UNIVERSIDAD DE INVESTIGACIÓN DE TECNOLOGÍA EXPERIMENTAL YACHAY

Escuela de Ciencias Matemáticas y Computacionales

TÍTULO: Semilinear Neutral Differential Equations with Impulses and Nonlocal Conditions. Existence of solutions and Controllability.

Trabajo de integración curricular presentado como requisito para la obtención del título de Matemático

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Dedication

"To my family, especially to my mother and siblings for giving me the gift of time."

Acknowledgements

"Thanks to everyone who can acknowledge their contribution and impact on me and the production of this thesis."

Abstract

This thesis consists of two parts. In the first part, we study the following semilinear neutral differential equation in $\mathbb{R}^n$ with impulses and nonlocal conditions:

$$\begin{cases} \dfrac{d}{dt}\left[z(t) - g(t, z_t)\right] = A(t)z(t) + \mathfrak{F}(t, z_t), & t \in (0, \tau] \setminus \{t_k\}_{k \in I_p}, \\ z(\theta) = -[h(z_{\tau_1}, \dots, z_{\tau_q})](\theta) + \eta(\theta), & \theta \in [-r, 0], \\ z(t_k^+) = z(t_k^-) + J_k(t_k, z(t_k)), & k \in I_p, \end{cases}$$

where $z(t) \in \mathbb{R}^n$ is the state, $A(t)$ is a $n \times n$ continuous matrix, $0 < t_1 < \cdots < t_p < \tau$, $0 < \tau_1 < \cdots < \tau_q < r < \tau$, $I_p := \{1, \dots, p\}$, z_t is the function $[-r, 0] \ni \theta \mapsto z_t(\theta) = z(t + \theta) \in \mathbb{R}^n$, $h : PW_{qp} \to PW_r$, $J_k : [0, \tau] \times \mathbb{R}^n \to \mathbb{R}^n$, $g, \mathfrak{F} : [0, \tau] \times PW_r \to \mathbb{R}^n$, and $\eta \in PW_r$ are appropriate functions. We investigate the existence of solutions via Karakostas' fixed point theorem, the exact controllability by means of the Rothe's fixed point theorem and the Banach contraction theorem separately, and the approximate controllability using a technique developed by Bashirov *et al.*

In the second part, we extend the existence results of the previous system to an infinite-dimensional setting. That is, we study the following system in a general Banach space Z:

$$\begin{cases} \dfrac{d}{dt}[z(t) - g(t, z_t)] = -Az(t) + \mathfrak{F}(t, z_t), & t \in (0, \tau] \setminus \{t_k\}_{k \in I_p}, \\ z(\theta) = -[h(z_{\tau_1}, \dots, z_{\tau_q})](\theta) + \eta(\theta), & \theta \in [-r, 0], \\ z(t_k^+) = z(t_k^-) + J_k(z(t_k)), & k \in I_p, \end{cases}$$

where $z(t) \in Z$ is the state, $A : D(A) \subset Z \to Z$ is a sectorial operator, z_t is the function $[-r, 0] \ni \theta \mapsto z_t(\theta) = z(t + \theta) \in Z^\alpha$, Z^α is the fractional power space of A, $g, \mathfrak{F} : [0, \tau] \times PW_{r\alpha} \to Z$, $h : PW_{qp\alpha} \to PW_{r\alpha}$, $J_k : Z^\alpha \to Z^\alpha$, and $\eta \in PW_{r\alpha}$ are appropriate functions. We address the existence of solutions through Karakostas' fixed point theorem and provide an application to exemplify our results.

Keywords: neutral differential equations, impulses, nonlocal conditions, Karakostas' fixed point theorem, Rothe's fixed point theorem, exact controllability, approximate controllability.

Resumen

Esta tesis consta de dos partes. En la primera parte, estudiamos la siguiente ecuación diferencial semilineal de tipo neutral en $\mathbb{R}^n$ con impulsos y condiciones no locales:

$$\begin{cases} \dfrac{d}{dt}\left[z(t) - g(t, z_t)\right] = A(t)z(t) + \mathfrak{F}(t, z_t), & t \in (0, \tau] \setminus \{t_k\}_{k \in I_p}, \\ z(\theta) = -[h(z_{\tau_1}, \dots, z_{\tau_q})](\theta) + \eta(\theta), & \theta \in [-r, 0], \\ z(t_k^+) = z(t_k^-) + J_k(t_k, z(t_k)), & k \in I_p, \end{cases}$$

donde $z(t) \in \mathbb{R}^n$ es el estado, $A(t)$ es una matriz continua de dimensión $n \times n$, $0 < t_1 < \cdots < t_p < \tau$, $0 < \tau_1 < \cdots < \tau_q < r < \tau$, $I_p := \{1, \dots, p\}$, z_t es la función $[-r, 0] \ni \theta \mapsto z_t(\theta) = z(t+\theta) \in \mathbb{R}^n$, $h : PW_{qp} \to PW_r$, $J_k : [0, \tau] \times \mathbb{R}^n \to \mathbb{R}^n$, $g, \mathfrak{F} : [0, \tau] \times PW_r \to \mathbb{R}^n$, y $\eta \in PW_r$ son funciones adecuadas. Investigamos la existencia de soluciones a través del teorema del punto fijo de Karakostas, la controlabilidad exacta mediante el teorema del punto fijo de Rothe y el teorema de la contracción de Banach por separado, y la controlabilidad aproximada utilizando una técnica desarrollada por Bashirov *et al.*

En la segunda parte, extendemos los resultados de existencia del sistema anterior a un escenario de dimensión infinita. Es decir, estudiamos el siguiente sistema en un espacio de Banach general Z:

$$\begin{cases} \dfrac{d}{dt}[z(t) - g(t, z_t)] = -Az(t) + \mathfrak{F}(t, z_t), & t \in (0, \tau] \setminus \{t_k\}_{k \in I_p}, \\ z(\theta) = -[h(z_{\tau_1}, \dots, z_{\tau_q})](\theta) + \eta(\theta), & \theta \in [-r, 0], \\ z(t_k^+) = z(t_k^-) + J_k(z(t_k)), & k \in I_p, \end{cases}$$

donde $z(t) \in Z$ es el estado, $A : D(A) \subset Z \to Z$ es un operador sectorial, z_t es la función $[-r, 0] \ni \theta \mapsto z_t(\theta) = z(t+\theta) \in Z^\alpha$, Z^α es el espacio de potencia fraccionaria de A, $g, \mathfrak{F} : [0, \tau] \times PW_{r\alpha} \to Z$, $h : PW_{qp\alpha} \to PW_{r\alpha}$, $J_k : Z^\alpha \to Z^\alpha$, y $\eta \in PW_{r\alpha}$ son funciones adecuadas. Abordamos la existencia de soluciones a través del teorema del punto fijo de Karakostas y proporcionamos una aplicación para ejemplificar nuestros resultados.

Palabras clave: ecuaciones diferenciales neutrales, impulsos, condiciones no locales, teorema del punto fijo de Karakostas, teorema del punto fijo de Rothe, controlabilidad exacta, controlabilidad aproximada.

Contents

Chapter 1

Introduction

Differential equations arise naturally when the evolution of a real-life problem in science and engineering is described mathematically. This is not a mere coincidence, and Newton already knew it better than anyone in the late seventeenth century when he claimed that *"the laws of nature are expressed by differential equations"*.[1]

The description of a problem in mathematical language is referred to as a model (or system, interchangeably). The complexity of the model depends on the nature of the problem. The simplest model encountered in the literature consists of a law governing the dynamics of the problem and some pre-established conditions, which usually take the form $z'(t) = f(t, z(t))$ and $z(t_0) = z_0$, respectively, and together are also known as an initial value problem. The preceding model represents a beautiful abstraction, but unfortunately, it is not general enough to describe a wide range of intrinsic phenomena affecting the behavior of the system. To improve the model, we can follow several directions. Hereafter we briefly introduce three of them.

One direction is to replace the initial condition $z(t_0) = z_0$ of the system by the nonlocal condition $z(t_0) + h(\tau_1, \ldots, \tau_q, z(\cdot)) = z_0$. In this way, the model considers more than one initial measurement. We refer the reader to [30, 31, 32, 65, 90, 95, 96, 108, 110], where several authors have reported improvements in applications when considering nonlocal conditions.

Another direction is to include impulsive conditions of the form $z(t_k^+) = z(t_k^-) + J_k(z(t_k))$, which allow modeling instantaneous perturbations [94]. These conditions allow us to describe evolution processes undergoing abrupt changes such as shocks, harvesting, population dynamics, and the spread of diseases, to name a few. See [1, 2, 3, 10, 13, 22, 33, 46, 53, 77, 80, 86, 91, 93, 94, 97, 98, 104, 107, 123] for applications and more information.

A third direction is to consider a retarded argument $r > 0$ in the unknown function of the model. In this way, our system takes the form $z'(t) = f(t, z(t), z(t - r))$. The underlying meaning of this change is that the future state of the model depends on not only the present but also the past. If the retarded argument also affects the derivative of the system, then the differential equation is referred to as neutral. Previous results in this direction can be found in [4, 6, 7, 8, 21, 23, 36, 41, 59, 62, 64, 66, 81, 109] and

[1] Taken from [9, pp. 1].

references therein.

Once the model is properly described and accurately represents the problem as possible, the next step before attempting to solve it is to assess whether the model is well posed or not. That is, whether a solution exists or not (existence property), and if so, whether it is unique or not (uniqueness property). These two properties are essential when studying a differential equation. Another fundamental property also studied without having an explicit solution is controllability. These three properties will be covered in detail later.

In this work, we simultaneously follow the three directions described above and study the existence and uniqueness of solutions for the following semilinear neutral differential equation in $\mathbb{R}^n$ with impulses and nonlocal conditions.

$$\begin{cases} \dfrac{d}{dt}\left[z(t)-g(t,z_t)\right] = A(t)z(t) + \mathfrak{F}(t,z_t), & t \in (0,\tau] \setminus \{t_k\}_{k\in I_p}, \\ z(\theta) = -[h(z_{\tau_1},\dots,z_{\tau_q})](\theta) + \eta(\theta), & \theta \in [-r,0], \\ z(t_k^+) = z(t_k^-) + J_k(t_k, z(t_k)), & k \in I_p, \end{cases} \tag{1.1}$$

where $z(t) \in \mathbb{R}^n$ is the state, $A(t)$ is a $n \times n$ continuous matrix, $0 < t_1 < \cdots < t_p < \tau$, $0 < \tau_1 < \cdots < \tau_q < r < \tau$, $I_p := \{1,\dots,p\}$, z_t is the time history function $[-r,0] \ni \theta \mapsto z_t(\theta) = z(t+\theta) \in \mathbb{R}^n$, $h : PW_{qp} \to PW_r$ is the nonlocal function, $J_k : [0,\tau] \times \mathbb{R}^n \to \mathbb{R}^n$ is the impulsive function, $g, \mathfrak{F} : [0,\tau] \times PW_r \to \mathbb{R}^n$, and $\eta \in PW_r$ are appropriate functions to be specified later, as well as the spaces PW_r and PW_{qp}.

Once we know that system (1.1) has a solution, we address the associated control problem. For each u fixed, we let $\mathfrak{F}(t,z_t) = B(t)u(t) + f(t,z_t,u(t))$. Then the controllability problem is given by

$$\begin{cases} \dfrac{d}{dt}\left[z(t)-g(t,z_t)\right] = A(t)z(t) + B(t)u(t) + f(t,z_t,u(t)), & t \in (0,\tau] \setminus \{t_k\}_{k\in I_p}, \\ z(\theta) = -[h(z_{\tau_1},\dots,z_{\tau_q})](\theta) + \eta(\theta), & \theta \in [-r,0], \\ z(t_k^+) = z(t_k^-) + J_k(t_k, z(t_k)), & k \in I_p, \end{cases} \tag{1.2}$$

where $B(t)$ is a $n \times m$ continuous matrix, $f : [0,\tau] \times PW_r \times \mathbb{R}^m \longrightarrow \mathbb{R}^n$ is a suitable function to be specified later, the control function u belongs to the space $\mathcal{C}([0,\tau],\mathbb{R}^m)$, and the remaining terms satisfy the same conditions as in system (1.1).

In the last part of the present work, we extend the above existence results[2] to an infinite-dimensional setting. This means that we no longer work in $\mathbb{R}^n$, but in a general Banach space Z. In this case, system (1.1) becomes

$$\begin{cases} \dfrac{d}{dt}[z(t)-g(t,z_t)] = -Az(t) + \mathfrak{F}(t,z_t), & t \in (0,\tau] \setminus \{t_k\}_{k\in I_p}, \\ z(\theta) = -[h(z_{\tau_1},\dots,z_{\tau_q})](\theta) + \eta(\theta), & \theta \in [-r,0], \\ z(t_k^+) = z(t_k^-) + J_k(z(t_k)), & k \in I_p, \end{cases} \tag{1.3}$$

[2] To be more precise, we extend the above investigation when $A(t) := A$. If $A(t)$ is not constant, then using the verb *to extend* is not appropriate.

where $z(t) \in Z$ is the state, $A : D(A) \subset Z \to Z$ is a sectorial operator such that its resolvent operator is compact, z_t is the time history function $[-r, 0] \ni \theta \mapsto z_t(\theta) = z(t+\theta) \in Z^\alpha$, Z^α is the fractional power space of A, $g, \mathfrak{F} : [0, \tau] \times PW_{r\alpha} \to Z$, $h : PW_{qp\alpha} \to PW_{r\alpha}$, $J_k : Z^\alpha \to Z^\alpha$, and $\eta \in PW_{r\alpha}$ are appropriate smooth functions. The spaces $PW_{r\alpha}$ and $PW_{qp\alpha}$ are described below.

The difference between systems (1.1) and (1.3) may look inessential at first glance. However, we assure the reader that this is not the case. The differences arise from multiple viewpoints, such as the problem dimension, the approach, and techniques used, to point out a few. While in problem (1.1) the dimension of the state of the system is $\dim(\mathbb{R}^n) = n < \infty$, in problem (1.3), it is $\dim(Z) = \infty$.[3] Both systems also differ in that (1.3) allows, in essence, studying partial (functional) differential equations while (1.1) does not because it is basically an ordinary (functional) differential equation. The last distinction we mention here is that investigating (1.3) requires some knowledge of Strongly Continuous Semigroup (SCS) Theory while (1.1) does not.

This manuscript is organized as follows.[4]

Summary of Chapter 2

This chapter provides a compilation of concepts and results that are fundamental to our work. In Section 2.1, we present an overview of some basic results of Functional Analysis and Operator Theory. Here we introduce properties of topological, normed, Banach, and Hilbert spaces. We then review definitions and properties of linear, bounded, closed, and compact operators. At the end of the section, we list the Banach contraction theorem, Karakostas' fixed point theorem, and Rothe's fixed point theorem. These theorems will be used to transform an existence and controllability problem into a fixed point one.

Section 2.2 supplies a thorough and constructive literature review on the area of differential equations (DEs) that will lead us to our problem statement (systems (1.1) and (1.3)). For the sake of conciseness, this section covers both finite-dimensional and infinite-dimensional cases. We begin with the simplest initial value problem (IVP) studied by Peano and then escalate to system (1.1). Along the way, we recapitulate linear differential equations and their properties. We also consider retarded differential equations (RDEs) and provide a literature review on that topic. Likewise, we review systems with nonlocal conditions only and then RDEs with nonlocal conditions. Following a constructive approach in the sense of complexity, we similarly investigate impulsive differential equations (IDEs) only and then RDEs with impulses and nonlocal conditions. Finally, we also study neutral differential equations (NDEs) only and then NDEs with impulses and nonlocal conditions.

In Section 2.3, we give a summary of Control Theory and present some results on the controllability of linear systems. We also define exact and approximate controllability.

Section 2.4 is devoted to Semigroup Theory. Here we first motivate strongly continuous semigroups as an abstract description of a well-posed dynamical system. Then we introduce the concept of infinitesimal generator of a SCS and list some of its

[3] Although both problems are infinite-dimensional.

[4] Some ideas for the presentation and organization of this manuscript were inspired by [34] and [33].

properties. After a brief recapitulation about Spectral Theory, we introduce sectorial operators and analytic semigroups. Finally, we define fractional powers of sectorial operators and fractional power spaces.

Summary of Chapter 3

In this chapter, we present our research results in a finite-dimensional setting. This chapter consists of two sections. The first one provides existence results. Here we propose a formula for the solutions of system (1.1) and state an existence theorem. This theorem is later proved using the Karakostas' fixed point theorem. In the second section, we present our controllability results. We use the Banach contraction theorem and Rothe's fixed point theorem separately to prove the exact controllability of system (1.2). The difference between the two approaches relies on the given assumptions for each case. The approximate controllability is shown employing a technique developed by Bashirov *et al.*

The results presented in this chapter can be thought as an extension of the results obtained by Cabada [33]. If $g \equiv 0$ in (1.1) and (1.2), then these systems are no longer neutral and become the case studied by Cabada [33].

Summary of Chapter 4

This chapter extends the existence results presented in Chapter 3. As pointed out before, this extension is in the sense of dimension. Again we use Karakostas' fixed point theorem to prove an existence theorem for system (1.3).

Summary of Chapter 5

In this chapter, we present our conclusions and recommendations.

Chapter 2

Theoretical framework

In this chapter, we provide a compilation of concepts and results intended to make this manuscript self-contained. No proof is provided unless it is strictly necessary. For example, when it is not easily encountered in the literature. Nevertheless, for the sake of completeness, most of the time, we do include a reference containing a proof.

We start with an overview on some necessary concepts and results of Functional Analysis and Operator Theory, mainly taken from Kreyszig [78]. Then, we provide a thorough and constructive literature review on the area of differential equations that will lead us to our problem statement (systems (1.1) and (1.3)). We also review the notion of controllability, which will be core in Section 3.2. Finally, we present some more advanced concepts about Semigroup theory.

2.1 Topics on Functional Analysis and Operator Theory

This section gives concepts and results of Functional Analysis and Operator Theory that are fundamental for the subsequent chapters.

2.1.1 Topological spaces

Our starting point is the simple but abstract notion of *topology*. This concept provides the building blocks to construct a solid theory in Analysis.

Definition 1. *Let Λ be an index set. Let Z be a nonempty set and $\mathcal{T}$ a family of subsets of Z. We say that $\mathcal{T}$ is a topology on Z if and only if*

(i) Both $\emptyset$ and Z belong to $\mathcal{T}$.

(ii) If $A_1, A_2 \in \mathcal{T}$, then $A_1 \cap A_2 \in \mathcal{T}$.

(iii) If $(A_\lambda)_{\lambda\in\Lambda}$ is an arbitrary family of elements of $\mathcal{T}$, then $\bigcup_{\lambda\in\Lambda} A_\lambda \in \mathcal{T}$.

The pair $(Z, \mathcal{T})$ *is called a* topological space *and the elements of* $\mathcal{T}$ *are referred to as* open sets. *The complement of every open set is said to be a* closed set.

Given a topological space $(Z, \mathcal{T})$ and $z \in Z$, we say that $V \subseteq Z$ is a *neighborhood* of z if and only if

$$\exists O \in \mathcal{T} : z \in O \subseteq V.$$

Let A, B, C, D, and E be subsets of Z. We define the *closure* of A, denoted by $\overline{A}$, as the smallest closed set containing A. B is said to be *compact* if and only if for every open covering[1] $(A_\lambda)_{\lambda \in \Lambda}$ of B there is a finite subset I of Λ such that $(A_\lambda)_{\lambda \in I}$ still covers B. C is called *relatively compact* if and only if $\overline{C}$ is compact. D is said to be *dense* in E if and only if $E \subset \overline{D}$. For a deeper understanding of these topological concepts, we refer the reader to [52, Chapters 1-3,9].

2.1.2 Normed spaces

We now turn our attention to the notion of *norm* and *normed space*.

Definition 2. *A* (real) normed (linear) space *is a pair* $(Z, \|\cdot\|_Z)$, *where* Z *is a linear space and* $\|\cdot\|_Z : Z \to \mathbb{R}$ *is a functional satisfying*

(i) **[Non-negativity]** $\forall z \in Z : \|z\|_Z \geq 0.$

(ii) **[Point-separating]** $\forall z \in Z : \|z\|_Z = 0 \iff z = 0.$

(iii) **[Homogeneity]** $\forall z \in Z, \forall \lambda \in \mathbb{R} : \|\lambda z\|_Z = |\lambda| \|z\|_Z.$

(iv) **[Triangle inequality]** $\forall z, y \in Z : \|z + y\|_Z \leq \|z\|_Z + \|y\|_Z.$

The functional $\|\cdot\|_Z$ *is called a* norm. *When there is no risk of ambiguity, we simply denote* $\|\cdot\| := \|\cdot\|_Z$[2] *and refer to the normed space only as* Z.

When working with normed spaces we can define a special open set called *open ball*. These sets are important because they provide an useful characterization for open sets.

Definition 3. *Let* Z *be a normed space and* $z \in Z$. *For* $r > 0$, *the set*

$$B_r(z) = \left\{ y \in Z \;\middle|\; \|z - y\|_Z < r \right\}$$

is called an open ball. A subset A *of* Z *is open if and only if for every* $z \in A$ *there exists* $r > 0$ *such that* $B_r(z) \subseteq A$.

[1] A covering of B is a family $(A_\lambda)_{\lambda \in \Lambda}$ such that $B \subseteq \bigcup_{\lambda \in \Lambda} A_\lambda$. It is called open if each A_λ is open.

[2] We will follow this convention mostly in the next chapter. But, for now, we will make clear the set for which $\|\cdot\|_Z$ is a norm for the sake of completeness.

A *closed ball* is intuitively defined as

$$\overline{B}_r(z) = \left\{ y \in Z \,\middle|\, \|z - y\|_Z \leq r \right\}.$$

Another special set is given in the following definition.

Definition 4. *A subset C of a vector space Z is a* convex set *if and only if*

$$\forall z, y \in C, \forall \lambda \in [0,1] : \lambda z + (1-\lambda) y \in C$$

Remark 1. *Open balls and closed balls are convex sets.*

We next consider a sequence of elements in a normed space Z, usually denoted as $(z_n)_{n\in\mathbb{N}} \subseteq Z$. We say that $(z_n)_{n\in\mathbb{N}}$ (strongly) converges to $z \in Z$[3] if and only if

$$\forall \epsilon > 0, \exists N \in \mathbb{N} : n > N \implies \|z_n - z\|_Z < \epsilon.$$

We call $\left(z_{\varphi(n)}\right)_{n\in\mathbb{N}}$[4] a *subsequence* of $(z_n)_{n\in\mathbb{N}}$ whenever $\varphi : \mathbb{N} \to \mathbb{N}$ is a strictly increasing function. A sequence $(z_n)_{n\in\mathbb{N}}$ is called a *Cauchy sequence* if and only if

$$\forall \epsilon > 0, \exists N \in \mathbb{N} : n, m > N \implies \|z_n - z_m\|_Z < \epsilon.$$

Clearly, any convergent sequence is a Cauchy sequence. The special normed spaces such that all their sequences satisfy the converse are called *Banach spaces*.

Let $\|\cdot\|_1$ and $\|\cdot\|_2$ be two norms on a normed space Z. They are said to be *equivalent* if and only if

$$\exists c_1, c_2 > 0, \forall z \in Z : c_1\|z\|_1 \leq \|z\|_2 \leq c_2\|z\|_1.$$

In case Z is *finite dimensional*, all norms are equivalent. For a better understanding of normed spaces and their properties, we refer the reader to [78, Ch. 2].

2.1.3 Hilbert spaces

As we have seen, the concept of norm generalizes the notion of distance in the real line $\mathbb{R}$. We next define a functional $(\cdot,\cdot)$[5], called *inner product*, which is a generalization of the notion of scalar product in $\mathbb{R}^n$[6].

Definition 5. *A (real) linear space Z is called an inner product space if and only if there is a functional $(\cdot,\cdot) : Z \times Z \to \mathbb{R}$ satisfying*

[3] The phrase $(z_n)_{n\in\mathbb{N}}$ converges to z will be used interchangeably with $\lim\limits_{n\to\infty} z_n = z$, $\lim\limits_{n\to\infty} \|z_n - z\|_Z = 0$, $z_n \xrightarrow[n\to\infty]{} z$, or $z_n \to z$ as $n \to \infty$.

[4] If we let $n_k = \varphi(k)$, then we refer to the subsequence as $\left(z_{n_k}\right)_{k\in\mathbb{N}}$.

[5] The notation $\langle\cdot,\cdot\rangle$ is also used to denote an inner product.

[6] $n \in \mathbb{N}$.

(i) **[Non-negativity]** $\forall z \in Z : (z, z) \geq 0$.

(ii) **[Point-separating]** $\forall z \in Z : (z, z) = 0 \iff z = 0$.

(iii) **[Linearity]** $\forall z, y, w \in Z, \forall \lambda \in \mathbb{R} : (\lambda z + y, w) = \lambda(z, w) + (y, w)$.

(iv) **[Symmetry]** $\forall z, y \in Z : (z, y) = (y, z)$.

It is easy to see that $\|\cdot\|_Z = (\cdot, \cdot)^{1/2}$ defines a norm on Z. Thus an inner product space becomes a normed space. When Z is a Banach space, it is called a *Hilbert space*. Any two elements z and y in a Hilbert space Z satisfy

$$|(z, y)| \leq \|z\|_Z \|y\|_Z. \tag{2.1}$$

This inequality is known as the Cauchy-Bunyakovsky-Schwarz inequality (CBS) and its proof is a standard problem that can be found in, *e.g.*, [118, Lem. 6.20, pp. 180].

Let $T : Z \to Y$ be a bounded operator[7], and let Z and Y be Hilbert spaces. Then the *(Hilbert) adjoint* operator T^* of T is the operator $T^* : Y \to Z$ such that

$$\forall z \in Z, y \in Y : \left(Tz, y\right) = \left(z, T^* y\right).$$

It can be proven that the adjoint operator T^* of T exists, is unique and is a bounded linear operator with norm

$$\|T\|_{\mathcal{B}(Z,Y)} = \|T^*\|_{\mathcal{B}(Y,Z)}. \tag{2.2}$$

Moreover, if $T : Z \to Z$ has a bounded inverse T^{-1}, so has T^* with $(T^*)^{-1} = (T^{-1})^*$. For further understanding of Hilbert spaces and their properties, we refer the reader to [78, Ch. 3].

2.1.4 Bounded linear operators

Let Z, Y, W be normed spaces. A function $T : Z \to Y$ is referred to as an *operator*. It is further called a *linear operator* if and only if

$$\forall z, y \in Z, \forall \lambda \in \mathbb{R} : T(\lambda z + y) = \lambda T(z) + T(y).$$

We denote by $\mathcal{L}(Z, Y)$ the collection of such operators. If a linear operator $T : Z \to Y$ is such that

$$\exists c > 0, \forall z \in Z : \|Tz\|_Y \leq c\|z\|_Z, \tag{2.3}$$

then it is called a *bounded operator*. The set of all bounded linear operators is denoted by $\mathcal{B}(Z, Y)$[8]. This is a normed space (see, *e.g.*, [78, Lem. 2.7-2, pp. 92]) when endowed with the *operator norm*

$$\|T\|_{\mathcal{B}(Z,Y)} := \inf\left\{c > 0 \;\middle|\; \forall z \in Z : \|Tz\|_Y \leq c\|z\|_Z\right\} = \sup_{z \neq 0} \frac{\|Tz\|_Y}{\|z\|_Z} = \sup_{\|z\|=1} \|Tz\|_Y.$$

[7] What it means to be a bounded operator will be defined in the next subsection.

[8] When $Y = Z$, we simply write $\mathcal{B}(Z, Z) = \mathcal{B}(Z)$.

Obviously, for any c such that (2.3) holds,

$$\|T\|_{\mathcal{B}(Z,Y)} \leq c. \tag{2.4}$$

If T is bounded, we can write the following useful inequality.

$$\|Tz\|_Y \leq \|T\|_{\mathcal{B}(Z,Y)} \|z\|_Z. \tag{2.5}$$

The composition of two bounded operators, say T as before and $S : Y \to W$, is also bounded, and moreover

$$\|ST\|_{\mathcal{B}(Z,W)} \leq \|S\|_{\mathcal{B}(Y,W)} \|T\|_{\mathcal{B}(Z,Y)}.$$

The following theorem gives sufficient conditions for $\mathcal{B}(Z,Y)$ to be complete.

Theorem 1. *If Y is Banach, so is $\mathcal{B}(Z,Y)$ when equipped with the operator norm.*

See Kreyszig [78, Th. 2.10-2, pp. 118] for a proof.

If $Y = \mathbb{R}$, then T is called a (*linear* or *bounded*, accordingly) *functional* and $\mathcal{B}(Z,\mathbb{R})$ is usually denoted as Z^*. Clearly, Z^* is a Banach space as a consequence of Theorem 1.

For a sequence of elements $\{T_n\}_{n\in\mathbb{N}}$ in the normed space $\mathcal{B}(Z,Y)$ we can define three types of convergence (see, *e.g.*, [78, Def. 4.9-1, pp. 263]).

Definition 6. *Let Z and Y be normed spaces, $T \in \mathcal{B}(Z,Y)$, and $\{T_n\}_{n\in\mathbb{N}}$ be a sequence in $\mathcal{B}(Z,Y)$. Then, $\{T_n\}_{n\in\mathbb{N}}$*

(i) uniformly converges *to T if and only if* $\|T_n - T\|_{\mathcal{B}(Z,Y)} \underset{n\to\infty}{\longrightarrow} 0.$

(ii) strongly converges *to T if and only if* $\forall z \in Z : \|T_n z - Tz\|_Y \underset{n\to\infty}{\longrightarrow} 0.$

(iii) weakly converges *to T if and only if* $\forall z \in Z, \forall f \in Y^* : |f(T_n z) - f(Tz)| \underset{n\to\infty}{\longrightarrow} 0.$

Noting that

$$|f(T_n z) - f(Tz)| \leq \|f\|_{Y^*} \|T_n z - Tz\|_Y \leq \|f\|_{Y^*} \|T_n - T\|_{\mathcal{B}(Z,Y)} \|z\|_Z$$

as a consequence of (2.5), we easily deduce that *(i)* $\Longrightarrow$ *(ii)* $\Longrightarrow$ *(iii)*.

2.1.5 Closed linear operators

Definition 7. *Let Z and Y be normed spaces and $T : D(T) \subseteq Z \to Y$ a linear operator with domain $D(T) \subseteq Z$. Then T is called a* closed linear operator *if its* graph

$$G(T) = \left\{ (z,y) \;\middle|\; z \in D(T) \text{ and } y = Tz \right\}$$

is closed in the normed space $Z \times Y$ with the graph norm $\left\|(z,y)\right\|_T = \|z\|_Z + \|Tz\|_Y$.[9] *It is customary to only write $\|z\|_T := \left\|(z,y)\right\|_T$.*

[9]Both $\|z\| = \left(\|z\|_Z^2 + \|Tz\|_Y^2\right)^{1/2}$ and $\|z\| = \max\left\{\|z\|_Z, \|Tz\|_Y\right\}$ are equivalent norms.

Proposition 1. *T is a closed operator if and only if $D(T)$ is a Banach space with respect to the graph norm.*

See Berezansky *et al.* [25, pp. 5] for a proof.

The following theorem provides a sequential characterization for closed operators, which is sometimes more convenient in applications.

Theorem 2. *Let Z and Y be normed spaces, $T : D(T) \subset Z \to Y$ be a linear operator, and $(z_n)_{n\in\mathbb{N}} \subseteq D(T)$. Then T is a closed operator if and only if it has the following property. If $z_n \to z$ and $Tz_n \to y$, then $z \in D(T)$ and $Tz = y$.*

For a proof, see Kreyszig [78, Th. 4.13-3, pp. 293].

2.1.6 Compact linear operators

Let Z and Y be normed spaces. A linear operator is called a *compact operator* if and only if

$$\forall A \subseteq Z \text{ bounded } : T(A) \subseteq Y \textit{ is relatively compact}.$$

The space of all compact linear operators will be denoted by $\mathcal{K}(Z,Y)$. In practice, the following characterization is most useful when studying the compactness of an operator.

Theorem 3. *Let Z and Y be normed spaces and $T : Z \to Y$ a linear operator. Then $T \in \mathcal{K}(Z,Y)$ if and only if*

$$\forall\, (z_n)_{n\in\mathbb{N}} \subseteq Z \textit{ bounded} : (Tz_n)_{n\in\mathbb{N}} \subseteq Y \textit{ has a convergent subsequence}.$$

See Kreyszig [78, Th. 8.1-3, pp. 407] for a proof.

Lemma 1. *Let Z be a normed space, $T \in \mathcal{K}(Z)$, and $S \in \mathcal{B}(Z)$. Then $TS \in \mathcal{K}(Z)$ and $ST \in \mathcal{K}(Z)$.*

A proof of this lemma is given in [78, Th. 8.3-2, pp. 422].

2.1.7 More definitions

In this subsection, we introduce more concepts that will allow us to state Arzelà-Ascoli theorem. We start with the concept of Lipschitz continuity (see, *e.g.*, [56, pp. 9]). This concept is necessary to introduce the notion of equicontractivity (Definition 9) and the Banach contraction theorem (Theorem 6).

Definition 8. *Let Z and Y be normed spaces and $T : Z \to Y$ be a mapping. T is called* Lipschitz continuous *if and only if*

$$\exists k \geq 0, \forall z, y \in Z : \|Tz - Ty\|_Y \leq k\|z - y\|_Z.$$

The constant k is referred to as a Lipschitz constant *for T. If $k \in [0,1)$, then T is called a* contraction.

The next definition (see, *e.g.*, [76, Def. 2.1]) applies to a family of operators. It regards the family as *equicontractive* if all its members are contractions for the same Lipschitz constant $k \in [0,1)$.

Definition 9. *Let Z be a normed space. A family of operators $\{T_\lambda : Z \to Z\}_{\lambda \in \Lambda}$ is said to be* equicontractive *if and only if*

$$\exists k \in [0,1), \forall \lambda \in \Lambda, \forall z, y \in Z : \left\|T_\lambda z - T_\lambda y\right\|_Z \leq k\left\|z - y\right\|_Z.$$

The following definition can be found in [121, pp. 208].

Definition 10. *Let Z be a normed space. A family of functions $\mathfrak{F} = \{f : [a,b] \to Z\}$ is said to be (uniformly) equicontinuous if and only if*

$$\forall \epsilon > 0, \exists \delta > 0, \forall t, s \in [a,b], \forall f \in \mathfrak{F} : |t-s| < \delta \implies \left\|f(t) - f(s)\right\|_Z < \epsilon,$$

or equivalently, if $\left\|f(t) - f(s)\right\|_Z \to 0$ as $t \to s$ independently of $f \in \mathfrak{F}$.

Definition 11. *Let Z be a normed space. A family of functions $\{f_\lambda : [a,b] \to Z\}_{\lambda \in \Lambda}$ is said to be uniformly bounded if and only if*

$$\exists M \geq 0, \forall t \in [a,b], \forall \lambda \in \Lambda : \left\|f_\lambda(t)\right\|_Z \leq M.$$

The next theorem is the n dimensional generalization of the classical Arzelà-Ascoli theorem (see, *e.g.*, [112, Th. 1.3, pp. 3]).

Theorem 4. *If $\left(f_m\right)_{m \in \mathbb{N}} \subseteq \mathcal{C}\left([a,b], \mathbb{R}^n\right)$ is a uniformly bounded and equicontinuous sequence of functions, then it has a subsequence $\left(f_{m_k}\right)_{k \in \mathbb{N}}$ that converges uniformly on $[a,b]$ to a function $f \in \mathcal{C}\left([a,b], \mathbb{R}^n\right)$.*

A proof of an equivalent version of this theorem can be found in [45, Th. 3.1.2., pp. 62] or [106, Th. 45.4, pp. 278].

The following is another version of the classical Arzelà-Ascoli theorem and is regarded as the Arzelà-Ascoli theorem for abstract functions (see, *e.g.*, [79, Th. 1.1.1, pag. 3]).

Theorem 5. *Let Z be a Banach space and $\mathfrak{F} = \{f : [a,b] \to Z\}$ be an equicontinuous family of functions from $[a,b]$ into Z. Let $\left(f_n\right)_{n \in \mathbb{N}}$ be a sequence in $\mathfrak{F}$ such that for each $t \in [a,b]$ the set $\{f_n(t) : n \geqslant 1\}$ is relatively compact in Z. Then, there is a subsequence $\left(f_{n_k}\right)_{k \in \mathbb{N}}$ which is uniformly convergent on $[a,b]$.*

See Royden [120, Th. 33, pp. 179] for a proof.

2.1.8 Fixed point theorems

Here we list some fixed point theorems we used to prove our results. The more remarkable one is undoubtedly the Banach contraction theorem (see, *e.g.*, [56, Th. 1.1]).

Theorem 6. *Let Z be a Banach space and $T : Z \to Z$ be a contractive mapping. Then T has a unique fixed point $z \in Z$, and $T^n(y)$*[10] *$\to z$ as $n \to \infty$ for each $y \in Z$.*

See Smart [126, Th. 1.2.2, pp. 2] for a proof.

The next theorem is due to Karakostas [76, Th. 2.2].

