UNIVERSIDAD DE INVESTIGACIÓN DE TECNOLOGÍA EXPERIMENTAL YACHAY

Escuela de Ciencias Físicas y Nanotecnología.

TÍTULO: Regular Black Holes and Non-linear Electrodynamics

Trabajo de integración curricular presentado como requisito para la obtención del título de Físico

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Urcuquí, Diciembre, 2021

Urcuquí, 21 de diciembre de 2021

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Dedication

This graduate project is dedicated to my grandfather Hernán, thank you for everything.

Acknowledgements

I would like to express my deepest gratitude to Ernesto Contreras Ph.D., for his permanent guidance, his patience, motivation and for being an example to follow both professionally and personally for all of us who were his students.

I would also like to thank my beloved Yachay Tech University, my second home and where I found great teachers who knew how to give me their hand and guide me through my career. I also thank all my close friends who became my family at this stage of my life and were able to support me in the most difficult moments. I will carry all of them in my mind and in my heart.

And above all, I would like to thank my family who always supported me to follow my dreams and to give my all. To my parents who always helped me, to my mother who is my inspiration and the reason to maintain my strength and determination, to my grandparents who are my second parents and gave me everything I needed to continue my career, to my sister who is the light of my life, and especially to my two angels Hernán and Inés who guide and take care of me from a better place.

Resumen

Los agujeros negros son resultados de la teoría de la Relatividad General los cuales se ha demostrado su existencia, y en los últimos tiempos han sido el centro de interés en investigación para entender su dinámica. Estos objetos tienen comportamientos extremos cuando el observador cae en su interior. En el interior de estos objetos hay muchas preocupaciones e incertidumbres, siendo la más importante la presencia de singularidades. Cualquier teoría existente no ha podido explicar el problema de la presencia de singularidades en los agujeros negros. Por lo tanto, para superar esas dificultades, se han propuesto algunas teorías que conducen a soluciones de agujeros negros regulares. Estas teorías suelen funcionar en regímenes de escala muy pequeña, proporcional a la escala de Plank. Este trabajo se centrará en la Relatividad General acoplada a la electrodinámica no lineal, y mediante el formalismo dual F-P, proponemos un modelo de función estructural $\mathscr{H}(P)$ para caracterizar una familia de soluciones de agujeros negros regulares. Además, el campo eléctrico correspondiente se construye utilizando esta función estructural y la teoría de la electrodinámica no lineal. Además, se estudia la regularidad de la geometría resolviendo las ecuaciones de campo de Einstein.

Palabras Clave: Relatividad General, Agujeros Negros Regulares, Electrodinámicas no-lineales, Formalismo Dual F-P

Abstract

Black holes are the results of the theory of General Relativity which have been proven to exist, and in recent times have been the focus of research interest to understand their dynamics. These objects have extreme behaviors as the observer falls into their interior. Inside these objects, there are many concerns and uncertainties, the most important being the presence of singularities. Any existing theory could not explain the problem of the presence of singularities in black holes. Therefore, to overcome those difficulties, some theories leading to regular black hole solutions have been proposed. These theories often turn out to work on very small-scale regimes proportional to the Plank scale. This work will focus on General Relativity coupled to non-linear electrodynamics, and through F-P dual formalism, we propose a structural function model $\mathscr{H}(P)$ to characterize a family of solutions of regular black holes. In addition, the corresponding electric field is constructed using this structural function and the theory of non-linear electrodynamics. Additionally, the regularity of geometry is studied by solving Einstein field equations.

Keywords: General Relativity, Regular Black Holes, Non-linear Electrodynamics, F-P Dual Formalism

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Chapter 1

Introduction

General Relativity is one of the most influential theories in physics. Since it was published in 1915 by Albert Einstein[1], it has provided a more sophisticated way of understanding how geometry is related to matter and energy, consequently, it states a geometrical interpretation of gravity. All of this happens within a four-dimensional pseudo-Riemannian manifold called space-time[2].

The first results of this theory were immediate since they explained the problem with the perihelion anomaly of mercury[3] and deflection of light[4]. In the last decades, theoretical predictions from General Relativity such as the existence of black holes[5] and the gravitational waves[6] have been demonstrated experimentally[7,8]. The presence of singularities in solutions given by General Relativity are a typical particularity of this theory. Penrose and Hawking proved that the existence of singularities in Einstein's equations is inevitable under certain circumstances[9]. These singularities can not be naked, and event horizons must dress them[10], so it is not possible to connect causal effects in the interior of the black hole with external fields[11]. The presence of singularities is a motivation to establish that the theory is not valid and could break at minimal scale lengths, and is where quantum theories of gravity[12] appear to give a better view of small-scale lengths. Unfortunately, there is still no theory of quantum gravity consolidated and proven. Therefore, the study of classical theories of gravity is one of the most indicated tools to study black holes. In this case, it is necessary to focus on regular solutions[13].

In the middle of 1960, Sakharov[14] proposes that at high densities, the equation of state

$$p = -\rho,$$

is the proper manner to describe the matter-energy content. Followed by Gliner[15], who stipulates that at high energy densities also, all the particles lose their matter nature, this through a transition of state to a false vacuum state given by the energy-momentum tensor

$$T_{\mu\nu} = \Lambda g_{\mu\nu},$$

which describes a de Sitter geometry. Here, Gliner stipulates that if such state could occur by a gravitational collapse, the contraction can cease, and instead of a singularity, it results in a final vacuum state inside the horizons. Although, this idea was not taken into account as an exact solution to the Einstein equations, now can be thought as a mechanism leading to the formation of regular black holes.

Regular black holes, are proposed as solutions of the Einstein equations where singularities do not appear. In 1968, Bardeen [16] gave the first regular black hole solution, but in this solution appears a parameter that does not have a physical interpretation. It was not until Ayón-Beato and García [17] shows that the parameter of Bardeen solution corresponds to the monopole charge of a self-gravitating magnetic field described by nonlinear electrodynamics. Non-linear electrodynamics (NED) was proposed for the first time in 1943 by Born and Infield [18] with the aim to remove the divergence of the self-energy of the electron in classical electrodynamics. In the framework of NED, the dynamics of the system are governed by the equations obtained from a lagrangian $L(F)$, which is certain function of the the Maxwell scalar F. The interest on NED appears again since it can be coupled to Einstein's equations, and some nonlinear models appear in quantum electrodynamics [19] and string theory [20,21].
Although, the Lagrangian $L(F)$ plays an important role in defining the NED sector, is possible to use an alternative form of NED through the so called dual F-P formalism, which allows to define the electromagnetic field tensor in terms of an auxiliary tensor $P_{\mu\nu}$, via a Legendre transformation of the Lagrangian $L(F)$ that leads to the functional $\mathscr{H}(P)$ which, as we shall see later, is related to the mass function of the black hole.

1.1 Problem Statement

Although there exist several ways to find regular black hole solutions, one of the most interesting is the dual F-P formalism, where an adequate form of $\mathscr{H}(P)$ is proposed and studied through the theory of non-linear electrodynamics coupled to General Relativity. In this work, we propose a generalized form for an $\mathscr{H}(P)$ such that it corresponds to a family of regular black holes.

This work is organized as follows:

- In the chapter 2, it will be presented all the theoretical framework to explain the coupling of General Relativity and non-linear electrodynamics. This consists of a review of General Relativity, the non-linear electrodynamics theory coupled to General Relativity and the F-P dual formalism.
- In chapter 3, it will provided a review for the construction of the structural function using the F-P dual formalism, then a way to generalize the resulting structural function. Additionally, is presented a study of the horizons of the solutions achieved and their regularity, then a study of the energy conditions for the solutions.
- In chapter 4, we present the conclusions of the graduation project and as well a proposals to extend this work in the future.

1.2 General and Specific Objectives

In this project the general objective is to study the non-linear electrodynamics coupled to General Relativity and the F-P dual formalism in order to propose a $\mathscr{H}(P)$ from which a family of regular black hole solutions can be constructed.

1.2.1 Specific Objectives

- To study the theoretical framework to couple General Relativity and non-linear electrodynamics.
- To construct of the structural function using the F-P dual formalism.
- To present the horizons of the solutions found.
- To study the regularity of the curvature scalars and the electric field.
- To present the energy conditions of the solutions.

Chapter 2

Methodology

This chapter establishes the theoretical concepts that were used for the completion of this final work. It begins with a review of General Relativity, followed by the study of the Schwarzschild exterior solution to the Einstein field equations. Additionally, the Reissner-Nordström solution is summarized, a review on black holes, singularities, and the Bardeen solution is presented. Finally, the energy conditions and their implication for studying the solutions of regular black holes are shown.

2.1 General Relativity

The theory of General Relativity establishes a relationship between space-time local geometry and the local distribution of matter-energy. In General Relativity, the space-time is a 4-dimensional Riemannian manifold endowed with a Lorentzian metric tensor $g_{\mu\nu}$, which encodes its geometrical structure. The Einstein field equations determine how the presence of an energy-momentum source leads to curve the space-time, namely

$$G_{\mu\nu} = \kappa T_{\mu\nu}, \tag{2.1}$$

where $G_{\mu\nu}$ is the Einstein tensor[†], $T_{\mu\nu}$ is the energy-momentum tensor, and[‡] $\kappa = -\frac{8\pi G}{c^4}$.

The Einstein tensor is defined as

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R, \tag{2.2}$$

where $R_{\mu\nu}$ is the Ricci tensor, $g_{\mu\nu}$ the metric, and R the Ricci scalar. The Ricci tensor is given by

$$R_{\mu\nu} = g^{\alpha\beta} R_{\alpha\mu\beta\nu}, \tag{2.3}$$

where $R^{\alpha}_{\mu\beta\nu}$ is the Riemann tensor. The Ricci scalar also called curvature scalar, is the contraction of the Ricci tensor with the metric

$$R = g^{\mu\nu} R_{\mu\nu}. \tag{2.4}$$

[†]The greek letters are the index of to the components $\left(x^0, x^1, x^2, x^3\right)$.

[‡]In this work the constants for the speed of light c, and the universal gravitational constant G are expressed as natural units: $c = G = 1$.

The Riemann tensor, contains the information about the curvature of the manifold and is defined

$$R^{\rho}_{\mu\beta\nu} = \partial_{\beta}\Gamma^{\rho}_{\mu\nu} - \partial_{\nu}\Gamma^{\rho}_{\mu\beta} + \Gamma^{\lambda}_{\mu\nu}\Gamma^{\rho}_{\lambda\beta} - \Gamma^{\lambda}_{\mu\beta}\Gamma^{\rho}_{\lambda\nu}, \tag{2.5}$$

where the Christoffel symbols $\Gamma^{\theta}_{\gamma\eta}$ are given by

$$\Gamma^{\theta}_{\gamma\eta} = \frac{1}{2}g^{\theta\sigma}\left(\partial_{\gamma}g_{\sigma\eta} + \partial_{\eta}g_{\gamma\sigma} - \partial_{\sigma}g_{\gamma\eta}\right). \tag{2.6}$$

Note that, from (2.6), it is easy to show that $R_{\mu\nu}$ is symmetric.

