UNIVERSIDAD DE INVESTIGACIÓN DE TECNOLOGÍA EXPERIMENTAL YACHAY

Escuela de Matemáticas y Ciencias Computacionales

TÍTULO: Stochastic Volatility Models in Finance: Measurement of Financial Stress in Ecuador

Trabajo de integración curricular presentado como requisito para la obtención del título de Matemático

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Dedication

“Con infinito amor y gratitud este trabajo está dedicado a mis padres Ana y Oswaldo quienes con su esfuerzo y paciencia me apoyaron para cumplir mis metas“

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Quiero agradecer a todos mis profesores que contribuyeron a mi formación profesional en especial a mi tutor Saba Infante por apoyarme y guiarme con su conocimiento en el transcurso del proyecto. Me gustaría agradecer a la universidad por continuar con este gran proyecto del cual todos formamos parte. También un agradecimiento a mis amigos especialmente Alexis, Andrés, Diana, Pedro, Jonathan y Steeven quienes con su confianza y ocurrencias pasé grandes momentos en la universidad. No podría dejar pasar en alto aquellas personas que me apoyaron durante la realización del presente trabajo Ana, Alex, Ismael, Katherine, Karola, Lilia, Oswaldo, Pamela, Marco y Wilson.

Resumen

Este trabajo desarrolla un marco para el análisis de modelos de espacio de estados combinados con los filtros de Kalman, Kalman suavizado, Gibbs y partículas para la estimación de estados y parámetros desconocidos, determinando la precisión de los algoritmos, con el propósito de analizar algunas series de tiempo de la macroeconomía del Ecuador. Esta metodología juega un papel importante en el área de la economía y las finanzas ademas tiene muchas ventajas porque permite describir cómo las variables macroeconómicas observadas se pueden relacionar con variables de estado potencialmente no observadas, determinando la evolución en tiempo real, estimando tendencias no observadas, cambios de estructuras y pronósticos en tiempos futuros. Para lograr los objetivos se proponen tres modelos: el primero se utiliza para estimar el producto interno bruto del Ecuador. El segundo modelo combina un modelo de espacio de estados con el modelo clásico ARIMA (p, q, r) para ajustar la tasa del PIB y finalmente se considera un modelo para el análisis simultáneo de series temporales de estrés relacionado con: índice de precios al consumidor, índice de producción industrial y tasa de interés activa. En todos los casos estudiados, las estimaciones obtenidas reflejan el comportamiento real de la economía ecuatoriana. La raíz cuadrada del error cuadrático medio se utilizó como medida de bondad de ajuste para medir la calidad de estimación de los algoritmos, obteniendo pequeños errores.

Palabras Clave: Sistemas dinamicos, Filtro de Kalman , Modelos de espacio estado, Muestreador de Gibbs, Monte Carlo Samples, Modelos ARIMA and Producto interno bruto.

Abstract

This work develops a framework for the analysis of state-space models combined with Kalman, Kalman smoothed, Gibbs and particle filters for the estimation of unknown states, and parameters, determining the accuracy of the algorithms, to analyze some time series of the macroeconomy of Ecuador. This methodology plays an important role in the area of economics and finance and has many advantages because it allows describing how observed macroeconomic variables can be related to potentially unobserved state variables, determining the evolution in real time, estimating unobserved trends, changes of structures and make forecasts in future times. To achieve the objectives, three models are proposed: the first model is used to estimate Ecuador's gross domestic product. The second model combines a state space model with the classic ARIMA (p, q, r) model to adjust the GDP rate and finally, it is considered a model for the simultaneous stress time series analysis related to: consumer price index, industrial production index and active interest rate. In all the cases studied, the estimates obtained reflect the real behavior of the Ecuadorian economy. The square root of the mean square error was used as a measure of goodness of fit to measure the quality of estimation of the algorithms, obtaining small errors.

Keywords: Dynamic system, Kalman filter, State space model, Gibbs sampler, Monte Carlo Samples, ARIMA model and Gross domestic product

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Chapter 1

Introduction

In this work, a methodology based on filtering algorithms is applied to estimate states and parameters in time series models that are used to model dynamic phenomena that evolve over time. It is interesting to study the behavior of stochastic processes with partially observed dynamics measured with errors; It is particularly interesting to study financial time series of macroeconomic variables such as gross domestic product, unemployment rate, prices of shares in the stock market, volatility of interest rates in the short and medium term, commodity prices of primary products, and the neutral density of active risk among other financial series.

In particular, this work presents an adjustment of a state space model to estimate macroeconomic indicators of Ecuador. To achieve the objectives, three models are proposed: the first model analyzes the gross domestic product of Ecuador corresponding to the period 2000-2020. The second model is a combination of a state space model with the ARIMA (p, q, r) models which will be used to adjust the GDP rate and finally it is considered a model for the simultaneous analysis of several time series related to: consumer price index, industrial production index and active interest rate. Each of these models are estimated using the Kalman filter and Smoothed Kalman filter.
The rest of the thesis is as follows: in Section 2, the state space models and the Kalman filter are described; Section 3 specifies the variables, models, and parameters to be used; In section 4 the results and discussion of the studies carried out are presented; section 5 ends with the conclusions.

1.1 Background

There is an extensive literature on state space models beginning with the works of [1, 2, 3, 4, 5, 6] classical time series models such as those studied in [7], model of Markov hidden discrete [8, 9, 10, 11]; stochastic volatility model used to model the time variance of logarithmic returns on assets [12]; and the point change models used to model stock prices [13]. The methodology allows estimating smoothed states with linear and non-linear structures by implementing efficient computational algorithms.

Related works to this research highlight: In [14] was implemented the algorithms:

Gibbs, Kalman filter, extended Kalman filter and particles filters, they analyze series of oil gross domestic product (GDP), and not oil; and the dollar to bolivar exchange rate of the Venezuelan economy; They also conducted a simulation study, demonstrating that the algorithms estimate adequately.

In [15] describes a general procedure to make Bayesian inference based on the evaluation of the plausibility of stochastic general equilibrium models through the Markov Chain Monte Carlo methods, they implemented the Kalman filter to evaluate the likelihood function and finally apply the Metropolis Hastings algorithm to estimate the parameters of the posterior distribution. They illustrate the methodology by using the basic stochastic growth model, considering quarterly data for the Venezuelan economy from the first quarter of (1984) to the third quarter of (2004). The empirical analysis carried out allows us to conclude that the algorithms used works efficiently and at a low computational cost, the estimates obtained are consistent, and the estimates of the predictions adequately reflect the behavior of the product, employment, consumption and investment per capital of the country. In [16] propose a methodology based on the state-space structure applying filtering techniques such as the auxiliary particle filter to estimate the underlying volatility of the system.

Additionally, they used a Markov chain Monte Carlo algorithm to estimate the parameters. The methodology was illustrated using a series of returns from simulated data, and the series of returns corresponding to the Standard and Poor's 500 price index for the period 1999-2003. The results show that the proposed methodology allows to adequately explain the dynamics of volatility when there is an asymmetric response to a shock of a different sign. In [17] a methodology was applied based on state space models inspired by the Monte Carlo Markov Chain sampling schemes, which simplifies the estimation and prediction process of the Markov switching model. The general objective of this study was to simultaneously determine: non-linearity, structural changes, asymmetries and outliers that are characteristics present in many financial series. The methodology was empirically illustrated using series that measure the annual growth rate of industrial production in the MERCOSUR countries. The study concludes that there is no reduction in economic volatility, there is no reduction in the depth of economic cycles. At breakpoints, outliers and non-linearity are observed in the data. It is evident that there are no common economic cycles for the countries analyzed. In [18], two recursive filtering algorithms were implemented, the optimized particle filter, and the Viterbi algorithm, which allow the joint estimation of states and parameters of stochastic volatility models in continuous time, such as the Cox Ingersoll Ross and Heston model, using daily empirical data from the time series of the S & P500 stock index returns. Furthermore, these parameters prove that the Viterbi algorithm has less execution time than the optimized particle filter. In [19] an estimation methodology based on the Monte Carlo sequential algorithm is proposed, which jointly estimates the states and parameters, the relationship between the prices of futures contracts and the spot prices of primary products, they determined the evolution of prices and volatility of the historical data of the primary market (Gold and Soybeans), using three algorithms: the sampling algorithm of sequential importance with resampling (SISR), the Storvik algorithm, and the particle learning and smoothing algorithm (PLS). The results conclude that the prices of products for future delivery at different expiration dates with the spot price are highly correlated.

1.2 Problem statement

Filtering algorithms are applied to estimate states and parameters in time series models that are used to model dynamic phenomena that evolve over time. It is interesting to study the behavior of stochastic processes with partially observed dynamics measured with errors. Due to the number of variables that make up the time series such as GDP, the data contain errors that are a problem in their treatment. Elimination of errors allows us to extract components of economic interest within each time series. Models for filtering data have played a role in the economy, the Bayesian filters being notable for their precision. In the present work, we propose the implementation and construction of state space models that allow us to filter time series.

1.3 Objectives

1.3.1 General Objective

The general objective of this work is to implement Bayesian state space models that allow us to estimate unknown parameters in order to filter the time series of the macroeconomy of Ecuador.

1.3.2 Specific Objectives

- Bibliography review about time series, ARIMA models, space state models, Kalman filter, Gibbs sampling and particle filter.
- Propose a state space model that allows us to filter the GDP of Ecuador through the Kalman filter and Kalman smoothed.
- Implement an ARIMA model that allows us to filter the GDP rate of Ecuador through the Kalman filter and Kalman smoothed.
- Use a Gibbs sampler and a particle filter algorithm that allows us to filter the macroeconomic series.
- Compare the different proposed filters using the mean square error metric.

1.3.3 Contribution

This thesis consists of overview of an article presented for the Conference on Information and Communication Technologies of Ecuador.

- Bautista H., Saba I., Amaro I.: Estimation of the State Space Models: An Application in Macroeconomic Series of Ecuador. In Rodriguez G., Fonseca C., Salgado J., Peréz-Gosendo P., Orellana M. (eds)Information and Communication Technologies of Ecuador. TICEC2021. Communication in Computer and Information Science. Springer, Cham.

Chapter 2

Theoretical Framework

In this section, we present the main concepts and results that will be used to solve our problem.

2.1 Preliminaries

Definition 1. *The expected value of a random variable X is the average or average value of X, and is given by*

$$\begin{aligned} \mu_x = E(X) &= \sum_x xP(X=x), \quad if \quad X \quad is \quad discrete, \\ \mu_x = E(X) &= \int_{-\infty}^{\infty} xf(x), \qquad if \quad X \quad is \quad continuous. \end{aligned} \tag{2.1}$$

In general, let a function $h(x)$, the expected value of $h(x)$ is given by:

$$\begin{aligned} E(h(x))) &= \sum_x h(x)P(X=x), \quad if \quad X \quad is \quad discrete, \\ E(h(x)) &= \int_{-\infty}^{\infty} h(x)f(x), \qquad if \quad X \quad is \quad continuous. \end{aligned} \tag{2.2}$$

Theorem 1. *Let X and Y be to random variables having finite expectation*

- *If c is a constan and P(X=c)=1, them E(cX)=cE(X).*
- *$X+Y$ having finite expectation them E(X+Y)= E(X)+E(Y).*

Theorem 2. *Let X and Y two independent random variable that having finite expectation, them XY have finite expectation and*

$$E(XY) = E(X)E(Y). \tag{2.3}$$

Proof. Let X and Y independent randon variables, by definition we have that

$$f(x,y) = f_X(X)f_Y(Y) \tag{2.4}$$

therefore

$$\sum_x \sum_y |xy| f(x,y) = \sum_x \sum_y |xy| f_X(x) f_Y(y) = \sum_x |x| f_X(x) \sum_y |y| f_Y(y) < \infty. \tag{2.5}$$

□

Definition 2. *If X is a random variable with $E(X) = \mu_x$, the variance of a random variable X is defined as*

$$\sigma_x^2 = Var(X) = E(X - \mu_x)^2 = E(X^2) - E(X)^2. \tag{2.6}$$

Definition 3. *Let X and Y random vectors the covariance is defined*

$$Cov(X,Y) = E(X - E(X)(Y - E(Y))) = E(XY) - E(X)E(Y). \tag{2.7}$$

Note that if X and Y are independent them Cov(X,Y)=0.
Some other important properties are illustrated below.
Let X and Y random variables that have moments of finite second order them

- $Var(X + Y) = Var(X) + Var(Y) + 2Cov(X,Y)$.
- If c is a constant them $Var(c) = 0$.
- If c is a constant and X is a random variable, them $Var(X + c) = Var(X)$.
- $Var(aX + b) = a^2 Var(X)$, a and b are constants.

Remark 1. *In our study, we use Σ_{xy} to denote the valiance of two random variables.*

Definition 4. *We say that X is a normal random variable with parameters μ and σ^2 or $X \sim N(\mu, \sigma^2)$, if the density of X is given by*

$$f(x) = \frac{1}{\sqrt{2\pi\sigma}} e^{-(x-\mu)^2/2\sigma^2}. \tag{2.8}$$

If $\mathbf{X} = (x_1, ..., x_n)$ is a random vector, it is called normal random vector with parameters k-vector $\boldsymbol{\mu}$ and $k \times k$ matrix $\mathbf{\Sigma}$, if exist $\boldsymbol{\mu} \in \mathbb{R}^n$, $\mathbf{H} \in \mathbb{R}^{n \times k}$ such that $\mathbf{X} = \mathbf{HZ} + \boldsymbol{\mu}$ for $Z_n \sim N(0,1)$ i.i.d. where $\mathbf{Z} \in \mathbb{R}^k$ and $\mathbf{\Sigma} = \mathbf{HH}^T$. The join distribution function is

$$f_{\mathbf{x}}(x_1, ..., x_2) = \frac{exp\{-\frac{1}{2}(\mathbf{x} - \boldsymbol{\mu})^T \mathbf{\Sigma}^{-1} (\mathbf{x} - \boldsymbol{\mu})\}}{\sqrt{(2\pi)^k |\mathbf{\Sigma}|}}. \tag{2.9}$$

Definition 5. *We say that X has Generalized inverse Gaussian (GIG) distribution [20] or $x \sim GIG(\psi, \chi, \phi)$ if its probability density function is given by*

$$p_x = (x|\psi, \chi, \phi) = \frac{(\frac{\psi}{\chi}^{\phi/2}) x^{\phi-1}}{2K_\phi(\sqrt{\psi\chi})} exp\{-\frac{1}{2}(\chi x^{-1} + \psi x)\} \quad x > 0, \quad \psi > 0, \quad \chi > 0, \tag{2.10}$$

where $K_\phi(\zeta) = (1/2)\int_{-\infty}^{\infty} \cosh(\phi\zeta) \exp[-z\cosh(\zeta)] d\zeta$ is the Bessel function of third kind.

2.2 Bayesian inference

The results of Bayesian inference arise from the need to make probability about x given y [21], these are derived from join density function (2.11) and Bayes' rule (2.12)

$$p(x, y) = p(x)p(y|x), \tag{2.11}$$

$$p(x|y) = \frac{p(x, y)}{p(y)}. \tag{2.12}$$

If we join equation (2.11) and (2.12), we have the **posterior** density:

$$p(x|y) = \frac{p(x, y)}{p(y)} = \frac{p(x)p(y|x)}{p(y)}, \tag{2.13}$$

where $p(y) = \sum_x p(x)p(y|x)$ (discrete case) and $p(y) = \int p(x)p(y|\theta)dx$ in continuous case. If we fixed y, we can omits $p(y)$ since it don't depends of parameter x to obtain

$$p(x|y) \propto p(x)p(y|x), \tag{2.14}$$

where $p(x)$ is know as **prior** distribution of parameter x and $p(y|x)$ as the **likelihood function**.