Theorem 7. *Let Z and Y be Banach spaces and D be a closed convex subset of Z, and let $\mathcal{P} : D \to Y$ be a continuous operator such that $\mathcal{P}(D)$ is a relatively compact subset of Y, and*

$$\mathcal{Q} : D \times \overline{\mathcal{P}(D)} \to D$$

a continuous operator such that the family $\{\mathcal{Q}(\cdot, y) : y \in \overline{\mathcal{P}(D)}\}$ is equicontractive. Then, the operator equation

$$\mathcal{Q}(z, \mathcal{P}(z)) = z$$

admits a solution on D.

Now we state the Rothe's fixed point theorem. Actually, it is a general version of the finite dimensional Rothe's fixed point theorem. The latter can be found in [38, pp. 59].

Theorem 8. *Let Z be a Banach space and consider $D \subseteq Z$ a closed convex subset containing the zero of Z in its interior. Let $\Psi : D \to Z$ be a continuous function with $\Psi(D)$ relatively compact in Z and $\Psi(\partial D) \subset D$. Then*

$$\exists z^\star \in D : \Psi\left(z^\star\right) = z^\star.$$

A proof of Theorem 8 is given in [68, Th. 2, pp. 129].

2.2 Differential Equations

Recall from the classical theory of ordinary differential equations (ODEs) that the following IVP

$$\begin{cases} \dfrac{d}{dt} z(t) = f(t, z(t)), & t \in [0, \tau], \\ z(0) = z_0 \in \mathbb{R}^n, \end{cases} \tag{2.6}$$

or equivalently,

$$z(t) = z_0 + \int_0^t f(s, z(s))ds, \; t \in [0, \tau] \tag{2.7}$$

[10] $T^n(y)$ is defined inductively as $T^0(y) = y$ and $T^{n+1}(y) = T(T^n(y))$ for $n \in \mathbb{N} \cup \{0\}$.

describes the time evolution of a physical system. It is well known that if $f : D \subseteq \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ is a continuous function, where D is an open set, then (2.6) has at least one local solution defined on some neighborhood of $t = 0$. A proof of this result was originally given by Peano [114] in 1890 and is usually referred to as the Peano's existence theorem. For a proof with standard notation, we refer the reader to [81, Vol. I, Th. 1.1.2, pp. 4]. In the case that f is continuous in D and locally Lipschitz continuous in the second argument, system (2.6) has a unique local solution defined on some neighborhood of $t = 0$. This result is also classic and is known as the Picard-Lindelöf theorem. A proof of this theorem can be found in [61, Th. 3.1, pp. 18] or [129, Th. 2.2, pp. 38]

2.2.1 Linear Systems of Differential Equations

In this section we develop the theory of neutral differential equation. Our starting point is the non-autonomous inhomogeneous linear system of ordinary differential equations

$$\frac{d}{dt}z(t) = A(t)z(t) + b(t),\ t \in I, \tag{2.8}$$

where $z : I \to \mathbb{R}^n$ is the unknown, $A(t)$ is an $n \times n$ matrix function defined on some open interval $I \subseteq \mathbb{R}$ and b is a given function defined on I as well. If $A(t)$ is a constant matrix, say $A(t) := A$, then (2.8) is called *autonomous*. If $b \equiv 0$, then (2.8) is regarded as *homogeneous*. The initial value problem (IVP) associated with (2.8) is given by

$$\begin{cases} \dfrac{d}{dt}z(t) = A(t)z(t) + b(t), & t \in I, \\ z(t_0) = z_0 \in \mathbb{R}^n, & t_0 \in I. \end{cases} \tag{2.9}$$

The following result is classic.

Theorem 9. *If A and b are continuous on I, then for every pair (t_0, z_0) the solution of the system (2.9) is unique and is given by*

$$z(t) = \Phi(t)\Phi^{-1}(t_0)z_0 + \Phi(t)\int_{t_0}^{t} \Phi^{-1}(\theta)b(\theta)d\theta,\ t \in I, \tag{2.10}$$

where Φ is the fundamental matrix of the homogeneous linear system. Such matrix satisfies[11]

$$\begin{cases} \dfrac{d}{dt}\Phi(t) = A(t)\Phi(t), & t \in I, \\ \Phi(0) = I. \end{cases}$$

See Sideris [125, Th. 4.1, pp. 54] for a proof.

[11] In the second equality, I represents the $n \times n$ identity matrix.

Note that (2.10) generalizes the Variation of Parameters Formula (VPF) (see, *e.g.*, [28, Eq. 41, pp. 278]). Indeed, if A is a constant matrix, then $\Phi(t) = \exp(At)$ and (2.10) reduces to the familiar VPF

$$z(t) = \exp\left(A(t-t_0)\right) z_0 + \int_{t_0}^{t} \exp\left(A(t-\theta)\right) b(\theta) d\theta, \ t \in I, \tag{2.11}$$

where $\exp(\cdot)$ is defined by its series representation. For $t, \theta \in I$, we define $\mathbf{\Phi}(t,\theta) := \Phi(t)\Phi^{-1}(\theta)$. From this definition, $\mathbf{\Phi}$ immediately satisfies the properties given in the next proposition.

Proposition 2. *For all $t, \tau, \theta \in I$ we have that*

(i) $\mathbf{\Phi}(t,t) = I, \ t \geq 0.$

(ii) **[Cocycle property]** $\mathbf{\Phi}(\tau,t)\mathbf{\Phi}(t,\theta) = \mathbf{\Phi}(\tau,\theta), \ 0 \leq \theta \leq t \leq \tau.$

(iii) $\dfrac{\partial}{\partial t}\mathbf{\Phi}(t,\theta) = A(t)\mathbf{\Phi}(t,\theta).$

(iv) $\mathbf{\Phi}$ *is continuous.*

(v) [12] $\exists M \geq 1, \exists M_0, \omega > 0 : \|\mathbf{\Phi}(t,\theta)\| \leq M_0 \exp\left(\omega(t-\theta)\right) \leq M, \ 0 \leq \theta \leq t \leq \tau.$

(vi) $\mathbf{\Phi}^{-1}(\theta,t) = \mathbf{\Phi}(t,\theta).$

For more details about these properties, see [39, Prop. 2.12, pp. 133] and [28, Prop. 1 & Prop. 2, pp. 289-292].

2.2.2 Retarded Differential Equations

Before providing the definition of a retarded differential equation (RDE), we establish preliminary notation. For that purpose, we consider the following discussion developed by Hale in [8, Ch. 1, pp. 3].

For $r > 0$, we consider the set of continuous vector valued functions defined on $[-r, 0]$, denoted by $\mathcal{C}\left([-r,0], \mathbb{R}^n\right)$. If $\tau > 0$ and $z \in \mathcal{C}\left([-r,\tau], \mathbb{R}^n\right)$, then for any $t \in [0,\tau]$ we let $z_t \in \mathcal{C}\left([-r,0], \mathbb{R}^n\right)$ be defined by $z_t(\theta) = z(t+\theta)$, which is usually called the *time history function*. The number r is referred to as the *delay* of the system. We notice that $z_t \in \mathcal{C}\left([-r,0], \mathbb{R}^n\right)$ if and only if $z \in \mathcal{C}\left([-r,\tau], \mathbb{R}^n\right)$. The function z_t is defined as the section of z on the interval $[t-r, t]$ shifted to the interval $[-r, 0]$ [59, pp. 42]. See Figure 2.1.

Now we are ready to define a RDE.

Definition 12. *If $f : [0,\tau] \times \mathcal{C}\left([-r,0], \mathbb{R}^n\right) \to \mathbb{R}^n$ is a given function, a RDE is defined by the relation*

$$\frac{d}{dt} z(t) = f(t, z_t) \tag{2.12}$$

[12] Here, $\|\cdot\|$ represents any matrix norm.

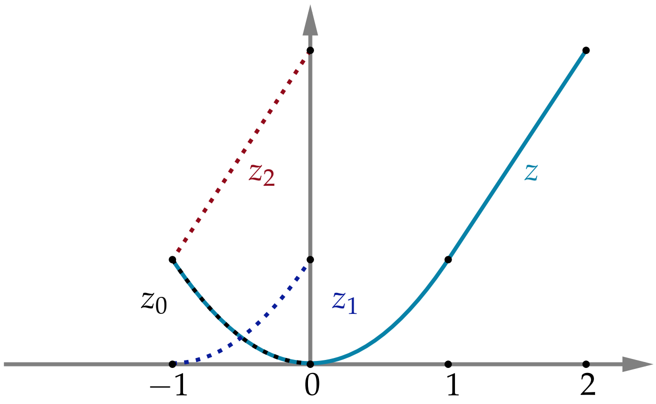

Figure 2.1: This graph represents (not to scale) the quadratic function $z(t) = t^2$, $t \in [-1, 2]$. Here, $\tau = 2$ and $r = 1$. For example, $z_2(\theta) = z(2+\theta) = (2+\theta)^2$, $\theta \in [-1, 0]$, and z_0 is the restriction of z to $\theta \in [-1, 0]$.

If $\eta \in \mathcal{C}\left([-r, 0], \mathbb{R}^n\right)$ is given, then a solution $z(t, \eta)$ of (2.12) with initial value η at $t = 0$ is a continuous function defined on $[-r, \tau]$ such that $z_0(\theta) = z(\theta, \eta) = \eta(\theta)$ for $\theta \in [-r, 0]$. Also, $z(t, \eta)$ has a continuous derivative on $(0, \tau)$, a right hand derivative at $t = 0$ and satisfies (2.12) for $t \in [0, \tau)$.

The IVP associated with (2.12) is given by

$$\begin{cases} \dfrac{d}{dt} z(t) = f(t, z_t), & t \in [0, \tau], \\ z(\theta) = \eta(\theta), & \theta \in [-r, 0], \end{cases} \tag{2.13}$$

or equivalently,

$$z(t) = \begin{cases} \eta(0) + \displaystyle\int_0^t f(s, z_s) ds, & t \in [0, \tau], \\ \eta(t), & t \in [-r, 0]. \end{cases}$$

We note that if $r = 0$, then system (2.13) reduces to (2.6). A proof concerning the existence and uniqueness of solutions for system (2.13) based on the well-known Schauder's fixed point theorem can be found in [81, Vol. II, Th. 6.1.1, pp. 5].

Equation (2.12) is also known in the literature as a *differential equation with a delay argument* and belongs to the wide category of *differential equations with deviating argument*. There are three types of differential equations with deviating argument. According to [4, pp. 674], the other two types are *advanced differential equations* of the form

$$\frac{d}{dt} z(t) = f(t, z(t), z(t + r)),\ t \geq t_0,\ r > 0$$

and *neutral differential equations* (NDEs) having the general structure

$$\frac{d}{dt} z(t) = f(t, z(t), z(t - r), z'(t - r)),\ t \geq t_0,\ r > 0. \tag{2.14}$$

Note that NDEs can be understood as a generalization of RDEs.

2.2.3 Differential Equations with Nonlocal Conditions

The expression $z(0) = z_0$ in (2.6) indicates that z_0 is the initial condition, sometimes also called *local condition*, of the system at time $t = 0$ and represents an initial measurement. As pointed out in [108] and [95], sometimes it is better to have more than one initial measurement. This can be achieved if instead of $z(0) = z_0 \in \mathbb{R}^n$ we consider the *nonlocal condition* $z(0) + h(\tau_1, \dots, \tau_q, z(\cdot)) = z_0 \in \mathbb{R}^n$, where h is a given function and $0 < \tau_1 < \cdots < \tau_q < \tau$. This way, system (2.6) becomes

$$\begin{cases} \dfrac{d}{dt} z(t) = f(t, z(t)), & t \in [0, \tau], \\ \quad z(0) = -h(\tau_1, \dots, \tau_q, z(\cdot)) + z_0 \in \mathbb{R}^n, \end{cases} \tag{2.15}$$

or equivalently,

$$z(t) = \Big[z_0 - h(\tau_1, \dots, \tau_q, z(\cdot)) \Big] + \int_0^t f(s, z(s)) ds, \ t \in [0, \tau]$$

under certain conditions. We note that if $h(\tau_1, \dots, \tau_q, z(\cdot)) = 0$, then system (2.15) reduces to (2.6). The symbol $h(\tau_1, \dots, \tau_q, z(\cdot))$[13] indicates that we can only replace $\cdot$ by the points $\{\tau_1, \dots, \tau_q\}$ as remarked by Byszewski and Lakshmikantham in [32]. For instance, $h(\tau_1, \dots, \tau_q, z(\cdot))$ may be given by $h(\tau_1, \dots, \tau_q, z(\cdot)) = C_1 z(\tau_1) + \cdots + C_q z(\tau_q)$, where $C_i, i \in I_q$ are given constants [95].

The existence and uniqueness of solutions for the general version[14] of system (2.15) were proved by Byszewski & Lakshmikantham [32] by means of the Banach contraction theorem. Remarks about the importance of nonlocal conditions can be found in [32, Sec. 3, pp. 16] and the references therein.

If we add the term $A(t)z(t)$ in (2.15), then we have

$$\begin{cases} \dfrac{d}{dt} z(t) = A(t)z(t) + f(t, z(t)), & t \in [0, \tau], \\ \quad z(0) = -h(z) + z_0 \in \mathbb{R}^n, \end{cases} \tag{2.16}$$

or equivalently,

$$z(t) = \mathbf{\Phi}(t, 0) \left[z_0 - h(z) \right] + \int_{t_0}^t \mathbf{\Phi}(t, s) f(s, z(s)) ds, \ t \in [0, \tau].$$

The existence and uniqueness of solutions for the infinite dimensional version of system (2.16) were studied by Byszewski [30, Th. 3.1] by using the Banach contraction theorem. Using the same theorem, Leiva & Sivoli [90, Th. 3.1] derived existence results for the infinite dimensional local version of system (2.16). In both cases, authors assumed that A is the infinitesimal generator of a C_0 semigroup. In contrast, Hernández

[13] Sometimes we simply write $h(z) := h(\tau_1, \dots, \tau_q, z(\cdot))$ to alleviate the notation.

[14] Instead of $\mathbb{R}^n$, Byszewski and Lakshmikantham considered E^n, where E is a Banach space.

et al. [65, Th. 2.1] did the same as Byszewski, but they supposed A to be the generator of an analytic semigroup. Liu & Chang [96] proposed the same model as Hernández *et al.*, but they used Schauder's fixed point theorem [81] or Sadovskii's fixed point theorem [122] instead of the Banach contraction theorem. In [110], Ntouyas & Tsamatos investigated the existence of global solutions for the infinite dimensional version of system (2.16) by means of the Leray-Schauder alternative theorem [130, Lem. 2.2].

If instead of $f(t,z(t))$ in (2.15) we consider $f(t,z_t)$ with the meaning given in Subsection 2.2.2, then we have[15]

$$\begin{cases} \dfrac{d}{dt}z(t) = f(t,z_t), & t \in [0,\tau], \\ z(\theta) = -[h(z_{\tau_1},\dots,z_{\tau_q})](\theta) + \eta(\theta), & \theta \in [-r,0], \end{cases}$$

or equivalently,

$$z(t) = \begin{cases} [\eta(0) - [h(z)](0)] + \displaystyle\int_0^t f(s,z_s)ds, & t \in [0,\tau], \\ \eta(t) - [h(z_{\tau_1},\dots,z_{\tau_q})](t), & t \in [-r,0]. \end{cases}$$

If we combine the above results, it is easy to see now that solving the following system[16]

$$\begin{cases} \dfrac{d}{dt}z(t) = A(t)z(t) + f(t,z_t), & t \in [0,\tau], \\ z(\theta) = -[h(z)](\theta) + \eta(\theta), & \theta \in [-r,0] \end{cases} \tag{2.17}$$

is equivalent to solving the integral equation

$$z(t) = \begin{cases} \mathbf{\Phi}(t,0)\left[\eta(0) - [h(z)](0)\right] + \displaystyle\int_0^t \mathbf{\Phi}(t,s)f(s,z_s)ds, & t \in [0,\tau], \\ \eta(t) - [h(z)](t), & t \in [-r,0]. \end{cases}$$

Byszewski and Akça [31] derived the existence and uniqueness of solutions for the infinite dimensional version of system (2.17) by using the Banach contraction theorem.

2.2.4 Impulsive Differential Equations

Let $p \in \mathbb{N}$. The basic form of an impulsive differential equation (IDE) is given by

$$\begin{cases} \dfrac{d}{dt}z(t) = f(t,z(t)), & t \in [0,\tau] \setminus \{t_k\}_{k\in I_p}, \\ z(t_k^+) = z(t_k^-) + J_k(z(t_k)), & k \in I_p, \end{cases} \tag{2.18}$$

[15] In the context of RDEs, it is customary to use $[h(z_{t_1},\dots,z_{t_q})](\theta)$ instead of $h(\tau_1,\dots,\tau_q,z(\cdot))$.

[16] For the sake of convenience, we will also use the notation $[h(z)](\theta) =: [h(z_{t_1},\dots,z_{t_q})](\theta)$.

where $J_k : \mathbb{R}^n \to \mathbb{R}^n$ are suitable functions called *jump functions* and the *impulsive moments* t_k are such that $0 < t_1 < \cdots < t_p < \tau$. As usual, $z(t_k^-) := \lim_{t \to t_k^-} z(t)$ and $z(t_k^+) := \lim_{t \to t_k^+} z(t)$.

According to [2, Sec. 1.4, pp. 16], the theory of IDE was first introduced by Milman & Myshkis [104] in 1960. Since then, IDE have been developed in modeling impulsive problems in physics, mathematical economy, population dynamics, mechanics, optimal control, engineering, pharmcokinetics, ecology, chemistry, spread of disease, and biotechnology to name a few [22, Sec. 2.1, pp. 11]. Remarkable in the area of IDE are the works of Samoilenko & Perestyuk [123], Bainov & Simeonov [10], and Lakshmikantham *et al.* [80].

An advantage of IDEs over ODEs is that IDEs allow to describe evolution processes that at certain moments in time experience abrupt perturbations of state [107].

Let us consider the corresponding IVP associated with (2.18)

$$\begin{cases} \dfrac{d}{dt}z(t) = f(t, z(t)), & t \in [0, \tau] \setminus \{t_k\}_{k \in I_p}, \\ z(0) = z_0 \in \mathbb{R}^n, & \\ z(t_k^+) = z(t_k^-) + J_k(z(t_k)), & k \in I_p, \end{cases} \tag{2.19}$$

or equivalently,

$$z(t) = z_0 + \int_0^t f(s, z(s))ds + \sum_{0 < t_k < t} J_k(z(t_k)), \; t \in [0, \tau]. \tag{2.20}$$

We note that if $z(t_k^+) - z(t_k^-) = 0, k \in I_p$, then system (2.19) reduces to system (2.6), and the solution (2.20) becomes (2.7)[17], as expected.

Li *et al.* [93, Th. 3.1] used the Banach contraction theorem to show that (2.19) has a unique periodic solution. The existence and uniqueness of solutions for the retarded version of system (2.19) were studied by Ballinger and Liu [13, Cor. 3.1] without using fixed point techniques. Liu [98, Th. 2,3,4] established some stability criteria for system (2.19) using Lyapunov's direct method. In [22, Sec. 3.2, Th. 3.3, pp. 68], Benchohra *et al.* derived the existence of solutions for the retarded infinite dimensional version of (2.19) by using a nonlinear alternative of Leray-Schauder type given in [56, Th. 4.1, pp. 14]

If instead of $z(0) = z_0 \in \mathbb{R}^n$ in (2.19) we consider nonlocal conditions, then we have

$$\begin{cases} \dfrac{d}{dt}z(t) = f(t, z(t)), & t \in [0, \tau] \setminus \{t_k\}_{k \in I_p}, \\ z(0) = -h(z) + z_0 \in \mathbb{R}^n, & \\ z(t_k^+) = z(t_k^-) + J_k(z(t_k)), & k \in I_p, \end{cases} \tag{2.21}$$

[17] Each $J_k(z(t_k)) = 0, k \in I_p$ and consequently $\sum_{0 < t_k < t} J_k(z(t_k)) = 0$.

or equivalently,

$$z(t) = \left[z_0 - h(z)\right] + \int_0^t f(s, z(s))ds + \sum_{0<t_k<t} J_k(z(t_k)),\ t \in [0, \tau].$$

The existence and uniqueness of solutions for system (2.21) were studied by Knapik [77] by means of the Banach contraction theorem.

If we combine (2.17) with (2.19), then we readily see that solving the following system

$$\begin{cases} \dfrac{d}{dt}z(t) = A(t)z(t) + f(t, z_t), & t \in [0, \tau] \setminus \{t_k\}_{k\in I_p}, \\ z(\theta) = -[h(z)](\theta) + \eta(\theta), & \theta \in [-r, 0], \\ z(t_k^+) = z(t_k^-) + J_k(z(t_k)), & k \in I_p, \end{cases} \tag{2.22}$$

is equivalent to solve the integral equation

$$z(t) = \begin{cases} \mathbf{\Phi}(t, 0)\left[\eta(0) - [h(z)](0)\right] + \displaystyle\int_0^t \mathbf{\Phi}(t, s)f(s, z_s)ds & \\ + \displaystyle\sum_{0<t_k<t} \mathbf{\Phi}(t, t_k)J_k(z(t_k)), & t \in [0, \tau], \\ \eta(t) - [h(z)](t), & t \in [-r, 0]. \end{cases}$$

In [33], Cabada studied the existence and uniqueness of system (2.22). The infinite dimensional local version of system (2.22) were studied by Leiva & Sundar [91]. Later, Leiva [86] investigated the same case but with nonlocal conditions, i.e., the infinite dimensional version of (2.22). In these three papers, the authors used a fixed point theorem developed by Karakostas [76, Th. 2.2] to show the existence. Akça *et al.* [3] derived the existence and uniqueness by using the Banach contraction theorem. In [46], Diagana & Leiva investigated the existence of bounded solutions for the non retarded infinite dimensional version of system (2.22). They also used the Banach contraction theorem. Liang *et al.* [94] and Fan & Li [53] studied the non retarded infinite dimensional case with special emphasis on the nonlocal function h. The local and non retarded infinite dimensional version of (2.22) was examined by Lui [97, Th. 2.1]. He used the Banach contraction theorem to show existence and uniqueness of solutions for system (2.22). Abada *et al.* [1] derived existence results for both the local and nonlocal infinite dimensional version of system (2.22) by means of a fixed point theorem of Krasnoselskiii-Schaefer type developed by Burton & Kirk [29].

2.2.5 Neutral Differential Equations

In Subsection 2.2.2, we briefly introduced NDEs. In this Subsection, we provide a more in-depth discussion. Our starting point is Equation (2.14). In the literature, there are many ways to write or formulate Equation (2.14). For instance, Guo & Wu [59, Sec. 2.3, pp. 58] regarded the relation

$$\frac{d}{dt}g(z_t) = f(z_t) \tag{2.23}$$

as a NDE, where $f, g : \mathcal{C}\left([-r, 0], \mathbb{R}^n\right) \to \mathbb{R}^n$ are suitable functions. Equations (2.14) and (2.23) do not seem to be related at all. However, they are related because they share the most important characteristic of a NDE: the derivative on the retarded term. In the same direction, the direction we are going to follow as well, Hale & Cruz [62, Sec. 3] considered a general version of (2.23) given by

$$\frac{d}{dt}\left[z(t) - g(t, z_t)\right] = f(t, z_t). \tag{2.24}$$

The IVP associated with (2.24) is given by

$$\begin{cases} \dfrac{d}{dt}\left[z(t) - g(t, z_t)\right] = f(t, z_t), & t \in [0, \tau], \\ z(\theta) = \eta(\theta), & \theta \in [-r, 0], \end{cases} \tag{2.25}$$

or equivalently,

$$z(t) = \begin{cases} [\eta(0) - g(0, \eta)] + g(t, z_t) + \displaystyle\int_0^t f(s, z_s) ds, & t \in [0, \tau], \\ \eta(t), & t \in [-r, 0]. \end{cases}$$

The existence of solutions for system (2.25) was studied by Arino *et al.* [7, Th. 3] while Ntouyas & Sficas [109, Th. 2] obtained results on continuation of solutions.

Adding impulses to system (2.24) yield

$$\begin{cases} \dfrac{d}{dt}\left[z(t) - g(t, z_t)\right] = f(t, z_t), & t \in [0, \tau] \setminus \{t_k\}_{k \in I_p}, \\ z(\theta) = \eta(\theta), & \theta \in [-r, 0], \\ z(t_k^+) = z(t_k^-) + J_k(z(t_k)), & k \in I_p, \end{cases} \tag{2.26}$$

or equivalently,

$$z(t) = \begin{cases} [\eta(0) - g(0, \eta)] + g(t, z_t) + \displaystyle\int_0^t f(s, z_s) ds + \sum_{0 < t_k < t} J_k(z(t_k)), & t \in [0, \tau], \\ \eta(t), & t \in [-r, 0]. \end{cases}$$

In [21, Th. 3.1], Benchohra *et al.* used Schaefer's fixed point theorem (see, *e.g.*, [126, pp. 29]) to show the existence of solutions for system (2.26). The same fixed point theorem was applied by Benchohra & Ouahab [23, Th. 3.2] to obtain existence results for a version of system (2.26) where the impulsive effects occur at variable times.

We finally add nonlocal conditions and the term $A(t)z(t)$ to obtain

$$\begin{cases} \dfrac{d}{dt}\left[z(t) - g(t, z_t)\right] = A(t)z(t) + f(t, z_t), & t \in [0, \tau] \setminus \{t_k\}_{k \in I_p}, \\ z(\theta) = -[h(z)](\theta) + \eta(\theta), & \theta \in [-r, 0], \\ z(t_k^+) = z(t_k^-) + J_k(z(t_k)), & k \in I_p, \end{cases} \tag{2.27}$$

or equivalently,

$$z(t) = \begin{cases} \mathbf{\Phi}(t,0)\left[\eta(0) - [h(z)](0) - g(0,\eta - h(z))\right] \\ + \displaystyle\int_0^t \mathbf{\Phi}(t,s)\left[A(s)g(s,z_s) + f(s,z_s)\right] ds + g(t,z_t) \\ + \displaystyle\sum_{0<t_k<t} \mathbf{\Phi}(t,t_k)J_k(z(t_k)), & t \in [0,\tau], \\ \eta(t) - [h(z)](t), & t \in [-r,0]. \end{cases}$$

We note that if $r = 0$, $h(z) = 0$, $z(t_k^+) - z(t_k^-) = 0, k \in I_p$, and $g = 0$, then system (2.27) reduces to (2.6).

Anguraj & Karthikeyan [6] used the Banach contraction theorem to show the existence of the infinite-dimensional version of (2.27). The non impulsive and non retarded infinite-dimensional version of (2.27) was studied by [36] by means of a fixed point theorem developed by Sadovskii [122]. In [41], Cuevas *et al.* investigated the existence of solutions for the infinite dimensional local version of system (2.27) by means of the Leray-Schauder alternative theorem [130, Lem. 2.2]. Hernández & Henríquez [66] derived existence results for the local and non impulsive infinite dimensional version of (2.27) using Sadovskii's fixed point theorem [122]. Later, Hernández [64] studied the same case, but he did consider impulses.

System (2.27) is the one we investigate in this project.

2.3 Controllability of Linear Systems

In this section, we briefly introduce notions about control theory. We mainly focus on the concept of controllability and its characterization for linear systems. The main references are [40, 43, 82, 127].

2.3.1 Control Theory

Control theory is the area of applied mathematics that deals with the behaviour of dynamical systems. The main objective of control theory is to answer the question of whether or not it is possible to reach a desired state from an initial state in a controlled, stable, and optimal fashion [42, Ch. 11, pp. 220]. Surprisingly, the underlying meaning of this question was addressed by Aristotle (384-322 BC) [5, pp. 1] as shown in the following excerpt. According to Bennett [24], Aristotle wrote

> "... if every instrument could accomplish its own work, obeying or anticipating the will of others ... if the shuttle weaved and the pick touched the lyre without a hand to guide them, chief workmen would not need servants, nor masters slaves."
>
> Aristotle, *Politics*, Book 1, chapter 3

The modern control theory was started by Kálmán [72, 73, 74] in 1960. He introduced the concept of controllability. This is the area of control theory that we mainly focus on in this project.

2.3.2 Characterization of the Controllability of Linear Systems

In order to define the concept of controllability we consider the following linear system

$$\frac{d}{dt}z(t) = A(t)z(t) + B(t)u(t),\ t \in [0,\tau], \tag{2.28}$$

where $A(t)$, $B(t)$, and u are as in system (1.2).

The following definition can be found in [72, Def. 5.1].

Definition 13. *The system (2.28) is said to be (exactly) controllable on $[0,\tau]$ if for every $z_0, z_1 \in \mathbb{R}^n$, there exists a control $u \in L^2\left([0,\tau], \mathbb{R}^m\right)$ such that the corresponding solution z of (2.28) with initial condition $z(0) = z_0$ satisfies $z(\tau) = z_1$.*

When speaking of controllability, we can distinguish two concepts: exact controllability as in Definition 13 and approximate controllability. To appreciate the difference between the two, we use system (1.2) to rephrase Definition 13 and present the definition of approximate controllability.

Definition 14. *System (1.2) is said to be (exactly) controllable on $[0,\tau]$ if and only if for all $\eta \in PW_r$[18] and $z_1 \in \mathbb{R}^n$ there exists $u \in L^2\left([0,\tau], \mathbb{R}^m\right)$ such that the solution z of (1.2) corresponding to u verifies*

$$z(0) + [h(z_{\tau_1}, \ldots, z_{\tau_q})](0) = \eta(0) \text{ and } z(\tau) = z_1.$$

Definition 15. *System (1.2) is said to be approximately controllable on $[0,\tau]$ if and only if for all $\eta \in PW_r$, $z_1 \in \mathbb{R}^n$, and $\epsilon > 0$ there exists $u \in L^2\left([0,\tau], \mathbb{R}^m\right)$ such that the solution z of (1.2) corresponding to u verifies*

$$z(0) + [h(z_{\tau_1}, \ldots, z_{\tau_q})](0) = \eta(0) \text{ and } \left\|z(\tau) - z_1\right\|_{\mathbb{R}^n} < \epsilon.$$

In what follows we give some useful characterizations for the controllability of system (2.28). From Theorem 9, we know that system (2.28) has a unique solution given by

$$z(t) = \mathbf{\Phi}(t,0)z_0 + \int_0^t \mathbf{\Phi}(t,\theta)B(\theta)u(\theta)d\theta,\ t \in [0,\tau] \tag{2.29}$$

when subjected to the initial condition $z(0) = z_0 \in \mathbb{R}^n$. This result leads us to the definition of three particular operators (see, *e.g.*, [43, Def. 4.1.3, pp. 143]).

[18] The meaning of PW_r will be given in Section 3.1.

Definition 16. *The mapping*

$$\begin{array}{rccl} C: & L^2\left([0,\tau],\mathbb{R}^m\right) & \longrightarrow & \mathbb{R}^n \\ & u & \longmapsto & C(u) = \int_0^\tau \mathbf{\Phi}(\tau,\theta)B(\theta)u(\theta)d\theta \end{array}$$

is called the controllability operator, *whose adjoint operator C^* is the mapping*

$$\begin{array}{rccl} C^*: & \mathbb{R}^n & \longrightarrow & L^2\left([0,\tau],\mathbb{R}^m\right) \\ & z & \longmapsto & C^*(z) \end{array}$$

given by $[C^*(z)](t) = B^*(t)\mathbf{\Phi}^*(\tau,t)z,\ t \in [0,\tau]$. The third operator is the Gramian operator $W := CC^*$ *defined as*

$$\begin{array}{rccl} W: & \mathbb{R}^n & \longrightarrow & \mathbb{R}^n \\ & z & \longmapsto & W(z) = \int_0^\tau \mathbf{\Phi}(\tau,\theta)B(\theta)B^*(\theta)\mathbf{\Phi}^*(\tau,\theta)zd\theta \end{array}$$

If $z(\tau) = z_1$, then by (2.29) and Definition 16 we obtain

$$z_1 - \mathbf{\Phi}(\tau,0)z_0 = Cu.$$

From this expression, we see that the controllability of system (2.28) is closely related with the surjectivity of C. In fact, the next lemma (see [69, Sec. 4, Th. 1 & Th. 2]) confirms such relation and provides others.

Lemma 2. *The following statements are equivalent.*

(i) *The system (2.28) is controllable on $[0,\tau]$.*

(ii) $\operatorname{Ran}(C) = \mathbb{R}^n$.

(iii) $\ker(C^*) = \{0\}$.

(iv) $\exists \gamma > 0, \forall z \in \mathbb{R}^n \setminus \{0\} : (Wz,z) \geq \gamma \|z\|^2$.

(v) *W is invertible.*

See Curtain & Zwart [43, Th. 4.1.7, pp. 147] for a proof.

By Lemma 2, the operator

$$\begin{array}{rccl} S: & \mathbb{R}^n & \longrightarrow & L^2\left([0,\tau],\mathbb{R}^m\right) \\ & z & \longmapsto & S(z) \end{array}$$

given by $[S(z)](t) = B^*(t)\mathbf{\Phi}^*(\tau,t)W^{-1}z = C^*\left(CC^*\right)^{-1}z,\ t \in [0,\tau]$ is well defined. It is called the *steering operator* and it is a right inverse of C in the sense that $CS = I$. Moreover,

$$\|W^{-1}z\| = \|(CC^*)^{-1}z\| \leq \gamma^{-1}\|z\|,\ z \in \mathbb{R}^n, \tag{2.30}$$ [19]

[19] This inequality follows from Lemma 2*(iv)* and CBS inequality (2.1).

$$\|z\|\|W^{-1}z\| \geq (z, W^{-1}z) \geq \gamma\|W^{-1}z\|^2 \implies (2.30)$$

and a control steering system (2.28) from z_0 to z_1 at time $\tau > 0$ is given by

$$u(t) = B^*(t)\mathbf{\Phi}^*(\tau,t)W^{-1}(z_1 - \mathbf{\Phi}(\tau,0)z_0) = [S(z_1 - \mathbf{\Phi}(\tau,0)z_0)](t), \ t \in [0,\tau]. \quad (2.31)$$

The following lemma allows us to apply the theory developed in this subsection to control functions in $\mathcal{C}([0,\tau],\mathbb{R}^m)$.

Lemma 3. *Let D be any dense subspace of $L^2([0,\tau],\mathbb{R}^m)$. Then, system (2.28) is controllable with control $u \in L^2([0,\tau],\mathbb{R}^m)$ if and only if it is controllable with control $u \in D$, i.e.,*

$$\mathrm{Ran}(C) = \mathbb{R}^n \iff \mathrm{Ran}(C|_D) = \mathbb{R}^n,$$

where $C|_D$ is the restriction of C to D.

See Leiva [83, Lem. 2.3] for a proof.

When A and B are constant matrices, say $A(t) := A$ and $B(t) := B$, we have the following characterization for the controllability of system (2.28). It is known as the Kálmán's rank condition.

Theorem 10. *The system* (2.28) *is controllable if and only if*

$$\mathrm{rank}[B|AB|A^2B|\cdots|A^{n-1}B] = n$$

See Kálmán [72, Cor. 5.5] for a proof.

Several authors have addressed the problem of controllability. For instance, Chang *et al.* [37] investigated the controllability of the system (2.16) by means of Sadovskii's fixed point theorem [122]. They did so without requiring the compactness of the semigroup. In [84], Rothe's fixed point theorem was used to prove the exact controllability of the local version of system (2.16). Tomar & Sukavanam [131] addressed the approximate controllability of the infinite-dimensional local version of system (2.17). Leiva [83] derived the exact controllability of the local and non retarded version of system (2.22) by means of Rothe's fixed point theorem. Later, Leiva & Rojas [89] did the same, but this time they included nonlocal conditions. In [87], the exact controllability of system (2.22) was studied using Rothe's fixed point theorem, while in [88], the approximately controllability of the same system was assessed following the aforementioned scheme developed by Bashirov *et al.* The exact controllability of the infinite-dimensional local version of system (2.22) (with infinite delay) was shown in [136] via Schauder's fixed point theorem. Chalishajar [35] analyzed the exact controllability of the infinite-dimensional local version of system (2.27) (with infinite delay) without assuming the compactness of the associated semigroup. This assumption is not trivial since, otherwise, it is not possible to study exact controllability [26, 44, 132, 133].

.

2.4 Semigroup Theory

In this section, we briefly introduce the theory of semigroups of linear operators. Our main references are [43, 48, 55, 75, 79, 101, 113, 124, 135]. We will see that the notion of semigroup of linear operators is a quite natural extension of the exponential of a matrix to the exponential of a possible unbounded operator [135, pp. 35]. For now, we discuss the concept of a dynamical system to motivate that of a semigroup of bounded linear operators.

The evolution of a well-posed physical system in time is usually described by an IVP of the form

$$\begin{cases} \dfrac{d}{dt}z(t) =Az(t), \quad t \geq 0, \\ z(0) =z_0, \end{cases} \tag{2.32}$$

where $A : D(A) \to Z$ is a time-independent linear operator with domain $D(A) \subset Z$, Z is a Banach space, $z : \mathbb{R}_+ \to Z$ is the state of the system ($z(t)$ is the state at time t), and $z_0 \in D(A)$ is the initial state. The time-invariance of A reflects that of the underlying physical mechanism. The well-posedness assumption is in the sense of Hadamard [60]: there is a unique solution to the problem for some given class of initial data and the solution varies continuously with the initial data [79, pp. 21-22].

