Observe that, if $k=p$, then $z(t_{p+1})=z^{t_{p+1}}=z^1$, and since $t_{p+1}=\tau$, we get that $z(\tau)=z^1$. This complete the proof.

Chapter 4

Conclusions and final remarks

In this work we have successfully proved the existence and uniqueness of solutions of retarded equations with infinite delay, infinite many non-instantaneous impulses, and non-local conditions; by using Karakosta's fixed point theorem, and after showing that the phase space that we choose satisfies the axioms proposed by Hale and Kato to study retarded equations with unbounded delay. The choice made for the phase space was a subspace of the piecewise continuous functions due to impulses and non-local conditions. Once we have proved the existence of solutions for this type of equation, we opened the door to study another aspect related to this type of problem, such as the controllability, the stability, the existence of bounded solutions, periodic solutions, almost periodic solutions, and in general other topics of dynamical systems.

We focused on controllability, where by using the fixed point approach and Bashirov techniques, we have proved the approximate and the exact controllability of the system, which contains infinitely many non-instantaneous impulses, non-local conditions, and infinite delay. The showed proofs confirmed that impulses, delays, and non-local conditions are, under some conditions, intrinsic phenomena that do not destroy the controllability of a system. That is, if we consider these elements as disturbances of the system, it turns out that the controllability is robust under these influences not taken into account in many mathematical models that represent extremely important problems in real life.

Our future research will focus on studying the same results for evolution equations in infinite-dimensional Banach. Those are the existence of bounded solutions of such equations, uniqueness, stability, controllability, as well as, other aspects of dynamical systems.

Bibliography

[1] L. Ferguson, "Control systems as used by the ancient world," 2015.

[2] J. C. Maxwell, "On governors," *Proceedings of the Royal Society of London*, vol. 16, pp. 270–283, 1868.

[3] I. Stamova and G. T. Stamov, *Applied impulsive mathematical models.* Springer, 2016, vol. 318.

[4] M. Li, M. Wang, and F. Zhang, "Controllability of impulsive functional differential systems in banach spaces," *Chaos, Solitons & Fractals*, vol. 29, no. 1, pp. 175–181, 2006.

[5] R. Sakthivel, N. Mahmudov, and J. Kim, "Approximate controllability of nonlinear impulsive differential systems," *Reports on Mathematical Physics*, vol. 60, no. 1, pp. 85–96, 2007.

[6] M. Fečkan, J. Wang, and Y. Zhou, "Periodic solutions for nonlinear evolution equations with non-instantaneous impulses," *Nonautonomous Dynamical Systems*, vol. 1, no. 1, 2014.

[7] M. Malik, R. Dhayal, S. Abbas, and A. Kumar, "Controllability of non-autonomous nonlinear differential system with non-instantaneous impulses," *Revista de la Real Academia de Ciencias Exactas, Físicas y Naturales. Serie A. Matemáticas*, vol. 113, no. 1, pp. 103–118, 2019.

[8] L. Byszewski and V. Lakshmikantham, "Theorem about the existence and uniqueness of a solution of a nonlocal abstract cauchy problem in a banach space," *Applicable analysis*, vol. 40, no. 1, pp. 11–19, 1991.

[9] N. I. Mahmudov, "Approximate controllability of evolution systems with nonlocal conditions," *Nonlinear Analysis: Theory, Methods & Applications*, vol. 68, no. 3, pp. 536–546, 2008.

[10] Y. Chang, J. J. Nieto, and W. Li, "Controllability of semilinear differential systems with nonlocal initial conditions in banach spaces," *Journal of Optimization Theory and Applications*, vol. 142, no. 2, pp. 267–273, 2009.

[11] S. Kumar and N. Sukavanam, "Approximate controllability of fractional order semilinear systems with bounded delay," *Journal of Differential Equations*, vol. 252, no. 11, pp. 6163–6174, 2012.

[12] X.-f. Su and X.-l. Fu, "Approximate controllability of second-order semilinear evolution systems with finite delay," *Acta Mathematicae Applicatae Sinica, English Series*, vol. 37, no. 3, pp. 573–589, 2021.

[13] F. Wang and Z. Yao, "Approximate controllability of fractional neutral differential systems with bounded delay," *Fixed Point Theory*, vol. 17, no. 2, pp. 495–508, 2016.

[14] J. Hale and J. Kato, "Phase space for retarded equations with infinite delay," *Funkcial. Ekvac*, no. 1, pp. 11–41, 1978.

[15] S. Ji, G. Li, and M. Wang, "Controllability of impulsive differential systems with nonlocal conditions," *Applied Mathematics and Computation*, vol. 217, no. 16, pp. 6981–6989, 2011.

[16] L. Chen and G. Li, "Approximate controllability of impulsive differential equations with nonlocal conditions," *International Journal of Nonlinear Science*, vol. 10, no. 4, pp. 438–446, 2010.

[17] H. Leiva and R. A. Rojas, "Controllability of semilinear nonautonomous systems with impulses and nonlocal conditions," *Rev. Decienc. Nat*, vol. 1, pp. 23–38, 2016.

[18] S. Selvi and M. M. Arjunan, "Controllability results for impulsive differential systems with finite delay," *J. Nonlinear Sci. Appl*, vol. 5, no. 3, pp. 206–219, 2012.

[19] Y.-K. Chang, "Controllability of impulsive functional differential systems with infinite delay in banach spaces," *Chaos, Solitons & Fractals*, vol. 33, no. 5, pp. 1601–1609, 2007.

[20] H. Leiva, D. Cabada, and R. Gallo, "Roughness of the controllability for time varying systems under the influence of impulses, delay, and nonlocal conditions," *Nonautonomous Dynamical Systems*, vol. 7, no. 1, pp. 126–139, 2020.

[21] H. Leiva, "Karakostas fixed point theorem and the existence of solutions for impulsive semilinear evolution equations with delays and nonlocal conditions," *Communications in Mathematical Analysis*, vol. 21, no. 2, pp. 68–91, 2018.

