# UNIVERSIDAD DE INVESTIGACIÓN DE TECNOLOGÍA EXPERIMENTAL YACHAY

**Escuela de Ciencias Matemáticas y Computacionales**


# TÍTULO: SOME GENERALIZATIONS COMING FROM THE STUDY OF THE DISCRETE NAGUMO EQUATION

Trabajo de integración curricular presentado como requisito para la obtención del título de Matemática


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Urcuquí, Diciembre, 2021

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# Dedication

*"To my family."*

# Acknowledgments

I would like to express my gratitude and my love to my family: mother, Mónica; father, Miguel; brother, Miguel; grandparents, uncles, aunts, and cousins, for supporting me in this journey and always watching over me and my well-being. Each of them have been of great help, both supportive and inspiring in order to achieve this goal.

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# Resumen

La ecuación discreta de Nagumo corresponde a:

$$\dot{u}_n = d(u_{n-1} - 2u_n + u_{n+1}) + f(u_n), \qquad n \in \mathbb{Z}$$

y en este trabajo se obtienen resultados concernientes a la siguiente generalización:

$$\dot{u}_n = d(au_{n-1} + bu_n + cu_{n+1}) + f(u_n), \qquad n \in \mathbb{Z}$$

siendo a, b y c parámetros tales que a + b + c = 0 con a ≥ c ≥ 0. Se han obtenido resultados que generalizan parte del trabajo desarrollado por Bertram Zinner [1] y estos constituyen un punto de partida para posterior obtención de lo que sería existencia de soluciones del tipo ondas viajeras en la ecuación que consideramos.



**Palabras Clave:**

Ecuación discreta de Nagumo, Solución de Onda Viajera, Teorema de Banach de Punto fijo, Teorema de Schauder de Punto fijo.

# Abstract

The discrete Nagumo equation corresponds to:

$$\dot{u}_n = d(u_{n-1} - 2u_n + u_{n+1}) + f(u_n), \qquad n \in \mathbb{Z}$$

and in this work we obtain results concerning the following generalization:

$$\dot{u}_n = d(au_{n-1} + bu_n + cu_{n+1}) + f(u_n), \qquad n \in \mathbb{Z}$$

With a, b and c being parameters such that a + b + c = 0 with a ≥ c ≥ 0. We have obtained results that generalize part of the work developed by Bertram Zinner [1] and these constitute a starting point for later obtaining what would be the existence of solutions of the traveling wave type in the equation that we consider.

**Keywords:**

Discrete Nagumo's equation, Traveling wave solution, Banach Fixed Point Theorem, Schauder Fixed Point Theorem.

# Contents

# Chapter 1

# Introduction

The continuous Nagumo equation [1]

$$\frac{\partial u}{\partial t} = D\frac{\partial^2 u}{\partial x^2} + f(u) \tag{1.1}$$

appears as a model of excitation and propagation in nerve where $u$ is associated with the membrane voltage, see for instance [2], [3], [4], [5]. This equation is well studied in [6] and it has been shown, for example, that there is a function $U$, with $U(-\infty) = 0, U(\infty) = 1$, and a constant $c > 0$ such that

$$u(x,t) := U\left(\frac{x}{\sqrt{D}} + ct\right) \tag{1.2}$$

is a solution of (1.1) for all $D > 0$. Such a function $U(z)$ satisfies the ordinary differential equation

$$U'' + cU' + f(U) = 0.$$

The kind of solution giving by (1.2) is called traveling wavefront, or simply traveling wave.

Two discretizations of the Nagumo equation (1.1) stand out in the literature, see for example [7],[8],

$$\dot{u}_n = d\left(u_{n-1} - 2u_n + u_{n+1}\right) + f\left(u_n\right), \quad n \in \mathbb{Z} \tag{1.3}$$

the spatially discretization and

$$u_n(j+1) = u_n(j) + h\left(d\left(u_{n-1}(j) - 2u_n(j) + u_{n+1}(j)\right) + f\left(u_n(j)\right)\right), \tag{1.4}$$

which is the Euler time-discretization of (1.3).

In both equations the existence of traveling wave solutions has been studied, but unlike the continuous case in these scenarios this type of solution is guaranteed for a large value of $d$. In particular for equation (1.3), which is perhaps the most studied discretization of (1.1), Bertram Zinner [7] has established the existence of traveling waves for this equation. Specifically he showed that if $d$ is a positive real number and $f$ denotes a Lipschitz continuous function satisfying

$$f(0) = f(a) = f(1), \quad f(x) < 0, \quad \text{for} \quad 0 < x < a$$
$$f(x) > 0, \quad \text{for } a < x < 1, \text{ and } \int_0^1 f(x)dx > 0.$$

Then there exists some positive number $d^*$ such that for $d > d^*$ the discrete Nagumo equation (1.3) admits a solution $u_n(t) = U(n + ct)$, where $c > 0$, $U \in C^1(\mathbb{R}, (0, 1)), U(-\infty) = 0, U(\infty) = 1$, and $U'(x) > 0$ for all $x \in \mathbb{R}$.
In this work we obtain results concerning to the following generalization of (1.3)

$$\dot{u}_n = d\left(au_{n-1} + bu_n + cu_{n+1}\right) + f\left(u_n\right), \quad n \in \mathbb{Z}, \tag{1.5}$$

where $a, b$ and $c$ are parameters such that $a+b+c = 0$. We pay attention to a finite array of equations, i.e. $n \in \{0, 1, \cdots, N\}$ and then our goal is to generalize part of the work developed by Bertram Zinner [7]. Concretely, we focus on studying the following initial value problem

$$\begin{aligned} \dot{v}_n &= d\left(au_{n-1} + bu_n + cu_{n+1}\right) + h\left(u_n\right) \\ u_n &= P\left(v_n\right) \\ v_n(0) &= x_n \quad \text{ with } \quad 0 \leqslant x_n \leqslant 1 \text{ for } n = 0, \ldots, N; \end{aligned} \tag{1.6}$$

where $P$ is defined by

$$P\left(v_n\right) := \begin{cases} 0 & \text{for} \quad v_n < 0 \\ v_n & \text{for} \quad 0 \leqslant v_n \leqslant 1 \\ 1 & \text{for } 1 < v_n \end{cases}$$

and $h : [0, 1] \to \mathbb{R}$ is a Lipschitz continuous function that has a unique zero in the interval $(0, 1)$ and also satisfies the following conditions:

$$h(0) < 0, h(1) > 0 \text{ and } \int_0^1 h(s)ds > 0.$$

In particular, we establish invariance results, a priori estimates and other relevant properties of the problem (1.6). Thus our work leaves the door open for what would correspond to the study of the existence of traveling waves for the equation (1.5).
This work is organized as follow.

- In Chapter 2, we present the mathematical framework in which we will work. First, we mentioned basic concepts of Normal and Banach Space with the defined norms as well as providing examples for a better understanding of the concepts and help for the next steps. Then some fixed point theorems are considered so that we can figure out which one to use later. In the end, we will mention definitions and important theorems for the solutions of ordinary differential equations.

- In Chapter 3, we obtain generalizations from Zinner's work [7] that correspond to the initial value problem (1.6). In particular, we have monotonically invariant solutions, we also define an important operator, and in this regard, we establish a priori estimators.

- In Chapter 4, we present some conclusion and recommendation. In particular, in addition to Zinner's work, there is a discretization of the Nagumo equation; In our work, with some configurations in the parameters, we believe that somehow it is possible to study the following non linear partial differential equation of first order $\dfrac{\partial u}{\partial t} = D\dfrac{\partial u}{\partial x} + f(u)$. Also, to mention a possible work where some generalizations are made for discrete Fisher equation.

# Chapter 2

# Mathematical Framework

In this chapter we will point out definitions and results that are fundamental for the development of this work. The chapter is divided into three sections. The first one includes aspects concerning Banach spaces, then some fixed point theorems are highlighted and finally fundamental theorems on existence and uniqueness of solutions for ordinary differential equations are stated. For this chapter our main references are [9], [10],[11],[12],[13],[14], [15] and [16].

## 2.1 Banach spaces and examples

We start recalling that a linear vector space (or linear space) $X$ over $\mathbb{R}$ (or $\mathbb{C}$) is a set $\{x, y, z, \ldots\}$ such that for each $x, y, z \in X$, the sum $x + y$ is defined, $x + y \in X$ $x + y = y + x, (x + y) + z = x + (y + z)$, there is an element $0 \in X$ such that for every $x \in X : x + 0 = x$ and for a given $x \in X$ there is an element $\bar{x} \in X$ such that $x + \bar{x} = 0$ Also, for each $\alpha, \beta \in \mathbb{R}$ (or $\mathbb{C}$) and for each $x, y \in X$, scalar multiplication $\alpha x$ is defined, $\alpha x \in X$ and $1x = x, (\alpha\beta)x = \alpha(\beta x) = \beta(\alpha x), (\alpha+\beta)x = \alpha x+\beta x$ and $\alpha(x+y) = \alpha x+\alpha y$, we can see with more detail this definition in [9].
In this work we restrict our attention to linear spaces over $\mathbb{R}$ (real linear spaces). let $X$ be a real linear space, a **norm** on $X$ is a map $\| \cdot \| : X \to [0, \infty)$ which satisfies

i) $\forall x \in X : \|x\| > 0$ if $x \neq 0, \|0\| = 0$.

ii) $\forall \alpha \in \mathbb{R}, \forall x \in X : \|\alpha x\| = |\alpha|\|x\|$.

iii) $\forall x, y \in X : \|x + y\| \leq \|x\| + \|y\|$ (triangle inequality).

In this scenario we say that the pair $(X, \| \cdot \|)$ is a **normed linear space** or simply, in case there is no confusion, that $X$ is a **normed space**.
Now we highlight some important facts and definitions associated with normed linear spaces

- A **sequence** $\{x_n\}$ in a normed linear space $X$ converges to $x$ in $X$ if

$$\lim_{n\to\infty} \|x_n - x\| = 0$$

We shall write this as

$$\lim_{n\to\infty} x_n = x.$$

- A sequence $\{x_n\}$ in $X$ is a **Cauchy sequence** if for every $\varepsilon > 0$, there is an $N(\varepsilon) > 0$ such that $\|x_n - x_m\| < \varepsilon$ if $n, m \geq N(\varepsilon)$. The space $X$ is **complete** if every Cauchy sequence in $X$ converges to an element of $X$. A complete normed linear space is a **Banach space**.
- The **$\varepsilon$-neighborhood** of an element $x$ of a normed linear space $X$ is

$$\{y \in X : \|y - x\| < \varepsilon\}.$$

- A set $S$ in $X$ is **open** if for every $x \in S$, there exists an $\varepsilon$ - neighborhood of $x$ which is contained in $S$.
- An element $x$ is a **limit point** of a set $S$ in $X$ if each $\varepsilon$-neighborhood of $x$ contains points of $S$.
- A set $S$ in $X$ is **closed** if it contains its limit points.
- $S$ is closed if and only if $X - S$ is open.
- A Cauchy sequence $\{x_n\}$, in a Banach space $X$, which is contained in a closed set $S$ converges to an element of $S$.
- The **closure** of a set $S$ is the union of $S$ and its limit points.
- A set $S$ is **dense** in $X$ if the closure of $S$ is $X$.
- If $S$ is a subset of a normed space $X$, $A$ is a subset of $\mathbb{R}$ and $V_a, a \in A$, is a collection of open sets of $X$ such that $S \subset \bigcup_{a\in A} V_a$, then the collection $V_a$ is called an **open covering** of $S$.
- If $X$ is a Banach space, a subset $S$ is compact if and only if every sequence $\{x_n\}, x_n \in S$, contains a subsequence which converges to an element of $S$ .
- A set $S$ in $X$ is **bounded** if there exists an $r > 0$ such that $S \subset \{x \in X : \|x\| < r\}$.