Let $T(t)$ transfer the state $z(s)$ at time s to the state $z(t + s)$ at time $t + s$. The assumption that A does not depend on time implies that $T(t)$ is independent of s. The solution $z(t + s)$ at time $t + s$ can be computed as $T(t + s)z_0$ or, alternatively, we can solve for $z(s) = T(s)z_0$, take this as initial data, and t units of time later the solution becomes $z(t + s) = T(t)T(s)z_0$. The uniqueness of the solution implies the semigroup property $T(t + s) = T(t)T(s),\ t, s \geq 0$ [55, pp. 5]. The requirement that the state varies continuously with the initial state z_0 implies that $T(t)$ is a continuous map on Z. For the initial condition $z(0) = z_0$ to be satisfied we must have $T(0) = I$ [43, pp. 15].

2.4.1 Strongly Continuous Semigroups

The foregoing discussion shows how the concept of a dynamical system leads naturally to the concept of a semigroup of bounded linear operators [43]. The following definition can be found in [79, Def. 2.1.1, pp. 23].

Definition 17. *Let Z be a Banach space and $T(t) := \{T(t)\}_{t\geq 0}$ be a family of operators in $\mathcal{B}(Z)$. $T(t)$ is called a* semigroup of bounded linear operators *on Z if and only if*

(i) $T(0) = I$,

(ii) **[Semigroup property]** $\forall t, s \geq 0 : T(t + s) = T(t)\,T(s)$.

If, in addition, $T(t)$ satisfies

(iii) $\forall z \in Z : \lim\limits_{t \to 0} \left\| T(t)z - z \right\|_Z = 0,$[20]

then $T(t)$ is called a strongly continuous semigroup *(or C_0 semigroup) on Z. Furthermore, if*

(iv) $\lim\limits_{t \to 0} \left\| T(t) - I \right\|_{\mathcal{B}(Z)} = 0,$[21]

then $T(t)$ is referred to as a uniformly continuous semigroup *in Z.*

By (2.5), it is clear that *(iv)* implies *(iii)*. This means that a uniformly continuous semigroup is always a strongly continuous semigroup.

The following definition can be found in [113, Def. 3.1, pp. 48].

Definition 18. *A strongly continuous semigroup $\{T(t)\}_{t \geq 0}$ is said to be compact on Z if and only if $T(t)$ is a compact operator (see Subsection 2.1.6) for every $t > 0$.*

Theorem 11. *Let $T(t)$ be a strongly continuous semigroup and $t_0 > 0$. If $T(t)$ is compact for $t > t_0$, then $T(t)$ is continuous in the uniform operator topology for $t > t_0$.*

See Pazy [113, Th. 3.2, pp. 48] for a proof.

The operator A in (2.32) plays an important role in the theory of semigroups.

Definition 19. *The* (infinitesimal) generator $A : D(A) \subset Z \longrightarrow Z$ *of a strongly continuous semigroup $\{T(t)\}_{t \geq 0}$ on a Banach space Z is the linear operator*

$$Az = \lim_{h \to 0} \frac{T(h)z - z}{h}$$

defined for every z in its domain

$$D(A) = \left\{ z \in Z \;\middle|\; \lim_{h \to 0} \frac{T(h)z - z}{h} \text{ exists} \right\}.$$

Some useful properties of strongly continuous semigroup are listed below.

Theorem 12. *Let $T(t)$ be a strongly continuous semigroup and let A be its infinitesimal generator with domain $D(A)$ in Z. Then the following properties hold.*

(i) *There exist constant $\omega \geq 0$ and $M \geq 1$ such that $\left\| T(t) \right\|_{\mathcal{B}(Z)} \leq M \exp(\omega t)$, $t \geq 0$.*

(ii) $\overline{D(A)} = Z$.

(iii) *A is a closed linear operator on $D(A)$.*

For $z \in Z$

[20] Or equivalently, the map $\mathbb{R}_+ \ni t \longmapsto T(t)z \in Z$ is right continuous at zero for every $z \in Z$.

[21] It is equivalent to say that the mapping $\mathbb{R}_+ \ni t \longmapsto T(t) \in \mathcal{B}(Z)$ is right continuous at zero.

(iv) The map $t \longrightarrow T(t)z$ is continuous from $[0, \infty)$ into Z.

For $z \in D(A)$

(v) $T(t)z \in D(A), t \geq 0$ is (strongly) differentiable in t and

$$\frac{d}{dt}T(t)z = AT(t)z = T(t)Az, \ t \geq 0.$$

For a proof of *(i)* in Theorem 12, see [113, Th. 2.2, pp. 4]. Item *(iv)* in Theorem 12 is proven in [113, Cor. 2.3, pp. 4]. The remaining items are proven in [79, Th. 2.2.1, pp. 27].

It can be shown (see, *e.g.*, [48, Th. 3.7, pp. 17]) that if A is a constant $n \times n$ matrix, then $T(t)$ is an uniformly continuous semigroup and

$$T(t) = \exp(At) = \sum_{k=0} \frac{(At)^k}{k!}.$$

In fact, $T(t) = \exp(At)$ is a uniformly continuous semigroup if and only if $A \in \mathcal{B}(Z)$. The notation $\exp(At)$ is usually kept even when A is not a bounded operator. Besides, it is useful for making explicit the generator. However, to avoid possible confusion, we will not follow this convention (except in Theorem 18).

Remark 2. *If $\omega = 0$ in Theorem 12(i), then $T(t)$ is called* uniformly bounded *since*

$$\|T(t)\|_{\mathcal{B}(Z)} \leq M, \ t \geq 0. \tag{2.33}$$

Remark 3. *Let $T(t)$ be a strongly continuous semigroup with infinitesimal generator A (and hence (i) in Theorem 12 holds). Then $S(t) = \exp(-\omega t)T(t)$ is a uniformly bounded C_0 semigroup with infinitesimal generator $A - \omega I$.*

Remark 4. *Problem (2.32) is usually referred to as an abstract Cauchy problem.*

Theorem 13. *Let A on $D(A)$ be the infinitesimal generator of a strongly continuous semigroup $T(t)$. Let $f : [0, \infty) \longrightarrow Z$ be a strongly continuously differentiable function. Then the Cauchy problem (2.32) has a unique solution $z(t) = T(t)z_0$. Also, the inhomogeneous Cauchy problem*

$$\begin{cases} \dfrac{d}{dt}z(t) = Az(t) + f(t), & t \geq t_0, \\ z(t_0) = z_0 \in D(A), \end{cases} \tag{2.34}$$

has the unique solution

$$z(t) = T(t - t_0)z_0 + \int_{t_0}^{t} T(t - s)f(s)ds, \ t \geq 0. \tag{2.35}$$

See Ladas & Lakshmikantham [79, Th. 2.2.2, pp. 29 & Th. 2.2.3, pp. 30].

From the comment preceding Remark 2, if A is a constant $n \times n$ matrix, then (2.35) is exactly (2.11).

2.4.2 Sectorial Operators and Analytic Semigroups

Recall that the *resolvent set* $\rho(A)$ of a linear operator A acting on a Banach space Z is the set of all complex numbers λ for which $(\lambda I - A)^{-1}$ exists as a bounded operator with dense domain in Z. The operator $(\lambda I - A)^{-1}$ is referred to as the *resolvent of* A and the set $\sigma(A) := \mathbb{C} \setminus \rho(A)$ as the *spectrum of* A. These three concepts come from one of the main branches of modern functional analysis called *spectral theory of linear operators*. For a deeper understanding of these concepts, we refer the reader to [78, Ch. 7, pp. 363].

The following theorems will be useful in the sequel.

Theorem 14. *Let A be a linear operator acting on a Banach space X. If $\rho(A) \neq \emptyset$, then A is closed.*

Proof. Let $(z_n)_{n\in\mathbb{N}} \subseteq D(A)$ be such that $z_n \to z$ and $Az_n \to y$. Since $\rho(A) \neq \emptyset$, it has at least one element, say λ. By the continuity of $(\lambda I - A)^{-1}$, we can write

$$z = \lim_{n\to\infty} (\lambda I - A)^{-1}(\lambda I - A)z_n = (\lambda I - A)^{-1} \lim_{n\to\infty} (\lambda z_n - Az_n) = (\lambda I - A)^{-1}(\lambda z - y)$$

Hence $z \in D(T)$ since $(\lambda I - A)^{-1}$ maps X into $D(A)$. Applying $(\lambda I - A)$ to the last expression yields $\lambda z - Az = \lambda z - y$, whence $Az = y$. The result then follows from Theorem 2. □

Theorem 15. *Let A be a linear operator. Then*

$$\forall \lambda, \mu \in \rho(A) : (\mu I - A)^{-1} - (\lambda I - A)^{-1} = (\mu - \lambda)(\mu I - A)^{-1}(\lambda I - A)^{-1}$$[22].

A proof of this theorem is provided in [78, Th. 7.4-1, pp. 379].

Theorem 16. *Let $A : D(A) \subset Z \longrightarrow Z$ be a linear operator. If the resolvent $(\lambda I - A)^{-1}$ of A is compact for some $\lambda \in \rho(A)$, then it is compact for all $\lambda \in \rho(A)$.*

Proof. Let $\lambda \in \rho(A)$ such that $(\lambda I - A)^{-1}$ is compact. By the resolvent equation (see Theorem 15),

$$(\mu I - A)^{-1} = (\lambda I - A)^{-1} + (\mu - \lambda)(\mu I - A)^{-1}(\lambda I - A)^{-1}$$

for any $\mu \in \rho(A)$. The result then follows from Lemma 1. □

We now introduce sectorial operators and analytic semigroups [63, Def. 1.3.1, pp. 18 & Def. 1.3.3, pp. 20].

Definition 20. *A closed densely defined operator A acting on a Banach space Z is a* sectorial operator *if and only if there exists $\phi \in (0, \pi/2), M \geq 1$, and $a \in \mathbb{R}$ such that*

$$S_{a,\phi} = \{\lambda \in \mathbb{C} \mid \phi \leq |\arg(\lambda - a)| \leq \pi, \lambda \neq a\} \subset \rho(A)$$

and

$$\forall \lambda \in S_{a,\phi} : \|(\lambda I - A)^{-1}\|_{\mathcal{B}(Z)} \leq \frac{M}{|\lambda - a|}.$$

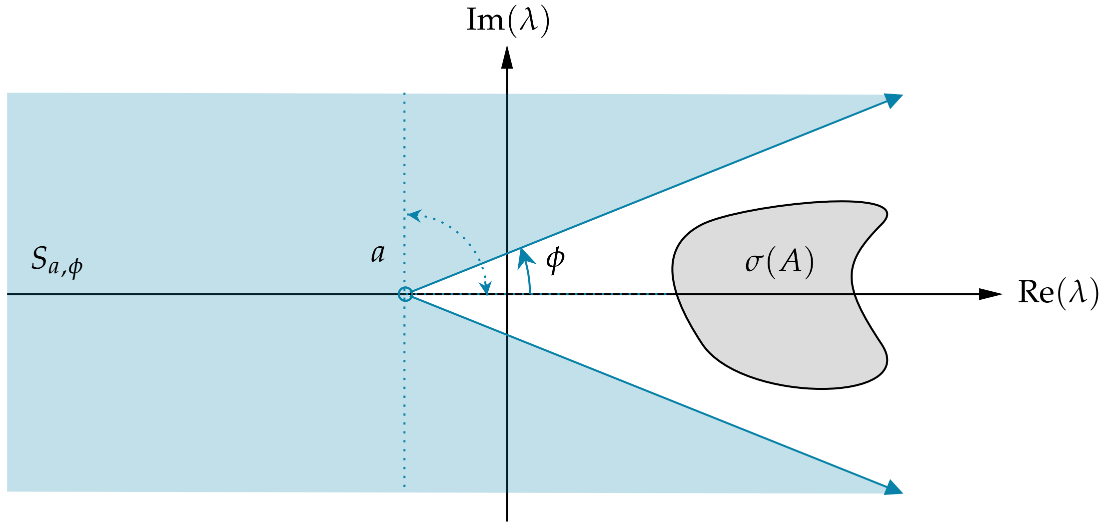

Figure 2.2: Sector $S_{a,\phi}$ in the complex plane $\mathbb{C}$. The dotted line is intended to show that ϕ ranges from 0 to $\pi/2$ but never reaches those points. We remark that the spectrum of A is not necessarily bounded.

See Figure 2.2 for a graphical representation of $S_{a,\phi}$.

Definition 21. *A strongly continuous semigroup $T(t)$ on a Banach space Z is an* analytic semigroup *if and only if*

$$\forall z \in Z : \text{the map } t \longrightarrow T(t)z \text{ is real analytic for } t > 0.$$

Theorem 17. *If A is a sectorial operator, then $-A$ is the infinitesimal generator of an analytic semigroup $T(t)$.*

See Henry [63, Th. 1.3.4, pp. 20] for a proof.

The following theorem essentially states that without loss of generality, we can always assume that $a = 0$ in Definition 20.

Theorem 18. *If A is a sectorial operator as in Definition 20, so is the operator*

$$\begin{aligned} B : \quad D(A) \subset Z &\longrightarrow Z \\ z &\longmapsto Bz = Az - az. \end{aligned}$$

and the following properties hold.

(i) If $\lambda + a \in \rho(A)$, then $\lambda \in \rho(B)$ and $(\lambda I - B)^{-1} = ((\lambda + a)I - A)^{-1}$.

(ii) $S_{0,\phi} \subset \rho(B)$.

(iii) If $\lambda \in S_{0,\phi}$, then $\|(\lambda I - B)^{-1}\|_{\mathcal{B}(Z)} \leq \dfrac{M}{|\lambda|}$.

[22]This identity is known as the *resolvent equation*.

(iv) *For* $t \geq 0$, $\exp(tB) \in \mathcal{B}(Z)$ *and* $\exp(tB) = e^{-at}\exp(tA)$.

See [102, Prop. 2.3.4, pp. 38] for a proof.

2.4.3 Fractional powers of Sectorial Operators

Having introduced the concept of sectorial operators, we now turn to the concept of fractional powers of sectorial operators. For doing so, we shall consider an operator $A^{-\alpha}$ and then define the fractional powers A^{α} of a sectorial operator A as $(A^{-\alpha})^{-1}$.

The following definition can be found in [63, Def. 1.4.1, pp. 24].

Definition 22. *For a sectorial operator* $A : D(A) \subset Z \longrightarrow Z$ *with* $\operatorname{Re}\sigma(A) > 0$[23] *and* $\alpha > 0$, *we define the following operator*

$$\begin{aligned} A^{-\alpha}: \quad D(A^{-\alpha}) \subset Z &\longrightarrow Z \\ z &\longmapsto A^{-\alpha}z = \frac{1}{\Gamma(\alpha)}\int_0^{\infty} t^{\alpha-1}T(t)zdt \end{aligned} \tag{2.36}$$

Theorem 19. *If* A *is a sectorial operator in* Z *with* $\operatorname{Re}\sigma(A) > 0$, *then*

(i) $\forall \alpha > 0 : A^{-\alpha} \in \mathcal{B}(Z)$.

(ii) $\forall \alpha > 0 : A^{-\alpha}$ *is one to one.*

(iii) $\forall \alpha, \beta > 0 : A^{-\alpha}A^{-\beta} = A^{-(\alpha+\beta)}$.

(iv) $\forall \alpha \in (0,1) : A^{-\alpha} = \dfrac{\sin(\pi\alpha)}{\pi}\displaystyle\int_0^{\infty} \lambda^{-\alpha}(\lambda I + A)^{-1}d\lambda$

See Henry [63, Th. 1.4.2, pp. 25] for a proof.

Definition 23. *Let* $\alpha > 0$. *Define the fractional powers of a sectorial operator* A *as* $A^{\alpha} = (A^{-\alpha})^{-1}$ *and* $A^0 = I$ *with domain* $D(A^{\alpha}) = R(A^{-\alpha})$. *If* $\alpha \in (0,1)$ *and* $z \in D(A) \subset D(A^{\alpha})$, *then* A^{α} *has explicit formulation*

$$A^{\alpha}z = \frac{\sin(\pi\alpha)}{\pi}\int_0^{\infty} \lambda^{\alpha-1}A(\lambda I + A)^{-1}zd\lambda$$

A proof of the last part of this definition can be found in [113, Th. 6.9, pp. 72].

Theorem 20. *Let* A^{α} *be defined as above. Then*

(i) $\forall \alpha > 0 : A^{\alpha}$ *is a closed operator.*

(ii) $\forall \alpha \geq \beta > 0 : D(A^{\alpha}) \subset D(A^{\beta})$.

(iii) $\forall \alpha \geq 0 : \overline{D(A^{\alpha})} = Z$.

[23] By $\operatorname{Re}\sigma(A) > 0$, we mean that $\forall \lambda \in \sigma(A) : \operatorname{Re}(\lambda) > 0$.

(iv) $\forall \alpha, \beta \in \mathbb{R} : A^\alpha A^\beta = A^\beta A^\alpha = A^{\alpha+\beta}$ *on* $D(A^\gamma)$, *where* $\gamma = \max(\alpha, \beta, \alpha + \beta)$.

See Pazy [113, Th. 6.8, pp. 72] for a proof. In particular, item *(i)* in Theorem 20 follows from Theorem 14. Indeed, since A^α is invertible with bounded inverse $A^{-\alpha}$, $0 \in \rho(A)$. The conclusion is now immediate.

Theorem 21. *Let* $-A$ *be the infinitesimal generator of an analytic semigroup* $T(t)$. *If* $0 \in \rho(A)$, *then*

(i) $\forall t > 0, \forall \alpha \geq 0 : T(t) : Z \to D\left(A^\alpha\right)$.

(ii) $\forall z \in D\left(A^\alpha\right) : T(t)A^\alpha z = A^\alpha T(t)z$.

A proof of this theorem is provided in [113, Th. 6.13, pp. 74].

Remark 5. *Without loss of generality we can always assume that* $0 \in \rho(A)$. *If* $0 \notin \rho(A)$, *then instead of* A *we work with* $A - \sigma I$ *since* $0 \in \rho(A - \sigma I)$ *is always possible for* σ *large enough.*

Theorem 22. *Suppose* $A : D(A) \subset Z \longrightarrow Z$ *is a sectorial operator with* $\operatorname{Re}\sigma(A) > \delta > 0$. *For* $\alpha \geq 0, t > 0$, *there exists a finite constant* M_α *such that*

$$\|A^\alpha T(t)\| \leq M_\alpha t^{-\alpha} \exp(-\delta t) \leq M_\alpha t^{-\alpha} \tag{2.37}$$

i.e., the operator $A^\alpha T(t)$ *is bounded.*

A proof of this theorem can be found in [63, Th. 1.4.3, pp. 26].

Proposition 3. *Suppose* A *is sectorial with* $\operatorname{Re}\sigma(A) > 0$. *Then the following are equivalent.*

(i) A^{-1} *is compact .*

(ii) $A^{-\alpha}$ *is compact for all* $\alpha > 0$.

(iii) $T(t)$ *is compact for* $t > 0$.

Proof. *(i)* $\Longrightarrow$ *(iii)*. By Theorem 22, $AT(t), t > 0$ is bounded. Since A^{-1} is compact and $T(t) = A^{-1}AT(t), t > 0$, by Lemma 1, $T(t)$ is compact for $t > 0$. *(iii)* $\Longrightarrow$ *(ii)*. This can be seen from (2.36) since the integral converges in the uniform operator topology [51]. *(ii)* $\Longrightarrow$ *(i)*. This follows immediately with $\alpha = 1$. □

Definition 24. *If* A *is a sectorial operator in a Banach space* Z, *we define, for* $\alpha \geq 0$

$$Z^\alpha = D\left(A^\alpha\right)$$

with the graph norm,

$$\|z\|_{A^\alpha} = \|z\| + \|A^\alpha z\|, \; z \in Z^\alpha, \tag{2.38}$$

such that $\operatorname{Re}\sigma\left(A\right) > 0$.

It is customary to endow Z^α with the norm

$$|z|_\alpha = \|A^\alpha z\|, \; z \in Z^\alpha \tag{2.39}$$

(also referred to as the graph norm) instead of the graph norm (2.38). This convention (which we will adopt) is based on the equivalence[24] of both norms. Indeed, since A^α has bounded inverse $A^{-\alpha}$ (and hence (2.5) holds), we have that

$$|z|_\alpha \leq \|z\| + \|A^\alpha z\| = \|A^{-\alpha} A^\alpha z\| + \|A^\alpha z\| \leq (1 + \|A^{-\alpha}\|)|z|_\alpha,$$

and therefore both norms are equivalent. The space Z^α is usually referred to as the *fractional power spaces of* A. Sometimes, it is also called as the *interpolation space between* $D(A)$ *and* Z since

$$D(A) \subset Z^\alpha \subset Z^\beta \subset Z \tag{2.40}$$

for $0 \leq \beta < \alpha \leq 1$ (with $Z^0 = Z$).

Theorem 23. *If A is a sectorial operator in a Banach space Z, then Z^α is a Banach space with norm $|\cdot|_\alpha$ for $\alpha \geq 0, Z^0 = Z$ and, for $\alpha \geq \beta \geq 0, Z^\alpha$ is a dense subspace of Z^β with continuous inclusion. If the resolvent of A is compact and $0 \in \rho(A)$, then the inclusion $Z^\alpha \subset Z^\beta$ is compact when $\alpha > \beta \geq 0$.*

Proof. Let A be a sectorial operator acting on a Banach space Z. First, Z^α endowed with the norm (2.38) (and hence with (2.39)) is a Banach space as a consequence of Proposition 1. Second, the density[25] of Z^α in Z^β follows from Theorem 20*(iii)* and (2.40) since $Z^\beta \subset Z = \overline{Z^\alpha}$. Third, by Theorem 20*(iv)*, $A^\beta z = A^{\beta-\alpha} A^\alpha z$, $z \in Z^\alpha$. Since $A^{\beta-\alpha}$ is bounded, it follows that

$$|Iz|_\beta = |z|_\beta = \|A^\beta z\| \leq \|A^{\beta-\alpha}\| \|A^\alpha z\| = c|z|_\alpha, \tag{2.41}$$

where $c = \|A^{\beta-\alpha}\|$ and $I : Z^\alpha \to Z^\beta$ is the identity operator. Inequality (2.41) shows that I is bounded[26] and therefore the inclusion $Z^\alpha \subset Z^\beta$ is continuous. For the last part, the compactness inclusion, we need to prove that (see Theorem 3)

$$\forall\, (z_n)_{n\in\mathbb{N}} \subseteq Z^\alpha \text{ bounded} : (Iz_n)_{n\in\mathbb{N}} \subseteq Z^\beta \text{ has a convergent subsequence.}$$

Let $(z_n)_{n\in\mathbb{N}} \subseteq Z^\alpha \subset Z^\beta$ be a bounded sequence, say $\exists M \geq 0, \forall n \in \mathbb{N} : |z_n|_\alpha \leq M$. By (2.41), $\|A^\beta z_n\| \leq cM$ and so the sequence $(A^\beta z_n)_{n\in\mathbb{N}}$ is also bounded. Since the resolvent of A is compact and $0 \in \rho(A)$, we have that A^{-1} is compact. Therefore, by Proposition 3, $A^{-\beta}$ is compact and hence $A^{-\beta} A^\beta z_n = Iz_n$ has a convergent subsequence. This concludes the proof. □

[24]See at the end of Subsection 2.1.2.
[25]See at the end of Subsection 2.1.1.
[26]See Subsection 2.1.4.

Chapter 3

Results in finite-dimensional systems

In this chapter, we present our research results in a finite-dimensional setting.

3.1 Existence results

This section is devoted to study the existence and uniqueness of solutions for the following semilinear neutral differential equation in $\mathbb{R}^n$ with impulses and nonlocal conditions.

$$\begin{cases} \dfrac{d}{dt}\left[z(t) - g(t, z_t)\right] = A(t)z(t) + f(t, z_t), & t \in [0,\tau] \setminus \{t_k\}_{k \in I_p}, \\ z(\theta) = -[h(z_{\tau_1}, \dots, z_{\tau_q})](\theta) + \eta(\theta), & \theta \in [-r, 0], \\ z(t_k^+) = z(t_k^-) + J_k(t_k, z(t_k)), & k \in I_p, \end{cases} \tag{3.1}$$

where $A(t)$ is a $n \times n$ continuous matrix, $0 < t_1 < \cdots < t_p < \tau$, $0 < \tau_1 < \cdots < \tau_q < r < \tau$, z_t is the function $[-r, 0] \ni \theta \mapsto z_t(\theta) = z(t+\theta) \in \mathbb{R}^n$, $h : PW_{qp} \to PW_r$ is the nonlocal function, $J_k : [0,\tau] \times \mathbb{R}^n \to \mathbb{R}^n$ is the impulsive function, $g, f : [0,\tau] \times PW_r \to \mathbb{R}^n$ are appropriate functions to be specified later, and η belongs to the Banach space (see, *e.g.*, [58, 97, 98])

$$\begin{aligned} PW\left([-r,0], \mathbb{R}^n\right) = \Big\{ \eta : [-r,0] \to \mathbb{R}^n \;\Big|\; & \eta \text{ is continuous except at the points } \theta_k, \\ & \text{where the one-sided limits } \eta(\theta_k^-) \text{ and } \eta(\theta_k^+) \text{ exist with} \\ & \eta(\theta_k^+) = \eta(\theta_k) \text{ for all } k \in I_p \Big\} \end{aligned} \tag{3.2}$$

provided with the norm

$$\|\eta\|_r = \sup_{\theta \in [-r,0]} \|\eta(\theta)\|_{\mathbb{R}^n}.$$

In a similar way as (3.2) was defined, we consider the space $PW_\tau\left([0,\tau], \mathbb{R}^n\right)$ equipped with the supremum norm $\|\cdot\|_\tau$. In the sequel, for the sake of simplicity we will write

$$PW_r := PW\left([-r,0], \mathbb{R}^n\right) \text{ and } PW_\tau := PW\left([0,\tau], \mathbb{R}^n\right).$$

Now, we define the natural Banach space where the solutions of problem (1.1) will take place [88].

$$PW_p := \left\{ z : [-r, \tau] \to \mathbb{R}^n \;\middle|\; z|_{[-r,0]} \in PW_r \text{ and } z|_{[0,\tau]} \in PW_\tau \right\}$$

endowed with the supremum norm $\| \cdot \|_p$. We will also consider

$$(\mathbb{R}^n)^q = \underbrace{\mathbb{R}^n \times \mathbb{R}^n \times \cdots \times \mathbb{R}^n}_{q-\text{times}}$$

equipped with the norm

$$\|y\|_q = \sum_{i=1}^{q} \|y_i\|_{\mathbb{R}^n}, \; y = \left(y_1, \ldots, y_q\right) \in (\mathbb{R}^n)^q .$$

Similarly to PW_r and PW_τ, we define the Banach space $PW_{qp} := PW\left([-r,0], (\mathbb{R}^n)^q\right)$ endowed with the norm

$$\|\eta\|_{qp} = \sup_{t\in[-r,0]} \|\eta(t)\|_q = \sup_{t\in[-r,0]} \left(\sum_{i=1}^{q} \|\eta_i(t)\|_{\mathbb{R}^n} \right), \; \eta = \left(\eta_1, \ldots, \eta_q\right) \in PW_{qp}.$$

3.1.1 Formula for the solutions of system (3.1).

We devote this subsection to find a formula for solutions of the semilinear neutral differential equations with impulses and nonlocal conditions (3.1). Specifically, we transform problem (3.1) into an integral differential equation problem, which allows us to apply Karakosta's fixed point theorem to prove the existence of solutions for (3.1) in the next section.

From now on, we adopt the notation (introduced in Subsection 2.2.3) $[h(z)](t) = [h(z_{\tau_1}, \ldots, z_{\tau_q})](t)$ to indicate the value of the function in $\mathbb{R}^n$ and $h(z) = h(z_{\tau_1}, \ldots, z_{\tau_q})$ to denote the function in PW_r.

Following the ideas presented in Section 2.2, specifically at the end, we state and prove the following proposition.

Proposition 4. *The system (3.1) has solution z on $[-r, \tau]$ if and only if z is a solution of the following integral equation*

$$z(t) = \begin{cases} \mathbf{\Phi}(t,0)[\eta(0) - [h(z)](0) - g\left(0, \eta - h(z)\right)] \\ + \displaystyle\int_0^t \mathbf{\Phi}(t,\theta) \left[A(\theta) g(\theta, z_\theta) + f(\theta, z_\theta)\right] d\theta + g(t, z_t) \\ + \displaystyle\sum_{0<t_k<t} \mathbf{\Phi}(t, t_k) J_k(t_k, z(t_k)), & t \in [0, \tau], \\ \eta(t) - [h(z)](t), & t \in [-r, 0]. \end{cases} \tag{3.3}$$

Proof.
($\implies$) Suppose that z is a solution for system (3.1) on $[-r,\tau]$. Let $z_0 = \eta(0) - [h(z)](0)$. On $[0,t_1)$, z is the solution of the system

$$\begin{cases} \dfrac{d}{dt}[z(t) - g(t,z_t)] = A(t)z(t) + f(t,z_t), \quad t \in [0,t_1), \\ z(0) = z_0 \end{cases}$$

and by the VPF (see (2.27)) we therefore obtain

$$z(t) = g(t,z_t) + \mathbf{\Phi}(t,0)[z_0 - g\left(0,\eta - h(z)\right)] + \int_0^t \mathbf{\Phi}(t,\theta)[A(\theta)g(\theta,z_\theta) + f(\theta,z_\theta)]d\theta$$

for $t \in [0,t_1)$. As $t \to t_1^-$,

$$\begin{aligned} z(t_1^-) =& g(t_1,z_{t_1}) + \mathbf{\Phi}(t_1,0)[z_0 - g\left(0,\eta - h(z)\right)] \\ &+ \int_0^{t_1} \mathbf{\Phi}(t_1,\theta)\left[A(\theta)g(\theta,z_\theta) + f(\theta,z_\theta)\right] d\theta. \end{aligned} \tag{3.4}$$

In the same way, on $[t_1,t_2)$ z is the solution of the system

$$\begin{cases} \dfrac{d}{dt}[z(t) - g(t,z_t)] = A(t)z(t) + f(t,z_t), \quad t \in [t_1,t_2), \\ z(t_1) = z(t_1^+) \end{cases}$$

and again the VPF yields

$$z(t) = g(t,z_t) + \mathbf{\Phi}(t,t_1)[z(t_1) - g\left(t_1,\eta - h(z)\right)] + \int_{t_1}^t \mathbf{\Phi}(t,\theta)[A(\theta)g(\theta,z_\theta) + f(\theta,z_\theta)]d\theta$$

for $t \in [t_1,t_2)$. Now, since $z(t_1^+) = z(t_1^-) + J_1(t_1,z(t_1))$, we obtain that

$$\begin{aligned} z(t) =& g(t,z_t) + \mathbf{\Phi}(t,t_1)\Big\{z(t_1^+) - g\left(t_1,\eta - h(z)\right)\Big\} \\ &+ \int_{t_1}^t \mathbf{\Phi}(t,\theta)[A(\theta)g(\theta,z_\theta) + f(\theta,z_\theta)]d\theta \\ =& g(t,z_t) + \mathbf{\Phi}(t,t_1)\Big\{z(t_1^-) + J_1(t_1,z(t_1)) - g\left(t_1,\eta - h(z)\right)\Big\} \\ &+ \int_{t_1}^t \mathbf{\Phi}(t,\theta)[A(\theta)g(\theta,z_\theta) + f(\theta,z_\theta)]d\theta. \end{aligned}$$

As a consequence of (3.4),

$$\begin{aligned}
z(t) =& g(t, z_t) + \mathbf{\Phi}(t, t_1)\Big\{ g(t_1, z_{t_1}) + \mathbf{\Phi}(t_1, 0)[z_0 - g\left(0, \eta - h(z)\right)] \\
&+ \int_0^{t_1} \mathbf{\Phi}(t_1, \theta)[A(\theta)g(\theta, z_\theta) + f(\theta, z_\theta)]d\theta + J_1(t_1, z(t_1)) \\
&- g\left(t_1, \eta - h(z)\right) \Big\} + \int_{t_1}^{t} \mathbf{\Phi}(t, \theta)[A(\theta)g(\theta, z_\theta) + f(\theta, z_\theta)]d\theta. \\
=& g(t, z_t) + \mathbf{\Phi}(t, t_1)\Big\{ \mathbf{\Phi}(t_1, 0)[z_0 - g\left(0, \eta - h(z)\right)] \\
&+ \int_0^{t} \mathbf{\Phi}(t_1, \theta)[A(\theta)g(\theta, z_\theta) + f(\theta, z_\theta)]d\theta + J_1(t_1, z(t_1)) \Big\} \\
&+ \int_{t_1}^{t} \mathbf{\Phi}(t, \theta)[A(\theta)g(\theta, z_\theta) + f(\theta, z_\theta)]d\theta, \ t \in [t_1, t_2).
\end{aligned}$$

Using the cocycle property of $\mathbf{\Phi}$ yields

$$\begin{aligned}
z(t) =& g(t, z_t) + \mathbf{\Phi}(t, 0)[z_0 - g\left(0, \eta - h(z)\right)] \\
&+ \int_0^{t_1} \mathbf{\Phi}(t, \theta)[A(\theta)g(\theta, z_\theta) + f(\theta, z_\theta)]d\theta + \mathbf{\Phi}(t, t_1)J_1(t_1, z(t_1)) \\
&+ \int_{t_1}^{t} \mathbf{\Phi}(t, \theta)[A(\theta)g(\theta, z_\theta) + f(\theta, z_\theta)]d\theta \\
=& g(t, z_t) + \mathbf{\Phi}(t, 0)[z_0 - g\left(0, \eta - h(z)\right)] \\
&+ \int_0^{t} \mathbf{\Phi}(t, \theta)[A(\theta)g(\theta, z_\theta) + f(\theta, z_\theta)]d\theta + \mathbf{\Phi}(t, t_1)J_1(t_1, z(t_1)), \ t \in [t_1, t_2). \quad (3.5)
\end{aligned}$$

As $t \to t_2^-$,

$$z(t_2^-) = g(t_2, z_{t_2}) + \mathbf{\Phi}(t_2, 0)[z_0 - g\left(0, \eta - h(z)\right)] \quad (3.6)$$

$$+ \int_0^{t_2} \mathbf{\Phi}(t_2, \theta)[A(\theta)g(\theta, z_\theta) + f(\theta, z_\theta)]d\theta + \mathbf{\Phi}(t_2, t_1)J_1(t_1, z(t_1)). \quad (3.7)$$

Accordingly, on $[t_2, t_3)$, z satisfies the system

$$\begin{cases} \dfrac{d}{dt}[z(t) - g(t, z_t)] = A(t)z(t) + f(t, z_t), \quad t \in [t_2, t_3), \\ z(t_2) = z(t_2^+) \end{cases}$$

and once again the VPF gives

$$\begin{aligned}
z(t) =& g(t, z_t) + \mathbf{\Phi}(t, t_2)[z(t_2) - g\left(t_2, \eta - h(z)\right)] \\
&+ \int_{t_2}^{t} \mathbf{\Phi}(t, \theta)[A(\theta)g(\theta, z_\theta) + f(\theta, z_\theta)]d\theta, \quad t \in [t_2, t_3).
\end{aligned}$$

In the same way as before, since $z(t_2^+) = z(t_2^-) + J_2(t_2, z(t_2))$, we have that

$$
\begin{aligned}
z(t) =& g(t, z_t) + \mathbf{\Phi}(t, t_2)\Big\{ z(t_2^+) - g\left(t_2, \eta - h(z)\right) \Big\} \\
&+ \int_{t_2}^{t} \mathbf{\Phi}(t,\theta)[A(\theta)g(\theta, z_\theta) + f(\theta, z_\theta)]d\theta \\
=& g(t, z_t) + \mathbf{\Phi}(t, t_2)\Big\{ z(t_2^-) + J_2(t_2, z(t_2)) - g\left(t_2, \eta - h(z)\right) \Big\} \\
&+ \int_{t_2}^{t} \mathbf{\Phi}(t,\theta)[A(\theta)g(\theta, z_\theta) + f(\theta, z_\theta)]d\theta).
\end{aligned}
$$

By (3.7),

$$
\begin{aligned}
z(t) =& g(t, z_t) + \mathbf{\Phi}(t, t_2)\Big\{ g(t_2, z_{t_2}) + \mathbf{\Phi}(t_2, 0)[z_0 - g\left(0, \eta - h(z)\right)] \\
&+ \int_0^{t_2} \mathbf{\Phi}(t_2,\theta)[A(\theta)g(\theta, z_\theta) + f(\theta, z_\theta)]d\theta + \mathbf{\Phi}(t_2, t_1)J_1(t_1, z(t_1)) + J_2(t_2, z(t_2)) \\
&- g\left(t_2, \eta - h(z)\right) \Big\} + \int_{t_2}^{t} \mathbf{\Phi}(t,\theta)[A(\theta)g(\theta, z_\theta) + f(\theta, z_\theta)]d\theta \\
=& g(t, z_t) + \mathbf{\Phi}(t, t_2)\Big\{ \mathbf{\Phi}(t_2, 0)[z_0 - g\left(0, \eta - h(z)\right)] \\
&+ \int_0^{t_2} \mathbf{\Phi}(t_2,\theta)[A(\theta)g(\theta, z_\theta) + f(\theta, z_\theta)]d\theta + \mathbf{\Phi}(t_2, t_1)J_1(t_1, z(t_1)) + J_2(t_2, z(t_2)) \Big\} \\
&+ \int_{t_2}^{t} \mathbf{\Phi}(t,\theta)[A(\theta)g(\theta, z_\theta) + f(\theta, z_\theta)]d\theta, \ t \in [t_2, t_3).
\end{aligned}
$$

