UNIVERSIDAD DE INVESTIGACIÓN DE TECNOLOGÍA EXPERIMENTAL YACHAY

Escuela de Ciencias Matemáticas y Computacionales

TÍTULO: Semilinear Neutral Differential Equations with non-instantaneous impulses, non-local conditions and infinite delay: Existence of solutions and Controllability

Trabajo de integración curricular presentado como requisito para la obtención
del título de Matemático

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Dedication

"To my family."

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Resumen

En esta tesis se estudia la existencia de soluciones y la controlabilidad de un sistema semilineal de ecuaciones diferenciales de tipo neutral con impulsos no instantáneos, condiciones no locales y retardo infinito. Primero, fijamos nuestro problema en un espacio de fase que satisface la teoría axiomática de Hale-Kato para ecuaciones diferenciales con retardo infinito. Luego, asumimos que las funciones no lineales de nuestro sistema son localmente Lipschitz y aplicamos el teorema de punto fijo de Karakostas para obtener la existencia de soluciones. Adicionalmente, bajo nuevas condiciones, probamos la unicidad. Posteriormente, asumiendo que los términos no lineales son globalmente Lipschitz, consideramos un sistema más simple en el cual aplicamos el teorema contractivo de Banach para demostrar la existencia de soluciones. Finalmente, estudiamos la controlabilidad de nuestro sistema. Por un lado, investigamos la controlabilidad aproximada aplicando la técnica desarrollada por Bashirov y Ghahramanlou, la cual no usa teoremas de punto fijo. Por otro lado, demostramos la controlabilidad exacta del mismo sistema. Para ello, transformamos el problema de controlabilidad en un problema de punto fijo. Entonces, bajo ciertas condiciones sobre las funciones no lineales de nuestro sistema, usamos el teorema de punto fijo de Rothe para obtener el resultado deseado.
Palabras Clave: ecuaciones diferenciales neutrales, impulsos no instantáneos, condiciones no locales, retardo infinito, teorema de punto fijo de Karakostas, teorema de punto fijo de Rothe , controlabilidad.

Abstract

In this thesis, we study the existence of solutions and controllability for retarded semilinear neutral differential equations with non-instantaneous impulses, non-local conditions, and infinite delay. First, we set the problem in a phase space satisfying the Hale-Kato axiomatic theory for retarded differential equations with infinite delay. Second, we assume that the nonlinear functions are locally Lipschitz, and Karakostas's fixed point theorem is applied to obtain the existence of solutions. Additionally, under some additional conditions, the uniqueness is proved as well. Next, assuming that the nonlinear terms are globally Lipschitz, we consider a more simplified system that allows us to apply the Banach contraction theorem to prove the existence of solutions. Subsequently, we study the associated control problem. On the one hand, we investigate the approximate controllability by using the technique employed by Bashirov and Ghahramanlou, which avoids the use of fixed point theorems. On the other hand, we prove the exact controllability of the same system. To this end, we transform the controllability problem into a fixed point problem. Then, under some conditions on the nonlinear terms, we use Rothe's fixed point theorem to obtain the desired result.
Keywords: neutral differential equations, non-instantaneous impulses, non-local conditions, infinite delay, Karakostas's fixed point theorem, Rothe's fixed point theorem, controllability.

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Chapter 1

Introduction

1.1 Background

Whenever one desires to examine a problem arising in the real world mathematically, the first step is selecting a mathematical model that best represents the problem. Usually, the mathematical models chosen are differential equations, i.e., equations concerning the derivative of some unknown function. For instance, elementary examples have the form $v'(t) = f(t, v(t))$, where v is the unknown function and f is a given continuous function.

Real-world problems or systems in physics, engineering, biology, ecology, and economics are often represented by simple differential equations. However, there are intrinsic phenomena governing the behavior of the problem that must be considered. These phenomena could vary and depend on the problem to be modeled. It is of recent interest to study differential equations with non-instantaneous impulses, non-local conditions, and infinite delay.

Non-instantaneous impulsive differential equations are characterized by abrupt changes occurring at some points that remain active over a finite time interval. Hernández and O'Regan [57] introduced this new class of differential equations motivated for the study of the hemodynamical equilibrium of a person. For example, in the case of decompensation, the injection of drugs in the bloodstream and their consequent absorption in the body are gradual and continuous processes. One can interpret this situation as a non-instantaneous impulsive action. Some recent results on non-instantaneous impulsive differential equations have been reported in [8, 40, 41, 76, 90, 91, 97, 100] and in monographs [1, 101].

To determine a particular solution of a system, some additional data is needed. This often takes the form of an initial condition, say $v(t_0) = v_0$. The previous formulation is referred to as an initial value problem or a Cauchy problem. Byszewski [21] introduced the study of the non-local Cauchy problem as a generalization of the classic initial value problem. The advantage of using non-local conditions is that measurements at more places are considered to get more realistic models. For a more detailed description of non-local conditions and applications in physics, see [22, 23, 99] and references therein.

It is well known that the future state of realistic models in the natural sciences, economics, and engineering depends not only on the present but on the past state and the derivative of the past state. Such models that contain past information are called delay differential equations (DDEs). There are simple examples in control theory, physics, biology, ecology, economics, and inventory control (see, e.g., [38] and [28]). Differential equations with infinite (or unbounded) delay are a particular class of DDEs with important applications in mechanics, physics, and engineering. The literature related to these equations is extensive, and we refer the reader to [59, 67] and the references therein.

Once the model representing the real problem is selected, the next step is to study the existence and uniqueness of solutions for the differential equations governing the system. Additionally, qualitative properties such as controllability, observability, and invertibility could be addressed. In this work, we are interested in studying the existence of solutions and controllability of semilinear neutral differential equations with non-instantaneous impulses, non-local conditions, and infinite delay.

Neutral differential equations naturally arise in various applications, such as control systems, mechanics, distributed networks, neural networks, the interaction of species, epidemiology, and many others [42]. In addition, neutral equations with infinite delay appear in the description of heat conduction in materials with fading memory developed by Gurtin & Pipkin [44] and Nunziato [88]. The theory of Neutral differential equations has become an independent trend, and the literature on this subject is extensive. We shall mention the survey on the theory of neutral equations by Akhmerov et al. [2], where a classification is made and a statement of the main problems is given, as well as the books by Chukwu [28], Bainov & Mishev [10] and Hale & Lunel [50].

There are some works on semilinear neutral equations with infinite delay [53], with infinite delay and instantaneous impulses [55, 54], with instantaneous impulses and finite delay [3, 81], with infinite delay and non-local conditions [52], with infinite delay, instantaneous impulses and non-local conditions [7]. However, to our knowledge, there are no studies considering the three phenomena simultaneously: non-instantaneous impulses, non-local conditions, and infinite delay. This fact and the several applications of neutral differential equations are the main motivations for this work. The results presented in this note can be thought as an extension of the results obtained by Lalvay *et al.* [68] and Riera-Segura [92].

1.2 Problem statement

In this manuscript, we are concerned with the existence of solutions and controllability of the following first-order non-autonomous semilinear differential equations of neutral type,

$$\begin{aligned} \frac{d}{dt}[v(t)-g(t,v_t)] &= \mathbf{A}(t)v(t)+\mathfrak{F}(t,v_t), \quad t\in J_k^1, k=0,1,\dots,\\ v(t) &= \Gamma_k(t,v(t_k^-)), \quad t\in J_k^2, k=1,\dots,\\ v(s)+\zeta(v_{\lambda_1},v_{\lambda_2},\dots,v_{\lambda_q})(s) &= \phi(s), \quad s\in\mathbb{R}_-=(-\infty,0], \end{aligned} \tag{1.1}$$

where the function $v(\cdot)$ takes values in $\mathbb{R}^n$, $s_0=0, s_{k-1}<t_k<s_k<t_{k+1}\to\infty$, as $k\to\infty$, $J_0^1=[0,t_1], J_k^1=(s_k,t_{k+1}], J_k^2=(t_k,s_k]$. Letting $T>0$, the interval $(0,T]$ is the maximal interval of local existence of solutions to (1.1), furthermore, there is $\xi>0$ fixed such that $\lambda_q\le\min\{\xi,T\}$. Here $0\le\lambda_1<\lambda_2<\cdots<\lambda_q$ are prefixed numbers selected conveniently according to the phenomenon to be modelled. The matrix $\mathbf{A}(t)$ is continuous of order $n\times n$. The function $v_t:(-\infty,0]\longrightarrow\mathbb{R}^n$ given by $v_t(\theta)=v(t+\theta),\theta\le 0$, represents the history of v up to t and belongs to the axiomatically defined phase space $\mathscr{C}_m$ to be specified later. The functions $\mathfrak{F},g:[0,T]\times\mathscr{C}_m\longrightarrow\mathbb{R}^n$, $\phi\in\mathscr{C}_m$, $v_t\in\mathscr{C}_m$, $\Gamma_k:(t_k,s_k]\times\mathbb{R}^n\to\mathbb{R}^n$

and $\zeta : \mathscr{C}_m^q \to \mathscr{C}_m$ are appropriate functions. In particular, $\Gamma_k(t, v(t_k^-)), k = 1, ...,$ describes the non-instantaneous impulses in the model and the function ζ denotes the non-local conditions.

For each u fixed, we let $\mathfrak{F}(t, v_t) = \mathbf{B}(t)u(t) + f(t, v_t, u(t))$. Then, the control problem associated to (1.1) is given by

$$\begin{aligned} \frac{d}{dt}[v(t) - g(t, v_t)] &= \mathbf{A}(t)v(t) + \mathbf{B}(t)u(t) + f(t, v_t, u(t)), \quad t \in \bigcup_{k=0}^{N} J_k^1, \\ v(t) &= \Gamma_k(t, v(t_k^-)), \quad t \in J_k^2, k = 1, \ldots, N, \\ v(s) + \zeta(v_{\lambda_1}, \ldots, v_{\lambda_q})(s) &= \phi(s), \quad s \in (-\infty, 0], \end{aligned} \tag{1.2}$$

where $s_0 = 0 < t_1 < s_1 < t_2 < \cdots < t_N < s_N < t_{N+1} = T$, $J_0^1 = [0, t_1], J_k^1 = (s_k, t_{k+1}]$ and $J_k^2 = (t_k, s_k]$ for $k = 1, ..., N$. The control $u(\cdot)$ belongs to a space of admissible control functions. The matrices $\mathbf{A}(t)$ and $\mathbf{B}(t)$ are continuous of order $n \times n$ and $n \times m$, respectively. The function $f : [0, T] \times \mathscr{C}_m \times \mathbb{R}^m \to \mathbb{R}^n$ is smooth enough and the remaining terms are the same as in equation (1.1).

The rest of this note is organized in the following manner:

- **Chapter 2:** This chapter provides the theoretical framework and the mathematical tools needed to prove our results. In Section 2.1, we review the basic concepts of metric, normed, Banach, and Hilbert spaces as well as some results on Operators defined on normed spaces. Additionally, we give a characterization of dense range operators in Hilbert spaces, which is useful to prove the controllability of the linear system of differential equations. At the end of this section, we present the fixed point theorems to be used in this work. Section 2.2 is devoted to the basic material on differential equations. Here, we develop a systematic description of differential equations with non-instantaneous impulses, non-local conditions, and infinite delay. Also, we introduce neutral differential equations and the current research on this area. In Section 2.3 , a brief review on control theory is given. Furthermore, a characterization of the controllability of the linear system is provided.

- **Chapter 3:** In this chapter, we present our main results. In section 3.1, we apply Karakostas's fixed point theorem to prove the existence and uniqueness of solutions to problem (1.1). Also, we provide an alternative proof of the existence of solutions employing the Banach contraction theorem. In section 3.2, the approximate controllability of system (1.2) is proved by applying the technique developed by Bashirov and Ghahramanlou. Finally, under some additional conditions, the exact controllability of the same system is proved by using Rothe's fixed point theorem.

- **Chapter 4:** In this chapter, we present our conclusions and final remarks.

Chapter 2

Theoretical Framework

In this chapter we present elementary concepts and results of Functional Analysis, Differential Equations and Control Theory to be used in the next chapters.

2.1 Preliminary results of Functional Analysis

This section is essentially devoted to introduce some notations, definitions, and preliminary facts of Functional Analysis and Operator Theory that are used throughout the next chapters.

2.1.1 Metric spaces

Definition 1 (Metric)**.** *Let X be a non-void set. A metric on X is a real-valued function d defined on $X \times X$ such that for all $x, y, z \in X$ we have:*

(i) $d(x, y) \geq 0$;

(ii) (Symmetry) $d(x, y) = d(y, x)$;

(iii) $d(x, y) = 0$ *iff* $x = y$;

(iv) (Triangle Inequality) $d(x, y) \leq d(z, x) + d(z, y)$.

Properties (i) and (ii) are derived from (iii) and (iv). For any three points $x, y, z \in X$ one has the following useful inequality:

$$|d(x, y) - d(x, z)| \leq d(y, z).$$

The pair (X, d) is referred as a *metric space*. When no confusion can arise, we will denote the metric space (X, d) by X.

Remark 1. *Any subset A of a metric space X is a metric subspace when it is equipped with the restriction of d to $A \times A$.*

In a metric space X, given a point $a \in X$ and radius $r > 0$, we shall write

$$B(a, r) = \{y \in X | d(y, a) < r\},$$
$$\overline{B}(a, r) = \{y \in X | d(y, a) \leq r\},$$
$$S_r(a, r) = \{y \in X | d(y, a) = r\},$$

referred to as *open ball, closed ball* and *sphere* of center a and radius r, respectively. Observe that a set $A \subseteq X$ is bounded if and only if A is contained in some ball.

A set $O \subseteq X$ is said to be d-open iff

$$\forall x \in O, \exists r > 0 : \quad B(x, r) \subseteq O.$$

We denote by $\mathscr{T}_d$ the set of d-open sets of X. Clearly, any open ball is a d-open set (see, e.g., [31, Ch. 3]). Note that the last definition of open sets makes sense in the context of metric spaces. A more general notion of open sets is considered in topology as follows:

Definition 2. *Let $Y \neq \emptyset$ and $\mathscr{T}$ a family of subsets of Y. We say that $\mathscr{T}$ is a topology on Y iff*

(*i*) *The void set $\emptyset$ and Y belong to $\mathscr{T}$;*

(*ii*) *If $U_1, U_2 \in \mathscr{T}$, then $U_1 \cap U_2 \in \mathscr{T}$;*

(*iii*) *If $(U_\alpha)_{\alpha \in I}$ is a family of elements of $\mathscr{T}$, then $\bigcup_{\alpha \in I} U_\alpha \in \mathscr{T}$.*

The pair $(Y, \mathscr{T})$ is called a topological space. The elements of $\mathscr{T}$ are called open sets and their complements, closed sets.

The set $\mathscr{T}_d$ is a topology on X and is referred to as the topology induced by the metric d. A proof of this result can be found in [18, Th. 2.1.3].

In the context of Definition 2, we say that a set $\mathscr{E} \subset \mathscr{T}$ is a basis of $\mathscr{T}$ if every set in $\mathscr{T}$ is the union of a family of sets belonging to $\mathscr{E}$. It is not difficult to show that

$$\mathscr{E}_d = \{B(x, r) | x \in X, r > 0\}$$

is a basis of $\mathscr{T}_d$. This result leads to characterize the topological concepts of interior point and adherent point in metric spaces as follows (see [39, Ch. 3, Ch. 13]):

Definition 3 (Interior point, Adherent point)**.** *Let X be a metric space, $A \subset X$ and $x \in X$. Then*

(*i*) *x is an interior point of A iff $\exists r > 0 : \quad B(x, r) \subset A$.*

(*ii*) *x is an adherent point of A iff $\forall r > 0 : \quad B(x, r) \cap A \neq \emptyset$.*

The set of all interior points of A, denoted by $\text{int}(A)$, is an open set in $\mathscr{T}_d$. The set of all adherent points of A, denoted by $\overline{A}$(Closure of A), is a closed set in $\mathscr{T}_d$.

The introduction of metric spaces allows us to generalize the concept of convergent sequences in a fashion analogous to real numbers.

Definition 4. *A sequence of points $(x_n)_{n \in \mathbb{N}}$ in a metric space X is said to converge to a point $x_0 \in X$ iff*

$$\forall \epsilon > 0, \exists N(\epsilon) \in \mathbb{N} : \quad d(x_n, x_0) < \epsilon \text{ whenever } n \geq N. \tag{2.1}$$

The point x_0 is called the limit of the sequence. This is sometimes written $\lim_{n \to \infty} x_n = x_0$ or $x_n \to x_0$ as $n \to \infty$. In metric spaces, the limit of a convergent sequence is unique. Indeed, if $(x_n)_{n \in \mathbb{N}}$ converges to x_0 and y_0, we should have

$$0 \leq d(x_0, y_0) \leq d(x_n, x_0) + d(x_n, y_0) \to 0 \text{ as } n \to \infty,$$

which from *(iii)* implies that $x_0 = y_0$. If $(k_n)_{n\in\mathbb{N}}$ is a strictly increasing sequence of positive integers, the sequence $(x_{k_n})_{n\in\mathbb{N}}$ is called a **subsequence** of $(x_n)_{n\in\mathbb{N}}$.