Five examples are considered here and all of them come into play during the development of this work.

**Example 2.1.1.** *Let $X = \mathbb{R}^n$ be the space of real $n$-dimensional column vectors. For a particular coordinate system, elements $x$ in $\mathbb{R}^n$ will be written as $x = (x_1, \ldots, x_n)$ where each $x_j$ is in $\mathbb{R}$. If $x = (x_1, \ldots, x_n), y = (y_1, \ldots, y_n)$ are in $\mathbb{R}^n$, then $\alpha x + \beta y$ for $\alpha, \beta$ in $\mathbb{R}$ is defined to be $(\alpha x_1 + \beta y_1, \ldots, \alpha x_n + \beta y_n)$. Under the previous operations $\mathbb{R}^n$ is a linear space. Moreover, it is a Banach space if we choose $\|x\|$ to be either*

$$\sup\{|x_i| : i = 1, \ldots, n\}, \sum_{i=1}^{n} |x_i| \quad or \quad \left(\sum_{i=1}^{n} x_i^2\right)^{\frac{1}{2}}.$$

*Each of these norms is equivalent in the sense that a sequence converging in one norm converges in any of the other norms.*
*A very useful characterization for compact sets in this Banach space is the following: A set $S$ in $X = \mathbb{R}^n$ is compact if and only if it is closed and bounded.*

**Example 2.1.2.** *Let $X = C\left([a,b], \mathbb{R}^n\right)$ be the linear space of continuous functions which take the closed interval $[a,b]$ into $\mathbb{R}^n$. If we define for a given $x \in X$*

$$\|x\|_\infty = \sup_{t\in[a,b]} \|x(t)\|$$

*then $\|\cdot\|_\infty$ is a norm on $X$ and also $X$ is complete with this norm. Thus, the pair $\Big(C\left([a,b], \mathbb{R}^n\right), \|\cdot\|_\infty\Big)$ is a Banach space.*

**Example 2.1.3.** *Let $X$ as in the previous example. Given a positive number $K$. If we define for a given $x \in X$*

$$\|x\|_K = \sup_{t\in[a,b]} e^{-K(t-a)}\|x(t)\|$$

*it follows that $\|\cdot\|_K$ is a norm on $X$. Also $X$ is complete with this norm. Therefore, the pair $(X, \|\cdot\|_D)$ is a Banach space.*

*We remark that the norms $\|\cdot\|_\infty$ and $\|\cdot\|_K$ are equivalents.*

**Example 2.1.4.** *Given $K > 0$. We denote the set of all functions $x \in C(\mathbb{R}, \mathbb{R}^n)$ such that*

$$\sup_{t\in\mathbb{R}} e^{-K|t|}\|x(t)\| < \infty$$

*by $BC^K(\mathbb{R}, \mathbb{R}^n)$. If we define for a given $x \in BC^K(\mathbb{R}, \mathbb{R}^n)$*

$$\|x\|_K = \sup_{t\in\mathbb{R}} e^{-K|t|}\|x(t)\|$$

*then $\|\cdot\|_K$ is a norm on $BC^K(\mathbb{R}, \mathbb{R}^n)$ and the pair $\Big(BC^K(\mathbb{R}, \mathbb{R}^n), \|\cdot\|_K\Big)$ is a Banach space.*

For the last example, let us recall that a scalar-valued function $h$ on the interval $[0,1]$ is said to satisfy a **Lipschitz** condition, we refer to this saying that $h$ is Lipschitz on $[0,1]$, if there is a nonnegative real number $L$ such that

$$|h(s) - h(t)| \leq L|s-t| \text{ whenever } s,t \in [0,1].$$

**Example 2.1.5.** *Denote by* $\mathrm{Lip}([0,1], \mathbb{R}) = \{h : [0,1] \to \mathbb{R} : h \textit{ is Lipschitz on } [0,1]\}$ *the linear space of all Lipschitz functions from $[0,1]$ to $\mathbb{R}$ and let*

$$X := \mathrm{BLip}([0,1], \mathbb{R}) = \{h \in \mathrm{Lip}([0,1], \mathbb{R}) : h \textit{ is bounded on } [0,1]\}.$$

*From the fact that the interval* $[0,1]$ *is a compact subset of* $\mathbb{R}$ *it is obvious that* $\mathrm{Lip}([0,1],\mathbb{R}) = \mathrm{BLip}([0,1],\mathbb{R})$. *However, in order to be closer to the presentation given in [10], we will keep the notation* $X := \mathrm{BLip}([0,1],\mathbb{R})$. *Two equivalent norms on* $X$ *are the following*

$$\begin{aligned} \|h\|_{\max} &= \max\left\{\sup_{s\neq t}|\frac{h(s)-h(t)}{s-t}|, \|h\|_\infty\right\} - \text{ the max-norm} \\ \|h\|_{sum} &= \sup_{s\neq t}|\frac{h(s)-h(t)}{s-t}| + \|h\|_\infty - \text{ the sum-nom} \end{aligned} \tag{2.1}$$

**Theorem** $X = \mathrm{BLip}([0,1],\mathbb{R})$ *is a Banach space with respect to any of the norms (2.1). The proof of this result can be found in [10], Theorem 8.1.3 , page 368.*

## 2.2 Fixed Point Theory

Let $A$ be a subset of a Banach space $X$. A **fixed point** of a transformation $T : A \to A$ is a point $x$ in $A$ such that $Tx = x$. Among the many fixed point theorems in the literature, here we consider three that are classics: Banach's fixed point theorem (also known as the contraction mapping theorem), Brouwer's fixed point theorem and Schauder's fixed point theorem.

If $A$ is a subset of a Banach space $X$ and $T$ is a transformation taking $A$ into a Banach space $Y$, then $T$ is a **contraction** on $A$ if there is a $\lambda, 0 \leqq \lambda < 1$, such that

$$\|Tx_1 - Tx_2\|_Y \leq \lambda\|x_1 - x_2\|_X, \quad x_1, x_2 \in A.$$

The constant $\lambda$ is called the **contraction constant** for $T$ on $A$.

**Theorem 2.2.1** (Banach Fixed Point Theorem)**.** *If* $A$ *is a closed subset of a Banach space* $X$ *and* $T : A \to A$ *is a contraction on* $A$*, then* $T$ *has a unique fixed point* $\bar{x}$ *in* $A$*. Also, if* $x_0$ *in* $A$ *is arbitrary and* $\lambda$ *is the contraction constant for* $T$ *on* $A$*, then the sequence* $\{x_{n+1} = Tx_n, n = 0, 1, 2, \ldots\}$ *converges to* $\bar{x}$ *as* $n \to \infty$ *and*

$$\|x_n - \bar{x}\| \leq \frac{\lambda^n \|x_1 - x_0\|}{1-\lambda}.$$

*The proof of this result can be found in [11], Theorem 3.1, page 5; or in [12], in the section Banach Contraction Theorem page 159.*

**Theorem 2.2.2** (Brouwer Fixed Point Theorem)**.** *Let* $T$ *a continuous mapping of the closed unit ball in* $\mathbb{R}^n$ *into itself. Then there is at least one fixed point for* $T$*.*

*Note that for* $n = 1$ *the proof of the theorem results as a straightforward application of the mean value theorem to the function*

$$\bar{T}(t) = T(t) - t, \quad t \text{ in the interval } [0,1].$$

*For* $n > 1$ *the proof can be found in [13], Theorem 3.3.2, page 76; or [14], the section The Brouwer Fixed-Point Theorem in the page 253.*

The Brouwer fixed point theorem has been generalized to Banach spaces by Schauder. First, let us recall that a subset $A$ of a Banach space $X$ is **convex** if for $x_1, x_2$ in $A$ it follows $tx_1 + (1-t)x_2$ is in $A$ for any $t$ in the interval $[0,1]$. This means that $A$ contains the line segment joining $x_1$ and $x_2$.

**Theorem 2.2.3** (Schauder Fixed Point Theorem)**.** *If $A$ is a convex, compact subset of a Banach space $X$ and $T : X \to X$ is continuous, then $T$ has a fixed point in $A$.*
*The proof of this result can be found in [15], the subsection The Proof in page 77; or [16], Lecture 09 in the page 43.*

## 2.3 Fundamental Theorems of Differential Equations

In this section we present some results concerning the existence of solutions of ordinary differential equations having the form $\dot{x}(t) = f(t, x(t))$. The first of these is a simple application of Banach's fixed point theorem and is often referred to as the Picard-Lindelöf theorem. Before presenting this theorem, let us be more precise about the equation $\dot{x}(t) = f(t, x(t))$ and also give a discussion that will put us in a framework to apply the Banach's fixed point theorem.
In the equation

$$\dot{x}(t) = f(t, x(t)) \ ,$$

$t$ is a scalar, $(t, x) \in I \times \mathbb{R}^n$ with $I$ an interval, $f : I \times \mathbb{R}^n \to \mathbb{R}^n$ is a continuous function and $\dot{x} = \dfrac{dx}{dt}$.
We begin with the assumption that $I = [a, b]$. Let $x_a \in \mathbb{R}^n$. Here, we are looking to find, if possible, a continuously differentiable function $x : [a, b] \to \mathbb{R}^n$ such that

$$\dot{x}(t) = f(t, x(t)) \quad \text{for all } t \in [a, b]$$

and such that $x(a) = x_a$. We write this type of problem as

$$\begin{cases} \dot{x}(t) = f(t, x(t)), \\ x\,(a) = x_a, \quad t \in [a, b] \end{cases} \tag{2.2}$$

and refer to it as an **initial value problem**.
It is clear that problem (2.2) is equivalent to find a function $x$ such that

$$x(t) = x_a + \int_a^t f(\tau, x(\tau))d\tau, \tag{2.3}$$

provided $f(t, x)$ is continuous.
Now, inspired by (2.3) we define for $x \in C\,([a, b], \mathbb{R}^n)$ an operator $T$ by means of

$$(Tx)(t) = x_a + \int_a^t f(\tau, x(\tau))d\tau.$$

It is clear that $Tx \in C\,([a, b], \mathbb{R}^n)$ and also, provided $f(t, x)$ is continuous, is continuously differentiable. Moreover, a solution of the integral equation (2.3) is a fixed point of $T$.