[22] S. Allauca, C. Herrera, and H. Leiva, "Approximate controllability of time-dependent impulsive semilinear retarded differential equations with infinite delay and nonlocal conditions," *Journal of Mathematical Control Science and Applications*, vol. 6, no. 2, 2020.

[23] A. E. Bashirov, N. Mahmudov, N. Şemı, and H. Etıkan, "Partial controllability concepts," *International Journal of Control*, vol. 80, no. 1, pp. 1–7, 2007.

[24] A. E. Bashirov and M. Jneid, "On partial complete controllability of semilinear systems," in *Abstract and Applied Analysis*, vol. 2013. Hindawi, 2013.

[25] A. E. Bashirov and N. Ghahramanlou, "On partial approximate controllability of semilinear systems," *Cogent Engineering*, vol. 1, no. 1, p. 965947, 2014.

[26] V. Lakshmikantham, P. S. Simeonov *et al.*, *Theory of impulsive differential equations*. World scientific, 1989, vol. 6.

[27] E. Hernández and D. O'Regan, "On a new class of abstract impulsive differential equations," *Proceedings of the American Mathematical Society*, vol. 141, no. 5, pp. 1641–1649, 2013.

[28] M. Fečkan and J. Wang, "A general class of impulsive evolution equations," *Topological Methods in Nonlinear Analysis*, vol. 46, no. 2, pp. 915–933, 2015.

[29] J. Uzcátegui, H. Leiva, and A. Carrasco, *Controlabilidad de Ecuaciones de Evolución Semilineal*, 08 2015.

[30] H. Leiva, "Controllability of semilinear impulsive nonautonomous systems," *International Journal of Control*, vol. 88, no. 3, pp. 585–592, 2015.

[31] J. H. Liu, "Periodic solutions of infinite delay evolution equations," *Journal of Mathematical Analysis and Applications*, vol. 247, no. 2, pp. 627–644, 2000.

[32] T. D. Ayala Maria José, Leiva Hugo, "Existence of solutions for retarded equations with infinite delay, impulses, and nonlocal conditions," *Journal of Mathematical Control Science and Applications*, vol. 6, no. 1, 2020.

[33] S. Abbas, N. Al-Arifi, M. Benchohra, and J. Graef, "Periodic mild solutions of infinite delay evolution equations with non-instantaneous impulses," *Journal of Nonlinear Functional Analysis. Article ID7*, 2020.

[34] J. Liu, T. Naito, and N. Van Minh, "Bounded and periodic solutions of infinite delay evolution equations," *Journal of Mathematical Analysis and Applications*, vol. 286, no. 2, pp. 705–712, 2003.

[35] G. L. Karakostas, "An extension of krasnoselskii's fixed point theorem for contractions and compact mappings," *Topological Methods in Nonlinear Analysis*, vol. 22, no. 1, pp. 181–191, 2003.

[36] G. Isac, "On rothe's fixed point theorem in general topological vector space," *An. St. Univ. Ovidius Constanta*, vol. 12, no. 2, pp. 127–134, 2004.

Appendices

Appendix A

Some bounds

In Chapter 3, section 3, some bounds in some steps are presented. This chapter's aim is to show the computations made in those steps. Considering the first step, we got that

$$\|\mathfrak{L}_0(z,u)-\mathfrak{L}_0(w,v)\| \leq M d_q q \|z-w\| + M t_1 \sup_{s\in(0,t_1]} \|f(s,z_s,u(s)) - f(s,w_s,v(s))\|$$

and

$$\begin{aligned}\|\mathfrak{L}_k(z,u)-\mathfrak{L}_k(w,v)\| \leq M d_k \|z-w\| + M t_1 \sup_{s\in(s_k,t_{k+1}]} &\|f(s,z_s,u(s)) \\ &- f(s,w_s,v(s))\| \,(t_{k+1}-s_k)\end{aligned}$$

In fact, for the first inequality, we get

$$\begin{aligned}\|\mathfrak{L}_0(z,u)(s)-\mathfrak{L}_0(w,v)(s)\| &= \Bigg\| z^{t_1} - \mathcal{U}(t_1,0)G_0(0,z(t_1^-)) - \int_0^{t_1}\mathcal{U}(t_1,s)f(s,z_s,u(s))ds \\ &\quad - w^{t_1} + \mathcal{U}(t_1,0)G_0(0,w(t_1^-)) + \int_0^{t_1}\mathcal{U}(t_1,s)f(s,w_s,v(s))ds \Bigg\| \\ &\leq \|\mathcal{U}(t_1,0)\| \left\| -h(z_{\pi_1},\dots,z_{\pi_q})(0) + h(w_{\pi_1},\dots,w_{\pi_q})(0)\right\|_{\mathbb{R}^n} \\ &\quad + \int_0^{t_1} \|\mathcal{U}(t_1,s)\| \, \|f(s,z_s,u(s)) - f(s,w_s,v(s)\| \, ds \\ &\leq M \left\| h(z_{\pi_1},\dots,z_{\pi_q}) - h(w_{\pi_1},\dots,w_{\pi_q})\right\|_{\mathfrak{H}} \\ &\quad + M \int_0^{t_1} \sup_{s\in(0,t_1]} \|f(s,z_s,u(s)) - f(s,w_s,v(s)\| \, ds \\ &\leq M d_q \|z-w\|_{\mathfrak{H}^q} + M t_1 \sup_{s\in(0,t_1]} \|f(s,z_s,u(s)) - f(s,w_s,v(s))\| \\ &\leq M d_q q \|z-w\|_{\mathfrak{H}} + M t_1 \sup_{s\in(0,t_1]} \|f(s,z_s,u(s)) - f(s,w_s,v(s)\| \\ &\leq M d_q q \|z-w\| + M t_1 \sup_{s\in(0,t_1]} \|f(s,z_s,u(s)) - f(s,w_s,v(s))\|,\end{aligned}$$