The next Theorem is useful to prove that a subset of a metric space is closed.

Theorem 1. *Let X be a metric space, $A \subseteq X$ and $a \in X$. Then*

(i) $a \in \overline{A}$ iff there exists a sequence $(x_n)_{n\in\mathbb{N}} \subseteq A$ that converges to a.

(ii) A is closed iff the sequence $(x_n)_{n\in\mathbb{N}} \subseteq A$, $x_n \to a$ implies that $a \in A$.

A proof of this theorem can be found in [39, Ch. 14].

Definition 5 (Cauchy sequence)**.** *A sequence of points $(x_n)_{n\in\mathbb{N}}$ in a metric space X is said to be a Cauchy sequence iff*

$$\forall \epsilon > 0, \exists N(\epsilon) \in \mathbb{N} : \quad d(x_n, x_m) < \epsilon \text{ whenever } n, m \geq N. \tag{2.2}$$

Clearly, every convergent sequence is a Cauchy sequence; however, the converse is not valid. A metric space X is said to be **complete** if every Cauchy sequence of points of X converges to a point of X.

We now introduce the concept of *compactness* in metric spaces. There are at least four equivalent ways of defining compactness in metric spaces. The definition we choose is based on the notion of sequential compactness.

Definition 6. *A metric space X is said to be sequentially compact iff for every sequence $(x_n)_{n\in\mathbb{N}} \subset X$ there exists a subsequence that converges to some $a \in X$.*

Definition 7. *Let A be a set in a metric space X. A collection of open sets $\{M_\alpha\}$ in X is said to be an open covering of A iff $A \subset \bigcup_\alpha M_\alpha$.*

Definition 8. *A metric space X is said to have the Heine-Borel property or to be compact if from every open covering of the space it is possible to select a finite open covering.*

Theorem 2. *Let X be a metric space. Then the following statements are equivalent:*

(i) X is compact.

(ii) X is sequentially compact.

We consider a compact metric space in Theorem 2-*(i)* as one which has the Heine-Borel property. The reason for this is that Heine-Borel compactness can easily be generalized to topological spaces that are not metrizable[1]. For a better understanding of all the equivalent definitions of compactness and the proof of Theorem 2, we refer the reader to [31, Ch. 6].

Remark 2.

(i) If a subset A of a metric space is such that $\overline{A}$ is compact, we say that A is relatively compact.

(ii) In finite dimension, a metric space X is compact iff it is closed and bounded.

[1] A topological space $(X, \mathscr{T})$ is metrizable iff there exist a metric d on X such that $\mathscr{T} = \mathscr{T}_d$

2.1.2 Normed and Inner product spaces

Normed linear spaces can be thought of as a generalization of the n-dimensional vector space $\mathbb{R}^n$ together with its length function.

Definition 9 (Norm)**.** *Let V be a* (real) *linear space. A norm on V is a function $\|\cdot\|_V : V \longrightarrow \mathbb{R}$ satisfying the following conditions:*

(*i*) $\forall x \in V : \|x\|_V = 0$ *iff* $x = 0$*;*

(*ii*) $\forall x \in V, \forall \lambda \in \mathbb{R} : \|\lambda x\|_V = |\lambda| \|x\|_V$*;*

(*iii*) (Triangle inequality) $\forall x, y \in V : \|x + y\|_V \leq \|x\|_V + \|y\|_V$.

The couple $(V, \|\cdot\|_V)$ is called a normed space.

When there is no confusion, we shall say the normed space V instead of $(V, \|\cdot\|_V)$ and the norm $\|\cdot\|$ instead of $\|\cdot\|_V$. The function $d : V \times V \longrightarrow \mathbb{R}$ defined by

$$d(x, y) = \|x - y\|, \quad x, y \in V, \tag{2.3}$$

is a metric on V. As a result, every normed space becomes a metric space and, consequently, a topological space. Note that a normed space V is complete if the corresponding metric space (V, d), where d is the metric defined in (2.3), is complete. In this case, we say that V is a **Banach space**.

Once we generalize the length of a vector in normed spaces, it is easy to generalize the concept of continuity from calculus.

Definition 10. *Let V, W be normed spaces. A function $f : V \longrightarrow W$ is said to be:*

(*i*) *Continuous at the point $x_0 \in V$ iff*

$$\forall \epsilon > 0, \exists \delta = \delta(x_0, \epsilon) : \quad x \in V \text{ and } \|x - x_0\|_V < \delta \implies \|f(x) - f(x_0)\|_W < \epsilon,$$

f is continuous on V if it is continuous at every point in V.

(*ii*) *Uniformly continuous iff*

$$\forall \epsilon > 0, \exists \delta = \delta(\epsilon) : \quad x, y \in V \text{ and } \|x - y\|_V < \delta \implies \|f(x) - f(y)\|_W < \epsilon.$$

Another important notion of continuity used throughout this work is the *Lipschitz continuity.*

Definition 11. *Let V and W be normed spaces and $T : V \longrightarrow W$. We say that T is Lipschitz continuous iff*

$$\exists \kappa \geq 0, \forall x, y \in V : \quad \|Ty - Tx\|_W \leq \kappa \|y - x\|_V.$$

If $\kappa < 1$, T is called a contraction.

A Banach space generalizes the notion of $\mathbb{R}^n$ as a linear space with a length function, but in order to generalize the useful geometry property of orthogonality, we need some extra structure.

Definition 12 (Inner product)**.** *Let V be a* (real) *linear space. An inner product on V is a function $\langle \cdot, \cdot \rangle : V \times V \longrightarrow \mathbb{R}$ satisfying the following conditions:*

(*i*) $\forall x, y, z \in V, \forall \lambda \in \mathbb{R} : \quad \langle \lambda x + y, z \rangle = \lambda \langle x, z \rangle + \langle y, z \rangle;$

(*ii*) (Symmetry) $\forall x, y \in V : \quad \langle x, y \rangle = \langle y, x \rangle;$

(*iii*) (Non-negativity) $\forall x \in V : \quad \langle x, x \rangle \geq 0;$

(*iv*) $\forall x \in V : \quad \langle x, x \rangle = 0$ *iff* $x = 0$.

The pair $(V, \langle \cdot, \cdot \rangle)$ is called an inner product space.

Lemma 1 (Cauchy-Bunyakovsky-Schwarz (CBS) inequality)**.** *Let V be an inner product space. Then*

$$\forall x, y \in V : \quad |\langle x, y \rangle| \leq \sqrt{\langle x, x \rangle} \cdot \sqrt{\langle y, y \rangle}.$$

For the proof of this Lemma, one can see [65, Ch. 3, Lem. 3.2-1]. By using CBS inequality, we can prove that every inner product space induces the norm:

$$\|x\| = \sqrt{\langle x, x \rangle}.$$

In this setting, we shall say that an inner product space is a **Hilbert Space** if it is a Banach space with respect to the norm induced by the inner product.

2.1.3 Operators on Normed spaces

Let $(E, \|\cdot\|_E)$ and $(F, \|\cdot\|_F)$ be normed spaces. We say that $T : E \longrightarrow F$ is a *linear operator* iff

$$\forall x, y \in E, \forall \alpha \in \mathbb{R} : \quad T(\alpha u + v) = \alpha Tx + Ty.$$

The *range* of a linear operator $T : E \longrightarrow F$ is the set

$$\text{Ran}(T) = \{y \in F | y = Tx \text{ for some } x \in E\},$$

and the *null space* or *kernel* of T is the set

$$\text{Ker}(T) = \{x \in E | Tx = 0\}.$$

A linear operator $T : E \longrightarrow F$ is further called *bounded* iff

$$\exists c > 0, \forall x \in E : \quad \|Tx\|_F \leq c\|x\|_E. \tag{2.4}$$

We denote by $\mathcal{L}(E, F)$[2] the space of all bounded linear operators from E to F. If $F = \mathbb{R}$, then T is called a bounded linear functional and $\mathcal{L}(E, \mathbb{R})$ is denoted by $E^\star$(Dual space of E). The term “bounded operator” is motivated by the following proposition.

Proposition 1. *Let E, F be normed spaces. A linear operator $T : E \longrightarrow F$ is bounded if and only if T maps bounded sets into bounded sets*

[2]When $F = E$, we write $\mathcal{L}(E, E) = \mathcal{L}(E)$.

Proof. Assume that T is bounded. Then, there exists $c > 0$ such that $\|Tx\| \leq c\|x\|$ for all $x \in E$. If $\|x\| \leq k$, for some constant k, then $\|Tx\| \leq c\|x\| \leq kc$. That is, T maps a bounded set into a bounded set. Conversely, assume that T maps bounded sets into bounded sets. Then T maps the unit closed ball $\overline{B}(0,1) = \{x \in E : \|x\| \leq 1\}$ into a bounded set. That is, there exists a constant $c > 0$ such that $\|Tx\| \leq c$ for all $x \in \overline{B}(0,1)$. Therefore, for any nonzero $x \in E$,

$$\frac{\|Tx\|}{\|x\|} = \left\| T\left(\frac{x}{\|x\|}\right)\right\| \leq c.$$

Hence, $\|Tx\| \leq c\|x\|$. This concludes the proof. □

It leads naturally to define the *operator norm*:

$$\|T\|_{\mathcal{L}(E,F)}{}^{3} := \inf\left\{c > 0 \,\middle|\, \forall x \in E : \|Tx\|_F \leq c\|x\|_E\right\} = \sup_{x\neq 0} \frac{\|Tx\|_F}{\|x\|_E} = \sup_{\|x\|=1} \|Tx\|_F.$$

It is not difficult to show that $\|T\|_{\mathcal{L}(E,F)}$ is a norm on $\mathcal{L}(V,E)$, see for instance [65, Lem. 2.7-2]. Furthermore, we have that

$$\forall x \in E : \quad \|Tx\|_F \leq \|T\|\|x\|_E.$$

Theorem 3. *Let E and F be normed spaces. A linear operator $T : E \longrightarrow F$ is continuous if and only if it is bounded.*

Proof. Assume that T is continuous. Then T is continuous at 0, that is,

$$\forall \epsilon > 0, \exists \delta > 0, \forall x \neq 0 \in E : \quad \|x\| < \delta \implies \|Tx\| < \epsilon. \tag{2.5}$$

Take $x = \delta y/2\|y\|$. Hence $\|x\| = \delta/2 < \delta$, so that from (2.5), we have that

$$\|Tx\| = \left\| T\left(\frac{\delta}{2\|y\|}y\right)\right\| = \frac{\delta}{2\|y\|}\|Ty\| < \epsilon.$$

Setting $c = 2\epsilon/\delta$; since x was arbitrary, last shows that T is bounded. Conversely, assume that T is bounded. Let $x_0 \in E$ and let $\epsilon > 0$. Since T is bounded and linear, we have that

$$\|Tx_0 - Tx\| \leq \|T\|\|x_0 - x\|, \quad \forall x \in E.$$

By setting $\delta = \epsilon/\|T\|$ and assuming that $\|x_0 - x\| < \delta$ we obtain that

$$\|Tx_0 - Tx\| < \|T\|\delta < \epsilon.$$

Since x_0 was arbitrary, T is continuous. □

The following Theorem summarizes some important properties of the space $\mathcal{L}(E,F)$.

Theorem 4. *Let E, F and V be normed spaces. Then*

[3] Usually, we will write $\|T\|$ instead of $\|T\|_{\mathcal{L}(E,F)}$.

(i) *If F is a Banach space, then $\mathcal{L}(E,F)$ is a Banach space.*

(ii) *If $T \in \mathcal{L}(E,F)$ and $S \in \mathcal{L}(F,V)$, then the composition $S \circ T = ST$ is in $\mathcal{L}(E,V)$ and $\|ST\| \leq \|S\|\|T\|$.*

(iii) *For the case that $E = F$, $\mathcal{L}(E)$ is an algebra; that is, αT_1, $T_1 + T_2$ and $T_1 T_2$ are in $\mathcal{L}(E)$ for every T_1, T_2 in $\mathcal{L}(E)$.*

For a proof of this result, see, e.g., [65, Ch. 2].

There are three very important theorems in functional analysis about bounded linear operators on Banach spaces: the *Uniform Boundedness Principle*, the *Open Mapping Theorem* and the *Closed Graph Theorem*. Here, it is of our interest the open mapping theorem.

Theorem 5 (Open Mapping Theorem)**.** *Let E and F be two Banach spaces and let $T \in \mathcal{L}(E,F)$ surjective. Then there exists $c > 0$ such that*

$$T(B(0,1)) \supset B(0,c). \tag{2.6}$$

For the proof of Theorem 5, we refer to [17, Ch. 2, Th. 2.6]. Note that relation in (2.6) implies that T maps open sets into open sets. One of the most important consequences of Theorem 5 is the following Corollary.

Corollary 1. *Let E and F be two Banach spaces and let $T \in \mathcal{L}(E,F)$ bijective. Then $T^{-1} \in \mathcal{L}(F,E)$.*

A proof of Corollary 1 can be found in [17, Ch. 5, Cor. 2.7].

Finally, we define a special subclass of bounded linear operators with valuable properties.

Definition 13. *Let E and F be normed spaces. A linear operator $T : E \longrightarrow F$ is said to be compact iff*

$$\forall A \subseteq E, \text{ bounded: } \quad T(A) \subseteq F \text{ is relatively compact.}$$

Equivalently, T is compact iff

$$\forall (x_n)_{N\in\mathbb{N}} \subseteq E \text{ bounded: } \quad (Tx_n)_{n\in\mathbb{N}} \text{ has a convergent subsequence.}$$

2.1.4 Properties of Hilbert spaces.

In this part, we exploit the structure of the inner product to derive important properties of Hilbert spaces. First, we give a definition of convex sets.

Definition 14 (Convex set)**.** *A subset M of a linear space V is said to be convex iff*

$$\forall x, y \in M, \forall \lambda \in [0,1] : \quad \lambda x + (1-\lambda)y \in M.$$

The next Theorem has theoretical and practical applications in minimization problems.

Theorem 6 (Projection on a closed convex set)**.** *Let H be a Hilbert space and let $K \subset H$ be a nonempty closed convex set. Then for every $f \in H$ there exists a unique $u \in K$ such that*

$$\|f - u\| = \min_{v \in K} \|f - v\|.$$

A proof of this Theorem is in [17, Ch.5, Th.5.2]. The element $u = P_K f$ above is called the projection of f onto K.

Corollary 2. *Let H be a Hilbert space and M be a closed linear subspace of H. Let $f \in H$. Then $u = P_M f$ is characterized by*

$$u \in M \text{ and } \langle f - u, v \rangle = 0, \quad \forall v \in M.$$

A proof of Corollary 2 can be found in [17, Ch.5, Cor.5.4].

One of the most important properties of a Hilbert space is that there is a particularly simple representation of its dual space.

Theorem 7. (Riesz-Fréchet representation theorem). *Let H be a Hilbert space and $\varphi \in H^\star$. Then there exists a unique $f_\varphi \in H$ such that*

$$\forall u \in H: \quad \varphi(u) = \langle f_\varphi, u \rangle, \text{ and } \|\varphi\|_{H^\star} = \left\|f_\varphi\right\|_H.$$

For the proof of this theorem, one can see [17, Ch.5, Th.5.5]. Thanks to Theorem 7, we can prove that H and $H^\star$ are isomorphic.

Now, let H_1, H_2 be Hilbert spaces, $T \in \mathcal{L}(H_1, H_2)$ and $y \in H_2$. We consider $\varphi_y(x) = \langle Tx, y \rangle_{H_2}$ as a linear functional on H_1. Furthermore, from the CBS inequality we have $|\varphi(x)| = |\langle Tx, y \rangle_{H_2}| \leq \|T\|\|y\|\|x\|$, that is, φ_y is bounded for fixed $y \in H_1$. Therefore by Theorem 7, there exists a unique $T^* y \in H_1$ such that

$$\langle Tx, y \rangle_{H_2} = \langle x, T^* y \rangle_{H_1}, \quad \forall x \in H_1. \tag{2.7}$$

Thus, given a $y \in H_2$, there is a unique $T^* y$ associated with it.

Theorem 8. *Let $T \in \mathcal{L}(H_1, H_2)$, where H_1 and H_2 are Hilbert spaces. Then there exists a unique operator $T^* \in \mathcal{L}(H_2, H_1)$ called the adjoint of T that satisfies*

$$\langle Tx, y \rangle_{H_2} = \langle x, T^* y \rangle_{H_1}, \quad \forall x \in H_1, \forall y \in H_2, \text{ and } \|T\| = \|T^*\|.$$

The existence of T^* comes by relation (2.7). The rest of the proof can be found in [65, Ch.3, Th.3.9-2]. The following Theorem states the general properties of Hilbert adjoint operators.