**Remark 2.3.1.** *The set of states of $S \subseteq \mathbb{R}^n$ (2.2) is called an* **invariant set** *of (2.2) if for all $x_0 \in S$ and for all $t \geq 0, x(t) \in S$.*

**Theorem 2.3.2** (Picard-Lindelöf)**.** *If $f : [a, b] \times \mathbb{R}^n \to \mathbb{R}^n$ is a continuous function and satisfies the Lipschitz condition*

$$\|f(t, x^1) - f(t, x^2)\| \leqslant L\|x^1 - x^2\|$$

*for all $t \in [a, b]$ and $x^1, x^2 \in \mathbb{R}^n$; where $L$ is a positive constant. Then the initial value problem*

$$\begin{cases} \dot{x}(t) = f(t, x(t)) \\ x(a) = x_a, \quad t \in [a, b] \end{cases}$$

*has a unique solution.*

*Proof.* Consider the Banach space $\left(X = C\left([a, b], \mathbb{R}^n\right), \|\cdot\|_K\right)$, example 2.1.4, with $K > L$ a fixed constant.
We define for $x \in X$

$$(Tx)(t) = x_a + \int_a^t f(\tau, x(\tau))d\tau.$$

If $x_1, x_2 \in X$, then

$$\begin{aligned} e^{-K(t-a)}\|(Tx_1)(t) - (Tx_2)(t)\| &\leqslant \int_a^t e^{-K(t-a)}\|f\left(\tau, x_1(\tau)\right) - f\left(\tau, x_2(\tau)\right)\|d\tau \\ &\leqslant L\|x_1 - x_2\|_K \int_a^t e^{-K(t-\tau)}d\tau \\ &\leqslant \left(\frac{L}{K}\right)\|x_1 - x_2\|_K \ . \end{aligned}$$

Hence

$$\|Tx_1 - Tx_2\|_K \leqslant \frac{L}{K}\|x_1 - x_2\|_K.$$

Thus, as $\frac{L}{K} < 1$, it turns out that $T$ is a contraction and there exists a unique fixed point $x$ which is the only solution of the initial value problem. □

Now, we consider an initial value problem where the ordinary differential equation is autonomous and $f$ is globally Lipschitz. In this scenario we have the following

**Theorem 2.3.3.** *If $f : \mathbb{R}^n \to \mathbb{R}^n$ is a continuous function and satisfies the Lipschitz condition*

$$\|f(x^1) - f(x^2)\| \leqslant L\|x^1 - x^2\|$$

*for all $x^1, x^2 \in \mathbb{R}^n$; where $L$ is a positive constant. Then the initial value problem*

$$\begin{cases} \dot{x}(t) = f(x(t)) \\ x(0) = x_0 \end{cases}$$

*has a unique solution defined for all $t \in \mathbb{R}$.*

*Proof.* Consider $K > L$ a fixed constant and the Banach space given in the example 2.1.4, i.e $\left(X = BC^K\left(\mathbb{R}, \mathbb{R}^n\right), \|\cdot\|_K\right)$. Now, define for $x \in X$

$$(Tx)(t) = x_0 + \int_0^t f(\tau, x(\tau))d\tau.$$

The remainder of the proof is obtained, except for obvious modifications, following the proof of the Picard-Lindelöf theorem. □

If in the previous theorem we remove the Lipschitz condiction of $f$ and we say that $f$ is a continuous function then the existence of solutions can be proved but not the uniqueness, this leads to the Cauchy-Peano theorem, however for our purposes it does not it is necessary to appeal to that result.

# Chapter 3

# Results

## 3.1 Monotonically invariant solutions

Consider the initial value problem

$$
\begin{aligned}
\dot{v}_n &= d\left(au_{n-1} + bu_n + cu_{n+1}\right) + h\left(u_n\right) \\
u_n &= P\left(v_n\right) \\
v_n(0) &= x_n \quad \text{ with } \quad 0 \leqslant x_n \leqslant 1 \text{ for } n = 0, \dots, N;
\end{aligned}
\tag{3.1}
$$

where we set $u_{-1} \equiv 0$, $u_{N+1} \equiv 1$, $P$ is defined by

$$
P\left(v_n\right) := \begin{cases} 0 & \text{for} \quad v_n < 0 \\ v_n & \text{for} \quad 0 \leqslant v_n \leqslant 1 \\ 1 & \text{for } 1 < v_n \end{cases},
$$

$a, b, c, d$ are real numbers such that $d > 0$, $a + b + c = 0$ with $a > 0$, $c > 0$; and $h \in \mathcal{B}$, where $\mathcal{B}$ is the subset of $\mathrm{BLip}([0,1], \mathbb{R})$ defined by

$$
\begin{aligned}
\mathcal{B} := \Big\{ h \in \mathrm{BLip}([0,1], \mathbb{R}) : & \ h \text{ is continuous, } h(0) < 0, \\
& h(1) > 0, h \text{ has a unique zero in } (0,1) \\
& \text{and } \int_0^1 h(s) ds > 0 \Big\}.
\end{aligned}
$$

According that we seen in the previous chapter about the Lipschitz functions (Theorem 2.3.3), the initial value problem (3.1) has a unique solution which is defined for all $t$ in $\mathbb{R}$.

Three invariance results associated with the initial value problem (3.1) are established in this section. The first result shows that certain conditions on the initial values imposed on (3.1) are preserved in the corresponding solutions.
Let $u(t) = \{u_n\}_{n=0}^{N}$ be the unique solution of the initial value problem (3.1).

**Theorem 3.1.1.** *Suppose that:* $x = \{x_n\}_{n=0}^{N} \in \mathbb{R}^{N+1}$ *satisfies:*

*($P_1$)* $0 \leqslant x_0 \leq \cdots \leqslant x_N \leqslant 1$

*($P_2$)* $x_n < x_{n+1}$ *when* $0 < x_n < 1$

*($P_3$)* $d(ax_{n-1} + bx_n + cx_{n+1}) + h(x_n) > 0$ *when* $0 < x_n$, *where* $x_{N+1} := 1$ *and* $x_{-1} := 0$

*Furthermore suppose*

$$x_0 = 0, \quad x_1 = -\frac{h(0)}{cd}, \quad cd > -h(0) \quad and \quad h(1) > ad.$$

*Thus* $u(t)$ *satisfies (*$P_1$*), (*$P_2$*) and (*$P_3$*)* $\forall t \geqslant 0$

*Proof.* We have that

$$t_1 := sup\{t \geq 0 : u(s) \text{ satisfies } (P_1), (P_2) \text{ and } (P_3), \forall s \in [0, t]\}$$

The hypothesis on the initial condition implies that $t_1$ is well defined. Now, to establishes the result will be enough to show that $t_1 = \infty$. First, we show that $t_1$ is strictly bigger than zero. Using that $x_1 > 0$ and (P1) it is obtained $x_n > 0$ for $n = 1, 2, \ldots, N$ . Thus, by (P3)

$$\dot{v}_n(0) = d\left(ax_{n-1} + bx_n + cx_{n+1}\right) + h\left(x_n\right) > 0 \quad \text{for} \quad n = 1, 2, \ldots, N. \tag{3.2}$$

With relation to $\dot{v}_0(0)$ we have that $\dot{v}_0(0) = 0$, indeed:

$$\dot{v}_0(t) = d(au_{-1}(t) + bu_0(t) + cu_1(t)) + h(u_n(t))$$

by continuity

$$\begin{aligned}
\lim_{t\to 0}(\dot{v}_0(t)) =& \lim_{t\to 0} d(bu_0(t) + cu_1(t)) + h(u_n(t)) \\
=& d(bu_0(0) + cu_1(0)) + h(u_n(0)) \\
=& d(b \cdot 0 + cx_1) + h(0) \\
=& dcx_1 + h(0)
\end{aligned}$$

By the definition of $x_1$ we have that:

$$\dot{v}_0(0) = dcx_1 + h(0) = 0$$

Next, let us see that $\dot{u}_0(0)$ and $\left.\dfrac{d}{dt}h\left(u_0(t)\right)\right|_{t=0} = 0$ exist and are equal to zero: For sufficiently small $\Delta t \neq 0$ , it turns out that

$$\frac{u_0(h) - u_0(0)}{h} = \frac{u_0(h)}{h} = \begin{cases} \frac{v_0(h)}{h} & \text{if } v_0(h) > 0 \\ 0 & \text{if } v_0(h) \leq 0 \end{cases}$$

The expression, due to $\dot{v}_0(0) = 0$, tends to zero as $\Delta t \to 0$. Hence, $\dot{u}_0(0) = 0$. To establish that $\left.\dfrac{d}{dt} h\left(u_0(t)\right)\right|_{t=0} = 0$ , we observe that

$$\left|h\left(u_0(\Delta t)\right) - h\left(u_0(0)\right)\right| \leq K_h \left|u_0(\Delta t) - u_0(0)\right|,$$

where $K_h$ is the Lipschitz constant of $h$. Thus, for $\Delta t \neq 0$ we have

$$\left|\frac{h\left(u_0(\Delta t)\right) - h\left(u_0(0)\right)}{\Delta t}\right| \leqslant K_h \left|\frac{u_0(\Delta t) - u_0(0)}{\Delta t}\right| \longrightarrow 0 \text{ as } \Delta t \to 0.$$

Now, since $v_1(t) = u_1(t)$ for all $t$ small enough and $v_1(0) = u_1(0)$ we have at $t = 0$ that

$$\ddot{v}_0(0) = dc\dot{u}_1(0) = dc\dot{v}_1(0) = dc\left(d\left(ax_0 + bx_1 + cx_2\right) + h(x_1)\right) > 0\ .$$

Thus, for $n = 0$ we have obtained

$$\dot{v}_0(0) = 0 \text{ and } \ddot{v}_0(0) = 0. \tag{3.3}$$

Locally, the graph of $v_0$ looks as follow

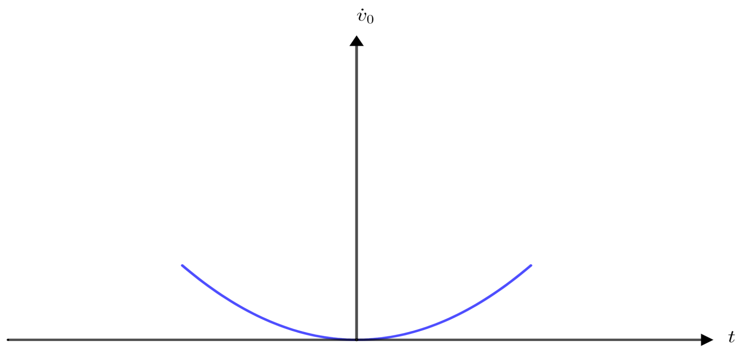


Figure 3.1: $v_0$

By continuity of $\dot{v}_n$ and $u_n$ with $n = 0, 1, \ldots, N$; we can conclude that certainly $t_1 > 0$. To finalize the proof of the theorem we assume that $t_1 < \infty$ and we will obtain a contradiction. If $t_1 < \infty$, then either

i) $u_n(t_1) = u_{n+1}(t_1)$ for some $0 < u_n(t_1) < 1$

ii) $\dot{v}_n(t_1) = 0$ for some $u_n(t_1) > 0$

Let us start supposing that (i) occurs. From the fact that $u_{N+1}(t) \equiv 1$ , we may choose $n$ such that $u_n(t_1) = u_{n+1}(t_1) < u_{n+2}(t_1)$. Now,

$$\frac{d}{dt}(v_{n+1} - v_n)|_{t=t_1} = d[a(u_n(t_1) - u_{n-1}(t_1)) + c(u_{n+2}(t_1) - u_n(t_1))] > 0$$