The information of local matter-energy distribution is encoded in the energy-moment tensor $T_{\mu\nu}$, which for a perfect fluid it is defined

$$T_{\mu\nu} = (\rho + p)\,U_{\mu}U_{\nu} - pg_{\mu\nu}, \tag{2.7}$$

where:

- ρ is the density.
- p is the pressure.
- U^{μ} its the fluid four-velocity.

Using equations (2.7) and (2.2) we can construct the complete set of Einstein's field equations which, given the symmetry of both the Einstein and the energy-momentum tensor, represent a set of 10 second-order partial differential equations.
When the matter content is associated to a fundamental field, the components of the energy momentum-tensor are given by

$$T_{\mu\nu} = -\frac{2}{\sqrt{-g}}\frac{\delta S_M}{\delta g^{\mu\nu}}. \tag{2.8}$$

For a more detailed description of equation (2.8) see Appendix A.

2.1.1 Schwarzschild exterior solution

In 1916 Karl Schwarzschild gives the first exact solution[22] for the Einstein's field equations. This solution was constructed based on the following assumptions

- The space-time is vacuum, namely
$$T_{\mu\nu} = 0. \tag{2.9}$$
- Space-time is static, so the metric does not depend on time explicitly
$$\partial_t g_{\mu\nu} = 0. \tag{2.10}$$
- The space-time is spherically symmetric.

- Space-time is asymptotically flat

$$ds^2 = -c^2dt^2 + dr^2 + r^2\left(d\theta^2 + \sin^2\theta d\phi^2\right). \tag{2.11}$$

Based on the above assumptions the line element can be parametrized as

$$ds^2 = -e^{u(r)}dt^2 + e^{v(r)}dr^2 + r^2\left(d\theta^2 + \sin^2\theta d\phi^2\right). \tag{2.12}$$

Using (2.12) the Christoffel symbols read

$$\Gamma^0_{01} = \frac{\partial_r u(r)}{2}, \qquad \Gamma^1_{00} = \frac{1}{2}e^{u(r)-v(r)}\partial_r u(r), \tag{2.13}$$

$$\Gamma^1_{11} = \frac{\partial_r v(r)}{2}, \qquad \Gamma^1_{22} = r\left[-e^{-v(r)}\right], \tag{2.14}$$

$$\Gamma^1_{33} = r\sin^2(\theta)\left[-e^{-v(r)}\right], \qquad \Gamma^2_{12} = \frac{1}{r}, \tag{2.15}$$

$$\Gamma^2_{33} = -\cos(\theta)\sin(\theta), \qquad \Gamma^3_{13} = \frac{1}{r}, \tag{2.16}$$

$$\Gamma^3_{23} = \frac{\cot(\theta)}{2}. \tag{2.17}$$

Now, as we are assuming a vacuum solution, it is easy to show that $R = 0$, so the Einstein field equations are given by $R_{\mu\nu} = 0$. More precisely, using the above results we find

$$R_{00} = \frac{e^{u(r)-v(r)}}{4r}\left\{\partial_r u(r)\left[4 - r\partial_r v(r)\right] + r\partial_r u(r)^2 + 2r\partial_r^2 u(r)\right\} = 0, \tag{2.18}$$

$$R_{11} = \frac{1}{4}\left[\partial_r u(r)\partial_r v(r) - \partial_r u(r)^2 - 2\partial_r^2 u(r)\right] + \frac{\partial_r v(r)}{r} = 0, \tag{2.19}$$

$$R_{22} = \frac{e^{-v(r)}}{2}\left[-r\partial_r u(r) + r\partial_r v(r) + 2e^{v(r)} - 2\right] = 0, \tag{2.20}$$

$$R_{33} = \frac{\sin^2(\theta)e^{-v(r)}}{2}\left[-r\partial_r u(r) + r\partial_r v(r) + 2e^{v(r)} - 2\right] = 0, \tag{2.21}$$

which, given that $R_{33} = R_{22}\sin^2\theta$, corresponds to a system of three equations with two unknowns.

To make a further simplification of the system is possible to multiply R_{tt} by $e^{v(r)-u(r)}$, and sum with (2.18), to get

$$\frac{\partial_r u(r) + \partial_r v(r)}{r} = 0, \tag{2.22}$$

from where

$$v(r) = c_1 - u(r). \tag{2.23}$$

Replacing the resulting expression (2.23) in (2.20) yield

$$1 - e^{u(r)-c_1}\left(r\frac{du(r)}{dr} + 1\right) = 0, \tag{2.24}$$

from where

$$u(r) = \log\left(\frac{e^{c_2}}{r} + e^{c_1}\right). \tag{2.25}$$

The constant c_1 can be obtained from the asymptotic behaviour condition, namely

$$\lim_{r\to\infty} e^{u(r)} = e^{c_1} = 1. \tag{2.26}$$

Now, using (2.26) in (2.23) is possible verify that

$$\lim_{r\to\infty} e^{v(r)} = 1. \tag{2.27}$$

To give a value for the constant c_2, we recall that in the weak field limit

$$g_{00} = \eta_{00} + h_{00}, \tag{2.28}$$

where h_{00} is defined as

$$h_{00} = 2\Phi, \tag{2.29}$$

with $\Phi = \frac{M}{r}$, is the gravitational potential for Newtonian theory[23]. Then, as η_{00} corresponds to the Minkowski metric[24], this makes $\eta_{00} = -1$, from where

$$e^{c_2} = -2M. \tag{2.30}$$

Therefore, the resulting functions are

$$e^{u(r)} = 1 - \frac{2M}{r}, \tag{2.31}$$

and

$$e^{v(r)} = \left(1 - \frac{2M}{r}\right)^{-1}. \tag{2.32}$$

Replacing this functions in (2.12), the line element reads

$$ds^2 = -\left(1 - \frac{2M}{r}\right)dt^2 + \left(1 - \frac{2M}{r}\right)^{-1} dr^2 + r^2 d\Omega^2, \tag{2.33}$$

which corresponds to the Schwarzschild[¶] exterior solution for empty and spherically symmetric space-time. With this solution was possible to explain the advance of the perihelion in the orbit of mercury[3]. Also, it has a characteristic that makes it interesting, when r takes minimal values ($r \sim 0$), the time component of the metric diverges, just as when $r = 2M$ the radial component diverges. In section 2.1.3 we describe these divergences and its implications.

[¶]Here, $d\Omega^2 = \left(d\theta^2 + \sin^2\theta d\phi^2\right)$ is the angular part of the line element.

2.1.2 Reissner-Nordström exterior solution

Postulated for the first time in 1918 by H. Reissner[25] and years latter by G. Nördstrom[26], this solution to Einstein's field equations is of great importance since it describes the spherically symmetric and static space-time, but in this case, the electromagnetic field is the source. To present the Reissner-Nordström solution first, let us use a general, spherically symmetric, and static line element

$$ds^2 = -A(r) + B(r) + r^2 d\Omega^2. \tag{2.34}$$

The energy-momentum tensor from electromagnetic field as source (see Appendix B) is given by

$$T_{\mu\nu} = -\frac{1}{4\pi}\left(F_{\mu\rho}F^{\rho}_{\nu} - \frac{1}{4}g_{\mu\nu}F_{\rho\sigma}F^{\rho\sigma}\right), \tag{2.35}$$

where $F_{\mu\nu} = \partial_\mu \mathbf{A}_\nu - \partial_\nu \mathbf{A}_\mu$ is the electromagnetic tensor, and $\mathbf{A}_\mu$ is the gauge field which for static and spherically symmetric solutions reads $\mathbf{A}_\mu = (-\Phi, 0, 0, 0)$.
The non-vanishing components of $T_{\mu\nu}$ are

$$T_{00} = \frac{\partial_r \Phi(r)^2}{8\pi B(r)}, \tag{2.36}$$

$$T_{11} = -\frac{\partial_r \Phi(r)^2}{8\pi A(r)}, \tag{2.37}$$

$$T_{22} = \frac{r^2 \partial_r \Phi(r)^2}{8\pi A(r) B(r)}, \tag{2.38}$$

$$T_{33} = \frac{r^2 \sin^2\theta \partial_r \Phi(r)^2}{8\pi A(r) B(r)}. \tag{2.39}$$

Additionally, as

$$A(r)T_{11} + B(r)T_{00} = 0, \tag{2.40}$$

and

$$A(r)R_{11} + B(r)R_{00} = 0, \tag{2.41}$$

from where, the functions $A(r)$ and $B(r)$ can be expressed as a general $f(r)$ as

$$f(r) = A(r) = \frac{1}{B(r)}. \tag{2.42}$$

At this point it is necessary to recall the Maxwell's equations in vacuum[27], given by

$$\nabla_\nu F^{\mu\nu} = 0, \tag{2.43}$$

and

$$\partial_{[\mu}F_{\nu\lambda]} = 0, \tag{2.44}$$

where square brackets means cyclic permutations of the index. Using (2.34) and (2.43), we obtain

$$E_r = \frac{q}{r^2}, \tag{2.45}$$

where q is the electric charge[28]. With these results at hand, $T_{\mu\nu}$ reads

$$[T_{\mu\nu}] = \begin{pmatrix} f(r) & 0 & 0 & 0 \\ 0 & -\frac{1}{f(r)} & 0 & 0 \\ 0 & 0 & r^2 & 0 \\ 0 & 0 & 0 & r^2\sin^2\theta \end{pmatrix} \left(\frac{q^2}{8\pi r^4}\right). \tag{2.46}$$

Additionally, using (2.34) and (2.42), the component of the Ricci tensor R_{22} is

$$R_{22} = 1 - f(r) - r\partial_r f(r), \tag{2.47}$$

from where

$$8\pi T_{22} = 1 - f(r) - r\partial_r f(r). \tag{2.48}$$

Comparing with the corresponding component of (2.46), is found

$$f(r) = 1 + \frac{q^2}{r^2} + \frac{c_1}{r}. \tag{2.49}$$

Note that, when $q = 0$, the Schwarzschild solution must be recovered, so the constant c_1 is given by

$$c_1 = -2M. \tag{2.50}$$

Finally, the line element of the Reissner-Nordström solution reads

$$ds^2 = -\left(1 - \frac{2M}{r} + \frac{q^2}{r^2}\right)dt^2 + \left(1 - \frac{2M}{r} + \frac{q^2}{r^2}\right)^{-1} dr^2 + r^2 d\Omega^2. \tag{2.51}$$

In this case, in contrast to the Schwarzschild solution, the metric function $f(r)$ has two roots given by

$$r_\pm = M \pm \sqrt{M^2 - q^2}. \tag{2.52}$$

Additionally, as in the Schwarzschild case, the solution present a singularity at $r = 0$ which consequences will be discussed in the next section.