Theorem 9. *Let H_1, H_2 be Hilbert spaces, $S \in \mathcal{L}(H_1, H_2), T \in \mathcal{L}(H_1, H_2)$, and $\alpha \in \mathbb{R}$. Then*

(i) $(S + T)^* = S^* + T^*$;

(ii) $(\alpha T)^* = \alpha T^*$;

(iii) $(T^*)^* = T$;

(*iv*) $(ST)^* = T^*S^*$ $(H_2 = H_1)$;

(*v*) $\|T^*T\| = \|TT^*\| = \|T\|^2$;

(*vi*) *If* T *has a bounded inverse* T^{-1}, *then* T^* *has a bounded inverse and* $(T^*)^{-1} = (T^{-1})^*$.

Proof. (*i*)-(*v*) are easily checked and can be found in [65, Ch.3, Th.3.9-4]. Suppose that T has a bounded inverse T^{-1}, then by (*iv*)

$$T^*(T^{-1})^* = (T^{-1}T)^* = I^* = (T^{-1})^*T^*.$$

Moreover, since T^{-1} is bounded, $(T^*)^{-1}$ is bounded. □

To finish this subsection, we present a useful Theorem that uses the adjoint T^* to characterize the range and kernel space of T and vice versa. Before that, we recall the concept of orthogonal complement. Let Z be an inner product space and $x, y \in Z$. We say that x is orthogonal to y and write $x \perp y$ iff $\langle x, y\rangle = 0$. If $M \subset Z$, we define $M^\perp$, the *orthogonal complement* of M, by

$$M^\perp = \{x \in Z : \quad \langle x, y\rangle = 0 \text{ for all } y \in M\}.$$

Theorem 10. *Let* H_1 *and* H_2 *be Hilbert spaces and* $T \in \mathcal{L}(H_1, H_2)$, *then*

(*i*) $\text{Ran}(T)^\perp = \text{Ker}(T^*)$;

(*ii*) $\overline{\text{Ran}(T)} = \text{Ker}(T^*)^\perp$;

(*iii*) $\text{Ran}(T^*)^\perp = \text{Ker}(T)$;

(*iv*) $\overline{\text{Ran}(T^*)} = \text{Ker}(T)^\perp$.

Proof. Since $(T^*)^* = T$, it will be suffice to prove (*i*) and (*ii*). Let $z \in \text{Ran}(T)^\perp$, then $\langle z, Tx\rangle = 0$ for all $x \in H_1$. But by Theorem 8, it follows

$$\langle T^*z, x\rangle = 0, \quad \forall x \in H_1.$$

So $T^*z = 0$, i.e., $z \in \text{Ker}(T^*)$. Similarly, $z \in \text{Ker}(T^*)$, implies that $\langle T^*z, x\rangle = 0$ for all $x \in H_1$. Again, by Theorem 8, we obtain that $z \in \text{Rang}(T)^\perp$. Now, $\text{Ran}(T)$ may not be closed, so it is not necessarily the case that $\text{Ran}(T)^{\perp\perp} = \text{Ran}(T)$; but, we do always have that $\text{Ran}(T)^{\perp\perp} = \overline{\text{Ran}(T)}$ (see [84, Ch. 5, Th. 5.15.4]). This completes the proof. □

2.1.5 Characterization of Dense Range Operators.

Let Y, Z be Hilbert spaces. This subsection will show an important characterization of bounded linear operators with dense range on Hilbert spaces. This result will play a substantial role in the controllability of linear differential equations.

Theorem 11. (Curtain & Pritchard [35]). *Let* $T \in \mathcal{L}(Y, Z)$. *Then*

(*i*) $\text{Ran}(T) = Z \iff \exists \gamma > 0 : \quad \|T^*z\|_Y \geq \gamma \|z\|_Z, \quad \forall z \in Z.$

(*ii*) $\overline{\text{Ran}(T)} = Z \iff \text{Ker}(T^*) = \{0\}$.

Proof.

(i) Assume that $\operatorname{Ran}(T) = Z$, we shall consider the cases:

(a) T is a one-to-one mapping, that is, T is a bijection. By Corollary 1, $T^{-1} \in \mathcal{L}(Z, Y)$. Moreover by Theorem 9-$(vi)$, $(T^{-1})^* = (T^*)^{-1} \in \mathcal{L}(Y, Z)$, and there exists $\beta > 0$, such that

$$\left\|(T^*)^{-1}y\right\|_Z \leq \beta\|y\|_Y, \quad \forall y \in Y.$$

Now, for any $z = (T^*)^{-1}y \in Z$, we let $y = T^*z$, then

$$\|T^*z\|_Y \geq \gamma\|z\|_Z, \quad \forall z \in Z, \gamma = \frac{1}{\beta}.$$

(b) For the general case we consider the closed linear subspace $W = \operatorname{Ker}(T)^\perp \subset Y$, which is a Hilbert space. Now, define the linear operator $\tilde{T} : W \longrightarrow Z$ by

$$\tilde{T}w = Tw.$$

Clearly, $\tilde{T}$ is one-to-one. Thus, $\tilde{T}$ is a bijection and there exists $\gamma > 0$ such that

$$\left\|\tilde{T}^*z\right\|_W \geq \gamma\|z\|_Z, \quad \forall z \in Z. \tag{2.8}$$

From Theorem 7, we have that

$$\begin{aligned}
\left\|\tilde{T}^*z\right\|_W &= \sup_{\substack{w\in W\\ \|w\|_Y\leq 1}} \left|\langle w, \tilde{T}^*z\rangle_Y\right| = \sup_{\substack{w\in W\\ \|w\|_Y\leq 1}} \left|\langle \tilde{T}w, z\rangle_Z\right| \\
&= \sup_{\substack{w\in Y\\ \|w\|_Y\leq 1}} \left|\langle Tw, z\rangle_Z\right| = \sup_{\substack{w\in Y\\ \|w\|_Y\leq 1}} \left|\langle w, T^*z\rangle_Y\right| \\
&= \|T^*z\|_Y.
\end{aligned} \tag{2.9}$$

Hence, from (2.8) and (2.9), the result follows.

Conversely, assume that there exists $\gamma > 0$ such that

$$\|T^*z\|_Y \geq \gamma\|z\|_Z, \quad \forall z \in Z,$$

then we have that

$$\langle T^*z, T^*z\rangle_Z \geq \gamma^2\|z\|^2, \quad \forall z \in Z,$$

$$\langle TT^*z, z\rangle_Z \geq \gamma^2\|z\|^2, \quad \forall z \in Z, \text{ (Theorem 7)} \tag{2.10}$$

$$\|TT^*z\|_Z \geq \gamma^2\|z\|_Z, \quad \forall z \in Z. \text{ (CBS inequality)} \tag{2.11}$$

Moreover $\operatorname{Ran}(TT^*) = Z$. In fact, we first see that $\operatorname{Ran}(TT^*)$ is closed (see Theorem 1-(ii)). Let $\tilde{z} \in \overline{\operatorname{Ran}(TT^*)}$, then there exists $(z_n)_{n\in\mathbb{N}} \subset Z$ such that $TT^*z_n \to \tilde{z}$ as $n \to \infty$. From (2.11), we get that

$$\|z_n - z_m\|_Z \leq \frac{1}{\gamma^2}\|TT^*z_n - TT^*z_m\|_Z.$$

Since $(TT^*z_n)_{n\in\mathbb{N}}$ is a Cauchy sequence, $\|z_n - z_m\| \to 0$ as $n, m \to \infty$. Hence $z_n \to z$ as $n \to \infty$. Last implies that $TT^*z_n \to TT^*z = \tilde{z}$ as $n \to \infty$, that is, $\tilde{z} \in \operatorname{Ran}(TT^*)$. Now, suppose that $\operatorname{Ran}(TT^*) \subsetneq Z$. By the Perpendicular Theorem, there exists $z_0 \in Z$, with $\|z_0\|_Z = 1$ such that

$$\langle TT^*z, z_0\rangle = 0, \quad \forall z \in Z.$$

In particular, $\langle TT^*z_0, z_0\rangle_Z = 0$, so by (2.10), $\|z_0\|^2 = 0$, wich is a contradiction. Hence $\operatorname{Ran}(TT^*) = Z$. Clearly $\operatorname{Ran}(TT^*) \subset \operatorname{Ran}(T)$; therefore $\operatorname{Ran}(T) = Z$.

(ii) The proof of this item is an immediate consequence of Theorem 10-(ii).

□

Corollary 3. *Let $T \in \mathcal{L}(Y, Z)$. Then, the following assertions are equivalent*

(i) $\operatorname{Ran}(T) = Z$;

(ii) $\exists (TT^*)^{-1} \in \mathcal{L}(Z)$;

(iii) $\exists \alpha > 0, \forall z \in Z \setminus \{0\} : \langle TT^*z, z\rangle \geq \alpha \|z\|^2$.

Proof. Let us show that (i) implies (ii). Assume that $\operatorname{Ran}(T) = Z$, then by Theorem 11, there exists $\gamma > 0$ such that

$$\|TT^*z\|_Z \geq \gamma \|z\|_Z, \quad \forall z \in Z. \tag{2.12}$$

If $z \neq 0$, then $TT^*z \neq 0$, which implies that $\operatorname{Ker}(TT^*) = \{0\}$, i.e., TT^* is one-to-one. Also, from the proof of Theorem 11, it follows that $\operatorname{Ran}(TT^*) = Z$. Therefore, TT^* is bijective and there exists $(TT^*)^{-1}$. From (2.12), let $\gamma^* = 1/\gamma$, then we have that

$$\left\|(TT^*)^{-1}z\right\| \leq \gamma^* \|z\|, \quad z \in Z. \tag{2.13}$$

□

Corollary 4. *Let $T \in \mathcal{L}(Y, Z)$ such that* $\operatorname{Ran}(T) = Z$. *Then, $y_z = T^*(TT^*)^{-1}z$ is the solution of the equation*

$$Ty = z$$

with minimum norm, i.e.,

$$\|y_z\|_Y = \inf\{y \in Y : Ty = z\}.$$

Proof. Let $\mathscr{S} := T^*(TT^*)^{-1} : Z \longrightarrow Y$, and let $y_z = \mathscr{S}z$ for some $z \in Z$. Consider the following equation

$$Ty = z \tag{2.14}$$

Clearly, y_z is a solution of (2.14). Moreover, for $y \neq y_z$, such that $Ty = z$, we have that

$$\langle y - y_z, y_z\rangle = \langle y, y_z\rangle - \|y_z\|^2 = 0,$$

so that

$$\|y\|^2 = \|y - y_z\|^2 + \|y_z\|^2.$$

In fact, it follows that

$$\begin{aligned}\langle y, y_z\rangle &= \langle y, T^*(TT^*)^{-1}z\rangle = \langle Ty, (TT^*)^{-1}z\rangle = \langle z, (TT^*)^{-1}z\rangle \\ &= \langle TT^*(TT^*)^{-1}z, (TT^*)^{-1}z\rangle = \langle T^*(TT^*)^{-1}z, T^*(TT^*)^{-1}z\rangle \\ &= \|\mathscr{S}z\|^2 = \|y_z\|^2.\end{aligned}$$

This completes the proof. □

Corollary 5. *If* $\dim(Z) < \infty$ *and* $T \in \mathcal{L}(Y, Z)$. *Then the following statements are equivalent*

- (i) $\mathrm{Ran}(T) = Z$;
- (ii) *There exists* $\gamma > 0$ *such that* $\|T^*z\|_Y \geq \gamma\|z\|_Z, \quad \forall z \in Z$;
- (iii) $\mathrm{Ker}(T^*) = \{0\}$;
- (iv) $\exists (TT^*)^{-1} \in \mathcal{L}(Z)$.

Proof. Since $\dim(Z) < \infty$, it follows that $\mathrm{Ran}(T) = \overline{\mathrm{Ran}(T)}$(see [84, Th. 5.10.3]). Thus, from Theorem 11 we obtain the desired result. □

2.1.6 Fixed point theorems

Now, we present the fixed point theorems that will be used to prove our main results.

Theorem 12. *Let* (Y, d) *be a complete metric space and* $F : Y \longrightarrow Y$ *be a contractive mapping. Then* F *has a unique fixed point* $u \in Y$, *and* $F^n(y) \to u$ *for each* $y \in Y$.

This theorem is referred to as the Banach contraction theorem, and its proof can be found in [43, Th. 1.1, pp. 10].

The next theorem is an extension of Krasnosel'skii's fixed point theorem and was proved by Karakostas in [62, Th. 2.2, pp. 183].

Theorem 13. *Let* Z *and* Y *be Banach spaces and* D *be a closed convex subset of* Z, *and let* $\mathcal{C} : D \to Y$ *be a continuous operator such that* $\mathcal{C}(D)$ *is a relatively compact subset of* Y, *and*

$$\mathcal{F} : D \times \overline{\mathcal{C}(D)} \to D$$

a continuous operator such that the family $\{\mathcal{F}(\cdot, y) : y \in \overline{\mathcal{C}(D)}\}$ *is equicontractive. Then, the operator equation*

$$\mathcal{F}(z, \mathcal{C}(z)) = z$$

admits a solution on D.

Finally, we state the Rothe's fixed theorem, which is a general version of the finite dimensional Rothe's fixed theorem.

Theorem 14. *Let X be a Banach space and consider $D \subseteq X$ a closed convex subset containing the zero of X in its interior. Let $\mathcal{B}: D \to X$ be a continuous function with $\mathcal{B}(D)$ relatively compact in X and $\mathcal{B}(\partial D) \subset D$. Then*

$$\exists x^* \in D : \mathcal{B}(x^*) = x^*.$$

A proof of Rothe's fixed point theorem is given in [60, Th. 2, pp. 129].

2.2 Differential Equations

In this chapter, we present a brief review of the general theory of differential equations as well as a description of differential equations with impulses, delay, and non-local conditions.

2.2.1 Existence and Uniqueness

The foundations of the theory of differential equations are the general theorems of existence and uniqueness. Therefore, we begin by recalling these theorems. First, we define the concept of a Differential Equation (DE).

Definition 15. *Let $D \subseteq \mathbb{R}^{n+1}$ be an open set; let $f : D \longrightarrow \mathbb{R}^n$ a continuous function and $v' = dv(t)/dt$. A differential equation is a relation of the form*

$$v'(t) = f(t, v(t)). \tag{2.15}$$

We say that v is a solution of (2.15) on $I \subseteq \mathbb{R}$ if v is a continuously differentiable function defined on I, $(t, v(t)) \in D$ for $t \in I$ and v satisfies (2.15) on I. Suppose $(t_0, v_0) \in D$ is given. An initial value problem (IVP) for (2.15) consist of finding $I \subseteq \mathbb{R}$ containing t_0 and a solution v of (2.15) satisfying $v(t_0) = v_0$. The IVP formulated above is equivalent to

$$v(t) = v_0 + \int_{t_0}^{t} f(s, v(s))ds, \quad t \in I. \tag{2.16}$$

If f is continuous in D, then for any $(t_0, v_0) \in D$, there is at least one solution of (2.15) passing through (t_0, v_0). This results is known as the Peano Existence Theorem (see [47, Ch. 1]).

Theorem 15 (Picard-Lindelöf)**.** *If f is continuous in D and locally lipschitzian with respect to v in D, there exists a unique solution $v(t, t_0, v_0)$ of the IVP passing through (t_0, v_0) and defined in some neighborhood of t_0.*

Theorem 15 can be found in [47, Ch. 1, Th. 3.1]. For additional information on the classical theory of existence-uniqueness for DEs we refer the reader to [45, 51].

2.2.2 Linear Systems

Suppose that $f(t, v(t)) = A(t)v(t)$ in (2.15); where $A(t)$ is a continuous $n \times n$ matrix on $I \subseteq \mathbb{R}$. Then, given $(t_0, v_0) \in D$, the IVP becomes

$$v'(t) = A(t)v(t), \quad t \in I, \tag{2.17}$$
$$v(t_0) = v_0. \tag{2.18}$$

We refer to (2.17) as a *Linear Homogeneous System* (LHS). From the continuity of $A(t)$, Theorem 15 implies that the IVP (2.17)-(2.18) has a unique solution on I. In addition, the set of all solutions of (LHS) on I form an n-dimensional vector space (see, e.g., [29, Ch. 3, Th. 2.1]).

Definition 16. *If Φ is a matrix whose n columns are n linearly independent solutions of* (LHS) *on I, then Φ is called a fundamental matrix of* (LHS). *A principal matrix of* (LHS) *at initial time t_0 is a fundamental matrix Φ such that $\Phi(t_0) = I_{n\times n}$.*[4]

Evidently, Φ satisfies the matrix differential equation

$$\frac{d}{dt}\Phi(t) = A(t)\Phi(t), \quad t \in I.$$

Note that a general solution of the IVP (2.17)-(2.18) is given by $W(t,t_0)v_0 := \Phi(t)\Phi^{-1}(t_0)v_0$, where $W(t,t_0)$ is known as the *evolution operator(or transition matrix)*. If we consider $W(t,s)$ for $t,s \in I$, then the following proposition holds.