Again, using the cocycle property of $\mathbf{\Phi}$ yields

$$
\begin{aligned}
z(t) =& g(t, z_t) + \mathbf{\Phi}(t, 0)[z_0 - g\left(0, \eta - h(z)\right)] \\
&+ \int_0^{t_2} \mathbf{\Phi}(t,\theta)[A(\theta)g(\theta, z_\theta) + f(\theta, z_\theta)]d\theta + \mathbf{\Phi}(t, t_1)J_1(t_1, z(t_1)) \\
&+ \mathbf{\Phi}(t, t_2)J_2(t_2, z(t_2)) + \int_{t_2}^{t} \mathbf{\Phi}(t,\theta)[A(\theta)g(\theta, z_\theta) + f(\theta, z_\theta)]d\theta \\
=& g(t, z_t) + \mathbf{\Phi}(t, 0)[z_0 - g\left(0, \eta - h(z)\right)] \\
&+ \int_0^{t} \mathbf{\Phi}(t,\theta)[A(\theta)g(\theta, z_\theta) + f(\theta, z_\theta)]d\theta + \sum_{k=1}^{2} \mathbf{\Phi}(t, t_k)J_k(t_k, z(t_k)), \ t \in [t_2, t_3).
\end{aligned}
$$

Proceeding inductively as above, for $t \in [t_p, t_{p+1})$ we have that

$$
\begin{aligned}
z(t) =& g(t, z_t) + \mathbf{\Phi}(t, 0)[z_0 - g\left(0, \eta - h(z)\right)] \\
&+ \int_0^{t} \mathbf{\Phi}(t,\theta)[A(\theta)g(\theta, z_\theta) + f(\theta, z_\theta)]d\theta + \sum_{k=1}^{p} \mathbf{\Phi}(t, t_k)J_k(t_k, z(t_k)), \ t \in [t_p, t_{p+1}).
\end{aligned}
$$

Therefore

$$\begin{aligned} z(t) =& g(t,z_t) + \mathbf{\Phi}(t,0)[\eta(0) - [h(z)](0) - g\left(0,\eta - h(z)\right)] \\ &+ \int_0^t \mathbf{\Phi}(t,\theta)[A(\theta)g(\theta,z_\theta) + f(\theta,z_\theta)]d\theta + \sum_{0<t_k<t} \mathbf{\Phi}(t,t_k)J_k(t_k,z(t_k)),\ t \in [0,\tau]. \end{aligned}$$

This shows that z satisfies (3.3).
($\Longleftarrow$) Assume that z is solution of the integral equation (3.3). On the one hand, we first show that $(3.1)_3$ is satisfied. For doing so, we notice that at t_1 we have that

$$\begin{aligned} z(t_1^-) =& g(t_1,z_{t_1}) + \mathbf{\Phi}(t_1,0)[z_0 - g\left(0,\eta - h(z)\right)] \\ &+ \int_0^{t_1} \mathbf{\Phi}(t_1,\theta)[A(\theta)g(\theta,z_\theta) + f(\theta,z_\theta)]d\theta, \end{aligned}$$

$$\begin{aligned} z(t_1^+) =& g(t_1,z_{t_1}) + \mathbf{\Phi}(t_1,0)[z_0 - g\left(0,\eta - h(z)\right)] \\ &+ \int_0^{t_1} \mathbf{\Phi}(t_1,\theta)[A(\theta)g(\theta,z_\theta) + f(\theta,z_\theta)]d\theta + \mathbf{\Phi}(t_1,t_1)J_1(t_1,z(t_1)) \end{aligned}$$

by (3.4) and taking $t \to t_1^+$ in (3.5), respectively. Summing up both expressions above we get that

$$z(t_1^+) = z(t_1^-) + J_1(t_1,z(t_1))$$

since (see Proposition 2(*i*)) $\mathbf{\Phi}(t_1,t_1) = I$. Similarly, at t_2 we have that

$$\begin{aligned} z(t_2^-) =& g(t_2,z_{t_2}) + \mathbf{\Phi}(t_2,0)[z_0 - g\left(0,\eta - h(z)\right)] \\ &+ \int_0^{t_2} \mathbf{\Phi}(t_2,\theta)[A(\theta)g(\theta,z_\theta) + f(\theta,z_\theta)]d\theta + \mathbf{\Phi}(t_2,t_1)J_1(t_1,z(t_1)), \end{aligned}$$

$$\begin{aligned} z(t_2^+) =& g(t_2,z_{t_2}) + \mathbf{\Phi}(t_2,0)[z_0 - g\left(0,\eta - h(z)\right)] \\ &+ \int_0^{t_2} \mathbf{\Phi}(t_2,\theta)[A(\theta)g(\theta,z_\theta) + f(\theta,z_\theta)]d\theta + \mathbf{\Phi}(t_2,t_1)J_1(t_1,z(t_1)) \\ &+ \mathbf{\Phi}(t_2,t_2)J_2(t_2,z(t_2)), \end{aligned}$$

which implies that

$$z(t_2^+) = z(t_2^-) + J_2(t_2,z(t_2)).$$

Proceeding inductively as above, we get that

$$z(t_k^+) = z(t_k^-) + J_k(t_k,z(t_k)),\ k \in I_p.$$

On the other hand, differentiating z with respect to t, for $t \in [0,\tau) \setminus \{t_k\}_{k\in I_p}$, we obtain that

$$\begin{aligned}\frac{d}{dt}\left(z(t)\right) =&\frac{d}{dt}\Bigg(g(t,z_t)+\mathbf{\Phi}(t,0)\left[z_0-g(0,\eta-h(z))\right]\\&+\int_0^t \mathbf{\Phi}(t,\theta)\left[A(\theta)g(\theta,z_\theta)+f(\theta,z_\theta)\right]d\theta+\sum_{0<t_k<t}\mathbf{\Phi}(t,t_k)J_k(t_k,z(t_k))\Bigg)\\=&\frac{d}{dt}g(t,z_t)+A(t)\mathbf{\Phi}(t,0)[z_0-g\left(0,\eta-h(z)\right)]\\&+A(t)\int_0^t\mathbf{\Phi}(t,\theta)[A(\theta)g(\theta,z_\theta)+f(\theta,z_\theta)]d\theta+A(t)g(t,z_t)+f(t,z_t)\\&+A(t)\sum_{0<t_k<t}\mathbf{\Phi}(t,t_k)J_k(t_k,z(t_k)),\end{aligned}$$

where we have used Proposition 2(*iii*) and the Leibniz product rule to differentiate the integral term. By rearranging terms it finally follows that

$$\begin{aligned}\frac{d}{dt}\left[z(t)-g(t,z_t)\right] =&A(t)\Bigg\{g(t,z_t)+\mathbf{\Phi}(t,0)\left[z_0-g(0,\eta-h(z))\right]\\&+\int_0^t\mathbf{\Phi}(t,\theta)\left[A(\theta)g(\theta,z_\theta)+f(\theta,z_\theta)\right]d\theta\\&+\sum_{0<t_k<t}\mathbf{\Phi}(t,t_k)J_k(t_k,z(t_k))\Bigg\}+f(t,z_t)\\=&A(t)z(t)+f(t,z_t),\end{aligned}$$

that is to say, z is a solution of (3.1). □

3.1.2 Existence Theorems

In this section we shall prove our main result about the existence and uniqueness of solutions for the semilinear neutral equation with impulses and nonlocal conditions (3.1). To achieve this, we consider the following hypotheses[1] on the terms involving the system (3.1).

[H1] There exist positive constants L_g, γ, and $d_k, k \in I_p$ such that

(*i*) $L_g qM < \gamma + M\sum_{k=1}^{p} d_k < \frac{1}{2}$,

(*ii*) $J_k(t,0)=0$ and $\left\|J_k(t,y)-J_k(t,z)\right\|_{\mathbb{R}^n} \le d_k\left\|y-z\right\|_{\mathbb{R}^n},\ y,z\in\mathbb{R}^n,\ t\in[0,\tau]$,

[1]Each set of hypotheses is independent for each section.

(*iii*) $h(0) \equiv 0$ and

$$\left\|[h(u)](t) - [h(v)](t)\right\|_{\mathbb{R}^n} \leq L_g \sum_{i=1}^{q} \left\|u_i(t) - v_i(t)\right\|_{\mathbb{R}^n},\ t \in [-r, 0],\ u, v \in PW_p,$$

where M is given in Proposition 2(*v*).

[H2] The function g satisfies

(*i*) $\|A(t)g(t,\eta_1) - A(t)g(t,\eta_2)\|_{\mathbb{R}^n} \leq \mathcal{K}\left(\|\eta_1\|_r, \|\eta_2\|_r\right)\|\eta_1 - \eta_2\|_r, \eta_1, \eta_2 \in PW_r,$

(*ii*) $\|g(t,\eta_1) - g(t,\eta_2)\|_{\mathbb{R}^n} \leq \gamma\|\eta_1 - \eta_2\|_r, \eta_1, \eta_2 \in PW_r,$

(*iii*) $\|A(t)g(t,\eta)\|_{\mathbb{R}^n} \leq \Psi\left(\|\eta\|_r\right), \eta \in PW_r,$

(*iv*) $\|g(t,\eta)\|_{\mathbb{R}^n} \leq \Psi\left(\|\eta\|_r\right), \eta \in PW_r$

and f satisfies

(*v*) $\|f(t,\eta_1) - f(t,\eta_2)\|_{\mathbb{R}^n} \leq \mathcal{K}\left(\|\eta_1\|_r, \|\eta_2\|_r\right)\|\eta_1 - \eta_2\|_r, \eta_1, \eta_2 \in PW_r,$

(*vi*) $\|f(t,\eta)\|_{\mathbb{R}^n} \leq \Psi\left(\|\eta\|_r\right), \eta \in PW_r,$

where $\mathcal{K} \in \mathcal{C}(\mathbb{R}_+ \times \mathbb{R}_+, \mathbb{R}_+)$ and $\Psi \in \mathcal{C}(\mathbb{R}_+, \mathbb{R}_+)$ are non decreasing functions. Items (*i*) and (*v*) are essentially local Lipschitz conditions because for each ball $B_R(\cdot)$ in PW_r containing η_1 and η_2, $\mathcal{K}(\|\eta_1\|, \|\eta_2\|)$ is bounded by the constant value $\mathcal{K}(\|\cdot\| + R, \|\cdot\| + R)$. The reason for using these conditions will be evident when applying our infinite-dimensional results (Chapter 4), which is an extension of the results in this chapter, to the Burgers equation (4.20).

[H3] There exists $\rho > 0$ such that

$$M\Psi\left(\|\eta\| + L_g q\left(\|\tilde{\eta}\| + \rho\right)\right) + \left(ML_g q + M\sum_{k=1}^{p} d_k\right)\left(\|\tilde{\eta}\| + \rho\right)$$
$$+ (2M\tau + 1)\Psi\left(\|\tilde{\eta}\| + \rho\right) < \rho,$$

where $\tilde{\eta}$ is a function given by

$$\tilde{\eta}(t) = \begin{cases} \mathbf{\Phi}(t,0)\eta(0), & t \in [0,\tau], \\ \eta(t), & t \in [-r,0]. \end{cases}$$

[H4] Assume the following relation holds

$$M\left\{L_g q\,(1+\gamma)+2\tau\mathcal{K}\left(\|\tilde{\eta}\|+\rho,\|\tilde{\eta}\|+\rho\right)\right\}<\frac{1}{2}.$$

Theorem 24. *Suppose that* **[H1]**,**[H2]**, *and* **[H3]** *hold. Then, the system* (3.1) *has at least one solution in* PW_p.

Proof. We shall transform the problem of proving the existence of solutions for system (3.1) into a fixed point problem. For this, we define two operators

$$\begin{array}{rccc}\mathcal{Q}: & PW_p\times PW_p & \longrightarrow & PW_p\\ & (z,y) & \longmapsto & \mathcal{Q}(z,y)\end{array}$$

defined by

$$[\mathcal{Q}(z,y)](t)=\begin{cases} y(t)+g(t,z_t)+\displaystyle\sum_{0<t_k<t}\mathbf{\Phi}(t,t_k)J_k(t_k,z(t_k)), & t\in[0,\tau],\\ \eta(t)-[h(z)](t), & t\in[-r,0]\end{cases}$$

and

$$\begin{array}{rccc}\mathcal{P}: & PW_p & \longrightarrow & PW_p\\ & y & \longmapsto & \mathcal{P}(y)\end{array}$$

given by

$$[\mathcal{P}(y)](t)=\begin{cases}\mathbf{\Phi}(t,0)[\eta(0)-[h(y)](0)-g\left(0,\eta-h(y)\right)]\\ +\displaystyle\int_0^t\mathbf{\Phi}(t,\theta)\left[A(\theta)g(\theta,y_\theta)+f(\theta,y_\theta)\right]d\theta, & t\in[0,\tau],\\ \eta(t), & t\in[-r,0].\end{cases}$$

We also consider the following closed and convex set

$$D=D(\rho,\tau,\eta)=\left\{y\in PW_p\;\middle|\;\|y-\tilde{\eta}\|_p\le\rho\right\}. \tag{3.8}$$

With this setting, the problem of finding solutions for system (3.1) has been reduced to the problem of finding solutions of the following operator equation

$$\mathcal{Q}\left(z,\mathcal{P}(z)\right)=z.$$

The rest of the proof will be given by steps as follows.

Step 1. *$\mathcal{P}$ is a continuous mapping.*

For any $z, y \in PW_p$ and $t \in [0, \tau]$ let us denote the difference $\|[\mathcal{P}(z)](t) - [\mathcal{P}(y)](t)\|$ as Π_1. From the definition of $\mathcal{P}$ we have that

$$\begin{aligned}\Pi_1 \leq & \|\mathbf{\Phi}(t,0)\|\Big\{\|[h(y)](0) - [h(z)](0)\| + \|g\left(0, \eta - h(y)\right) - g\left(0, \eta - h(z)\right)\|\Big\} \\ & + \int_0^t \|\mathbf{\Phi}(t,\theta)\|\Big\{\|A(\theta)g(\theta, z_\theta) - A(\theta)g(\theta, y_\theta)\| + \|f(\theta, z_\theta) - f(\theta, y_\theta)\|\Big\}d\theta.\end{aligned}$$

Using Proposition 2(v) and applying hypotheses **[H1]**(iii) and **[H2]**(i)(ii)(v) we obtain that

$$\begin{aligned}\Pi_1 & \leq M\Big[L_g q\|z-y\| + \gamma\|h(z) - h(y)\|\Big] \\ & \quad + M\tau\Big[\mathcal{K}(\|z\|, \|y\|)\|z-y\| + \mathcal{K}(\|z\|, \|y\|)\|z-y\|\Big] \\ & \leq M\Big[L_g q\|z-y\| + \gamma L_g q\|z-y\|\Big] + 2M\tau\mathcal{K}(\|z\|, \|y\|)\|z-y\|.\end{aligned}$$

It follows that

$$\|\mathcal{P}(z) - \mathcal{P}(y)\|_p \leq M\Big\{L_g q\left(1+\gamma\right) + 2\tau\mathcal{K}\left(\|z\|, \|y\|\right)\Big\}\|z-y\|_p$$

by taking supremum over $t \in [-r, \tau]$. Hence $\mathcal{P}$ is locally Lipschitz, which implies the continuity of $\mathcal{P}$.

Step 2. *$\mathcal{P}$ maps bounded sets of PW_p into bounded sets of PW_p.*

In order to prove this statement, we will show that

$$\forall R > 0, \exists \zeta > 0, \forall y \in B_R : \|\mathcal{P}(y)\|_p \leq \zeta,$$

where $B_R = \{y \in PW_p : \|y\|_p \leq R\}$. Let $R > 0$ and consider $\zeta = \max\{\xi, \|\eta\|\}$, where ξ is a positive constant to be determined later. Let $y \in B_R$. Then, on the one hand, we have that

$$\|[\mathcal{P}(y)](t)\| = \|\eta(t)\| \leq \|\eta\|,$$

if $t \in [-r, 0]$. While, on the other hand,

$$\begin{aligned}\|[\mathcal{P}(y)](t)\| & \leq \|\mathbf{\Phi}(t,0)\|\|\eta(0) - [h(y)](0) - g(0, \eta - h(y))\| \\ & \quad + \int_0^t \|\mathbf{\Phi}(t,\theta)\|\Big[\|A(\theta)g(\theta, y_\theta)\| + \|f(\theta, y_\theta)\|\Big]d\theta \\ & \leq M\Big\{\|\eta(0)\| + \|[h(y)](0)\| + \|g(0, \eta - h(y))\|\Big\} \\ & \quad + M\int_0^t \Big[\Psi(\|y_\theta\|) + \Psi(\|y_\theta\|)\Big]d\theta \\ & \leq M\Big\{\|\eta(0)\| + L_g q\|y\| + \Psi\left(\|\eta - h(y)\|\right)\Big\} + \tau M 2\Psi(\|y\|) \\ & \leq M\Big\{\|\eta(0)\| + L_g q\|y\| + \Psi\left(\|\eta\| + \|h(y)\|\right)\Big\} + \tau M 2\Psi(\|y\|) \\ & \leq M\Big\{\|\eta(0)\| + L_g q\|y\| + \Psi\left(\|\eta\| + L_g q\|y\|\right)\Big\} + \tau M 2\Psi(\|y\|) \\ & \leq M\Big\{\|\eta(0)\| + L_g qR + \Psi\left(\|\eta\| + L_g qR\right) + \tau 2\Psi(R)\Big\} = \xi\end{aligned}$$

if $t \in [0,\tau]$. Here we have used **[H1]***(iii)* and **[H2]***(iii)(iv)(vi)*. Now, taking supremum over $t \in [-r,\tau]$, we have that

$$\|\mathcal{P}(y)\| \leq \zeta.$$

Step 3. *$\mathcal{P}$ maps bounded sets of PW_p into equicontinuous sets of PW_p.*

Let us consider B_R as above and let us show that $\mathcal{P}(B_R)$ is equicontinuous on $[-r,\tau]$. On $[-r,0]$, the continuity of η immediately implies the result. Let us denote the difference $\|[\mathcal{P}(y)](s_1) - [\mathcal{P}(y)](s_2)\|$ as Π_2 for $s_1, s_2 \in (0,\tau]$. From the definition of $\mathcal{P}$ we have that

$$\begin{aligned}
\Pi_2 \leq\ & \|\mathbf{\Phi}(s_1,0) - \mathbf{\Phi}(s_2,0)\| \|\eta(0) - [h(y)](0) - g\left(0, \eta - h(y)\right)\| \\
& + \int_0^{s_2} \|\mathbf{\Phi}(s_1,\theta) - \mathbf{\Phi}(s_2,\theta)\| \|A(\theta)g(\theta,y_\theta) + f(\theta,y_\theta)\| d\theta \\
& + \int_{s_2}^{s_1} \|\mathbf{\Phi}(s_1,\theta)\| \|A(\theta)g(\theta,y_\theta) + f(\theta,y_\theta)\| d\theta \\
\leq\ & \|\mathbf{\Phi}(s_1,0) - \mathbf{\Phi}(s_2,0)\| \left\{ \|\eta(0)\| + L_g q \|y\| + \|g\left(0,\eta - h(y)\right)\| \right\} \\
& + \int_0^{s_2} \|\mathbf{\Phi}(s_1,\theta) - \mathbf{\Phi}(s_2,\theta)\| \left[\|A(\theta)g(\theta,y_\theta)\| + \|f(\theta,y_\theta)\| \right] d\theta \\
& + \int_{s_2}^{s_1} \|\mathbf{\Phi}(s_1,\theta)\| \left[\|A(\theta)g(\theta,y_\theta)\| + \|f(\theta,y_\theta)\| \right] d\theta \\
\leq\ & \|\mathbf{\Phi}(s_1,0) - \mathbf{\Phi}(s_2,0)\| \left\{ \|\eta(0)\| + L_g q \|y\| + \Psi\left(\|\eta\| + L_g q \|y\| \right) \right\} \\
& + 2\Psi\left(\|y\|\right) \int_0^{s_2} \|\mathbf{\Phi}(s_1,\theta) - \mathbf{\Phi}(s_2,\theta)\| d\theta + 2M\Psi\left(\|y\|\right)(s_1 - s_2) \\
\leq\ & \|\mathbf{\Phi}(s_1,0) - \mathbf{\Phi}(s_2,0)\| \left\{ \|\eta(0)\| + L_g q R + \Psi\left(\|\eta\| + L_g q R \right) \right\} \\
& + 2\Psi\left(R\right) \int_0^{s_2} \|\mathbf{\Phi}(s_1,\theta) - \mathbf{\Phi}(s_2,\theta)\| d\theta + 2M\Psi\left(R\right)(s_1 - s_2) \to 0
\end{aligned}$$

as $s_1 \to s_2$ by the continuity of $\mathbf{\Phi}$ (see Proposition 2*(iv)*) and the fact that $\|\eta(0)\| + L_g q R + \Psi\left(\|\eta\| + L_g q R\right)$ is bounded. Here we have considered **[H2]***(iii)(iv)(vi)* and **[H1]***(iii)*. This shows that $\mathcal{P}(B_R)$ is equicontinuous since the convergence to zero is independent of y (see Definition 10).

Let D be as in (3.8) for the subsequent steps.

Step 4. *The subset $\mathcal{P}(D)$ is relatively compact in PW_p.*

Let D be a bounded subset of PW_p. By Steps 2 and 3, $\mathcal{P}(D)$ is bounded and equicontinuous in PW_p. Let $\left(y_n\right)_{n\in\mathbb{N}} \subseteq \mathcal{P}(D)$, then

$$y_n|_{[-r,0]} = \eta, \ \forall n \in \mathbb{N}$$

Hence, $y_n|_{[-r,0]}$ converges uniformly on $[-r,0]$.

Now, putting $\varphi_n = y_n|_{[0,\tau]}$, we get that $(\varphi_n)_{n\in\mathbb{N}} \subseteq PW_\tau$. Let us put $t_0 = 0$ and $t_{p+1} = \tau$. Then, applying Arzelà-Ascoli Theorem 4, the sequence $(\varphi_n)_{n\in\mathbb{N}}$ contains a subsequence $(\varphi_n^1)_{n\in\mathbb{N}}$ that converges in the interval $[t_0, t_1]$. Now, applying Arzelà-Ascoli Theorem 4 again, we get that the sequence $(\varphi_n^1)_{n\in\mathbb{N}}$ contains a subsequence $(\varphi_n^2)_{n\in\mathbb{N}}$ that converges in the interval $[t_1, t_2]$. Continuing with this process we find a subsequence $(\varphi_n^{p+1})_{n\in\mathbb{N}}$ of $(\varphi_n)_{n\in\mathbb{N}}$ that converges in each interval $[t_k, t_{k+1}]$, with $k \in I_p$. Therefore, $\varphi_n^{p+1} = y_n^{p+1}|_{[0,\tau]}$ converges on $[0, \tau]$. Consequently, $(\varphi_n^{p+1})_{n\in\mathbb{N}} = (y_n^{p+1})_{n\in\mathbb{N}}$ converges uniformly on $[-r, \tau]$. Thus, $\mathcal{P}(D)$ is relatively compact, and the proof of Step 4 is complete.

Step 5. *The family* $\left\{\mathcal{Q}(\cdot, y) : y \in \overline{\mathcal{P}(D)}\right\}$ *is equicontractive.*

On the one hand, for any $u, v \in PW_p$ and $t \in [-r, 0]$ we get that

$$\begin{aligned}\left\|[\mathcal{Q}(u,\mathcal{P}(y))](t) - [\mathcal{Q}(v,\mathcal{P}(y))](t)\right\| &\leq \left\|[h(u)](t) - [h(v)](t)\right\| \\ &\leq L_g q\|u-v\| \\ &\leq ML_g q\|u-v\|.\end{aligned}$$

While on the other hand, by using [H1](*ii*) and [H2](*ii*), for all $t \in (0, \tau]$ we obtain that

$$\begin{aligned}\left\|[\mathcal{Q}(u,\mathcal{P}(y))](t) - [\mathcal{Q}(v,\mathcal{P}(y))](t)\right\| &\leq \|g(t,u_t) - g(t,v_t)\| \\ &\quad + \sum_{0<t_k<t} \left\|\mathbf{\Phi}(t,t_k)\left[J_k(t_k,u(t_k)) - J_k(t_k,v(t_k))\right]\right\| \\ &\leq \gamma\|u-v\| + M\sum_{k=1}^{p}\left\|J_k(t_k,u(t_k)) - J_k(t_k,v(t_k))\right\| \\ &\leq \gamma\|u-v\| + M\sum_{k=1}^{p} d_k\left\|u(t_k) - v(t_k)\right\| \\ &\leq \gamma\|u-v\| + M\|u-v\|\sum_{k=1}^{p} d_k \\ &\leq \left(\gamma + M\sum_{k=1}^{p} d_k\right)\|u-v\|.\end{aligned}$$

It follows that

$$\left\|\mathcal{Q}(u,\mathcal{P}(y)) - \mathcal{Q}(v,\mathcal{P}(y))\right\| \leq \left(\gamma + M\sum_{k=1}^{p} d_k\right)\|u-v\| \leq \frac{1}{2}\|u-v\|$$

by taking supremum over $t \in [-r, \tau]$ and using [H1](*i*). This shows that $\mathcal{Q}(\cdot, \mathcal{P}(y))$ is a contraction which does not depend on $y \in \overline{\mathcal{P}(D)}$.

Let us consider the operator $\mathcal{H} = \mathcal{Q}\left(\cdot, \mathcal{P}(\cdot)\right)$ for the next step.

Step 6. *The inclusion* $\mathcal{H}(D) \subset D$ *holds.*

Let z be a generic element in D. We have to prove that $\mathcal{H}(z) \in D$. By (3.8), this is equivalent to prove that $\|\mathcal{H}(z) - \tilde{\eta}\| \leq \rho$. For that purpose, from the definition of $\mathcal{Q}$ and $\mathcal{P}$, we notice that $[\mathcal{H}(z)](t)$ can be written as

$$[\mathcal{H}(z)](t) = \begin{cases} \mathbf{\Phi}(t,0)\left[\eta(0) - [h(z)](0) - g(0,\eta - h(z))\right] & \\ + \displaystyle\int_0^t \mathbf{\Phi}(t,\theta)\left[A(\theta)g(\theta,z_\theta) + f(\theta,z_\theta)\right] d\theta + g(t,z_t) & \\ + \displaystyle\sum_{0<t_k<t} \mathbf{\Phi}(t,t_k)J_k(t_k,z(t_k)), & t \in [0,\tau], \\ \eta(t) - [h(z)](t), & t \in [-r,0], \end{cases}$$

which is exactly Equation (3.3).

Now, let us consider the difference $\|[\mathcal{H}(z)](t) - \tilde{\eta}(t)\|$ and denote it as Π_3. On the one hand, for $t \in [-r,0]$, we have that

$$\Pi_3 \leq \|[h(z)](t)\| \leq L_g q\|z\| \leq ML_g q\|z\| \leq ML_g q\left(\|\tilde{\eta}\| + \rho\right) < \rho$$

by **[H1]**(*iii*) and **[H3]**. While on the other hand, for $t \in [0,\tau]$, we have that

$$\begin{aligned}
\Pi_3 &\leq M\|[h(z)](0) - g(0,\eta - h(z))\| \\
&\quad + \int_0^t \|\mathbf{\Phi}(t,\theta)\left[A(\theta)g(\theta,z_\theta) + f(\theta,z_\theta)\right]\| d\theta + \|g(t,z_t)\| \\
&\quad + \sum_{0<t_k<t} \|\mathbf{\Phi}(t,t_k)J_k(t_k,z(t_k))\| \\
&\leq M\left\{L_g q\|z\| + \|g(0,\eta - h(z))\|\right\} \\
&\quad + 2M\tau\Psi(\|z\|) + \Psi(\|z\|) + M\sum_{0<t_k<t} \|J_k(t_k,z(t_k))\| \\
&\leq M\left\{L_g q\|z\| + \Psi\left(\|\eta\| + L_g q\|z\|\right)\right\} \\
&\quad + 2M\tau\Psi(\|z\|) + \Psi(\|z\|) + \left(M\sum_{k=1}^{p} d_k\right)\|z\| \\
&\leq M\left\{L_g q\left(\|\tilde{\eta}\| + \rho\right) + \Psi\left(\|\tilde{\eta}\| + L_g q\left(\|\tilde{\eta}\| + \rho\right)\right)\right\} \\
&\quad + 2M\tau\Psi\left(\|\tilde{\eta}\| + \rho\right) + \Psi\left(\|\tilde{\eta}\| + \rho\right) + \left(M\sum_{k=1}^{p} d_k\right)\left(\|\tilde{\eta}\| + \rho\right) \\
&\leq M\Psi\left(\|\tilde{\eta}\| + L_g q\left(\|\tilde{\eta}\| + \rho\right)\right) + \left(ML_g q + M\sum_{k=1}^{p} d_k\right)\left(\|\tilde{\eta}\| + \rho\right) \\
&\quad + (2M\tau + 1)\Psi\left(\|\tilde{\eta}\| + \rho\right) < \rho.
\end{aligned}$$

Here we have used **[H1]**(*ii*)(*iii*), **[H2]**(*iii*)(*iv*)(*vi*), and **[H3]**. Now, by taking supremum over $t \in [-r, \tau]$, we get that

$$\|\mathcal{H}(z) - \tilde{\eta}\| \leq \rho.$$

By the arbitrariness of $z \in D$, we therefore conclude that $\mathcal{H}(z) \in D$.

Finally, taking into account Steps 1, 4, 5, and 6 we note that the hypotheses of Theorem 7 are satisfied and therefore we conclude that the operator equation

$$\mathcal{H}(z) = z$$

admits a solution on D. This finishes the proof of Theorem 24. □

Theorem 25. *System* (3.1) *has a unique solution if* **[H4]** *is additionally assumed.*

Proof. Suppose u and v are two solutions of system (3.1). Now, considering **[H1]** and **[H2]** we have that

$$\begin{aligned}
\big\|u(t) - v(t)\big\| &\leq \big\|\mathbf{\Phi}(t,0)\big\|\big\|[h(u)](0) - [h(v)](0)\big\| \\
&\quad + \big\|\mathbf{\Phi}(t,0)\big\|\big\|g\left(0, \eta - h(u)\right) - g\left(0, \eta - h(v)\right)\big\| \\
&\quad + \int_0^t \big\|\mathbf{\Phi}(t,\theta)\big\|\big\|A(\theta)g(\theta, u_\theta) - A(\theta)g(\theta, v_\theta)\big\| d\theta \\
&\quad + \int_0^t \big\|\mathbf{\Phi}(t,\theta)\big\|\big\|f(\theta, u_\theta) - f(\theta, v_\theta)\big\| d\theta + \big\|g(t, u_t) - g(t, v_t)\big\| \\
&\quad + \sum_{0<t_k<t} \big\|\mathbf{\Phi}(t, t_k)\big\|\big\|J_k(t_k, u(t_k)) - J_k(t_k, v(t_k))\big\| \\
&\leq M\left\{L_g q\,(1+\gamma) + 2\tau\mathcal{K}\left(\|u\|, \|v\|\right)\right\}\|u - v\| + \left(\gamma + M\sum_{k=1}^{p} d_k\right)\|u - v\| \\
&\leq M\left\{L_g q\,(1+\gamma) + 2\tau\mathcal{K}\left(\big\|\tilde{\eta}\big\| + \rho, \big\|\tilde{\eta}\big\| + \rho\right)\right\}\|u - v\| + \frac{1}{2}\|u - v\|
\end{aligned}$$

Bearing in mind the hypothesis **[H4]**, and taking supremum over $t \in [-r, \tau]$ we get that

$$\|u - v\| \leq \omega\|u - v\|$$

with $0 \leq \omega < 1$. This implies $\|u - v\| = 0$, and therefore $u = v$. □

3.2 Controllability results

This section is devoted to prove that, under certain conditions on the nonlinear terms, the controllability of the associated ordinary differential equation to a semilinear neutral differential equations with impulses, delay and nonlocal conditions is robust. To be more specific, we give a sufficient condition for the exact controllability of the following neutral differential equation with impulses, delay and nonlocal conditions

$$\begin{cases} \dfrac{d}{dt}\left[z(t) - g(t, z_t)\right] = A(t)z(t) + B(t)u(t) + f(t, z_t, u(t)), & t \in [0, \tau] \setminus \{t_k\}_{k \in I_p} \\ z(\theta) = -[h(z)](\theta) + \eta(\theta), & \theta \in [-r, 0], \\ z(t_k^+) = z(t_k^-) + J_k(t_k, z(t_k)), & k \in I_p, \end{cases} \tag{3.9}$$

where $B(t)$ is a $n \times m$ continuous matrix, $f : [0, \tau] \times PW_r \times \mathbb{R}^m \longrightarrow \mathbb{R}^n$ is a suitable function to be specified later, the control function u belongs to the space $\mathcal{C}([0, \tau], \mathbb{R}^m)$, and the remaining terms satisfy the same conditions as in system (3.1).

3.2.1 Exact controllability using the Rothe's fixed point theorem

In this section, we present our controllability result for system (3.9). For doing so, we need to impose the following hypotheses on system (2.28) and the nonlinear terms in system (3.9).

[h1] The system (2.28) is controllable on $[0, \tau]$.

[h2] The nonlinear terms in system (3.9) are globally Lipschitz, i.e.,

(i) $\|h(z) - h(w)\|_r \leq L_g \|z - w\|_{qp}$, $z, w \in PW_{qp}$,

(ii) $\|g(t, \eta) - g(t, \psi)\|_{\mathbb{R}^n} \leq L_{-1} \|\eta - \psi\|_r$, $\eta, \psi \in PW_r, t \in [0, \tau]$,

(iii) $\|f(t, \eta, u) - f(t, \psi, v)\|_{\mathbb{R}^n} \leq L_1 \left\{ \|\eta - \psi\|_r + \|u - v\|_{\mathbb{R}^m} \right\}$, $\eta, \psi \in PW_r$, $u, v \in \mathbb{R}^m$, $t \in [0, \tau]$,

(iv) $\|J_k(t, z) - J_k(t, w)\|_{\mathbb{R}^n} \leq d_k \|z - w\|_{\mathbb{R}^n}$, $z, w \in \mathbb{R}^n$, $t \in [0, \tau]$.

For all bounded set $\mathfrak{B}$ in PW_p there exists a continuous function $\rho : [0, \tau] \to \mathbb{R}_+$ depending on $\mathfrak{B}$ such that $\rho(0) = 0$, and for all $z \in \mathfrak{B}$ we have that

(v) $\|g(t_2, z_{t_2}) - g(t_1, z_{t_1})\|_{\mathbb{R}^n} \leq \rho\left(|t_2 - t_1|\right) \|z\|_p$, $t_2, t_1 \in [0, \tau]$,

(vi) $\|[h(z)](t_2) - [h(z)](t_1)\|_{\mathbb{R}^n} \leq \rho\left(|t_2 - t_1|\right) \|z\|_{pq}$, $t_2, t_1 \in [-r, 0]$.

[h3] *(i)* $\|f(t, \eta, u)\|_{\mathbb{R}^n} \leq a_0 \|\eta(-r)\|_{\mathbb{R}^n}^{\alpha_0} + \|u\|_{\mathbb{R}^m}^{\beta_0} + c_0$, $\eta \in PW_r$, $t \in [0, \tau]$,

(ii) $\|J_k(t, z)\|_{\mathbb{R}^n} \leq a_k \|z\|_{\mathbb{R}^n}^{\alpha_k} + c_k$, $k \in I_p$, $z \in \mathbb{R}^n$, $t \in [0, \tau]$,

(iii) $\|h(z)\|_r \leq e \|z\|_{qp}^{\eta_1}$, $z \in PW_{qp}$,

(iv) $\|g(t, \eta)\| \leq \|\eta(-r)\|_{\mathbb{R}^n}^{\omega_1}$, $\eta \in PW_r$, $t \in [0, \tau]$,

where $0 \leq \alpha_k < 1,\ k \in I_p \cup \{0\}$, $0 \leq \beta_0 < 1$, $0 \leq \omega_1 < 1$, and $0 \leq \eta_1 < 1$.

Remark 6. *Obviously, every bounded and globally Lipschitz function chosen conveniently, satisfies the hypotheses* **[h2]** *and* **[h3]**.