2.1.3 Black holes and singularities

To begin the analysis of the solutions that we have found in sections 2.1.1 and 2.1.2, let us define the curvature scalars: R, $R_{\mu\nu}R^{\mu\nu}$, and $R_{\sigma\lambda\mu\nu}R^{\sigma\lambda\mu\nu}$. In this work, we assume that the geometry has real singularities as long as one of these scalars are singular. Then, for a line element of the form

$$ds^2 = -f(r)dt^2 + f(r)^{-1}dr^2 + r^2 d\Omega^2, \tag{2.53}$$

the Ricci scalar R gives

$$R = -\frac{2}{r^2}\left[-1 + f(r) + 2rf'(r)\right] - f''(r), \tag{2.54}$$

the Ricci squared $R_{\mu\nu}R^{\mu\nu}$ is

$$R_{\mu\nu}R^{\mu\nu} = \frac{1}{2r^4}\left\{4 + 4f(r)\left[-2 + f(r) + 2rf'(r)\right] + 4rf'(r)\left[-2 + 2rf'(r) + r^2 f''(r)\right] + r^4 f''(r)^2\right\}, \tag{2.55}$$

and the Kretschmann scalar $R_{\sigma\lambda\mu\nu}R^{\sigma\lambda\mu\nu}$ reads

$$R_{\sigma\lambda\mu\nu}R^{\sigma\lambda\mu\nu} = \frac{4}{r^4}\left[(f(r)-1)^2 + r^2 f'(r)^2\right] + f''(r)^2. \tag{2.56}$$

For both Schwarzschild and Reissner-Nordström solutions the resulting expressions for the Ricci scalar vanishes, and for the Reisner-Nordström metric, the scalar $R_{\mu\nu}R^{\mu\nu}$ give us a way to arrive an expression for the Electric field E_r corresponding with (2.45). In this regard, all the information about the geometry is encoded in the Kretschmann scalar which we shall discuss in detail for both the Schwarzschild and Reisner-Nordström solution in what follows.

Schwarzschild Black Hole

The expression for the scalar is

$$K_S = \frac{48M^2}{r^6}, \tag{2.57}$$

which diverges at $r = 0$ so this point corresponds to a real singularity. However, at $r = 2M$ the Kretschmann is regular and, although the metric is singular there, this point does not represent a real singularity but an inadequate choice of coordinates. Indeed, as we shall illustrate in what follows, these coordinates lead to incorrect conclusions about to what happen to an observer that falls into the black hole. In order to do so, let us consider radial null geodesics, namely

$$-\left(1 - \frac{2M}{r}\right)dt^2 + \left(1 - \frac{2M}{r}\right)^{-1} dr^2 = 0, \tag{2.58}$$

that leads to

$$t_\pm = \pm\left[r + 2M\log(r - 2M)\right] + c, \tag{2.59}$$

with c as a constant of integration.

Notice that for $r > 2M$, results $\frac{dt}{dr} > 0$. The *outgoing* radial null geodesics are t_+, and t_- the *ingoing* radial null geodesics as shown in Fig 2.1. Note that, when $r \to \infty$, the null geodesics and the r axis are in an angle of $\frac{\pi}{4}$, which corresponds to the asymptotically flat space-time. In the region $r > 2M$, the light cones become narrow as they approach $2M$, making that the null geodesics can not pass through this surface. For $r < 2M$, the light cones turn to point the causal future towards the singularity located at $r = 0$.

The diagram with these coordinates gives that neither light nor any observer can go through $r = 2M$. However, a better choice of coordinates reveals that the $r = 2M$ can be traversed.

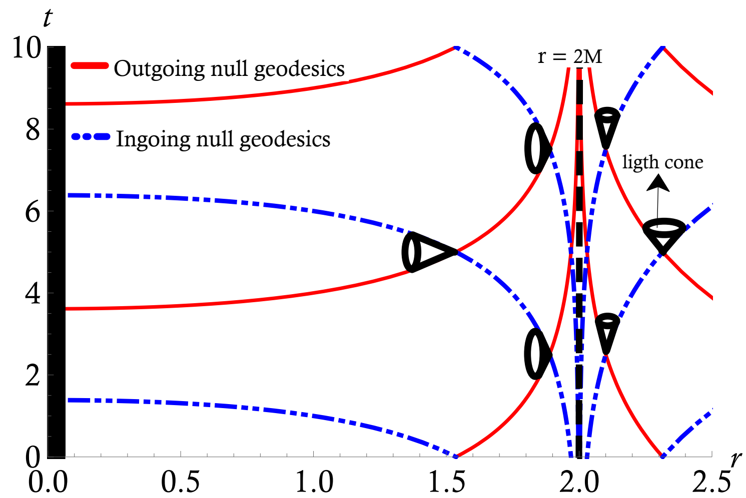

Figure 2.1: The blue lines correspond to the outgoing null geodesics and the red lines are the ingoing null geodesics, the black dashed line marks the event horizon located at $r = 2M$, the thick black line at $r = 0$ is the singularity.

With this in mind, Eddington[4] and Finkelstein[29] postulate a change of coordinates that could connect the trajectory of an observer in both regions. This change of coordinates is given by

$$t_+^* = t - 4M \log\left|\frac{r}{2M} - 1\right| + k, \tag{2.60}$$

and the new coordinate system leads to

$$t_-^* = -r + k, \tag{2.61}$$

with k as a constant.

The space-time diagram in Eddington-Finkelstein coordinates is shown in Figure 2.2. Notice that, light cones in the region $r > 2M$ points towards the causal future. For large values of r, the outgoing null trajectories are in an angle of $\frac{\pi}{4}$ with the r axis corresponding to a flat space-time. Even so, they allow trajectories towards smaller values of r, so that when they reach the point $r = 2M$, they can traverse the surface, and indeed, trajectories on this null surface can only be maintained by light beams. In the region $r < 2M$, the light cones fall, pointing towards the singularity at $r = 0$, making sense that the geodesics converge towards it. In the same way, it can be noted that the point at $r = 2M$ acts as a one-way membrane since it allows time-like and null trajectories to cross it from the outside ($r > 2M$). However, after crossing it, it cannot be returned to the anterior region, being trapped in the interior ($r < 2M$), so neither time nor null trajectory can escape again, heading unavoidably towards the singularity. For this reason, the null surface located at $r = 2M$ is known as "Event Horizon"[28] since, on this border, all causal events from the outside are disconnected, making any observer in the outer region unable to receive any information about the inside of the event horizon.

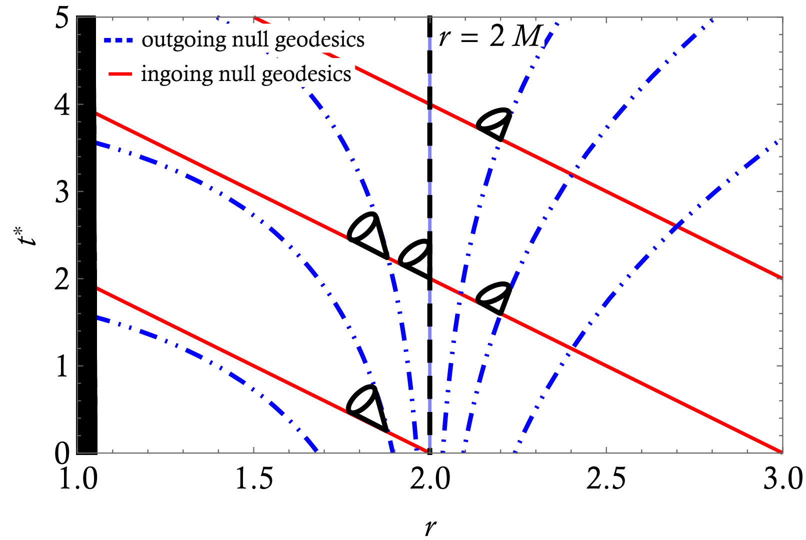

Figure 2.2: The blue lines correspond to the outgoing null geodesics and the red lines are the ingoing null geodesics, the black dashed line marks the event horizon located at $r = 2M$, the thick black line at $r = 0$, labels the singularity.

Reissner-Nordström Black Hole

The Reissner-Nordstöm metric has two roots given by equation (2.52), which can represent three different scenarios where the metric has different behaviours. These cases are: $(M^2 - q^2) < 0$, $(M^2 - q^2) = 0$, and $(M^2 - q^2) > 0$.

Case 1: $M^2 - q^2 < 0$.
In this case, the solution for $r_\pm$ is a complex number, and the metric is regular everywhere except for $r = 0$. In Figure 2.3, is easy to see that the corresponding $f(r)$ for this case does not have roots, thereby event horizons do not appear. This case is a non-physical situation since the total energy of the system is less than the energy of the electromagnetic field, and this scenario would imply that the mass of the charged object in the system is negative[30–32]. Also, as there is no event horizon it violates Penrose's cosmic censorship conjecture[10].

Case 2: $M^2 - q^2 > 0$.
In this solution, there are two roots, one r_+ that is the analogous of the event horizon in the Schwarzschild case, and the other at r_- is the "Cauchy Horizon"[33]. The behaviour of the metric function is shown in Figure 2.3. Note that when $r \to 0$, the function $f(r) \to \infty$.

Case 3: $M^2 - q^2 = 0$.
This case occurs when the total mass of the system is equal to the total charge, which is unstable because adding a small amount of mass to the system would became the second case. In Figure 2.3, it is possible to see that for $f(r)$ in

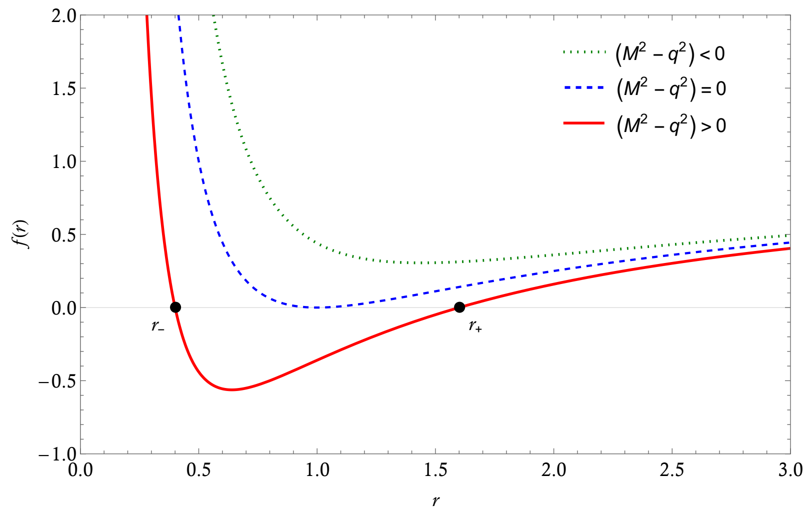

Figure 2.3: Plot of the three situations of the expression $r_\pm$, the green line represent the first case: $(M^2 - q^2) < 0$, the blue line is the second case: $(M^2 - q^2) = 0$, and the red line is a representation of the third case: $(M^2 - q^2) > 0$.

this scenario, it has a unique root, so there is only one horizon at $r = M$, and for this reason, it is called an "Extremal black hole"[28].