Proposition 2. *For all $t, s, r \in I$, it follows that*

(*i*) $W(t,t) = I$;

(*ii*) $W(t,r)W(r,s) = W(t,s)$;

(*iii*) $\dfrac{d}{dt}W(t,s) = A(t)W(t,s)$;

(*iv*) $W(t,s)$ *is continuous;*

(*v*) *There exist $M \geq 1$ and $K, \alpha > 0$ such that*

$$\|W(t,s)\| \leq Ke^{\alpha(t-s)} \leq M, \quad 0 \leq s \leq t \leq T.$$

(*vi*) $W^{-1}(s,t) = W(t,s)$.

Proof. (i)-(iv) are immediate by the definition of $W(t,s)$. For the proof of (v) and (vi), see [45, pp. 42-44]. □

Suppose now that h is a continuous vector function on I. Then the IVP

$$v'(t) = A(t)v(t) + h(t), \quad t \in I, \tag{2.19}$$

$$v(t_0) = v_0. \tag{2.20}$$

admits a unique solution on I. In fact, suppose that v_1 and v_2 are two solutions, then $u = v_1 - v_2$ would be a solution of (LHS) on I and $u(t_0) = 0$. But, by the uniqueness theorem for (LHS), $u = 0$ on I, and thus $v_1 = v_2$. The equation (2.19) is called a *Nonhomogeneous Linear System*(NHS).

Theorem 16. *If Φ is a fundamental matrix of* (LHS)*, then every solution of the* IVP(2.19)-(2.20) *is given by*

$$v(t) = \Phi(t)\Phi^{-1}(t_0)v_0 + \Phi(t)\int_{t_0}^{t}\Phi^{-1}(s)h(s)ds. \tag{2.21}$$

Formula (2.21) is referred to as the *variation of constants formula*. See [29, Ch. 3, Th. 3.1] for the proof of Theorem 16.

[4] $I_{n\times n}$ denotes the Identity matrix.

2.2.3 Differential Equations with Delay

This subsection introduces the concept of differential equations with delay. In the literature, these equations also are known as differential equations with retarded argument or delay differential equations. For this part, our primary reference is Driver [38].

By a *Delay Differential Equation*(DDE) we mean an equation of the form

$$v'(t) = f(t, v(t), v(t-\tau)), \tag{2.22}$$

where $0 < \tau < \infty$ is called the *delay*. Existence results of the IVP for (2.22) are studied in [45, Ch. 4]. We shall consider a more general form of (2.22) by adding different delays. Let $I = [0, T]$, $D \subseteq \mathbb{R}^n$ an open set and $f : [0, T] \times D^m \longrightarrow \mathbb{R}^n$. Consider the differential system

$$v'(t) = f(t, v(d_1(t)), ..., v(d_m(t))), \quad t \in [0, T], \tag{2.23}$$

where each $d_i(t)$[5] is a retarded argument, i.e., $d_i(t) \le t$ for $i = 1, ..., m$. We shall assume that

$$t - r \le d_j(t) \le t, \quad t \in [0, T], j = 1, ..., m,$$

for some $0 \le r < \infty$. Given $\phi : [-r, 0] \longrightarrow D$, an initial condition for (2.23) takes the form

$$v(t) = \phi(t), \quad t \in [-r, 0]. \tag{2.24}$$

A solution of the last IVP is a continuous function $v : [-r, T]$ such that $v(t) = \phi(t)$ for $t \in [-r, 0]$, and v satisfies (2.23) for $t \in [0, T]$.

Example 1 (Smith [95])**.** *Consider the scalar delay differential equation given by*

$$v'(t) = -v(t-\tau) \tag{2.25}$$

where $0 < \tau < \infty$*. When* $\tau = 0$*, we obtain the differential equation* $v'(t) = -v(t)$*, whose general solution is* $v(t) = v(0)e^{-t}$*. Suppose we set*

$$v(t) = 1, \quad -\tau \le t \le 0 \tag{2.26}$$

as initial condition for (2.25)*. Then,* $t - \tau \le 0$ *for* $0 \le t \le \tau$ *so* $v'(t) = -v(t-\tau) = -1$*, and therefore*

$$v(t) = v(0) + \int_0^t (-1) ds = 1 - t, \quad 0 \le t \le \tau. \tag{2.27}$$

On $\tau \le t \le 2\tau$*, we have* $0 \le t - \tau \le \tau$ *so by* (2.27)*,* $v'(t) = -v(t-\tau) = -[1 - (t-\tau)]$ *and thus*

$$v(t) = v(\tau) + \int_\tau^t -[1 - (s-\tau)] ds = 1 - t + (t-\tau)^2/2, \quad \tau \le t \le 2\tau.$$

Following this procedure, we can verify that

$$v(t) = 1 + \sum_{k=1}^{n} (-1)^k \frac{[t - (k-1)\tau]^k}{k!}, \quad (n-1)\tau \le t \le n\tau, \ n \ge 1.$$

Thus, $v(t)$ *is a polynomial of degree* n *on* $[(n-1)\tau, n\tau)$ (See Fig. 2.1)*. Observe that* $v(t)$ *is a smooth function, except at each* $n\tau, n \ge 0$*.*

[5] Some authors write $t - \tau_i(t)$ instead of $d_i(t)$, where $\tau_i(t) \ge 0, j = 1, ..., m$ are the delays.

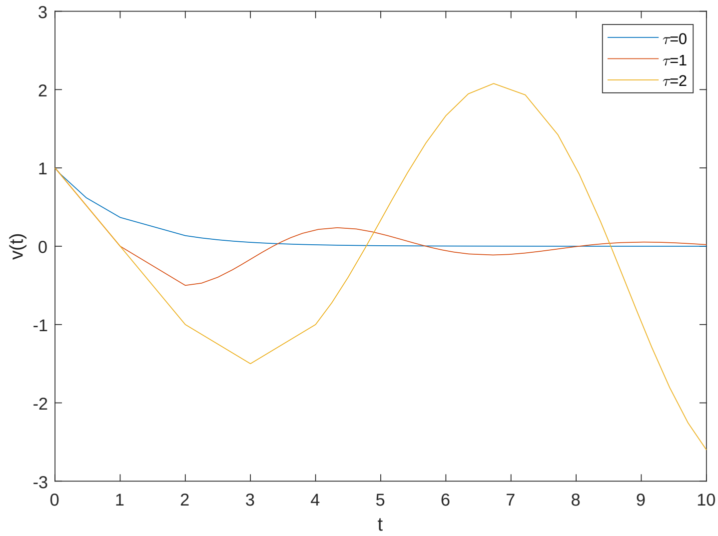

Figure 2.1: Solution of equation (2.25)-(2.26) for various τ

Let $v \in C([-r,T];\mathbb{R}^n)$[6], then for $t \in [0,T]$ we let $v_t \in C([-r,0];\mathbb{R}^n)$ be defined by $v_t(s) = v(t+s)$. Define the map F from $[0,T] \times C([-r,0;\mathbb{R}^n])$ to $\mathbb{R}^n$ and consider the IVP

$$v'(t) = F(t, v_t), \quad t \in [0,T], \tag{2.28}$$

$$v(t) = \phi(t), \quad t \in [-r,0]. \tag{2.29}$$

Note that F is defined on a function space, usually called the *phase space*, and can conceivably depend on any or all values of $v_t(s) = v(t+s), -r \leq s \leq 0$. Equation (2.28) is known as a *Retarded Functional Differential Equation*(RFDE) and generalizes DDEs. In fact, for F defined by

$$F(t, v_t) = f(t, v_t(d_1(t) - t), ..., v_t(d_m(t) - t))$$

we recover (2.23). If F is continuous with respect to $t \in [0,T]$ for each $v \in C([-r,T];\mathbb{R}^n)$, then a function $v \in C([-r,T];\mathbb{R}^n)$ is a solution of (2.28)-(2.29) iff

$$v(t) = \begin{cases} \phi(0) + \displaystyle\int_0^t F(s, v_s)\, ds, & t \in [0,T], \\ \phi(t), & t \in [-r,0]. \end{cases} \tag{2.30}$$

The existence theorem for problem (2.28)-(2.29) generalizes the one of subsection 2.2.1. This is obtained by assuming that F satisfies a Lipschitz condition. A detailed proof of existence and uniqueness theorems for FDEs can be consulted in [38, Ch. VI].

2.2.4 Differential Equations with Infinite Delay

In the last subsection we consider FDEs with finite delay, where the function v_t belongs to the phase space $C([-r,0];\mathbb{R}^n)$, which is characterized by some axioms. The crucial assumption is that the motion of v_t in the phase space is continuous for t. If the phase

[6] $C([-r,T];\mathbb{R}^n)$ is the space of continuous functions from $[-r,T]$ to $\mathbb{R}^n$.

space $C([-r,0];\mathbb{R}^n)$ is endowed with the topology induced by the supremum norm, then last assumption holds. Thus, we usually take this space as the phase space for equations with finite delay. Here, we study this assumption for the case where the delay is infinite. Our main reference in this part is Hino *et al.* [59].

In the case of infinite delay, we have several possibilities for the choice of the phases spaces taken properly according to the problem. However, there are many facts which hold independently of each concrete phase space.

The first axiomatic approach for equations with infinite delay was given by Coleman & Mizel [30]. Later, Hale [46] introduced some other axioms, and contributions to these axioms have been brought by Hino [58] and Naito [83]. Furthermore, Hale & Kato [49], and Schumacher [94] gave a more systematic development of this subject, independently.

Suppose E is a Banach space, for a function f mapping a topological space S into E, and for a subset K of S, we set

$$\|f\|_K = \sup\left\{\|f(x)\| : x \in K\right\}.$$

If $\|f\|_K < \infty$ for every compact set $K \subset S$, f is said to be locally bounded on S. For a function $x : (-\infty, a) \longrightarrow E$ and for $t < a$, we define the function $x_t : (-\infty, 0] \longrightarrow E$ by

$$x_t(s) = x(t+s), \quad -\infty < s \leq 0.$$

The phase space $\mathscr{B}$ for equations with infinite delay is a linear space, with a seminorm $\|\cdot\|_{\mathscr{B}}$, consisting of functions mapping $(-\infty,0]$ into E. Hale & Kato [49] provided the following fundamental axioms on $\mathscr{B}$

(A1) If x is a function mapping $(-\infty, \sigma + a)$ into E, $a > 0$, such that $x \in \mathscr{B}$ and x is continuous on $[\sigma, \sigma + a)$, then for every $t \in [\sigma, \sigma + a)$ the following conditions hold;

- (i) x_t is in $\mathscr{B}$,
- (ii) $\|x(t)\|_E \leq H\|x_t\|_{\mathscr{B}}$,
- (iii) $\|x_t\| \leq K(t-\sigma)\sup\{\|x(s)\|_E : \sigma \leq s \leq t\} + M(t-\sigma)\|x_\sigma\|_{\mathscr{B}}$,

where H is a constant, $K, M : [0,\infty) \longrightarrow [0,\infty)$, K is continuous, M is locally bounded, and both are independent of x.

(A2) For the function x in **(A1)**, x_t is a $\mathscr{B}$-valued continuous function for $t \in [\sigma, \sigma + a)$.

(A3) The space $\mathscr{B}$ is complete.

We can mention two standard spaces used in the study of some differential equations with infinite delay.

$$BC = \left\{\varphi \in C((-\infty,0];\mathbb{R}^n) : \|\varphi\| := \sup_{-\infty<\theta\leq 0} \|\varphi(\theta)\| < \infty\right\}.$$

For a positive continuous function g on $(-\infty, 0]$, let us consider the phase space C_g introduced by Burton and Haddock [5]:

$$C_g = \left\{\varphi \in C((-\infty,0];\mathbb{R}^n) : \|\varphi\|_g := \sup_{-\infty<\theta\leq 0} \frac{\|\varphi(\theta)\|}{g(\theta)} < \infty\right\}.$$

The requirement that the motion of x_t is continuous in the phase space is satisfied by defining the new spaces

$$BC^0 = \left\{ \varphi \in BC : \lim_{\theta \to -\infty} \varphi(\theta) = 0 \right\} \text{ and } C_g^0 = \left\{ \varphi \in C_g : \lim_{\theta \to -\infty} \frac{\varphi(\theta)}{g(\theta)} = 0 \right\}.$$

The space BC^0 satisfies the axioms **(A1)**-**(A3)**. Additionally, if the function g fulfills the following condition

$$G(t) := \sup_{-\infty < \theta \leq -t} \frac{g(t+\theta)}{g(\theta)} \text{ is locally bounded for } t \geq 0$$

Then, the function spaces C_g, C_g^0 satisfy the axioms **(A1)**-**(A3)**. This result can be found in [59, Th. 3.1, Th. 3.2].

Recently, some attention have been focused on the space C_g as a phase space for functional differential equations with infinite delay; especially for many types of integro-differential equations. Here, we will adopt some ideas from Liu [77] and consider a continuous positive function $m : \mathbb{R} \longrightarrow \mathbb{R}_+$ such that

(M1) $m(0) = 1$;

(M2) $m(s) \to \infty$ as $s \to -\infty$;

(M3) m is decreasing.

For the function m given above, we define the space of functions

$$C_m = \left\{ \varphi \in C((-\infty, 0]; \mathbb{R}^n) : \|\varphi\|_{C_m} := \sup_{-\infty < \theta \leq 0} \frac{\|\varphi(\theta)\|}{m(\theta)} < \infty \right\}. \tag{2.31}$$

The space C_m satisfies axioms **(A1)**-**(A3)** (see, e.g., [20], [59]).

2.2.5 Differential Equations with Non-local Conditions

The non-local condition is a generalization of the classical initial condition and was motivated by physical problems such as the position of a material point at different moments. Byszewski & Lakshmikantham [23] originally introduced the non-local problem

$$\begin{aligned} v'(t) &= f(t, v(t)), \quad t \in [0, T], \\ v(0) &= v_0 - \zeta(\lambda_1, ..., \lambda_p, v(\cdot)) \in E^n, \end{aligned} \tag{2.32}$$

where E is a Banach space, $v(t) \in \Omega \subseteq E^n$, $0 < \lambda_1 < \ldots < \lambda_p \leq T$,$f$ and ζ are suitable functions. The symbol $\zeta(\lambda_1, ..., \lambda_p, v(\cdot))$ is meant in the sense that in the place of $\cdot$ we can substitute only elements of the set $\{\lambda_1, ..., \lambda_p\}$. For instance, ζ can be defined by

$$\zeta(\lambda_1, ..., \lambda_p, v(\cdot)) = C_1 v(\lambda_1) + \ldots + C_p v(\lambda_p) \tag{2.33}$$

where $C_i, i = 1, ..., p$ are constants. To prove the existence and uniqueness of solutions for (2.32), authors in [23] used the Banach contraction theorem.

The semilinear non-local problem in finite dimension associated to (2.32) takes the form

$$\begin{aligned} v'(t) &= \mathbf{A}(t)v(t) + f(t, v(t)), \quad t \in [0, T], \\ v(0) &= v_0 - \zeta(\lambda_1, ..., \lambda_p, v(\cdot)) \in \mathbb{R}^n, \end{aligned} \tag{2.34}$$

where $\mathbf{A}(t)$ is a continuous $n \times n$ matrix on $[0, T]$. Byszewski [21] give three theorems on the existence and uniqueness of solutions to (2.34) in infinite dimension using Banach contraction theorem and semigroup theory. This result generalizes the local problem given by Pazy [89, Sec. 6.1, Th. 1.4, Th. 1.6]. Balachandran & Ilamaran [12] studied (2.34) with $f(t, u(\sigma(t)))$ in place of $f(t, u(t))$, with σ an absolutely continuous function.

If we consider the non-local semilinear problem with delay, system (2.34) becomes

$$\begin{aligned} v'(t) &= A(t)v(t) + f(t, v_t)), \quad t \in [0, T], \\ v(s) &= \phi(s) - \zeta(v_{\lambda_1}, ..., v_{\lambda_p})(s), \quad s \in [-r, 0]. \end{aligned} \tag{2.35}$$

Here $\phi(s) - \zeta(v_{\lambda_1}, ..., v_{\lambda_p})(s)$ incorporates the historical information of the solution on $[-r, 0]$. Byszewski & Akca [22] discuss the existence, uniqueness and continuous dependence on initial data of solutions for system (2.35). For a complementary bibliography on differential equations with non-local conditions and applications in physics, refer to [19, 99, 86].