Remark 2. *We do not have information on the parameters x. The correct selection of a prior distribution will be of vital importance in the results of the posterior distribution, we can guide with the likelihood distribution. To deepen the subject we recommended [22].*

Definition 6. *The likelihood function of x is a mapping that associates the value $p(y|x)$ to each x. A common notacion is $l(x; y)$ and it is defined as follows*

$$\begin{aligned} l(\cdot; x) :&\Omega \to \mathbf{R}^+, \\ &x \to l(x; y) = p(y|x). \end{aligned} \tag{2.15}$$

Suppose that we have two observation y_1, y_2, where y_2 is not depending of y_1, the posterior function of x given y_1 is

$$p(x|y_1) = l_1(x; y_1)p(x), \tag{2.16}$$

Now, let y_2, we have that the posterior distribution of x given y_1, y_2 is

$$\begin{aligned} p(x|y_1, y_2) &= \frac{p(y_2|x, y_1)p(x|y_1)}{p(y_2|y_1)} \\ &\propto p(y_2|x)p(x|y_1) \\ &\propto l_2(x; y_2)l_1(x; y_1)p(x). \end{aligned} \tag{2.17}$$

Note that, if we have $y_1, y_2, ..., y_n$ independent observation and proceeding in the same way as (2.17), we have that the posterior function is

$$\begin{aligned} p(x|y_n, ..., y_1) &\propto l(x; y_n)p(x|y_{n-1}, ..., y_1) \\ &\propto \prod_{i=1}^{n} l_i(x; y_i)p(x) \end{aligned} \tag{2.18}$$

Remark 3. *The result of (2.18) is due to observations are independent, in (2.38) we show what happens if observations are not independent.*

Example 1. *Suppose that we have an independent and identically normal distributed* $\mathbf{y}_t = \{y_1, ..., y_t\}$ *with unknown mean* μ*, know* σ^2 *and distribution* $f(y|\mu, \sigma^2)$*. In order to make inference about parameter* $x = \mu$ *assume that the prior distribution on* μ *is* $N(0, \sigma^2)$*, following the approximation in (2.18), the posterior distribution is*

$$\begin{aligned}\pi(\mu|\mathbf{y_t}) &\propto \pi(\mu) l(\mathbf{y}_t; \mu, \sigma) \\ &\propto \pi(\mu) \prod_{i=1}^{t} f(\mu, \sigma|x_i) \\ &\propto \exp\{\mu^2/2\sigma^2\} \prod_{i=1}^{t} \frac{\exp\{-(y_i-\mu)^2/2\sigma^2\}}{\sqrt{2\pi}\sigma} \\ &\propto \exp\{\mu^2/2\sigma^2\} \exp\{-[t(\mu-\bar{y}) + s^2]/2\sigma^2\}\sigma^t \\ &\propto \exp\{-(t+1)\mu^2/2\sigma^2 + 2t\mu\bar{y}/2\sigma^2\} \\ &\propto \exp\{-(t+1)[\mu - t\bar{y}/(t+1)]^2/2\sigma^2\},\end{aligned} \tag{2.19}$$

as result have that the posterior distribution in μ *is a normal distribution with mean* $t\bar{y}/(t+1)$ *and variance* $\sigma^2/(t+1)$*, which is different to the classical estimator* $\bar{y}$*.*

Definition 7. *The maximum likelihood estimation (MLE) is the value* $\mathbf{x}$ *that maximize* $l(\mathbf{x}; \mathbf{y})$*.*

2.3 Stochastic Processes

A stochastic process is a collection or family of random variables $\{X_t, \quad with \quad t \in T\}$, ordered according to the subscript t which in general is usually identified with the time. If T is a continuous set, for example $\mathbb{R}^+$, we say that X_t is a stochastic process of continue parameter on the other hand, if T is discrete, for example $\mathbb{N}$.e say that X_t is a stochastic process of discrete parameter, moreover if for each t the random variable X_t is of continuous type, we will say that the stochastic process is continuous state and if for each t the random variable X_t is of discrete type, we will say that the stochastic process is a discrete state.

Remark 4. *During our study, we are using a stochastic process of the discrete state.*

Definition 8. ***(Marcovian)*** *[23] A Markov chain is a sequence of dependent random variables* $\{X_t\}_{t\in\mathbf{N}}$ *such that*

$$p(X_t|X_1, ..., X_{t-1}) = p(X_t|X_{t-1}). \tag{2.20}$$

We can interpret (2.20) in such a way that the current state (X_t) only depends on the previous state X_{t-1}.

2.4 Time series

Time series are stochastic processes which principal idea is build a mathematical model that provide plausible descriptions for sample data. Time series are a collection of random variables $\{x_t, \quad t \in \mathbb{Z}\}$ where x_1 denotes the first value in the time period, x_2 denote the second value in the period time. Using the definition of stochastic process we can say that time series is a stochastic process of discrete state and discrete time.

Example 2. ***Gross Domestic Product of Ecuador***

Gross Domestic Product of Ecuador (GDP) is a sum of all the goods and services produced within the national territory over a period of time. In figure 2.1 shows the GDP's Ecuador of the quarterly period.

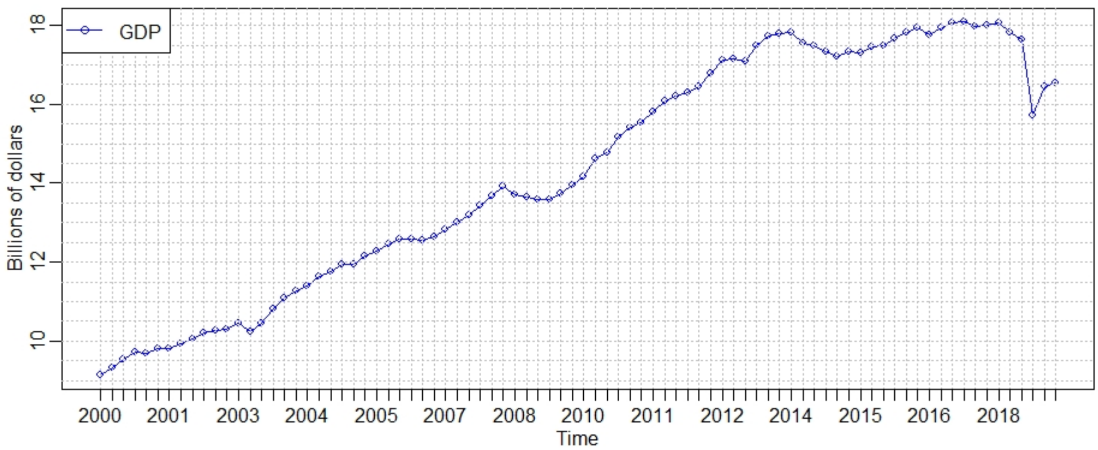

Figure 2.1: Gross Domestic Product of Ecuador

2.5 ARIMA models

Autoregressive integrated moving average (ARIMA) models is a complementing of classical regression which is not enough to explain to explain the behavior of dynamic series. The introduction of ARIMA models implies the use of correlation generated through lagged linear relations that help us to make better interpretation of data.

2.5.1 Autoregressive model

The Autoregressive model is a mathematical structure that makes use of previous states to interpret the present state through a linear combination.

Definition 9. *An Autoregressive model of order p, AR(p), is defined as*

$$x_t = \phi_1 x_{t-1} + \phi_2 x_{t-2} + \dots + \phi_p x_{t-p} + w_t, \tag{2.21}$$

where $\phi_1, \phi_2, ..., \phi_p$ are constants, x_t is stationary and $w_t \sim N(0, \sigma^2)$. The mean of (2.21) is zero. If the mean of (2.21) is not zero, we can replace x_t by $x_t - \mu$ and rewrite as

$$x_t = \alpha + \phi_1 x_{t-1} + \phi_2 x_{t-2} + ... + \phi_p x_{t-p} + w_t, \tag{2.22}$$

where

$$\alpha = \mu(1 - \phi_1 - \phi_2 ... - \phi_p).$$

Proposition 1. *The mean of (2.21) is zero.*

Proof.

$$\begin{aligned} E(x_t) &= \phi_1 E(x_{t-1}) + E(\phi_2 x_{t-2}) + ... + E(\phi_p x_{t-p}) + E(w_t) \\ \mu &= \phi_1 \mu + \phi_2 \mu + ... + \phi_p \mu + E(w_t) \\ \mu &= \frac{E(w_t)}{1 - \phi_1 - \phi_2 - ... - \phi_p} = 0. \end{aligned} \tag{2.23}$$

Example 3. ***Random walk*** *is clear example of AR(1)-model and it is defined as*

$$x_t = x_{t-1} + w_t, \tag{2.24}$$

where w_t is a withe noise and the initial condition $x_0 = 0$. Clearly the mean of (2.24) is zero. If we add a constant, $\delta \neq 0$, to system:

$$x_t = \delta + x_{t-1} + w_t, \tag{2.25}$$

*we say that (2.25) is a **random walk with drift**, see A.1.1.*

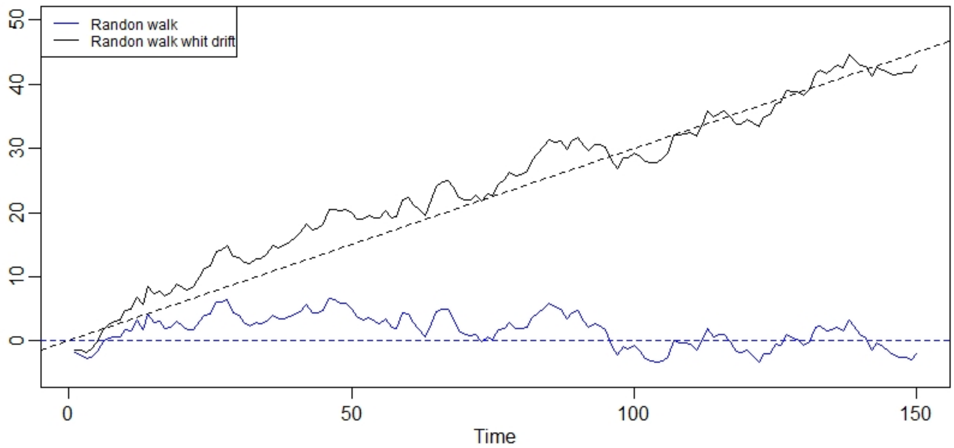

Figure 2.2: Random Walk.

2.5.2 Vector autoregressive model

Vector autoregressive model is similar to AR model, only in this case we have k-variables. We introduce the use of boldface letters in order to refer to vectors and matrices.

Definition 10. *Vector autoregressive model, VAR(p), of p order is defined*

$$\mathbf{x_t} = \alpha + \Phi_1 \mathbf{x}_{t-1} + ... + \Phi_p \mathbf{x}_{t-p} + w_t, \tag{2.26}$$

where Φ_i *is a* $k \times k$ *transition matrix,* $\mathbf{w_t}$ *is a vector white nose,* $\mathbf{x}_t = (x_{t1}, ..., x_{tk})'$ *is a* $k \times 1$ *vector-value.*

Follow the previous definition, we have VAR(1) model as

$$\mathbf{x}_t = \alpha + \mathbf{\Phi} \mathbf{x}_{t-1} + \mathbf{w}_t, \tag{2.27}$$

where Φ is a $k \times k$ transition matrix, $\mathbf{w_t}$ is a vector white nose, $\mathbf{x}_t = (x_{t1}, ..., x_{tk})'$ is a $k \times 1$ vector-value.

2.5.3 Mixed frequency model

In economics, the time series are normally recorded with different time variations. We have, for example, that the gross domestic product is found on a quarterly basis and the consumer price index on a monthly basis.

2.6 Monte Carlo Methods

Monte Carlo Methods is a special numerical techniques computing that allow us approximate integrals and relies in the use of random variables. these approximations are made by law of large numbers, that is, if $x_1, ..., x_N$ are distributed from g and independent, them the empirical average

$$\hat{\mathbb{J}} = \frac{(h(x_1) + ... + h(x_N))}{N} \tag{2.28}$$

converges (almost surely) to the integral

$$\mathcal{E}_g(h(X)) = \int h(x) g(x) dx \tag{2.29}$$

Example 4. *Consider that we want to integrate* $h(x) = x^3 \exp^{-x}$ *on (0,∞). We can generate 50000 random samples from uniform distribution* $\mathcal{U}(0, 4000)$ *them following (2.29) we have that*

$$\begin{aligned} \mathbb{E}_g(h(X)) &= \int h(x) g(x) dx \\ &= \int x^3 e^{-x} \frac{1}{4000} dx, \end{aligned} \tag{2.30}$$

continuing whit the approximation (2.28), we have that

$$\int_0^\infty x^3 e^{-x} dx \approx 4000 * \mathbb{E}_g(h(X)) \approx 4000 * \frac{\sum_{i=1}^{5000} x_i^3 e^{-x_i}}{50000}; \quad x_i \sim N(0, 4000), \tag{2.31}$$

you can see the implementation of this example in A.1.2.

The computational cost is large and less accurate compared to other methods such as Importance sampling.

2.6.1 Importance sampling

Importance sampling continue using random variable to compute the approximation, but in these case another join distribution function, $q(\cdot)$, is added

$$\mathbb{E}_g(h(X)) = \int h(x)g(x)dx = \int \frac{h(x)g(x)}{q(x)}q(x)dx = \mathbb{E}_q(\frac{h(x)g(x)}{q(x)}). \tag{2.32}$$

Using the law of large numbers, we have that

$$\hat{\mathbb{E}}_q(\frac{h(x)g(x)}{q(x)}) = \frac{1}{N}\sum_{i=1}^{N}\frac{h(X_i)g(X_i)}{q(X_i)}, \qquad X_i \sim q. \tag{2.33}$$

The adjustment factor $w_i(X_i) = g_i(X_i)/q_i(X_i)$ is called the likelihood ratio.

2.7 State Space Models

State space models (SSM) are mathematical structures customized to the study of stochastic processes, especially when data are contaminated with error. Some uses are the localization of an airplane, cellphone signal, economic indicators such as the gross domestic product (GDP).
A generalized SSM is in the form:

$$State\ \ equation: \qquad x_t = h(\mathbf{x}_{t-1}, \epsilon_t)\ \ or\ \ x_t \sim q_t(\cdot \mid \mathbf{x}_{t-1}). \tag{2.34}$$

$$Observation\ \ equation: \qquad y_t = g(\mathbf{x}_t, e_t)\ \ or\ \ y_t \sim f_t(\cdot \mid \mathbf{x}_t), \tag{2.35}$$

where y_t is the observation, x_t is the (unobservable) state variable. Let $\mathbf{y_t} = (y_1, \ldots, y_t)'$ denote the entire past sequence of the observations at time t and $\mathbf{x_j} = (x_1, \ldots, x_j)'$ denotes the entire history of the state before and at time j. Let's recall that y_t can be multi-dimensional, moreover x_t evolves through the conditional distribution $q_t(\cdot)$ and underlying states evolve with the function $h_t(\cdot)$. Conditional distribution $q(\cdot)$ and state innovation ϵ_t (or equivalent the function $h_t(\cdot)$ are assumed be known.