The Kretschmann scalar for the Reissner-Nordström solution is

$$K_{RN} = \frac{8\left(6M^2r^2 - 12Mq^2r + 7q^4\right)}{r^8}, \tag{2.62}$$

which is non-singular at the points $r_\pm = M \pm \sqrt{M^2 - q^2}$. This null surfaces are singularities of the metric for the same reason discussed above on the Schwarzschild solution, and they do not correspond to a real singularity. The unique singularity in this space-time is when $r \to 0$, where the Kretschmann scalar diverges. It is possible to use Eddington and Finkelstein coordinates to explore the behavior of null geodesics near the $r_\pm$ surfaces. In this case, the new time coordinate system is given by

$$t^* = r + \frac{r_+^2}{r_+ + r_-}\log|r - r_+| - \frac{r_-^2}{r_+ - r_-}\log|r - r_-|, \tag{2.63}$$

from where the outgoing null geodesics have the form

$$t_+^* = t^* + c, \tag{2.64}$$

and the ingoing null geodesics are given by

$$t_-^* = -r + c, \tag{2.65}$$

where c is a constant.

The resulting space-time diagram in Eddington-Filkenstein advanced coordinates defined with t_+^* and t_-^* is shown in Figure 2.4. In this space-time there are three regions delimited by r_+ and r_-. In region **A**, the light cones are in the same way as in the Schwarzschild solution. When an observer is closer to $r = r_+$, the radial coordinate changes from a space-like coordinate to be a time-like coordinate, so it is an Event Horizon, and inevitably the observer continues to fall in region **B** towards $r = r_+$. In region **B**, the infalling observer inevitably keeps falling towards a smaller radius until it reaches another null surface located at $r = r_-$. After the observer passes through $r = r_-$, then is located in region **C**, where the radial component changes back from a time-like to a space-like coordinate. Trajectories for decreasing r can be avoided, and the observer does not need to reach the singularity at $r = 0$. Curiously, the observer could choose between continue the trajectory into $r = 0$, or take a trajectory increasing the value of r towards r_-.

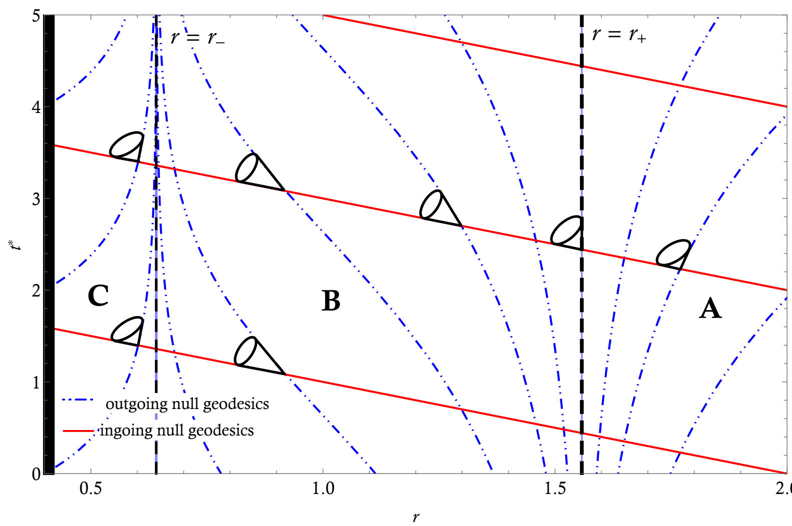

Figure 2.4: The blue lines correspond to the outgoing null geodesics (2.64) and the red lines are the ingoing null geodesics given by (2.65), the black dashed line marks the event horizon located at $r = r_+$, and the Cauchy Horizon located at: $r = r_-$, the thick black line at $r = 0$, labels the singularity in the origin.

2.1.4 Bardeen solution for regular black hole

J. Bardeen[16] postulates for the first time a model of space-time where it is possible to describe a black hole solution without a singularity. It means that a region is trapped by a null surface or event horizon, but a singularity does not appear inside, leading to the metric being regular everywhere. The Bardeen model is given by

$$ds^2 = -\left[1 - \frac{2Mr^2}{(r^2 + g^2)^{\frac{3}{2}}}\right] dt^2 + \left[1 - \frac{2Mr^2}{(r^2 + g^2)^{\frac{3}{2}}}\right]^{-1} dr^2 + r^2 d\Omega^2. \quad (2.66)$$

This solution was constructed as an extension of the Schwarzschild space-time, which can be recognized directly if in (2.66) we set $g = 0$. The solution is asymptotically flat, namely in the limit $r \to \infty$

$$f_B(r) = 1 - \frac{2M}{r} + \frac{3Mg^2}{r^3} + O\left(\frac{1}{r^5}\right). \tag{2.67}$$

Now, the expansion for $r \to 0$, leads

$$f_B(r) = 1 - \frac{2M}{g^3} r^2 + O\left(r^4\right), \tag{2.68}$$

which corresponds to a asymptotic de Sitter space-time[5] with cosmological constant

$$\Lambda = \frac{6M}{g^3}. \tag{2.69}$$

Then, note that the horizon condition $f(r) = 0$ leads to two roots whenever

$$g^2 \leq \frac{16}{27} M^2. \tag{2.70}$$

The Ricci, Ricci squared and Kretschmann scalars are given by

$$R_B = \frac{6M\left(4g^4 - g^2 r^2\right)}{(g^2 + r^2)^{7/2}}, \tag{2.71}$$

$$R_{\mu\nu}R^{\mu\nu} = \frac{18g^4 M^2\left(-4g^2 r^2 + 8g^4 + 13r^4\right)}{(g^2 + r^2)^7}, \tag{2.72}$$

$$K_B = \frac{12M^2\left(-4g^6 r^2 + 47g^4 r^4 - 12g^2 r^6 + 8g^8 + 4r^8\right)}{(g^2 + r^2)^7}, \tag{2.73}$$

so the solution is regular everywhere. Additionally, in the case that $g^2 = \frac{16}{27}M^2$, both horizons $r_\pm$ joins into single one corresponding to an extremal black hole as in the Reissner-Nordström case.

The model proposed by Bardeen has the parameter g, which had no physical relevance, so it was not until Ayón-Beato and García demonstrated[17] that g is the charge corresponding to a self-gravitating magnetic monopole.

2.1.5 The Energy Conditions

In this section we summarize some of the energy conditions that a suitable solution should satisfies.

- **Null Energy Condition**

  The null energy condition (NEC) stipulates that for all null vectors u^μ the following inequality must be satisfied

$$T_{\mu\nu}u^\mu u^\nu \geq 0, \tag{2.74}$$

which for a perfect fluid reads

$$\rho + p \geq 0. \tag{2.75}$$

- **Weak Energy Condition**

  The weak energy condition (WEC) requires that for all time-like vectors t^μ

$$T_{\mu\nu} t^\mu t^\nu \geq 0, \tag{2.76}$$

  which for perfect fluid results in

$$\rho \geq 0, \tag{2.77}$$

$$\rho + p \geq 0. \tag{2.78}$$

  From expressions (2.77) and (2.78) it is possible to notice that the WEC implies the NEC.

- **Dominant Energy Condition**

  The dominant energy condition (DEC) says that for any time-like vector t^μ equation (2.76), and the relation

$$T_{\mu\nu} T^\nu_\rho t^\mu t^\rho \leq 0, \tag{2.79}$$

  which for a perfect fluid reads

$$\rho \geq |p| \,. \tag{2.80}$$

  Note that the DEC implies the WEC.

- **Strong Energy Condition**

  The strong energy condition (SEC) establish that for all time-like vectors t^μ it must be fulfilled that

$$T_{\mu\nu} t^\mu t^\nu - \frac{1}{2} T^\mu_\mu t^\nu t_\nu \geq 0, \tag{2.81}$$

  which for a perfect fluid is equivalent to the following inequalities

$$\rho + p \geq 0, \tag{2.82}$$

$$\rho + 3p \geq 0. \tag{2.83}$$

  Note that the SEC implies the NEC, additionally the SEC implies that the gravitational force is always attractive[34].

In the construction of regular black holes, Zaslaviski showed that for static and spherically symmetric black hole solutions, the SEC is violated in some sector within the horizon[35]. Nevertheless, Dymnikova shows that regular black hole solutions with a de Sitter metric at the origin could satisfy the weak energy condition[36]. The following sections will describe that regular black hole solutions can be constructed by imposing that the WEC is fulfilled.

2.2 General Relativity coupled with Non-linear Electrodynamics

The equations of the Einstein - NED system can be obtained from the action

$$S = \int d^4x \frac{\sqrt{-g}}{4\pi}\left(\frac{1}{2}S_{EH} + S_M\right), \tag{2.84}$$

where S_{EH} is the the Einstein-Hilbert action, and S_M the matter sector[17], namely

$$S_M = -L(F), \tag{2.85}$$

where $L(F)$ is a function of the invariant F. It is necessary to set the requirement that $L(F)$ in the weak field limit corresponds to the Maxwell Lagrangian[37], or $L(F) = F$. Replacing (2.85) in (2.84) and applying the definition (2.8), results a general expression for the energy-momentum tensor for an non-linear electrodynamics source

$$T_{\mu\nu} = \frac{1}{4\pi}\left(L_F F_\mu^\rho F_{\nu\rho} - g_{\mu\nu}L(F)\right), \tag{2.86}$$

with $L_F = \frac{dL}{dF}$, (to see the complete formulation of (2.86) see Appendix C). Applying the conservation property reads

$$\nabla^\mu T_{\mu\nu} = \frac{1}{4\pi}F_\nu^\rho \nabla^\mu \left(L_F F_{\mu\rho}\right) = 0. \tag{2.87}$$

Using the Maxwell equations (see Appendix C) leads to

$$E(r)L_F = -\frac{q}{4\pi r^2}. \tag{2.88}$$

For the line element (2.53), we have

$$-\frac{f(r)}{4\pi}\left[L_F E(r)^2 - L(F)\right] = -\frac{f(r)}{r^2}\left[-1 + f(r) + rf'(r)\right], \tag{2.89}$$

$$\frac{1}{f(r)4\pi}\left[L_F E(r)^2 - L(F)\right] = -\frac{1}{r^2 f(r)}\left[-1 + f(r) + rf'(r)\right], \tag{2.90}$$

$$-\frac{r^2}{4\pi}L(F) = \frac{r}{2}\left[2f'(r) + rf''(r)\right], \tag{2.91}$$

$$\sin^2\theta T_{22} = \sin^2\theta G_{22}. \tag{2.92}$$

It can be shown that

$$E(r) = \frac{2 - 2f(r) + r^2 f''(r)}{4q}. \tag{2.93}$$

Now, from (2.55) and (2.54), is possible to write the following equation

$$4R_{\mu\nu}R^{\mu\nu} - R^2 = \frac{1}{r^4}\left[2 - 2f(r) + r^2 f''(r)\right]^2, \tag{2.94}$$

so, with (2.94) and (2.93) we obtain

$$E(r) = \frac{r^2\sqrt{4R_{\mu\nu}R^{\mu\nu} - R^2}}{4q}. \tag{2.95}$$

From the expression above and (2.94) one can establish that the the electric field $E(r)$ is is regular everywhere whenever $R_{\mu\nu}R^{\mu\nu}$ and R are also regular.