Thus, the function $v_{n+1} - v_n$ is increasing on the open interval $(t_1 - \varepsilon, t_1)$, for $\varepsilon > 0$. This implies that we can find $t \in (t_1 - \varepsilon, t_1)$ such that:

$$0 = v_{n+1}(t_1) - v_n(t_1) > v_{n+1}(t_1 - \varepsilon) - v_n(t_1 - \varepsilon)$$

In $t_1$

$$\begin{aligned} v_{n+1}(t_1) =& u_{n+1}(t_1) \\ v_n(t_1) =& u_n(t_1) \end{aligned}$$

Then

$$v_{n+1}(t_1 - \varepsilon) > v_n(t_1 - \varepsilon) \Rightarrow u_{n+1}(t_1 - \varepsilon) > u_n(t_1 - \varepsilon)$$

This leads to the contradiction $u_n(t) > u_{n+1}(t)$.
Now, we pay attention to $u_n(t)$ on the interval $[0, t_1]$. If exist $t_0 \in (0, t_1)$ such that $u_n(t_0) = 1$, then $v_n(t)$ decreasing on $[0, t_1]$, we have that $u_n(t) = 1$ $\forall t \in [t_0, t_1]$. one of the following cases occurs:

$$\begin{aligned} &\text{(a)} \quad u_n(t) = v_n(t) \quad \text{for all } t \in [0, t_1] \\ &\text{(b)} \quad u_n(t) = \begin{cases} v_n(t) & \text{for all } t \in [0, t_0] \\ 1 & \text{for all } t \in (t_0, t_1] \end{cases} \\ &\text{(c)} \quad u_n(t) = 1 \quad \text{for all } t \in [0, t_1]\,. \end{aligned}$$

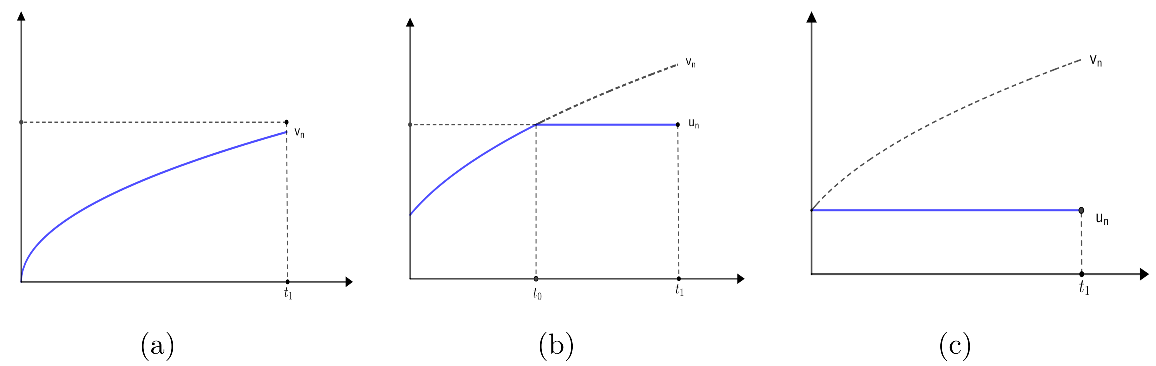


Figure 3.2: Possible Scenarios

In any of these case, the derivative by left of $u_n(t)$ exist for all $t \in (0, t_1]$ and we denote by $\dot{u}_n(t^-)$.

Suppose that $\dot{v}_n(t_1) = 0$ for any $u_n(t_1) > 0$.

We have that $\dot{u}_n(t^-) = 0$

$$\dot{u}_n(t) \quad \Rightarrow \quad \ddot{v}_n(t_1^-) = d(a\dot{u}_{n-1}(t) + c\dot{u}_{n+1}(t))$$

Then:

$$\ddot{v}_n(t_1^-) = d(a\dot{u}_{n-1}(t) + c\dot{u}_{n+1}(t)) \tag{3.4}$$

where $\dot{u}_{n-1}(t_1-) \geqslant 0$ and $\dot{u}_{n+1}(t_1-) \geqslant 0$. Let us see that $\ddot{v}_n(t_1-) = 0$. In fact, if $\ddot{v}_n(t_1-) > 0$, then $\dot{v}_n(t)$ would be increasing on an interval of the form $(t_1 - \varepsilon, t_1)$ , with $\varepsilon > 0$ sufficiently small. This implies that we can find $t \in (t_1 - \varepsilon, t_1)$ such that $\dot{v}_n(t) < \dot{v}_n(t_1) = 0$, which is impossible.

From the equation (3.4), we obtained

$$\dot{u}_{n-1}(t_1^-) = 0 \quad \text{and} \quad \dot{u}_{n+1}(t_1^-) = 0 \tag{3.5}$$

We know that $a + c = -b$, and note that $v_{n-1}(t_1) \geqslant 1$ would imply that $u_{n-1}(t_1) = u_n(t_1) = u_{n+1}(t) = 1$ and thus

$$\begin{aligned}\dot{v}_n(t_1) &= d(a + b + c) + h(1)\\ &= h(1) > 0,\end{aligned}$$

Which is not possible. This leads us to conclude that $v_{n-1}(t_1) < 1$.
If $0 < v_{n-1}(t_1) < 1$, then $v_{n-1}(t_1) = u_{n-1}(t_1)$ and from (3.5) results $\dot{v}_{n-1}(t_1) = 0$. We are as at the beginning, when it was assumed that (ii) occurs, only now instead of having $\dot{v}_n(t_1) = 0$ with $u_n(t_1) > 0$, the situation is $v_{n-1}(t_1) = 0$ with $u_{n-1}(t_1) > 0$. Thus, reasoning as before, analogous to (3.5), we obtain that

$$\dot{u}_{n-2}(t_1-) = 0 \ \text{ and } \ u_n(t_1-) = 0.$$

If $v_{n-1}(t_1) \leq 0$, then $u_{n-1}(t_1) = 0$ and this equality, from the definition of $t_1$, can only occur when $n = -1$.
Now, by induction we can conclude that

$$\dot{u}_j(t_1-) = 0 \quad \text{for} \quad j = 0, 1, \ldots, n+1. \tag{3.6}$$

If $v_{n+1}(t_1) > 1$, then $u_{n+1}(t) = 1$ for $t$ sufficiently close to $t_1$ and therefore by (P1)

$$\dot{u}_{n+1}(t_1-) = \cdots = \dot{u}_N(t_1-) = 0. \tag{3.7}$$

If $v_{n+1}(t_1) \leqslant 1$, then $\dot{v}_{n+1}(t_1) = \dot{u}_{n+1}(t_1-) = 0$ and we may repeat the arguments which lead to (3.5). In either case we will arrive at (3.7) which together with (3.6) gives

$$\dot{u}_n(t_1-) = 0 \quad \text{for} \quad n = 0, 1, \ldots, N. \tag{3.8}$$

We now consider the number $n_1$ given by

$$n_1 := \max\{n \in \{0, 1, \ldots, N\} : u_n(t) < 1 \quad \text{for all } t \in [0, t_1)\}.$$

Let us see that $n_1$ is well defined, in fact: If for all $n$ in the set $\{0,1,\ldots,N\}$ one has that $u_n\left(\tilde{t}_n\right)=1$ for some $\tilde{t}_n\in(0,t_1)$, then $u_n(t)=1$ for all $t\in\left(\tilde{t}_n,t_1\right)$. It follows that $u_n(t)=1$ for all $n\in\{0,1,\ldots,N\}$ and all $t\in\left(\tilde{t},t_1\right)$, where $\tilde{t}:=\max\left\{\tilde{t}_0,\tilde{t}_1,\ldots,\hat{t}_N\right\}$. Consequently, for all $t\in\left(\tilde{t},t_1\right)$ it follows that

$$v_0(t)=d(b+c)+h(1)\quad\text{and}\quad \dot{v}_1(t)=\ldots..=\dot{v}_N(t)=h(1).$$

But this, because $v_n\left(t_1\right)=0$ for some $n$, is not possible. Thus, $n_1$ is well defined.
Now, from the definition of $n_1$ it follows that $u_n(t)=v_n(t)$ for $n\in\{0,1,\ldots,n_1\}$ and furthermore there exists $t_0\in[0,t_1)$ such that $u_{n_1+1}(t)=1$ for all $t\in[t_0,t_1]$. Then, for $n=0,1,\ldots,n_1$ the equation

$$\dot{v}_n(t)=d\left(au_{n-1}(t)+bu_n(t)+cu_{n+1}(t)\right)+h\left(u_n(t)\right)$$

becomes

$$\dot{u}_n(t)=d\left(au_{n-1}(t)+bu_n(t)+cu_{n+1}(t)\right)+h\left(u_n(t)\right)$$

and by (3.8) it follows that $\dot{u}_n\left(t_1\right)=0$ for $n=0,1,\ldots,n_1$.
Finally, because of the uniqueness of the solutions of the initial value problem (3.1), it follows that $u_n(t)=u_n\left(t_1\right)$ for all $t\in\left[t_0,t_1\right], n=0,1,\ldots,n_1$. This contradicts the choice of $t_1$, one should have that if $t<t_1$ and $u_n(t)>0$, then $\dot{v}_n(t)>0$. ☐

The proof of theorem 3.1.1 leads us to define two sets that are associated with the initial value problem.