Definition 11. ***Marcovian SSM*** *assumes that* $h_t(\mathbf{x}_{t-1}, \epsilon_t)$ *only depends on* x_{t-1} *and* $g_t(\mathbf{x}_t, e_t)$ *only depends on* x_t*, that is,*

- *State equation:*

$$x_t = h(x_{t-1}, \epsilon_t)\ \ or\ \ x_t \sim q_t(\cdot \mid x_{t-1}). \tag{2.36}$$

- *Observation equation:*

$$y_t = g(x_t, e_t)\ \ or\ \ y_t \sim f_t(\cdot \mid x_t). \tag{2.37}$$

The following diagram gives a graphic notion of the system.

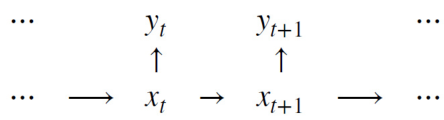

Figure 2.3: Markovian SSM.

We can see that state variable, x_t, gives information to obtain y_t through $g(\cdot)$. x_{t+1} is obtained using x_t and provides information to obtain y_{t+1}. Using statistical inference that at any time t the states $x_1, \dots, x_t$ can be found given the observation $y_1, \dots, y_t$, up to time t, we can obtain the posteriors distribution

$$\begin{aligned} p(x_1, ..., x_t|\mathbf{y}_t) &\propto p(x_1, ..., x_t, y_1, ..., y_t) \\ &\propto \prod_{i=1}^{t} p(y_i|x_1, ..., x_i, y_1, ..., y_{i-1})p(x_i|x_1, ...x_{i-1}, y_1, ..., y_{i-1}) \\ &\propto \prod_{i=1}^{t} f_i(y_i|x_i)q_i(x_i|x_{i-1}). \end{aligned} \tag{2.38}$$

2.7.1 Statistical inference

Principal objective of statistical inference are

- Filtering: Obtain the marginal posterior distribution $p(x_t|\mathbf{y}_t)$ and $E[\phi(x_t|\mathbf{y}_t)]$.
- Prediction: Obtain the posterior distribution $p(x_{t+1}|y_1, ..., y_t)$ and $E[\phi(x_{t+1}|\mathbf{y}_t)]$.
- Smoothing:Found the posterior distribution$p(x_1, ..., x_{t-1}|y_1, ..., y_t)$ and estimate a value that maximize $p(x_1, ..., x_t|y_1, ..., y_t)$.
- Likelihood and parameter estimation.

Let θ be a collection unknown parameter's in the model. Given all observation $\mathbf{y}_T = \{y_t, t = 1, ..., T\}$ likelihood function is

$$L(\theta) = p(\mathbf{y}_T|\theta) = \int p(y_1, ..., y_T, x_1, ..., x_T|\theta)dx_1...dx_T \tag{2.39}$$

Another formulation is

$$L(\theta) = p(\mathbf{y}_T|\theta) = \prod_{t=1}^{T} p(y_t|\mathbf{y}_{t-1}, \theta) \tag{2.40}$$

where

$$p(y_t|\mathbf{y}_{t-1}, \theta) = \int p(y_t|x_t\mathbf{y}_{t-1}, \theta)p(x_t|\mathbf{y}_{t-1}, \theta)dx_t. \tag{2.41}$$

The Kalman filter needs a specific model to work and all theory developed is supported by the following model.

2.7.2 Linear Gaussian state space models

If the function are linear and the noise is Gaussian, we can rewrite (2.36) and (2.37) as

$$\begin{aligned} x_t &= \mathbf{H}_t x_{t-1} + \mathbf{B}_t b_t + \mathbf{W}_t w_t \\ y_t &= \mathbf{G}_t x_t + \mathbf{C}_t c_t + \mathbf{V}_t v_t, \end{aligned} \tag{2.42}$$

where $\mathbf{H}_t$, $\mathbf{G}_t$, $\mathbf{B}_t$, $\mathbf{C}_t$, $\mathbf{W}_t$ and $\mathbf{V}_t$ are matrices, the input series (c_t and b_t) are known and $w_t \sim N(0, \mathbf{I})$ and $v_t \sim N(0, \mathbf{I})$. In literature model (2.42) is known as dynamic linear model see [5].

In order to maximize the understanding of this model, we will introduce the following example which is a AR(1) with state space.

Example 5. *(* **1-Dimensional Gaussian random walk***)*

$$\begin{aligned} x_t &= x_{t-1} + w_t, \qquad w_t \sim N(0, W) \\ y_t &= x_t + v_t, \qquad w_t \sim N(0, V) \end{aligned} \tag{2.43}$$

Using the R code A.1.3, we can plot a random walk with $x_0 = 2$ as initial state, $W = 0.25$ and $V = 1, 25$. The is result joint the Kalman filter is presented in Fig. 2.4.

Example 6. *Time Series With Observational Noises. It is a representation of the arima models by the state space models.*

If $\mathbf{B}$ *and* $\mathbf{C}$ *are equal to zero,*

$$H = \begin{pmatrix} \phi_1 & \phi_2 & \cdots & \phi_{p-1} & \phi_p \\ 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & 0 \\ 0 & 0 & \cdots & 1 & 0 \end{pmatrix},$$

$$\mathbf{W}' = \mathbf{V}' = \mathbf{G} = (1, 0, ..., 0)$$

and the state variable is $x_t = (z_t, z_{t-1}, ..., z_{t-p+1})$, we have that z_t follows an AR(p) where $y_t = z_t + v_t$ is the noise observation. Replacing everything in (2.42), we have that

$$\begin{pmatrix} z_t \\ z_{t-1} \\ \vdots \\ z_{t-p+1} \end{pmatrix} = \begin{pmatrix} \phi_1 z_{t-1} + \phi_2 z_{t-2} + ... + \phi_p z_{t-p} \\ z_{t-1} \\ \vdots \\ z_{t-p+1} \end{pmatrix} + \begin{pmatrix} w_t \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

and

$$y_t = z_t + e_t \quad \textit{where} \quad e_t = w_t + v_t$$

Before to continue is indispensable the use of lemma 1 and lemma 2:

Lemma 1. *If* $X \sim N(\mu_x, \mathbf{\Sigma}_x)'$ *and* $\mathbf{Y} = c + \mathbf{G}X + \mathbf{V}v$*, where* $v \sim N(0, \mathbf{I})$ *and is independent with* X*, them the join distribution of* (X, Y) *is*

$$\begin{pmatrix} X \\ Y \end{pmatrix} \sim N\left(\begin{pmatrix} \mu_X \\ \mu_Y \end{pmatrix}, \begin{bmatrix} \mathbf{\Sigma}_{xx} & \mathbf{\Sigma}_{xy} \\ \mathbf{\Sigma}_{yx} & \mathbf{\Sigma}_{yy} \end{bmatrix}\right) \tag{2.44}$$

where

$$\begin{aligned}
\mathbf{\Sigma}_{xx} &= \mathbf{\Sigma}_x \\
\mu_y &= E[Y] = E[c + \mathbf{G}x + \mathbb{V}v] \\
&= c + \mathbf{G}E[X] = c + \mathbf{G}\mu_x \\
\mathbf{\Sigma}_{xy} &= \mathbf{\Sigma}_x \mathbf{G}', \\
\mathbf{\Sigma}_{yx} &= \mathbf{G}\mathbf{\Sigma}_x, \\
\mathbf{\Sigma}_{yy} &= \mathbf{G}\mathbf{\Sigma}_x \mathbf{G}' + \mathbf{V}\mathbf{V}'
\end{aligned} \tag{2.45}$$

Proof. The first equation, $\mathbf{\Sigma}_{xx} = \mathbf{\Sigma}_x$, follows from the definition of variance. Let's prove third equation

$$\begin{aligned}
\mathbf{\Sigma}_{xy} &= E[(X - \mu_x)(Y - \mu_y)'] \\
&= E[(X - \mu_x)(c + \mathbf{G}X + \mathbf{V}v - c - \mathbf{G}\mu_x)'] \\
&= E[(X - \mu_x)((X - \mu_x)'\mathbf{G}' + v'V')] \\
&= \mathbf{\Sigma}_x \mathbf{G}' + E[(X - \mu_x)v'V')] \\
&= \mathbf{\Sigma}_x \mathbf{G}', \quad X \ \ and \ \ v \ \ are \ \ independent.
\end{aligned} \tag{2.46}$$

Fourth equation is similar to third. Let's prove fifth equation

$$\begin{aligned}
\mathbf{\Sigma}_{yy} &= E[(Y - \mu_y)(Y - \mu_y)'] \\
&= E[\mathbf{G}(X - \mu_x)(X - \mu_x)'\mathbf{G}' + \mathbf{V}vv'\mathbf{V}'] \\
&= \mathbf{G}E[(X - \mu_x)(X - \mu_x)']\mathbf{G}' + \mathbf{V}E[vv']\mathbf{V}'] \\
&= \mathbf{G}\mathbf{\Sigma}_x \mathbf{G}' + \mathbf{V}\mathbf{I}\mathbf{V}' = \mathbf{G}\mathbf{\Sigma}_x \mathbf{G}' + \mathbf{V}\mathbf{V}'
\end{aligned} \tag{2.47}$$

□

Lemma 2. *If*

$$\begin{pmatrix} X \\ Y \end{pmatrix} \sim N\left(\begin{pmatrix} \mu_X \\ \mu_Y \end{pmatrix}, \begin{bmatrix} \mathbf{\Sigma}_{xx} & \mathbf{\Sigma}_{xy} \\ \mathbf{\Sigma}_{yx} & \mathbf{\Sigma}_{yy} \end{bmatrix}\right) \tag{2.48}$$

and we assume that $\mathbf{\Sigma}_{yy}^{-1}$ *exist, them*

$$\begin{aligned}
E(X|Y = y) &= \mu_x - \mathbf{\Sigma}_{xy}\mathbf{\Sigma}_{yy}^{-1}(y - \mu_y) \\
Var(X|Y = y) &= \mathbf{\Sigma}_{xx} - \mathbf{\Sigma}_{xy}\mathbf{\Sigma}_{yy}^{-1}\mathbf{\Sigma}_{yx}
\end{aligned} \tag{2.49}$$

2.8 Kalman Filter

In statistical inference of Kalman filter, the use of lemma 1 and lemma 2 are essential to find $p(x_t|\mathbf{y}_t)$, where $\mathbf{y}_t = (y_1, ..., y_t)$. The process is done recursively, following the steps below:
1) Suppose at time $t-1$ we have obtained μ_{t-1} and $\mathbf{\Sigma}_{t-1}$,

$$p(x_{t-1}|\mathbf{y}_{t-1}) \sim N(\mu_{t-1}, \mathbf{\Sigma}_{t-1}) \tag{2.50}$$

. Before we observe $\mathbf{y}_t$, we can predict $\mathbf{x_t}$ using

$$\mathbf{x_t} = \mathbf{H}_t x_{t-1} + \mathbf{B}_t b_t + \mathbf{W}_t w_t \tag{2.51}$$

moreover, since $p(x_t|\mathbf{y}_{t-1})$ and w_t are normal, we have that

$$p(x_t|\mathbf{y}_{t-1}) \sim N(\mu_{t|t-1}, \mathbf{\Sigma}_{t|t-1}) \tag{2.52}$$

where

$$\begin{aligned} \mu_{t|t-1} &= E[\mathbf{H}_t x_{t-1} + \mathbf{B}_t b_t + \mathbf{W}_t w_t|\mathbf{y}_{t-1}] \\ &= \mathbf{H}_t E[x_{t-1}|\mathbf{y}_{t-1}] + \mathbf{B}_t b_t \\ &= \mathbf{H}_t \mu_{t-1} + \mathbf{B}_t b_t \end{aligned} \tag{2.53}$$

$$\begin{aligned} \mathbf{\Sigma}_{t|t-1} &= Var[\mathbf{H}_t x_{t-1} + \mathbf{B}_t b_t + \mathbf{W}_t w_t|\mathbf{y}_{t-1}] \\ &= Var[\mathbf{H}_t x_{t-1}|\mathbf{y}_{t-1}] + \mathbf{W}_t \mathbf{W}_t \\ &= \mathbf{H}_t \mathbf{\Sigma}_{t-1} \mathbf{H}'_t + \mathbf{W}_t \mathbf{W}'_t \end{aligned} \tag{2.54}$$

Now using $y_t = \mathbf{G}_t x_t + \mathbf{C}_t c_t + \mathbf{V}_t v_t$, (2.53) and (2.54), lemma 1 provides $p(x_t, y_t|y_1, ..., y_{t-1})$. Finally, from lemma 2, we get $p(x_t|y_1, ...y_{t-1}, y_t)$.

Using the previous information, we can expose the Kalman filter algorithm

Algorithm 1 Kalman filter

$\mu_{t|t-1} = \mathbf{H}_t \mu_{t-1} + \mathbf{B}_t b_t$

$\mathbf{\Sigma}_{t|t-1} = \mathbf{H}_t \mathbf{\Sigma}_{t-1} \mathbf{H}'_t + \mathbf{W}_t \mathbf{W}'_t$

$\mu_t = \mu_{t|t-1} + \mathbf{K}_t (y_t - \mathbf{C}_t c_t - \mathbf{G}_t \mu_{t|t-1})$

$\mathbf{\Sigma}_t = \mathbf{\Sigma}_{t|t-1} - \mathbf{K}_t \mathbf{G}_t \mathbf{\Sigma}_{t|t-1},$

where

$\mathbf{K}_t = \mathbf{\Sigma}_{t|t-1} \mathbf{G}'_t [\mathbf{G}_t \mathbf{\Sigma}_{t|t-1} \mathbf{G}'{}_t + \mathbf{V}'{}_t \mathbf{V}_t]^{-1}$

In the literature, the matrix $\mathbf{K}_t$ is called the Kalman gain matrix.

Continuing with the example of 5, the observations of the Gaussian random walk were filtered and presented below, the code used is detailed in A.1.3.