2.2.6 Impulsive Differential Equations

Impulsive systems were introduced in the 1960s by Millman & Mishkis [80]. After that, the theory of impulsive differential equations has increased rapidly. According to Agarwal *et al.* [1], impulsive differential equations consist of two parts:

(*i*) Differential equation that describes the continuous part of the solutions;

(*ii*) Impulsive part that defines the rapid change and the discontinuity of the solution.

The first part could consist of ordinary differential equations, fractional differential equations, partial differential equations, integro-differential equations, etc. The points at which the impulses occur are called *moments of impulses*, and the functions that define the amount of the impulses are called *impulsive functions*.

In general, two types of impulses are described by impulsive differential equations. The first type is concerned with *instantaneous impulses* [82], which are abrupt changes with a relatively short duration compared to the overall duration of the whole process. The fundamental theory of instantaneous impulsive differential equations is provided by [15, 11, 93, 9, 66].

On the other hand, the second impulsive action are the so-called non-instantaneous impulses, which start at fixed points and remain active over a finite time interval. Hernández and O'Regan [57] introduced this new class of differential equations with non-instantaneous impulses. Specifically, they study the existence of mild and classical solutions for an abstract impulsive problem of the form

$$\begin{aligned} v'(t) &= Av(t) + f(t, v(t)), \quad t \in (s_i, t_{i+1}], i = 0, ..., N, \\ v(t) &= g_i(t, v(t)), \quad t \in (t_i, s_i], i = 1, ..., N, \\ v(0) &= v_0. \end{aligned} \tag{2.36}$$

They used strongly continuous semigroup theory and fixed point methods to prove their existence results.

Example 2 (Agarwal *et al.* [1]). *Let* $\{t_i\}_{i=1}^{\infty}$ *and* $\{s_i\}_{i=1}^{\infty}$ *be two increasing sequences of real numbers such that* $0 < s_0 < t_k \leq s_k < t_{k+1}$ *for* $i = 1, 2, ..., \infty$, $\lim_{k\to\infty} t_k = \infty$. *Let* $t_0 \in [0, s_0)$ *and consider the following IVP with non-instantaneous impulses*

$$\begin{aligned} v'(t) &= f(t, v(t)), \quad t \in (t_k, s_k], k = 0, 1, 2, ..., \\ v(t) &= g_k(t, v(t)), \quad t \in (s_k, t_{k+1}], k = 0, 1, 2, ..., \\ v(t_0) &= v_0, \end{aligned} \tag{2.37}$$

where $v(t), v_0 \in \mathbb{R}^n$ *and* $f : \cup_{k=0}^{\infty}[t_k, s_k] \times \mathbb{R}^n \longrightarrow \mathbb{R}^n$, $g_k : [s_k, t_{k+1}] \times \mathbb{R}^n \longrightarrow \mathbb{R}^n, k = 1, 2, ...,$ *are suitable functions. Then, the solution* $v(\cdot)$ *of* (2.37) *satisfies the following integral equation*

$$v(t) = \begin{cases} v_0 + \displaystyle\int_{t_0}^{t} f(s, v(s))ds, \quad t \in [t_0, s_0], \\ g_k(t, v(t)), \quad t \in (s_k, t_{k+1}], k = 0, 1, 2, ..., \\ g_{k-1}(t_k, v(t_k)) + \displaystyle\int_{t_k}^{t} f(s, v(s))ds \quad t \in [t_k, s_k], k = 1, 2, ... \end{cases} \tag{2.38}$$

Let $g_k(t, v(t)) = 2t - v(t), k = 0, 1, ...,$ *and* $f(t, v(t)) = a_k v(t)$, *where* a_k *are constants for* $k = 0, 1, ...$ *Hence, since* $v(t) = t$ *is the unique solution of the equation* $v(t) = 2t - v(t)$, *the solution of* (2.37) *is*

$$v(t) = \begin{cases} v_0 e^{a_0(t-t_0)}, \quad t \in [t_0, s_0], \\ t, \quad t \in (s_k, t_{k+1}], k = 0, 1, 2, ..., \\ t_k e^{a_k(t-t_k)} \quad t \in [t_k, s_k], k = 1, 2, ... \end{cases} \tag{2.39}$$

Several investigations for non-instantaneous impulsive equations have been considered in the literature. Pierri *et al.* [91, 90] extended new existence results of equation (2.36) by using analytic semigroup theory and fixed point techniques in fractional power space.

Later, Wang & Feckan [100] have a remark on the condition of the impulsive function in (2.36), where $g_i \in C([t_i, s_i] \times X, X)$, with X a Banach space. Indeed, it follows from Theorem 2.1 and 2.2 in [57, 91] that Banach fixed point theorem gives $y_i \in C([t_i, s_i], X)$ so that $y = g_i(t, y)$ if and only if $y = y_i$. So (2.36) is equivalent to

$$v(t) = y_i(t), \quad t \in (t_i, s_i], i = 1, ..., N,$$

which does not depend on the state $v(\cdot)$. Thus, authors in [100] recommend to modify (2.36) by

$$v(t) = g_i(t, v(t_i^-)), \quad t \in (t_i, s_i], i = 1, ..., N.$$

This new conditions is a better generalization of abrupt impulses to non-instantaneous ones.

2.2.7 Neutral Differential Equations

Neutral Differential Equations (NDEs) are frequently encountered in the literature for systems of the form

$$v'(t) = f(t, v(t), v(t - \tau), v'(t - \tau)). \tag{2.40}$$

Here, the past history and derivatives of the past history are involved as well as the present state of the system. The general form of (2.40) can be written as

$$v'(t) = f(t, v_t, v'_t). \tag{2.41}$$

For the general theory of neutral differential equations, one can see, for instance, Akmerov *et al.* [2], Kolmanovskii & Myshkis [64], and Hale & Lunel [50].

In recent years a majority of authors prefer the general form of NDEs proposed by Hale & Cruz [48]

$$\begin{aligned} \frac{d}{dt}[v(t) - g(t, v_t)] &= f(t, v_t), \quad t \in [0, T] \\ v(t) &= \phi(t), \quad t \in [-r, 0], \end{aligned} \tag{2.42}$$

where $f, g : [0, T] \times C([-r, 0]; \mathbb{R}^n) \longrightarrow \mathbb{R}^n$ are appropriate functions. Arino *et al.* [4] studied the existence of solutions for system (2.42), while Ntouyas & Sficas [87] derived results on continuation of solutions. In [81], authors use Schaefer's fixed point theorem to prove the existence of solutions for system (2.42) under the influence of instantaneous impulses.

The semilinear version of problem (2.42) stands for

$$\begin{aligned} \frac{d}{dt}[v(t) - g(t, v_t)] &= \mathbf{A}(t)v(t) + f(t, v_t), \quad t \in [0, T] \\ v(t) &= \phi(t), \quad t \in [-r, 0]. \end{aligned} \tag{2.43}$$

Anguraj & Karthikeyan [3] showed the existence of solutions of problem (2.43) with instantaneous impulses and non-local conditions in infinite dimension. The local version of problem (2.42) with instantenous impulses has been considered in Cuevas *et al.* [33] and Hernández [56].

Now, we state the semilinear neutral problem with infinite delay

$$\begin{aligned} \frac{d}{dt}[v(t) - g(t, v_t)] &= \mathbf{A}(t)v(t) + f(t, v_t), \quad t \in [0, T] \\ v(t) &= \phi(t), \quad t \in (-\infty, 0]. \end{aligned} \tag{2.44}$$

Many authors investigate the existence of solutions of system (2.44) by employing an axiomatic definition for the phase space of retarded functions with infinite delay. Such space satisfies the axioms proposed by Hale & Kato [49]. Hernández & Henríquez [53] established the existence of mild and strong solutions for system (2.44) in Banach spaces. After that, Hernández & Henríquez [54] used analytic semigroup theory and Banach contraction principle in fractional power space to prove the existence of mild solutions for system (2.44) with instantaneous impulses. Hernández *et al.* [55] did the same, but using the Leray-Schauder alternative. Chang *et al.* [27] studied the same problem using Krasnoselski-Schaefer fixed point theorem. The infinite dimensional non-local version of (2.44) is discussed in Hernández [52].

2.3 Preliminary Control Theory

In this section, we present a brief review of control theory and a characterization for the controllability of the linear system, which is helpful for the proof of the semilinear neutral case.

A *control system* is a dynamical system on which one can act by using suitable controls. According to Barnett [13], the main features of a control system can be represented as in Fig. 2.2

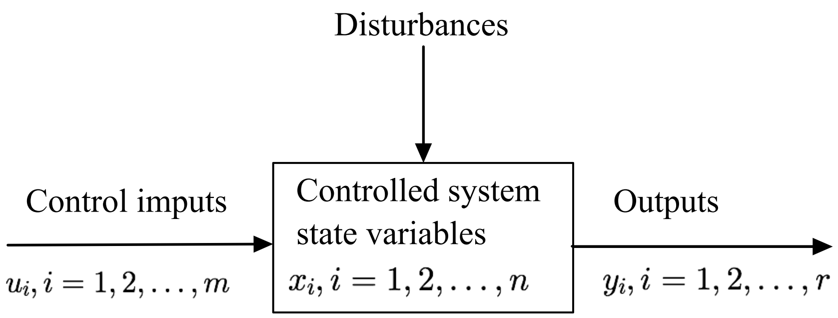

Figure 2.2: Scheme of a control system

The state variables x_i describe the condition of the system and provide the information needed to calculate the future behavior from the knowledge of the inputs. In practice, it is usually not possible to determine the values of the state variables directly; instead, only a set of output variables is measured. Also, systems are often subjected to external disturbances of an unpredictable nature. In general, the object is to make a system perform in some required way by suitably manipulating the control variables u_i.

There are many exciting applications of control theory in science and engineering, for example, aircraft, spacecraft, chemical, industrial processes; such as distillation columns and rolling mills, quantum systems theory, electric bulk power systems, etc., [16]. The literature related to control theory is extensive; we refer the reader to [32, 13, 34, 69] and the references therein.

An essential first step in dealing with many control problems is to study the controlled linear system:

$$\begin{aligned} x'(t) &= \mathbf{A}(t)x(t) + \mathbf{B}(t)u(t), \quad t \in (0, T], \\ x(0) &= x_0, \end{aligned} \tag{2.45}$$

where $\mathbf{A}(t)$ and $\mathbf{B}(t)$ are as in eq. (1.2). By formula (2.21), system (2.45) has a unique solution given by

$$x(t) = W(t,0)x_0 + \int_0^t W(t,s)\mathbf{B}(s)u(s)ds, \quad \forall t \in [0, T]. \tag{2.46}$$

Definition 17. *We say that system (2.45) is exactly controllable on $[0, T]$ if for any pair $x_0, x_1 \in \mathbb{R}^n$, there exists a control $u \in L^2([0, T]; \mathbb{R}^m)$ such that :*

$$x(0) = x_0 \quad \text{and} \quad x(T) = x_1.$$

In 1960, Kalman [61] established a purely algebraic criterion for controllability of the linear autonomous system; that is, whenever $\mathbf{A}(t)$ and $\mathbf{B}(t)$ are constant matrices in (2.45). We state this result in the next theorem.

Theorem 17. *The linear autonomous system is controllable if and only if*

$$\operatorname{rank}(\mathbf{B}|\mathbf{AB}|\ldots|\mathbf{A}^{n-1}\mathbf{B}) = n.$$

A proof of this result can be found in [69, Th. 5, pp. 81].

For the non-autonomous case, exact controllability can not be checked via rank condition. Instead we can use results of subsection 2.1.5 to characterize the exact controllability of system (2.45). In this regard, we define the controllability operator

$$\begin{aligned}\mathscr{S}_{[0,T]} : L^2\left([0,T],\mathbb{R}^m\right) &\longrightarrow \mathbb{R}^n\\ u &\longmapsto \mathscr{S}_{[0,T]}(u) = \int_0^T W(T,s)\mathbf{B}(s)u(s)ds.\end{aligned}$$

Proposition 3. *The system* (2.45) *is controllable on* $[0,T]$ *if, and only if, Ran*$(\mathscr{S}_{[0,T]}) = \mathbb{R}^n$*.*

A detailed proof of this result can be found in [24, Prop. 3.2, pp.19]. Clearly, operator $\mathscr{S}_{[0,T]}$ is linear and bounded and its adjoint is given by (see [25, Th. 2.1.1])

$$(\mathscr{S}^*_{[0,T]}x)(t) = \mathbf{B}^*(t)W^*(T,t)x, \quad \forall t \in [0,T], \forall x \in \mathbb{R}^n.$$

Operators $\mathscr{S}_{[0,T]}$ and $\mathscr{S}^*_{[0,T]}$ lead us to define the *Controllability Gramian*:

$$\begin{aligned}\Theta_{[0,T]} := \mathscr{S}_{[0,T]}\mathscr{S}^*_{[0,T]} : \mathbb{R}^n &\longrightarrow \mathbb{R}^n\\ x &\longmapsto \Theta_{[0,T]}x = \int_0^T W(T,s)\mathbf{B}(s)\mathbf{B}^*(s)W^*(T,s)xds.\end{aligned}$$

We apply the results of subsection 2.1.5 to $\mathscr{S}_{[0,T]}$, to obtain the following Lemma.

Lemma 2. *The following statements are equivalent:*

(*i*) $\operatorname{Ran}(\mathscr{S}_{[0,T]}) = \mathbb{R}^n$;

(*ii*) $\operatorname{Ker}(\mathscr{S}^*_{[0,T]}) = \{0\}$;

(*iii*) $\exists\gamma > 0, \forall x \in \mathbb{R}\setminus\{0\} : \langle \mathscr{S}_{[0,T]}\mathscr{S}^*_{[0,T]}x, x\rangle \geq \gamma\|x\|^2$;

(*iv*) $\Theta_{[0,T]}$ *is invertible.*

Lemma 2 allows to define the steering operator $\mathscr{G}_{[0,T]} : \mathbb{R}^n \longrightarrow L^2([0,T];\mathbb{R}^m)$ by

$$\mathscr{G}_{[0,T]}x(t) = \mathscr{S}^*_{[0,T]}(\mathscr{S}_{[0,T]}\mathscr{S}^*_{[0,T]})^{-1}x(t) = \mathbf{B}^*(t)W^*(T,t)\Theta^{-1}_{[0,T]}x, \quad t \in [0,T].$$

Observe that $\mathscr{G}_{[0,T]}$ is a right inverse of $\mathscr{S}_{[0,T]}$, i.e., $\mathscr{S}_{[0,T]}\mathscr{G}_{[0,T]} = I$. Hence, a control u steering the system (2.45) from x_0 to x_1 is given by

$$u(t) = \mathbf{B}^*(t)W^*(T,t)\Theta^{-1}_{[0,T]}(x_1 - W(T,0)x_0), \quad t \in [0,T].$$

Moreover, by relation (2.13),

$$\left\|\Theta_{[0,T]}x\right\| = \left\|(\mathscr{S}\mathscr{S}^*)^{-1}x\right\| \leq \gamma^*\|x\|, \quad x \in \mathbb{R}^n,$$

where $\gamma^* = 1/\gamma$.

Lemma 3. *Let S be any dense subspace of $L^2([0,T];\mathbb{R}^n)$. Then, system (2.45) is controllable on $[0,T]$, with control $u \in L^2([0,T];\mathbb{R}^n)$ iff it is controllable with control $u \in S$, i.e.,*

$$\mathrm{Ran}(\mathscr{S}_{[0,T]}) \iff \mathrm{Ran}(\mathscr{S}_{[0,T]}|_S) = \mathbb{R}^n$$

A proof of this Lemma can be found in Leiva [71, Lem. 2.3].

Remark 3. *According to the previous Lemma, the spaces $C([0,T];\mathbb{R}^n)$ and $C^\infty([0,T];\mathbb{R}^n)$ are admissible space of controls. Furthermore, the operator $\mathscr{S}_{[0,T]}$ is well defined in these spaces.*

The natural extension of the finite dimensional concept of controllability to infinite dimensions may be too strong for many infinite dimensional systems [35]. For this reason, the weaker notion of approximate controllability was defined. Here, we present the definition of approximate controllability for the non-autonomous linear system.

Definition 18. *We say that system (2.45) is approximate controllable on $[0,T]$ if for any $\varepsilon > 0$ and any pair $x_0, x_1 \in \mathbb{R}^n$, there exists a control $u \in L^2([0,T];\mathbb{R}^m)$ such that*

$$x(0) = x_0 \quad \text{and} \quad \|x(T) - x_1\| < \varepsilon.$$

The controllability of the linear system in finite and infinite dimensions is well known and has been studied in [36, 32, 96]. On the other hand, the controllability of nonlinear systems has been considered in [6, 37, 98]. The common direction followed by these authors was to impose some conditions on the nonlinear terms and assume the exact controllability of the linear system so that controllability is preserved under perturbation. Likewise, the controllability of systems governed by instantaneous impulses has been treated by many authors [73, 70, 85, 72, 71, 74, 26]. In [74], Rothe's fixed point theorem is used to prove the controllability of a semilinear system with instantaneous impulses and non-local conditions. Nieto & Tisdell [85] use the Schaefer's theorem to establish the controllability of an impulsive system without delay and non-local conditions. The controllability of a neutral equation with instantaneous impulses, delay, and non-local conditions is considered in [26].