**Definition 3.1.2.** *For $dc>-h(0)$ and $N\in\mathbb{N}$ define*

$$\mathbf{C}\left(a,b,c,h,d,N\right):=\{x\in\mathbb{R}^{N+1}:x_0=0,x_1=\frac{-h(0)}{cd}\leq x_2\leq\cdots\leq x_N\leq 1,\textit{ and } x_n=\frac{n}{N},\ \textit{for } n=1,2,\cdots,N\}$$

$$\mathbf{O}\left(a,b,c,h,d,N\right):=\{x\in\mathbb{R}^{N+1}:d(ax_{n-1}+bx_n+cx_{n+1})+h(x_n)>0\ \textit{for } n=1,2,\cdots,N\ \textit{where } x_{N+1}:=1\}$$

*We will write $\mathbf{C}$,$\mathbf{O}$ instead $\mathbf{C}\left(c,h,d,N\right)$,$\mathbf{O}\left(a,b,c,h,d,N\right)$. The space we are interested in is $\overline{\mathbf{C}\cap\mathbf{O}}$ for suitable chosen $a,b,c,h,d$ and $N$.*

**Definition 3.1.3.** *Define $t^*:\overline{\mathbf{C}\cap\mathbf{O}}\to(0,\infty]$ by*

$$t^*:=sup\left\{t:u_0(x;t)<\frac{h(0)}{cd}\right\}$$

*and define*

$$\mathbf{T}:\left\{x\in\overline{\mathbf{C}\cap\mathbf{O}}:t^*<\infty\right\}\to\mathbb{R}^{N+1}$$

*by*

$$(\mathbf{T}x)_n:=\left\{\begin{array}{cl}0 & \textit{for } n=0\\ u_{n-1} & \textit{for } n=1,\cdots,N.\end{array}\right.$$

**Theorem 3.1.4.** $\mathbf{T}\{x \in \overline{\mathbf{C} \cap \mathbf{O}} : t^*(x) < \infty\} \subset \mathbf{O}$

*Proof.* Let us first observe that $u_0\left(x; t^*(x)\right) > 0$. Indeed: If $u_0\left(x; t^*(x)\right) = 0$, then for all $t \geq 0$, $u_0\left(x; t\right) = 0$ . This would imply that $\sup\left\{t : u_0(x;t) < -\dfrac{h(0)}{cd}\right\} = \infty$ and consequently we would have $x$ is not in the domain of $\mathbf{T}$.
Consider the case where $x \in \mathbf{C} \cap \mathbf{O}$ and $x$ satisfies $(P_2)$ of theorem 3.1.1. Then by theorem 3.1.1 and the definition of $t^*(x)$ we have that

$$\begin{aligned}\mathbf{T}x &= \Big(0, u_0\left(x; t^*(x)\right), u_1\left(x; t^*(x)\right), \ldots\ldots, u_{N-1}\left(x; t^*(x)\right)\Big) \\ &= \left(0, -\frac{h(0)}{cd}, u_1\left(x; t^*(x)\right), \ldots\ldots, u_{N-1}\left(x; t^*(x)\right)\right) \in O.\end{aligned}$$

Suppose that $x \in \mathbf{C} \cap \mathbf{O}$. Let $n_1$ be the largest $n \in \{0, 1, \cdots, N\}$ for $x_n < 1$ and define for $\varepsilon > 0$

$$x_n^\varepsilon := \begin{cases} x_n & \text{for } n = 0, 1 \\ x_n + \varepsilon n & \text{for } n = 2, \cdots, n_1 \\ x_n & \text{for } n = n_1 + 1, \cdots, N. \end{cases}$$

For $\varepsilon$ sufficiently small $x^\varepsilon \in \mathbf{C} \cap \mathbf{O}$ and $x^\varepsilon$ satisfies $(P_2)$ of theorem 3.1.1. Thus $u(x^\varepsilon; t^*(x))$ satisfies $(P_3)$ of theorem 3.1.1.

Since $u$ depends continuously on its initial conditions, $x^\varepsilon \to x$ as $\varepsilon \to 0$, one conclude that :

$$y := u(x; t^*) \qquad \text{satisfies } d(ay_{n-1} + by_n + cy_{n+1}) + h(y_n) \geq 0 \qquad \text{whenever } y_n > 0$$

Using the same argument as in the proof of theorem 3.1.1 $(ii)$, we suppose that $d(ay_{n-1} + by_n + cy_{n+1}) + h(y_n) = 0$ and we obtain a contradiction.

Therefore $d(ay_{n-1} + by_n + cy_{n+1}) + h(y_n) > 0$ whenever $y_n > 0$, thus $\mathbf{T}x \in \mathbf{O}$.

For the general case where $x \in \overline{\mathbf{C} \cap \mathbf{O}}$ is also by approximation. There exists a sequence $\{x^l\}$ in $\mathbf{C} \cap \mathbf{O}$ such that $x^l \to x$ as $l \to 0$. Since $u(x^l; t^*(x))$ satisfies $(P_3)$ for each $l$, we conclude that $u(x; t^*(x))$ satisfies $(P_3)$ and $\mathbf{T}x \in \mathbf{O}$ □

From now on, for technical reasons, it was necessary to introduce the following condition:

$$\frac{a}{c} \geqslant 1 \qquad \text{or} \qquad a \geqslant c$$

**Theorem 3.1.5.** *For every $h \in \mathcal{B}$ there exists a positive $\delta$ such that*

$$\mathbf{T}\{x \in \overline{\mathbf{C} \cap \mathbf{O}} : t^*(x) < \infty\} \subset \mathbf{C} \qquad \text{for } N \geq \frac{1}{\delta}$$

*Proof.* Let $y := \mathbf{T}x$. By the definition of $t^*$ one has $y_0 = 0$ and $y_1 = -\frac{h(0)}{cd}$. By theorem 3.1.1 we have that

$$-\frac{h(0)}{cd} = u_0\left(x; t^*(x)\right) \leq u_1\left(x; t^*(x)\right) \leq \ldots \leq u_{N-1}\left(x; t^*(x)\right) \leq 1.$$

Therefore $y_1 \leq y_2 \leq \ldots \leq y_N$. Thus, $y = \mathbf{T}x$ satisfies the first condition defining $\mathbf{C}$. The task now focuses on proving that there exists $\delta > 0$ such that $y_n \geq \frac{n}{N}$ for $n = 1, \ldots, N$, whenever $N \geq \frac{1}{\delta}$. In the following we will construct $\delta$ explicitly. Let

$$\delta_1 := -\frac{h(0)}{cd},$$

$$\delta_2 := \max\{x - A : A < x \leq 1 \text{ and } B(x) \leq \frac{cd}{2}\delta_1\},$$

where $B(x) := \max\limits_{A \leqslant s \leqslant x} h(s)$, with $x \in [A, 1]$ and $A \in (0, 1)$ is the unique zero of $h$,

$$\delta_3 := \min\left\{\frac{1}{2}\delta_1, \delta_2\right\},$$

$$\delta_4 := \frac{\delta_3}{2},$$

$$\delta_5 := \min\left\{\frac{h(s)}{4ad} : A + \frac{\delta_3}{2} \leq s \leqslant 1\right\},$$

and

$$\delta_6 := ad\delta_5 \min\left\{\frac{\delta_1}{m_1}, \frac{ad\delta_5}{m_2}\right\},$$

where

$$m_1 := d(a+c) + \sup_{0 \leq s \leq 1} |h(s)| \text{ and } m_2 := m_1\left(2d(a+c) + \sup_{t \neq s} \frac{|h(s) - h(t)|}{|s - t|}\right).$$

Finally, let

$$\delta := \min\left\{\delta_1, \delta_2, \delta_3, \delta_4, \delta_5, \delta_6\right\}. \tag{3.9}$$

Now, the rest of the discussion of the theorem focuses on proving, under the assumption that $N \geq \frac{1}{\delta}$, that

$$y_n \geqslant \frac{n}{N} \quad \text{for} \quad n = 1, 2, \ldots, N.$$

Suppose that $y_n \leq A$. Then $h(y_i) \leq 0$ for $i = 1, \ldots, n$. Let us see that $y_{i+1} \geqslant \frac{i+1}{N}$ for $i = 0, \ldots, n$. Since $y \in \mathbf{O}$, we have that

$$d\left(ay_{n-1} + by_n + cy_{n+1}\right) + h\left(y_n\right) = d\left(ay_{n-1} - (a+c)y_n + cy_{n+1}\right) + h\left(y_n\right) > 0.$$

Then

$$a\left(y_{n-1}-y_n\right)+c\left(y_{n+1}-y_n\right)>-\frac{h(y_n)}{d}\geq 0$$

and thus

$$y_{n+1}-y_n>\frac{a}{c}\left(y_n-y_{n-1}\right)\geq y_n-y_{n-1}$$

which lead us to the following chain of inequalities

$$y_{n+1}-y_n>y_n-y_{n-1}>\cdots>y_2-y_1>y_1-y_0=-\frac{h(0)}{cd}.$$

Now, by induction it follows that

$$y_{i+1}\geqslant i\delta\geqslant\frac{i+1}{N}\quad\text{for}\quad i=0,\ldots,n.$$

If $y_{N-1}\leq A$, then $\mathbf{T}x\in\mathbf{C}$ and there is nothing more to prove. If that is not the case there is an index $n_0$ such that

$$y_{n_0-1}\leqslant A\quad\text{and}\quad y_{n_0}>A.$$

According to the above discussion, in order to establish that $\mathbf{T}x\in\mathbf{C}$ it would be necessary to prove that

$$y_n\geqslant\frac{n}{N}\quad\text{pasa } n=n_0+1,\ldots\ldots,N.$$

We have that

$$\begin{aligned}
y_{n_0+1}-y_{n_0} &> \frac{a}{c}\left(y_{n_0}-y_{n_0-1}\right)-\frac{1}{cd}h\left(y_{n_0}\right)\\
&\geq y_{n_0}-y_{n_0-1}-\frac{1}{cd}h\left(y_{n_0}\right)\\
&\vdots\\
&\geq y_1-y_0-\frac{1}{cd}h\left(y_{n_0}\right)=\delta_1-\frac{1}{cd}h\left(y_{n_0}\right).
\end{aligned}$$

Thus, if $A<y_{n_0}\leq A+\delta_2$, then $y_{n_0+1}-y_{n_0}>\frac{1}{2}\delta_1$. Therefore,

$$\begin{aligned}
y_{n_0+1} &\geq y_{n_0}+\frac{1}{2}\delta_1\geqslant n_0\delta_1+\frac{1}{2}\delta_1\\
&\qquad\geq n_0\delta_1+\delta_3\geq\left(n_0+1\right)\delta\geqslant\frac{n_0+1}{N}.
\end{aligned}$$

Then, as far as case $y_{n_0}>A+\delta_2$ is concerned, it is sufficient to show that

$$y_n\geq\frac{n}{N}\quad\text{for}\quad y_n\geqslant A+\delta_3.$$

Suppose the contrary and let $n_1$ be the smallest index for which

$$y_{n_1}\geqslant A+\delta_3\quad\text{and}\quad y_{n_1}<\frac{n_1}{N}.$$

Then

$$\frac{n_1-2}{N} \leqslant x_{n_1-2} \leqslant x_{n_1-1} \leqslant u_{n_1-1}\left(x; t^*(x)\right) = y_{n_1} < \frac{n_1}{N}$$

and

$$\begin{aligned} x_{n_1-1} \quad \geqslant \quad \frac{n_1-1}{N} = \frac{n_1}{N} - \frac{1}{N} > A + \delta_3 - \frac{1}{N} \\ = A + \frac{\delta_3}{2} + \left(\frac{\delta_3}{2} - \frac{1}{N}\right) \\ \geqslant A + \frac{\delta_3}{2}. \end{aligned}$$

Now, from the definitions of $\delta$ and $\delta_5$, it follows that

$$\frac{1}{N} \leqslant \delta \leqslant \delta_5 \leqslant \frac{h\left(x_{n_1-1}\right)}{4ad} .$$

Then

$$x_{n_1-1} - x_{n_1-2} \quad < \quad \frac{n_1}{N} - \frac{n_1-2}{N} = \frac{2}{N} \leqslant 2\delta_5 \leqslant \frac{h\left(x_{n_1-1}\right)}{2ad}.$$