By Section 3.1, we know that for all $\eta \in PW_r$ and $u \in \mathcal{C}([0,\tau],\mathbb{R}^m)$ the system (3.9) admits one solution $z(t) = z(t,\eta,u)$ given by

$$z(t) = \begin{cases} \mathbf{\Phi}(t,0)[\eta(0) - [h(z)](0) - g(0,\eta - h(z))] \\ + g(t,z_t) + \int_0^t \mathbf{\Phi}(t,\theta)\left[A(\theta)g(\theta,z_\theta) + f(\theta,z_\theta,u(\theta))\right] d\theta \\ + \int_0^t \mathbf{\Phi}(t,\theta)B(\theta)u(\theta)d\theta + \sum_{0<t_k<t} \mathbf{\Phi}(t,t_k)J_k(t_k,z(t_k)), & t \in [0,\tau], \\ \eta(t) - [h(z)](t), & t \in [-r,0]. \end{cases} \tag{3.10}$$

Now, let us suppose for a moment that system (3.9) is exactly controllable. That is to say (see Definition 14), for all $\eta \in PW_r$ and $z_1 \in \mathbb{R}^n$ there exists $u \in \mathcal{C}([0,\tau],\mathbb{R}^m)$ (see Lemma 3) such that the corresponding solution $z(t) = z(t,\eta,u)$ of (3.9) satisfies

$$z(0) + [h(z)](0) = \eta(0) \text{ and } z(\tau) = z_1,$$

i.e.,

$$\begin{aligned} z_1 = &\mathbf{\Phi}(\tau,0)\left[\eta(0) - [h(z)](0) - g(0,\eta - h(z))\right] \\ &+ g(\tau,z_\tau) + \int_0^\tau \mathbf{\Phi}(\tau,\theta)\left[A(\theta)g(\theta,z_\theta) + f(\theta,z_\theta,u(\theta))\right] d\theta \\ &+ \int_0^\tau \mathbf{\Phi}(\tau,\theta)B(\theta)u(\theta)d\theta + \sum_{0<t_k<\tau} \mathbf{\Phi}(\tau,t_k)J_k(t_k,z(t_k)). \end{aligned} \tag{3.11}$$

Recognizing the second integral of the right hand side in (3.11) as the controllability operator given in Definition 16 we can write

$$\begin{aligned} Cu = &z_1 - \mathbf{\Phi}(\tau,0)\left[\eta(0) - [h(z)](0) - g(0,\eta - h(z))\right] \\ &- g(\tau,z_\tau) - \int_0^\tau \mathbf{\Phi}(\tau,\theta)\left[A(\theta)g(\theta,z_\theta) + f(\theta,z_\theta,u(\theta))\right] d\theta \\ &- \sum_{0<t_k<\tau} \mathbf{\Phi}(\tau,t_k)J_k(t_k,z(t_k)). \end{aligned}$$

Then

$$u(t) = B^*(t)\mathbf{\Phi}^*(\tau,t)W^{-1}\mathfrak{L}(z,u),\ t \in [0,\tau],$$

where W is the Gramian operator (see Definition 16) and

$$\begin{aligned} \mathfrak{L}: \quad PW_p \times \mathcal{C}([0,\tau],\mathbb{R}^m) &\longrightarrow \mathbb{R}^n \\ (z,u) &\longmapsto \mathfrak{L}(z,u) \end{aligned}$$

is an operator given by

$$\begin{aligned}\mathfrak{L}(z,u) =& z_1 - \mathbf{\Phi}(\tau,0)\left[\eta(0) - [h(z)](0) - g(0,\eta - h(z))\right] \\ & - g(\tau, z_\tau) - \int_0^\tau \mathbf{\Phi}(\tau,\theta)\left[A(\theta)g(\theta,z_\theta) + f(\theta, z_\theta, u(\theta))\right] d\theta \\ & - \sum_{0<t_k<\tau} \mathbf{\Phi}(\tau,t_k) J_k(t_k, z(t_k)).\end{aligned}$$

Next, we consider the operator

$$\begin{aligned}\Omega: \quad PW_p \times \mathcal{C}([0,\tau],\mathbb{R}^m) &\longrightarrow PW_p \times \mathcal{C}([0,\tau],\mathbb{R}^m) \\ (z,u) &\longmapsto \Omega(z,u) = \left(\Omega_1(z,u), \Omega_2(z,u)\right),\end{aligned}$$

where

$$\begin{aligned}\Omega_1: \quad PW_p \times \mathcal{C}([0,\tau],\mathbb{R}^m) &\longrightarrow PW_p \\ (z,u) &\longmapsto \Omega_1(z,u)\end{aligned}$$

is defined by

$$[\Omega_1(z,u)](t) = \begin{cases} \mathbf{\Phi}(t,0)[\eta(0) - [h(z)](0) - g(0,\eta - h(z))] \\ + \displaystyle\int_0^t \mathbf{\Phi}(t,\theta)\left[A(\theta)g(\theta,z_\theta) + f(\theta,z_\theta,u(\theta))\right] d\theta \\ + \displaystyle\int_0^t \mathbf{\Phi}(t,\theta)B(\theta)u(\theta)d\theta + g(t,z_t) \\ + \displaystyle\sum_{0<t_k<t} \mathbf{\Phi}(t,t_k)J_k(t_k,z(t_k)), & t \in [0,\tau], \\ \eta(t) - [h(z)](t), & t \in [-r,0] \end{cases}$$

and

$$\begin{aligned}\Omega_2: \quad PW_p \times \mathcal{C}([0,\tau],\mathbb{R}^m) &\longrightarrow \mathcal{C}([0,\tau],\mathbb{R}^m) \\ (z,u) &\longmapsto \Omega_2(z,u)\end{aligned}$$

is given by

$$[\Omega_2(z,u)](t) = B^*(t)\mathbf{\Phi}^*(\tau,t)W^{-1}\mathfrak{L}(z,u), \; t \in [0,\tau].$$

Taking into account the discussion above, the following proposition is now obvious.

Proposition 5. *System* (3.9) *is controllable if and only if the operator* Ω *has a fixed point, i.e.,*

$$\exists (z,u) \in PW_p \times \mathcal{C}([0,\tau],\mathbb{R}^m) : \Omega(z,u) = (z,u).$$

Now we are in position to present the main theorem of this section.

Theorem 26. *Suppose conditions* **[h1]**, **[h2]** *and* **[h3]** *hold. Then, the semilinear neutral differential equation* (3.9) *is also controllable on* $[0,\tau]$. *Moreover, for* $\eta \in PW_r$ *and* $z_1 \in \mathbb{R}^n$

there exists $u \in \mathcal{C}([0,\tau],\mathbb{R}^m)$ *such that the corresponding solution* $z(t) = z(t,\eta,u)$ *of* (3.9) *satisfies*

$$
\begin{aligned}
z_1 =& \boldsymbol{\Phi}(\tau,0)\left[\eta(0) - [h(z)](0) - g(0,\eta - h(z))\right] \\
&+ g(\tau,z_\tau) + \int_0^\tau \boldsymbol{\Phi}(\tau,\theta)\left[A(\theta)g(\theta,z_\theta) + f(\theta,z_\theta,u(\theta))\right] d\theta \\
&+ \int_0^\tau \boldsymbol{\Phi}(\tau,\theta)B(\theta)u(\theta)d\theta + \sum_{0<t_k<\tau} \boldsymbol{\Phi}(\tau,t_k)J_k(t_k,z(t_k))
\end{aligned}
$$

and

$$u(t) = B^*(t)\boldsymbol{\Phi}^*(\tau,t)W^{-1}\mathfrak{L}(z,u),\ t \in [0,\tau].$$

Proof. The proof of this theorem will be given by steps.

Step 1. *The operator* Ω *is continuous.*

It is enough to prove that the operators Ω_1 and Ω_2 are continuous.

On the one hand, the continuity of Ω_1 is proved as follows.
For $t \in [0,\tau]$, we get

$$\left\|[\Omega_1(z,u)](t) - [\Omega_1(w,v)](t)\right\| \leq N_1\|z-w\| + N_2\|u-v\|,$$

where

$$
\begin{aligned}
N_1 &= M\left[L_g + L_{-1}L_g + L_{-1} + \tau L_{-1}\|A\| + L_1\tau + d\right] \\
N_2 &= M\tau\left[L_1 + \|B\|\right]
\end{aligned}
$$

with $d = \sum\limits_{0<t_k<t} d_k$, $\|B\| = \sup\limits_{\theta\in[0,\tau]} \left\|B(\theta)\right\|$, and $\|A\| = \sup\limits_{\theta\in[0,\tau]} \left\|A(\theta)\right\|$.
For $t \in [-r,0]$ we have that

$$\left\|[\Omega_1(z,u)](t) - [\Omega_1(w,v)](t)\right\| \leq L_g\|z-w\|.$$

These two inequalities imply the continuity of Ω_1.

On the other hand, the continuity of Ω_2 follows from the continuity of B, $\boldsymbol{\Phi}$, and $\mathfrak{L}$.

Step 2. *The operator* Ω *maps bounded sets of* $PW_p \times \mathcal{C}([0,\tau],\mathbb{R}^m)$ *into equicontinuous sets of* $PW_p \times \mathcal{C}([0,\tau],\mathbb{R}^m)$.

In fact, let D be a bounded set of $PW_p \times \mathcal{C}([0,\tau],\mathbb{R}^m)$, and consider the following inequalities.

On the one hand, for $0 < t_1 < t_2 < \tau$ and $(z,u) \in D$ we get

$$\begin{aligned}\left\|[\Omega_1(z,u)](t_2) - [\Omega_1(z,u)](t_1)\right\| \leq & \left\|\mathbf{\Phi}(t_2,0) - \mathbf{\Phi}(t_1,0)\right\| \Big[\left\|\eta(0)\right\| + \left\|h(z)\right\| \\ & + \left\|g(0,\eta - h(z))\right\|\Big] \\ & + \int_0^{t_1} \left\|\mathbf{\Phi}(t_2,\theta) - \mathbf{\Phi}(t_1,\theta)\right\| \left\|B(\theta)\right\| \left\|u(\theta)\right\| d\theta \\ & + \int_{t_1}^{t_2} \left\|\mathbf{\Phi}(t_2,\theta)\right\| \left\|B(\theta)\right\| \left\|u(\theta)\right\| d\theta \\ & + \rho\left(|t_2 - t_1|\right)\|z\| + \left\|\mathbf{\Phi}(t_2,\theta) - \mathbf{\Phi}(t_1,\theta)\right\| \\ & \times \left(\int_0^{t_1} \left\|A(\theta)g(\theta,z_\theta) + f(\theta,z_\theta,u(\theta))\right\| d\theta\right) \\ & + \int_{t_1}^{t_2} \left\|\mathbf{\Phi}(t_2,\theta)\right\| \left\|A(\theta)g(\theta,z_\theta) + f(\theta,z_\theta,u(\theta))\right\| d\theta \\ & + \sum_{0<t_k<t_1} \left\|\mathbf{\Phi}(t_2,t_k) - \mathbf{\Phi}(t_1,t_k)\right\| \left\|J_k(t_k,z(t_k))\right\| \\ & + \sum_{t_1<t_k<t_2} \left\|\mathbf{\Phi}(t_2,t_k)\right\| \left\|J_k(t_k,z(t_k))\right\|.\end{aligned}$$

For $-r < t_1 < t_2 < 0$, we have that

$$\begin{aligned}\left\|[\Omega_1(z,u)](t_2) - [\Omega_1(z,u)](t_1)\right\| \leq & \left\|\eta(t_2) - \eta(t_1)\right\| + \\ & \left\|[h(z)](t_2) - [h(z)](t_1)\right\| \\ \leq & \left\|\eta(t_2) - \eta(t_1)\right\| + \rho\left(|t_2 - t_1|\right)\|z\|_p.\end{aligned}$$

Since $\left\|\mathbf{\Phi}(t_2,\theta) - \mathbf{\Phi}(t_1,\theta)\right\| \to 0$, $\rho\left(|t_2 - t_1|\right) \to 0$ as $t_1 \to t_2$ and the above inequalities, we obtain that $\Omega_1(D)$ is equicontinuous.

On the other hand, for $0 < t_1 < t_2 < \tau$ and $(z,u) \in D$, the following estimate holds.

$$\left\|[\Omega_2(z,u)](t_2) - [\Omega_2(z,u)](t_1)\right\| \leq \left\|W^{-1}\mathfrak{L}(z,u)\right\| \left\|B^*(t_2)\mathbf{\Phi}^*(\tau,t_2) - B^*(t_1)\mathbf{\Phi}^*(\tau,t_1)\right\|$$

Analogously, since $\left\|B^*(t_2)\mathbf{\Phi}^*(\tau,t_2) - B^*(t_1)\mathbf{\Phi}^*(\tau,t_1)\right\| \to 0$ as $t_2 \to t_1$ and $\mathfrak{L}(z,u)$ is bounded in D, we get that $\Omega_2(D)$ is equicontinuous.

Step 3. *The set $\Omega(D)$ is relatively compact on $PW_p \times \mathcal{C}([0,\tau],\mathbb{R}^m)$.*

Indeed, since the functions g, f, h, and J_k are smooth enough, there exist positive constants M_1, M_2, M_3, M_4, and M_{-1} such that for all $(z,u) \in D$ and all $t \in [-r,\tau]$ we have that

$$\begin{aligned}
\left\|g(t,z_t)\right\| &\leq M_{-1},\\
\left\|f(t,z_t,u(t))\right\| &\leq M_1,\\
\left\|W^{-1}\mathfrak{L}(z,u)\right\| &\leq M_2,\\
\left\|h(z)\right\| &\leq M_3,\\
\left\|J_k(t,z(t))\right\| &\leq M_4.
\end{aligned}$$

Hence $\Omega(D)$ is bounded.

Now, let $\left\{\varphi_i = (\varphi_{i1},\varphi_{i2}) : i \in \mathbb{N}\right\}$ be a sequence in $\Omega(D) \subset PW_p \times \mathcal{C}([0,\tau],\mathbb{R}^m)$. Since $(\varphi_{i2})_{i\in\mathbb{N}}$ is a sequence in $\Omega_2(D) \subset \mathcal{C}([0,\tau],\mathbb{R}^m)$, which is uniformly bounded and equicontinuous, we can apply the Arzelà-Ascoli theorem directly to ensure the existence of a convergent subsequence of $(\varphi_{i2})_{i\in\mathbb{N}}$ that, without loss of generality, we can keep calling $(\varphi_{i2})_{i\in\mathbb{N}}$.

On the other hand, we consider the sequence $(\varphi_{i1})_{i\in\mathbb{N}}$, which is in $\Omega_1(D) \subset PW_p$. Since $\Omega_1(D)$ is a uniformly bounded and equicontinuous family, on $[-r,t_1]$, there exists a convergent subsequence $(\varphi^1_{i1})_{i\in\mathbb{N}} \subseteq (\varphi_{i1})_{i\in\mathbb{N}}$ by applying the Arzelà-Ascoli theorem again. Now, consider $(\varphi^1_{i1})_{i\in\mathbb{N}}$ on $[t_1,t_2]$. Then $(\varphi^1_{i1})_{i\in\mathbb{N}}$ has a convergent subsequence $(\varphi^2_{i1})_{i\in\mathbb{N}}$ on $[t_1,t_2]$. Continuing with this process the subsequence $(\varphi^{p+1}_{i1})_{i\in\mathbb{N}}$ converges uniformly on each interval $[-r,t_1],[t_1,t_2],\ldots,[t_p,\tau]$. Therefore, the subsequence $\{\varphi^{p+1}_i = (\varphi^{p+1}_{i1},\varphi^{p+1}_{i2}) : i \in \mathbb{N}\}$ of $(\varphi_i)_{i\in\mathbb{N}}$ is uniformly convergent. Hence $\overline{\Omega(D)}$ is compact, i.e., $\Omega(D)$ is relatively compact.

Step 4. *The operator Ω satisfies the following condition.*

$$\lim_{\|\|(z,u)\|\|\to\infty} \frac{\|\|\Omega(z,u)\|\|}{\|\|(z,u)\|\|} = 0,$$

where

$$\|\|(z,u)\|\| = \|z\| + \|u\|$$

is the norm in the Banach space $PW_p \times \mathcal{C}([0,\tau],\mathbb{R}^m)$.

From the definition of $\mathfrak{L}$,

$$\begin{aligned}
\|\mathfrak{L}(z,u)\| \leq &\|z_1\| + \|\mathbf{\Phi}(\tau,0)\|\|\eta(0) - [h(z)](0) - g(0,\eta - h(z))\|\\
&+ \|g(\tau,z_\tau)\| + \int_0^\tau \|\mathbf{\Phi}(\tau,\theta)\|\|A(\theta)g(\theta,z_\theta) + f(\theta,z_\theta,u(\theta))\|d\theta\\
&+ \sum_{0<t_k<\tau} \|\mathbf{\Phi}(\tau,t_k)\|\|J_k(t_k,z(t_k))\|.
\end{aligned}$$

Hypotheses **[h2]** and **[h3]** imply that

$$\begin{aligned}\|\mathfrak{L}(z,u)\| \leq& \|z_1\| + M\|\eta(0)\| + M\left[e\|z\|^{\eta_1} + 2^{\omega_1}\|\eta\|^{\omega_1} + 2^{\omega_1}e^{\omega_1}\|z\|^{\omega_1\eta_1}\right] \\ &+ \|z\|^{\omega_1} + M\tau\left[\|A\|\|z\|^{\omega_1} + a_0\|z\|^{\alpha_0} + \|u\|^{\beta_0} + c_0\right] \\ &+ M\sum_{0<t_k<\tau}\left[a_k\|z\|^{\alpha_k} + c_k\right] \\ \leq& K + M\left[e\|z\|^{\eta_1} + 2^{\omega_1}e^{\omega_1}\|z\|^{\omega_1\eta_1}\right] + \|z\|^{\omega_1} \\ &+ M\tau\left[\|A\|\|z\|^{\omega_1} + a_0\|z\|^{\alpha_0} + \|u\|^{\beta_0}\right] + M\sum_{0<t_k<\tau}\left[a_k\|z\|^{\alpha_k}\right],\end{aligned}$$

where $K = \|z_1\| + M\left[\|\eta(0)\| + 2^{\omega_1}\|\eta\|^{\omega_1} + \tau c_0 + \sum_{0<t_k<\tau} c_k\right]$. As consequence of (2.2) and (2.30), we obtain

$$\left\|[\Omega_2(z,u)](t)\right\| \leq \|B^*(t)\|\|\mathbf{\Phi}^*(\tau,t)\|W^{-1}\mathfrak{L}(z,u)\| \leq \|B(t)\|\|\mathbf{\Phi}(\tau,t)\|\gamma^{-1}\|\mathfrak{L}(z,u)\|.$$

Hence,

$$\begin{aligned}\left\|[\Omega_2(z,u)](t)\right\| \leq& \|B\|M\gamma^{-1}K + \|B\|M^2\gamma^{-1}\left[e\|z\|^{\eta_1} + 2^{\omega_1}e^{\omega_1}\|z\|^{\omega_1\eta_1}\right] + \|B\|M\gamma^{-1}\|z\|^{\omega_1} \\ &+ \|B\|M^2\gamma^{-1}\tau\left[\|A\|\|z\|^{\omega_1} + a_0\|z\|^{\alpha_0} + \|u\|^{\beta_0}\right] \\ &+ \|B\|M^2\gamma^{-1}\sum_{0<t_k<\tau} a_k\|z\|^{\alpha_k}.\end{aligned} \tag{3.12}$$

Likewise,

$$\begin{aligned}\left\|[\Omega_1(z,u)](t)\right\| \leq& M\|\eta(0)\| + M\left[e\|z\|^{\eta_1} + 2^{\omega_1}\|\eta\|^{\omega_1} + 2^{\omega_1}e^{\omega_1}\|z\|^{\omega_1\eta_1}\right] \\ &+ \|z\|^{\omega_1} + M\tau\left[\|A\|\|z\|^{\omega_1} + a_0\|z\|^{\alpha_0} + \|u\|^{\beta_0} + c_0\right] \\ &+ M^2\tau\|B\|^2\gamma^{-1}\|\mathfrak{L}(z,u)\| + M\sum_{0<t_k<\tau}\left[a_k\|z\|^{\alpha_k} + c_k\right] \\ \leq& K_0 + K_1\Big(M\|\eta(0)\| + M\left[e\|z\|^{\eta_1} + 2^{\omega_1}\|\eta\|^{\omega_1} + 2^{\omega_1}e^{\omega_1}\|z\|^{\omega_1\eta_1}\right] + \|z\|^{\omega_1} \\ &+ M\tau\left[\|A\|\|z\|^{\omega_1} + a_0\|z\|^{\alpha_0} + \|u\|^{\beta_0} + c_0\right] + M\sum_{0<t_k<\tau}\left[a_k\|z\|^{\alpha_k} + c_k\right]\Big),\end{aligned} \tag{3.13}$$

where $K_0 = M^2\tau\|B\|^2\gamma^{-1}\|z_1\|$ and $K_1 = M^2\tau\|B\|^2\gamma^{-1} + 1$. Let $K_2 = K_1 + \|B\|M\gamma^{-1}$. Then, by (3.12) and (3.13),

$$\begin{aligned}\|\Omega(z,u)\| =& \left\|\Omega_1(z,u)\right\| + \left\|\Omega_2(z,u)\right\| \\ \leq& K_3 + K_4\|z\|^{\omega_1} + K_5\|z\|^{\omega_1\eta_1} + K_6\|z\|^{\eta_1} + \\ & K_7\|z\|^{\alpha_0} + K_8\|u\|^{\beta_0} + K_9\sum_{0<t_k<\tau} a_k\|z\|^{\alpha_k},\end{aligned}$$

where

$$K_3 = K_0 + M\Big[K_1\Big(\|\eta(0)\| + \tau c_0 + \sum_{0<t_k<\tau} c_k + 2^{\omega_1}\|\eta\|^{\omega_1}\Big) + \|B\|\gamma^{-1}K\Big],$$

$$K_4 = K_1 + \|B\|M\gamma^{-1} + K_1 M\tau\|A\| + \|B\|M^2\gamma^{-1}\tau\|A\|, \quad K_5 = M2^{\omega_1}e^{\omega_1}K_2,$$

and

$$K_6 = MeK_2, \quad K_7 = M\tau a_0 K_2, \quad K_8 = M\tau K_2, \quad K_9 = MK_2.$$

Consequently,

$$\begin{aligned}\frac{|||\Omega(z,u)|||}{|||(z,u)|||} =& \frac{\|\Omega_1(z,u)\| + \|\Omega_2(z,u)\|}{\|z\| + \|u\|} \\ \leq & \frac{K_3}{\|z\| + \|u\|} + K_4\|z\|^{\omega_1 - 1} + K_5\|z\|^{\omega_1\eta_1 - 1} + K_6\|z\|^{\eta_1 - 1} + \\ & K_7\|z\|^{\alpha_0 - 1} + K_8\|u\|^{\beta_0 - 1} + K_9 \sum_{0<t_k<\tau} a_k\|z\|^{\alpha_k - 1},\end{aligned}$$

whence

$$\lim_{|||(z,u)|||\to\infty} \frac{|||\Omega(z,u)|||}{|||(z,u)|||} = 0.$$

Step 5. *The operator Ω has at least one fixed point.*

Actually, by the previous step, we have that for $0 < \rho < 1$ there exists $R > 0$ such that

$$\frac{|||\Omega(z,u)|||}{|||(z,u)|||} < \rho \quad \text{if} \quad |||(z,u)||| \geq R.$$

Therefore, if $|||(z,u)||| = R$, then $|||\Omega(z,u)||| \leq \rho|||(z,u)||| \leq \rho R < R$. This implies that

$$\Omega\left(\partial B(0,R)\right) \subset B(0,R),$$

where $B(0,R)$ is the closed ball of radius R centered at zero. The foregoing Steps 1, 2, 3, and 4 together with Theorem 8 allow us to conclude that there exists $(z,u) \in PW_p \times \mathcal{C}([0,\tau],\mathbb{R}^m)$ such that

$$\Omega(z,u) = (z,u).$$

By Proposition 5 and Step 5, the system (3.9) is exactly controllable on $[0,\tau]$. Furthermore,

$$u(t) = B^*(t)\mathbf{\Phi}^*(\tau,t)W^{-1}\mathfrak{L}(z,u)$$

and

$$\begin{aligned} z_1 =& \mathbf{\Phi}(\tau,0)\left[\eta(0) - [h(z)](0) - g(0,\eta - h(z))\right] \\ & + g(\tau,z_\tau) + \int_0^\tau \mathbf{\Phi}(\tau,\theta)\left[A(\theta)g(\theta,z_\theta) + f(\theta,z_\theta,u(\theta))\right]d\theta \\ & + \int_0^\tau \mathbf{\Phi}(\tau,\theta)B(\theta)u(\theta)d\theta + \sum_{0<t_k<t} \mathbf{\Phi}(\tau,t_k)J_k(t_k,z(t_k)).\end{aligned}$$

This finishes the proof. □

3.2.2 Exact Controllability using the Banach contraction theorem

In this subsection, we will use the Banach contraction theorem to study the exact controllability of the following system

$$\begin{cases} \frac{d}{dt}\left[z(t)-g(t,z_t)\right] = A(t)z(t)+B(t)u(t)+f(t,z_t), & t\in[0,\tau]\setminus\{t_k\}_{k\in I_p} \\ z(\theta) = -[h(z)](\theta)+\eta(\theta), & \theta\in[-r,0], \\ z(t_k^+) = z(t_k^-)+J_k(t_k,z(t_k)), & k\in I_p, \end{cases} \tag{3.14}$$

which admits a solution given by

$$z(t) = \begin{cases} \mathbf{\Phi}(t,0)[\eta(0)-[h(z)](0)-g(0,\eta-h(z))] \\ +g(t,z_t)+\int_0^t \mathbf{\Phi}(t,\theta)\left[A(\theta)g(\theta,z_\theta)+f(\theta,z_\theta)\right]d\theta \\ +\int_0^t \mathbf{\Phi}(t,\theta)B(\theta)u(\theta)d\theta+\sum\limits_{0<t_k<t}\mathbf{\Phi}(t,t_k)J_k(t_k,z(t_k)), & t\in[0,\tau], \\ \eta(t)-[h(z)](t), & t\in[-r,0]. \end{cases} \tag{3.15}$$

We impose the following assumptions.

[A1] The system (2.28) is exactly controllable on $[0,\tau]$.

[A2] There exists constants $d_k, L_g>0, k=1,2,\ldots,p$ such that

(i) $\|J_k(t,y)-J_k(t,z)\|_{\mathbb{R}^n}\le d_k\|y-z\|_{\mathbb{R}^n},\ y,z\in\mathbb{R}^n, t\in[0,\tau],$

(ii) $\|[h(y)](t)-[h(v)](t)\|_{\mathbb{R}^n}\le L_g\sum\limits_{i=1}^{q}\|y_i(t)-v_i(t)\|_{\mathbb{R}^n},\ y,v\in PW_{qp}.$

[A3] The function g satisfies

(i) $\|g(t,\eta_1)-g(t,\eta_2)\|_{\mathbb{R}^n}\le L_{-1}\|\eta_1-\eta_2\|_r,\ \eta_1,\eta_2\in PW_r,$

and f satisfies

(ii) $\|f(t,\eta_1)-f(t,\eta_2)\|_{\mathbb{R}^n}\le L_1\|\eta_1-\eta_2\|_r,\ \eta_1,\eta_2\in PW_r.$

The following notations are introduced for convenience.

$$\|B\|_\infty=\sup_{\theta\in[0,\tau]}\|B(\theta)\|,\ \|S\|=\sup_{\theta\in[0,\tau]}\|B^*(\theta)\mathbf{\Phi}^*(\tau,\theta)W^{-1}\|,\ M_1=M\sup_{\theta\in[0,\tau]}\|A(\theta)\|,$$

$$M_2=L_{-1}+L_gMq+L_gL_{-1}Mq+M_1L_{-1}\tau+ML_1\tau+MT,\ \text{and } T=\sum_{k=1}^{q}d_k.$$

Lemma 4. *If* **[A1]**, **[A2]**, *and* **[A3]** *hold, then the control function*

$$u(t) = B^*(t)\mathbf{\Phi}^*(\tau,t)W^{-1}N(z) = C^*(CC^*)^{-1}N(z) = SN(z),\ t \in [0,\tau], \tag{3.16}$$

transfer the system (3.14) *from the initial state to* z_1 *at time* τ, *where* W *is the Gramian operator (see Definition* 16*) and*

$$\begin{array}{rccc} N: & \mathcal{C}([0,\tau],\mathbb{R}^m) & \longrightarrow & \mathbb{R}^n \\ & z & \longmapsto & N(z) \end{array}$$

is an operator given by

$$\begin{aligned} N(z) =& z_1 - \mathbf{\Phi}(\tau,0)\left[\eta(0) - [h(z)](0) - g(0,\eta - h(z))\right] \\ & - g(\tau,z_\tau) - \int_0^\tau \mathbf{\Phi}(\tau,\theta)\left[A(\theta)g(\theta,z_\theta) + f(\theta,z_\theta)\right]d\theta \\ & - \sum_{0<t_k<\tau} \mathbf{\Phi}(\tau,t_k)J_k(t_k,z(t_k)). \end{aligned}$$

Proof. Evaluating (3.15) at τ we obtain

$$\begin{aligned} z(\tau) =& \mathbf{\Phi}(\tau,0)[\eta(0) - [h(z)](0) - g(0,\eta - h(z))] \\ & + g(\tau,z_\tau) + \int_0^\tau \mathbf{\Phi}(\tau,\theta)\left[A(\theta)g(\theta,z_\theta) + f(\theta,z_\theta)\right]d\theta \\ & + \int_0^\tau \mathbf{\Phi}(\tau,\theta)B(\theta)u(\theta)d\theta + \sum_{0<t_k<\tau} \mathbf{\Phi}(\tau,t_k)J_k(t_k,z(t_k)) \\ =& \mathbf{\Phi}(\tau,0)[\eta(0) - [h(z)](0) - g(0,\eta - h(z))] \\ & + g(\tau,z_\tau) + \int_0^\tau \mathbf{\Phi}(\tau,\theta)\left[A(\theta)g(\theta,z_\theta) + f(\theta,z_\theta)\right]d\theta \\ & + Cu + \sum_{0<t_k<\tau} \mathbf{\Phi}(\tau,t_k)J_k(t_k,z(t_k)). \end{aligned}$$

Replacing the control (3.16) above yields

$$\begin{aligned} z(\tau) =& \mathbf{\Phi}(\tau,0)[\eta(0) - [h(z)](0) - g(0,\eta - h(z))] \\ & + g(\tau,z_\tau) + \int_0^\tau \mathbf{\Phi}(\tau,\theta)\left[A(\theta)g(\theta,z_\theta) + f(\theta,z_\theta)\right]d\theta \\ & + CC^*(CC^*)^{-1}N(z) + \sum_{0<t_k<\tau} \mathbf{\Phi}(\tau,t_k)J_k(t_k,z(t_k)) \\ =& z_1. \end{aligned}$$

□

Theorem 27. *Suppose that* **[A1]**, **[A2]**, *and* **[A3]** *hold. If*

$$L_{-1} + ML_g q + ML_{-1}L_g q + \tau M_1 L_{-1} + M\tau L_1 + \tau M\|B\|_\infty\|S\|M_2 + MT < 1, \tag{3.17}$$

then the system (3.14) *is exactly controllable on* $[0,\tau]$.

Proof. We transform the controllability problem into a fixed point problem. For that purpose, we consider the following operator

$$\begin{aligned} K: \quad \mathcal{C}([0,\tau],\mathbb{R}^n) &\longrightarrow \mathcal{C}([0,\tau],\mathbb{R}^n) \\ z &\longmapsto K(z) \end{aligned}$$

given by

$$\begin{aligned} [K(z)](t) =& \mathbf{\Phi}(t,0)[\eta(0) - [h(z)](0) - g(0,\eta - h(z))] \\ &+ g(t,z_t) + \int_0^t \mathbf{\Phi}(t,\theta)\left[A(\theta)g(\theta,z_\theta) + f(\theta,z_\theta)\right] d\theta \\ &+ \int_0^t \mathbf{\Phi}(t,\theta)B(\theta)SN(z)d\theta + \sum_{0<t_k<t} \mathbf{\Phi}(t,t_k)J_k(t_k,z(t_k)), \quad t \in [0,\tau], \end{aligned}$$

To apply Banach contraction theorem, we need to prove that K is a contraction mapping. For doing so, we estimate the difference $\Pi_4 := \big\|[K(y)](t) - [K(z)](t)\big\|, t \in [0,\tau]$ for any $z, y \in \mathcal{C}([0,\tau],\mathbb{R}^n)$ as follows.

On the one hand, by the definition of K we get

$$\begin{aligned} \Pi_4 \leq & M\|g(0,\eta - h(z)) - g(0,\eta - h(y))\| + M\|[h(z)](0) - [h(y)](0)\| \\ &+ \big\|g(t,z_t) - g(t,y_t)\big\| + \int_0^t \|\mathbf{\Phi}(t,\theta)A(\theta)[g(\theta,z_\theta) - g(\theta,y_\theta)]\|d\theta \\ &+ \int_0^t \|\mathbf{\Phi}(t,\theta)[f(\theta,z_\theta) - f(\theta,y_\theta)]\|d\theta + \int_0^t \|\mathbf{\Phi}(t,\theta)B(\theta)S[N(z) - N(y)]\|d\theta \\ &+ \sum_{0<t_k<t} \|\mathbf{\Phi}(t,t_k)\|\|J_k(t_k,z(t_k)) - J_k(t_k,y(t_k))\|. \end{aligned}$$

Then, using the assumptions [A2] and [A3], and the above notation, we obtain

$$\begin{aligned} \Pi_4 \leq & L_{-1}\big\|z-y\big\| + ML_g q\big\|z-y\big\| + ML_{-1}L_g q\big\|z-y\big\| + \tau M_1 L_{-1}\big\|z-y\big\| \\ &+ M\tau L_1\big\|z-y\big\| + \tau M\|B\|_\infty\|S\|\big\|N(z) - N(y)\big\| + MT\big\|z-y\big\|. \end{aligned}$$

On the other hand, we have the estimate

$$\big\|N(z) - N(y)\big\| \leq M_2\big\|z-y\big\|.$$

Taking supremum over $t \in [0,\tau]$ yields

$$\begin{aligned} \big\|K(z) - K(y)\big\| \leq & \Big\{L_{-1} + ML_g q + ML_{-1}L_g q + \tau M_1 L_{-1} \\ &+ M\tau L_1 + \tau M\|B\|_\infty\|S\|M_2 + MT\Big\}\big\|z-y\big\|. \end{aligned}$$

Since $L_{-1} + ML_g q + ML_{-1}L_g q + \tau M_1 L_{-1} + M\tau L_1 + \tau M\|B\|_\infty\|S\|M_2 + MT < 1$, then K is a contraction mapping (see Theorem 6), and consequently it has a fixed point. This finishes the proof. $\square$

3.2.3 Approximate controllability

In this section we shall study the approximate controllability of the following semi-linear neutral differential equation with impulses and nonlocal conditions.

$$\begin{cases} \dfrac{d}{dt}\left[z(t) - g(t,z_t)\right] = A(t)z(t) + B(t)u(t) + f(t,z_t,u(t)), & t \in [0,\tau] \setminus \{t_k\}_{k\in I_p} \\ z(\theta) = -[h(z)](\theta) + \eta(\theta), & \theta \in [-r,0], \\ z(t_k^+) = z(t_k^-) + J_k(t_k,z(t_k),u(t_k)), & k \in I_p, \end{cases} \tag{3.18}$$

For doing so, we will employ a technique developed by Bashirov *et al.* in [15, 16, 17, 18, 20, 19]. This technique uses the delayed feature of the system. The delay allows us to prove approximate controllability by pulling back the control solution to a fixed curve in a compressed period of time. From such fixed curve we can reach a neighborhood of the final state in time τ by utilizing the exact controllability of the associated linear system on any interval $[\tau-\delta,\tau]$ where $0<\delta<\tau$ [88]. This technique has been used, for instance, in [47, 57, 85, 88, 92].

Before using Bashirov *et al.* technique to address the approximate controllability of the system (3.18), we illustrate its usage in the following more manageable system without impulses and nonlocal conditions.

$$\begin{cases} \dfrac{d}{dt}[z(t) - Gz(t-r)] = Az(t) + Fz(t-r) + Bu(t), & t \in [0,\tau], \\ z(\theta) = \eta(\theta), & \theta \in [-r,0]. \end{cases} \tag{3.19}$$

Here, we also have considered $A(t) := A$, $B(t) = B$, $g(t,z_t) := Gz(t-r)$, and $f(t,z_t,u(t)) := Fz(t-r)$, where A, B, F, and G are constant matrices of appropriate dimension. Corresponding to the system (3.19)(when $t_0 = 0$), we have the following linear system

$$\begin{cases} \dfrac{d}{dt}y(t) = Ay(t) + Bu(t), & t \in [t_0,\tau], \\ y(t_0) = z_0 \in \mathbb{R}^n. \end{cases} \tag{3.20}$$

From Theorem (10), the system (3.20) is controllable on $[t_0,\tau]$ if and only if

$$\operatorname{rank}[B|AB|A^2B|\cdots|A^{n-1}B] = n.$$

We note that Kálmán's rank condition is purely algebraic and does not depend on time. This realization leads us to the following remark.