More recently, the study of controllability and approximate controllability of nonlinear non-instantaneous impulsive systems has drawn the attention of many researchers. For instance, Leiva *et al.* [76] investigated the approximate controllability of the semilinear Heat equation with non-instantaneous impulses by employing a technique that avoids fixed point theorems. In [78, 40], authors use Rothe's fixed point to prove the exact controllability of a semilinear system with non-instantaneous impulses. García & Leiva [41] addressed the approximate controllability of a semilinear system with non-instantaneous impulses, non-local conditions, and infinite delay. There are few works on the controllability of neutral differential equations with non-instantaneous impulses. Kavitha *et al.* [63] investigated the exact controllability of a neutral system with non-instantaneous impulses and non-local conditions in finite dimension. Malik & Kumar [79] documented existence and controllability results to a second-order neutral equation with non-instantaneous impulses in a Hilbert space.

Chapter 3

Results

3.1 Existence of Solutions

This section aims to prove the existence and uniqueness of solutions for the semilinear neutral problem (1.1). For achieving this goal, we apply two methods. First, Karakostas's fixed point theorem is applied to prove the existence and uniqueness of solutions for system (1.1). Later, the Banach contraction theorem is used to prove the existence of solutions for a simplified version of the system (1.1).

3.1.1 Existence and uniqueness of solutions

Let us consider the following semilinear neutral differential equation with non-instantaneous impulses, non-local conditions, and infinite delay

$$\begin{aligned} \frac{d}{dt}[v(t) - g(t, v_t)] &= \mathbf{A}(t)v(t) + \mathfrak{F}(t, v_t), \quad t \in J_k^1, k = 0, 1, \ldots, \\ v(t) &= \Gamma_k(t, v(t_k^-)), \quad t \in J_k^2, k = 1, \ldots, \\ v(s) + \zeta(v_{\lambda_1}, v_{\lambda_2}, \ldots, v_{\lambda_q})(s) &= \phi(s), \quad s \in \mathbb{R}_- = (-\infty, 0]. \end{aligned} \tag{3.1}$$

A thorough description of system (3.1) is given in (1.1). Now, we set the phase space for equation (3.1). To this end, we use some ideas from subsection 2.2.4. Denote by $\mathcal{PC} = \mathcal{PC}((-\infty, 0]; \mathbb{R}^n)$ the space of normalized piecewise continuous functions such that their restriction to any interval $[a, 0]$ is a piecewise continuous function, i.e.,

$$\mathcal{PC} = \left\{ \varphi : (-\infty, 0] \longrightarrow \mathbb{R}^n : \varphi\Big|_{[a,0]} \text{is a piecewise continuous function}, \forall a < 0 \right\}.$$

Consider a positive function m satisfying the conditions **(M1)-(M3)**, and define the following space of functions,

$$\mathscr{C}_m = \left\{ v \in \mathcal{PC} : \sup_{s \le 0} \frac{\|v(s)\|}{m(s)} < \infty \right\}.$$

The space $\mathscr{C}_m$ endowed with the norm

$$\|v\|_m = \sup_{s \le 0} \frac{\|v(s)\|}{m(s)}, \quad v \in \mathscr{C}_m,$$

is a Banach space. Now, for $T > 0$, define the larger space $\mathcal{P}C_m := \mathcal{P}C_m((-\infty, T]; \mathbb{R}^n)$:

$$\mathcal{P}C_m = \Big\{ v : (-\infty, T] \to \mathbb{R}^n : v\big|_{\mathbb{R}_-} \in \mathscr{C}_m \text{ and } v\big|_{(0,T]} \text{ is continuous, except at } t_k, k = 1, 2, ..., p \text{ with } s_{p-1} < T, \text{where the side limits } v(t_k^+), v(t_k^-) \text{ exist and } v(t_k^-) = v(t_k) \Big\}.$$

We shall consider the product space $(\mathbb{R}^n)^q = \prod_{i=1}^q \mathbb{R}^n$ endowed with the norm

$$\|y\|_q = \sum_{i=1}^q \|y_i\|_{\mathbb{R}^n}, \quad y = (y_1, \dots, y_q)^T \in (\mathbb{R}^n)^q,$$

and the Banach space $\mathscr{C}_m^q = \prod_{i=1}^q \mathscr{C}_m$ equipped with the norm

$$\|y\|_{mq} = \sum_{i=1}^q \|y_i\|_m, \quad y = (y_1, \dots, y_q)^T \in \mathscr{C}_m^q.$$

Throughout this note, the elements of $\mathscr{C}_m^q$ will be denoted by

$$z = (z^1, \dots, z^q)^T \in \mathscr{C}_m^q \text{ and } \tilde{z} = (z_{\lambda_1}, ..., z_{\lambda_q})^T \in \mathscr{C}_m^q. \tag{3.2}$$

Lemma 4. *$\mathcal{P}C_m$ is a Banach space endowed with the norm*

$$\|v\| = \left\| v|_{\mathbb{R}_-} \right\|_m + \|v|_I\|_\infty ,$$

where $\|v|_I\|_\infty = \sup_{t \in I = (0,T]} \|v(t)\|$.

The phase space $\mathcal{P}C_m$ verifies the axioms **(A1)**-**(A3)** proposed by Hale & Kato [49]. The subsequent Lemma is fundamental to prove our existence theorem, and its proof uses the fact that the function m is defined on the entire real line.

Lemma 5. (see [7]) *For all function $v \in \mathcal{P}C_m$ the following estimate holds for all $s \in [0, T]$:*

$$\|v_s\|_m \le \|v\|_{\mathcal{P}C_m} = \|v\|.$$

The next proposition states without proof the characterization of the solutions to our problem.

Proposition 4. *Suppose that the nonlinear terms in (3.1) are smooth enough. Then the semilinear system (3.1) has a solution $v(\cdot)$ on $(-\infty, T]$ if, and only if, $v(\cdot)$ satisfies the following integral equation*

$$v(t) = \begin{cases} W(t,0)\left[\phi(0) - \zeta(v_{\lambda_1}, v_{\lambda_2}, \dots, v_{\lambda_q})(0) - g(0, \phi - \zeta(v_{\lambda_1}, v_{\lambda_2}, \dots, v_{\lambda_q}))\right] \\ + \displaystyle\int_0^t W(t,s)\left[\mathbf{A}(s)g(s, v_s) + \mathfrak{F}(s, v_s)\right] ds + g(t, v_t), \quad t \in [0, t_1], \\ W(t, s_k)\left[\Gamma_k(s_k, v(t_k^-)) - g(s_k, v_{s_k})\right] + g(t, v_t) \\ + \displaystyle\int_{s_k}^t W(t,s)\left[\mathbf{A}(s)g(s, v_s) + \mathfrak{F}(s, v_s)\right] ds, \quad t \in J_k^1, k = 1, \dots, p-1, \\ \Gamma_k(t, v(t_k^-)), \quad t \in J_k^2, \quad k = 1, \dots, p-1, \\ \phi(t) - \zeta(v_{\lambda_1}, v_{\lambda_2}, \dots, v_{\lambda_q})(t), \quad t \in (-\infty, 0]. \end{cases} \tag{3.3}$$

In order to prove the existence theorem, the subsequent hypotheses are assumed:

(H1) There exist constants $d_q, L > 0, \Theta \geq 0$, for all $k = 1, 2, ...$, such that:

(i) $\|\Gamma_k(t, y) - \Gamma_k(\ell, z)\|_{\mathbb{R}^n} \leq L\left\{|t - \ell| + \|y - z\|_{\mathbb{R}^n}\right\}, \quad y, z \in \mathbb{R}^n, \ell, t \in J_k^2;$

(ii) $\|\Gamma_k(t, 0)\| \leq \Theta, \quad t \in J_k^2;$

(iii) $\|\zeta(z) - \zeta(x)\|_m \leq d_q \|x - y\|_{mq}, \quad x, y \in \mathscr{C}_m^q$, with $\zeta(0) = 0$, and

$$M(L + d_q q + \gamma) < \frac{1}{2}.$$

(H2) The functions g and $\mathfrak{F}$ satisfies, for all $\varphi, \varphi_1, \varphi_2 \in \mathscr{C}_m$, the following

(i) $\|\mathbf{A}(t)g(t, \varphi_1) - \mathbf{A}(t)g(t, \varphi_2)\|_{\mathbb{R}^n} \leq \mathcal{K}\left(\|\varphi_1\|_m, \|\varphi_2\|_m\right)\|\varphi_1 - \varphi_2\|_m;$

(ii) $\|g(t, \varphi_1) - g(t, \varphi_2)\|_{\mathbb{R}^n} \leq \gamma\|\varphi_1 - \varphi_2\|_m;$

(iii) $\|\mathbf{A}(t)g(t, \varphi)\|_{\mathbb{R}^n} \leq \Psi\left(\|\varphi\|_m\right);$

(iv) $\|g(t, \varphi)\|_{\mathbb{R}^n} \leq \Psi\left(\|\varphi\|_m\right);$

(v) $\|\mathfrak{F}(t, \varphi_1) - \mathfrak{F}(t, \varphi_2)\|_{\mathbb{R}^n} \leq \mathcal{K}\left(\|\varphi_1\|_m, \|\varphi_2\|_m\right)\|\varphi_1 - \varphi_2\|_m;$

(vi) $\|\mathfrak{F}(t, \varphi)\|_{\mathbb{R}^n} \leq \Psi\left(\|\varphi\|_m\right),$

where $\mathcal{K} : \mathbb{R}_+ \times \mathbb{R}_+ \longrightarrow \mathbb{R}_+, \Psi : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ are continuous and non decreasing functions.

(H3) The following relation holds for $T, \sigma > 0$

$$\begin{aligned} &M\left\{(L + d_q q)(\|\tilde{\psi}\| + \sigma) + \Psi(\|\tilde{\psi}\| + \sigma) + \Psi(\|\tilde{\psi}\| + d_q q(\|\tilde{\psi}\| + \sigma)) + \Theta\right\} \\ &\quad + (2MT + 1)\Psi(\|\tilde{\psi} + \sigma\|) < \sigma, \end{aligned}$$

where the function $\tilde{\psi} \in \mathcal{P}C_m$ is defined by

$$\tilde{\psi} = \begin{cases} W(t, 0)\phi(0), & t \in J_0^1, \\ 0, & t \in J_k^1, \\ 0, & t \in J_k^2, \\ \phi(t), & t \in (-\infty, 0]. \end{cases} \tag{3.4}$$

(H4) The following relation hold for $T, \sigma > 0$

$$M\left\{d_q q(1 + \gamma) + L + \gamma + 2T\mathcal{K}(\|\tilde{\psi}\| + \sigma, \|\tilde{\psi}\| + \sigma)\right\} < \frac{1}{2}.$$

Now, we are in a position to state and prove our existence result for the problem (3.1)

Theorem 18. *Let the hypotheses* **(H1)**-**(H4)** *be satisfied. Then, system* (3.1) *has at least one solution on* $(-\infty, T]$.

Proof. State the operators $\mathcal{J} : \mathcal{PC}_m \times \mathcal{PC}_m \longrightarrow \mathcal{PC}_m$ and $\mathcal{S} : \mathcal{PC}_m \longrightarrow \mathcal{PC}_m$, given by

$$\mathcal{J}(z,w)(t) = \begin{cases} w(t) + g(t, z_t), \quad t \in J_0^1, \\ w(t) + W(t, s_k)[\Gamma_k(s_k, z(t_k^-)) - g(s_k, z_{s_k})] \\ +g(t, z_t), \quad t \in J_k^1, k = 1, ..., p, \\ \Gamma_k(t, z(t_k^-)), \quad t \in J_k^2, k = 1, ..., p, \\ \phi(t) - \zeta(z_{\lambda_1}, \dots, z_{\lambda_q})(t), \quad t \in (-\infty, 0]. \end{cases}$$

$$\mathcal{S}(z)(t) = \begin{cases} W(t,0)\left[\phi(0) - \zeta(z_{\lambda_1}, z_{\lambda_2}, \dots, z_{\lambda_q})(0) - g(0, \phi - \zeta(z_{\lambda_1}, z_{\lambda_2}, \dots, z_{\lambda_q}))\right] \\ + \displaystyle\int_0^t W(t,s)\left[\mathbf{A}(s) g(s, z_s) + \mathfrak{F}(s, z_s)\right] ds, \quad t \in J_0^1, \\ \displaystyle\int_{s_k}^t W(t,s)\left[\mathbf{A}(s) g(s, z_s) + \mathfrak{F}(s, z_s)\right] ds, \quad t \in J_k^1, k = 1, ..., p, \\ 0, \quad t \in J_k^2, k = 1, ..., p, \\ \phi(t), \quad t \in (-\infty, 0]. \end{cases}$$

From the definition of $\mathcal{J}$ and $\mathcal{S}$, solving the fixed-point equation $\mathcal{J}(z, \mathcal{S}(z)) = z$ is equivalent to find a solution of problem (3.1). In agreement with Theorem 13, let $D_\sigma \subset \mathcal{PC}_m$ be a closed and convex set, namely,

$$D_\sigma := D(\sigma, T, \psi) = \left\{ w \in \mathcal{PC}_m : \|w - \tilde{\psi}\| \leq \sigma \right\}, \tag{3.5}$$

where the function $\tilde{\psi}$ is given by (3.4), and $\sigma > 0$. For better readability, the rest of the proof is divided in six statements:

Statement 1: $\mathcal{S}$ is a continuous operator.

Let $z, w \in \mathcal{PC}_m$. Trivially, for $t \in (-\infty, 0]$,

$$\|\mathcal{S}(z)(t) - \mathcal{S}(w)(t)\|_{\mathbb{R}^n} = \|\phi(t) - \phi(t)\|_{\mathbb{R}^n} = 0. \tag{3.6}$$

Thus,

$$\| \left(\mathcal{S}(z) - \mathcal{S}(w)\right)|_{\mathbb{R}_-} \|_m = 0.$$

Let $t \in (0, t_1]$. By **(H1)**-*(iii)*, **(H2)**-*(i)*, *(ii)*, *(v)* and Lemma 5, the following estimate holds:

$$\begin{aligned}\|\mathcal{S}(z)(t)-\mathcal{S}(w)(t)\| &\leq \|W(t,0)\|\Big\{\left\|\zeta(w_{\lambda_1},\dots,w_{\lambda_q})-\zeta(z_{\lambda_1},\dots,z_{\lambda_q})\right\|_m \\ &\quad+\left\|g(0,\phi-\zeta(w_{\lambda_1},...,w_{\lambda_q}))-g(0,\phi-\zeta(z_{\lambda_1},...,z_{\lambda_q}))\right\|\Big\} \\ &\quad+\int_0^t\|W(t,s)\|\Bigg\{\|\mathbf{A}(s)g(s,z_s)-\mathbf{A}(s)g(s,w_s)\| \\ &\quad+\|\mathfrak{F}(s,z_s)-\mathfrak{F}(s,w_s)\|\Bigg\}ds \\ &\leq M[d_q\|\tilde{z}-\tilde{w}\|_{mq}+\gamma d_q\|\tilde{z}-\tilde{w}\|_{mq}] \\ &\quad+\int_0^t 2M\mathcal{K}(\|z_s\|_m,\|w_s\|_m)\|z_s-w_s\|_m ds \\ &\leq M\left[d_q q\|z-w\|+\gamma d_q q\|z-w\|\right] \\ &\quad+\int_0^t 2M\mathcal{K}(\|z\|,\|w\|)\|z-w\|ds \\ &\leq M\left[d_q q\|z-w\|+\gamma d_q q\|z-w\|\right]+2MT\mathcal{K}(\|z\|,\|w\|)\|z-w\|.\end{aligned}$$

Hence, on J_0^1 we get that

$$\|\mathcal{S}(z)(t)-\mathcal{S}(w)(t)\|\leq M\{d_q q(1+\gamma)+2T\mathcal{K}(\|z\|,\|w\|)\}\|z-w\|. \tag{3.7}$$

Now, consider $t\in J_k^1$ for $k=1,2,...,p$. Again, By **(H2)**-(i), (v) and Lemma 5,

$$\begin{aligned}\|\mathcal{S}(z)(t)-\mathcal{S}(w)(t)\|_{\mathbb{R}^n} &\leq \int_{s_k}^t\|W(t,s)\|\Bigg\{\|\mathbf{A}(s)g(s,z_s)-\mathbf{A}(s)g(s,w_s)\| \\ &\quad+\|\mathfrak{F}(s,z_s)-\mathfrak{F}(s,w_s)\|\Bigg\}ds \\ &\leq M\int_{s_k}^t[2\mathcal{K}(\|z_s\|_m,\|w_s\|_m)\|z_s-w_s\|_m]ds \\ &\leq \int_{s_k}^t 2M\mathcal{K}(\|z\|,\|w\|)\|z-w\|ds \\ &\leq 2MT\mathcal{K}(\|z\|,\|w\|)\|z-w\|.\end{aligned}$$

Therefore, on J_k^1 we get that

$$\|\mathcal{S}(z)(t)-\mathcal{S}(w)(t)\|\leq 2MT\mathcal{K}(\|z\|,\|w\|)\|z-w\|. \tag{3.8}$$

Taking the supremum in (3.7), (3.8), and since $\|\mathcal{S}(z)(t)-\mathcal{S}(w)(t)\|_{\mathbb{R}^n}=0$ for $t\in J_k^2$, $k=1,2,...,p$, it yields that there exists $N_{w,z}>0$ such that

$$\|\mathcal{S}(z)-\mathcal{S}(w)\|\leq N_{w,z}\left\|z-w\right\|.$$

Hence $\mathcal{S}$ is continuous. In fact, it is Lipschitz continuous.