Thus,

$$x_{n_1-1} - x_{n_1-2} \leq \frac{h\left(x_{n_1-1}\right)}{2ad}. \tag{3.10}$$

Next, we give a general discussion that later will be applied to $n = n_1 - 1$. Suppose $0 < u_n(0) < 1$ and $0 < u_n(t) < 1$. Then

$$\dot{u}_n(t) = d\left(au_{n-1}(t) + bu_n(t) + cu_{n+1}(t)\right) + h\left(u_n(t)\right),$$

$$\begin{aligned} |\dot{u}_n(t)| \quad &= \quad \left|d\Big(a\left(u_{n-1}(t) - u_n(t)\right) + c\left(u_{n+1}(t) - u_n(t)\right)\Big) + h\left(u_n(t)\right)\right| \\ &\leqslant \quad da\left|u_n(t) - u_{n-1}(t)\right| + dc\left|u_{n+1}(t) - u_n(t)\right| + \Big|h\left(u_n(t)\right)\Big| \\ &\leqslant \quad d(a+c) + \sup_{0\leqslant s\leqslant 1}|h(s)| = m_1, \end{aligned}$$

$$\begin{aligned} \dot{u}_n(t) - \dot{u}_n(0) \quad &= \quad da\left(u_{n-1}(t) - u_{n-1}(0)\right) + db\left(u_n(t) - u_n(0)\right) \\ &\quad\quad +dc\left(u_{n+1}(t) - u_{n+1}(0)\right) + h\left(u_n(t)\right) - h\left(u_1(0)\right) \\ &= \quad dat\dot{u}_{n-1}\left(\xi_{n-1}\right) + dbt\dot{u}_n\left(\xi_n\right) + dct\dot{u}_{n+1}\left(\xi_{n+1}\right) \\ &\quad\quad +h\left(u_n(t)\right) - h\left(u_n(0)\right), \qquad \text{where } \ \xi_{n-1}, \xi_n, \xi_{n+1} \in (0,t), \end{aligned}$$

and

$$\begin{aligned}
|\dot{u}_n(t)-\dot{u}_n(0)| &\leqslant dat|\dot{u}_{n-1}(\xi_{n-1})|-dbt|\dot{u}_n(\xi_n)|+dct|\dot{u}_{n+1}(\xi_{n+1})| \\
&\quad+\left(\sup_{t\neq s}\frac{|h(s)-h(t)|}{|s-t|}\right)|u_n(t)-u_n(0)| \\
&= dt\left(a|\dot{u}_{n-1}(\xi_{n-1})|-b|\dot{u}_n(\xi_n)|+c|\dot{u}_{n+1}(\xi_{n+1})|\right) \\
&\quad+t\left(\sup_{t\neq s}\frac{|h(s)-h(t)|}{|s-t|}\right)|\dot{u}_n(\xi_n)| \\
&\leqslant tm_1\left(d(a-b+c)+\sup_{t\neq s}\frac{|h(s)-h(t)|}{|s-t|}\right)=tm_2.
\end{aligned}$$

It follows from the above that

$$\dot{u}_n(t)\geqslant\dot{u}_n(0)-tm_2$$

and thus

$$\dot{u}_n(t)\geqslant\frac{1}{2}\dot{u}_n(0)\quad\text{for}\quad 0\leqslant t\leqslant\frac{\dot{u}_n(0)}{2m_2}.$$

This gives

$$u_n(t)-u_n(0)=\int_0^t\dot{u}_n(s)ds\geqslant\frac{1}{2}\dot{u}_n(0)\int_0^t ds=\frac{1}{2}t\dot{u}_n(0) \tag{3.11}$$

for $0\leqslant t\leqslant\frac{\dot{u}_n(0)}{2m_2}$.

We now continue the discussion with $n=n_1-1$. We have that

$$\begin{aligned}
\dot{u}_{n_1-1}(0) &= da\left(u_{n_1-2}(0)-u_{n_1-1}(0)\right)+dc\left(u_{n_1}(0)-u_{n_1-1}(0)\right)+h\left(u_{n_1-1}(0)\right) \\
&= -da\left(x_{n_1-1}-x_{n_1-2}\right)+dc\left(x_{n_1}-x_{n_1-1}\right)+h\left(x_{n_1-1}\right) \\
&\geqslant -da\left(x_{n_1-1}-x_{n_1-2}\right)+h\left(x_{n_1-1}\right)
\end{aligned}$$

and using (3.10) we obtain

$$\dot{u}_{n_1-1}(0)\geqslant-\frac{1}{2}h\left(x_{n_1-1}\right)+h\left(x_{n_1-1}\right)=\frac{1}{2}h\left(x_{n_1-1}\right).$$

Now, from this estimate, using (3.11) , we obtain that

$$u_{n_1-1}(t)-x_{n_1-1}\geqslant\frac{1}{4}th\left(x_{n_1-1}\right)\ \text{for}\ 0\leqslant t\leqslant\frac{1}{4m_2}h(x_{n_1-1}). \tag{3.12}$$

Next, we obtain an estimate for $t^*$. Using that

$$\begin{aligned}
-\frac{h(0)}{cd} &= u_0\left(t^*\right)-u_0(0) \\
&= \int_0^{t^*}\dot{u}_0(s)ds\leqslant t^*\sup_{0\leqslant s\leq 1}\dot{u}_0(s)
\end{aligned}$$

and

$$
\begin{aligned}
\dot{u}_0(s) &= d\left(bu_0(s)+cu_1(s)\right)+h\left(u_0(s)\right) \\
&\leqslant dcu_1(s)+h\left(u_0(s)\right) \\
&\leqslant dc+h\left(u_0(s)\right)\leqslant m_1,
\end{aligned}
$$

it follows that

$$t^* \geqslant \frac{\delta_1}{m_1}.$$

Since $u_{n_1-1}\left(t^*\right)\geqslant u_{n_1-1}(t)$, we infer now from (3.11) that

$$
\begin{aligned}
u_{n_1-1}\left(t^*\right)-x_{n_1-1} &\geqslant \frac{1}{4}th\left(x_{n_1-1}\right) \\
&= adt\left(\frac{h(x_{n_1-1})}{4ad}\right) \\
&\geqslant adt\delta_5.
\end{aligned}
$$

Now, setting $t=\min\left\{\frac{\delta_1}{m_1},\frac{ad\delta_5}{m_2}\right\}$ it is obtained that

$$u_{n_1-1}\left(t^*\right)-x_{n_1-1}\geqslant\delta_6\geqslant\delta$$

and thus

$$y_{n_1}=u_{n_1-1}\left(t^*\right)\geqslant x_{n_1-1}+\delta\geqslant\frac{n_1-1}{N}+\frac{1}{N}=\frac{n_1}{N}.$$

This is a contradiction to $y_{n_1}<\dfrac{n_1}{N}$. □

## 3.2 Priori Estimates

In this section, we will obtain a priori estimates and properties that are associated with the operator $\mathbf{T}$ already defined in the Definition 3.1.3.

**Theorem 3.2.1.** *Suppose $x$ is a fixed point of $\mathbf{T}$, and let $\tau := t^*(x)$. Let*

$$e(h):=\min_{\frac{a}{4}\leq s\leq\frac{a}{2}}\frac{-h(s)}{cd} \qquad m(h):=\min\left\{\frac{a(h)}{4},e(h)\right\}$$

*and* $$M(h):=d(a+c)+\sup_{0\leq s\leq 1}||h(s)||.$$

*Then* $\tau\geq\tau_0(h):=\dfrac{m(h)}{M(h)}>0$

*Proof.* In order to establish this result, we paid attention to two cases. First suppose that there exists an integer $n$ such that $\frac{A}{4} \leqslant x_n \leqslant \frac{A}{2}$. Since $x = \mathbf{T}x \in \mathbf{O}$ by theorem 3.1.4,we have that

$$d\left(ax_{n-1} + bx_n + cx_{n+1}\right) + h\left(x_n\right) > 0.$$

Then,

$$\begin{aligned} x_{n+1} - x_n &> -\frac{h\left(x_n\right)}{cd} + \frac{a}{c}\left(x_n - x_{n-1}\right) \\ &\geqslant -\frac{h\left(x_n\right)}{cd}. \end{aligned}$$

Thus, in this case

$$x_{n+1} - x_n \geqslant e(h) \geqslant m(h).$$

The second case corresponds to the fact that there is no index $n$ such that $\frac{A}{4} \leqslant x_n \leqslant \frac{A}{2}$. In this scenario there is $n$ such that $x_{n+1} - x_n > \frac{A}{4} \geqslant m(h)$. In either case there exists an integer $n$ such that

$$x_n \leqslant \frac{A}{2} \quad \text{and} \quad x_{n+1} - x_n \geqslant m(h). \tag{3.13}$$

Suppose now that $x_n, x_{n+1}$ are in the interval $(0,1)$. Then

$$v_n(0) = u_n(0) = x_n \ , \ v_n(\tau) = u_n(\tau) = x_{n+1} \ \text{ and } \ u_n(t) = v_n(t) \ , \ 0 \leq t \leq \tau.$$

Thus, $\dot{u}_n(t) = \dot{v}_n(t)$ and

$$\begin{aligned} 0 \ < \ \dot{u}_n(t) &= d\left(au_{n-1}(t) + bu_n(t) + cu_{n+1}(t)\right) + h\left(u_n(t)\right) \\ &\leqslant d\left(a + c\right) + h\left(u_n(t)\right) \\ &= -bd + h\left(u_n(t)\right) \leqslant M(h). \end{aligned}$$

Since $\dot{u}_n(t) \leqslant M(h)$ for all $t$ in the interval $[0, \tau]$ we obtain

$$\begin{aligned} m(h) &\leqslant x_{n+1} - x_n \\ &= u_{n+1}(0) - u_n(0) \\ &= u_n(\tau) - u_n(0) \leqslant \int_0^\tau \dot{u}_n(t)dt \leqslant \tau M(h). \end{aligned}$$

This implies that

$$\tau \geqslant \frac{m(h)}{M(h)} := \tau_0(h).$$

Finally, in the extreme cases $x_n = 0$ and $x_{n+1} = 1$, $x_n = 0$ and $x_{n+1} < 1$ or $x_n > 0$ and $x_{n+1} = 1$, it is also observed that $\dot{u}_n(t) \leqslant M(h)$ for all $t$ in the interval $[0, \tau]$. Then, proceeding as before, the desired result is obtained.

□

**Theorem 3.2.2.** *Let* $\varepsilon_0 := \frac{1}{2} min\{\alpha, 1-\alpha\}$*. For* $x = \{x_n\}_0^N$*, let* $\#(x,\varepsilon)$ *the number of components* $x_n$ *of* $x$ *such that* $\varepsilon < x_n < 1-\varepsilon$*. We have that for all* $\varepsilon \in [0, \varepsilon_0]$ *there exists a bound* $S(\varepsilon, h)$*, independent of* $N$*, such that*

$$\#(x,\varepsilon) \leq S(\varepsilon, h)$$

*for all* $x \in \{x \in \bigcup_N \overline{\mathbf{C} \cap \mathbf{O}} : \mathbf{T}x = x\}$

*Proof.* Let $h \in \mathcal{B}, \varepsilon \in [0, \varepsilon_0]$ and $x$ a fixed point of $\mathbf{T}$. We construct $S(\varepsilon, h)$ in following steps.