and for the last one, we have that

$$\begin{aligned}\|\mathfrak{L}_k(z,u)(s)-\mathfrak{L}_k(w,v)(s)\| = &\Big\|z^{t_{k+1}} - \mathcal{U}(t_{k+1},s_k)G_k(s_k,z(t_k^-))\\
&-\int_{s_k}^t \mathcal{U}(t_{k+1},s)f(s,z_s,u(s))ds\\
&-w^{t_{k+1}} + \mathcal{U}(t_{k+1},s_k)G_k(s_k,w(t_k^-))\\
&+\int_{s_k}^t \mathcal{U}(t_{k+1},s)f(s,w_s,v(s))ds\Big\|\\
&\le \|\mathcal{U}(t_{k+1},s_k)\|\Big\|G_k(s_k,z(t_k^-))-G_k(s_k,w(t_k^-))\Big\|\\
&+\int_{s_k}^t \|\mathcal{U}(t_{k+1},s)\|\|f(s,z_s,u(s))-f(s,w_s,v(s)\|\\
&\le Md_k\|z-w\|\\
&+M\sup_{s\in(s_k,t_{k+1}]}\|f(s,z_s,u(s))-f(s,w_s,v(s)\|\,(t_{k+1}-s_k)\end{aligned}$$

Once, we have compute these two bound, consider the bounds on the first step of the exact controllability proof.

i) In the case of $t\in(s_k,t_{k+1})$, we had that

$$\begin{aligned}\|\mathcal{S}_1(z,u)(t)-\mathcal{S}_1(w,v)(t)\|_{\mathbb{R}^n} = &\Bigg\|\mathcal{U}(t,s_k)G_k(t,z(t_k^-)) + \int_{s_k}^t \mathcal{U}(t,s)f(s,z_s,u(s))ds\\
&+\int_{s_k}^t \mathcal{U}(t,s)\mathcal{B}(s)(\Upsilon_k\mathfrak{L}_k(z,u))(s)ds\\
&-\mathcal{U}(t,s_k)G_k(t,w(t_k^-)) - \int_{s_k}^t \mathcal{U}(t,s)f(s,w_s,v(s))ds\\
&-\int_{s_k}^t \mathcal{U}(t,s)\mathcal{B}(s)(\Upsilon_k\mathfrak{L}_k(w,v))(s)ds\Bigg\|\\
&\le \|\mathcal{U}(t,s_k)\|\Big\|G_k(t,z(t_k^-))-G_k(t,w(t_k^-))\Big\|\\
&+\int_{s_k}^t \|\mathcal{U}(t,s)\|\|f(s,z_s,u(s))-f(s,w_s,v(s))\|\,ds\\
&+\int_{s_k}^t \|\mathcal{U}(t,s)\|\|\mathcal{B}(s)\|\|\Upsilon_k\mathfrak{L}_k(z,u)(s)-\Upsilon_k\mathfrak{L}_k(w,u)(s)\|\\
&\le Md_k\|z-w\|\\
&+M\int_{s_k}^t \sup_{s\in(s_k,t_{k+1}]}\|f(s,z_s,u(s))-f(s,w_s,v(s)\|]ds\\
&+M\|\mathcal{B}\|\int_{s_k}^t \|\mathcal{B}^*(s)\|\|\mathcal{U}^*(t_{k+1},s)\|\Big\|\mathcal{W}^{-1}_{[s_k,t_{k+1}]}\Big\|\\
&\times\|\mathfrak{L}_k(z,u)(s)-\mathfrak{L}_k(w,u)(s)\|\,ds\\
&\le Md_k\|z-w\|\\
&+M\sup_{s\in(s_k,t_{k+1}]}\|f(s,z_s,u(s))-f(s,w_s,v(s)\|](t_{k+1}-s_k)\\
&+\frac{M^2\|\mathcal{B}\|^2}{\delta_k}\int_{s_k}^t \|\mathfrak{L}_k(z,u)(s)-\mathfrak{L}_k(w,u)(s)\|\end{aligned}$$

$$
\begin{aligned}
&\leq Md_k\|z-w\| + \frac{M^2\|\mathcal{B}\|^2}{\delta_k}\int_{s_k}^{t}\bigg[Md_k\|z-w\| \\
&+ M\sup_{s\in(s_k,t_{k+1}]}\|f(s,z_s,u(s))-f(s,w_s,v(s)\|](t_{k+1}-s_k)\bigg] \\
&+ M\sup_{s\in(s_k,t_{k+1}]}\|f(s,z_s,u(s))-f(s,w_s,v(s)\|](t_{k+1}-s_k) \\
&\leq Md_k\|z-w\| + \frac{M^2\|\mathcal{B}\|^2(t_{k+1}-s_k)}{\delta_k}\bigg[Md_k\|z-w\| \\
&+ M\sup_{s\in(s_k,t_{k+1}]}\|f(s,z_s,u(s))-f(s,w_s,v(s)\|](t_{k+1}-s_k)\bigg] \\
&+ M\sup_{s\in(s_k,t_{k+1}]}\|f(s,z_s,u(s))-f(s,w_s,v(s)\|](t_{k+1}-s_k)
\end{aligned}
$$