Remark 7. *The system (3.20) is controllable on any interval, particularly, on $[t_0,\tau]$ with $t_0<\tau$.*

In the literature, a similar controllability characterization can be found for system (3.19). See, for example, [14, 70, 103, 111, 115, 117, 119, 134] and references therein.

Remark 8. *The system* (3.19) *is controllable if and only if*

$$\operatorname{rank}[\Delta(\lambda)B] = n \textit{ and } \operatorname{rank}[B|GB|G^2B|\cdots|G^{n-1}B] = n,$$

where $\Delta(\lambda) = \lambda I - \lambda G e^{-\lambda r} - A - F e^{-\lambda r}$.

This summarizes the results on exact controllability for the system (3.19). Now, we turn our attention to the approximate controllability.

Lemma 5. *If system* (3.20) *is controllable, then system* (3.19) *is approximately controllable on* $[0, \tau]$.

Proof. Suppose that the system (3.20) is exactly controllable. Then, from Remark 7, it is exactly controllable on any interval $[t_0, \tau]$, with $0 \leq t_0 < \tau$. Therefore, for any initial state z_0 and a final state z_1, there exists a control $u_{t_0} \in L^2\left([t_0, \tau], \mathbb{R}^m\right)$ such that the corresponding solution of the initial value problem (3.20) satisfies $y(\tau) = z_1$. Moreover, u_{t_0} can be taken (see (2.31)) as follows

$$u_{t_0}(t) = B^* e^{A^*(\tau - t)} W_{t_0}^{-1}\left(z_1 - e^{A(\tau - t_0)} z_0\right), \ t \in [t_0, \tau],$$

where

$$W_{t_0} = \int_{t_0}^{\tau} e^{A(\tau-\theta)} B B^* e^{A^*(\tau-\theta)} d\theta.$$

On the other hand, the solution of the initial value problem (3.19) is given by

$$\begin{aligned} z(t) =& Gz(t-r) + e^{At}[\eta(0) - G\eta(-r)] \\ &+ \int_0^t e^{A(t-\theta)}[AG+F]z(\theta - r)d\theta + \int_0^t e^{A(t-\theta)} Bu(\theta) d\theta. \end{aligned}$$

Let η, z_1 be the initial and the final state for system (3.19), respectively. Given $\epsilon > 0$. consider any fixed control $u \in L^2\left([0, \tau], \mathbb{R}^m\right)$ and the corresponding solution z of (3.19) evaluated at $t = \tau - d$,

$$\begin{aligned} z(\tau - d) =& Gz(\tau - d - r) + e^{A(\tau-d)}[\eta(0) - G\eta(-r)] \\ &+ \int_0^{\tau-d} e^{A(\tau-d-\theta)}[AG+F]z(\theta - r)d\theta + \int_0^{\tau-d} e^{A(\tau-d-\theta)} Bu(\theta) d\theta, \end{aligned} \tag{3.21}$$

where $0 < d < \min\{r, \tau - r, \epsilon/M\}$ and

$$M = \max_{0 \leq \theta \leq \tau}\left\{\left\| e^{A(\tau-d)}(AG+F)\right\| \|z(\theta)\|\right\}.$$

Define the control

$$u^d(t) = \begin{cases} u(t), & t \in [0, \tau - d], \\ u_{\tau-d}(t), & t \in (\tau - d, \tau], \end{cases} \tag{3.22}$$

where

$$u_{\tau-d}(t) = B^* e^{A^*(\tau-t)} G_{\tau-d}^{-1}(z_1 - e^{Ad} z_0)$$

and

$$z_0 = e^{-Ad} G z(\tau - d) - G z(\tau - d - r) + F z(\tau - d). \tag{3.23}$$

Regard $z^d(t) := z^d(t, \eta, u^d)$ as the corresponding solution of (3.19) for the control u^d, which we evaluate at $t = \tau$.

$$\begin{aligned} z^d(\tau) =& G z^d(\tau - r) + e^{A\tau}[\eta(0) - G\eta(-r)] \\ &+ \int_0^\tau e^{A(\tau-\theta)}[AG + F] z^d(\theta - r) d\theta + \int_0^\tau e^{A(\tau-\theta)} B u^d(\theta) d\theta. \end{aligned}$$

By Proposition 2(*ii*),

$$\begin{aligned} z^d(\tau) =& G z^d(\tau - r) + e^{Ad}\Big\{ e^{A(\tau-d)}[\eta(0) - G\eta(-r)] \\ &+ \int_0^{\tau-d} e^{A(\tau-d-\theta)}[AG + F] z^d(\theta - r) d\theta + \int_0^{\tau-d} e^{A(\tau-d-\theta)} B u^d(\theta) d\theta \Big\} \\ &+ \int_{\tau-d}^{\tau} e^{A(\tau-\theta)}[AG + F] z^d(\theta - r) d\theta + \int_{\tau-d}^{\tau} e^{A(\tau-\theta)} B u^d(\theta) d\theta. \end{aligned} \tag{3.24}$$

Adding $e^{Ad} G z(\tau - d - r) - e^{Ad} G z(\tau - d - r) = 0$ to the right hand side of (3.24) yields

$$\begin{aligned} z^d(\tau) =& G z^d(\tau - r) - e^{Ad} G z(\tau - d - r) + e^{Ad}\Big\{ G z(\tau - d - r) + e^{A(\tau-d)}[\eta(0) - G\eta(-r)] \\ &+ \int_0^{\tau-d} e^{A(\tau-d-\theta)}[AG + F] z(\theta - r) d\theta + \int_0^{\tau-d} e^{A(\tau-d-\theta)} B u(\theta) d\theta \Big\} \\ &+ \int_{\tau-d}^{\tau} e^{A(\tau-\theta)}[AG + F] z^d(\theta - r) d\theta + \int_{\tau-d}^{\tau} e^{A(\tau-\theta)} B u^d(\theta) d\theta \end{aligned}$$

Therefore,

$$\begin{aligned} z^d(\tau) =& G z^d(\tau - r) - e^{Ad} G z(\tau - d - r) + e^{Ad} z(\tau - d) \\ &+ \int_{\tau-d}^{\tau} e^{A(\tau-\theta)}[AG + F] z^d(\theta - r) d\theta + \int_{\tau-d}^{\tau} e^{A(\tau-\theta)} B u^d(\theta) d\theta. \end{aligned}$$

On $(\tau - d, \tau]$, $u^d(t) = u_{\tau-d}(t)$ (see (3.22)) and hence

$$\begin{aligned} z^d(\tau) =& G z^d(\tau - r) - e^{Ad} G z(\tau - d - r) + e^{Ad} z(\tau - d) \\ &+ \int_{\tau-d}^{\tau} e^{A(\tau-\theta)}[AG + F] z^d(\theta - r) d\theta + \int_{\tau-d}^{\tau} e^{A(\tau-\theta)} B u_{\tau-d}(\theta) d\theta. \end{aligned} \tag{3.25}$$

On the other hand, if we consider

$$z_0 = e^{-Ad} G z^d(\tau - r) - G z(\tau - d - r) + z(\tau - d),$$

then the solution of the initial value problem (3.20), with $t_0 = \tau - d$, evaluated at τ takes the form

$$\begin{aligned} z_1 =& y_d(\tau) \\ =& e^{Ad} z_0 + \int_{\tau-d}^{\tau} e^{A(t-\theta)} B u_{\tau-d}(\theta) d\theta \\ =& Gz^d(\tau - r) - e^{Ad} Gz(\tau - d - r) + e^{Ad} z(\tau - d) + \int_{\tau-d}^{\tau} e^{A(\tau-\theta)} B u_{\tau-d}(\theta) d\theta \end{aligned} \tag{3.26}$$

Hence, from (3.25) and (3.26),

$$\|z^d(\tau) - z_1\| \leq \int_{\tau-d}^{\tau} \|e^{A(\tau-d)}\| \|AG + F\| \|z^d(\theta - r)\| d\theta.$$

From the way we choose d, it turns out that $z^d(\theta - r) = z(\theta - r)$. Thus,

$$\|z^d(\tau) - z_1\| \leq \int_{\tau-d}^{\tau} \|e^{A(\tau-d)}\| \|AG + F\| \|z(\theta - r)\| d\theta < dM < \epsilon.$$

□

Having introduced Bashirov *et al.* technique for studying the approximate controllability (3.19), we now dedicate to proving the approximate controllability of the system (3.18).

From Section 3.1, the system (3.18) admits a solution given by

$$z(t) = \begin{cases} \mathbf{\Phi}(t,0)[\eta(0) - [h(z)](0) - g(0, \eta - h(z))] \\ + g(t, z_t) + \int_0^t \mathbf{\Phi}(t,\theta) \left[A(\theta) g(\theta, z_\theta) + f(\theta, z_\theta, u(\theta)) \right] d\theta \\ + \int_0^t \mathbf{\Phi}(t,\theta) B(\theta) u(\theta) d\theta + \sum_{0<t_k<t} \mathbf{\Phi}(t, t_k) J_k(t_k, z(t_k), u(t_k)), & t \in [0, \tau], \\ \eta(t) - [h(z)](t), & t \in [-r, 0]. \end{cases} \tag{3.27}$$

From Subsection 2.3.2, we know that the corresponding linear system

$$\frac{d}{dt} y(t) = A(t) y(t) + B(t) v(t), \; t \in [\tau - \delta, \tau] \tag{3.28}$$

admits only one solution (see Theorem 9 and (2.29)) given by

$$y^\delta(t) = \mathbf{\Phi}(t, \tau - \delta) z_0 + \int_{\tau-\delta}^{t} \mathbf{\Phi}(t,\theta) B(\theta) v^\delta(\theta) d\theta, \; t \in [\tau - \delta, \tau]. \tag{3.29}$$

We also know that a control (see (2.31)) steering system (3.28) from z_0 to $y^\delta(\tau) = z_1$ at time $\tau > 0$ is given by

$$v^\delta(t) = B^*(t) \mathbf{\Phi}^*(\tau, t) W^{-1}(z_1 - \mathbf{\Phi}(\tau, \tau - \delta) z_0), \; t \in [\tau - \delta, \tau].^2 \tag{3.30}$$

[2] Here $W = CC^*$ is defined by $W(x) = \int_{\tau-\delta}^{\tau} \mathbf{\Phi}(\tau,\theta) B(\theta) B^*(\theta) \mathbf{\Phi}^*(\tau,\theta) x d\theta$ and C is given by $C(v^\delta) = \int_{\tau-\delta}^{\tau} \mathbf{\Phi}(\tau,\theta) B(\theta) v^\delta(\theta) d\theta$.

In order to prove the approximate controllability of the system (3.18) we need to impose the following assumptions.

[a1] The linear system (3.28) is exactly controllable in any interval $[t_0, \tau]$ with $0 < t_0 < \tau$.

[a2] The functions g and f satisfy

$$\|g(t,\eta)\| \leq \rho\left(\|\eta(-r)\|\right) \text{ and } \|f(t,\eta,\nu)\| \leq \varrho\left(\|\eta(-r)\|\right),$$

respectively, where $\rho, \varrho : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ are continuous functions.

The next theorem provides the approximate controllability of the system (3.18) through Bashirov *et al.* technique.

Theorem 28. *Under the hypotheses* **[a1]** *and* **[a2]***, the semilinear neutral differential equation with impulses and nonlocal conditions* (3.18) *is approximately controllable on* $[0,\tau]$.

Proof. Given $\epsilon > 0$, consider any fixed control $u \in L^2\left([0,\tau], \mathbb{R}^m\right)$ and the corresponding solution $z(t) = z(t,\eta,u)$ of system (3.18). Also consider a number $\delta > 0$ such that $0 < \delta < \min\{r, \tau - r, \tau - t_p, \epsilon/MN\}$, where

$$\begin{aligned} M &= \sup_{\theta \in [0,\tau]} \left\{\|\mathbf{\Phi}(\tau,\theta)\| \|A(\theta)\|\right\}, \\ N &= \max_{\theta \in [0,\tau]} \left\{\rho\left(\|z(\theta - r)\|\right) + \varrho\left(\|z(\theta - r)\|\right)\right\}. \end{aligned}$$

We define a control $u^\delta \in L^2\left([0,\tau], \mathbb{R}^m\right)$ as follows.

$$u^\delta(t) = \begin{cases} u(t), & t \in [0, \tau - \delta], \\ v^\delta(t), & t \in (\tau - \delta, \tau], \end{cases} \tag{3.31}$$

where v^δ is given in (3.30).

Now, on the one hand, let $z^\delta(t) = z^\delta(t,\eta,u^\delta)$ be the corresponding solution of (3.18) for the control u^δ defined above. At time τ, we have that

$$\begin{aligned} z^\delta(\tau) =& \mathbf{\Phi}(\tau,0)[\eta(0) - [h(z^\delta)](0) - g(0, \eta - h(z^\delta))] \\ &+ g(\tau, z^\delta_\tau) + \int_0^\tau \mathbf{\Phi}(\tau,\theta)\left[A(\theta) g(\theta, z^\delta_\theta) + f(\theta, z^\delta_\theta, u^\delta(\theta))\right] d\theta \\ &+ \int_0^\tau \mathbf{\Phi}(\tau,\theta) B(\theta) u^\delta(\theta) d\theta + \sum_{0 < t_k < \tau} \mathbf{\Phi}(\tau, t_k) J_k(t_k, z^\delta(t_k), u^\delta(t_k)). \end{aligned} \tag{3.32}$$

By the cocycle property of $\mathbf{\Phi}$ (see Proposition 2(*ii*)), we obtain

$$\begin{aligned} z^\delta(\tau) =& g(\tau, z^\delta_\tau) + \mathbf{\Phi}(\tau, \tau-\delta)\Big\{ \mathbf{\Phi}(\tau-\delta, 0)[\eta(0) - [h(z^\delta)](0) - g(0, \eta - h(z^\delta))] \\ &+ \int_0^{\tau-\delta} \mathbf{\Phi}(\tau-\delta, \theta)\left[A(\theta)g(\theta, z^\delta_\theta) + f(\theta, z^\delta_\theta, u^\delta(\theta))\right] d\theta \\ &+ \int_0^{\tau-\delta} \mathbf{\Phi}(\tau-\delta, \theta)B(\theta)u^\delta(\theta)d\theta + \sum_{0<t_k<\tau} \mathbf{\Phi}(\tau-\delta, t_k)J_k(t_k, z^\delta(t_k), u^\delta(t_k))\Big\} \\ &+ \int_{\tau-\delta}^{\tau} \mathbf{\Phi}(\tau, \theta)\left[A(\theta)g(\theta, z^\delta_\theta) + f(\theta, z^\delta_\theta, u^\delta(\theta))\right] d\theta \\ &+ \int_{\tau-\delta}^{\tau} \mathbf{\Phi}(\tau, \theta)B(\theta)u^\delta(\theta)d\theta. \end{aligned} \tag{3.33}$$

If we add $\mathbf{\Phi}(\tau, \tau-\delta)g(\tau-\delta, z^\delta_{\tau-\delta}) - \mathbf{\Phi}(\tau, \tau-\delta)g(\tau-\delta, z^\delta_{\tau-\delta}) = 0$ to the right hand side of (3.33), then it becomes

$$\begin{aligned} z^\delta(\tau) =& g(\tau, z^\delta_\tau) + \mathbf{\Phi}(\tau, \tau-\delta)\Big\{ \mathbf{\Phi}(\tau-\delta, 0)[\eta(0) - [h(z)](0) - g(0, \eta - h(z))] \\ &+ \int_0^{\tau-\delta} \mathbf{\Phi}(\tau-\delta, \theta)\left[A(\theta)g(\theta, z_\theta) + f(\theta, z_\theta, u(\theta))\right] d\theta + g(\tau-\delta, z_{\tau-\delta}) \\ &+ \int_0^{\tau-\delta} \mathbf{\Phi}(\tau-\delta, \theta)B(\theta)u(\theta)d\theta + \sum_{0<t_k<\tau} \mathbf{\Phi}(\tau-\delta, t_k)J_k(t_k, z(t_k), u(t_k))\Big\} \\ &+ \int_{\tau-\delta}^{\tau} \mathbf{\Phi}(\tau, \theta)\left[A(\theta)g(\theta, z^\delta_\theta) + f(\theta, z_\theta, u^\delta(\theta))\right] d\theta - \mathbf{\Phi}(\tau, \tau-\delta)g(\tau-\delta, z^\delta_{\tau-\delta}) \\ &+ \int_{\tau-\delta}^{\tau} \mathbf{\Phi}(\tau, \theta)B(\theta)u^\delta(\theta)d\theta. \end{aligned}$$

Hence,

$$\begin{aligned} z^\delta(\tau) =& g(\tau, z^\delta_\tau) + \mathbf{\Phi}(\tau, \tau-\delta)z(\tau-\delta) - \mathbf{\Phi}(\tau, \tau-\delta)g(\tau-\delta, z^\delta_{\tau-\delta}) \\ &+ \int_{\tau-\delta}^{\tau} \mathbf{\Phi}(\tau, \theta)\left[A(\theta)g(\theta, z^\delta_\theta) + f(\theta, z^\delta_\theta, u^\delta(\theta))\right] d\theta + \int_{\tau-\delta}^{\tau} \mathbf{\Phi}(\tau, \theta)B(\theta)u^\delta(\theta)d\theta. \end{aligned}$$

Since $u^\delta(\theta) = v^\delta(\theta), \theta \in (\tau-\delta, \tau]$ (see (3.31)), we get

$$\begin{aligned} z^\delta(\tau) =& g(\tau, z^\delta_\tau) + \mathbf{\Phi}(\tau, \tau-\delta)z(\tau-\delta) - \mathbf{\Phi}(\tau, \tau-\delta)g(\tau-\delta, z^\delta_{\tau-\delta}) \\ &+ \int_{\tau-\delta}^{\tau} \mathbf{\Phi}(\tau, \theta)\left[A(\theta)g(\theta, z^\delta_\theta) + f(\theta, z^\delta_\theta, v^\delta(\theta))\right] d\theta + \int_{\tau-\delta}^{\tau} \mathbf{\Phi}(\tau, \theta)B(\theta)v^\delta(\theta)d\theta. \end{aligned}$$

On the other hand, the corresponding solution $y(t) = y(t, z_0, v^\delta)$ of the linear system (3.28) at time τ is given by (see (3.29))

$$y^\delta(\tau) = \mathbf{\Phi}(\tau, \tau-\delta)z_0 + \int_{\tau-\delta}^{\tau} \mathbf{\Phi}(\tau, \theta)B(\theta)v^\delta(\theta)d\theta. \tag{3.34}$$

Letting $z_0 = z(\tau - \delta) + \mathbf{\Phi}(\tau - \delta, \tau) g(\tau, z^\delta_\tau) - g(\tau - \delta, z^\delta_{\tau-\delta})$ and considering $z_1 = y^\delta(\tau)$, we then have that

$$\begin{aligned} \|z^\delta(t) - z_1\| &= \left\| \int_{\tau-\delta}^{\tau} \mathbf{\Phi}(\tau,\theta) \left[A(\theta) g(\theta, z^\delta_\theta) + f(\theta, z^\delta_\theta, v^\delta(\theta)) \right] d\theta \right\| \\ &\leq \int_{\tau-\delta}^{\tau} \|\mathbf{\Phi}(\tau,\theta)\| \left[\|A(\theta)\| \|g(\theta, z^\delta_\theta)\| + \|f(\theta, z^\delta_\theta, v^\delta(\theta))\| \right] d\theta \\ &\leq \int_{\tau-\delta}^{\tau} M \left[\rho \left(\|z^\delta(\theta - r)\| \right) + \varrho \left(\|z^\delta(\theta - r)\| \right) \right] d\theta. \end{aligned}$$

Now, we observe that $0 < \delta < r$ and $\tau - \delta \leq \theta \leq \tau$ implies $\theta - r \leq \tau - r < \tau - \delta$ and consequently $z^\delta(\theta - r) = z(\theta - r)$. Thus,

$$\|z^\delta(t) - z_1\| \leq \int_{\tau-\delta}^{\tau} M \left[\rho \left(\|z(\theta - r)\| \right) + \varrho \left(\|z(\theta - r)\| \right) \right] d\theta \leq \delta M N < \epsilon.$$

Last inequality gives the approximate controllability (Definition 15) of system (3.18). This finishes the proof. □

Chapter 4

Results in infinite-dimensional systems

This chapter extends the existence results presented in Chapter 3. This extension is in the sense of dimension. While in Chapter 3, we worked in a finite-dimensional setting ($\dim(\mathbb{R}^n) = n < \infty$), in this chapter, we work in an infinite-dimensional Banach space Z ($\dim(Z) = \infty$). As we shall see, this apparently minor change poses a significant complication in the mathematical techniques and tools used to address the general problem in the infinite-dimensional setting.

4.1 Existence results

In this section, we study the existence and uniqueness of solutions for the following semilinear neutral evolution equation in a Banach space Z with impulses and nonlocal conditions

$$\begin{cases} \dfrac{d}{dt}[z(t) - g(t, z_t)] = -Az(t) + f(t, z_t), & t \in (0, \tau] \setminus \{t_k\}_{k \in I_p}, \\ z(\theta) + [h(z_{\tau_1}, \dots, z_{\tau_q})](\theta) = \eta(\theta), & \theta \in [-r, 0], \\ z(t_k^+) = z(t_k^-) + J_k(z(t_k)), & k \in I_p, \end{cases} \tag{4.1}$$

where $A : D(A) \subset Z \to Z$ is a sectorial operator such that its resolvent operator is compact, $0 < t_1 < \cdots < t_p < \tau$, $0 < \tau_1 < \cdots < \tau_q < r < \tau$, z_t is the time history function $[-r, 0] \ni \theta \mapsto z_t(\theta) = z(t + \theta) \in Z^\alpha$, Z^α is the fractional power space of A, $g, f : [0, \tau] \times PW_{r\alpha} \to Z$, $h : PW_{qp\alpha} \to PW_{r\alpha}$, $J_k : Z^\alpha \to Z^\alpha$, $\eta \in PW_{r\alpha}$ are appropriate smooth functions, and the spaces $PW_{qp\alpha}$ and $PW_{r\alpha}$ are defined below. We assume without loss of generality (see Remark 5) that $0 \in \rho(A)$. By Theorem 17, $-A$ is the infinitesimal generator of an analytic semigroup $T(t)$, which is compact as a consequence of Proposition 3. We further assume without loss of generality (see Remark 3) that $T(t)$ is uniformly bounded. From Subsection 2.4.3, we can see that the fractional power spaces $Z^\alpha, 0 < \alpha \leq 1$ are well defined. Moreover, Z^α is dense in Z (see Theorem 20(*iii*)), Z^α is a Banach space (see Theorem 23) when endowed with the norm $|\cdot|_\alpha$ (see (2.39)), and the embedding $Z^\alpha \hookrightarrow Z^\beta$ is compact (see Theorem 23) for $0 < \beta < \alpha \leq 1$.

Here, $PW_{r\alpha}$ is the Banach space

$$PW([-r,0],Z^{\alpha}) = \left\{ \eta : [-r,0] \longrightarrow Z^{\alpha} \;\middle|\; \eta \text{ is piecewise continuous} \right\} \tag{4.2}$$

equipped with the supremum norm $\|\cdot\|_{r\alpha}$. Let $[0,\tau]' = [0,\tau] \setminus \{t_k\}_{k\in I_p}$. A suitable Banach space to work with impulsive differential systems is the following.

$$\begin{aligned} PW_{p\alpha} := \Big\{ z : [-r,\tau] \longrightarrow Z^{\alpha} \;\Big|\; & z|_{[-r,0]} \in PW_{r\alpha},\ z|_{[0,\tau]} \in C\left([0,\tau]',Z^{\alpha}\right) \text{ and the} \\ & \text{one-sided limits } z(t_k^-), z(t_k^+) \text{ exist with } z(t_k^-) = z(t_k) \text{ for all } k \in I_p \Big\} \end{aligned}$$

equipped with the supremum norm $\|\cdot\|_{p\alpha}$. Also, we shall consider the Banach space

$$(Z^{\alpha})^q = \underbrace{Z^{\alpha} \times Z^{\alpha} \times \cdots \times Z^{\alpha}}_{q-\text{times}},$$

endowed with the norm

$$\|z\|_{q\alpha} = \sum_{i=1}^{q} |z_i|_{\alpha},\ z = (z_1, z_2, \ldots, z_q) \in (Z^{\alpha})^q .$$

Similarly to $PW_{r\alpha}$, we define the Banach space

$$PW_{qp\alpha} := PW([-r,0],(Z^{\alpha})^q) \tag{4.3}$$

with norm

$$\|\eta\|_{qp\alpha} = \sup_{t\in[-r,0]} \|\eta(t)\|_{q\alpha} = \sup_{t\in[-r,0]} \left(\sum_{i=1}^{q} |\eta_i(t)|_{\alpha} \right),\ \eta = \left(\eta_1, \ldots, \eta_q\right) \in PW_{qp\alpha}.$$

In order to prove the existence of solutions for system (4.1) we establish the following definition to characterize the compactness in $C\left([0,\tau]',Z^{\alpha}\right)$.

Definition 25. *Let z be a function belonging to $C\left([0,\tau]',Z^{\alpha}\right)$. For $i \in I_p$, we define $\bar{z}_i \in C([t_i,t_{i+1}],Z^{\alpha})$ by*

$$\bar{z}_i(t) = \begin{cases} z(t), & t \in (t_i, t_{i+1}], \\ z(t_i^+), & t = t_i, \end{cases} \tag{4.4}$$

and the set $\overline{H}_i = \{\bar{y}_i : y \in H\}$, where H is any subset of $C\left([0,\tau]',Z^{\alpha}\right)$.

The next lemma provides the aforementioned compactness characterization.

Lemma 6. *A set $H \subset C\left([0,\tau]',Z^{\alpha}\right)$ is relatively compact in $C\left([0,\tau]',Z^{\alpha}\right)$ if, and only if, each set $\overline{H}_i, i \in I_p$, with $t_0 = 0$ and $t_{p+1} = \tau$, is relatively compact in $C([t_i,t_{i+1}],Z^{\alpha})$.*

For more details regarding this characterization, we refer the reader to [67].

4.1.1 Existence Theorems

This subsection is devoted to prove the main results of this chapter, which concerns with the existence and uniqueness of mild solutions for the system (4.1). The following definition characterizes such solutions (see, *e.g.*, [6, 11, 12, 86]).

Definition 26. *A function $z \in PW_{p\alpha}$ is said to be a mild solution of problem (4.1) if it satisfies the integral equation*

$$z(t) = \begin{cases} T(t)[\eta(0) - [h(z)](0) - g(0, \eta - h(z))] \\ - \int_0^t AT(t-\theta)g(\theta, z_\theta)d\theta + \int_0^t T(t-\theta)f(\theta, z_\theta)d\theta + g(t, z_t) \\ + \sum_{0<t_k<t} T(t-t_k)J_k(z(t_k)), & t \in [0,\tau], \\ \eta(t) - [h(z)](t), & t \in [-r, 0]. \end{cases} \tag{4.5}$$

Let us consider the following hypotheses.

[P1] There exist positive constants L_h, Υ, and $d_k, k \in I_p$ such that

(*i*) $L_h q M < \Upsilon + M \sum_{k=1}^{p} d_k < \frac{1}{2}$,

(*ii*) $J_k(0) = 0$ and $|J_k(y) - J_k(z)|_\alpha \leq d_k |y - z|_\alpha,\ y, z \in Z^\alpha$,

(*iii*) $h(0) = 0$ and

$$|[h(y)](t) - [h(v)](t)|_\alpha \leq L_h \sum_{i=1}^{q} |y_i(t) - v_i(t)|_\alpha,\ y, v \in PW_{p\alpha},$$

where M is given in (2.33).

[P2] The map $g : [0,\tau] \times PW_{r\alpha} \to D(A)$ satisfies

(*i*) [1] $\|Ag(t,\eta_1) - Ag(t,\eta_2)\| \leq \mathcal{K}\left(\|\eta_1\|_{r\alpha}, \|\eta_2\|_{r\alpha}\right)\|\eta_1 - \eta_2\|_{r\alpha},\ \eta_1, \eta_2 \in PW_{r\alpha}$,

(*ii*) $\|Ag(t,\eta)\| \leq \Psi\left(\|\eta\|_{r\alpha}\right),\ \eta \in PW_{r\alpha}$,

(*iii*) $\|g(t,\eta_1) - g(t,\eta_2)\| \leq \Upsilon \|\eta_1 - \eta_2\|_{r\alpha},\ \eta_1, \eta_2 \in PW_{r\alpha}$

and the mapping $f : [0,\tau] \times PW_{r\alpha} \to Z$ satisfies

[1] **[P2]**(*i*) and **[P2]**(*ii*) also work for A^α instead of A when $0 < \alpha \leq 1$ by the continuous embedding (see (2.41)). The constant c that appears in (2.41) will be deliberately omitted because it is irrelevant in practical terms.

(*iv*) $\|f(t,\eta_1) - f(t,\eta_2)\| \le \mathcal{K}(\|\eta_1\|_{r\alpha}, \|\eta_2\|_{r\alpha})\|\eta_1 - \eta_2\|_{r\alpha}$, $\eta_1, \eta_2 \in PW_{r\alpha}$,

(*v*) $\|f(t,\eta)\| \le \Psi(\|\eta\|_{r\alpha})$, $\eta \in PW_{r\alpha}$,

where $\mathcal{K} \in \mathcal{C}(\mathbb{R}_+ \times \mathbb{R}_+, \mathbb{R}_+)$ and $\Psi \in \mathcal{C}(\mathbb{R}_+, \mathbb{R}_+)$ are non-decreasing functions.

[P3] There exists $\rho > 0$ such that

$$M\Psi\left(\|\eta\| + L_g q(\|\tilde{\eta}\| + \rho)\right) + \left(ML_g q + M\sum_{k=1}^{p} d_k\right)(\|\tilde{\eta}\| + \rho)$$
$$+\left(\frac{2M_\alpha}{1-\alpha}\tau^{1-\alpha} + 1\right)\Psi(\|\tilde{\eta}\| + \rho) < \rho$$

where the function $\tilde{\eta}$ is defined as follows

$$\tilde{\eta}(t) = \begin{cases} T(t)\eta(0), & t \in [0,\tau], \\ \eta(t), & t \in [-r,0]. \end{cases} \tag{4.6}$$

[P4] Assume the following relation holds.

$$ML_g q(1+\Upsilon) + 2M_\alpha \mathcal{K}\left(\|\tilde{\eta}\| + \rho, \|\tilde{\eta}\| + \rho\right)\frac{\tau^{1-\alpha}}{1-\alpha} < \frac{1}{2}.$$

Theorem 29. *Suppose that* **[P1]**, **[P2]**, *and* **[P3]** *hold. Then problem (4.1) has a least one mild solution in $PW_{p\alpha}$.*

Proof. We base the argument on Theorem 7. To that end, we introduce two operators $\mathcal{Q}$ and $\mathcal{P}$ as follows.

$$\begin{array}{rccc} \mathcal{Q}: & PW_{p\alpha} \times PW_{p\alpha} & \longrightarrow & PW_{p\alpha} \\ & (z,y) & \longmapsto & \mathcal{Q}(z,y) \end{array}$$

defined by

$$[\mathcal{Q}(z,y)](t) = \begin{cases} y(t) + g(t,z_t) + \sum\limits_{0<t_k<t} T(t-t_k)J_k(z(t_k)), & t \in [0,\tau], \\ \eta(t) - [h(z)](t), & t \in [-r,0], \end{cases}$$

and

$$\begin{array}{rccc} \mathcal{P}: & PW_{p\alpha} & \longrightarrow & PW_{p\alpha} \\ & y & \longmapsto & \mathcal{P}(y) \end{array}$$

given by

$$[\mathcal{P}(y)](t) = \begin{cases} T(t)[\eta(0) - [h(y)](0) - g(0,\eta - h(y))] & \\ -\displaystyle\int_0^t AT(t-\theta)g(\theta,y_\theta)d\theta + \int_0^t T(t-\theta)f(\theta,y_\theta)d\theta, & t \in [0,\tau], \\ \eta(t), & t \in [-r,0]. \end{cases} \tag{4.7}$$

In what follows, we show that $\mathcal{Q}$ and $\mathcal{P}$ satisfy the hypotheses of Theorem 7.

Step 1. *$\mathcal{P}$ is a continuous mapping.*

Let $z, y \in PW_{p\alpha}$ and consider the difference

$$\begin{aligned}
\Pi_5 := &\, |[\mathcal{P}(z)](t) - [\mathcal{P}(y)](t)|_\alpha \\
= &\, \Big| T(t)[[h(y)](0) - [h(z)](0)] + \int_0^t AT(t-\theta)[g(\theta, y_\theta) - g(\theta, z_\theta)]d\theta \\
&+ T(t)[g(0, \eta - h(y)) - g(0, \eta - h(z))] + \int_0^t T(t-\theta)[f(\theta, z_\theta) - f(\theta, y_\theta)]d\theta \Big|_\alpha \\
\leq &\, |T(t)[[h(y)](0) - [h(z)](0)]|_\alpha + \int_0^t |AT(t-\theta)[g(\theta, y_\theta) - g(\theta, z_\theta)]|_\alpha d\theta \\
&+ |T(t)[g(0, \eta - h(y)) - g(0, \eta - h(z))]|_\alpha + \int_0^t |T(t-\theta)[f(\theta, z_\theta) - f(\theta, y_\theta)]|_\alpha d\theta.
\end{aligned}$$

By definition of $|\cdot|_\alpha$ and the boundedness of $T(t)$ we have that

$$\begin{aligned}
\Pi_5 \leq &\, \|T(t)\| \cdot |[h(y)](0) - [h(z)](0)|_\alpha + \int_0^t \|A^\alpha AT(t-\theta)[g(\theta, y_\theta) - g(\theta, z_\theta)]\| d\theta \\
&+ \|T(t)\| \cdot |g(0, \eta - h(y)) - g(0, \eta - h(z))|_\alpha + \int_0^t \|A^\alpha T(t-\theta)[f(\theta, z_\theta) - f(\theta, y_\theta)]\| d\theta.
\end{aligned}$$

By Theorem 12(*v*)[2] (A and $T(t)$ commute on $D(A)$), by Theorem 22 ($A^\alpha T(t-\theta)$ is bounded), by (2.33) ($\|T(t)\| \leq M$), and by hypothesis **[P1]**(*iii*) and **[P2]**(*iii*), we obtain

$$\begin{aligned}
\Pi_5 \leq &\, ML_g q\|y - z\| + + \int_0^t \|A^\alpha T(t-\theta)\| \cdot \|A[g(\theta, y_\theta) - g(\theta, z_\theta)]\| d\theta \\
&+ MY\|h(y) - h(z)\| + \int_0^t \|A^\alpha T(t-\theta)\| \cdot \|[f(\theta, z_\theta) - f(\theta, y_\theta)]\| d\theta.
\end{aligned}$$

Hypotheses **[P2]**(*i*)(*iv*) and (2.37) yield the following estimate.

$$\begin{aligned}
\Pi_5 \leq &\, ML_g q\|y - z\| + + \int_0^t \frac{M_\alpha}{(t-\theta)^\alpha} \|A[g(\theta, y_\theta) - g(\theta, z_\theta)]\| d\theta \\
&+ MYL_g q\|y - z\| + \int_0^t \frac{M_\alpha}{(t-\theta)^\alpha} \|[f(\theta, z_\theta) - f(\theta, y_\theta)]\| d\theta \\
\leq &\, ML_g q\|y - z\| + + M_\alpha \int_0^t \frac{1}{(t-\theta)^\alpha} \mathcal{K}\left(\|z_\theta\|, \|y_\theta\|\right) \cdot \|z_\theta - y_\theta\| d\theta \\
&+ MYL_g q\|y - z\| + M_\alpha \int_0^t \frac{1}{(t-\theta)^\alpha} \mathcal{K}\left(\|z_\theta\|, \|y_\theta\|\right) \cdot \|z_\theta - y_\theta\| d\theta \\
\leq &\, ML_g q\|y - z\| + MYL_g q\|y - z\| + 2M_\alpha \mathcal{K}\left(\|z\|, \|y\|\right) \|z - y\| \int_0^t \frac{1}{(t-\theta)^\alpha} d\theta.
\end{aligned}$$

After computing the integral, we finally obtain

$$\Pi_5 \leq ML_g q\|y - z\| + MYL_g q\|y - z\| + 2M_\alpha \mathcal{K}\left(\|z\|, \|y\|\right) \|z - y\| \frac{\tau^{1-\alpha}}{1-\alpha}. \tag{4.8}$$

[2]Or Theorem 21(*ii*) with $\alpha = 1$.

Therefore,

$$\|\mathcal{P}(z)-\mathcal{P}(y)\|_{p\alpha} \le \left[ML_g q\,(1+\Upsilon)+2M_\alpha\mathcal{K}\left(\|z\|,\|y\|\right)\frac{\tau^{1-\alpha}}{1-\alpha}\right]\cdot\|y-z\|_{p\alpha}, \tag{4.9}$$

which means that $\mathcal{P}$ is continuous. In fact, (4.9) shows that $\mathcal{P}$ is locally Lipschitz.