2.3 F-P Dual Formalism

The action of the non-linear electrodynamics coupled to General Relativity has the Lagrangian $L(F)$ as the matter sector responsible for coupling a non-linear electrodynamics theory with the Einstein field equations. The F-P dual formalism is an alternative formulation of the non-linear electrodynamics. Let us start by defining the invariant P as

$$P = \frac{1}{4}P_{\mu\nu}P^{\mu\nu}, \tag{2.96}$$

and the auxiliary electromagnetic tensor field as

$$P_{\mu\nu} = L_F F_{\mu\nu}. \tag{2.97}$$

Then, it is possible to obtain an alternative description of the non-linear electrodynamics by means of a Legendre transformation

$$\mathscr{H}(P) = 2FL_F - L, \tag{2.98}$$

where the function $\mathscr{H}(P)$ is the so-called structural function. Additionally, the Lagrangian $L(F)$ and the electromagnetic field tensor $F_{\mu\nu}$ can be recovered using an inverse Legendre transformation

$$L = 2P\mathscr{H}_P - \mathscr{H}(P), \tag{2.99}$$

where $\mathscr{H}_P = \frac{d\mathscr{H}}{dP}$, and the electromagnetic field tensor $F_{\mu\nu}$ with

$$F_{\mu\nu} = \mathscr{H}_P P_{\mu\nu}. \tag{2.100}$$

Similar to $L(F)$, the structural function $\mathscr{H}(P)$ should describe Maxwell's theory for weak fields or $\mathscr{H}(P) = P$. Using the definitions (2.96) - (2.100) and replacing in the energy-momentum tensor of the non-linear electrodynamics (2.86) leads to

$$T_{\mu\nu} = -\frac{1}{4\pi}\left[(2P\mathscr{H}_P - \mathscr{H}(P))\, g_{\mu\nu} - \mathscr{H}_P P^{\rho}_{\mu} P_{\nu\rho}\right], \tag{2.101}$$

which corresponds to the energy-momentum tensor for NED in the F-P dual formalism. Now, we want to relate this canonical expression of $T_{\mu\nu}$ obtained above with the geometry of the space-time. Let us define the metric function (2.53)

$$f(r) = 1 - \frac{2m(r)}{r}, \tag{2.102}$$

where $m(r)$ is so called as mass function. It can be noticed that in order to recover solutions (2.33), (2.51) and (2.66), the mass function must take the form

$$m_S(r) = M, \tag{2.103}$$

$$m_{RN}(r) = M - \frac{q^2}{2r}, \tag{2.104}$$

$$m_B(r) = \frac{Mr^3}{(r^2+g^2)^{\frac{3}{2}}}, \tag{2.105}$$

respectively for each one of the metrics. For the matter content, let us define the electromagnetic tensor field $F_{\mu\nu}$ is given by

$$F_{\mu\nu} = E(r)\left(\delta^0_\mu \delta^1_\nu - \delta^1_\mu \delta^0_\nu\right), \tag{2.106}$$

from where it is straightforward to show that

$$P = \frac{1}{2} P_{01} P^{01}. \tag{2.107}$$

Note that with (2.107), the resulting non vanishing component of the energy-momentum tensor T_μ^ν leads

$$T^0_0 = \frac{1}{4\pi}\mathscr{H}(r), \tag{2.108}$$

from where

$$m'(r) = -r^2 \mathscr{H}(r). \tag{2.109}$$

This equation is a significant result of the dual F-P formalism since it directly relates the structural function (a canonical representation of the system) and the mass function with space-time geometry information.

Chapter 3

Results & Discussion

This chapter will focus on using non-linear electrodynamics coupled to General Relativity and dual F-P formalism to propose a structural function based on a continuous distribution function. This distribution function will be expanded to a general version, providing a family of solutions for regular black holes. Next, a representation for the constant of the general structural function will be constructed based on the assumption that the space-time solution must behave asymptotically as a Reissner-Nordström solution. To check the regularity of the proposed solutions, we will present the curvature scalars and verify that these scalars do not diverge. Likewise, the electric fields of the solutions will be calculated, and it will be shown that these electric fields correspond to Maxwell's theory in the weak field limit. Finally, the energy conditions for the solutions found will be studied, verifying that the weak energy condition is met and that the strong energy condition is violated. These conditions will arise a set of constraints for each solution, providing broad relationships between the quantities present in the solutions, the geometric behavior, and the energy-matter distribution.

3.1 Structural function given by Non-linear Electrodynamics and the F-P dual formalism

In this section, we review the work by Balart and Vagenas[38], focusing on continuous distribution to get an expression for the structural function associated with them. To begin with, Balart and Vagenas define the mass function as

$$m(r) = \frac{\sigma(r)}{\sigma_\infty} M, \tag{3.1}$$

where $\sigma(r)$ is a representation of a continuous distribution function, and σ_∞ is a normalization factor of the distribution function for all the space given by

$$\sigma_\infty = \lim_{r\to\infty} \sigma(r). \tag{3.2}$$

Using equation (3.1), it is possible to test this definition for different forms of distribution functions. It should be noted that in the work, it is proposed a list of functions that have fixed different values of the essential quantities for the metric such as the charge q and the total mass M. In this work, we are going to generalize the structural function $\mathscr{H}(P)$ found for a distribution function like Maxwell-Boltzmann type, which reads

$$\sigma(r) = e^{-\frac{q^2}{2Mr}}, \tag{3.3}$$

from where

$$\sigma_\infty = 1. \tag{3.4}$$

Replacing (3.3) and (3.4) in (3.1), the derivative with respect to r of the mass function reads

$$m'(r) = \sigma'(r)M = \left(\frac{q^2}{2r^2}\right)e^{-\frac{q^2}{2Mr^2}}. \tag{3.5}$$

The invariant scalar P can be defined as

$$P = (L_F)^2\, F, \tag{3.6}$$

from where

$$P = -\frac{q^2}{2r^4}, \tag{3.7}$$

and as a consequence

$$\mathscr{H}(P) = Pe^{-\frac{q}{2M}\sqrt[4]{-2q^2P}}. \tag{3.8}$$

Defining

$$U(P) = \left(\frac{q}{2M}\sqrt[4]{-2q^2P}\right), \tag{3.9}$$

we finally arrive at

$$\mathscr{H}(P) = Pe^{-U(P)}. \tag{3.10}$$

3.2 Generalized structural function

An expansion of (3.10) reveals that

$$\mathscr{H}(P) = Pe^{-U(0)} - P^2e^{-U(0)}U'(0) + \frac{1}{2}P^3e^{-U(0)}\left[U'(0)^2 - U''(0)\right] + O\left(P^4\right). \tag{3.11}$$

In order to generalize the structural function (3.10), it is necessary to use an adequate form of $U(r)$ such that fulfills the conditions that lead to the first term of (3.11) be equal to P corresponding to Maxwell theory. Additionally, $U(P)$ has to be a differentiable function and well defined in $P = 0$.

Let us define the function $U(P)$ in terms of r with the general form

$$U(r) = Ar^\alpha, \tag{3.12}$$

from where

$$\mathscr{H}(r) = -\frac{q^2}{2r^4}e^{-Ar^\alpha}. \tag{3.13}$$

The mass function $m(r)$ can be obtained by the integration of (2.109) with $\mathscr{H}(P)$ from (3.13), resulting on

$$m(r) = -\frac{q^2 A^{1/\alpha}\Gamma\left(-\frac{1}{\alpha}, Ar^\alpha\right)}{2\alpha}, \tag{3.14}$$

where $\Gamma(a, b)$ is the incomplete Gamma function. Replacing (3.14) in (2.102) the resulting metric function is

$$f(r) = \frac{q^2 A^{1/\alpha}\Gamma\left(-\frac{1}{\alpha}, Ar^\alpha\right)}{\alpha r} + 1, \tag{3.15}$$

which has two parameters α and A that must be fixed to ensure a regular solution. When the parameter α is a positive integer, it does not correspond to a black hole solution since

$$\lim_{r\to\infty} m(r) = 0. \tag{3.16}$$

Additionally, from (3.11), the condition that $\mathscr{H}(P)$ has to correspond to Maxwell theory it will be achieved only if in (3.12) the value of α is any negative integer. Now for convergence of the solution we define the parameter $\alpha = -c$, resulting on the mass function

$$m(r) = \frac{q^2 A^{-1/c}\Gamma\left(\frac{1}{c}, Ar^{-c}\right)}{2c}, \tag{3.17}$$

from where

$$\lim_{r\to\infty} m(r) = \frac{q^2 A^{-1/c}\Gamma\left(\frac{1}{c}\right)}{2c}, \tag{3.18}$$

with the following conditions

$$q^2 \in \mathbb{R} \wedge A > 0 \wedge c > 0. \tag{3.19}$$

Then, with (3.18) is possible to obtain an expression for the constant A as

$$A = \left[\frac{2Mc}{q^2\Gamma\left(\frac{1}{c}\right)}\right]^{-c}, \tag{3.20}$$

thus, replacing (3.17) and (3.20) in the metric function (2.102) leads to

$$f(r) = 1 - \frac{2M\Gamma\left[\frac{1}{c}, 2^{-c}c^{-c}M^{-c}q^{2c}r^{-c}\Gamma\left(\frac{1}{c}\right)^c\right]}{r\Gamma\left(\frac{1}{c}\right)}, \tag{3.21}$$

so, the line element reads

$$ds^2 = -\left\{1 - \frac{2M\Gamma\left[\frac{1}{c}, 2^{-c}c^{-c}M^{-c}q^{2c}r^{-c}\Gamma\left(\frac{1}{c}\right)^c\right]}{r\Gamma\left(\frac{1}{c}\right)}\right\}dt^2 + \left\{1 - \frac{2M\Gamma\left[\frac{1}{c}, 2^{-c}c^{-c}M^{-c}q^{2c}r^{-c}\Gamma\left(\frac{1}{c}\right)^c\right]}{r\Gamma\left(\frac{1}{c}\right)}\right\}^{-1} dr^2 + r^2 d\Omega^2. \tag{3.22}$$

The expression (3.22) corresponds to a family of solutions for regular black hole solutions given by the structural function $\mathscr{H}(P)$. Note that the limit $r \to \infty$, leads to

$$f(r) = 1 + \Gamma\left[\frac{1}{c}, 2^{-c}c^{-c}M^{-c}q^{2c}r^{-c}\Gamma\left(\frac{1}{c}\right)^{c}\right]\left[-\frac{2M}{\Gamma\left(\frac{1}{c}\right)r} + O\left(\frac{1}{r^4}\right)\right], \tag{3.23}$$

which, regardless of the value of $c > 0$, the metric function will behave asymptotically flat. The asymptotic limit $r \to 0$ yield

$$f(r) = 1 + r^c e^{-2^{-c}c^{-c}M^{-c}q^{2c}r^{-c}\Gamma\left(\frac{1}{c}\right)^c}\left\{cq^{2c}\Gamma\left(\frac{1}{c}\right)^c\left[cq^{2c}\Gamma\left(\frac{1}{c}\right)^c - 2^c(c-1)c^cM^cr^c\right] + 4^c(c-1)c^{2c}(2c-1)M^{2c}r^{2c}\right\}$$
$$\left[-\frac{2^cc^{c-3}M^cq^{2-6c}\Gamma\left(\frac{1}{c}\right)^{-3c}}{r^2} + O\left(r^1\right)\right]. \tag{3.24}$$

Although (3.24) is not a friendly expression, it can be recognized that for values of c greater than 2, the asymptotic metric (3.24) has at least two terms, which vary in their representation depending on the value of c is even or odd.