Finally getting that

$$
\begin{aligned}
\|\mathcal{S}_1(z,u)(t)-\mathcal{S}_1(w,v)(t)\|_{\mathbb{R}^n} \leq{}& Md_k\|z-w\|\left[1+\frac{M^2\|\mathcal{B}\|^2(t_{k+1}-s_k)}{\delta_k}\right] \\
&+ M\sup_{s\in(s_k,t_{k+1}]}\|f(s,z_s,u(s))-f(s,w_s,v(s)\|](t_{k+1}-s_k) \\
&\times\left[1+\frac{M^2\|\mathcal{B}\|^2(t_{k+1}-s_k)}{\delta_k}\right]
\end{aligned}
$$

ii) Consider $t\in J_k$, then we had

$$
\begin{aligned}
\|\mathcal{S}_1(z,u)(t)-\mathcal{S}_1(w,v)(t)\|_{\mathbb{R}^n} \leq{}& \left\|G_k(t,z(t_k^-))-G_k(t,w(t_k^-))\right\| \\
\leq{}& d_k\|z-w\|
\end{aligned}
$$

iii) Let $t\in(-\infty,0]$, then we get

$$
\begin{aligned}
\|\mathcal{S}_1(z,u)(t)-\mathcal{S}_1(w,v)(t)\|_{\mathbb{R}^n} \leq{}& \left\|h(y_{\pi_1},\ldots,y_{\pi_q})(t)-h(z_{\pi_1},\ldots,z_{\pi_q})(t)\right\|_{\mathbb{R}^n} \\
\leq{}& \left\|h(y_{\pi_1},\ldots,y_{\pi_q})-h(z_{\pi_1},\ldots,z_{\pi_q})\right\|_{\mathfrak{H}} \\
\leq{}& d_q\|z-w\|_{\mathfrak{H}^q} \\
\leq{}& d_q q\|z-w\|_{\mathfrak{H}} \\
\leq{}& d_q q\|z-w\|
\end{aligned}
$$

iv) For $t\in(0,t_1]$, we got

$$
\begin{aligned}
\|\mathcal{S}_1(z,u)(t)-\mathcal{S}_1(w,v)(t)\|_{\mathbb{R}^n} ={}& \Big\|\mathcal{U}(t,0)\{\varphi(0)-h(z_{\pi_1},\ldots,z_{\pi_q})(0)\} \\
&+ \int_0^t \mathcal{U}(t,s)f(s,z_s,u(s))ds \\
&+ \int_0^t \mathcal{U}(t,s)\mathcal{B}(s)(\Upsilon_0\mathfrak{L}_0(z,u))(s)ds
\end{aligned}
$$

$$
\begin{aligned}
&- \mathcal{U}(t,0)\{\varphi(0) - h(y_{\pi_1},\dots,y_{\pi_q})(0)\} \\
&- \int_0^t \mathcal{U}(t,s) f(s,w_s,v(s))ds \\
&- \int_0^t \mathcal{U}(t,s)\mathcal{B}(s)(\Upsilon_0\mathfrak{L}_0(w,v))(s)ds\Bigg\|_{\mathbb{R}^n} \\
&\leq \Bigg\|\mathcal{U}(t,s)[h(y_{\pi_1},\dots,y_{\pi_q})(0) - h(z_{\pi_1},\dots,z_{\pi_q})(0)] \\
&+ \int_0^t \mathcal{U}(t,s)\mathcal{B}(s)[\Upsilon_0\mathfrak{L}_0(z,u)(s) - \Upsilon_0\mathfrak{L}_0(w,v)(s)]ds \\
&+ \int_0^t \mathcal{U}(t,s)[f(s,z_s,u(s)) - f(s,w_s,v(s))]ds\Bigg\|_{\mathbb{R}^n} \\
&\leq \|\mathcal{U}(t,s)\|\Big\|h(y_{\pi_1},\dots,y_{\pi_q})(0) - h(z_{\pi_1},\dots,z_{\pi_q})(0)\Big\|_{\mathbb{R}^n} \\
&+ \int_0^t \|\mathcal{U}(t,s)\|\|\mathcal{B}(s)\|\|\Upsilon_0\|\|\mathfrak{L}_0(z,u) - \mathfrak{L}_0(w,v)\|\, ds \\
&+ \int_0^t \|\mathcal{U}(t,s)\|\|f(s,z_s,u(s)) - f(s,w_s,v(s))\|\, ds \\
&\leq M\Big\|h(y_{\pi_1},\dots,y_{\pi_q}) - h(z_{\pi_1},\dots,z_{\pi_q})\Big\|_{\mathfrak{H}} \\
&+ M\|B\|\int_0^t \|\mathcal{B}^*(s)\|\|\mathcal{U}^*(t_1,s)\|\Big\|\mathcal{W}^{-1}_{[0,t_1]}\Big\|\,\|\mathfrak{L}_0(z,u) \\
&- \mathfrak{L}_0(w,v)\|\, ds + M\int_0^t \|f(s,z_s,u(s)) - f(s,w_s,v(s))\|\, ds \\
&\leq Md_q\|z-w\|_{\mathfrak{H}^q} + \frac{M^2\|\mathcal{B}\|^2}{\delta_0}\int_0^t \|\mathfrak{L}_0(z,u) - \mathfrak{L}_0(w,v)\|\, ds \\
&\leq Md_q q\|z-w\|_{\mathfrak{H}} + \frac{M^2\|\mathcal{B}\|^2}{\delta_0}\Big[Md_q q\|z-w\| \\
&+ Mt_1 \sup_{s\in(0,t_1]}\|f(s,z_s,u(s)) - f(s,w_s,v(s)\|\Bigg] t_1 \\
&\leq Md_q q\|z-w\| + \frac{M^2\|\mathcal{B}\|^2}{\delta_0}\Big[Md_q q\|z-w\| \\
&+ Mt_1 \sup_{s\in(0,t_1]}\|f(s,z_s,u(s)) - f(s,w_s,v(s)\|\Bigg] t_1
\end{aligned}
$$

Which implies that, for $t \in I_k$:

$$
\begin{aligned}
\|\mathcal{S}_1(z,u)(t) - \mathcal{S}_1(w,v)(t)\|_{\mathbb{R}^n} &\leq Md_q q\|z-w\|\left[1 + \frac{M^2\|\mathcal{B}\|^2 t_1}{\delta}\right] \\
&+ Mt_1 \sup_{s\in(0,t_1]}\|f(s,z_s,u(s)) - f(s,w_s,v(s)\|] \\
&\times \left[1 + \frac{M^2\|\mathcal{B}\|^2 t_1}{\delta}\right]
\end{aligned}
$$