Step 1. Let

$$p(\varepsilon, h) := \max_{\varepsilon \leq s \leq \frac{a}{2}} \frac{cd}{-h(s)}$$

and suppose that

$$\varepsilon < x_i \leq x_{i+1} \leq \cdots \leq x_j \leq \frac{A}{2}$$

We have that:

$$1 \geq x_{j+1} - x_i = \sum_{n=i}^{j} (x_{n+1} - x_n)$$

By the inequality $d(ax_{n-1} + bx_n + cx_{n+1}) + h(x_n) > 0$, result

$$\begin{aligned} ax_{n-1} + bx_n + cx_{n+1} + \frac{h(x_n)}{d} &> 0 \\ c(x_{n+1} - x_n) - a(x_n - x_{n-1}) + \frac{h(x_n)}{d} &> 0 \\ x_{n+1} - x_n &> \frac{a}{c}(x_n - x_{n-1}) - \frac{h(x_n)}{cd} \end{aligned}$$

Then,

$$\begin{aligned} 1 \geq x_{j+1} - x_i &= \sum_{n=i}^{j} (x_{n+1} - x_n) \\ &\geq \sum_{n=i}^{j} \frac{a}{c}(x_n - x_{n-1}) - \frac{h(x_n)}{cd} \\ &= \frac{a}{c} \sum_{n=i}^{j} (x_n - x_{n-1}) - \frac{1}{d} \sum_{n=i}^{j} h(x_n) \\ &\geq -\frac{1}{d} \sum_{n=i}^{j} h(x_n) \end{aligned}$$

By definition of $p(\varepsilon, h)$, we have that

$$p(\varepsilon, h) - \geq \frac{cd}{h(x_n)}, \quad n \in i, \cdots, j$$

Then $-h(x_n) \geq \frac{cd}{p(\varepsilon,h)}$
Therefore

$$1 \geq \sum_{n=i}^{j} \frac{1}{p(\varepsilon,h)} = \frac{j-i+1}{p(\varepsilon,h)} \Rightarrow j-i+1 \geq p(\varepsilon,h)$$

That implies that the number of components $x_n$ of vector $x$, with $\varepsilon < x_n \leq \dfrac{A}{2}$ is bounded above by $p(\varepsilon,h)$

Step 2. Now, we have to estimate the number of components that are in the interval $[\dfrac{A}{2}, A]$. In the proof of the theorem 3.2.1 it is shown that there exists an integer $n_0$ such that:

$$x_{n_0} \leq \frac{A}{2}, \quad \text{and} \quad x_{n_0+1} - x_{n_0} \geq m(h) \tag{3.14}$$

Where $m(h)$ is defined in theorem 3.2.1. Let $n_0$ the largest index such that the above occurs 3.14. As in the previous theorem, starting from inequality $d(ax_{n-1} + bx_n + cx_{n+1}) + h(x_n) > 0$, result that

$$x_{n+1} - x_n \leq x_n - x_{n-1}$$

We take $n = n_0+1, n_0+2, \cdots, n_0+k$ as $x_{n_0+1} - x_{n_0} \leq m(h)$ for all $x_n \in [\dfrac{A}{2}, A]$. Then:

$$\frac{A}{2} \geq x_{n_0+k} - x_{n_0+1} = \sum_{i=1}^{k-1} (x_{n_0+i+1} - x_{n_0+i}) \geq (k-1)m(h)$$

Therefore $k$ that is the number of elements $x_n$ in the interval $[\dfrac{A}{2}, A]$ satisfies

$$k \leq 1 + \frac{A}{2m(h)}$$

Step 3. Let

$$b(x,h) := \max_{A \leq s \leq x} \frac{h(s)}{cd}$$
$$b_1(h) := max\left\{x - A : A \leq x \leq 1, b(x,h) \leq \frac{m(h)}{2}\right\}$$
$$b_2(h) := max\left\{b_1(h), \frac{m(h)}{2}\right\}$$

Note that $b_2(h) > 0$, indeed:
If $b_2(h) = 0$, then $b_1(h) = 0 = max\left\{x - A : A \leq x \leq 1, b(x,h) \leq \dfrac{m(h)}{2}\right\}$. Now, exist an integer $n_0$ such that $x_{n_0-1} \geq A$ and $x_{n_0} > A$. We have that $x_{n_0} \geq \alpha + b_1(h)$ or $x_{n_0} > \alpha + b_1(h)$.

Suppose that $x_{n_0} \geq \alpha + b_1(h)$, then:

$$\begin{aligned} x_{n_0+1} - x_{n_0} &\geq x_{n_0} - x_{n_0-1} - \frac{h(x_{n_0})}{cd} \\ &\geq m(h) - \frac{h(x_{n_0})}{cd} \\ &\geq m(h) - \frac{m(h)}{2} \\ &= \frac{m(h)}{2} \end{aligned}$$

In this case

$$\begin{aligned} x_{n_0+1} &\geq \frac{m(h)}{2} + x_{n_0} \\ &> \frac{m(h)}{2} + \alpha \\ &\geq b_2(h) + \alpha \end{aligned}$$

As $b_1(h) \geq b_2(h)$, there is at least one element in a interval $(\alpha, \alpha + b_2(h)]$. Suppose that $x_{n_0} > \alpha + b_1(h)$. In this case $x_{n_0} > \alpha + b_2(h)$ and $x_{n_0} \notin (\alpha, \alpha + b_2(h)]$. In the interval $(\alpha, \alpha + b_2(h)]$ there is at least one $x_n$.

Step 4. Suppose that $\alpha + b_2(h) \leq x_n \leq x_{n+1} \leq 1 - \varepsilon$. Then for $\tau := t^*(x)$

$$x_{n+1} - x_{n-1} \geq x_{n+1} - x_n = u_n(\tau) - u_n(0) = \int_0^\tau \dot{u}_n(s)ds \tag{3.15}$$

For $s \in [0, \tau]$, we estimate $\dot{u}_n(s)$ as follows:

$$\begin{aligned} \dot{u}_n(s) &= d(au_{n-1} + bu_n + cu_{n+1}) + h(u_n) \\ &= d[a(u_{n-1} - u_n) + c(u_{n+1} - u_n)] + h(u_n) \\ &\geq da(u_{n-1} - u_n) + h(u_n) \\ &= -da(u_{n-1} - u_n) + h(u_n) \\ &\geq -da(u_{n-1} - u_n) + \min_{x \leq S \leq x_{n+1}} h(s) \end{aligned}$$

Using this estimate in (3.15), we obtains

$$x_{n+1} - x_{n-1} \geq \frac{\tau}{1 + ad\tau}(\min_{x_n \leq s \leq x_{n+1}} h(s))$$

Since that $\dfrac{\tau}{1 + ad\tau} \geq \dfrac{\tau_0}{1 + ad\tau_0}$ for $\tau \geq \tau_0$, then for the theorem 3.2.1

$$x_{n+1} - x_{n-1} \geqslant b_3(h) := \frac{\tau_0(h)}{1 + ad\tau_0(h)} \left( \min_{\alpha + b_2 \leqslant s \leqslant 1 - \varepsilon} h(s) \right)$$

Therefore, if we suppose that

$$\alpha + b_2 \leq x_i \leq \cdots \leq x_{j+1} < 1 - \varepsilon$$

then

$$2 \geq \sum_{n=i}^{j}(x_{n+1} - x_{n-1}) \geq \sum_{n=i}^{j} b_3 = (j - i + 1)b_3(h)$$

That means, the number of components $x_i$ of $x$ in the interval $[\alpha + b_2, 1 - \varepsilon)$ is bounded above by $1 + \dfrac{2}{b_3(h)}$. Summarizing:

$$\#(x, \varepsilon) \leqslant p(\varepsilon, h) + \left(1 + \frac{\alpha(h)}{2m(h)}\right) + 1 + \left(1 + \frac{2}{b_3(h)}\right) := S(\varepsilon, h)$$

□

We know that $\dot{u}(t) = \dot{u}(x, t)$ is the solution of the (3.1) with any $h \in \mathcal{B}$.
In definition 3.1.3, $\mathbf{T}$ is acting on those $x$ such that $t^*(x) < \infty$. The following two theorems will be used to show that for $d$ sufficiently large $t^*(x) < \infty$ for all $x \in \overline{\mathbf{C} \cap \mathbf{O}}$.

**Theorem 3.2.3.** *Let $x \in \mathbf{C} \cap \mathbf{O}$, $\dot{u}(t) = \dot{u}(x, t)$, where $u(x;t)$ stands for the unique solution of the initial value problem (3.1) with $h$ being any element in $\mathcal{B}$, and $D > 0$. Suppose that for all $n \in \{1, 2, \dots, N\}$ such that $0 < u_n(t) < 1$ we have $\dot{u}_n(t) < D$. Then for all $k \in \mathbb{N}$ such that $cd \geqslant k^2(D + \sup|h|)$, we have that $\Delta_n := u_n(t) - u_{n-1}(t) \leqslant \dfrac{2}{k}$ for $n = 1, 2, \dots, N$.*

*Proof.* For $0 < u_n(t) < 1$ we have

$$\begin{aligned}\dot{u}_n(t) &= d\left(au_{n-1}(t) + bu_n(t) + cu_{n+1}(t)\right) + h\left(u_n(t)\right)\\ &= d\left(au_{n-1}(t) - (a + c)u_n(t) + cu_{n+1}(t)\right) + h\left(u_n(t)\right)\\ &= d\left(c\Delta_{n+1} - a\Delta_n\right) + h\left(u_n(t)\right).\end{aligned}$$

Now, under the assumption $\dot{u}_n(t) < D$ and $cd \geqslant k^2(D + \sup|h|)$ we obtain

$$\begin{aligned}|\, c\Delta_{n+1} - a\Delta_n| &= \frac{1}{d}|\dot{u}_n(t) - h\left(u_n(t)\right)|\\ &\leq \frac{1}{d}(D + \sup|h|)\\ &\leq \frac{c}{k^2}.\end{aligned}$$

Thus,

$$\left|\Delta_{n+1} - \frac{a}{c}\Delta_n\right| \leqslant \frac{1}{k^2} \quad \text{if} \quad 0 < u_n(t) < 1.$$

If $u_n(t) = 0$, then $n = 0, t = 0$, and $u_1(0) = -\dfrac{h(0)}{cd}$. This implies that $\dot{u}_0(0) = 0$. Now, we rewrite $\dot{u}_0(0) = 0$ as

$$\begin{aligned}0 &= d\left(au_{-1}(0) - (a + c)u_0(0) + cu_1(0)\right) + h(0)\\ &= d\left(c\Delta_1 - a\Delta_0\right) + h(0).\end{aligned}$$

Then,

$$\begin{aligned}|c\Delta_1 - a\Delta_0| = \left|\frac{h(0)}{d}\right| &\leq \frac{1}{d}(D + \sup|h|) \\ &\leqslant \frac{c}{k^2}\end{aligned}$$

Thus,

$$\left|\Delta_1 - \frac{a}{c}\Delta_0\right| \leq \frac{1}{k^2} \quad \text{ if } \quad u_n(t) = 0.$$

If $u_n(t) = 1$, then by theorem 3.1.1

$$d\left(au_{n-1}(t) + bu_n(t) + cu_{n+1}(t)\right) + h\left(u_n(t)\right) > 0.$$