- **Even value of** c
  For c be an even number the metric function (3.24) has the form

  $$f(r) \sim 1 + h(r)r^2, \tag{3.25}$$

  where $h(r)$ is some function of r which can be described generically like

  $$h(r) \sim P(r)e^{-\frac{1}{r^c}}, \tag{3.26}$$

  here $P(r)$ is a polynomial of r.

- **Odd value of** c
  For c be an odd number, we obtain that as $r \to 0$

  $$f(r) \sim 1 + h(r)r, \tag{3.27}$$

  where, $h(r)$ is the same function as the previous case.

Note that for $c = 1$ in (3.21) results directly the solution from Ref. [38]

$$f(r) = 1 - \frac{2Me^{-\frac{q^2}{2Mr}}}{r}. \tag{3.28}$$

3.3 Horizons and Regularity

Case: $c = 1$

The case for $c = 1$ corresponds to the Balart and Vagenas solution[38]. With (3.28) the resulting horizons

$$r_+ = -\frac{q^2}{2MW\left(-\frac{q^2}{4M^2}\right)}, \tag{3.29}$$

$$r_- = -\frac{q^2}{2MW_{-1}\left(-\frac{q^2}{4M^2}\right)}, \tag{3.30}$$

where W is the Lambert W function. This solutions $r_\pm$ arises with the condition

$$q \leq \frac{2M}{\sqrt{e}}, \tag{3.31}$$

where the case $q = \frac{2M}{\sqrt{e}}$, correspond to the extremal black hole. The behavior of the metric is shown in Figure 3.1.

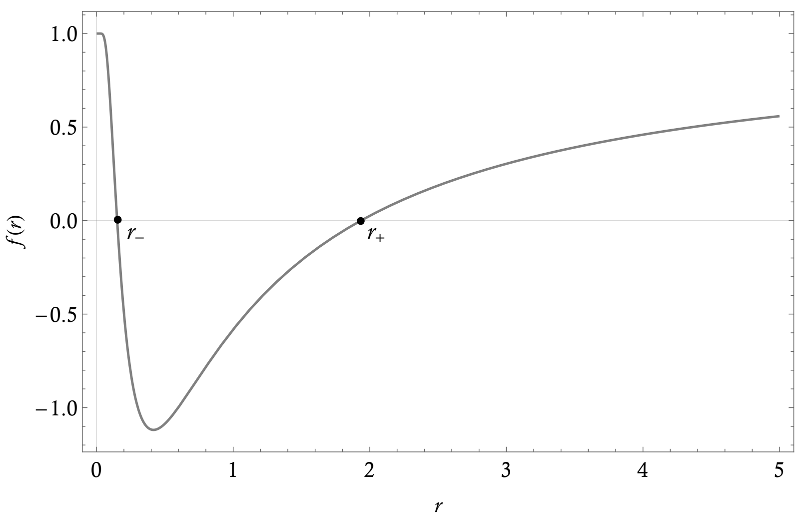

Figure 3.1: Plot of the metric function $f(r)$ for the case $c = 1$, the dots in the function corresponds to each value of $r_\pm$ respectively.

Case: $c \geq 2$

In this case, the condition $f(r) = 0$ does not admit an analytical solution, so we fix the values of the parameters c, q, and M to analyze the resulting plot. In Figure 3.2 we show the metric functions for $M = 1$, $q = 0.8$, and different values of c.

The roots r_+ and r_- have been obtained numerically and shown in Table 3.1.

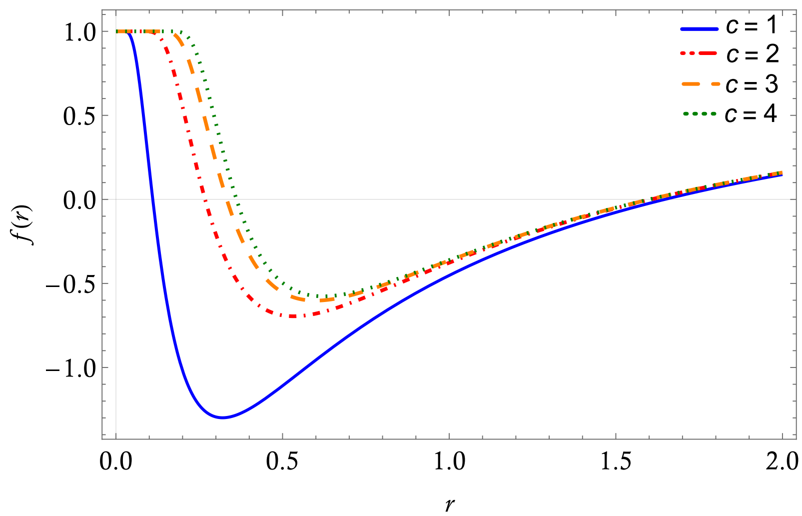

Figure 3.2: Plot of metric functions $f(r)$ for different values of c, this functions have fixed values of $q = 0.8$, $M = 1$, $c = 1$ (blue line), $c = 2$ (red line), $c = 3$ (orange line), and $c = 4$ (green line).

c	r_+	r_-
2	1.606	0.268
3	1.601	0.333
4	1.600	0.368

Table 3.1: Table for the horizons found fixing $q = 0.8$ and $M = 1$, the values of c selected are $c = 2$, $c = 3$, and $c = 4$.

3.3.1 Curvature scalars and Electric field

The curvature scalars (2.54) - (2.56) for the metric function (3.21) are

$$R = 2^{-c}c^{1-c}M^{-c}q^{2c+2}r^{-c-4}\Gamma\left(\frac{1}{c}\right)^{c} e^{\left[-2^{-c}c^{-c}M^{-c}q^{2c}r^{-c}\Gamma\left(\frac{1}{c}\right)^{c}\right]}, \tag{3.32}$$

$$R_{\mu\nu}R^{\mu\nu} = 2^{-2c-1}c^{-2c}q^{4}M^{-2c}r^{-2(c+4)}e^{\left[-2^{1-c}c^{-c}M^{-c}q^{2c}r^{-c}\Gamma\left(\frac{1}{c}\right)^{c}\right]}\left\{cq^{2c}\Gamma\left(\frac{1}{c}\right)^{c}\left[cq^{2c}\Gamma\left(\frac{1}{c}\right)^{c} - 2^{c+2}c^{c}M^{c}r^{c}\right] + 2^{2c+3}c^{2c}M^{2c}r^{2c}\right\}, \tag{3.33}$$

and

$$K = \left\{2^{-2c}c^{-2c}q^4M^{-2c}r^{-2c-8}e^{-2^{1-c}c^{-c}M^{-c}q^{2c}r^{-c}\Gamma\left(\frac{1}{c}\right)^c}\left[-2^{c+3}c^{c+1}M^cq^{2c}r^c\Gamma\left(\frac{1}{c}\right)^c + (5)\,2^{2c+2}c^{2c}M^{2c}r^{2c} + c^2q^{4c}\Gamma\left(\frac{1}{c}\right)^{2c}\right]\right\} - \left\{\frac{2^{3-c}c^{-c}q^2M^{1-c}r^{-c-7}e^{-2^{-c}c^{-c}M^{-c}q^{2c}r^{-c}\Gamma\left(\frac{1}{c}\right)^c}}{\Gamma\left(\frac{1}{c}\right)}\left[(3)\,2^{c+1}c^cM^cr^c - cq^{2c}\Gamma\left(\frac{1}{c}\right)^c\right]\Gamma\left[\frac{1}{c}, 2^{-c}c^{-c}M^{-c}q^{2c}r^{-c}\Gamma\left(\frac{1}{c}\right)^c\right]\right\} + \left\{\frac{48M^2}{r^6\Gamma\left(\frac{1}{c}\right)^2}\Gamma\left[\frac{1}{c}, 2^{-c}c^{-c}M^{-c}q^{2c}r^{-c}\Gamma\left(\frac{1}{c}\right)^c\right]^2\right\}. \quad (3.34)$$

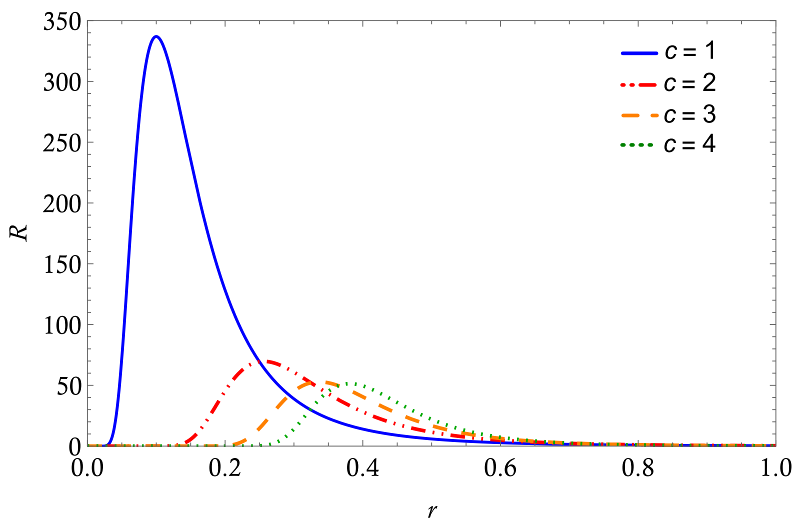

Figure 3.3: In the plot is possible to see different lines corresponding to the values of the Ricci scalar, the blue line is for $c = 1$, the red line is for $c = 2$, for $c = 3$ is the orange line, and for $c = 4$ is the green line.

It can be shown that

$$\lim_{r\to 0^+} R = 0, \quad (3.35)$$

$$\lim_{r\to 0^+} R_{\mu\nu}R^{\mu\nu} = 0, \quad (3.36)$$

$$\lim_{r\to 0^+} K = 0. \quad (3.37)$$

The behavior of each scalar is shown in Figures 3.3, 3.4, and 3.5.

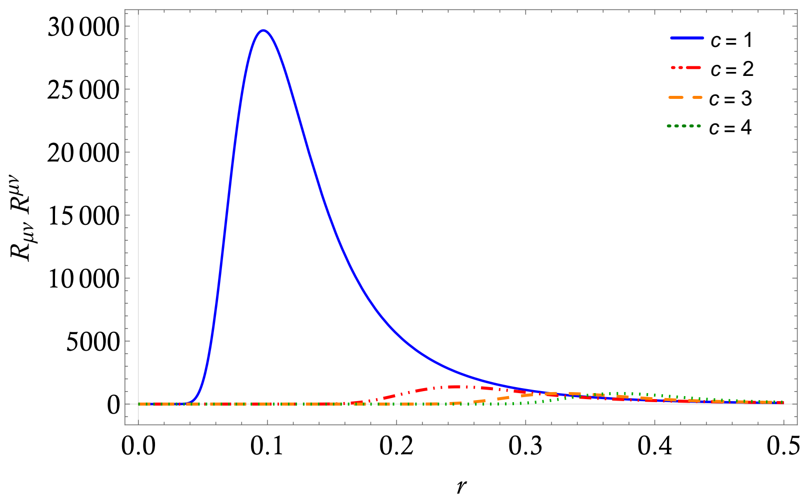

Figure 3.4: In the plot the lines corresponds to the Ricci squared for different c. $c = 1$ (blue line), $c = 2$ (red line), $c = 3$ (orange line), and $c = 4$ (green line).