We finally let

$$\hat{C}_k = C_k[1+\hat{K}_k], \qquad \hat{D}_k = D_k[1+\hat{D}], \qquad \hat{K}_k = \frac{(t_{k+1}-s_k)\|\mathcal{B}\|^2 M^2}{\delta_k},$$
$$C_0 = Md_q q, \qquad C_k = Md_k, \qquad D_k = M(t_{k+1}-s_k)$$

and we obtain the results showed in the main proof.
Let's consider now, the bounds on the step four of the proof of exact controllability. We will first compute the bounds of each operator, then the bound of $\frac{\|\mathcal{S}(z,u)\|}{\|(z,u)\|}$.

i) Consider $t \in (0, t_1]$, then we have by using E1),E3) and Lemma (2.3) that the bounds are given by

$$\begin{aligned}
\|\mathfrak{L}_0(z,u)\| =& \left\| z^{t_1} - \mathcal{U}(t_1,0)G_0(0,z(t_1^-)) - \int_0^{t_1} \mathcal{U}(t_1,s)f(s,z_s,u(s))ds \right\| \\
\leq& \left\| z^{t_1} \right\| + \|\mathcal{U}(t_1,0)\| \left\| G_0(0,z(t_1^-) \right\| + \int_0^{t_1} \|\mathcal{U}(t_1,s)\| \|f(s,z_s,u(s))\| \, ds \\
\leq& \left\| z^{t_1} \right\| + \left\| \varphi(0) - h(z_{\pi_1},\ldots,z_{\pi_q})(0) \right\| + M\{a_0\|z_t\|_{\mathfrak{H}}^{\alpha_0} + b_0\|u\|_{\mathbb{R}^m}^{\beta_0} + c_0\}t_1 \\
\leq& \left\| z^{t_1} \right\| + M\|\varphi\| + Mc\|z\|_{\mathfrak{H}^q}^{\eta} + Mt_1\{a_0\|z\|^{\alpha_0} + b_0\|u\|_{\mathbb{R}^m}^{\beta_0} + c_0\} \\
\leq& \left\| z^{t_1} \right\| + M\|\varphi\| + Mcq\|z\|_{\mathfrak{H}}^{\eta} + Mt_1\{a_0\|z\|^{\alpha_0} + b_0\|u\|_{\mathbb{R}^m}^{\beta_0} + c_0\} \\
\leq& \left\| z^{t_1} \right\| + M\|\varphi\| + Mcq\|z\|^{\eta} + Mt_1\{a_0\|z\|^{\alpha_0} + b_0\|u\|_{\mathbb{R}^m}^{\beta_0} + c_0\}
\end{aligned}$$

which implies that

$$\begin{aligned}
\|\mathcal{S}_2(z,u)(t)\| =& \|\Upsilon_0 \mathfrak{L}_0(z,u)(t)\| \\
\leq& \left\| \mathcal{B}^*(t)\mathcal{U}(t_{k+1},t)(\mathcal{W}_{[0,t_1]})^{-1} \right\| \|\mathfrak{L}_0(z,u)(t)\| \\
\leq& \frac{\|\mathcal{B}\| M}{\delta_0} \left[\left\| z^{t_1} \right\| + M\|\varphi\| + Mcq\|z\|^{\eta} + Mt_1\{a_0\|z\|^{\alpha_0} + b_0\|u\|_{\mathbb{R}^m}^{\beta_0} + c_0\} \right] \\
\leq& \frac{M\|\mathcal{B}\| \left\| z^{t_1} \right\|}{\delta_0} + \frac{M^2\|\mathcal{B}\|\|\varphi\|}{\delta_0} + \frac{M^2\|\mathcal{B}\| cq\|z\|^{\eta}}{\delta_0} \\
&+ \frac{M^2\|\mathcal{B}\| t_1}{\delta_0}\{a_0\|z\|^{\alpha_0} + b_0\|u\|_{\mathbb{R}^m}^{\beta_0} + c_0\}
\end{aligned}$$

and that,

$$\begin{aligned}
\|\mathcal{S}_1(z,u)(t)\| =& \left\| \mathcal{U}(t,0)\{\varphi(0) - h(z_{\pi_1},\ldots,z_{\pi_q})(0)\} + \int_0^t \mathcal{U}(t,s)f(s,z_s,u(s))ds \right. \\
&\left. + \int_0^t \mathcal{U}(t,s)\mathcal{B}(s)(\Upsilon_0\mathfrak{L}_0(z,u))(s)ds \right\| \\
&\leq \|\mathcal{U}(t,0)\| \, \|\varphi(0)\| + \|\mathcal{U}(t,0)\| \left\| h(z_{\pi_1},\ldots,z_{\pi_q})(0) \right\| \\
&+ \int_0^t \|\mathcal{U}(t,s)\| \, \|f(s,z_s,u(s))\| \, ds \\
&+ \int_0^t \|U(t,s)\| \, \|\mathcal{B}(s)\| \, \|(\Upsilon_0\mathfrak{L}_0(z,u))(s)\| \, ds
\end{aligned}$$