Statement 2: $\mathcal{S}$ maps bounded sets of $\mathcal{PC}_m$ into bounded sets of $\mathcal{PC}_m$.

Without loss of generality, set $R > 0$ arbitrarily and prove that there exists $r > 0$ such that, for each $w \in B_R = \{v \in \mathcal{PC}_m : \|v\| \le R\}$, it follows that $\|\mathcal{S}(w)\| \le r$. Let $w \in B_R$ and $t \in (-\infty, 0]$, it gives that

$$\|\mathcal{S}(w)(t)\|_{\mathbb{R}^n} = \|\phi(t)\|_{\mathbb{R}^n},$$

whence,

$$\left\|(\mathcal{S}(w))|_{\mathbb{R}_-}\right\| = \sup_{t\le 0} \frac{\|\mathcal{S}(w)(t)\|_{\mathbb{R}^n}}{m(t)} = \sup_{t\le 0} \frac{\|\phi(t)\|_{\mathbb{R}^n}}{m(t)} = \|\phi\|_m := r_1. \tag{3.9}$$

For $t \in (0, t_1]$, **(H1)**-*(iii)*, **(H2)**-*(iii)*, *(iv)*, *(vi)* yields

$$\begin{aligned}
\|\mathcal{S}(w)(t)\| \;&\le \|W(t,0)\| \left\|\phi(0) - \zeta(w_{\lambda_1},\dots,w_{\lambda_q})(0) - g(0, \phi - \zeta(w_{\lambda_1},\dots,w_{\lambda_q}))\right\| \\
&\quad + \int_0^t \|W(t,s)\| \Big[\|\mathbf{A}(s)g(s,w_s)\| + \|\mathfrak{F}(s,w_s)\|\Big]\, ds \\
&\le M\Big\{ \|\phi(0)\| + \left\|\zeta(w_{\lambda_1},\dots,w_{\lambda_q})\right\|_m + \left\|g(0,\phi - \zeta(w_{\lambda_1},\dots,w_{\lambda_q}))\right\| \Big\} \\
&\quad + \int_0^t M\left[\Psi(\|w_s\|_m) + \Psi(\|w_s\|_m)\right] ds \\
&\le M\Big\{ \|\phi(0)\| + d_q\|\tilde{w}\|_{mq} + \Psi\left(\|\phi - \zeta(\tilde{w})\|_m\right) \Big\} + TM2\Psi(\|w\|_m) \\
&\le M\Big\{ \|\phi(0)\| + d_q q\|w\| + \Psi\left(\|\phi\|_m + \|\zeta(\tilde{w})\|_m\right) \Big\} + TM2\Psi(\|w\|) \\
&\le M\Big\{ \|\phi(0)\| + d_q q\|w\| + \Psi\left(\|\phi\|_m + d_q q\|w\|\right) \Big\} + TM2\Psi(\|w\|) \\
&\le M\Big\{ \|\phi(0)\| + d_q R + \Psi\left(\|\phi\|_m + d_q qR\right) + T2\Psi(R)\Big\} := r_2.
\end{aligned} \tag{3.10}$$

Similarly, for $t \in J_k^1$, we have that

$$\begin{aligned}
\|\mathcal{S}(w)(t)\| \;&\le \int_{s_k}^t \|W(t,s)\| \Big[\|\mathbf{A}(s)g(s,w_s)\| + \|\mathfrak{F}(s,w_s)\|\Big]\, ds \\
&\le \int_{s_k}^t M\left[\Psi(\|w_s\|) + \Psi(\|w_s\|))\right] ds \\
&\le 2MT\Psi(\|w\|) \le 2MT\Psi(R) := r_3.
\end{aligned} \tag{3.11}$$

Taking the supremum in (3.10), (3.11) and letting $r = r_1 + r_2 + r_3$, boundedness is proved.

Statement 3:$\mathcal{S}$ maps bounded sets onto equicontinuous sets.

Let B_R as in Statement 2, and $w \in B_R$. It is enough to prove that $\mathcal{S}(B_R)$ is equicontinuous on $(0, T]$. For some $0 < \nu_1 < \nu_2 \le t_1$, by **(H1)**-*(iii)*, **(H2)**-*(iii)*, *(iv)*, *(vi)*, and

Lemma 5, the following estimate holds:

$$\begin{aligned}
\|\mathcal{S}(w)(\nu_2)-\mathcal{S}(w)(\nu_1)\| \ \le\ & \|W(\nu_2,0)-W(\nu_1,0)\|\Big\{\|\phi(0)\|+\|\zeta(w_{\lambda_1},\ldots,w_{\lambda_q})(0)\| \\
& +\|g\left(0,\phi-\zeta(w_{\lambda_1},\ldots,w_{\lambda_q})\right)\|\Big\} \\
& +\int_0^{\nu_1}\|W(\nu_2,s)-W(\nu_1,s)\|\Big\{\ \|\mathbf{A}(s)g(s,w_s)\| \\
& +\|\mathfrak{F}(s,w_s)\|\ \Big\}ds \\
& +\int_{\nu_1}^{\nu_2}\|W(\nu_2,s)\|\left[\|\mathbf{A}(s)g(s,w_s)\|+\|\mathfrak{F}(s,w_s)\|\right]ds \\
\le\ & \|W(\nu_2,0)-W(\nu_1,0)\|\Big\{\|\phi(0)\|+d_q q\|w\| \\
& +\Psi(\|\phi\|_m+d_q q\|w\|)\Big\}+2M\Psi(\|w\|)(\nu_2-\nu_1) \\
& +2\Psi(\|w\|)\int_0^{\nu_1}\|W(\nu_2,s)-W(\nu_1,s)\|\,ds \\
\le\ & \|W(\nu_2,0)-W(\nu_1,0)\|\Big\{\|\phi(0)\|+d_q qR \\
& +\Psi(\|\phi\|_m+d_q qR)\Big\}+2M\Psi(R)(\nu_2-\nu_1) \\
& +2\Psi(R)\int_0^{\nu_1}\|W(\nu_2,s)-W(\nu_1,s)\|\,ds. \qquad (3.12)
\end{aligned}$$

Similarly, for every ν_1, ν_2 such that $s_k < \nu_1 < \nu_2 < t_{k+1}, k = 1, ..., p$, it follows that

$$\begin{aligned}
\|\mathcal{S}(w)(\nu_2)-\mathcal{S}(w)(\nu_1)\| \ \le\ & \int_{s_k}^{\nu_1}\|W(\nu_2,s)-W(\nu_1,s)\|\Big\{\ \|\mathbf{A}(s)g(s,w_s)\| \\
& +\|\mathfrak{F}(s,w_s)\|\ \Big\}ds \\
& +\int_{\nu_1}^{\nu_2}\|W(\nu_2,s)\|\left[\|\mathbf{A}(s)g(s,w_s)\|+\|\mathfrak{F}(s,w_s)\|\right]ds \\
\le\ & 2\Psi(\|w\|)\int_{s_k}^{\nu_1}\|W(\nu_2,s)-W(\nu_1,s)\|\,ds \\
& +2M\Psi(\|w\|)(\nu_2-\nu_1) \\
\le\ & 2\Psi(R)\int_{s_k}^{\nu_1}\|W(\nu_2,s)-W(\nu_1,s)\|\,ds \\
& +2M\Psi(R)(\nu_2-\nu_1). \qquad (3.13)
\end{aligned}$$

In (3.12) and (3.13), the continuity and boundedness of $W(t,s)$ yield that, as ν_2 approaches to ν_1, $\|\mathcal{S}(w)(\nu_2)-\mathcal{S}(w)(\nu_1)\|$ goes to zero, independently of w. Therefore $\mathcal{S}(B_R)$ is an equicontinuous family.

Statement 4: The subset $\mathcal{S}(D_\sigma)$ is relatively compact in $\mathcal{PC}_m$.

Without loss of generality we can assume that $t_p \le T$. Let $D_\sigma \subset \mathcal{PC}_m$ be the bounded set defined in (3.5) and let us take a sequence $(w_n)_{n\in\mathbb{N}} \subseteq \mathcal{S}(D_\sigma)$. By Statements 2 and 3, it is bounded and equicontinuous in $\mathcal{PC}_m$. Note that $w_n|_{(-\infty,0]} = \phi$, then by Arzelà-Ascoli theorem applied to $\left(w_n|_{(0,t_1]}\right)_{n\in\mathbb{N}} \subset \mathcal{PC}_m$, there exist an uniformly convergent subsequence

$(w_n^1)_{n\in\mathbb{N}}$ on $(-\infty, t_1]$. Let's consider now the sequence $(w_n^1)_{n\in\mathbb{N}}$ on the interval $(t_1, t_2]$, which is also bounded and equicontinuous. Then, applying Arzelà-Ascoli theorem, it has a convergent subsequence $(w_n^2)_{n\in\mathbb{N}}$ over $(t_1, t_2]$. This sequence is actually an uniformly convergent subsequence of $(w_n)_{n\in\mathbb{N}}$ over $(-\infty, t_2]$. We continue this process iteratively over each interval $(t_2, t_3], \cdots, (t_p, T]$ and finally arrived to the conclusion that the subsequence $(w_n^p)_{n\in\mathbb{N}} \subseteq (w_n)_{n\in\mathbb{N}}$ is uniformly convergent on the whole interval $(-\infty, T]$. This implies that $\overline{\mathcal{S}(D_\sigma)}$ is compact.

Statement 5:The set $\{\mathcal{J}(\cdot, w) : w \in \overline{\mathcal{S}(D_\sigma)}\}$ is comprised of equicontractive operators.

Let $\sigma > 0$, $z, x \in \mathcal{PC}_m$, $w \in \overline{\mathcal{S}(D_\sigma)}$ and $t \in (-\infty, 0]$. Thus, **(H1)**-(*iii*) yields

$$\begin{aligned}\frac{1}{m(t)}\|\mathcal{J}(z,\mathcal{S}(w))(t)-\mathcal{J}(x,\mathcal{S}(w))(t)\|_{\mathbb{R}^n} \leq & \frac{1}{m(t)}\Big\|\zeta\left(z_{\lambda_1},...,z_{\lambda_q}\right)(t)\\ & -\zeta\left(x_{\lambda_1},...,x_{\lambda_q}\right)(t)\Big\|_{\mathbb{R}^n}\\ \leq & \Big\|\zeta\left(z_{\lambda_1},z_{\lambda_2},...,z_{\lambda_q}\right)\\ & -\zeta\left(x_{\lambda_1},x_{\lambda_2},...,x_{\lambda_q}\right)\Big\|_m\\ \leq & d_q\|\tilde{z}-\tilde{x}\|_{mq} \leq Md_q q\|z-x\|. \end{aligned} \tag{3.14}$$

If $t \in J_0^1$, we have that

$$\begin{aligned}\|\mathcal{J}(z,\mathcal{S}(w))(t)-\mathcal{J}(x,\mathcal{S}(w))(t)\|_{\mathbb{R}^n} &= \|g(t,z_t)-g(t,x_t)\|\\ &\leq \gamma\|z_t-x_t\|_m \leq \gamma\|z-x\|.\end{aligned} \tag{3.15}$$

Moreover, if $t \in J_k^1$, we have

$$\begin{aligned}\|\mathcal{J}(z,\mathcal{S}(w))(t)-\mathcal{J}(x,\mathcal{S}(w))(t)\|_{\mathbb{R}^n} \leq & M\Big\{\left\|\Gamma_k(s_k,z(t_k^-)-\Gamma_k(s_k,x(t_k^-)))\right\|\\ & +\left\|g(s_k,z_{s_k})-g(s_k,x_{s_k})\right\|\Big\}\\ & +\|g(t,z_t)-g(t,x_t)\|\\ \leq & ML\left\|z(t_k^-)-x(t_k^-)\right\|_{\mathbb{R}^n}+M\gamma\left\|z_{s_k}-x_{s_k}\right\|_m\\ & +\gamma\left\|z_t-x_t\right\|_m\\ \leq & ML\|z-x\|+M\gamma\|z-x\|+\gamma\|z-x\|\\ = & [M(L+\gamma)+\gamma]\|z-x\|. \end{aligned} \tag{3.16}$$

Finally, for $t \in J_k^2$,

$$\begin{aligned}\|\mathcal{J}(z,\mathcal{S}(w))(t)-\mathcal{J}(x,\mathcal{S}(w))(t)\|_{\mathbb{R}^n} = &\leq \left\|\Gamma_k(t,z(t_k^-))-\Gamma_k(t,x(t_k^-))\right\|\\ &\leq L\|z(t_k^-)-x(t_k^-)\| \leq L\|z-x\|.\end{aligned} \tag{3.17}$$

Combining (3.14)-(3.17), by **(H1)**-(*iii*) it follows that

$$\|\mathcal{J}(z,\mathcal{S}(w))-\mathcal{J}(x,\mathcal{S}(w))\| < \frac{1}{2}\|z-x\|.$$

Hence, $\mathcal{J}$ is a contraction on the first variable, independently of w.

Statement 6: For $\sigma > 0$ and D_σ given as (3.5), $\mathcal{J}(\cdot, \mathcal{S}(\cdot))(D_\sigma) \subseteq D_\sigma$.
Take an arbitrary $z \in D_\sigma$, for $t \in (-\infty, 0]$, **(H3)** yields

$$\begin{aligned} \frac{1}{m(t)} \|\mathcal{J}(z, \mathcal{S}(z))(t) - \tilde{\psi}(t)\|_{\mathbb{R}^n} &= \frac{1}{m(t)} \|\zeta(z_{\lambda_1}, \dots, z_{\lambda_q})(t)\|_{\mathbb{R}^n} \\ &\leq d_q \|\tilde{z}\|_{mq} \leq d_q q \|z\| \leq d_q q(\|\tilde{\psi}\| + \sigma) < \sigma. \end{aligned} \tag{3.18}$$

Similarly, $t \in J_0^1$ imply

$$\begin{aligned} \|\mathcal{J}(z, \mathcal{S}(z))(t) - \tilde{\psi}(t)\|_{\mathbb{R}^n} \leq & \|W(t,0)\| \Big\{ \|\zeta(z_{\lambda_1}, \dots, z_{\lambda_q})\|_{\mathfrak{S}} \\ & + \|g(0, \phi - \zeta(z_{\lambda_1}, \dots, z_{\lambda_q}))\| \Big\} \\ & + \int_0^t \|W(t,s)\| \Big[\|\mathbf{A}(s) g(s, z_s)\| + \|\mathfrak{F}(s, z_s)\| \Big] \, ds \\ & + \|g(t, z_t)\| \\ \leq & M \Big\{ d_q q \|z\| + \Psi(\|\phi\| + d_q q \|z\|) \Big\} + 2MT\Psi(\|z\|) + \Psi(\|z\|) \\ \leq & M \Big\{ d_q q(\|\tilde{\psi}\| + \sigma) + \Psi \Big(\|\tilde{\psi}\| + d_q q(\|\tilde{\psi}\| + \sigma) \Big) \Big\} \\ & + (2MT + 1) \Psi \Big(\|\tilde{\psi}\| + \sigma \Big) < \sigma. \end{aligned} \tag{3.19}$$

Likewise, $t \in J_k^1$, gives

$$\begin{aligned} \|\mathcal{J}(z, \mathcal{S}(z))(t) - \tilde{\psi}(t)\|_{\mathbb{R}^n} \leq & \|W(t,0)\| \Big\{ \|\Gamma_k(s_k, z(t_k^-))\| + \Big\| g(s_k, z_{s_k}) \Big\| \Big\} \\ & + \int_0^t \|W(t,s)\| \Big[\|\mathbf{A}(s) g(s, z_s)\| + \|\mathfrak{F}(s, z_s)\| \Big] \, ds \\ & + \|g(t, z_t)\| \\ \leq & M \Big\{ L(\|\tilde{\psi}\| + \sigma) + \Theta + \Psi(\|\tilde{\psi}\| + \sigma) \Big\} \\ & + (2MT + 1) \Psi(\|\tilde{\psi}\| + \sigma). \end{aligned} \tag{3.20}$$

Additionally, if $t \in J_k^2$, then

$$\begin{aligned} \|\mathcal{J}(z, \mathcal{S}(z))(t) - \tilde{\psi}(t)\|_{\mathbb{R}^n} &= \|\Gamma_k(t, z(t_k^-))\| \leq L\|z\| + \Theta \\ &\leq L(\|\tilde{\psi}\| + \sigma) + \Theta < \sigma. \end{aligned} \tag{3.21}$$

Thus, by taking the supremum in equations (3.18)-(3.21) gives

$$\Big\| \mathcal{J}(z, \mathcal{S}(z)) - \tilde{\psi} \Big\| \leq \sigma.$$

Applying Theorem 13, it follows $\mathcal{J}(z, \mathcal{S}(z)) = z$, i.e., there exists a fix-point solution $z \in D_\sigma \subset \mathcal{PC}_m$, equivalent to the system solution (3.1) given by Proposition 3.3. □

The following theorem proves the uniqueness of the solution for system (3.1).