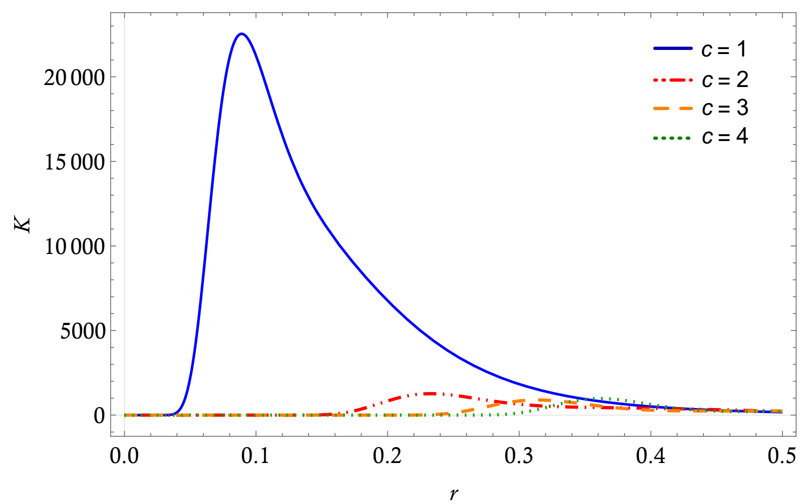

Figure 3.5: In the figure are plotted the Kretschmann scalars for different values of c. $c = 1$ (blue line), $c = 2$ (red line), $c = 3$ (orange line), and $c = 4$ (green line).

The electric field defined by (2.95) for the solutions (3.21) results in

$$E(r) = \frac{r^2}{2q\Gamma\left(\frac{1}{c}\right)} \left\{ 4^{-c} c^{2-2c} M^{2-2c} r^{-2(c+3)} e^{-2^{1-c}c^{-c}M^{-c}q^{2c}r^{-c}\Gamma\left(\frac{1}{c}\right)^c} \left[2^{-c}c^{-c}M^{-c}q^{2c}r^{-c}\Gamma\left(\frac{1}{c}\right)^c \right]^{2/c} \left[2^{c+2}c^c M^c r^c - cq^{2c}\Gamma\left(\frac{1}{c}\right)^c \right]^2 \right\}^{\frac{1}{2}} . \tag{3.38}$$

A series expansion of (3.38) in the weak field limit reveals

$$E(r) \approx \frac{q}{r^2}, \tag{3.39}$$

for any value of c. In Figure 3.6 is shown the behaviour of the electric field $E(r)$.

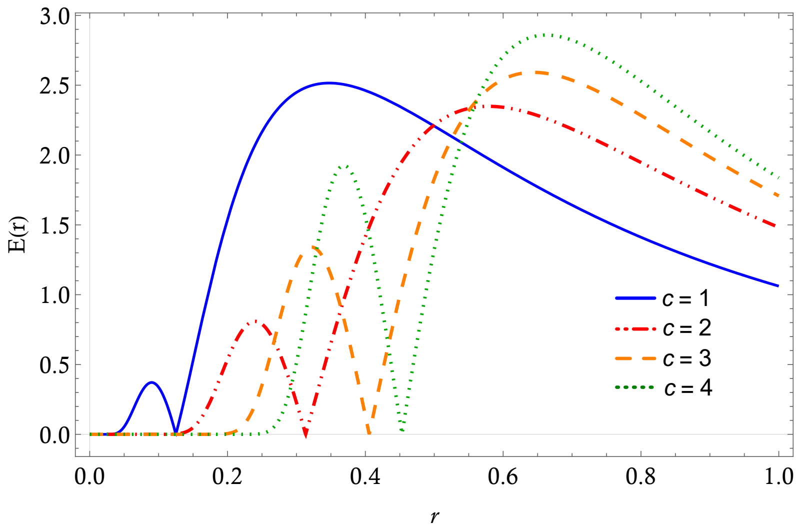

Figure 3.6: In the plot is possible to see different representations of the electric field obtained for different values of c. The blue line correspond to $c = 1$, the red line for $c = 2$, the orange line to $c = 3$, and the green line to $c = 4$.

3.3.2 Energy Conditions

The energy conditions provide important information to understand the behavior of matter-energy in the proposed space-time solution. In this work, the components of the energy-momentum tensor are defined as

$$T^0_0 = -\rho, \tag{3.40}$$

$$T^1_1 = p_r, \tag{3.41}$$

$$T^2_2 = T^3_3 = p_\perp, \tag{3.42}$$

with (2.102), and replacing in (3.40) - (3.42) leads to

$$\rho = \frac{2m'(r)}{r^2}, \tag{3.43}$$

$$p_r = -\frac{2m'(r)}{r^2}, \tag{3.44}$$

$$p_\perp = -\frac{m''(r)}{r}, \tag{3.45}$$

this leads to a direct calculation that

$$\rho + p_r = 0. \tag{3.46}$$

The result above is valid for all $m(r)$, now to see what energy conditions are fulfilled it is necessary to give a value to c, since the energy conditions are in some cases non analytic expressions to reduce. For simplicity, the first two solutions ($c = 1$ and $c = 2$) are taken as samples to get a perspective of the behaviour of all the set of solutions.

Null Energy Condition

Using (2.75) the NEC can be written as

$$\rho + p_r \geq 0, \tag{3.47}$$

$$\rho + p_\perp \geq 0, \tag{3.48}$$

since (3.46) is true, and with (3.43) - (3.45), results that the NEC is satisfied everywhere.

Weak Energy Condition

The equation (2.77) is true everywhere, and with the fulfillment of the NEC, the WEC is satisfied.

Dominant Energy Condition

The DEC defined by (2.80) with the components of $T_{\mu\nu}$ in (3.40) - (3.42) can be written as the conditions

$$\rho \geq |p_r|, \tag{3.49}$$

$$\rho \geq |p_\perp|. \tag{3.50}$$

For $c = 1$ (3.49) is fulfilled for all $r \geq 0$, and (3.50) results that the DEC is fulfilled in the region

$$r \geq \frac{q^2}{8r}. \tag{3.51}$$

Now, for $c = 2$, similarly (3.49) is achieved for all $r \geq 0$, and (3.50) results in the constrain that the DEC is fulfilled for

$$r \geq \frac{\sqrt{\frac{\pi}{2}}q^2}{4M}. \tag{3.52}$$

Strong Energy Condition

The SEC can be written as

$$\rho + p_r + 2p_\perp \geq 0, \tag{3.53}$$

from where, for $c = 1$

$$r \geq \frac{q^2}{4M}, \tag{3.54}$$

implying that the SEC is violated in the region $r < \frac{q^2}{4M}$. Now, for $c = 2$, we obtain that in the region $r < \frac{\sqrt{\pi}q^2}{4M}$ the SEC is violated. The violation of the SEC in these regions inside the horizons follows the result of Zaslavskii [35] for regular black holes. Additionally, the solutions presented (3.21) is an example of regular black hole solutions that fulfills the WEC everywhere.

In a recent work [39], the study of the energy conditions in the regular black hole solutions take great importance since the authors present the relation between the change of topology and the sign of the Tolman mass.

Chapter 4

Conclusions & Outlook

In this work, it was possible to obtain a family of solutions for regular black holes by generalizing a structural function $\mathscr{H}(P)$ obtained from the coupling of the non-linear electrodynamics theory and General Relativity. In addition, it was possible to show that the solutions presented are regular black holes characterized by a parameter c since there are regions where the metric function vanishes, which corresponds to the location of the horizons $r_{\pm}$. Moreover, in a specific case ($c = 1$), the resulting space-time corresponds to the Balart and Vagenas solution[38]. Additionally, it was shown that the solutions obtained are regular everywhere, which was demonstrated in the extreme behavior of both the metric function $f(r)$ and the scalars of curvature.

The asymptotic behavior of all the solutions characterized by this parameter c corresponds, to flat space-time solution in the limit $r \to \infty$. Likewise, it was shown that the family of solutions found correspond to two cases of regular functions, which depend on the parity of the parameter c. It was also possible to show through a numerical calculation the values of the horizons for selected cases of the parameter c: ($c = 2$, $c = 3$ and $c = 4$).

In addition, the energy conditions were studied. It was found that the weak energy condition is fulfilled everywhere, and the strong energy condition is violated in a sector located inside the horizons of such solutions.

Finally, the function $U(r)$ proposed can be further improved. For example, it can be proposed an $U(r)$ as a linear combinations of analytical functions that meet the condition of weak-field limit, we can generalize and couple solutions of regular black holes existing in the literature in a single expression characterized by one or more parameters.

Appendix A

The Energy-Momentum tensor by Lagrangian formulation of General Relativity

The energy-momentum tensor can be obtained via small variations with respect the metric in the General Relativity action defined as

$$S = \int d^4x \sqrt{-g}\left(\frac{1}{16\pi}\sqrt{-g}R + S_M\right), \tag{A.1}$$

where the Einstein-Hilbert action S_{EH} is

$$S_{EH} = \int d^4x \sqrt{-g}R, \tag{A.2}$$

here R is the Ricci scalar.Applying small variations δ to A.2 with respect the metric $g^{\mu\nu}$

$$\delta S_{EH} = \int d^4x \left(\sqrt{-g}R_{\mu\nu}\delta g^{\mu\nu} + g^{\mu\nu}\sqrt{-g}\delta R_{\mu\nu} + R\delta\sqrt{-g}\right). \tag{A.3}$$

We want to the expressions in (A.3) be in the form of a term multiplied by $\delta g^{\mu\nu}$, so it is necessary to develop the last two terms to get this form. In this way let us take the variations of the covariant derivative of the Christoffel symbols

$$\nabla_\lambda\left(\delta\Gamma^\rho_{\mu\nu}\right) = \partial_\lambda\delta\Gamma^\rho_{\mu\nu} + \Gamma^\rho_{\lambda\eta}\delta\Gamma^\eta_{\mu\nu} - \Gamma^\eta_{\lambda\nu}\delta\Gamma^\rho_{\mu\eta} - \Gamma^\eta_{\lambda\mu}\delta\Gamma^\rho_{\eta\nu}, \tag{A.4}$$

then, lets use the variations in the Riemann tensor, rearranging and using (A.4) leads to

$$\delta R^\sigma_{\mu\rho\nu} = \nabla_\rho(\delta\Gamma^\sigma_{\nu\mu}) - \nabla_\nu(\delta\Gamma^\sigma_{\rho\mu}) + \Gamma^\lambda_{\rho\nu}\delta\Gamma^\sigma_{\lambda\mu} - \Gamma^\lambda_{\nu\rho}\delta\Gamma^\sigma_{\lambda\mu} \tag{A.5}$$

.