$$
\begin{aligned}
&\leq M\|\varphi\| + Mc\|z\|_{\mathfrak{H}^q}^{\eta} + t_1 M\{a_0\|z\|_{\mathfrak{H}}^{\alpha_0} + b_0\|u\|_{\mathbb{R}^m}^{\beta_0} + c_0\} \\
&+ M\|\mathcal{B}\|\int_0^t \|\mathcal{B}^*(s)\|\,\|\mathcal{U}^*(t_1,s)\|\,\left\|(\mathcal{W}_{[0,t_1]})^{-1}\right\|\,\|\mathfrak{L}_0(z,u)(t)\| \\
&\leq M\|\varphi\| + Mcq\|z\|_{\mathfrak{H}}^{\eta} + t_1 M\{a_0\|z\|^{\alpha_0} + b_0\|u\|_{\mathbb{R}^m}^{\beta_0} + c_0\} \\
&+ \frac{M^2\|\mathcal{B}\|^2}{\delta_0}\int_0^t \Big[\left\|z^{t_1}\right\| + M\|\varphi\| + Mcq\|z\|^{\eta} \\
&+ Mt_1\{a_0\|z\|^{\alpha_0} + b_0\|u\|_{\mathbb{R}^m}^{\beta_0} + c_0\}\Big] \\
&\leq M\|\varphi\| + Mcq\|z\|^{\eta} + t_1 M\{a_0\|z\|^{\alpha_0} + b_0\|u\|_{\mathbb{R}^m}^{\beta_0} + c_0\} \\
&+ \frac{M^2\|\mathcal{B}\|^2 t_1}{\delta_0}\Big[\left\|z^{t_1}\right\| + M\|\varphi\| + Mcq\|z\|^{\eta} + Mt_1\{a_0\|z|^{\alpha_0} \\
&+ b_0\|u\|_{\mathbb{R}^m}^{\beta_0} + c_0\}\Big] \\
&\leq M\|\varphi\|\left[1 + \frac{M^2\|\mathcal{B}\|^2 t_1}{\delta_0}\right] + Mcq\|z\|^{\eta}\left[1 + \frac{M^2\|\mathcal{B}\|^2 t_1}{\delta_0}\right] \\
&+ \left\|z^{t_1}\right\|\left[\frac{M^2\|\mathcal{B}\|^2 t_1}{\delta_0}\right] + t_1 M\{a_0\|z\|^{\alpha_0} + b_0\|u\|_{\mathbb{R}^m}^{\beta_0} + c_0\} \\
&\times\left[1 + \frac{M^2\|\mathcal{B}\|^2 t_1}{\delta_0}\right]
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
\frac{\|\mathcal{S}(z,u)\|}{\|(z,u)\|} &= \frac{\|\mathcal{S}_1(z,u)\|}{\|(z,u)\|} + \frac{\|\mathcal{S}_2(z,u)\|}{\|(z,u)\|} \\
&\leq \frac{\left\|z^{t_1}\right\|}{\|(z,u)\|}\cdot\left[\frac{M^2\|\mathcal{B}\|^2 t_1}{\delta_0} + \frac{M\|\mathcal{B}\|}{\delta_0}\right] + \frac{M\|\varphi\|}{\|(z,u)\|}\cdot\left[1 + \frac{M^2\|\mathcal{B}\|^2 t_1}{\delta_0} + \frac{M\|\mathcal{B}\|}{\delta_0}\right] \\
&+ \frac{Mcq\|z\|^{\eta}}{\|(z,u)\|}\cdot\left[1 + \frac{M^2\|\mathcal{B}\|^2 t_1}{\delta_0} + \frac{M\|\mathcal{B}\|}{\delta_0}\right] \\
&+ \frac{Mt_1\{a_0\|z\|^{\alpha_0} + b_0\|u\|_{\mathbb{R}^m}^{\beta_0} + c_0\}}{\|(z,u)\|}\cdot\left[1 + \frac{M^2\|\mathcal{B}\|^2 t_1}{\delta_0} + \frac{M\|\mathcal{B}\|}{\delta_0}\right] \\
&\leq \frac{\left\|z^{t_1}\right\|}{\|z\| + \|u\|}\cdot\left[\frac{M^2\|\mathcal{B}\|^2 t_1}{\delta_0} + \frac{M\|\mathcal{B}\|}{\delta_0}\right] + \frac{M\|\varphi\|}{\|z\| + \|u\|}\cdot\Big[1 + \frac{M^2\|\mathcal{B}\|^2 t_1}{\delta_0} \\
&+ \frac{M\|\mathcal{B}\|}{\delta_0}\Big] + \frac{Mcq\|z\|^{\eta}}{\|z\| + \|u\|}\cdot\left[1 + \frac{M^2\|\mathcal{B}\|^2 t_1}{\delta_0} + \frac{M\|\mathcal{B}\|}{\delta_0}\right] \\
&+ \frac{Mt_1\{a_0\|z\|^{\alpha_0} + b_0\|u\|_{\mathbb{R}^m}^{\beta_0} + c_0\}}{\|z\| + \|u\|}\cdot\left[1 + \frac{M^2\|\mathcal{B}\|^2 t_1}{\delta_0} + \frac{M\|\mathcal{B}\|}{\delta_0}\right] \\
&\leq \frac{\left\|z^{t_1}\right\|}{\|z\| + \|u\|}\cdot\left[\frac{M^2\|\mathcal{B}\|^2 t_1}{\delta_0} + \frac{M\|\mathcal{B}\|}{\delta_0}\right] + \frac{\|\varphi\|}{\|z\| + \|u\|}\cdot\Big[M + \frac{M^3\|\mathcal{B}\|^2 t_1}{\delta_0}
\end{aligned}
$$

$$
\begin{aligned}
&+ \frac{M^2\|\mathcal{B}\|}{\delta_0}\Bigg] + \frac{\|z\|^\eta}{\|z\|}\cdot\left[Mcq + \frac{M^3cq\|\mathcal{B}\|^2\,t_1}{\delta_0} + \frac{M^2cq\|\mathcal{B}\|}{\delta_0}\right]\\
&+ \left[\frac{a_0\|z\|^{\alpha_0}}{\|z\|} + \frac{b_0\|u\|_{\mathbb{R}^m}^{\beta_0}}{\|u\|} + \frac{c_0}{\|z\|+\|u\|}\right]\cdot\left[Mt_1 + \frac{M^3\|\mathcal{B}\|^2\,t_1^2}{\delta_0} + \frac{M^2t_1\|\mathcal{B}\|}{\delta_0}\right]\\
&\le H_0\frac{\left\|z^{t_1}\right\|}{\|z\|+\|u\|} + E_0\frac{\|\varphi\|}{\|z\|+\|u\|} + D_0\|z\|^{\eta-1}\\
&+ F_0\left[a_0\|z\|^{\alpha_0-1} + b_0\|u\|_{\mathbb{R}^m}^{\beta_0-1} + \frac{c_0}{\|z\|+\|u\|}\right]
\end{aligned}
$$