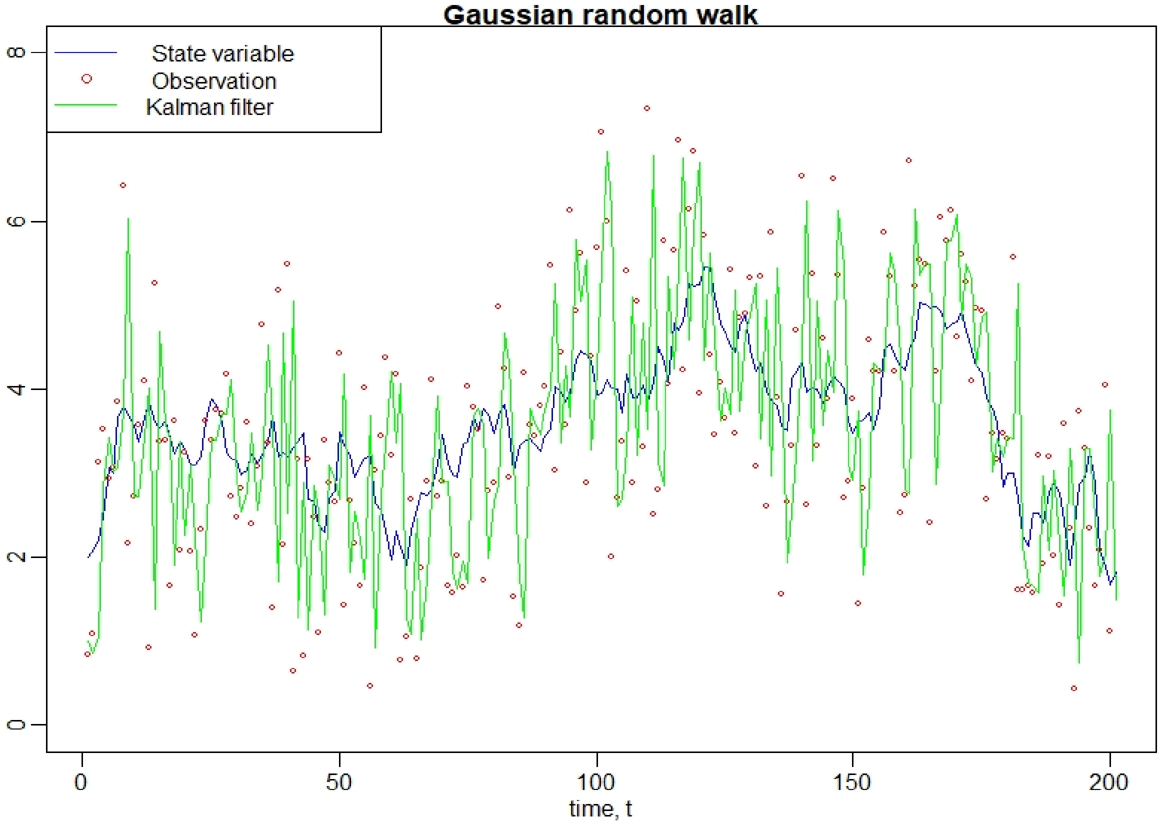

Figure 2.4: Gaussian random walk

2.9 Smoothing

Kalman Smoothing aims to find $p(x_1, ..., x_T|\mathbf{y}_T)$ given the entire observed sequence $\mathbf{y}_T = (y_1, ..., y_T)$ and recursively obtains $\mu_{t|T}$ and $\mathbf{\Sigma}_{t|T}$ in a forward and backward two-pass algorithm.

$$\begin{aligned} E(x_t|\mathbf{y}_T) &= E(E(x_t|x_{t+1}, \mathbf{y}_T)) \\ &= E(E(x_t|x_{t+1}, \mathbf{y}_t)|\mathbf{y}_T), \\ Var(x_t|\mathbf{y}_T) &= E[Var(x_t|x_{t+1}, \mathbf{y}_T)|\mathbf{y}_T] + Var[E(x_t + \mathbf{y}_T)|\mathbf{y}_T] \\ &= E[Var(x_t|x_{t+1}, \mathbf{y}_t)|\mathbf{y}_T] + Var[E(x_t + \mathbf{y}_t)|\mathbf{y}_T], \end{aligned} \tag{2.55}$$

to obtain $E(x_t|x_{t+1}, \mathbf{y}_t)$, we use

$$\begin{bmatrix} x_t|\mathbf{y}_t \\ x_{t+1}|\mathbf{y}_t \end{bmatrix} \sim N\left(\begin{bmatrix} \mu_{t|t} \\ \mu_{t+1|t} \end{bmatrix}, \begin{bmatrix} \mathbf{\Sigma}_{t|t} & \mathbf{\Sigma}_{t|t}\mathbf{H'}_{t+1} \\ \mathbf{H}_{t+1}\mathbf{\Sigma}_{t|t} & \mathbf{\Sigma}_{t+1|t} \end{bmatrix} \right), \tag{2.56}$$

and Lemma 2 provides

$$\begin{aligned} E(x_t|x_{t+1}, \mathbf{y}_t) &= \mu_{t|t} + \mathbf{J}_t(x_{t+1} - \mu_{t+1|t}), \\ Var(x_t|x_{t+1}, \mathbf{y}_t) &= \mathbf{\Sigma}_{t|t} - \mathbf{\Sigma}_{t|t}\mathbf{H'}_{t+1}[\mathbf{\Sigma}_{t+1|t}]^{-1}\mathbf{H}_{t+1}\mathbf{\Sigma}_t \\ &= \mathbf{\Sigma}_{t|t} - \mathbf{J}_t\mathbf{\Sigma}_{t+1|t}\mathbf{J'}_t, \end{aligned} \tag{2.57}$$

where $\mathbf{J}_t = \mathbf{\Sigma}_t\mathbf{H}_{t+1}[\mathbf{\Sigma}_{t+1|t}]^{-1}$. Hence

$$\begin{aligned} E[E(x_t|x_{t+1}, \mathbf{y}_t)] &= E[\mu_{t|t} + \mathbf{J}_t(x_{t+1} - \mu_{t+1|t})] \\ &= \mu_{t|t} + \mathbf{J}_t(E(x_{t+1}|\mathbf{y}_T - \mu_{t+1|t})) \\ &= \mu_{t|t} + \mathbf{J}(\mu_{t+1|T} - \mu_{t+1|t}) \end{aligned} \tag{2.58}$$

and

$$\begin{aligned} Var[E(x_t|x_{t+1}, \mathbf{y}_t)|\mathbf{y}_T] &= Var[\mu_{t|t} + \mathbf{J}_t(x_{t+1})|\mu_{t+1}|\mathbf{y}_T] \\ &= \mathbf{J}_t[V(x_{t+1}|\mathbf{y}_T)]\mathbf{J'}_t \\ &= \mathbf{J}_t\mathbf{\Sigma}_{t+1|T}\mathbf{J'}_t, \end{aligned} \tag{2.59}$$

moreover

$$\begin{aligned} E[Var(x_t|x_{t+1}, \mathbf{y}_t)|\mathbf{y}_T] &= E[\mathbf{\Sigma}_{t|t} - \mathbf{J}\mathbf{\Sigma}_{t+1|t}\mathbf{J'}_t|\mathbf{y}_t] \\ &= \mathbf{\Sigma}_{t|t} - \mathbf{J}_t\mathbf{\Sigma}_{t+1|t}\mathbf{J'}_t. \end{aligned} \tag{2.60}$$

Finally, putting everything together, we have Algorithm 5.

Prediction and Missing Data

Prediction is to get $p(\mu_{t+d}|y_1, ..., y_t)$, we can do it of recursive way

$$\begin{aligned} \mu_{t+d|t} &= E(x_{t+d}|\mathbf{y}_t) \\ &=E[E(x_{t+d}|x_{t+d-1}, \mathbf{y}_t)|\mathbf{y}_t] \\ &=\mathbf{H}_{t+d}\mu_{t+d-1|d} + \mathbf{B}_{t+d}b_{t+d}, \end{aligned} \tag{2.61}$$

and

$$\begin{aligned}\mathbf{\Sigma}_{t+d|t} &= Var(x_{t+d}|\mathbf{y}_t)\\ &= Var[E(x_{t+d}|x_{t+d-1},\mathbf{y}_t)|\mathbf{y}_t]\\ &+ E[Var(x_{t+d}|x_{t+d-1},\mathbf{y}_t)|\mathbf{y}_t]\\ &= \mathbf{H}'_{t+d}\mathbf{\Sigma}_{t+d-1}\mathbf{H}_{t+d} + \mathbf{W}'_{t+d}W_{t+d}.\end{aligned} \tag{2.62}$$

For missing data, y_t, just do a substitution $\mu_{t|t} = \mu_{t|t-1}$ and $\mathbf{\Sigma}_{t|t} = \mathbf{\Sigma}_{t|t-1}$.

Algorithm 2 Kalman smoothing

For $t = T-1, T-2, ..., 1$

$\mu_{t|T} = \mu_t + \mathbf{J}_t(\mu_{t+1|T} - \mu_{t+1|t})$

$\mathbf{\Sigma}_{t|T} = \mathbf{\Sigma}_t + \mathbf{J}_t(\mathbf{\Sigma}_{t+1|T} - \mathbf{\Sigma}_{t+1|t})\mathbf{J'}_t$

Where

$\mathbf{J}_t = \mathbf{\Sigma}_t\mathbf{H}_{t+1}[\mathbf{\Sigma}_{t+1|+t}]^{-1}.$

A second method for estimating state space models is using the Gibbs sampler technique.

2.10 Gibbs sample

Suppose that we have a distribution function $\pi(x) = p(x_1, ..., x_p)$, Gibbs sampler is the ability to generate a sequence of observation from marginal distribution $\pi(x_1)$ and the partial conditional distribution $p(x_i|x_1, ..., x_{i-1})$. The algorithm is presented below

Algorithm 3 Gibbs Sampler

Start with an initial value $x^{(1)} = (x_1^{(1)}, ..., x_t^{(1)})$

At iteration $j+1$

sample $x_1^{(j+1)} \sim \pi(x_1|x_2^{(j)}, ..., x_t^{(j)})$

sample $x_2^{(j+1)} \sim \pi(x_2|x_1^{(j+1)}, x_3^{(j)}, ..., x_t^{(j)})$

$\vdots$

sample $x_t^{(j+1)} \sim \pi(x_t|x_1^{(j+1)}, x_2^{(j+1)}, ..., x_{t-1}^{(j+1)})$.

To apply the Gibbs sampler, it is essential to define the likelihood of the model, and the prior distributions of the parameters. We can rewrite (2.42) and add two variable, ψ and ω, to obtain the model in (2.63).

$$\begin{aligned}x_t &= \mathbf{H}_t x_{t-1} + \mathrm{w}_t,\\ y_t &= \mathbf{G}_t x_t + v_t,\end{aligned} \tag{2.63}$$

where $w_t \sim N(0, \psi_t\mathbf{\Psi})$ and $v_t \sim N(0, \omega_t\mathbf{\Omega})$. The inference is made in the posterior distribution that is computed as

$$p(\mathbf{\Psi}, \mathbf{\Omega}|\mathbf{x_t}, \mathbf{y}_t, \psi_t, \omega_t) \propto \pi(\mathbf{x}_t, \mathbf{y}_t|\mathbf{\Psi}, \mathbf{\Omega}, \psi, \omega)p(\mathbf{\Psi}, \mathbf{\Omega}, \psi, \omega) \tag{2.64}$$

Using the probability chain ruler as (2.38), we have that the likelihood function is

$$\pi(\mathbf{x}_t, \mathbf{y}_t|\boldsymbol{\Psi}, \boldsymbol{\Omega}, \psi, \omega) \propto q(x_0|\mu_0, \boldsymbol{\Psi}) \prod_{t=1}^{n} q(x_t|x_{t-1}, \boldsymbol{\Psi}) \prod_{t=1}^{n} f(y_t|x_t, \boldsymbol{\Omega}) \tag{2.65}$$

where

$$q(x_t|x_{t-1}, \boldsymbol{\Psi}) = \int p(x_t|x_{t-1}, \psi_t, \boldsymbol{\Psi}) p_1(\psi_t) d\psi_t \quad t = 1, ..., n$$

and

$$f(y_t|x_t, \boldsymbol{\Omega}) = \int p(y_t|x_t, \omega) p_2(w_t) dw_t, \quad t = 1, ..., n.$$

Since u_t and v_t are normal, we have that $(x_t|x_{t-1}, \psi_t, \boldsymbol{\Psi}) \sim N(H_t x_{t-1}, \psi_t \boldsymbol{\Psi})$ and $(y_t|x_t, \omega_t, \boldsymbol{\Omega}) \sim N(G_t x_t, \omega_t \boldsymbol{\Omega})$ for $t = 1, ..., n$ therefore the likelihood of (2.65) is

$$\pi(\mathbf{x}_t, \mathbf{y}_t|\boldsymbol{\Psi}, \boldsymbol{\Omega}, \psi, \omega) \propto exp\{-\frac{1}{2\boldsymbol{\Psi}_0}(x_0 - \mu_0)^2 - \frac{1}{2\boldsymbol{\Psi}} \sum_{t=1}^{n} \frac{1}{\psi_t}(x_t - H_t x_{t-1})^2 - \frac{1}{2\boldsymbol{\Omega}} \sum_{t=1}^{n} \frac{1}{\omega_t}(y_t - G_t x_t)^2\}. \tag{2.66}$$

Let's suppose that the prior distributions, for the parameters in (2.66), are given by $\boldsymbol{\Psi} = \sigma^2 \sim Ig(a_0, b_0)$ and $\boldsymbol{\Omega} = \tau^2 \sim Ig(c_0, d_0)$, where Ig denotes the Inverse Gamma distribution and that σ^2 and τ^2 are independent, that is, $\pi(\sigma^2, \tau^2) = \pi(\sigma^2)\pi(\tau^2)$, considering that the hyperparameters ψ and ω are known. Them the joint priori distribution of the parameters is given by

$$\pi(\sigma^2, \tau^2) \propto (\sigma^2)^{-(a_0+1)}(\tau^2)^{-(c_0+1)} exp[-(\frac{b_0}{\sigma^2} + \frac{d_0}{\tau^2})]. \tag{2.67}$$

Once the likelihood is defined in (2.66) and the prior in (2.67), we proceed to calculate the posterior distribution

$$\begin{aligned} \pi(\sigma^2, \tau^2|\mathbf{x_t}, \mathbf{y}_t, \psi_t, \omega_t) &\propto \pi(\mathbf{x}_t, \mathbf{y}_t|\sigma^2, \tau^2, \psi, \omega)\pi(\sigma^2, \tau^2, \psi, \omega) \\ &\propto (\sigma)^{-\frac{n}{2}}(\tau^2)^{-\frac{n}{2}} exp[-\frac{1}{2\sigma^2} \sum_{t=1}^{n} \frac{1}{\psi_t}(x_t - H_t x_{t-1})^2] \\ &\times exp\{-\frac{1}{2\sigma_0^2}(x_0 - \mu_0)^2 - \frac{1}{2\tau^2} \sum_{t=1}^{n} \frac{1}{\omega_t}(y_t - G_t x_t)^2] \\ &\times (\sigma^2)^{-(a_0+1)}(\tau^2)^{-(c_0+1)} exp[-(\frac{b_0}{\sigma^2} + \frac{d_0}{\tau^2})]\}. \end{aligned} \tag{2.68}$$

The posterior distribution has no known distribution to solve this we proceed to obtain the complete conditional distributions using the Gibbs sampler. The process is iterative using the equation (2.68), we can obtain the complete conditional posterior distribution using the parameters of [24], is that

$$\pi(x_t|x_{j\neq t}, \lambda_t, \omega_t, \boldsymbol{\Psi}, \boldsymbol{\Omega}, \mathbf{y}_t) \sim N(a_t, B_t) = N(B_t, b_t, B_t) \tag{2.69}$$

where a_t, b_t and B_t are defined in [24]. The complete conditional posterior distribution a for σ^2 is given by

$$\begin{aligned}\pi(\sigma^2|\psi, \mathbf{x}_t, \mathbf{y}_t) &\propto (\sigma^2)^{-(a_0\frac{n}{2}+1)} \\ &\times exp\{-\frac{1}{\sigma^2}[b_0 + \frac{1}{2}(x_0-\mu_0)^2 + \frac{1}{2}\sum_{t=1}^{n}\frac{(x_t - H_t x_{t-1})^2}{\psi_t}]\} \\ &\sim Ig(a_0 + \frac{n}{2}, b_0 + \frac{1}{2}(x_0-\mu_0)^2 + \frac{1}{2}\sum_{t=1}^{n}\frac{x_t - H_t x_{t-1})^2}{\psi_t}\end{aligned} \tag{2.70}$$