Now,

$$\begin{aligned}d\left(au_{n-1}(t) + b + c\right) + h(1) &= da\left(u_{n-1}(t) - 1\right) + h(1) \\ &= -da\Delta_n + h(1) > 0\end{aligned}.$$

and

$$\begin{aligned}|c\Delta_{n+1} - a\Delta_n| &= a\Delta_n \\ &< \frac{h(1)}{d} \leqslant \frac{1}{d}(D + \sup|h|).\end{aligned}.$$

Thus,

$$\left|\Delta_{n+1} - \frac{a}{c}\Delta_n\right| \leqslant \frac{1}{k^2} \text{ if } u_n(t) = 1.$$

So far, we can conclude that in any case

$$-\frac{1}{k^2} \leq \Delta_{n+1} - \frac{a}{c}\Delta_n \leq \frac{1}{k^2}.$$

Then

$$\Delta_n - \Delta_{n+1} \leqslant \frac{a}{c}\Delta_{n-}\Delta_{n+1} \leqslant \frac{1}{k^2}.$$

Now assume contrary to the conclusion that there exists $n_0$ such that

$$\Delta_{n_0} > \frac{2}{k}.$$

Using that $\sum_{i=1}^{m}\left(\Delta_{n_0+i-1} - \Delta_{n_0+i}\right)$ is a telescoping sum we can rewrite $\Delta_{n_0+m}$ as

$$\Delta_{n_0+m} = \Delta_{n_0} - \sum_{i=1}^{m}\left(\Delta_{n_0+i-1} - \Delta_{n_0+i}\right).$$

It follows because of $\Delta_{n_0} > \frac{2}{k}$ and $\Delta_{n_0+i-1} - \Delta_{n_0+i} \leqslant \frac{1}{k^2}$ that

$$\Delta_{n_0+m} > \frac{2}{k} - \frac{m}{k^2}.$$

Thus, for $m = 0, 1, \dots, k$, it turns out that

$$\Delta_{n_0+m} > \frac{1}{k}.$$

This implies that $n_0 + k \leqslant N$ because $\Delta_{N+1} = 0$. Then

$$1 \geqslant u_{n_0+k}(t) - u_{n_0-1}(t) = \sum_{m=0}^{k} \Delta_{n_0+m} > \frac{k+1}{k} = 1 + \frac{1}{k},$$

which is a contradiction. □

**Theorem 3.2.4.** *There exists a number $d_1$ which depends only on* $\sup|h|$, $\sup_{s\neq 1}\left|\dfrac{h(s)-h(t)}{s-t}\right|$, *and* $\int_0^1 h(s)ds$, *such that for all $x \in \mathbf{C} \cap \mathbf{O}, t \in [0, t^*), d > d_1$, the following holds:*

$$\sup_n \dot{u}_n(t) \geqslant \frac{1}{2}\int_0^1 h(s)ds$$

*The* sup *here is taken over all $n$ for which $0 < u_n < 1$.*

*Proof.* Let $n_1 > 0$ such that $u_{n_1}(t) < 1$ In the following, $t$ is fixed and $u_n(t)$ is written as $u_n$, we have that:

$$\dot{u}_n = d(au_{n-1} + bu_n + cu_{n-1}) + h(u_n), \qquad n = 0, \cdots, n_1.$$

That results:

$$h(u_n)(u_{n+1} - u_n) + d(au_{n-1} + bu_n + cu_{n-1})(u_{n+1} - u_n) = \dot{u}_n(u_{n+1} - u_n) \tag{3.16}$$

and

$$h(u_n)(u_n - u_{n-1}) + d(au_{n-1} + bu_n + cu_{n-1})(u_{n+1} - u_n) = \dot{u}_n(u_{n+1} - u_n) \tag{3.17}$$

Adding (3.16) and (3.17), we have:

$$\begin{aligned} h(u_n)(u_{n+1} - u_n) + h(u_n)(u_n - u_{n-1}) + d(au_{n-1} + bu_n + cu_{n-1})(u_{n+1} - u_{n-1}) \\ = \dot{u}_n(u_{n+1} - u_{n-1}) \leqslant \left(\max_{0\leqslant n\leqslant n_1} \dot{u}_n\right)(u_{n+1} - u_{n-1}) \end{aligned}$$

From this result, adding over n from 0 to $n_1$, we obtain the following:

$$\begin{aligned} \sum_{n=0}^{n_1} h(u_n)(u_{n+1} - u_n) + \sum_{n=0}^{n_1} h(u_n)(u_n - u_{n-1}) + \sum_{n=0}^{n_1} d(au_{n-1} + bu_n + cu_{n-1})(u_{n+1} - u_{n-1}) \\ \leqslant \left(\max_{0\leqslant n\leqslant n_1} \dot{u}_n\right)(u_{n+1} - u_{n-1} - u_0) \end{aligned}$$

That is equivalent:

$$\begin{aligned}
&\sum_{n=0}^{n_1} h(u_n)(u_{n+1}-u_n) + \sum_{n=0}^{n_1} h(u_n)(u_n - u_{n-1}) \\
&\quad + d\left(a\left(\sum_{n=0}^{n_1-1} u_n u_{n+2} \sum_{n=0}^{n_1-1} u_n^2 - u_{n_1}u_{n_1+1}\right) + c\left(\sum_{n=0}^{n_1+1} u_n^2 \sum_{n=0}^{n_1+1} u_n u_{n+2} - u_{n_1}u_{n_1+1}\right)\right) \\
&\quad \leqslant \left(\max_{0\leqslant n\leqslant n_1} \dot{u}_n\right)(u_{n+1} - u_{n-1} - u_0)
\end{aligned}$$

Recall that $u_0 \leqslant -\dfrac{h(0)}{cd}$, therefore:

$$\sum_{n=0}^{n_1} h(u_n)(u_{n+1}-u_n) + \sum_{n=0}^{n_1} h(u_n)(u_n - u_{n-1}) \leqslant \left(\max_{0\leqslant n\leqslant n_1} \dot{u}_n\right) + \frac{h^2(0)}{cd} + d(a-c)$$

By the theorem 3.2.3 $\forall k \in \mathbb{N} : cd \leqslant k^2(D + \sup|h|)$ we have that:

$$u_n(t) - u_{n-1}(t) \leqslant \frac{2}{k}, \qquad n = 1, 2, \cdots, n_1(a,c,d)$$

Note that due of theorem 3.2.3 that

$$\begin{aligned}
u_0(t) - u_{-1}(t) = u_0(t) - 0 \leqslant \frac{2}{k} \\
u_{n_1+1}(t) - u_{n_1}(t) = 1 - u_{n_1}(t) \leqslant \frac{2}{k}
\end{aligned}$$

The $t$ is fixed and if we change the $t$, the $n_1$ change.
Then:

$$\left| h(u_0)(u_0 - 0) + \sum_{n=0}^{n_1(a,c,d)} h(u_n)(u_{n+1}-u_n) - \int_0^1 h(s)ds \right| < \varepsilon \tag{3.18}$$

And

$$\left| \sum_{n=0}^{n_1(a,c,d)} h(u_n)(u_n - u_{n-1}) + h(u_1)(1 - u_{n_1}) - \int_0^1 h(s)ds \right| < \varepsilon \tag{3.19}$$

From (3.18) and (3.19), we have:

$$\begin{aligned}
2\left(\int_0^1 h(s)ds - \varepsilon - \frac{1}{k}\sup|h|\right) &< \sum_{n=0}^{n_1(a,c,d)} h(u_n)(u_{n+1}-u_n) + \sum_{n=0}^{n_1(a,c,d)} h(u_n)(u_n - u_{n-1}) \\
&\leqslant \left(\max_{0\leqslant n\leqslant n_1} \dot{u}_n\right) + \frac{h^2(0)}{cd} + d(a-c)
\end{aligned}$$

□

# Chapter 4

# Conclusions and recommendations

In this work we have studied the following initial value problem

$$\begin{aligned} \dot{v}_n &= d\left(au_{n-1} + bu_n + cu_{n+1}\right) + h\left(u_n\right) \\ u_n &= P\left(v_n\right) \\ v_n(0) &= x_n \quad \text{ with } \quad 0 \leqslant x_n \leqslant 1 \text{ for } n = 0, \ldots, N; \end{aligned} \tag{4.1}$$

where we set $u_{-1} \equiv 0$, $u_{N+1} \equiv 1$, $P$ is defined by

$$P\left(v_n\right) := \begin{cases} 0 & \text{for} \quad v_n < 0 \\ v_n & \text{for} \quad 0 \leqslant v_n \leqslant 1 \\ 1 & \text{for } 1 < v_n \end{cases}.$$

We have obtained results that correspond to generalizations of part of the [7] work in which the parameters are $a = c = 1$ and $b = -2$. In our case $a + b + c = 0$ with $a \geq c \geq 0$. Concretely the seven lemmas, which are the fundamental basis for obtaining traveling waves in the discrete Nagumo equation, have been established and proved, emphasizing that the condition

$$a \geq c$$

had to be imposed. Here we put into consideration as future works, the following:

1 Fully generalize everything that is presented in Zinner's work and thus obtain what would be traveling wave type solutions for an equation of the type

$$\dot{u}_n = d\left(au_{n-1} + bu_n + cu_{n+1}\right) + f\left(u_n\right), \quad n \in \mathbb{Z}, \tag{4.2}$$

where $f$ denotes a Lipschitz continuous function satisfying

$$\begin{aligned} &f(0) = f(A) = f(1), \quad f(x) < 0, \quad \text{for} \quad 0 < x < A \\ &f(x) > 0, \quad \text{for } A < x < 1, \text{ and } \int_0^1 f(x)dx > 0. \end{aligned}$$

The idea for dealing with this problem is as follows. First, consider a simpler problem where $f$ is replaced by a linear function $h$ whose only zero is in the interval $(0, 1)$.

Then, the initial value problem considered in this work is considered together with all the developed machinery and the existence of solutions for the simplified problem is addressed via the application of Schauder's fixed point theorem. To conclude, more sophisticated tools such as homotopy theory and degree theory should be resorted to.

2 If we place in (4.2) $c = 0$, we obtain the equation

$$\dot{u}_n = -ad\left(u_{n+1} - u_n\right) + f\left(u_n\right), \quad n \in \mathbb{Z}$$

This can be viewed as a discretization of the nonlinear partial differential equation of first order

$$\frac{\partial u}{\partial t} = D\frac{\partial u}{\partial x} + f(u).$$

Thus, the problem here consists of studying (4.2) when one has a configuration of the parameters such that $a + b + c = 0$ and $c$ is small. This partial differential equation of order one this is of interest in the study of some types of hyperbolic conservation laws, see for instance [17].

3 To know the behavior of the generalized equation and compare it with the discrete Nagumo equation from numerical analysis.

4 Just as in this work obtains some generalizations of the discrete Nagumo equation, we can have similar results for the discrete Fisher equation:

$$\dot{u}_n = d\left(u_{n-1} - 2u_n + u_{n+1}\right) + f\left(u_n\right), \quad n \in \mathbb{Z}, \tag{4.3}$$

Where $f$ in (4.3) is $f(u) = uk(1-u)$, $k > 0$. That is different from Nagumo equation because Nagumo is a cubic whereas in Fisher is a quadratic function [18].

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