ii) Let $t \in (s_k, t_{k+1}]$, then we have by using E1), E2) and Lemma (2.3) that the bounds for this case are the following:

$$
\begin{aligned}
\|\mathfrak{L}_k(z,u)\| =& \left\| z^{t_{k+1}} - \mathcal{U}(t_{k+1},s_k)G_k(s_k,z(t_k^-)) - \int_{s_k}^{t_{k+1}}\mathcal{U}(t_{k+1},s)f(s,z_s(u(s)))ds\right\|\\
\le& \left\|z^{t_{k+1}}\right\| + \|\mathcal{U}(t_{k+1},s_k)\|\left\|G_k(s_k,z(t_k^-))\right\|\\
&+ \int_{s_k}^{t_{k+1}}\|\mathcal{U}(t_{k+1},s)\|\|f(s,z_s(u(s)))\|\,ds\\
\le& \left\|z^{t_{k+1}}\right\| + Ma_k\|z\|_{\mathbb{R}^n}^{\alpha_k} + Mc_k + M(t_{k+1}-s_k)\{a_0\|z\|_{\mathfrak{H}}^{\alpha_0} + b_0\|u\|_{\mathbb{R}^m}^{\beta_0} + c_0\}\\
\le& \left\|z^{t_{k+1}}\right\| + Ma_k\|z\|_{\mathbb{R}^n}^{\alpha_k} + Mc_k + M(t_{k+1}-s_k)\{a_0\|z\|^{\alpha_0} + b_0\|u\|_{\mathbb{R}^m}^{\beta_0} + c_0\}
\end{aligned}
$$

which helps us to get operator $\mathcal{S}_2$:

$$
\begin{aligned}
\|\mathcal{S}_2(z,u)(t)\| =& \|\Upsilon_k\mathfrak{L}_k(z,u)(t)\|\\
\le& \frac{\|\mathcal{B}\|\,M}{\delta_k}\|\mathfrak{L}_k(z,u)(t)\|\\
\le& \frac{\|\mathcal{B}\|\,M}{\delta_k}\Big[\left\|z^{t_{k+1}}\right\| + Ma_k\|z\|_{\mathbb{R}^n}^{\alpha_k} + Mc_k + M(t_{k+1}-s_k)\{a_0\|z\|^{\alpha_0}\\
&+ b_0\|u\|_{\mathbb{R}^m}^{\beta_0} + c_0\}\Big]
\end{aligned}
$$

and also operator $\mathcal{S}_1$, as it is showed below:

$$
\begin{aligned}
\|S_1(z,u)(t)\| =& \Bigg\|\mathcal{U}(t,s_k)G_k(t,z(t_k^-)) + \int_{s_k}^{t}\mathcal{U}(t,s)f(s,z_s,u(s))ds\\
&+ \int_{s_k}^{t}\mathcal{U}(t,s)\mathcal{B}(s)(\Upsilon_k\mathfrak{L}_k(z,u))(s)ds\Bigg\|\\
\le& \|\mathcal{U}(t,s_k)\|\left\|G_k(t,z(t_k^-))\right\| + \int_{s_k}^{t}\|\mathcal{U}(t,s)\|\|f(s,z_s,u(s))\|\,ds\\
&+ \int_{s_k}^{t}\|\mathcal{U}(t,s)\|\|\mathcal{B}(s)\|\|(\Upsilon_k\mathfrak{L}_k(z,u))(s)\|\,ds\\
\le& M[a_k\|z\|_{\mathbb{R}^n}^{\alpha_k} + c_k] + M(t_{k+1}-s_k)\{a_0\|z\|_{\mathfrak{H}}^{\alpha_0} + b_0\|u\|_{\mathbb{R}^m}^{\beta_0} + c_0\}\\
&+ \frac{M^2\|\mathcal{B}\|^2}{\delta_k}(t_{k+1}-s_k)\left[Mc_k + M(t_{k+1}-s_k)\{a_0\|z\|_{\mathfrak{H}}^{\alpha_0} + b_0\|u\|_{\mathbb{R}^m}^{\beta_0} + c_0\}\right.
\end{aligned}
$$

$$
\begin{aligned}
&+\left\|z^{t_{k+1}}\right\| + Ma_k\|z\|_{\mathbb{R}^n}^{\alpha_k}\Bigg] \\
&\leq Ma_k\|z\|_{\mathbb{R}^n}^{\alpha_k}\left[\frac{M^2\|\mathcal{B}\|^2\left(t_{k+1}-s_k\right)}{\delta_k}+1\right] \\
&+ Mc_k\left[\frac{M^2\|\mathcal{B}\|^2\left(t_{k+1}-s_k\right)}{\delta_k}+1\right] \\
&+ M(t_{k+1}-s_k)\{a_0\|z\|^{\alpha_0}+b_0\|u\|_{\mathbb{R}^m}^{\beta_0}+c_0\}\left[\frac{M^2\|\mathcal{B}\|^2\left(t_{k+1}-s_k\right)}{\delta_k}+1\right] \\
&+\left\|t^{k+1}\right\|\left[\frac{M^2\|\mathcal{B}\|^2\left(t_{k+1}-s_k\right)}{\delta_k}\right]
\end{aligned}
$$