Similarly, the complete posterior conditional distribution for τ^2 is given by

$$\begin{aligned}\pi(\tau^2|\omega, \mathbf{x}_t, \mathbf{y}_t) &\propto (\tau^2)^{-c_0+\frac{n}{2}+1} exp\{-\frac{1}{\tau^2}[d_0 + \frac{1}{2}\sum_{t=1}^{n}\frac{(y_t - G_t x_t)^2}{\omega_t}]\} \\ &\sim Ig[c_0 + \frac{n}{2}, d_0 + \frac{1}{2}\sum_{t=1}^{n}\frac{(y_t - G_t x_t)}{\omega_t}]\end{aligned} \tag{2.71}$$

Suppose that ψ and ω are aleatory variables distributed following an exponential model $\psi_t \sim \exp(\theta)$ (we can consider $\theta = 2$ to identify the complete conditional distribution as known distribution) ; them, if it is known a prior that $(x_t|x_{t-1}, \psi_t, \sigma^2) \sim N(H_t x_{t-1}, \psi_t \sigma^2)$, them the complete conditional posterior for ψ_t is given by

$$\pi(\psi_t|\sigma^2, \mathbf{x_t}, \mathbf{y}_t) \propto \psi_t - \frac{1}{2} exp\{-\frac{1}{2}[\psi_t + \frac{(x_t - H_t x_{t-1})^2}{\psi_t \sigma^2}]\}. \tag{2.72}$$

At same way, we can assume that $\omega_t \sim exp(\theta = 2)$, and as $(y_t|x_t, \omega_t, \tau^2) \sim N(G_t x_t, \omega_t \tau^2)$, we have that the conditional posterior for ω_t is

$$\pi(\omega_t|\tau^2, \mathbf{x}_t, \mathbf{y}_t) \propto \omega_t^{-\frac{1}{2}} exp\{-\frac{1}{2}(w_t + \frac{(y_t - G_t x_t)^2}{\omega_t \tau^2})\} \tag{2.73}$$

with

$$\pi(\psi_t|\sigma^2, \mathbf{x}_t, \mathbf{y}_t) \sim GIG(1/2, 1, (x_t - H_t x_{t-1}/\sigma^2)) \tag{2.74}$$

$$\pi(\omega_t|\tau^2, \mathbf{x}_t, \mathbf{y}_t) \sim GIG(1/2, 1, (x_t - G_t x_t/\tau^2)). \tag{2.75}$$

Finally, we present the last filter

2.11 Particle samples

A different way of estimating the unknown states in the general model given in (2.36), is to use the Monte Carlo method by Sequential Sampling known as particle filter [25]. To develop the particle filter algorithm in detail, consider $\{\mathbf{x}_t^{(i)}, w_t^{*(i)}\}_{i=1}^{N}$ be a random sample characterizes by the probability posteriori density function $\pi(\mathbf{x}_t|\mathbf{y}_t)$, where $\{\mathbf{x}_t^{(i)}, i = 1, ..., N\}$ is a set of points obtained by the weight $w_t^{*(i)}$. Moreover, the weight are normalized such that $\sum_{i=1}^{N} w_t^{*(i)} = 1$. Our goal is that at time t, we want to obtain a set of samples $\{x_t^{(1)}, ..., x_t^{(N)}\}$ following the distribution $\pi(\mathbf{x}_t|\mathbf{y}_t)$, it distribution at time t can be approximated by an empirical distribution formed by the points of mass or particles

$$\pi_N(\mathbf{x}_t|\mathbf{y}_t) \approx \sum_{i=1}^{N} w_t^{*(i)}\delta(\mathbf{x}_t - \mathbf{x}_t^{(i)}) \tag{2.76}$$

where $\delta(.)$ is the Dirac function, using the approximation of posteriori distribution (2.76), we can estimate the expected values of some function $g_n(\mathbf{x}_n)$ associate to the filtered distribution $\pi(\mathbf{x_t}|\mathbf{y_y})$, that is

$$E[g_n(\mathbf{x}_t)] = \int g_n(\mathbf{x}_t)\pi(\mathbf{x}_t|\mathbf{y}_t)d\mathbf{x}_t. \tag{2.77}$$

The weights $w_t^{*(i)}$ are obtained by the importance sampling principle (2.33).

Suppose that $\pi(x) \propto \gamma(x)$ is a probability density from which it is difficult to sample, but $\gamma(x)$ can be evaluated and consequently $\pi(x)$ can also be evaluated. Them we proceed as follows: let $x^{(i)} \sim q(x)$, $i = 1, ..., N$ be a sample generated from a proposed distribution $q(\cdot)$, called the importance density. Them a good approximation of the density $\pi(\cdot)$ is given by

$$\pi(x) \approx \sum_{i=1}^{N} w^{*(i)}\delta(x - x^{(i)}),$$

where

$$w^{*(i)} \propto \frac{\gamma(x^{(i)})}{q(x^{(i)})} \tag{2.78}$$

is the normalized weight of the i-th particle. If the samples $\{\mathbf{x}_t^{(i)}, i = 1, ..., N\}$ are chosen to use a density of importance $q(\mathbf{x}_t|\mathbf{y}_t)$ them the weights used to approximate the equation on (2.76) are obtained using equation (19), that is

$$w_t^{*(i)} \propto \frac{\pi(\mathbf{x}_t^{(i)}|\mathbf{y}_t)}{q(\mathbf{x}_t^{(i)}|\mathbf{y}_t)}. \tag{2.79}$$

If the importance density can be factored such that

$$q(\mathbf{x}_t|\mathbf{y}_t) = q(x_t|\mathbf{x}_{t-1}, \mathbf{y}_t)q(\mathbf{x}_{t-1}|\mathbf{y}_{t-1}) \tag{2.80}$$

them we can get the samples $\mathbf{x}_t^{(i)}$ starting from $q(\mathbf{x}_t|\mathbf{y}_t)$ increasing each of the samples that already exist $\mathbf{x}_{t-1}^{(i)}$ obtained from $q(\mathbf{x}_{t-1}^{(i)}|\mathbf{y}_{t-1})$ generating the new state $x_t^{(i)}$ from $q(x_t|\mathbf{x}_{t-1}, \mathbf{y}_t)$. To get the updated weights, the filtered distribution $\pi(\mathbf{x}_t|\mathbf{y}_t)$ is expressed in terms of $\pi(\mathbf{x}_{t-1}|\mathbf{y}_{t-1})$, $\pi(y_t|x_t)$ and $\pi(x_t|x_{t-1})$, is that

$$\pi(\mathbf{x}_t|\mathbf{y}_t) \propto \pi(y_t|x_t)\pi(x_t|x_{t-1})\pi(\mathbf{x}_{t-1}|\mathbf{y}_{t-1}). \tag{2.81}$$

For the justification of equation (2.81) see [24]. On the other hand, substituting equation (2.80) and (2.81) in equation (2.79), we find the equation for the weights updated

$$\begin{aligned} w_t^{*(i)} &\propto \frac{\pi(y_t|x_t^{(i)})\pi(x^{(i)}|x_{t-1}^{(i)})\pi(\mathbf{x}_{t-1}^{(i)}|\mathbf{y}_{t-1})}{q(x_t^{(i)}|\mathbf{x}_{t-1}^{(i)}, \mathbf{y}_t)q(\mathbf{x}_{t-1}^{(i)}|\mathbf{y}_{t-1})} \\ &= \frac{\pi(y_t|x_t^{(i)})\pi(x_t^{(i)}|x_{t-1}^{(i)})}{q(x_t^{(i)}|\mathbf{x}_{t-1}^{(i)}, \mathbf{y}_t)} w_{t-1}^{*(i)} \end{aligned} \tag{2.82}$$

where

$$w_{t-1}^{*(i)} = \frac{\pi(\mathbf{x}_{t-1}^{(i)}|\mathbf{y}_{t-1})}{q(\mathbf{x}_{t-1}^{(i)}|\mathbf{y}_{t-1})}.$$

In particular, if we consider that $q(x_t|\mathbf{x}_{t-1}, \mathbf{y}_t) = q(x_t|x_{t-1}, y_t)$, them the importance density depends only on x_{t-1} and y_t. This situation is suitable when it is necessary to obtain the filtered estimator $\pi(x_t|\mathbf{y}_t)$ in a real time t. them the modified weights are as follows

$$w_t^{*(i)} = \frac{\pi(y_t|x_t^{(i)})\pi(x_t^{(i)|x_{t-1}^{(i)}})}{q(x_t^{(i)}|x_{t-1})}, \tag{2.83}$$

and the posterior filtered density $\pi_N(x_t|\mathbf{y}_t)$ can be approximated by

$$\pi_N(x_t|\mathbf{y}_t) \approx \sum_{i=1}^{N} w_t^{*(i)} \delta(x_t - x_t^{(i)}). \tag{2.84}$$

Crisan and Doucet (2002) proved that when $N \to \infty$ the equation given in (2.84) approximates the true posterior distribution $\pi(x_t, \mathbf{y}_t)$. To implement the algorithm, suppose we have a set of random samples $\{\mathbf{x}_t^{(i)}, i = 1, ..., N\}$ generated from the known function $\pi(\mathbf{x}_{t-1}|\mathbf{y}_{t-1})$.

Chapter 3

Methodology

3.1 Analysis of the Problem

In the first part of this work, a model for the gross domestic product and another model for the GDP rate are presented. The quarterly time series that involves a total of 84 observations can be obtained from `https://contenido.bce.fin.ec/home1/estadisticas/bolmensual/IEMensual.jsp` and correspond to the period 2000-2020. The 100% of data was used for filtering models, corresponding to the period 2000-2017. In the forecast section 4.1.4, this 80% of the data was used as training and the rest as testing [26] corresponding to the period 2017-2020. In the second part of the work, a model for multivariate analysis is proposed in which we will include three monthly time series: consumer price index (CPI), industrial production index (IPI) and active interest rate (ACI). Due to the limited accessibility of the data, the study covers the period 2016-2019, the observations corresponding to CPI, IPI and ACI can be downloaded from `https://www.ecuadorencifras.gob.ec/estadisticas/`.

3.1.1 Kalman filter models

Model for gross domestic product of Ecuador

Using the time series corresponding to GDP, the model given in (3.1) was fitted. To initialize the Kalman filter, prior values ($\mu_{1|0}$ and $\mathbf{\Sigma}_{1|0}$) were taken as (3.2).

$$\begin{aligned} y_t &= \mathbf{G}_g x_t + \mathbf{V}_g v_t, \qquad v_t \sim N(0,1), \\ x_t &= \mathbf{F}_g x_{t-1} + \mathbf{W}_g w_t, \qquad w_t \sim N(0,1), \end{aligned} \tag{3.1}$$

where

$$\begin{aligned} \mu_{1|0} = (2,2), \qquad \mathbf{G}_g = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \qquad \mathbf{F}_g = \begin{bmatrix} 1 & 0 \end{bmatrix} \\ \mathbf{\Sigma}_{1|0} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \qquad \mathbf{V}_g = \begin{bmatrix} 0.05771 \end{bmatrix}, \quad \mathbf{W}_g = \begin{bmatrix} 0.02610 & 0.000 \\ 0.000 & 0.000249 \end{bmatrix}. \end{aligned} \tag{3.2}$$

For estimation of V and W the maximum likelihood (MLE) estimation method is used, provides by [27].

Model for gross domestic product of Ecuador rate

ARIMA models need the data to be stationary but the Kalman filter is an adequate methodology since allows us to work regardless of the stationarity of the data. After carrying out the respective study of the autocorrelation function (ACF) and (PACF) partial autocorrelation, the GDP rate data show stationarity and an AR (1) model with joined intersection is proposed. To be able to work with the ARIMA models, it is necessary to perform a representation in the state space models (3.4).

$$y_t - \mu = \phi(y_{t-1} - \mu) + \epsilon_t, \qquad \epsilon \sim N(0, \sigma^2). \tag{3.3}$$

In this model the observation and state equation are:

$$\begin{aligned} y_t &= [1, 1]x_t, \\ x_t &= \begin{bmatrix} \mu \\ y_t - \mu \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & \phi \end{bmatrix} \begin{bmatrix} \mu \\ y_{t-1} - \mu \end{bmatrix} + \begin{bmatrix} 0 \\ \epsilon_t \end{bmatrix} \end{aligned} \tag{3.4}$$

The missing parameters were estimated by MLE with the help of [27] and, the results are shown below

$$\begin{aligned} \mu_{1|0} &= (2, 2); \quad \mathbf{G}_p = \begin{bmatrix} 1 & 0 \\ 0 & 0.5254 \end{bmatrix}, \quad \mathbf{F}_p = \begin{bmatrix} 1 & 1 \end{bmatrix}, \\ \mathbf{\Sigma}_{1|0} &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}; \quad \mathbf{V}_p = \begin{bmatrix} 0 \end{bmatrix}, \quad \mathbf{W}_p = \begin{bmatrix} 0 & 0 \\ 0 & 1.35 * 10^{-8} \end{bmatrix}. \end{aligned} \tag{3.5}$$

Multiple variables

State space models allow us the possibility of working with several time series, together with the ARIMA(p,q,r) models there is a great variety of analyzes. In model (3.6) presented below, the matrix V[1,1] and V[2,2] obtained by MLE were altered in order to give the reader a demonstration on the accuracy of the filtering [28].

$$\begin{aligned} y_t &= \begin{bmatrix} y_{t1} \\ y_{t2} \\ y_{t3} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} x_t + \begin{bmatrix} 20 & 0 & 0 \\ 0 & 0.5 & 0 \\ 0 & 0 & 2.93 \end{bmatrix} \begin{bmatrix} v_{t1} \\ v_{t2} \\ v_{t3} \end{bmatrix} \qquad v_{ti} \sim N(0, 1) \\ x_t &= \begin{bmatrix} x_{t1} \\ x_{t2} \\ x_{t3} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} x_{t-1} + \begin{bmatrix} 8.88 & 0 & 0 \\ 0 & 7.81 & 0 \\ 0 & 0 & 3.31 \end{bmatrix} \begin{bmatrix} w_{t1} \\ w_{t2} \\ w_{t3} \end{bmatrix} \qquad w_t \sim N(0, 1). \end{aligned} \tag{3.6}$$

3.1.2 Algorithm Design

For the Kalman filter and Kalman smoothed, the adjustment of the models presented in (3.6),(3.4) and (3.2) by using the *dlm* library [27]. The results obtained are exposed in the results chapter 4 and the codes in annexes.

For the Gibbs filter, the results obtained in the section 2.10 were used, the algorithm is described below

Algorithm 4 Gibbs Sampler for GDP ans GDP rate

At iteration $j+1$

sample $x_t \sim \pi(x_t|x_{j\neq t}, \psi, \sigma^2, \tau^2, \mathbf{y}_t)$

sample $\sigma^2 \sim \pi(\sigma^2|\psi, \mathbf{x}_t, \mathbf{y}_t)$

sample $\tau^2 \sim \pi(\tau^2|\omega_t, \mathbf{x}_t, \mathbf{y}_t)$

sample $\psi_t \sim \pi(\psi|\sigma^2, \mathbf{x}_t, \mathbf{y}_t)$

sample $\omega_t \sim \pi(\omega_t|\tau^2, \mathbf{x}_t, \mathbf{y}_t)$.

For the Gibbs filter, the results obtained in the section 2.10 were used, the algorithm is described below

Algorithm 5 Particle filter for GDP and GDP rate

Step one

sample $\{x_0^{(j)}\} \sim N(0,1), \quad j = 1, ..., N$ where N is the number of particles.

For i=1,...N sample

Sample $\{x_t^{(i)}\} \sim q(x_t|\mathbf{x}_{t-1}, \mathbf{y}_t), \quad j = 1, ..., N$ where N is the number of particles.