Step 2. *For a bounded set $B\subset PW_{p\alpha}$, $\mathcal{P}(B)$ is bounded in $PW_{p\alpha}$.*

Let $R>0$ and consider $B:=\{z\in PW_{p\alpha}:\|z\|\le R\}$. Then, it is enough to show that there exists $\ell>0$ such that for all $y\in B$ we have that $\|\mathcal{P}(y)\|\le\ell$. Let us consider

$$\ell = M\left[|\eta(0)|_\alpha + L_g qR + \Psi(\|\eta\|+L_g qR\|)\right] + 2\Psi(\|y\|)M_\alpha\frac{\tau^{1-\alpha}}{1-\alpha}.$$

For any $y\in B$, we get

$$\begin{aligned}
|[\mathcal{P}(y)](t)|_\alpha &= \Big|T(t)[\eta(0)-[h(y)](0)-g(0,\eta-h(y))]\\
&\quad -\int_0^t AT(t-\theta)g(\theta,y_\theta)d\theta+\int_0^t T(t-\theta)f(\theta,y_\theta)d\theta\Big|_\alpha\\
&\le \|T(t)\|\cdot|\eta(0)-[h(y)](0)-g(0,\eta-h(y))|_\alpha\\
&\quad +\int_0^t |AT(t-\theta)g(\theta,y_\theta)|_\alpha d\theta+\int_0^t |T(t-\theta)f(\theta,y_\theta)|_\alpha d\theta
\end{aligned}$$

The definition of $|\cdot|_\alpha$ and hypothesis **[P2]***(ii)* allow us to write

$$\begin{aligned}
|[\mathcal{P}(y)](t)|_\alpha &\le M\left[|\eta(0)|_\alpha+|[h(y)](0)|_\alpha+\|A^\alpha g(0,\eta-h(y))\|\right]\\
&\quad+\int_0^t \|A^\alpha AT(t-\theta)g(\theta,y_\theta)\|d\theta+\int_0^t\|A^\alpha T(t-\theta)f(\theta,y_\theta)\|d\theta\\
&\le M\left[|\eta(0)|_\alpha+L_g q\|y\|+\Psi(\|\eta-h(y)\|)\right]\\
&\quad+\int_0^t\|A^\alpha T(t-\theta)\|\|Ag(\theta,y_\theta)\|d\theta+\int_0^t\|A^\alpha T(t-\theta)\|\|f(\theta,y_\theta)\|d\theta
\end{aligned}$$

By hypotheses **[P2]***(ii)(v)*, it follows that

$$\begin{aligned}
|[\mathcal{P}(y)](t)|_\alpha &\le M\left[|\eta(0)|_\alpha+L_g qR+\Psi(\|\eta\|+\|h(y)\|)\right]\\
&\quad+\int_0^t\frac{M_\alpha}{(t-\theta)^\alpha}\Psi(\|y_\theta\|)d\theta+\int_0^t\frac{M_\alpha}{(t-\theta)^\alpha}\Psi(\|y_\theta\|)d\theta\\
&\le M\left[|\eta(0)|_\alpha+L_g qR+\Psi(\|\eta\|+L_g q\|y\|)\right]+2\Psi(\|y\|)M_\alpha\frac{\tau^{1-\alpha}}{1-\alpha}\\
&\le M\left[|\eta(0)|_\alpha+L_g qR+\Psi(\|\eta\|+L_g R)\right]+2\Psi(R)M_\alpha\frac{\tau^{1-\alpha}}{1-\alpha},
\end{aligned}$$

and therefore $\|\mathcal{P}(y)\|\le\ell$.

Step 3. *For a bounded set $B \subset PW_{p\alpha}$, $\mathcal{P}(B)$ is an equicontinuous set in $PW_{p\alpha}$.*

Let B be as above and let us show that $\mathcal{P}(B)$ is an equicontinuous family for $t \in [-r, \tau]$. For $t \in [-r, 0]$ the result is clear. For $t \in (0, \tau]$, more attention is needed. Let $0 < \theta_1 < \theta_2 < \tau$ and consider the difference $\Pi_6 := |[\mathcal{P}(y)](\theta_2) - [\mathcal{P}(y)](\theta_1)|_\alpha$. From (4.7), we have that

$$
\begin{aligned}
&\Pi_6 \\
=&\Big| \left[T(\theta_2) - T(\theta_1)\right] \left[\eta(0) - [h(y)](0) - g(0, \eta - h(y))\right] \\
&+ \int_0^{\theta_1-\epsilon} A\left[T(\theta_1-\theta) - T(\theta_2-\theta)\right] g(\theta, y_\theta)d\theta + \int_{\theta_1-\epsilon}^{\theta_1} A\left[T(\theta_1-\theta) - T(\theta_2-\theta)\right] g(\theta, y_\theta)d\theta \\
&- \int_0^{\theta_1-\epsilon} \left[T(\theta_1-\theta) - T(\theta_2-\theta)\right] f(\theta, y_\theta)d\theta - \int_{\theta_1-\epsilon}^{\theta_1} \left[T(\theta_1-\theta) - T(\theta_2-\theta)\right] f(\theta, y_\theta)d\theta \\
&- \int_{\theta_1}^{\theta_2} AT(\theta_2-\theta)g(\theta, y_\theta)d\theta + \int_{\theta_1}^{\theta_2} T(\theta_2-\theta)f(\theta, y_\theta)d\theta\Big|_\alpha
\end{aligned}
$$

Hence,

$$
\begin{aligned}
\Pi_6 \le& \Big| \left[T(\theta_2) - T(\theta_1)\right] \left[\eta(0) - [h(y)](0) - g(0, \eta - h(y))\right]\Big|_\alpha \\
&+ \int_0^{\theta_1-\epsilon} |A\left[T(\theta_1-\theta) - T(\theta_2-\theta)\right] g(\theta, y_\theta)|_\alpha d\theta \\
&+ \int_0^{\theta_1-\epsilon} |\left[T(\theta_1-\theta) - T(\theta_2-\theta)\right] f(\theta, y_\theta)|_\alpha d\theta \\
&+ \int_{\theta_1-\epsilon}^{\theta_1} |A\left[T(\theta_1-\theta) - T(\theta_2-\theta)\right] g(\theta, y_\theta)|_\alpha d\theta \\
&+ \int_{\theta_1-\epsilon}^{\theta_1} |\left[T(\theta_1-\theta) - T(\theta_2-\theta)\right] f(\theta, y_\theta)|_\alpha d\theta \\
&+ \int_{\theta_1}^{\theta_2} |AT(\theta_2-\theta)g(\theta, y_\theta)|_\alpha d\theta + \int_{\theta_1}^{\theta_2} |T(\theta_2-\theta)f(\theta, y_\theta)|_\alpha d\theta.
\end{aligned}
$$

Let $E_i, i \in I_6$ denote each of the above six integrals, respectively. Thus,

$$
\Pi_6 \le \Big| \left[T(\theta_2) - T(\theta_1)\right] \left[\eta(0) - [h(y)](0) - g(0, \eta - h(y))\right]\Big|_\alpha + \sum_{i=1}^{6} E_i. \tag{4.10}
$$

By the semigroup property in Definition 17(*ii*) we get that

$$
T(\theta_1-\theta) - T(\theta_2-\theta) = \left[T(\epsilon) - T(\theta_2 - \theta_1 + \epsilon)\right]\left[T(\theta_1 - \theta - \epsilon)\right]
$$

Let us manipulate the first integral E_1. By definition of $|\cdot|_\alpha$ and the above equality we get

$$\begin{aligned} E_1 &= \int_0^{\theta_1-\epsilon} |A\left[T(\theta_1-\theta)-T(\theta_2-\theta)\right]g(\theta,y_\theta)|_\alpha d\theta \\ &= \int_0^{\theta_1-\epsilon} \|A^\alpha A\left[T(\theta_1-\theta)-T(\theta_2-\theta)\right]g(\theta,y_\theta)\|d\theta \\ &= \int_0^{\theta_1-\epsilon} \|A^\alpha A\left[T(\epsilon)-T(\theta_2-\theta_1+\epsilon)\right]\left[T(\theta_1-\theta-\epsilon)\right]g(\theta,y_\theta)\|d\theta. \end{aligned}$$

The boundedness of $T(\cdot)$ and hypothesis **[P2]***(ii)* imply that

$$\begin{aligned} E_1 &\leq \|T(\epsilon)-T(\theta_2-\theta_1+\epsilon)\| \int_0^{\theta_1-\epsilon} \|A^\alpha\left[T(\theta_1-\theta-\epsilon)\right]\|\|Ag(\theta,y_\theta)\|d\theta \\ &\leq \|T(\epsilon)-T(\theta_2-\theta_1+\epsilon)\| \int_0^{\theta_1-\epsilon} \|A^\alpha\left[T(\theta_1-\theta-\epsilon)\right]\|\Psi\left(\|y_\theta\|\right)d\theta. \end{aligned}$$

Theorem 22 yields

$$E_1 \leq \|T(\epsilon)-T(\theta_2-\theta_1+\epsilon)\| \int_0^{\theta_1-\epsilon} \frac{M_\alpha}{(\theta_1-\theta-\epsilon)^\alpha}\Psi\left(\|y\|\right)d\theta$$

and hence

$$\begin{aligned} E_1 &\leq \|T(\epsilon)-T(\theta_2-\theta_1+\epsilon)\|\Psi(R)\,M_\alpha \int_0^{\theta_1-\epsilon} \frac{1}{(\theta_1-\theta-\epsilon)^\alpha}d\theta \\ &= \|T(\epsilon)-T(\theta_2-\theta_1+\epsilon)\|\Psi(R)\,M_\alpha \frac{(\theta_1-\epsilon)^{1-\alpha}}{1-\alpha} \end{aligned} \tag{4.11}$$

Let us work with the second integral E_2. Since this case is similar to E_1, we omit the step-by-step explanation and present only the computations.

$$\begin{aligned} E_2 &= \int_0^{\theta_1-\epsilon} |\left[T(\theta_1-\theta)-T(\theta_2-\theta)\right]f(\theta,y_\theta)|_\alpha d\theta \\ &= \int_0^{\theta_1-\epsilon} \|A^\alpha\left[T(\theta_1-\theta)-T(\theta_2-\theta)\right]f(\theta,y_\theta)\|d\theta \\ &= \int_0^{\theta_1-\epsilon} \|A^\alpha\left[T(\epsilon)-T(\theta_2-\theta_1+\epsilon)\right]\left[T(\theta_1-\theta-\epsilon)\right]f(\theta,y_\theta)\|d\theta \\ &\leq \|T(\epsilon)-T(\theta_2-\theta_1+\epsilon)\| \int_0^{\theta_1-\epsilon} \|A^\alpha\left[T(\theta_1-\theta-\epsilon)\right]\|\|f(\theta,y_\theta)\|d\theta \\ &\leq \|T(\epsilon)-T(\theta_2-\theta_1+\epsilon)\| \int_0^{\theta_1-\epsilon} \|A^\alpha\left[T(\theta_1-\theta-\epsilon)\right]\|\Psi\left(\|y_\theta\|\right)d\theta \\ &\leq \|T(\epsilon)-T(\theta_2-\theta_1+\epsilon)\| \int_0^{\theta_1-\epsilon} \frac{M_\alpha}{(\theta_1-\theta-\epsilon)^\alpha}\Psi\left(\|y\|\right)d\theta \\ &\leq \|T(\epsilon)-T(\theta_2-\theta_1+\epsilon)\|\Psi(R)\,M_\alpha \int_0^{\theta_1-\epsilon} \frac{1}{(\theta_1-\theta-\epsilon)^\alpha}d\theta \\ &= \|T(\epsilon)-T(\theta_2-\theta_1+\epsilon)\|\Psi(R)\,M_\alpha \frac{(\theta_1-\epsilon)^{1-\alpha}}{1-\alpha}. \end{aligned} \tag{4.12}$$

By Theorem (22) and hypothesis **[P2]**(*ii*), E_3 can be estimated as follows.

$$\begin{aligned}
E_3 &= \int_{\theta_1-\epsilon}^{\theta_1} |A\left[T(\theta_1-\theta)-T(\theta_2-\theta)\right]g(\theta,y_\theta)|_\alpha d\theta \\
&\leq \int_{\theta_1-\epsilon}^{\theta_1} \|A^\alpha T(\theta_1-\theta)-A^\alpha T(\theta_2-\theta)\|\|Ag(\theta,y_\theta)\|d\theta \\
&\leq \int_{\theta_1-\epsilon}^{\theta_1} \|A^\alpha T(\theta_1-\theta)-A^\alpha T(\theta_2-\theta)\|\Psi\left(\|y_\theta\|\right)d\theta \\
&\leq \Psi\left(R\right)\int_{\theta_1-\epsilon}^{\theta_1} \|A^\alpha T(\theta_1-\theta)\|+\|A^\alpha T(\theta_2-\theta)\|d\theta \\
&\leq \Psi\left(R\right)M_\alpha\int_{\theta_1-\epsilon}^{\theta_1} \frac{1}{(\theta_1-\theta)^\alpha}+\frac{1}{(\theta_2-\theta)^\alpha}d\theta \\
&= \frac{M_\alpha\Psi(R)}{1-\alpha}\left\{(\theta_2-\theta_1+\epsilon)^{1-\alpha}-(\theta_2-\theta_1)^{1-\alpha}+\epsilon^{1-\alpha}\right\}. \qquad (4.13)
\end{aligned}$$

For E_4 we get the exact same bound.

$$E_4 \leq \frac{M_\alpha\Psi(R)}{1-\alpha}\left\{(\theta_2-\theta_1+\epsilon)^{1-\alpha}-(\theta_2-\theta_1)^{1-\alpha}+\epsilon^{1-\alpha}\right\}. \qquad (4.14)$$

For E_5, we obtain the following estimate.

$$\begin{aligned}
E_5 &= \int_{\theta_1}^{\theta_2} |AT(\theta_2-\theta)g(\theta,y_\theta)|_\alpha d\theta \\
&= \int_{\theta_1}^{\theta_2} \|A^\alpha AT(\theta_2-\theta)g(\theta,y_\theta)\|d\theta \\
&\leq \int_{\theta_1}^{\theta_2} \|A^\alpha T(\theta_2-\theta)\|\Psi\left(\|y_\theta\|\right)d\theta \\
&\leq \Psi(R)M_\alpha\int_{\theta_1}^{\theta_2} \frac{1}{(\theta_2-\theta)^\alpha}d\theta \\
&\leq \frac{M_\alpha\Psi(R)}{1-\alpha}(\theta_2-\theta_1)^{1-\alpha}. \qquad (4.15)
\end{aligned}$$

As expected, E_6 has the exact same bound as E_5.

$$E_6 \leq \frac{M_\alpha\Psi(R)}{1-\alpha}(\theta_2-\theta_1)^{1-\alpha}. \qquad (4.16)$$

From (4.10) and (4.11 – 4.16), we get the following estimate.

$$\begin{aligned}
\Pi_6 &\leq \Big| [T(\theta_2) - T(\theta_1)] \, [\eta(0) - [h(y)](0) - g(0, \eta - h(y))] \Big|_\alpha + \sum_{i=1}^{6} E_i \\
&\leq \|T(\theta_2) - T(\theta_1)\| \Big[|\eta(0)|_\alpha + L_g q R + \Psi(\|\eta\| + L_g q R) \Big] \\
&+ 2\|T(\epsilon) - T(\theta_2 - \theta_1 + \epsilon)\| \Psi\,(R)\, M_\alpha \frac{(\theta_1 - \epsilon)^{1-\alpha}}{1-\alpha} \\
&+ 2\frac{M_\alpha \Psi(R)}{1-\alpha} \left\{ (\theta_2 - \theta_1 + \epsilon)^{1-\alpha} + \epsilon^{1-\alpha} \right\}
\end{aligned}$$

Since $T(t)$ is a compact operator for $t > 0$, then, by Theorem 11, $T(t)$ is a uniformly continuous semigroup away from zero, which implies that $|[\mathcal{P}(y)](\theta_2) - [\mathcal{P}(y)](\theta_1)|_\alpha$ goes to zero uniformly on y as $\theta_2 - \theta_1 \to 0$, and therefore $\mathcal{P}(B)$ is equicontinuous.

Step 4. *The set $W = \{\mathcal{P}(y) : y \in B\}$ is relatively compact in $PW_{p\alpha}$.*

For $t \in [-r, 0]$, the result is trivial since $W(t) = \{\eta(t)\}$. For $t \in [0, \tau]$, we proceed as follows. According to Lemma 6, it is enough to prove that the corresponding set $\overline{W}_i$ is relatively compact in $C([t_i, t_{i+1}], Z^\alpha)$ for $i \in I_p$ with $t_0 = 0$ and $t_{p+1} = \tau$. By Arzelà-Ascoli theorem for abstract functions (see Theorem 5), this reduces to prove that $\overline{W}_i(t) = \{\overline{\mathcal{P}(y)}_i(t) : y \in B\}$ is relatively compact in Z^α for each $t \in [t_i, t_{i+1}]$.

For a fixed $i \in I_p$, we have that

$$\overline{W}_i(t) = T(t)\eta(0) + \overline{V}_i(t), \; t \in [t_i, t_{i+1}],$$

where

$$\begin{aligned}
\overline{V}_i(t) = \Big\{ & v_i(t) = T(t)[-[h(\overline{y}_i)](0) - g(0, \eta - h(\overline{y}_i))] \\
& - \int_0^t AT(t-\theta) g(\theta, \overline{y_{\theta}}_i) d\theta + \int_0^t T(t-\theta) f(\theta, \overline{y_{\theta}}_i) d\theta : y \in B \Big\}.
\end{aligned}$$

By the compactness of $T(t)$, it is sufficient to prove that $\overline{V}_i(t)$ is relatively compact in Z^α. We present two different methods of achieving this goal below.

First method. We consider $\epsilon \in (0, t)$ and the set

$$\begin{aligned}
\overline{V}_{i,\epsilon}(t) &= \Big\{ v_{i,\epsilon}(t) = T(t)[-[h(\overline{y}_i)](0) - g(0, \eta - h(\overline{y}_i))] \\
&\qquad - \int_0^{t-\epsilon} AT(t-\theta) g(\theta, \overline{y_{\theta}}_i) d\theta + \int_0^{t-\epsilon} T(t-\theta) f(\theta, \overline{y_{\theta}}_i) d\theta : y \in B \Big\} \\
&= \Big\{ v_{i,\epsilon}(t) = T(t)[-[h(\overline{y}_i)](0) - g(0, \eta - h(\overline{y}_i))] \\
&\qquad - T(\epsilon) \int_0^{t-\epsilon} AT(t-\epsilon-\theta) g(\theta, \overline{y_{\theta}}_i) d\theta \\
&\qquad + T(\epsilon) \int_0^{t-\epsilon} T(t-\epsilon-\theta) f(\theta, \overline{y_{\theta}}_i) d\theta : y \in B \Big\}
\end{aligned}$$

From the compactness of $T(\epsilon)$ for $\epsilon > 0$, we get that $\overline{V}_{i,\epsilon}(t)$ is relatively compact in Z^α for any ϵ. By Theorem 22, we obtain

$$\begin{aligned}|v_i(t) - v_{i,\epsilon}(t)|_\alpha &\leq \int_{t-\epsilon}^{t} \|A^\alpha A T(t-\theta) g(\theta, \overline{y_{\theta}}_i)\| d\theta + \int_{t-\epsilon}^{t} \|A^\alpha T(t-\theta) f(\theta, \overline{y_{\theta}}_i)\| d\theta \\ &\leq \int_{t-\epsilon}^{t} \|A^\alpha T(t-\theta)\| \|Ag(\theta, \overline{y_{\theta}}_i)\| d\theta + \int_{t-\epsilon}^{t} \|A^\alpha T(t-\theta)\| \|f(\theta, \overline{y_{\theta}}_i)\| d\theta\end{aligned}$$

Hypotheses **[P2]**$(ii)(v)$ and (2.37) yield

$$|v_i(t) - v_{i,\epsilon}(t)|_\alpha \leq \frac{2M_\alpha \Psi(R)}{1-\alpha} \epsilon^{1-\alpha}.$$

This shows that we have a sequence of relatively compact sets arbitrarily close to $\overline{V}_i(t)$, which implies that $\overline{V}_i(t)$ is relatively compact in Z^α.

Second method. For $0 < \alpha < \beta < 1$, we have that

$$\begin{aligned}\|A^\beta v_i(t)\| &\leq \|A^{\beta-\alpha} T(t)\| \cdot |[h(\overline{y}_i)](0) + g(0, \eta - h(\overline{y}_i))|_\alpha \\ &\quad + \int_0^t \|A^\beta T(t-\theta) A g(\theta, \overline{y_{\theta}}_i)\| d\theta + \int_0^t \|A^\beta T(t-\theta) f(\theta, \overline{y_{\theta}}_i)\| d\theta.\end{aligned}$$

Theorem 22 and hypothesis **[P2]**$(ii)(v)$ imply that

$$\begin{aligned}\|A^\beta v_i(t)\| &\leq \frac{M_{\beta-\alpha}}{t^{\beta-\alpha}} \Big[L_g q R + \Psi(\|\eta\| + \|h(y)\|)\Big] \\ &\quad + \int_0^t \|A^\beta T(t-\theta) A g(\theta, \overline{y_{\theta}}_i)\| d\theta + \int_0^t \|A^\beta T(t-\theta) f(\theta, \overline{y_{\theta}}_i)\| d\theta.\end{aligned}$$

Hence,

$$\|A^\beta v_i(t)\| \leq \frac{M_{\beta-\alpha}}{t_i^{\beta-\alpha}} \Big[L_g q R + \Psi(\|\eta\| + L_g q R)\Big] + \frac{2M_\beta \Psi(R)}{1-\beta} \tau^{1-\beta}.$$

This shows that $A^\beta \overline{V}_i(t)$ is bounded in Z. By Proposition 3, $A^{-\beta} : Z \to Z^\alpha$[3] is compact and hence $\overline{V}_i(t) = A^{-\beta} A^\beta \overline{V}_i(t)$ is relatively compact in Z^α.

Before continuing with the next step, we provide some necessary notation. Let D denote the following closed and convex set

$$D = D(\rho, \tau, \eta) = \Big\{ y \in PW_{p\alpha} \;\Big|\; \|y - \tilde{\eta}\| \leq \rho \Big\}, \tag{4.17}$$

where $\tilde{\eta}$ is given in (4.6).

Step 5. *The family* $\{\mathcal{Q}(\cdot, y) : y \in \overline{\mathcal{P}(D)}\}$ *is equicontractive.*

[3] $A^{-\beta}$ maps Z into Z^α because $A^{-\beta} = I \circ A^{-\beta}$ and $Z \xrightarrow{A^{-\beta}} D(A^\beta) = Z^\beta \xhookrightarrow{I} Z^\alpha$.

Let $z, x \in PW_{p\alpha}$. For $t \in [-r, 0]$, we get

$$\begin{aligned}
|[\mathcal{Q}(z,\mathcal{P}(y))](t) - [\mathcal{Q}(x,\mathcal{P}(y))](t)|_\alpha &\leq |[h(z)](t) - [h(x)](t)|_\alpha \\
&\leq L_h q\|z-x\| \\
&\leq L_h q M\|z-x\|.
\end{aligned}$$

On the other hand, for $t \in [0,\tau]$, we obtain the following estimate.

$$\begin{aligned}
|[\mathcal{Q}(z,\mathcal{P}(y))](t) - [\mathcal{Q}(x,\mathcal{P}(y))](t)|_\alpha \leq& |g(t,z_t) - g(t,x_t)|_\alpha \\
&+ \sum_{0<t_k<t} |T(t-t_k)\left[J_k(z(t_k)) - J_k(x(t_k))\right]|_\alpha \\
\leq& \Upsilon\|z-x\| + M\sum_{k=1}^{p} |J_k(z(t_k)) - J_k(x(t_k))|_\alpha \\
\leq& \Upsilon\|z-x\| + M\sum_{k=1}^{p} d_k|z(t_k) - x(t_k)|_\alpha \\
\leq& \Upsilon\|z-x\| + M\|z-x\|\sum_{k=1}^{p} d_k \\
\leq& \left(\Upsilon + M\sum_{k=1}^{p} d_k\right)\|z-x\|. \qquad (4.18)
\end{aligned}$$

Taking supremum over $t \in [-r, \tau]$ and using hypothesis **[P1]**(i) yield

$$\|\mathcal{Q}(z,\mathcal{P}(y)) - \mathcal{Q}(x,\mathcal{P}(y))\| \leq \frac{1}{2}\|z-x\|,$$

which implies that $\mathcal{Q}(\cdot,\mathcal{P}(y))$ is a contraction independently of $y \in \overline{\mathcal{P}(D)}$.

For the next step, we consider the operator $\mathcal{H} = \mathcal{Q}(\cdot,\mathcal{P}(\cdot))$.

Step 6. *The property* $\forall z \in D : \mathcal{H}(z) \in D$ *holds.*

For a generic element z in D, we have to prove that (see (4.17)) $\|\mathcal{H}(z) - \tilde{\eta}\| \leq \rho$. For doing so, we first notice that

$$[\mathcal{H}(z)](t) = \begin{cases} T(t)[\eta(0) - [h(z)](0) - g(0,\eta - h(z))] \\ -\int_0^t AT(t-\theta)g(\theta,z_\theta)d\theta + \int_0^t T(t-\theta)f(\theta,z_\theta)d\theta + g(t,z_t) \\ + \sum_{0<t_k<t} T(t-t_k)J_k(z(t_k)), & t \in [0,\tau], \\ \eta(t) - [h(z)](t), & t \in [-r,0]. \end{cases} \qquad (4.19)$$

For convenience, let $\Pi_7 := |[\mathcal{H}(z)](t) - \tilde{\eta}(t)|_\alpha$. On the interval $[-r,0]$ we have that

$$\Pi_7 = |[h(z)](t)|_\alpha \leq L_h q\|z\| \leq ML_h q\|z\|$$

as a consequence of **[P1]**(*iii*). Since $z \in D$, $\|z - \tilde{\eta}\| \leq \rho$ and hence the reverse triangle inequality yields $\|z\| \leq \|\tilde{\eta}\| + \rho$. This and hypothesis **[P3]** let us write $\Pi_7 \leq ML_h q(\|\tilde{\eta}\| + \rho) < \rho$. On the other hand, for $t \in [0, \tau]$, we proceed as follows. First, the definition of $|\cdot|_\alpha$ and the boundedness of $T(\cdot)$ yield

$$\begin{aligned}\Pi_7 \leq \|T(t)\| [|[h(z)](0)|_\alpha + \|A^\alpha g(0, \eta - h(z))\|] + \|A^\alpha g(t, z_t)\| \\ + \int_0^t \|A^\alpha T(t-\theta)\| \|Ag(\theta, z_\theta)\| d\theta + \int_0^t \|A^\alpha T(t-\theta)\| \|f(\theta, z_\theta)\| d\theta \\ + \sum_{0<t_k<t} \|T(t-t_k)\| |J_k(z(t_k))|_\alpha\end{aligned}$$

Second, Theorem 22, hypothesis **[P1]**(*ii*)(*iii*), and **[P2]**(*ii*)(*v*) let us write

$$\begin{aligned}\Pi_7 \leq M \left[L_g q \|z\| + \Psi \left(\|\eta\| + L_g q \|z\| \right) \right] + \Psi(\|z\|) \\ + \frac{2M_\alpha \Psi(\|z\|)}{1-\alpha} \tau^{1-\alpha} + M \sum_{k=1}^{p} d_k |z(t_k)|_\alpha\end{aligned}$$

Third, by the reverse triangle inequality, we further obtain

$$\begin{aligned}\Pi_7 \leq M \left[L_g q(\|\tilde{\eta}\| + \rho) + \Psi \left(\|\eta\| + L_g q(\|\tilde{\eta}\| + \rho) \right) \right] + \Psi \left(\|\tilde{\eta}\| + \rho \right) \\ + \frac{2M_\alpha \Psi(\|\tilde{\eta}\| + \rho)}{1-\alpha} \tau^{1-\alpha} + M(\|\tilde{\eta}\| + \rho) \sum_{k=1}^{p} d_k\end{aligned}$$

Lastly, after rearranging terms, we get

$$\begin{aligned}\Pi_7 \leq M\Psi \left(\|\eta\| + L_g q(\|\tilde{\eta}\| + \rho) \right) + \left(ML_g q + M \sum_{k=1}^{p} d_k \right) (\|\tilde{\eta}\| + \rho) \\ + \left(\frac{2M_\alpha}{1-\alpha} \tau^{1-\alpha} + 1 \right) \Psi(\|\tilde{\eta}\| + \rho)\end{aligned}$$

Hence, by hypothesis **[P3]**, $\Pi_7 < \rho$. Taking supremum over $t \in [-r, \tau]$ yields the desired result $\|\mathcal{H}(z) - \tilde{\eta}\| \leq \rho$.

Finally, Steps 1, 4, 5, and 6 satisfy the conditions of Theorem 7 and, consequently, the equation $\mathcal{H}(z) = z$ has a solution, which is a mild solution of problem (4.1). □

Theorem 30. *In addition to the conditions of Theorem 29, suppose that* **[P4]** *holds. Then problem (4.1) has only one mild solution in $PW_{p\alpha}$.*

Proof. Let z and x be two solutions of problem (4.1). Denote by Π_8 the difference $|z(t) - x(t)|_\alpha$. Notice that

$$z(t) = [\mathcal{P}(z)](t) + g(t, z_t) + \sum_{0<t_k<t} T(t-t_k) J_k(z(t_k)), \ t \in [0, \tau]$$

and

$$x(t) = [\mathcal{P}(x)](t) + g(t, x_t) + \sum_{0<t_k<t} T(t-t_k)J_k(x(t_k)),\ t \in [0,\tau].$$

Hence,

$$\begin{aligned}\Pi_8 \leq\ & |[\mathcal{P}(z)](t) - [\mathcal{P}(x)](t)|_\alpha \\ & + |g(t,z_t) - g(t,x_t)|_\alpha + \sum_{0<t_k<t} \|T(t-t_k)\| |J_k(z(t_k)) - J_k(x(t_k))|_\alpha\end{aligned}$$

By (4.8) and (4.18), we get

$$\begin{aligned}\Pi_8 \leq\ & ML_g q(1+\Upsilon)\|z-x\| + 2M_\alpha \mathcal{K}\left(\|z\|, \|x\|\right)\|z-x\|\frac{\tau^{1-\alpha}}{1-\alpha} \\ & + \left(\Upsilon + M\sum_{k=1}^{p} d_k\right)\|z-x\|\end{aligned}$$

Using the reverse triangle inequality again, we obtain

$$\begin{aligned}\Pi_8 \leq & \left\{ML_g q(1+\Upsilon) + 2M_\alpha \mathcal{K}\left(\|\tilde{\eta}\|+\rho, \|\tilde{\eta}\|+\rho\right)\frac{\tau^{1-\alpha}}{1-\alpha}\right\}\|z-x\| \\ & + \left(\Upsilon + M\sum_{k=1}^{p} d_k\right)\|z-x\|\end{aligned}$$

Hypotheses **[P1]**(*i*) and **[P4]** imply the desired result $z = x$. □

4.2 Applications

This section provides an example to illustrate the abstract results of this manuscript. We will investigate the existence of solutions for a class of Burgers equation of neutral type with impulses and nonlocal conditions of the form

$$\begin{cases}\dfrac{\partial}{\partial t}\left[y(t,x) + \displaystyle\int_0^x \gamma(t)y(t-r,s)ds\right] = y_{xx}(t,x) + y(t-r,x)y_x(t-r,x) \\ \qquad\qquad + p(t, y(t-r,x)),\ x \in \Omega,\ t \in (0,\tau] \setminus \{t_k\}_{k\in I_p}, \\ y(t,0) = y(t,\pi) = 0,\ t \in [0,\tau], \\ y(\theta,x) + h(y(\tau_1+\theta,x), \ldots, y(\tau_q+\theta,x)) = \eta(\theta,x),\ \theta \in [-r,0],\ x \in \Omega, \\ y(t_k^+,x) = y(t_k^-,x) + J_k(y(t_k,x)),\ x \in \Omega,\ k \in I_p,\end{cases} \tag{4.20}$$

where $\Omega = [0,\pi]$, $\gamma : \mathbb{R}_+ \to \mathbb{R}_+$ is a non-decreasing function such that $\gamma(0) = 0$ and $\gamma(t) \leq \min\{4\|\xi\|_{L^\infty[0,\tau]}, L\}$. Here, $\eta : [-r,0]\times\Omega \to \mathbb{R}$ is a piecewise continuous function. We assume that there exists $L > 0$, and $\xi, \beta \in L^\infty[0,\tau]$ such that

$$|p(t,u) - p(t,v)| \leq L|u-v|,\ t \in [0,\tau],\ u,v \in \mathbb{R}, \tag{4.21}$$

and

$$|p(t,w)| \leq \xi(t)|w| + \beta(t), \ t \in [0,\tau], \ w \in \mathbb{R}. \tag{4.22}$$

For simplicity, we also assume that $h : \mathbb{R}^q \to \mathbb{R}$ and $J_k : \mathbb{R} \to \mathbb{R}$, when formulated abstractly, satisfy **[P1]***(ii)* and **[P1]***(iii)*, respectively.

Let $Z = L^2(\Omega)$ and consider the linear operator $A : D(A) \subset Z \to Z$ defined by $A\phi = -\phi_{xx}$, where $D(A) = H_0^1(\Omega) \cap H^2(\Omega)$[4] [63, pp. 57 & pp. 119]. The properties of the operator A are well-known in the literature (see, *e.g.*, [36, 50, 54, 71, 96, 105, 116]). Hereafter we mention some of them. The spectrum of A consists of only discrete eigenvalues $\lambda_n = n^2, n \in \mathbb{N}$. Their corresponding normalized eigenvectors are given by $z_n(x) = (2/\pi)^{1/2} \sin(nx), x \in [0,\pi]$. The collection of these functions $\{z_n | n \in \mathbb{N}\}$ constitutes an orthonormal basis for Z. For all $z \in D(A)$, the operator A has representation

$$Az = \sum_{n=1}^{\infty} \lambda_n \langle z, z_n \rangle z_n,$$

where $\langle \cdot, \cdot \rangle$ is the inner product in Z. It is also well-known that A is a sectorial operator (see Henry [63, Ch. 1] or [99, Ch. 2]), and therefore (see Theorem 17), $-A$ generates a compact analytic semigroup $T(t)$ of uniformly bounded linear operators on Z given by

$$T(t)z = \sum_{n=1}^{\infty} \exp(-\lambda_n t)\langle z, z_n \rangle z_n,$$

and satisfying

$$\|T(t)\|_{\mathcal{B}(Z)} \leq \exp(-\lambda_1 t) \tag{4.23}$$

for $t \geq 0$. See [124, Exa. 36.3, pp. 83] for a proof of the analyticity of $T(t)$. The compactness of $T(t)$ can be seen from its formula since it can be understood as the uniform limit of a sequence of finite-rank operators (see [27, Cor. 6.2, pp. 157]). Inequality (4.23) follows from (2.4) and the estimation

$$\|T(t)z\|_Z^2 = \sum_{n=1}^{\infty} \exp(-2\lambda_n t)|\langle z, z_n \rangle|^2 \leq \exp(-2\lambda_1 t) \sum_{n=1}^{\infty} |\langle z, z_n \rangle|^2 = \exp(-2\lambda_1 t)\|z\|_Z^2,$$

which is a consequence of Parseval's identity (see, *e.g.*, [124, Eq. 32.4, pp. 66]).

From Subsection 2.4.3, since A is a sectorial operator and $0 \in \rho(A)$, we see that it is possible to define fractional powers of A. In particular, the operator $A^{1/2}$ is given by

$$A^{1/2}z = \sum_{n=1}^{\infty} \lambda_n^{1/2} \langle z, z_n \rangle z_n, \ z \in D(A^{1/2}),$$

where (see [105, Sec. 4, pp. 212])

$$D(A^{1/2}) = \left\{ z \in Z \,\middle|\, \sum_{n=1}^{\infty} \lambda_n^{1/2} \langle z, z_n \rangle z_n \in Z \right\} = \left\{ z \in Z \,\middle|\, z' \in Z \text{ and } z(0) = z(\pi) = 0 \right\}.$$

[4] To understand these spaces, we refer the reader to Brezis [27, Ch. 8 & 9] or Evans [49, Ch. 5].

This means (see also [63, Exe. 3, pp. 18 or pp. 57 or pp. 93]) that $Z^{1/2} = H_0^1(\Omega)$ with norm $|\cdot|_{1/2}$.