Thus, the final bound is given by:

$$
\begin{aligned}
\frac{\|\mathcal{S}(z,u)\|}{\|(z,u)\|} &= \frac{\|\mathcal{S}_1(z,u)\|}{\|(z,u)\|}+\frac{\|\mathcal{S}_2(z,u)\|}{\|(z,u)\|} \\
&\leq \frac{\left\|z^{t_{k+1}}\right\|}{\|(z,u)\|}\cdot\left[\frac{M^2\|\mathcal{B}\|^2\left(t_{k+1}-s_k\right)}{\delta_k}+\frac{M\|\mathcal{B}\|}{\delta_k}\right] \\
&+\frac{Ma_k\|z\|_{\mathbb{R}^n}^{\alpha_k}}{\|(z,u)\|}\left[\frac{M^2\|\mathcal{B}\|^2\left(t_{k+1}-s_k\right)}{\delta_k}+\frac{\|\mathcal{B}\|\,M}{\delta_k}+1\right] \\
&+\frac{Mc_k}{\|(z,u)\|}\left[\frac{M^2\|\mathcal{B}\|^2\left(t_{k+1}-s_k\right)}{\delta_k}+\frac{M\|B\|}{\delta_k}+1\right] \\
&+\frac{M(t_{k+1}-s_k)}{\|(z,u)\|}\{a_0\|z\|^{\alpha_0}+b_0\|u\|_{\mathbb{R}^m}^{\beta_0}+c_0\}\Bigg[\frac{M^2\|\mathcal{B}\|^2\left(t_{k+1}-s_k\right)}{\delta_k}+1 \\
&+\frac{M\|B\|}{\delta_k}\Bigg] \\
&\leq \frac{\left\|z^{t_{k+1}}\right\|}{\|z\|+\|u\|}\cdot\left[\frac{M^2\|\mathcal{B}\|^2\left(t_{k+1}-s_k\right)}{\delta_k}+\frac{M\|\mathcal{B}\|}{\delta_k}\right] \\
&+\frac{Ma_k\|z\|_{\mathbb{R}^n}^{\alpha_k}}{\|z\|+\|u\|}\left[\frac{M^2\|\mathcal{B}\|^2\left(t_{k+1}-s_k\right)}{\delta_k}+\frac{\|\mathcal{B}\|\,M}{\delta_k}+1\right] \\
&+\frac{Mc_k}{\|z\|+\|u\|}\left[\frac{M^2\|\mathcal{B}\|^2\left(t_{k+1}-s_k\right)}{\delta_k}+\frac{M\|B\|}{\delta_k}+1\right] \\
&+\frac{M(t_{k+1}-s_k)}{\|z\|+\|u\|}\{a_0\|z\|^{\alpha_0}+b_0\|u\|_{\mathbb{R}^m}^{\beta_0}+c_0\}\Bigg[\frac{M^2\|\mathcal{B}\|^2\left(t_{k+1}-s_k\right)}{\delta_k}+1 \\
&+\frac{M\|B\|}{\delta_k}\Bigg] \\
&\leq H_k\frac{\left\|z^{t_{k+1}}\right\|}{\|z\|+\|u\|}+\frac{a_k\|z\|_{\mathbb{R}^n}^{\alpha_k}}{\|z\|+\|u\|}\left[\frac{M^3\|\mathcal{B}\|^2\left(t_{k+1}-s_k\right)}{\delta_k}+\frac{\|\mathcal{B}\|\,M^2}{\delta_k}+M\right]
\end{aligned}
$$

$$
\begin{aligned}
&+ \frac{Mc_k}{\|z\| + \|u\|} \left[\frac{M^2 \|\mathcal{B}\|^2 (t_{k+1} - s_k)}{\delta_k} + \frac{M\|\mathcal{B}\|}{\delta_k} + 1 \right] \\
&+ \left\{ \frac{a_0 \|z\|^{\alpha_0}}{\|z\| + \|u\|} + \frac{b_0 \|u\|_{\mathbb{R}^m}^{\beta_0}}{\|z\| + \|u\|} + \frac{\|c_0\|}{\|z\| + \|u\|} \right\} \left[\frac{M^3 \|\mathcal{B}\|^2 (t_{k+1} - s_k)^2}{\delta_k} \right. \\
&\left. + M(t_{k+1} - s_k) + \frac{M^2 (t_{k+1} - s_k) \|\mathcal{B}\|}{\delta_k} \right] \\
&\leq H_k \frac{\left\| z^{t_{k+1}} \right\|}{\|z\| + \|u\|} + E_k \left[a_k \|z\|_{\mathbb{R}^n}^{\alpha_k - 1} + \frac{c_k}{\|z\| + \|u\|} \right] \\
&+ F_k \left\{ a_0 \|z\|^{\alpha_0 - 1} + b_0 \|u\|_{\mathbb{R}^m}^{\beta_0 - 1} + \frac{c_0}{\|z\| + \|u\|} \right\}
\end{aligned}
$$

iii) Next, considering hypothesis E2) for $t \in (t_k, s_k]$, we have

$$
\begin{aligned}
\|\mathcal{S}_1(z,u)(t)\| &= \left\| G_k(t, z(t_k^-)) \right\| \\
&\leq a_k \|z\|^{\alpha_k} + c_k
\end{aligned}
$$

which implies that

$$
\begin{aligned}
\frac{\|\mathcal{S}(z,u)\|}{\|(z,u)\|} &= \frac{\|\mathcal{S}_1(z,u)\|}{\|(z,u)\|} + \frac{\|\mathcal{S}_2(z,u)\|}{\|(z,u)\|} \\
&\leq \frac{a_k \|z\|^{\alpha_k}}{\|(z,u)\|} + \frac{c_k}{\|(z,u)\|} \\
&\leq \frac{a_k \|z\|^{\alpha_k}}{\|z\| + \|u\|} + \frac{c_k}{\|z\| + \|u\|} \\
&\leq a_k \|z\|^{\alpha - 1} + \frac{c_k}{\|z\| + \|u\|}
\end{aligned}
$$

where, H_k, E_k, D_0 and F_k were defined in the last section.