For i=1,...N evaluate the weights

$$w_t^{*(i)} \propto w_{t-1}^{*(i)} \frac{\pi(y_t|\tilde{x}_t^{(i)})\pi(\tilde{x}_t^{(i)}|\tilde{x}_{t-1}^{(i)})}{q(\tilde{x}_t|\tilde{\mathbf{x}}_{t-1}^{(i)}, \mathbf{y}_t)}.$$

For i=1,...N normalize the weights

$$\tilde{w}_t^{*(i)} = \frac{w_t^{*(i)}}{\sum_{k=1}^N w_t^{*(k)}}, \qquad \sum_{i=1}^N \tilde{w}_t^{(i)} = 1.$$

Chapter 4

Results and Discussion

4.1 Results and Discussions

Macroeconomic processes are usually described by mathematical models with linear and non-linear structures with Gaussian and non-Gaussian distributions that involve multiple parameters and partially observed dynamic processes measured with errors that must be estimated from data using classical statistics techniques such as the maximum likelihood estimator or methods of Bayesian statistics. State space models provide a general structure to study these stochastic processes. The filtering algorithms in the stage of the State space models involve the sequential calculation of the subsequent distribution of the unknown states x_t given the observations $y_1, \ldots, y_n$. For this, powerful computational algorithms such as Kalman filter and its variants are required when models are linear with Gaussian distributions, and when models are non-linear with non-Gaussian distribution, it is recommended to use particle filters and other approach techniques. In this work we focus on Gaussian linear models and are analyzed series of macroeconomics of Ecuador and the Kalman, Kalman smoothed, particle and Gibbs filters softened are implemented to estimate and predict unknown states.

The development of technologies and calculation capacity in recent years has made it possible to have massive sets of economic data and techniques to analyze these economic indices. Finance ministries and central banks need easy-to-interpret macroeconomic information to enable them to design policies to strengthen economic growth and preserve society's quality of life. Key economic indicators on which decision making is based are usually published late, information is incomplete and economists can only gauge economic conditions at the moment, information at a future time is scarce, which makes forecasting and predicting the economy difficult to understand. There are also interconnected factors in global economies, in which small disturbances that originate in one country spill over into other economies, resulting in low productivity levels, loss of employment and imbalance in the different economies.

This paper analyzes some economic indices in Ecuador to observe the behavior of these variables in the last decades. The variables analyzed are: GDP, GDP rate, CPI, IPI, and ACI, and to achieve this objective a dynamic Bayesian model and two learning algorithms were considered, this combination includes missing data information and allows evaluating the economic reaction to possible shocks and provides real time information and allows forecasting to market policy.

4.1.1 Gross domestic product of Ecuador

During the years 2000−2020, Fig. 4.1 shows the evolution of the average mean of the Gross Domestic Product series estimated by the Kalman filter, Kalman smoothed, Gibbs filters and particle filter together with true data. In the graph we can observe continuous growth over time, and a very similar adjustment between estimated states and true observations. It can also be appreciated that the algorithms mimic well the behavior of real data, these algorithms have the property of reducing noise and softening the series. The Kalman Smoothed filter captures the fluctuations of the economy and detects the peaks caused by the sudden jumps in the GDP and are characterized by being less pronounced.

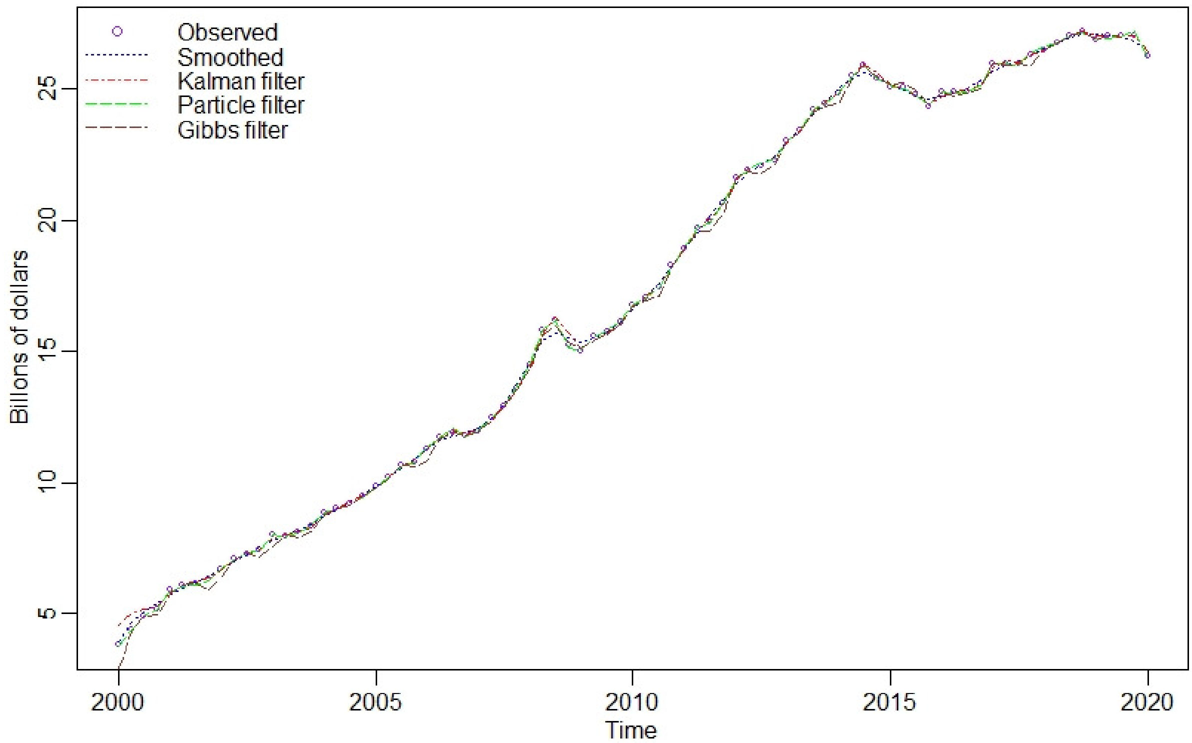

Figure 4.1: Evolution of the posteriori mean of the proposed model for the gross domestic product of Ecuador.

4.1.2 Gross domestic product rate of Ecuador

Using the model for gross domestic product of Ecuador rate (3.4) together with the values presented in (3.5), we can be show in Fig.4.2 the evolution of the posterior mean of the series of the GDP rate of Ecuador during the period between the years $2000 - 2020$. Time series of this style with constant mean and bounded variance as they are known in the literature [28] are usually very complex to filter. The results obtained, (see Tab. 4.2) show that the ARIMA models together with the Kalman filter are a good option in time series analysis.

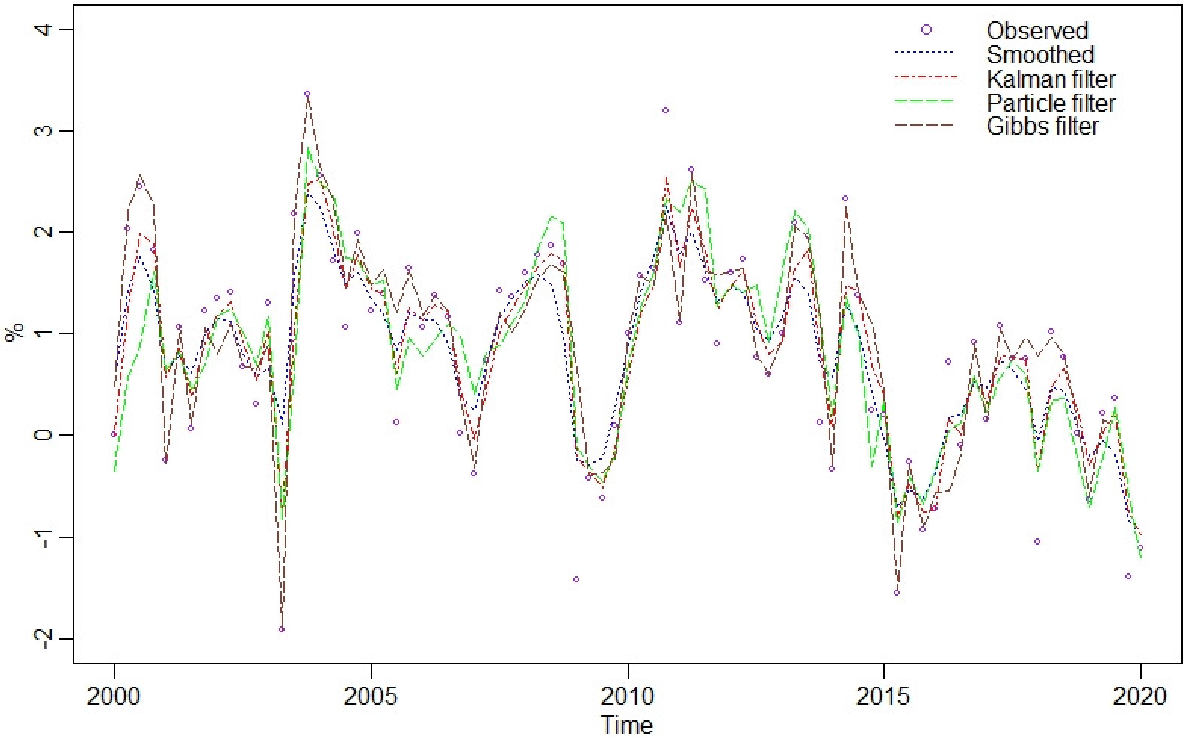

Figure 4.2: Evolution of the posteriori mean of the proposed model for the gross domestic product of Ecuador rate.

Kalman, Kalman smoothed, particle and Gibbs filters show good behavior in the simulation of observations. In the Fig. 4.3, we can see the residual of the proposed system to GDP and in Fig. 4.4 to GDP rate. The dynamics of the state space models allow us an independent adjustment of the W and V matrices. In the present work, we use algorithms that allowed us to obtain the values for mentioned matrices, but the researcher could change these parameters and thus obtain a smaller or larger error in filtering the data.

Using the mean square error metric, in Table 4.1 shows a measure of goodness of fit calculated to all filters. The results obtained show a good estimate for all filters, highlighting the particle filter for GDP and the Kalman filter for GDP rate.

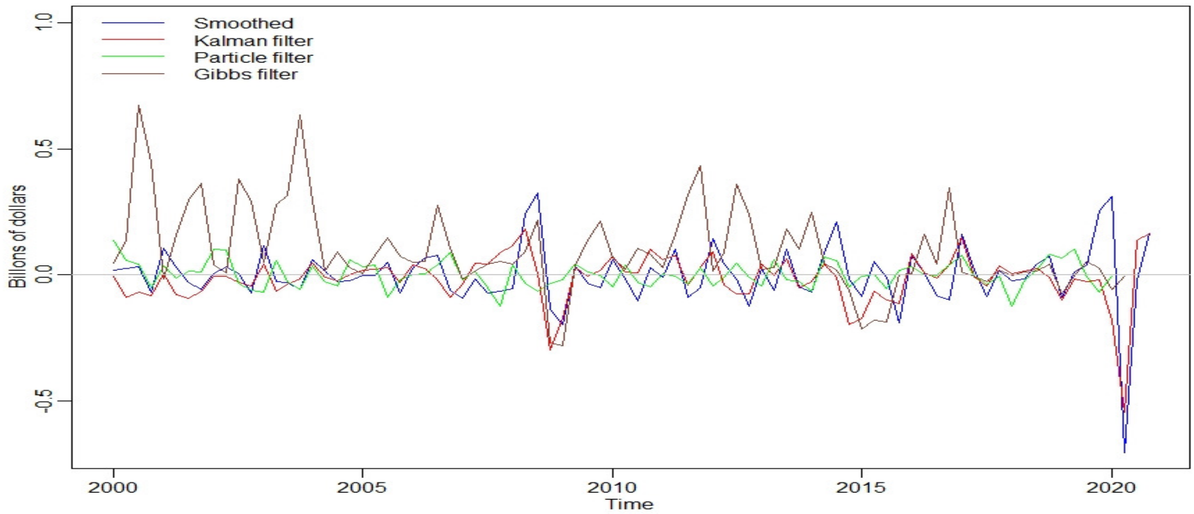

Figure 4.3: Estimation errors for the Kalman, Kalman smoothed, particle and Gibbs filters, models (3.2) and (3.1).

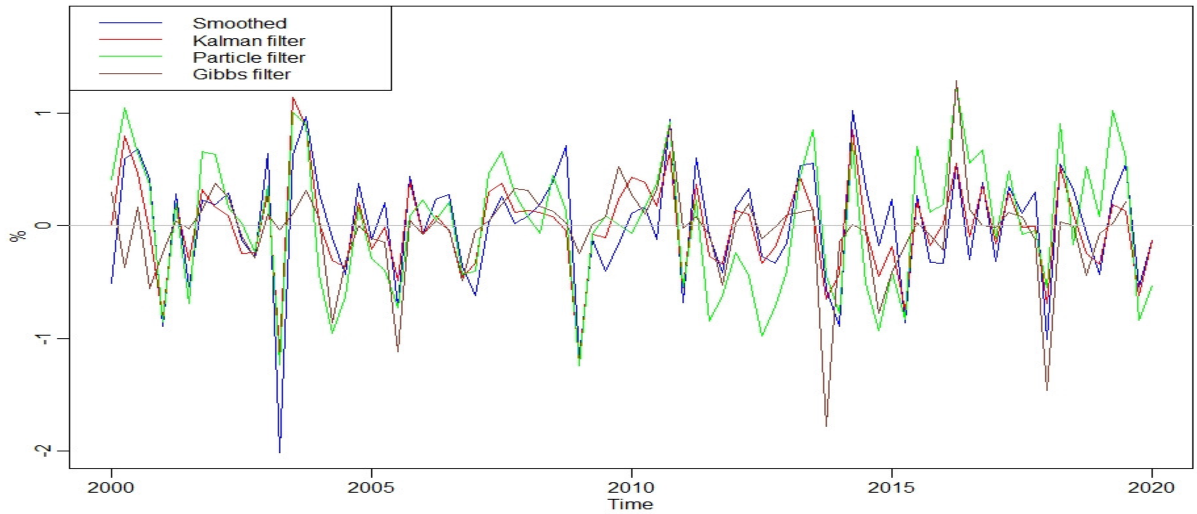

Figure 4.4: Estimation errors for the Kalman, Kalman smoothed, particle and Gibbs filters, models (3.4) and (3.5).

Table 4.1: Mean square error for GDP and GDP rate

	GDP	GDP rate
Kalman filter	0.0205	0.1744
Kalman smoothed	0.0230	0.2730
Particle filter	0.0027	0.3271
Gibbs filter	0.037	0.2095

4.1.3 Multiple variables

The use of matrices in state space models allows us to work with several time series, reducing the cost of analysis. In Fig. 4.5 the results obtained by the multiple variables model (3.6) are presented. As mentioned previously, the values of matrix V were modified, these would allow us to control the accuracy of the filtering. It can be seen in Fig. 4.5 that the filtering and smoothing values fit almost perfectly to the true observations while the CPI observations show a large error, finally the ACI value was the one obtained by MLE.