Since we are in a torsion free space-time, the lower indices of the Christoffel symbols can be interchanged, then it leads to (a4) be simplified to

$$\delta R^{\sigma}_{\mu\rho\nu} = \nabla_{\rho}(\delta\Gamma^{\sigma}_{\nu\mu}) - \nabla_{\nu}(\delta\Gamma^{\sigma}_{\rho\mu}), \tag{A.6}$$

then using definition (2.3) results the variation of the Ricci tensor as

$$\delta R_{\mu\nu} = \nabla_{\rho}(\delta\Gamma^{\rho}\nu\mu) - \nabla_{\nu}(\delta\Gamma^{\rho}_{\rho\mu}), \tag{A.7}$$

here it is necessary to recall the identity

$$\mathrm{Tr}(\ln M) = \ln(\det M). \tag{A.8}$$

Applying the variation to the last identity one gets

$$\frac{1}{\det M}\delta(\det M) = \mathrm{Tr}(M^{-1}\delta M). \tag{A.9}$$

Now, let $M = g^{\mu\nu}$, and $\det M = g^{-1}$, the above expression leads to

$$\delta\left(g^{-1}\right) = g^{-1}g_{\mu\nu}\delta g^{\mu\nu}, \tag{A.10}$$

with (A.10) is possible to get

$$\delta\left(\sqrt{-g}\right) = -\frac{1}{2}\left(\sqrt{-g}\right)g_{\mu\nu}\delta g^{\mu\nu}. \tag{A.11}$$

Replacing (A.11) and (A.7) in (A.3) one get

$$\delta S_{EH} = \int d^4x\left\{\sqrt{-g}R_{\mu\nu}\delta g^{\mu\nu} + \sqrt{-g}g^{\mu\nu}\left[\nabla_{\rho}(\delta\Gamma^{\rho}\nu\mu) - \nabla_{\nu}(\delta\Gamma^{\rho}_{\rho\mu})\right] - \frac{1}{2}R\sqrt{-g}g_{\mu\nu}\delta g^{\mu\nu}\right\}, \tag{A.12}$$

which can also be written as

$$\delta S_{EH} = \int d^4x\left\{\sqrt{-g}R_{\mu\nu}\delta g^{\mu\nu} + \sqrt{-g}\nabla_{\lambda}\left[g^{\mu\lambda}(\delta\Gamma^{\rho}\nu\mu) - g^{\mu\nu}(\delta\Gamma^{\rho}_{\rho\mu})\right] - \frac{1}{2}R\sqrt{-g}g_{\mu\nu}\delta g^{\mu\nu}\right\}. \tag{A.13}$$

The second term leads to an integral over the natural volume element of a total covariant derivative. The Stokes theorem in four dimension gives that this integral is equal to a boundary integral at the infinity, so the evaluating this term at the boundary results that it vanishes. This leads

$$\delta S_{EH} = \int d^4x\sqrt{-g}\left(R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu}\right)\delta g^{\mu\nu}. \tag{A.14}$$

Now, is possible to apply the variational principle to the whole expression (A.14), resulting

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = 0. \tag{A.15}$$

Using (A.14) in (A.1) and applying the variational principle gives

$$-\frac{16\pi}{\sqrt{-g}}\frac{\delta S_M}{\delta g^{\mu\nu}} = R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu}. \tag{A.16}$$

Replacing (2.1) in (A.16)

$$-\frac{16\pi}{\sqrt{-g}}\frac{\delta S_M}{\delta g^{\mu\nu}} = 8\pi T_{\mu\nu}. \tag{A.17}$$

Solving for $T_{\mu\nu}$, the energy-momentum tensor results in

$$T_{\mu\nu} = -\frac{2}{\sqrt{-g}}\frac{\delta S_M}{\delta g^{\mu\nu}}. \tag{A.18}$$

The expression (A.18) is the Lagrangian formulation of the energy-momentum tensor.

Appendix B

The Energy-Momentum tensor from an electromagnetic field source

The energy-momentum tensor for General Relativity coupled to electrodynamics can be obtained with definition (2.8), with the matter sector S_M defined as

$$S_M = \sqrt{-g}\frac{1}{4\pi}F. \tag{B.1}$$

Replacing (B.1) in (2.8) leads

$$T_{\mu\nu} = -\frac{1}{8\pi\sqrt{-g}}\frac{\delta}{\delta g^{\mu\nu}}\left(\sqrt{-g}F_{\mu\nu}F^{\mu\nu}\right), \tag{B.2}$$

and applying the Leibniz rule reads

$$T_{\mu\nu} = -\frac{1}{8\pi\sqrt{-g}}\left[F_{\mu\nu}F^{\mu\nu}\left(\frac{\delta}{\delta g^{\mu\nu}}\sqrt{-g}\right) + \sqrt{-g}\left(\frac{\delta}{\delta g^{\mu\nu}}F_{\mu\nu}F^{\mu\nu}\right)\right]. \tag{B.3}$$

Using (A.11) in the first term of (B.3) given any electromagnetic field, leads to

$$T_{\mu\nu} = -\frac{1}{8\pi\sqrt{-g}}\left[-\frac{1}{2}F_{\rho\sigma}F^{\rho\sigma}g^{\mu\nu}\sqrt{-g} + \sqrt{-g}\left(\frac{\delta}{\delta g^{\mu\nu}}F_{\mu\nu}F^{\mu\nu}\right)\right]. \tag{B.4}$$

Now, for the second term it is necessary to calculate the variation

$$\delta F_{\mu\nu}F^{\mu\nu} = \delta\left(F_{\mu\nu}F_{\alpha\beta}g^{\alpha\mu}g^{\beta\nu}\right), \tag{B.5}$$

applying the Leibniz rule and rearranging the expression it leads to

$$\delta F_{\mu\nu}F^{\mu\nu} = F_{\mu\nu}F_{\alpha\beta}\left(g^{\beta\nu}\delta g^{\alpha\mu} + g^{\alpha\mu}\delta g^{\beta\nu}\right), \tag{B.6}$$

which can be written as

$$\delta F_{\mu\nu}F^{\mu\nu} = F_{\mu\nu}F_{\alpha\beta}\left(g^{\beta\nu}\delta^{\alpha}_{\nu}\delta g^{\mu\nu} + g^{\alpha\mu}\delta^{\beta}_{\mu}\delta g^{\mu\nu}\right). \tag{B.7}$$

Expanding the resulting expression and contracting the indices gives

$$\delta F_{\mu\nu}F^{\mu\nu} = F_{\mu}^{\beta}F_{\nu\beta}\delta g^{\mu\nu} + F_{\nu}^{\alpha}F_{\mu\alpha}\delta g^{\mu\nu} = 2F_{\mu}^{\beta}F_{\nu\beta}\delta g^{\mu\nu}, \tag{B.8}$$

then, it implies that

$$\frac{\delta F_{\mu\nu}F^{\mu\nu}}{\delta g^{\mu\nu}} = 2F_{\mu}^{\beta}F_{\nu\beta}. \tag{B.9}$$

Replacing (B.9) in (B.4) gives

$$T_{\mu\nu} = -\frac{1}{4\pi}\left(F_{\mu}^{\beta}F_{\nu\beta} - \frac{1}{4}F_{\rho\sigma}F^{\rho\sigma}g_{\mu\nu}\right). \tag{B.10}$$

This expression obtained above is the energy-momentum tensor given by an electromagnetic field source.

Appendix C

The Energy-Momentum tensor for the Non-linear Electrodynamics

To calculate the energy-momentum tensor for Non-linear Electrodynamics coupled to General Relativity it is necessary to start with its action defined as

$$S = \int d^4x \sqrt{-g} \left(\frac{1}{16\pi} R - \frac{1}{4\pi} L(F) \right), \tag{C.1}$$

where $L(F)$ is some function of the electromagnetic scalar F, the second therm corresponds to the matter sector S_M, applying the definition of the energy-momentum tensor (2.8) results

$$T_{\mu\nu} = -\frac{2}{\sqrt{-g}} \frac{\delta}{\delta g^{\mu\nu}} \left(-\frac{\sqrt{-g}}{4\pi} L(F) \right). \tag{C.2}$$

Applying the variations to the expression inside the parenthesis gives

$$T_{\mu\nu} = \frac{2}{\sqrt{-g}} \frac{\delta}{\delta g^{\mu\nu}} \left[L(F) \left(\frac{\delta}{\delta g^{\mu\nu}} \sqrt{-g} \right) + \sqrt{-g} \left(\frac{\delta}{\delta g^{\mu\nu}} L(F) \right) \right]. \tag{C.3}$$

Using the result (A.11) in the first term in the above expression leads to

$$T_{\mu\nu} = -\frac{1}{4\pi} L(F) g_{\mu\nu} + \frac{1}{2\pi} \frac{\delta}{\delta g^{\mu\nu}} L(F), \tag{C.4}$$

applying the chain rule in the second term, it is necessary to evaluate the variation

$$\delta F = \delta \left(\frac{1}{4} F_{\mu\nu} F^{\mu\nu} \right) = \delta \left(\frac{1}{4} F_{\mu\nu} F_{\alpha\beta} g^{\mu\alpha} g^{\nu\beta} \right). \tag{C.5}$$

Comparing the result (B.8) with (C.5) the variation lead

$$\delta F = \frac{1}{2} F_{\nu}^{\beta} F_{\nu\beta} \delta g^{\mu\nu}, \tag{C.6}$$

from where

$$\frac{\delta}{\delta g^{\mu\nu}} L(F) = \frac{dL}{dF} \frac{\delta}{\delta g^{\mu\nu}} F. \tag{C.7}$$

With (C.6) and (C.7) is possible to rewrite (C.4) as

$$T_{\mu\nu} = \frac{1}{4\pi} \left(\frac{dL}{dF} F_{\nu}^{\beta} F_{\nu\beta} - L(F) g_{\mu\nu} \right). \tag{C.8}$$

The equations of motion can be calculated with the conservation property

$$\nabla T_{\mu\nu} = 0. \tag{C.9}$$

Replacing (C.8) results

$$\frac{1}{4\pi} \left[\frac{dL}{dF} F_{\mu\beta} \left(\nabla^{\mu} F_{\nu}^{\beta} \right) + F_{\nu}^{\beta} \left(\nabla^{\mu} \frac{dL}{dF} F_{\mu\beta} \right) + \frac{dL}{dF} g_{\mu\nu} \left(\nabla^{\mu} F \right) \right] = 0, \tag{C.10}$$

from where

$$F_{\nu}^{\beta} \left(\nabla^{\mu} \frac{dL}{dF} F_{\mu\beta} \right) = 0. \tag{C.11}$$

The electromagnetic tensor defined as

$$F_{\mu\nu} = E(r) \left(\delta_{\mu}^{1} \delta_{\beta}^{0} - \delta_{\beta}^{1} \delta_{\mu}^{0} \right), \tag{C.12}$$

with (C.11) and (C.12) results

$$E(r) \frac{dL}{dF} = -\frac{q^2}{4\pi r^2}. \tag{C.13}$$

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