To formulate (4.20) abstractly, we think of $y(t,x)$ as a $Z-$valued function of time which we denote by $z(t)$, i.e., $z(t) \in Z$ and $[z(t)](x) = y(t,x)$. See [124, pp. 104], [55, pp. 4], [100, pp. 3], and [63, pp. 16] for a more in-depth understanding of this method. In this way, (4.20) can be written as

$$\begin{cases} \dfrac{d}{dt}[z(t) - g(t,z_t)] = -Az(t) + f(t,z_t), & t \in (0,\tau] \setminus \{t_k\}_{k\in I_p}, \\ z(\theta) + [h(z)](\theta) = \eta(\theta), & \theta \in [-r,0], \\ z(t_k^+) = z(t_k^-) + J_k(z(t_k)), & k \in I_p, \end{cases} \tag{4.24}$$

where $0 < t_1 < \cdots < t_p < \tau$, $0 < \tau_1 < \cdots < \tau_q < r < \tau$, z_t is the time history function $[-r,0] \ni \theta \mapsto z_t(\theta) = z(t+\theta) \in Z^{1/2}$, and the functions $g, f : [0,\tau] \times PW_{r1/2} \to Z$, $h : PW_{qp1/2} \to PW_{r1/2}$, $J_k : Z^{1/2} \to Z^{1/2}$, $\eta \in PW_{r1/2}$ are defined by

$$\begin{aligned} [g(t,\phi)](x) &= -\int_0^x \gamma(t)\phi(-r,s)ds, \; x \in \Omega, \\ [f(t,\phi)](x) &= \phi(-r,x)\phi_x(-r,x) + p(t,\phi(-r,x)), \; x \in \Omega, \\ [[h(z)](\theta)](x) &= h(y(\tau_1+\theta,x),\ldots,y(\tau_q+\theta,x)), \; x \in \Omega, \\ [J_k(z(t_k))](x) &= J_k(y(t_k,x)), \; k \in I_p, \; x \in \Omega, \\ [\eta(\theta)](x) &= \eta(\theta,x), \; x \in \Omega, \; \theta \in [-r,0], \end{aligned}$$

accordingly. The spaces $PW_{qp1/2}$ and $PW_{r1/2}$ are defined by (4.2) and (4.3), respectively, with α replaced by $1/2$.

Proposition 6. *The functions f and g satisfy* **[P2]**.

Proof. Let us prove **[P2]***(iv)*. We first note that $f(t,\phi) = \phi(-r,\cdot)\phi_x(-r,\cdot) + p(t,\phi(-r,\cdot))$. Hence, by (4.21), we have

$$\begin{aligned} &\|f(t,\phi) - f(t,\mu)\|_Z \\ \leq &\|\phi(-r,\cdot)\phi_x(-r,\cdot) - \mu(-r,\cdot)\mu_x(-r,\cdot)\|_Z + \|p(t,\phi(-r,\cdot)) - p(t,\mu(-r,\cdot))\|_Z \\ \leq &\|\phi(-r,\cdot)\phi_x(-r,\cdot) - \mu(-r,\cdot)\mu_x(-r,\cdot)\|_Z + L\|\phi(-r,\cdot) - \mu(-r,\cdot)\|_Z. \end{aligned} \tag{4.25}$$

As in Tang & Wang [128] or Henry [63, pp. 57-58], we notice that

$$\begin{aligned} &\|\phi(-r,\cdot)\phi_x(-r,\cdot) - \mu(-r,\cdot)\mu_x(-r,\cdot)\|_Z \\ = &\|\phi(-r,\cdot)\phi_x(-r,\cdot) - \phi(-r,\cdot)\mu_x(-r,\cdot) + \phi(-r,\cdot)\mu_x(-r,\cdot) - \mu(-r,\cdot)\mu_x(-r,\cdot)\|_Z \\ \leq &\|\phi(-r,\cdot)[\phi_x(-r,\cdot) - \mu_x(-r,\cdot)]\|_Z + \|[\phi(-r,\cdot) - \mu(-r,\cdot)]\mu_x(-r,\cdot)\|_Z \\ \leq &\|\phi(-r,\cdot)\|_{L^\infty(\Omega)}\|\phi_x(-r,\cdot) - \mu_x(-r,\cdot)\|_Z + \|\phi(-r,\cdot) - \mu(-r,\cdot)\|_{L^\infty(\Omega)}\|\mu_x(-r,\cdot)\|_Z. \end{aligned} \tag{4.26}$$

For any $u \in Z^{1/2}$, we have that (see [128])

$$\|u\|_{L^\infty(\Omega)}^2 \leq 2\|u\|_Z\|u_x\|_Z \leq \|u\|_Z^2 + \|u_x\|_Z^2 = |u|_{1/2}^2 \tag{4.27}$$

as a consequence of Poincaré inequality (see, *e.g.*, [27, Prop. 8.13, pp. 218]). Hence,

$$\|u\|_{L^\infty(\Omega)} \leq |u|_{1/2} \text{ and } \|u_x\|_Z \leq |u|_{1/2}. \tag{4.28}$$

These inequalities and (4.26) imply that

$$\begin{aligned}
&\|\phi(-r,\cdot)\phi_x(-r,\cdot) - \mu(-r,\cdot)\mu_x(-r,\cdot)\|_Z \\
\leq &|\phi(-r,\cdot)|_{1/2}|\phi(-r,\cdot) - \mu(-r,\cdot)|_{1/2} + |\phi(-r,\cdot) - \mu(-r,\cdot)|_{1/2}|\mu(-r,\cdot)|_{1/2} \\
\leq &\Big(|\phi(-r,\cdot)|_{1/2} + |\mu(-r,\cdot)|_{1/2}\Big)|\phi(-r,\cdot) - \mu(-r,\cdot)|_{1/2}.
\end{aligned}$$

Therefore, (4.25) becomes

$$\begin{aligned}
\|f(t,\phi) - f(t,\mu)\|_Z \leq &\Big(|\phi(-r,\cdot)|_{1/2} + |\mu(-r,\cdot)|_{1/2}\Big)|\phi(-r,\cdot) - \mu(-r,\cdot)|_{1/2} \\
&+ L\|\phi(-r,\cdot) - \mu(-r,\cdot)\|_Z \\
\leq &\Big(|\phi(-r,\cdot)|_{1/2} + |\mu(-r,\cdot)|_{1/2}\Big)|\phi(-r,\cdot) - \mu(-r,\cdot)|_{1/2} \\
&+ L|\phi(-r,\cdot) - \mu(-r,\cdot)|_{1/2} \\
\leq &\Big(|\phi(-r,\cdot)|_{1/2} + |\mu(-r,\cdot)|_{1/2} + L\Big)|\phi(-r,\cdot) - \mu(-r,\cdot)|_{1/2} \\
\leq &\Big(\|\phi\|_{r1/2} + \|\mu\|_{r1/2} + L\Big)\|\phi - \mu\|_{r1/2}.
\end{aligned}$$

This shows **[P2]**(*iv*).

To show **[P2]**(*v*) we proceed as follows.

$$\|f(t,\varphi)\|_Z \leq \|\varphi(-r,\cdot)\varphi_x(-r,\cdot)\|_Z + \|p(t,\varphi(-r,\cdot))\|_Z. \tag{4.29}$$

The first term at the right-hand side of the inequality can be bounded by $\|\varphi\|_{r1/2}^2$ as a consequence of (4.28).

$$\begin{aligned}
\|\varphi(-r,\cdot)\varphi_x(-r,\cdot)\|_Z &\leq \|\varphi(-r,\cdot)\|_{L^\infty(\Omega)}\|\varphi_x(-r,\cdot)\|_Z \\
&\leq |\varphi(-r,\cdot)|_{1/2}|\varphi(-r,\cdot)|_{1/2} \\
&\leq \|\varphi\|_{r1/2}^2.
\end{aligned} \tag{4.30}$$

By (4.22), the second term squared satisfies

$$\begin{aligned}
\|p(t,\varphi(-r,\cdot))\|_Z^2 &= \int_0^\pi |p(t,\varphi(-r,s))|^2 ds \\
&\leq \int_0^\pi \left|\xi(t)|\varphi(-r,s)| + \beta(t)\right|^2 ds \\
&\leq \int_0^\pi \left|\|\xi\|_{L^\infty[0,\tau]}|\varphi(-r,s)| + \|\beta\|_{L^\infty[0,\tau]}\right|^2 ds.
\end{aligned}$$

Using the well-known inequality

$$\forall a,b \geq 0, \forall q \geq 1 : |a+b|^q \leq 2^q(|a|^q + |b|^q), \tag{4.31}$$

we get (with $q = 2$)

$$\begin{aligned}\|p(t,\varphi(-r,\cdot))\|_Z^2 &\leq \int_0^\pi 4\left(\|\xi\|^2_{L^\infty[0,\tau]}|\varphi(-r,s)|^2 + \|\beta\|^2_{L^\infty[0,\tau]}\right) ds \\ &= 4\|\xi\|^2_{L^\infty[0,\tau]}\int_0^\pi |\varphi(-r,s)|^2 ds + 4\pi\|\beta\|^2_{L^\infty[0,\tau]} \\ &= 4\|\xi\|^2_{L^\infty[0,\tau]}\|\varphi(-r,\cdot)\|_Z^2 + 4\pi\|\beta\|^2_{L^\infty[0,\tau]}.\end{aligned}$$

Hence

$$\|p(t,\varphi(-r,\cdot))\|_Z \leq 2\|\xi\|_{L^\infty[0,\tau]}\|\varphi(-r,\cdot)\|_Z + 2\sqrt{\pi}\|\beta\|_{L^\infty[0,\tau]}.$$

By (4.27),

$$\begin{aligned}\|p(t,\varphi(-r,\cdot))\|_Z &\leq 4\|\xi\|_{L^\infty[0,\tau]}|\varphi(-r,\cdot)|_{1/2} + 4\|\beta\|_{L^\infty[0,\tau]} \\ &\leq 4\|\xi\|_{L^\infty[0,\tau]}\|\varphi\|_{r1/2} + 4\|\beta\|_{L^\infty[0,\tau]}.\end{aligned} \tag{4.32}$$

Therefore, by (4.29), (4.30), and (4.32), we obtain

$$\|f(t,\varphi)\|_Z \leq \|\varphi\|^2_{r1/2} + 4\|\xi\|_{L^\infty[0,\tau]}\|\varphi\|_{r1/2} + 4\|\beta\|_{L^\infty[0,\tau]}.$$

Let us prove **[P2]**(*i*) and **[P2]**(*ii*). We begin noticing that

$$[Ag(t,\phi)](x) = \gamma(t)\phi_x(-r,x).$$

Hence

$$\|Ag(t,\phi) - Ag(t,\mu)\|_Z = \gamma(t)\|\phi_x(-r,\cdot) - \mu_x(-r,\cdot)\|_Z$$

By the second inequality in (4.28), we get

$$\|Ag(t,\phi) - Ag(t,\mu)\|_Z \leq \gamma(t)|\phi(-r,\cdot) - \mu(-r,\cdot)|_{1/2} \leq \gamma(t)\|\phi - \mu\|_{r1/2}.$$

Since $\gamma(t) \leq \min\{4\|\xi\|_{L^\infty[0,\tau]}, L\}$, we obtain

$$\|Ag(t,\phi) - Ag(t,\mu)\|_Z \leq L\|\phi - \mu\|_{r1/2} \leq \left(\|\phi\|_{r1/2} + \|\mu\|_{r1/2} + L\right)\|\phi - \mu\|_{r1/2}.$$

The second inequality in (4.28) also implies that

$$\|Ag(t,\varphi)\|_Z = \gamma(t)\|\varphi_x(-r,\cdot)\|_Z \leq \gamma(t)|\varphi(-r,\cdot)|_{1/2} \leq \gamma(t)\|\varphi\|_{r1/2}.$$

Consequently,

$$\|Ag(t,\varphi)\|_Z \leq 4\|\xi\|_{L^\infty[0,\tau]}\|\varphi\|_{r1/2} \leq \|\varphi\|^2_{r1/2} + 4\|\xi\|_{L^\infty[0,\tau]}\|\varphi\|_{r1/2} + 4\|\beta\|_{L^\infty[0,\tau]}.$$

This shows **[P2]**(*i*) and **[P2]**(*ii*).

Here, the functions $\mathcal{K}$ and Ψ are defined by

$$\mathcal{K}(u,v) = u + v + L \text{ and } \Psi(w) = w^2 + 4\|\xi\|_{L^\infty[0,\tau]}w + 4\|\beta\|_{L^\infty[0,\tau]},$$

respectively.

Our last task is to show **[P2]***(iii)*. For doing so, consider the following difference.

$$|[g(t,\phi)](x) - [g(t,\mu)](x)| \leq \gamma(t) \int_0^{\pi} |\phi(-r,s) - \mu(-r,s)| ds$$

Now, by Hölder's inequality (see, *e.g.*, [27, Th. 4.6, pp. 92]),

$$|[g(t,\phi)](x) - [g(t,\mu)](x)| \leq \sqrt{\pi}\gamma(t)\|\phi(-r,\cdot) - \mu(-r,\cdot)\|_Z$$

Inequality (4.27) implies that

$$|[g(t,\phi)](x) - [g(t,\mu)](x)| \leq \sqrt{\pi}\gamma(t)|\phi(-r,\cdot) - \mu(-r,\cdot)|_{1/2} \leq \sqrt{\pi}\gamma(t)\|\phi - \mu\|_{r1/2},$$

and hence,

$$\|g(t,\phi) - g(t,\mu)\|_Z \leq \pi\gamma(\tau)\|\phi - \mu\|_{r1/2}.$$

Therefore, **[P2]***(iii)* holds with $\Upsilon = \pi\gamma(\tau)$.

$\square$

We have the following result for system (4.20).

Theorem 31. *Suppose d_k is small enough for all $k \in I_p$. Then, system* (4.20) *has only one mild solution defined in $PW_{p1/2}$ for some $\tau > 0$.*

Chapter 5

Conclusions and Recommendations

5.1 Conclusions

In this manuscript, we studied a class of semilinear neutral differential equations with impulses and nonlocal conditions. In Chapter 3, we addressed the finite-dimensional case while in Chapter 4, the infinite-dimensional case. In the first case, we investigated the existence and uniqueness of solutions via Karakostas' fixed point theorem, the exact controllability by means of the Rothe's fixed point theorem and the Banach contraction theorem separately, and the approximate controllability using a technique developed by Bashirov *et al.* In the second case, we only assessed the existence and uniqueness of solutions via Karakostas' fixed point theorem and provided an example to apply our results. Controllability results for this case are reserved for future work due to time limitations. In advance, we know that it is only possible to study approximate controllability because having a compact semigroup is incompatible with exact controllability.

A part of the mathematical techniques and tools used in this manuscript are standard topics in an undergraduate program in mathematics. However, for the development of this project, it was required to study (independently) non-curricular topics such as IDEs, DEs with nonlocal conditions, NDEs, and Semigroup Theory. The last two topics can be regarded as part of what is known as the Theory of Abstract Semilinear Cauchy Problems.

5.2 Recommendations

We recommend extending this work to fractional differential equations and differential equations on time scales.

Bibliography

[1] Abada, N., Benchohra, M., and Hammouche, H. Existence results for semilinear differential evolution equations with impulses and delay. *CUBO. A Mathematical Journal 12*, 2 (2010), 1–17.

[2] Agarwal, R. P., Berezansky, L., Braverman, E., and Domoshnitsky, A. *Nonoscillation theory of functional differential equations with applications*. Springer Science & Business Media, 2012.

[3] Akça, H., Boucherif, A., and Covachev, V. Impulsive functional-differential equations with nonlocal conditions. *International Journal of Mathematics and Mathematical Sciences 29*, 5 (2002), 251–256.

[4] Akhmerov, R., Kamenskii, M., Potapov, A., Rodkina, A., and Sadovskii, B. Theory of equations of neutral type. *Journal of Soviet Mathematics 24*, 6 (1984), 674–719.

[5] Andrei, N. Modern control theory. *Studies in Informatics and Control 15*, 1 (2006), 51.

[6] Anguraj, A., and Karthikeyan, K. Existence of solutions for impulsive neutral functional differential equations with nonlocal conditions. *Nonlinear Analysis: Theory, Methods & Applications 70*, 7 (2009), 2717–2721.

[7] Arino, O., Benkhalti, R., and Ezzinbi, K. Existence results for initial value problems for neutral functional differential equations. *Journal of differential equations 138*, 1 (1997), 188–193.

[8] Arino, O., Hbid, M. L., and Dads, E. A. *Delay Differential Equations and Applications: Proceedings of the NATO Advanced Study Institute held in Marrakech, Morocco, 9-21 September 2002*, vol. 205. Springer Science & Business Media, 2006.

[9] Arnold, V. I. *Ordinary Differential Equations*. Springer-Verlag, 1992.

[10] Bainov, D. D., and Simeonov, P. S. *Systems with impulse effect: stability, theory, and applications*. Ellis Horwood, 1989.

[11] Balachandran, K., and Anandhi, E. R. Controllability of neutral functional integrodifferential infinite delay systems in Banach spaces. *Taiwanese Journal of Mathematics 8*, 4 (2004), 689–702.

[12] Balachandran, K., Sakthivel, R., and Dauer, J. P. Controllability of neutral functional integrodifferential systems in Banach spaces. *Computers & Mathematics with Applications 39*, 1 (2000), 117–126.

[13] Ballinger, G., and Liu, X. Existence, uniqueness and boundedness results for impulsive delay differential equations. *Applicable Analysis 74*, 1-2 (2000), 71–93.

[14] Banks, H., Jacobs, M. Q., and Langenhop, C. Characterization of the controlled states in $W_2^{(1)}$ of linear hereditary systems. *SIAM Journal on Control 13*, 3 (1975), 611–649.

[15] Bashirov, A. E., Etikan, H., and Şemi, N. Partial controllability of stochastic linear systems. *International Journal of Control 83*, 12 (2010), 2564–2572.

[16] Bashirov, A. E., and Ghahramanlou, N. On partial approximate controllability of semilinear systems. *Cogent Engineering 1*, 1 (2014), 965947.

[17] Bashirov, A. E., and Ghahramanlou, N. On partial S-controllability of semilinear partially observable systems. *International Journal of Control 88*, 5 (2015), 969–982.

[18] Bashirov, A. E., and Kerimov, K. R. On controllability conception for stochastic systems. *SIAM Journal on Control and Optimization 35*, 2 (1997), 384–398.

[19] Bashirov, A. E., Mahmudov, N., Şemi, N., and Etikan, H. Partial controllability concepts. *International Journal of Control 80*, 1 (2007), 1–7.

[20] Bashirov, A. E., and Mahmudov, N. I. On concepts of controllability for deterministic and stochastic systems. *SIAM Journal on Control and Optimization 37*, 6 (1999), 1808–1821.

[21] Benchohra, M., Henderson, J., and Ntouyas, S. Impulsive neutral functional differential equations in Banach spaces. *Applicable Analysis 80*, 3-4 (2001), 353–365.

[22] Benchohra, M., Henderson, J., and Ntouyas, S. *Impulsive differential equations and inclusions*, vol. 2. Hindawi Publishing Corporation, 2006.

[23] Benchohra, M., and Ouahab, A. Impulsive neutral functional differential equations with variable times. *Nonlinear Analysis: Theory, Methods & Applications 55*, 6 (2003), 679–693.

[24] Bennett, S. *A History of Control Engineering, 1800-1930*. Control, Robotics and Sensors Series. Peregrinus, 1986.

[25] Berezansky, Y. M., Sheftel, Z. G., and Us, G. F. *Functional Analysis*, vol. II. Birkhäuser Basel, 1996.

[26] Bárcenas, D., Leiva, H., and Sívoli, Z. A broad class of evolution equations are approximately controllable, but never exactly controllable. *IMA Journal of Mathematical Control and Information 22*, 3 (2005), 310–320.

[27] Brezis, H. *Functional analysis, Sobolev spaces and partial differential equations*. Springer, 2011.

[28] Bronson, R., and Costa, G. B. *Matrix methods: Applied linear algebra*. Academic Press, 2008.

[29] Burton, T., and Kirk, C. A fixed point theorem of Krasnoselskii-Schaefer type. *Mathematische Nachrichten 189*, 1 (1998), 23–31.

[30] Byszewski, L. Theorems about the existence and uniqueness of solutions of a semilinear evolution nonlocal Cauchy problem. *Journal of Mathematical analysis and Applications 162*, 2 (1991), 494–505.

[31] Byszewski, L., and Akça, H. On a mild solution of a semilinear functional-differential evolution nonlocal problem. *Journal of Applied Mathematics and Stochastic Analysis 10*, 3 (1997), 265–271.

[32] Byszewski, L., and Lakshmikantham, V. Theorem about the existence and uniqueness of a solution of a nonlocal abstract Cauchy problem in a Banach space. *Applicable analysis 40*, 1 (1991), 11–19.

[33] Cabada, D. Existence of solutions and controllability of time varying semilinear systems with impulses, delays, and nonlocal conditions. Bachelor's thesis, Yachay Tech University, Urcuquí, August 2019.

[34] Castillo-Jaramillo, J. Minimization of quantum free energy functionals defined on W^p Sobolev-like cones of operators. Bachelor's thesis, Yachay Tech University, Urcuquí, April 2021.

[35] Chalishajar, D. N. Controllability of impulsive partial neutral funcional differential equation with infinite delay. *International Journal of Mathematical Analysis 5*, 8 (2011), 369–380.

[36] Chang, J.-C., and Liu, H. Existence of solutions for a class of neutral partial differential equations with nonlocal conditions in the α-norm. *Nonlinear Analysis: Theory, Methods & Applications 71*, 9 (2009), 3759–3768.

[37] Chang, Y. K., Nieto, J. J., and Li, W. S. Controllability of Semilinear Differential Systems with Nonlocal Initial Conditions in Banach Spaces. *Journal of Optimization Theory and Applications 142*, 2 (2009), 267–273.

[38] Chichilnisky, G., and Kalman, P. Rothe's fixed point theorem and the existence of equilibria in monetary economies. *Journal of Mathematical Analysis and Applications 65*, 1 (1978), 56–65.

[39] Chicone, C. *Ordinary differential equations with applications*, vol. 34. Springer Science & Business Media, 2006.

[40] Chukwu, E. N. *Stability and time-optimal control of hereditary systems: with application to the economic dynamics of the US*, vol. 60. World Scientific, 2001.

[41] Cuevas, C., Hernández, E., and Rabelo, M. The existence of solutions for impulsive neutral functional differential equations. *Computers & Mathematics with Applications 58*, 4 (2009), 744–757.

[42] Curtain, R. F., and Pritchard, A. J. *Functional analysis in modern applied mathematics*. Academic Press, 1977.

[43] Curtain, R. F., and Zwart, H. *An introduction to infinite-dimensional linear systems theory*, vol. 21. Springer Science & Business Media, 2012.

[44] Dauer, J., and Mahmudov, N. Approximate controllability of semilinear functional equations in Hilbert spaces. *Journal of Mathematical Analysis and Applications 273*, 2 (2002), 310–327.

[45] de Guzmán, M. *Ecuaciones diferenciales ordinarias: teoría de estabilidad y control*. Editorial Alhambra, 1975.

[46] Diagana, T., and Leiva, H. Bounded Solutions for Impulsive Semilinear Evolution Equations with Non-local Conditions. *Mathematical Modeling of Random and Deterministic Phenomena* (2020), 141–159.

[47] Duque, C., Uzcátegui, J., Leiva, H., and Camacho, O. Controllability of the Burgers equation under the influence of impulses, delay, and nonlocal conditions. *International Journal of Applied Mathematics 33*, 4 (2020), 573–583.

[48] Engel, K.-J., and Nagel, R. *One-parameter semigroups for linear evolution equations*, vol. 194. Springer Science & Business Media, 1999.

[49] Evans, L. *Partial Differential Equations*. American Mathematical Society, 2010.

[50] Ezzinbi, K., Fu, X., and Hilal, K. Existence and regularity in the $\alpha-$norm for some neutral partial differential equations with nonlocal conditions. *Nonlinear Analysis: Theory, Methods & Applications 67*, 5 (2007), 1613–1622.

[51] Ezzinbi, K., and Rebey, A. Existence of solutions in the α-norm for partial functional differential equations with infinite delay. *Electronic Journal of Qualitative Theory of Differential Equations 2012*, 59 (2012), 1–16.

[52] Fairchild, W. W., and Tulcea, C. I. *Topology*. Saunders Co,. Philadelphia, Pa., 1971.

[53] Fan, Z., and Li, G. Existence results for semilinear differential equations with nonlocal and impulsive conditions. *Journal of Functional Analysis 258*, 5 (2010), 1709–1727.

[54] Fu, X., and Ezzinbi, K. Existence of solutions for neutral functional differential evolution equations with nonlocal conditions. *Nonlinear Analysis: Theory, Methods & Applications 54*, 2 (2003), 215–227.

[55] Goldstein, J. A. *Semigroups of linear operators and applications*. Oxford University Press, 1985.

[56] Granas, A., and Dugundji, J. *Fixed Point Theory*. Springer Monographs in Mathematics. Springer, 2003.

[57] Guevara, C., and Leiva, H. Approximated controllability of the strongly damped impulsive semilinear wave equation with memory and delay. *IFAC Journal of Systems and Control 4* (2018), 1–6.

[58] Guo, D., and Liu, X. Extremal solutions of nonlinear impulsive integrodifferential equations in Banach spaces. *Journal of Mathematical Analysis and Applications 177*, 2 (1993), 538–552.

[59] Guo, S., and Wu, J. *Bifurcation Theory of Functional Differential Equations*, vol. 184 of *Applied Mathematical Sciences*. Springer, 2013.

[60] Hadamard, J. Sur les problèmes aux dérivées partielles et leur signification physique. *Princeton University Bulletin 13*, 4 (1902), 49–52.

[61] Hale, J. *Ordinary Differential Equations*. Dover Books on Mathematics Series. Dover Publications, 2009.

[62] Hale, J., and Cruz, M. Existence, uniqueness and continuous dependence for hereditary systems. *Annali di matematica pura ed applicata 85*, 1 (1970), 63–81.

[63] Henry, D. *Geometric theory of semilinear parabolic equations*, vol. 840. Springer-Verlag, 1981.

[64] Hernández, E. Global solutions for abstract impulsive neutral differential equations. *Mathematical and computer modelling 53*, 1-2 (2011), 196–204.

[65] Hernández, E., Dos Santos, J. S., and Azevedo, K. A. Existence of solutions for a class of abstract differential equations with nonlocal conditions. *Nonlinear Analysis: Theory, Methods & Applications 74*, 7 (2011), 2624–2634.

[66] Hernández, E., and Henríquez, H. R. Existence results for partial neutral functional differential equations with unbounded delay. *Journal of Mathematical Analysis and Applications 221*, 2 (1998), 452–475.

[67] Hernández, E., Pierri, M., and Gonçalves, G. Existence results for an impulsive abstract partial differential equation with state-dependent delay. *Computers & mathematics with applications 52*, 3-4 (2006), 411–420.

[68] Isac, G. On Rothe's fixed point theorem in general topological vector space. *An. St. Univ. Ovidius Constanta 12*, 2 (2004), 127–134.

[69] Iturriaga, E., and Leiva, H. A characterization of semilinear surjective operators and applications to control problems. *Applied Mathematics 1*, 4 (2010), 265–273.

[70] Jacobs, M. Q., and Langenhop, C. Criteria for function space controllability of linear neutral systems. *SIAM Journal on Control and Optimization 14*, 6 (1976), 1009–1048.

[71] Jeet, K. Approximate controllability for finite delay nonlocal neutral integro-differential equations using resolvent operator theory. *Proceedings - Mathematical Sciences 130*, 62 (2020), 1–19.

[72] Kálmán, R. E. Contributions to the theory of optimal control. *Bol. Soc. Mat. Mexicana 5*, 2 (1960), 102–119.

[73] Kálmán, R. E. A new approach to Linear Filtering and Prediction Problems. *Journal of Basic Engineering 82*, 1 (1960), 35–45.

[74] Kálmán, R. E. On the general theory of control systems. In *Proceedings First International Conference on Automatic Control* (1960), pp. 481–492.

[75] Kantorovitz, S. *Semigroups of operators and spectral theory*, vol. 330. CRC Press, 1995.

[76] Karakostas, G. L. An extension of Krasnoselskii's fixed point theorem for contractions and compact mappings. *Topological Methods in Nonlinear Analysis 22*, 1 (2003), 181–191.

[77] Knapik, R. Impulsive differential equations with non-local conditions. *Morehead Electronic Journal of Applicable Mathematics* (2003), 1–6.

[78] Kreyszig, E. *Introductory functional analysis with applications*, vol. 1. John Wiley and Sons, 1978.

[79] Ladas, G. E., and Lakshmikantham, V. *Differential equations in abstract spaces*, vol. 85. Academic Press, 1972.

[80] Lakshmikantham, V., Bainov, D. D., and Simeonov, P. S. *Theory of impulsive differential equations*, vol. 6. World Scientific, 1989.

[81] Lakshmikantham, V., and Leela, S. *Differential and Integral Inequalities: Theory and Applications*, vol. I,II. Academic Press, 1969.

[82] Lee, E. B., and Markus, L. Foundations of optimal control theory. Tech. rep., John Wiley & Sons, 1967.

[83] Leiva, H. Controllability of semilinear impulsive nonautonomous systems. *International Journal of Control 88*, 3 (2014), 585–592.

[84] Leiva, H. Rothe's fixed point theorem and controllability of semilinear nonautonomous systems. *Systems & Control Letters 67* (2014), 14–18.

[85] Leiva, H. Controllability of the impulsive functional BBM equation with nonlinear term involving spatial derivative. *Systems & Control Letters 109* (2017), 12–16.

[86] Leiva, H. Karakostas fixed point theorem and the existence of solutions for impulsive semilinear evolution equations with delays and nonlocal conditions. *Communications in Mathematical Analysis 21*, 2 (2018), 68–91.

[87] Leiva, H., Cabada, D., and Gallo, R. Roughness of the Controllability for Time Varying Systems Under the Influence of Impulses, Delay, and Nonlocal Conditions. *Nonautonomous Dynamical Systems 7*, 1 (2020), 126–139.

[88] Leiva, H., Cabada, D., and Gallo, R. Controllability of time-varying systems with impulses, delays and nonlocal conditions. *Afrika Matematika 32*, 5 (2021), 959–967.

[89] Leiva, H., and Rojas, R. A. Controllability of semilinear nonautonomous systems with impulses and nonlocal conditions. *EQUILIBRIUM Journal of Natural Sciences 1* (2016), 23–38.

[90] Leiva, H., and Sivoli, Z. Smoothness of bounded solutions for semilinear evolution equations in Banach spaces and applications. *Far East Journal of Applied Mathematics 106*, 1-2 (2020), 1–24.

[91] Leiva, H., and Sundar, P. Existence of solutions for a class of semilinear evolution equations with impulses and delays. *Journal of Nonlinear Evolution Equations and Applications 2017*, 7 (2017), 95–108.

[92] Leiva, H., and Uzcátegui, J. Approximate controllability of discrete semilinear systems and applications. *Discrete & Continuous Dynamical Systems-B 21*, 6 (2016), 1803–1812.

[93] Li, X., Bohner, M., and Wang, C.-K. Impulsive differential equations: periodic solutions and applications. *Automatica 52* (2015), 173–178.

[94] Liang, J., Liu, J. H., and Xiao, T.-J. Nonlocal impulsive problems for nonlinear differential equations in Banach spaces. *Mathematical and Computer Modelling 49*, 3 (2009), 798–804.

[95] Lin, Y., and Liu, J. H. Semilinear integrodifferential equations with nonlocal Cauchy problem. *Nonlinear Analysis 26*, 5 (1996), 1023–1033.

[96] Liu, H., and Chang, J.-C. Existence for a class of partial differential equations with nonlocal conditions. *Nonlinear Analysis: Theory, Methods & Applications 70*, 9 (2009), 3076–3083.

[97] Liu, J. H. Nonlinear impulsive evolution equations. *Dynamics of Continuous Discrete and Impulsive Systems 6*, 1 (1999), 77–85.

[98] Liu, X. Stability results for impulsive differential systems with applications to population growth models. *Dynamics and stability of systems 9*, 2 (1994), 163–174.

[99] Lorenzi, L., Lunardi, A., Metafune, G., and Pallara, D. Analytic semigroups and reaction-diffusion problems. *Internet seminar* (2004-2005).

[100] Luo, Z.-H., Guo, B.-Z., and Morgül, Ö. *Stability and stabilization of infinite dimensional systems with applications*. Springer Science & Business Media, 2012.

[101] Magal, P., and Ruan, S. *Theory and applications of abstract semilinear Cauchy problems*. Springer, 2018.

[102] Manicom, G. T. Existence results for a class of semi-linear initial value problems. Master's thesis, University of Pretoria, Pretoria, July 2017.

[103] Manitius, A. Necessary and sufficient conditions of approximate controllability for general linear retarded systems. *SIAM Journal on Control and Optimization 19*, 4 (1981), 516–532.

[104] Milman, V. D., and Myshkis, A. D. On the stability of motion in the presence of impulses. *Sibirskii Matematicheskii Zhurnal 1*, 2 (1960), 233–237.

[105] Mokkedem, F. Z., and Fu, X. Approximate controllability of semi-linear neutral integro-differential systems with finite delay. *Applied Mathematics and Computation 242* (2014), 202–215.

[106] Munkres, J. R. *Topology*, 2 ed. Prentice Hall, 2000.

[107] Nieto, J., and Tisdell, C. On exact controllability of first-order impulsive differential equations. *Advances in Difference Equations 2010* (2010), 1–9.

[108] Ntouyas, S. Nonlocal initial and boundary value problems: a survey. In *Handbook of differential equations: ordinary differential equations*, vol. 2. Elsevier, 2006, pp. 461–557.

[109] Ntouyas, S., and Sficas, Y. On the asymptotic behavior of neutral functional differential equations. *Archiv der Mathematik 41*, 4 (1983), 352–362.

[110] Ntouyas, S., and Tsamatos, P. C. Global existence for semilinear evolution integrodifferential equations with delay and nonlocal conditions. *Applicable Analysis 64*, 1-2 (1997), 99–105.

[111] O'Connor, D. A., and Tarn, T. On the function space controllability of linear neutral systems. *SIAM Journal on Control and Optimization 21*, 2 (1983), 306–329.

[112] O'Regan, D. *Existence theory for nonlinear ordinary differential equations*, vol. 398. Springer Science & Business Media, 1997.

[113] Pazy, A. *Semigroups of linear operators and applications to partial differential equations*, vol. 44. Springer Science & Business Media, 2012.

[114] Peano, G. Démonstration de l'intégrabilité des équations différentielles ordinaires. In *Arbeiten zur Analysis und zur mathematischen Logik*. Springer, 1890, pp. 76–126.

[115] Rabah, R., and Sklyar, G. M. Exact controllability of linear neutral type systems by the moment problem approach. In *47th IEEE Conference on Decision and Control* (2008), IEEE, pp. 2734–2739.

[116] Radhakrishnan, B., Mohanraj, A., and Vinoba, V. Existence of solutions for nonlinear impulsive neutral integro-differential equations of Sobolev type with nonlocal conditions in Banach spaces. *Electronic Journal of Differential Equations 2013*, 18 (2013), 1–13.

[117] Reinbacher, H. New algebraic conditions for controllability of neutral differential equations. *Journal of optimization theory and applications 54*, 1 (1987), 93–111.

[118] Renardy, M., and Rogers, R. C. *An introduction to partial differential equations*, vol. 13. Springer Science & Business Media, 2006.

[119] Rivera Rodas, H., and Langenhop, C. A sufficient condition for function space controllability of a linear neutral system. *SIAM Journal on Control and Optimization 16*, 3 (1978), 429–435.

[120] Royden, H. L. *Real analysis*, 2 ed. Macmillan, 1968.

[121] Royden, H. L., and Fitzpatrick, P. *Real analysis*, 4 ed. Macmillan, 2010.

[122] Sadovskii, B. N. A fixed-point principle. *Functional Analysis and Its Applications 1*, 2 (1967), 151–153.

[123] Samoilenko, A. M., and Perestyuk, N. *Impulsive differential equations*, vol. 14. World Scientific, 1995.

[124] Sell, G. R., and You, Y. *Dynamics of evolutionary equations*, vol. 143. Springer Science & Business Media, 2013.

[125] Sideris, T. C. *Ordinary differential equations and dynamical systems*. Atlantis Press, 2013.

[126] Smart, D. R. *Fixed Point Theorems*. Cambrigde University Press, 1974.

[127] Sontag, E. D. *Mathematical control theory: deterministic finite dimensional systems*, vol. 6. Springer Science & Business Media, 2013.

[128] Tang, Y., and Wang, M. A remark on exponential stability of time-delayed burgers equation. *Discrete and Continuous Dynamical Systems - Series B 12*, 1 (2009), 219–225.

[129] Teschl, G. *Ordinary differential equations and dynamical systems*, vol. 140. American Mathematical Society, 2012.

[130] Tidke, H. L. Existence of global solutions to nonlinear mixed Volterra-Fredholm integrodifferential equations with nonlocal conditions. *Electronic Journal of Differential Equations 2009*, 55 (2009), 1–7.

[131] Tomar, N. K., and Sukavanam, N. Approximate controllability of non-densely defined semilinear delayed control systems. *Nonlinear studies 18*, 2 (2011), 229–234.

[132] Triggiani, R. A note on the lack of exact controllability for mild solutions in Banach spaces. *SIAM Journal on Control and Optimization 15*, 3 (1977), 407–411.

[133] Triggiani, R. Addendum: A note on the lack of exact controllability for mild solutions in Banach spaces. *SIAM Journal on Control and Optimization 18*, 1 (1980), 98–99.

[134] Underwood, R. G., and Chukwu, E. Null controllability of nonlinear neutral differential equations. *Journal of mathematical analysis and applications 129*, 2 (1988), 326–345.

[135] Vrabie, I. I. *Co-semigroups and applications*. Elsevier, 2003.

[136] Zabsonre, I. Controllability of some impulsive differential equation with infinite delay in Banach spaces. *Afrika Matematika 25*, 3 (2014), 723–738.