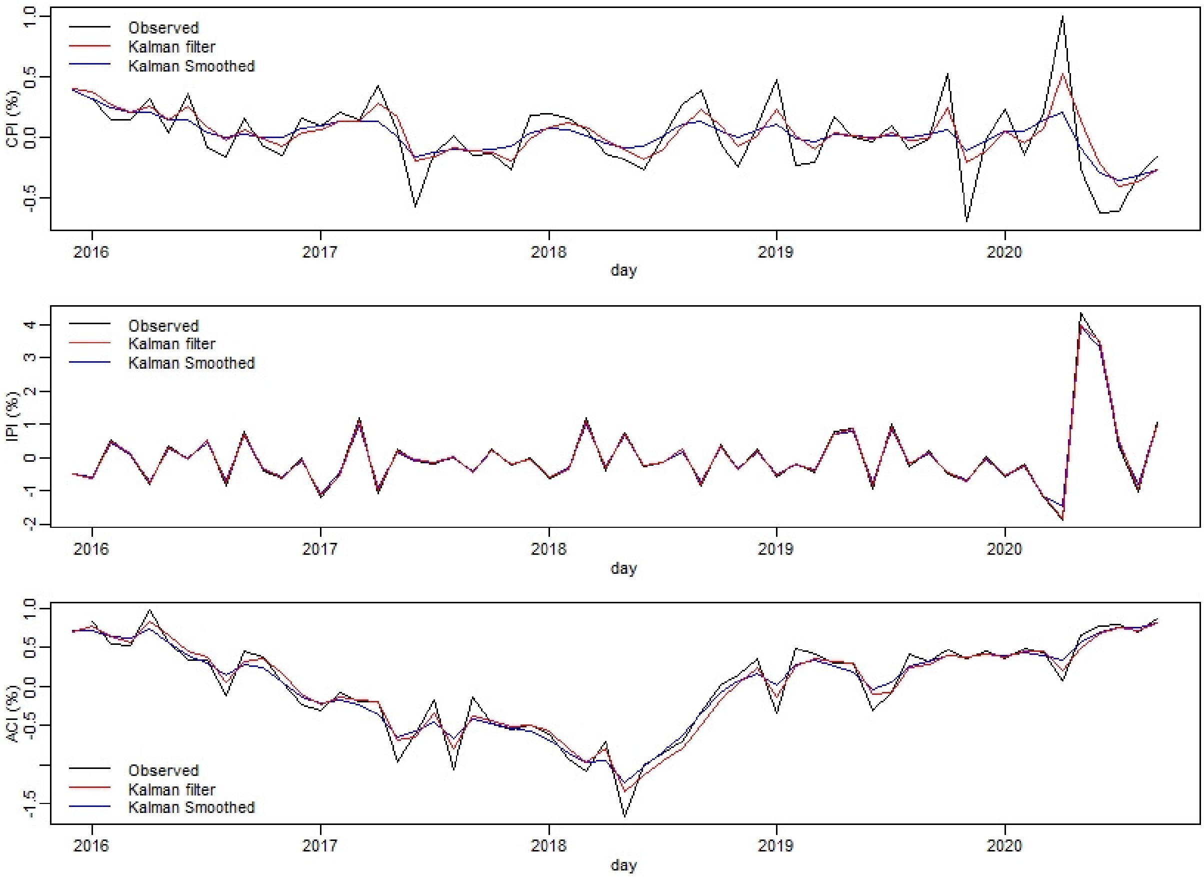

Figure 4.5: Filtered and smoothed time series, first the consumer price index; second, the industrial production index; third, activate interest rate.

Table 4.2 shows a measure of goodness of fit calculated in order to measure the quality of estimation of the algorithms for all the series considered in the study, the mean square error metric was evaluated for each filter used, obtaining small estimation errors.

Table 4.2: Mean square error for multivariate model

	CPI	IPI	ACI
Kalman filter	0.0299	0.0057	0.0154
Kalman smoothed	0.0437	0.014	0.0245

4.1.4 Forecast and New Observations

The dynamics of the state space models also allow us to make predictions, build confidence intervals and extract the new observations from (2.35). Using the equation proposed in (2.61) together with the model (3.1) the prediction of the posterior mean of GDP together with the new observations are presented in Fig. 4.6. Similarly, using the proposed adjustments for the GDP rate, the posterior mean prediction is shown in Fig. 4.7. The values with the MSE metric (see.Table 4.2) suggest an acceptable prediction of the data.

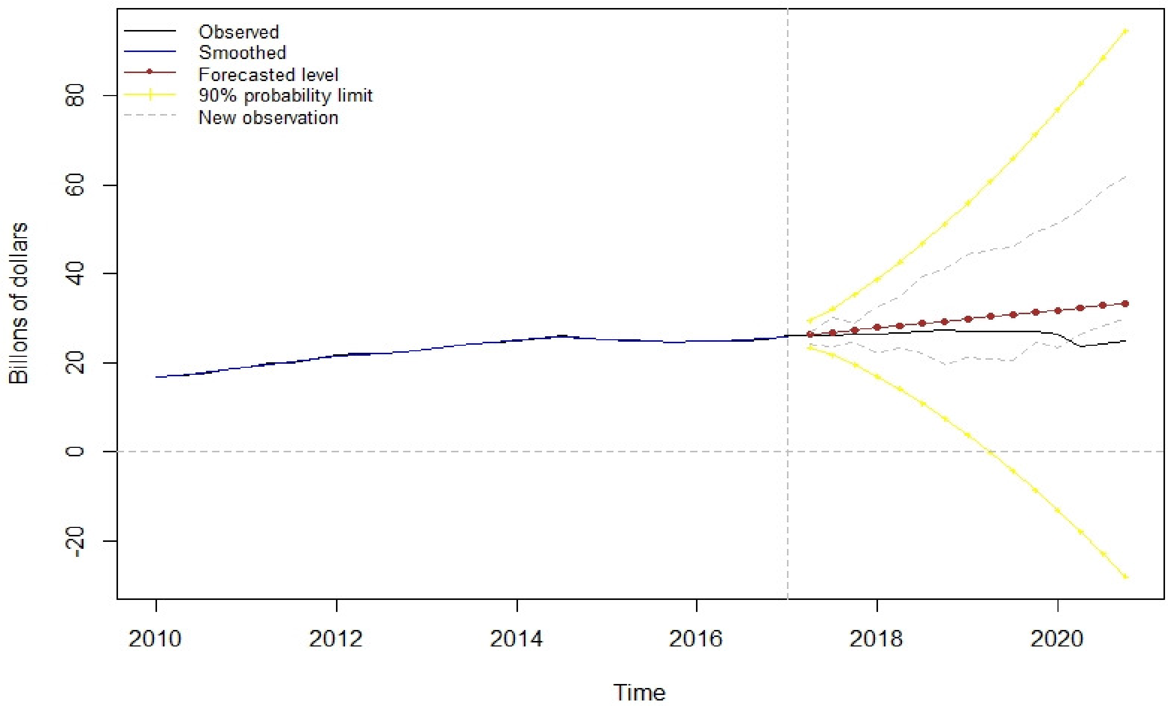

Figure 4.6: Forescast, model GDP.

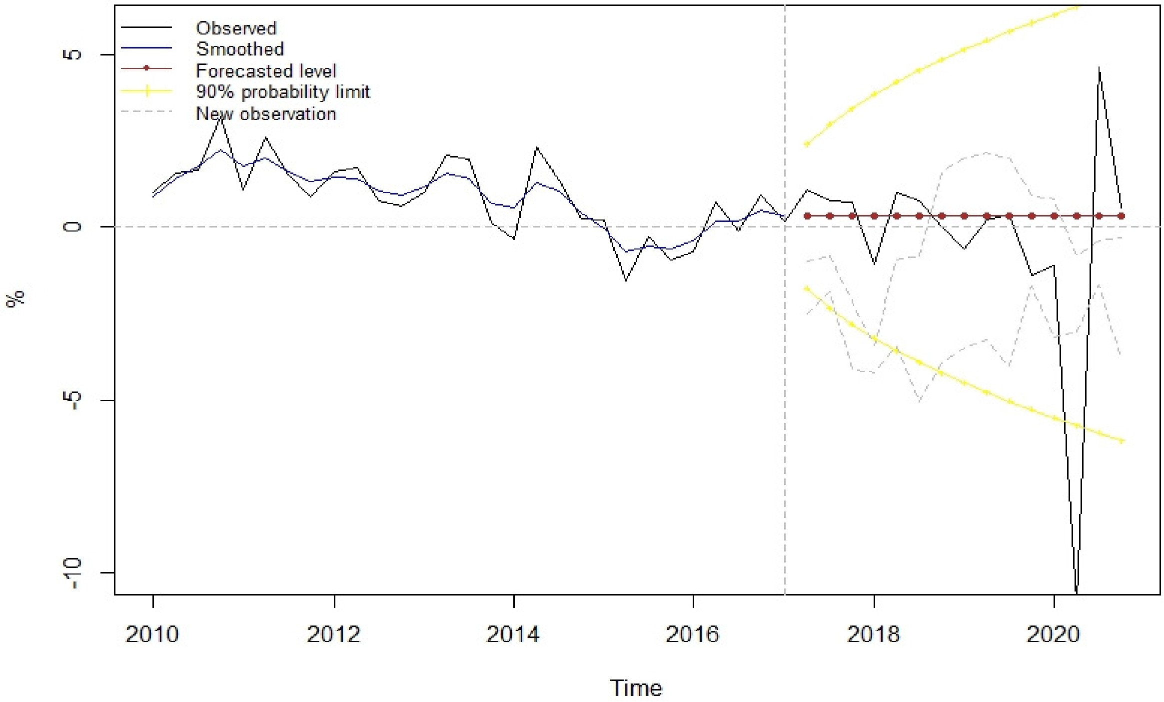

Figure 4.7: Forescast, model GDP rate.

Chapter 5

Conclusions

This thesis shows some applications of the space-state model in macroeconomics time series of Ecuador, considering filtering algorithm techniques under a Bayesian statistical approach. The objectives of the research are addressed as follows: the implementation of statistical tools in macroeconomic problems that have a lot of variability in time, non-linearity, non-stationary, structural changes, asymmetries and outliers that are characteristics present in many financial series. The estimation capacity of the algorithms to characterize and predict the nature of the stochastic phenomenon studied is compared, and the influence of external factors that may be causing fluctuations in the economic system in Ecuador is analyzed in real time. To illustrate the methodology, the macroeconomic series are analyzed: Gross domestic product, GDP rate, consumer price index (CPI), industrial production index (IPI) and active interest rate (ACI). An estimation of a linear Gaussian state space model and an ARIMA model with state space structure is performed using the Kalman, Kalman smoothed, particle and Gibbs filters and forecasts are obtained outside the range of the analyzed data with the purpose of validating the model. In the results, a linear growth in the GDP variable can be observed with a fall in the last period of the series studied, which agrees with the reality of the economies in the world, the existence of an economic pattern or atypical values is not detected, nor changes of structures. The variable GDP rate shows fluctuations with a downward trend at the end of the analyzed period, the same behavior shows the series CPI, IPI, and ACI. When the simultaneous analysis of the CPI, IPI, and ACI series is carried out, fluctuations can be observed in time, with a slight rebound around the year 2020. In reference to the predicted values, it can be observed that the filters maintain a linear estimation trend. Both filters offer relatively good predictive performance. It was used as a measure of goodness of fit to calibrate the estimation quality of the algorithms, the mean square error metric, obtaining small estimation errors. The study of time series has been a key factor in the economy, during this work a series of models was presented together with a series of Bayesian algorithms that allow series to be filtered, reducing their error. A joint study of the Marcovian state space models and artificial intelligence will allow machine learning models to improve their results.

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Appendices

.1 Appendix 1. R code

.1.1 Random Walk

```
# Henry Bautista
# 10/9/2021
# Yachay Tech University
# Build a simple  random walk and random walk with drift.

ranv = rnorm(150);     # Random values
rw = cumsum(ranv);     # Returns a vector whose elements are the
    cumulative sums
srwd = ranv +.3;
rwd = cumsum(srwd);    # Random walk drift=0.3
plot.ts(rwd, ylim=c(-5,50), main="", ylab='') # Plot data
lines(rw, col=4); abline(h=0, col=4, lty=2); abline(a=0, b=.3, lty=2)
legend("topleft", legend = c( "Random walk","Random walk whit drift" )
    ,cex=c(0.75), ncol=1, lwd = c(1,1), col = c("blue","black"), bty =
    "y")
```

.1.2 Monte Carlo $\gamma(4)$.

```
# Henry Bautista
# 10/9/2021
# Yachay Tech University
# Algorithm allows us compute the Gamma(4)  integral with important
    sampling method.
samples=c()
for(i in (1:200)){
u<-runif(50000,0,40000)
ga<-function(x){x^{4-1}*exp(-x)}
re=40000*sum(ga(u))/(50000)
samples=c(samples,re)}
summary(samples)
```

.1.3 State space random walk

```
# Henry Bautista
# 10/9/2021
# Yachay Tech University
# The program build a state space random walk and the values obtaining
     are filtered.
```

```

x_0 = 2 # Inital value
t = seq (0, 200, 1)
x = c(x_0)
x_aux = matrix (0, length (t), 1)
y = c()
for (k in 2: length(t)) {
x[k] = x[k -1] + rnorm (1, 0, .25)  # State equation and error
y[k-1] = x[k] + rnorm (1, 0, 1.25)  # Observation equation
  }
plot (t, x , type = "l", xlab = "time", col=" blue ",
ylab = "x(t), y(t)", ylim = c(-2,7),
    main = " Gaussian random walk ")
points (y, col = " red ", cex = .5)
legend ("topleft", lty=c(1, 0),
col = c(" blue ", "red"),
legend = c(" State variable ", " Observation "), pch = c(NA , 1),
bty = "o",  cex = .45 , ncol =1)
### Kalman Filter
data = y
ex = dlm(m0 = 1, C0 = 1.2 , FF = 1, V = .005 , GG = 1, W = .02) # Simple state space model
KFstate = dlmFilter (data , ex) # Filter
plot ( x, type = "l", xlab = "time , k",
 ylim = c(-2,7), col=" blue ",
 ylab = "", main = " Gaussian random walk ")
points (y, col = " red ", cex = .5)
lines ( KFstate $m, type = "l", col = " green ")
legend ("topleft", lty = c(1, 0, 1), pch = c(NA , 1, NA),
            col = c(" blue ", "red", " green "),
           legend = c(" State variable ", " Observation ", "Kalman filter "),
            bty = "o", cex = 0.5 , ncol =1)
```

.1.4 Data (GDP, GDP rate) and packages in R

```
# Henry Bautista
# 10/9/2021
# Yachay Tech University
# Treatment of the data used: # Data: https://drive.google.com/drive/folders/1Fuk1Cxvv6jn2cKu2toTkyZrZtzs9Cfzi?usp=sharing
library(dlm)       # Kalman filter package
library(MLmetrics)
```

```
library(tseries)
par(mfrow=c(2,1), mar=c(3,3,1,1), mgp=c(1.6,.6,0))
my_data<-read.csv("C:\\Users...\\GDPf.csv", header=T)
GDPi<-my_data$per_gdp        # GDP index
GDP_tri<-my_data$gdp_c       # GDP
GDP_tri<-GDP_tri/1000000     # GDP in billions
GDPall_tri=as.ts(GDP_tri)    # Time series GDP
stt=2000                     # Start year
edd=2020                     # Start year
GDPall_tri=ts(GDPall_tri,start = stt,end=edd  ,frequency = 4)
GDPi<-as.ts(GDPi)            # Time series GDP index
GDPi=ts(GDPi,start = stt,end= edd,  frequency = 4)
```

.1.5 Kalman filter for GDP model

```
# Henry Bautista
# 10/9/2021
# Yachay Tech University
# Program to filter time series using the Kalman filter, a model used
   for GDP.

dlmSri <- dlmModPoly()    # Model state space
m0(dlmSri)<-c(2,2)
diag(C0(dlmSri))<-c(1,1)
buildFun<-function(x){    # Function to compute the maximum likelihood
     (MLE)
diag(W(dlmSri))[1:2] <- 1/exp(x[1:2])
V(dlmSri) <- cos(x[3]*x[3])*0.0567
return(dlmSri)
}
fit <- dlmMLE(c(GDPall_tri), parm = rep(0, 3), build = buildFun) #MLE
modelgdp<-buildFun(fit$par)                      # Model GDP
filtered<-dlmFilter(GDPall_tri, mod = modelgdp) # Filtering
fore<-dlmForecast(filtered,nAhead = 15, sampleNew = 2 )
smoothed <- dlmSmooth(filtered)                  # Smoothed
resid.gdp_f3 <- residuals(filtered, sd = FALSE)
GDPf3_s <- dropFirst(smoothed$s)[,1]
GDPf3_f <- dropFirst(filtered$m)[,1]
```

.1.6 Kalman filter GDP rate model

```
# Henry Bautista
```

```
# 10/9/2021
# Yachay Tech University
# Program to filter time series using the Kalman filter, a model used
   for GDP.

parm_rest = function(parm){
  return( c(parm[1],exp(parm[2])) )
}
V=abs(rnorm(1,mean = 1,sd=1))
dlm1 = function(parm){       # Function to compute the maximum
   likelihood  (MLE)
  parm = parm_rest(parm)
  dlm = dlmModPoly(1) +
    dlmModARMA(ar=parm[1], ma=NULL, sigma2=parm[2])
  # set initial state distribution
  dlm$C0[2,2] <- solve(1-parm[1]^2)*parm[2]
  return(dlm)
}
fit1 = dlmMLE(y=GDPi,parm=c(0,0),build=dlm1,hessian=T)
mod1 = dlm1(fit1$par)    # Model GDP rate
C0(mod1)[1,1]<-0.25      # Initial value
mod=mod1
V(mod1)<-0
filtered <- dlmFilter(GDPi, mod = mod)  # Filtering
smoothed <- dlmSmooth(filtered)         # Smoothed
fore<-dlmForecast(filtered,nAhead = 15, sampleNew = 2 )
resid.gdp_f3 <- residuals(filtered, sd = FALSE)
GDPf3_s <- dropFirst(smoothed$s)[,1]
GDPf3_f <- dropFirst(filtered$m)[,1]
```

.1.7 Program to filter time series using Gibbs filter

```
# Henry Bautista
# 10/9/2021
# Yachay Tech University
# Gibbs function for estimation space state, This function was used to
    filter  GDP and GDP rate.

gibbs.filter <- function(data,kons,theta,FH){
  require(HyperbolicDist)
  a0 <- kons[1]       # Initial data
  b0 <- kons[2]
  c0 <- kons[3]
```

```
d0 <- kons[4]
mu0       <- theta[1]
sigma0.2 <- theta[2]
F <- FH[1]
H <- FH[2]
n <- length(data) # Length of the data. Simulation size.
mean.data <- mean(data)
sd.data   <- sd(data)
#  data <- (data - mean(data))/sd(data) # In case of standardizing the data
x        <- rep(0,n + 1)  # States vector.
sigma.2 <- rep(0,n + 1)  # Variance vector
tau.2    <- rep(0,n + 1)  # Accuracy vector.
lambda   <- rep(0,n + 1)  # Vector of weights: lambdas.
omega    <- rep(0,n + 1)  # Vector of weights: omegas.
B        <- rep(0,n + 1)
b        <- rep(0,n + 1)
x[1] <- rnorm(1,mean = mu0, sd = sqrt(sigma0.2))   # Initial random values
sigma.2[1] <- 1/rgamma(1,shape = a0,scale = b0)
tau.2[1]   <- 1/rgamma(1,shape = c0,scale = d0)
lambda[1] <- rgig(1,c(.5,1,1))
omega[1]  <- rgig(1,c(.5,1,1))
B[1] <- 1/(1/sigma0.2 + F^2/(sigma.2[1]*lambda[1]))
b[1] <- mu0/sigma0.2 + F^2*x[1]/(sigma.2[1]*lambda[1])
# Important sampling
for(t in 2:n){
  x[t] <- rnorm(1,mean = b[t - 1]*B[t - 1],sd = sqrt(B[t - 1]))
  sha <- a0 + t/2
  scl <- b0 + 0.5/lambda[t - 1]*sum((x[2:t] - F*x[1:(t - 1)])^2)
  sigma.2[t] <- 1/rgamma(1,shape = sha,scale = scl)
  sha <- c0 + t/2
  scl <- d0 + 0.5/omega[t - 1]*sum((data[1:(t - 1)] - H*x[2:t])^2)
  tau.2[t] <- 1/rgamma(1,shape = sha,scale = scl)
  psi <- (x[t] - F*x[t - 1])^2/sigma.2[t]
  lambda[t] <- rgig(1,c(.5,1,psi))
  psi <- (data[t - 1] - H*x[t])^2/tau.2[t]
  omega[t]  <- rgig(1,c(.5,1,psi))
  B[t] <- 1/((1/lambda[t - 1] + F^2/lambda[t])/sigma.2[t] + H^2/(tau.2[t]*omega[t]))
  b[t] <- F*(x[t - 1]/lambda[t - 1] + x[t]/lambda[t])/sigma.2[t] + H*data[t - 1]/(tau.2[t]*omega[t])
} # end for t
x[n + 1] <- rnorm(1,mean = b[n]*B[n],sd = sqrt(B[n]))
```

```
  sha <- a0 + (n + 1)/2
  scl <- b0 + 0.5/lambda[n]*sum((x[2:(n + 1)] - F*x[1:n])^2)
  sigma.2[n + 1] <- 1/rgamma(1,shape = sha,scale = scl)
  sha <- c0 + (n + 1)/2
  scl <- d0 + 0.5/omega[t - 1]*sum((data[1:n] - H*x[2:(n + 1)])^2)
  tau.2[n + 1] <- 1/rgamma(1,shape = sha,scale = scl)
  lambda[n + 1] <- rgig(1,c(.5,1,(x[n + 1] - F*x[n])^2/sigma.2[n + 1])
     )
  omega[n + 1]  <- rgig(1,c(.5,1,(data[n] - H*x[n + 1])^2/tau.2[n +
     1]))
  B[n + 1] <- 1/(1/(sigma.2[n + 1]*lambda[n + 1]) + H^2/(tau.2[n + 1]*
     omega[n + 1]))
  b[n + 1] <- F*x[n]/(sigma.2[n + 1]*lambda[n]) + H*data[n]/(tau.2[n +
      1]*omega[n])
  x[n + 1] <- rnorm(1,mean = b[n + 1]*B[n + 1],sd = sqrt(B[n + 1]))
  return(list(x = x[-1],
              B = B[-1],b = b[-1],
              sigma.2 = sigma.2[-1],tau.2 = tau.2[-1],
              lambda = lambda[-1],omega = omega[-1]))
GDPigF=GDPigFA$x[2:(length(GDPigFA$x)-1)]
GDPigF<-as.ts(GDPigF)  # Time series data
GDPigF=ts(GDPigF,start = stt,  frequency = 4)
```

.1.8 Program to filter time series using particle filer

```
# Henry Bautista
# 10/9/2021
# Yachay Tech University
# Particle filter used in GDP and GDP rate filtering.

T <-length(c(GDPi))
x_true <- rep(NA, T)
obs <- c(GDPi)
sy <- 8
sx<- 7
T <- length(obs)
N <- 1000
# Create x and weight matrices
x <- matrix(nrow =  N, ncol = T)
weights <- matrix(nrow =  N, ncol = T)
# Intial (at t=1):
# Draw X from prior distribution
x[, 1] <- rnorm(N, 0, sx)
```

```
# Calculate weights, i.e. probability of evidence given sample from X
weights[, 1] <- dnorm(obs[1], x[, 1], sy)
weights[, 1] <- weights[, 1]/sum(weights[, 1])   # Normalise weights
# Weighted re-sampling with replacement. This ensures that X will converge to the true distribution
x[, 1] <- sample(x[, 1], replace = TRUE, size = N, prob = weights[, 1])
for (t in seq(2, T)) {
  # Predict x_{t} from previous time step x_{t-1}
  # Based on process (transition) model
  x[, t] <- rnorm(N, x[, t-1], sx)
  # Calculate  and normalise weights
  weights[, t] <- dnorm(obs[t], x[, t], sy)
  weights[, t] <- weights[, t]/sum(weights[, t])
  # Weighted resampling with replacement
  x[, t] <- sample(x[, t], replace = TRUE, size = N, prob = weights[, t])
}
GDPipF <- apply(x, 2, mean)
x_quantiles <- apply(x, 2, function(x) quantile(x, probs = c(0.025, 0.975)))
GDPipF<-as.ts(GDPipF)
GDPipF=ts(GDPipF,start = stt,end= edd,  frequency = 4)
```

.1.9 Plot filters

```
# Henry Bautista
# 10/9/2021
# Yachay Tech University
# Program plot the results of time series filtered with the 4 models proposed.
dev.new()
# GDP
GDPpF<-as.ts(GDPpF)
GDPpF=ts(GDPpF,start = stt,end= edd,  frequency = 4)
GDPgF<-as.ts(GDPgF)
GDPgF=ts(GDPgF,start = stt,  frequency = 4)
dev.new()
par(mfrow=c(1,1), mar=c(3,3,1,1), mgp=c(1.6,.6,0))
plot.ts(GDPall_tri, type="p",col = "purple", xlab = "Time", ylab = "Billons of dollars", lwd = 1.5, cex=0.5)
lines(GDPf3_s, col = "blue",lty=3)
lines(GDPf3_f, col = "red",lty=4)
```

```
lines(GDPpF, col = "green",lty=5)
lines(GDPgF, col = "coral4",lty=5)
legend("topleft", legend = c("Observed", "Smoothed","Kalman filter","
    Particle filter","Gibbs filter"),
        lwd = c(1,1, 1,1,1),pch = c(1, NA, NA,NA,NA),lty = c(NA
            ,3,4,5,5), col = c("purple", "blue","red","green","coral4"),
             bty = "n")
# GDP rate
par(mfrow=c(1,1), mar=c(3,3,1,1), mgp=c(1.6,.6,0))
plot.ts(GDPi, ylim=c(-2,4), type="p",col = "purple", xlab = "Time",
    ylab = "%", lwd = 1.5, cex=0.5)
lines(GDPf3_s, col = "blue",lty=3)
lines(GDPf3_f, col = "red",lty=4)
lines(GDPipF, col = "green",lty=5)
lines(GDPigF, col = "coral4",lty=5)
legend("topright", legend = c("Observed", "Smoothed","Kalman filter",
    "Particle filter","Gibbs filter"), lwd = c(1,1, 1,1,1), pch = c(1,
    NA, NA, NA,NA),lty = c(NA,3,4,5,5), col = c("purple", "blue","red",
    "green","coral4"), bty = "n")
```

.1.10 Multivariate model

```
# Henry Bautista
# 10/9/2021
# Yachay Tech University
# Kalman filter for Multivariate model.

library(dlm)
library(MLmetrics)
my_data<-read.csv("C:\\Users...\\va_me_geall.csv", header=T)
CPI_me<-my_data$val_CPI
IPI_me<-my_data$value_IPI
AcIR_me<-my_data$A_IR
AcIR_me=(AcIR_me-mean(AcIR_me))
IPI_me=(IPI_me-mean(IPI_me))/sd(IPI_me)
# Transform to time series
CPI_me=as.ts(CPI_me)
IPI_me=as.ts(IPI_me)
AcIR_me=as.ts(AcIR_me)
GDP2016_tri=as.ts(GDP2016_tri)
CPI_me=ts(CPI_me,start = 2016,frequency = 12)
IPI_me=ts(IPI_me,start = 2016,frequency = 12)
AcIR_me=ts(AcIR_me,start = 2016,frequency = 12)
```

```r
GDP2016_tri=ts(GDP2016_tri[1:20],start = 2016,frequency = 4)
# Matrix definition
The=matrix(c(1,1,1), ncol = 1,nrow = 3)
FF=matrix(c(1,0,0,0,1,0,0,0,1), ncol = 3,nrow = 3)
GG=matrix(c(1,0,0,0,1,0,0,0,1), ncol = 3,nrow = 3)
VV=matrix(c(abs(rnorm(1)),0,0,0,abs(rnorm(1)),0,0,0,abs(rnorm(1))), ncol = 3,nrow = 3)
WW=matrix(c(abs(rnorm(1)),0,0,0,abs(rnorm(1)),0,0,0,abs(rnorm(1))), ncol = 3,nrow = 3)
CO=matrix(c(1,0,0,0,abs(rnorm(1)),0,0,0,abs(rnorm(1))), ncol = 3,nrow = 3)
MO=matrix(c(0.4,-0.5,abs(rnorm(1))), ncol = 1,nrow = 3)
me<-dlm(FF = FF, V = VV, GG =GG , W =WW , m0 = MO, CO = CO)
y = cbind(CPI_me, IPI_me, AcIR_me);
buildFun <- function(x) {     # Function to compute MLE.
        me<-dlm(FF = FF, V = VV, GG =GG , W =WW , m0 = MO, CO = CO)
        diag(W(me))[1:3] <- exp(x[1:3])
        diag(V(me))[1:3] <- exp(x[4:6])
        return(me) }
ma<-dlmMLE(y,parm = rep(0,6), build = buildFun)   # Model definition for Multivariate data
me=buildFun(log(abs(ma$par)))
W(me)[2,2]= 2    # Initial value
V(me)[2,2]=2
fil<-dlmFilter(y, mod = me)
mos=dlmSmooth(fil)
dev.new()     # Plot data
plot(CPI_me)
lines(fil$m[,1],col="red")
lines(mos$s[,1],col="blue")
plot(IPI_me)
lines(fil$m[,2],col="red")
lines(mos$s[,2], col="blue")
plot(AcIR_me)
lines(fil$m[,3],col="red")
lines(mos$s[,3],col="blue")
# Plot data
dev.new()
par(mfrow=c(3,1), mar=c(3,3,1,1), mgp=c(1.6,.6,0))
plot(CPI_me, type='l', pch=19,  xlab='day',ylab = "CPI (%)",col="black", cex=0.5)
lines(mos$s[,1],col="blue");
lines(fil$m[,1],col="red");
legend("topleft", legend = c("Observed","Kalman filter", "Kalman
```

```
Smoothed"), lwd = c(1,1,1), col = c("black","red", "blue"), bty = "
    n")
plot(IPI_me, type='l',  pch=19, xlab='day',ylab = "IPI (%)", col="
    black",cex=0.5)
lines(mos$s[,2],col="blue");
lines(fil$m[,2],col="red");
legend("topleft", legend = c("Observed","Kalman filter", "Kalman
    Smoothed"),
        lwd = c(1,1,1), col = c("black","red", "blue"), bty = "n")
plot(AcIR_me, type='l', pch=19,  xlab='day',ylab = "ACI (%)",col="
    black", cex=0.5)
lines(mos$s[,3],col="blue");
lines(fil$m[,3],col="red");
legend("bottomleft", legend = c("Observed","Kalman filter", "Kalman
    Smoothed"), lwd = c(1,1,1), col = c("black","red", "blue"), bty = "
    n") ;
```