Theorem 19. *Assuming* **(H1)-(H4)**, *system* (3.1) *has a unique solution on* $(-\infty, T]$.

Proof. Consider two solutions v^1 and v^2 to (3.1) which satisfy (3.3). Let $\sigma > 0$ such that $v^1, v^2 \in D_\sigma$. Then, for $t \in (-\infty, 0]$, we have

$$\begin{aligned}\frac{1}{m(t)}\|v^1(t) - v^2(t)\|_{\mathbb{R}^n} &= \frac{1}{m(t)}\|\zeta(v^2_{\lambda_1}, \cdots, v^2_{\lambda_q})(t) - \zeta(v^1_{\lambda_1}, \cdots, v^1_{\lambda_q})(t)\| \\ &\leq d_q\|\tilde{v^2} - \tilde{v^1}\|_{mq} \leq d_q q\|v^2 - v^1\| < \frac{1}{2}\|v^2 - v^1\|. \qquad (3.22)\end{aligned}$$

If $t \in (0, t_1]$, we get that

$$\begin{aligned}\left\|v^2(t) - v^1(t)\right\| &\leq \|W(t,0)\|\Big\{ \left\|\zeta(v^1_{\lambda_1}, \ldots, v^1_{\lambda_q}) - \zeta(v^2_{\lambda_1}, \ldots, v^2_{\lambda_q})\right\|_m \\ &\quad + \left\|g\left(0, \phi - \zeta(v^1_{\lambda_1}, \ldots, v^1_{\lambda_q})\right) - g\left(0, \phi - \zeta(v^2_{\lambda_1}, \ldots, v^2_{\lambda_q})\right)\right\| \Big\} \\ &\quad + \int_0^t \|W(t,s)\|\Big\{\|\mathbf{A}(s)g(s, v^2_s) - \mathbf{A}(s)g(s, v^1_s)\| \\ &\quad + \|\mathfrak{F}(s, v^2_s) - \mathfrak{F}(s, v^1_s)\|\Big\}ds + \left\|g(t, v^2_t) - g(t, v^1_t)\right\| \\ &\leq M[d_q\|\tilde{v}^2 - \tilde{v}^1\|_{mq} + \gamma d_q\|\tilde{v}^2 - \tilde{v}^1\|_{mq}] \\ &\quad + 2M\int_0^t \mathcal{K}(\|v^2_s\|_m, \|v^1_s\|_m)\|v^2_s - v^1_s\|_m ds + \gamma\left\|v^2_t - v^1_t\right\|_m \\ &\leq M\left[d_q q\|v^2 - v^1\| + \gamma d_q q\|v^2 - v^1\|\right] \\ &\quad + 2MT\mathcal{K}(\|v^2\|, \|v^1\|)\|v^2 - v^1\| + \gamma\|v^2 - v^1\| \\ &\leq M\left\{d_q q(1+\gamma) + 2T\mathcal{K}(\|\tilde{\psi}\| + \sigma, \|\tilde{\psi}\| + \sigma)\right\}\|v^2 - v^1\| + \gamma\|v^2 - v^1\|. \end{aligned} \qquad (3.23)$$

If $t \in J^1_k$, then

$$\begin{aligned}\left\|v^2(t) - v^1(t)\right\| &\leq \|W(t,s_k)\|\Big\{ \left\|\Gamma_k(s_k, v^2(t_k^-)) - \Gamma_k(s_k, v^1(t_k^-))\right\| \\ &\quad + \left\|g(s_k, v^1_{s_k}) - g(s_k, v^2_{s_k})\right\| \Big\} \\ &\quad + \int_{s_k}^t \|W(t,s)\|\Big\{\|\mathbf{A}(s)g(s, v^2_s) - \mathbf{A}(s)g(s, v^1_s)\| \\ &\quad + \|\mathfrak{F}(s, v^2_s) - \mathfrak{F}(s, v^1_s)\|\Big\}ds + \left\|g(t, v^2_t) - g(t, v^1_t)\right\| \\ &\leq M[L\|v^2(t_k^-) - v^1(t_k^-)\|_{\mathbb{R}^n} + \gamma\|v^2_{s_k} - v^1_{s_k}\|_m] \\ &\quad + 2MT\mathcal{K}(\|v^2_s\|_m, \|v^1_s\|_m)\|v^2_s - v^1_s\|_m + \gamma\left\|v^2_t - v^1_t\right\|_m \\ &\leq M\left[L\|v^2 - v^1\| + \gamma\|v^2 - v^1\|\right] \\ &\quad + 2MT\mathcal{K}(\|v^2\|, \|v^1\|)\|v^2 - v^1\| + \gamma\|v^2 - v^1\| \\ &\leq M\left\{L + \gamma + 2T\mathcal{K}(\|\tilde{\psi}\| + \sigma, \|\tilde{\psi}\| + \sigma)\right\}\|v^2 - v^1\| + \gamma\|v^2 - v^1\|. \end{aligned} \qquad (3.24)$$

Lastly, if $t \in J_k^2$, then

$$\begin{aligned}\|v^2(t) - v^1(t)\| &= \|\Gamma_k(t, v^2(t_k^-)) - \Gamma_k(t, v^1(t_k^-))\| \\ &\leq L\|v^2 - v^1\| < \frac{1}{2}\|v^2 - v^1\|. \end{aligned} \tag{3.25}$$

Therefore, taking the sup limit of equations (3.22)-(3.25) and **(H4)** imply that there exists a constant ω, with $0 < \omega < 1$, such that

$$\|v^2 - v^1\| \leq \omega\|v^2 - v^1\|.$$

Hence, $v^1 = v^2$. □

3.1.2 Global Lipschitz conditions

This subsection will assume stronger hypotheses on the nonlinear terms that allow us to apply the Banach contraction theorem. Specifically, we will suppose that the nonlinear functions in our system are globally Lipschitz. Moreover, we are going to consider the following more straightforward system

$$\begin{aligned}\frac{d}{dt}[v(t) - g(t, v_t)] &= \mathbf{A}(t)v(t) + \mathfrak{F}(t, v_t), \quad t \in J_k^1, k = 0, 1, ..., \\ v(t) &= \Gamma_k(t, v(t_k^-)), \quad t \in J_k^2, k = 1, 2, \ldots, \\ v(s) &= \eta(v)(s) + \phi(s), \quad s \in (-\infty, 0], \end{aligned} \tag{3.26}$$

where the non-local condition $v(s) = \eta(z)(s) + \phi(s), \quad s \in (-\infty, 0]$ means

$$v(s) = \eta\left(v\big|_{(-\infty,0]}\right)(s) + \phi(s), \quad s \in (-\infty, 0].$$

The remaining terms are the same as in system (3.1). Now, suppose that the following global Lipschitz conditions on the nonlinear terms hold

(L1) There exist positive constants L_g and $L_{\mathfrak{F}}$ such that for all $t \in [0, T]$, $\phi, \tilde{\phi} \in \mathscr{C}_m$

$$\begin{aligned}\|g(t, \phi) - g(t, \tilde{\phi})\| &\leq L_g\|\phi - \tilde{\phi}\|_m, \\ \|\mathfrak{F}(t, \phi) - \mathfrak{F}(t, \tilde{\phi})\| &\leq L_{\mathfrak{F}}\|\phi - \tilde{\phi}\|_m.\end{aligned}$$

(L2) There exists $L_G \geq 0$, for all $k = 1, 2, \ldots, p$ such that

$$\|\Gamma_k(t, z) - \Gamma_k(t, \tilde{z})\| \leq L_G\|z - \tilde{z}\|_{\mathbb{R}^n}, \quad t \in [0, \infty), z, \tilde{z} \in \mathbb{R}^n.$$

(L3) There exists $L_\eta \geq 0$ such that

$$\|\eta(\phi) - \eta(\psi)\|_m \leq L_\eta\|\phi - \psi\|_m, \quad \phi, \psi \in \mathscr{C}_m.$$

(L4)

$$L_g + M[L_\eta + L_G + L_g + L_g L_\eta + \|\mathbf{A}\|L_g T + L_{\mathfrak{F}} T] < 1,$$

where $\|\mathbf{A}\| = \max\{\|\mathbf{A}(t)\| : t \in [0, T]\}$.

Proposition 5. *Let $\phi \in \mathscr{C}_m$. Then v is solution of system (3.26) if and only if v satisfies the integral equation*

$$v(t) = \begin{cases} W(t,0)\left[\eta(v)(0) + \phi(0) - g(0, \eta(v)(0) + \phi(0))\right] \\ + \int_0^t W(t,s)\left[\mathbf{A}(s)g(s,v_s) + \mathfrak{F}(s,v_s)\right] ds + g(t,v_t), \quad t \in [0,t_1] \\ W(t,s_k)\left[\Gamma_k(s_k, v(t_k^-)) - g(s_k, v_{s_k})\right] \\ + \int_{s_k}^t W(t,s)\left[\mathbf{A}(s)g(s,v_s) + \mathfrak{F}(s,v_s)\right] ds + g(t,v_t), \quad t \in J_k^1, k = 1,\dots,p \\ \Gamma_k(t, v(t_k^-)), \quad t \in J_k^2, \quad k = 1,\dots,p \\ \eta(v)(t) + \phi(t), \quad t \in (-\infty, 0]. \end{cases} \tag{3.27}$$

Theorem 20. *Suppose that* **(L1)**-**(L4)** *hold. Then for $\phi \in \mathscr{C}_m$ the system (3.27) has a unique solution defined on $[0,T]$.*

Proof. Let us define the following operator $\mathcal{P} : \mathcal{P}C_m \longrightarrow \mathcal{P}C_m$, given by

$$(\mathcal{P}v)(t) = \begin{cases} W(t,0)\left[\eta(v)(0) + \phi(0) - g(0, \eta(v)(0) + \phi(0))\right] \\ + \int_0^t W(t,s)\left[\mathbf{A}(s)g(s,v_s) + \mathfrak{F}(s,v_s)\right] ds + g(t,v_t), \quad t \in [0,t_1] \\ W(t,s_k)\left[\Gamma_k(s_k, v(t_k^-)) - g(s_k, v_{s_k})\right] \\ + \int_{s_k}^t W(t,s)\left[\mathbf{A}(s)g(s,v_s) + \mathfrak{F}(s,v_s)\right] ds + g(t,v_t), \quad t \in I_k, k = 1,\dots,p \\ \Gamma_k(t, v(t_k^-)), \quad t \in J_k, \quad k = 1,\dots,p \\ \eta(v)(t) + \phi(t), \quad t \in (-\infty, 0]. \end{cases} \tag{3.28}$$

If $t \in (-\infty, 0]$, then

$$\begin{aligned} \|(\mathcal{P}v)(t) - (\mathcal{P}\tilde{v})(t)\| = \|\eta(v)(t) - \eta(\tilde{v})(t)\| &\le \left\|(\eta(v) - \eta(\tilde{v}))|_{(-\infty,0]}\right\|_m \\ &\le L_\eta \|(v - \tilde{v})|_{(-\infty,0]}\|_m \le L_\eta \|v - \tilde{v}\|. \end{aligned}$$

For $t \in (0, t_1]$, we have that

$$\begin{aligned} \|(\mathcal{P}v)(t) - (\mathcal{P}\tilde{v})(t)\| \le & \|g(t,v_t) - g(t,\tilde{v})\| + \|W(t,s)\|\Big[\|\eta(v)(0) - \eta(\tilde{v})(0)\| \\ & + \|g(0,\eta(v)(0) + \phi(0)) - g(0,\eta(\tilde{v})(0) + \phi(0))\|\Big] \\ & + \int_0^t \|W(t,s)\|\|\mathbf{A}\|\|g(s,v_s) - g(s,\tilde{v}_s)\| ds \\ & + \int_0^t \|W(t,s)\|\|\mathfrak{F}(s,v_s) - \mathfrak{F}(s,\tilde{v}_s)\| ds \\ \le & L_g\|v_t - \tilde{v}_t\|_m + M\left[L_\eta\|v - \tilde{v}\| + L_g\|\eta(v) - \eta(\tilde{v})\|_m\right] \\ & + M\|\mathbf{A}\|L_g \int_0^t \|v_s - \tilde{v}_s\| ds + M L_{\mathfrak{F}} \int_0^t \|v_s - \tilde{v}_s\| ds \\ \le & \left(L_g + M\left[L_\eta + L_g L_\eta + \|\mathbf{A}\|L_g T + L_{\mathfrak{F}} T\right]\right)\|v - \tilde{v}\|. \end{aligned}$$

If $t \in J_k^1$, then

$$\begin{aligned}\|(\mathcal{P}v)(t) - (\mathcal{P}\tilde{v})(t)\| \leq & \|g(t, v_t) - g(t, \tilde{v}_t)\| + \|W(t, s_k)\| \Big\{ \|\Gamma_k(s_k, v(t_k^-)) \\ & - \Gamma_k(s_k, \tilde{v}(t_k^-))\| + \|g(s_k, v_{s_k}) - g(s_k, \tilde{v}_{s_k})\| \Big\} \\ & + \int_{s_k}^t \|W(t,s)\| \|\mathbf{A}\| \|g(s, v_s) - g(s, \tilde{v}_s)\| ds \\ & + \int_{s_k}^t \|W(t,s)\| \|\mathfrak{F}(s, v_s) - \mathfrak{F}(s, \tilde{v}_s)\| ds \\ \leq & L_g \|v_t - \tilde{v}_t\|_m + M \left[L_G \|v - \tilde{v}\| + L_g \|v_{s_k} - \tilde{v}_{s_k}\|_m \right] \\ & + M \|\mathbf{A}\| L_g \int_0^t \|v_s - \tilde{v}_s\| ds + M L_{\mathfrak{F}} \int_0^t \|v_s - \tilde{v}_s\| ds \\ \leq & \Big(L_g + M \left[L_G + L_g + \|\mathbf{A}\| L_g T + L_{\mathfrak{F}} T \right] \Big) \|v - \tilde{v}\|.\end{aligned}$$

For $t \in J_k^2$, we obtain that

$$\|(\mathcal{P}v)(t) - (\mathcal{P}\tilde{v})(t)\| \leq L_G \|v(t_k^-) - \tilde{v}(t_k^-)\| \leq L_G \|v - \tilde{v}\|.$$

Therefore, from the preceding inequalities, the operator $\mathcal{P}$ satisfies all the assumptions of Theorem 12, and thus $\mathcal{P}$ has only one fixed point in the space $\mathcal{P}C_m$, which is the solution of problem (3.26). This completes the proof. □

Example 3. *Let $\phi \in \mathscr{C}_m$ and consider the following system*

$$\begin{cases} \frac{d}{dt}\left[v(t) - \left(1 + \frac{\tan v(t)}{8(t+10)^2} \right) \right] = -v(t) + e^{-\frac{v(t)}{10(t+5)^3}}, & t \in \bigcup_{k=0}^1 J_k^1, \\ v(t) = v(t_k^-) + 1 + \frac{\cos\left(v(t_k^-)\right)}{4(t+8)^4}, & t \in \bigcup_{k=1}^2 J_k^2, \\ v(s) = \left(1 + \frac{\sin v}{30^2}\right)(s) + \phi(s), & s \in (-\infty, 0]. \end{cases} \tag{3.29}$$

Here $J_0^1 = \left[0, \frac{3}{2}\right], J_1^2 = \left(\frac{3}{2}, \frac{5}{2}\right], J_1^1 = \left(\frac{5}{2}, \frac{9}{2}\right], J_2^2 = (\frac{9}{2}, \frac{11}{2}]$ and $T = 7$. Define the functions $g(t,v) = 1 + \frac{\tan(v)}{8(t+10)^2}$, $\mathfrak{F}(t,v) = e^{-\frac{v}{10(t+5)^3}}$, $\eta(v) = 1 + \frac{\sin(v)}{30^2}$, $\Gamma(t,v) = 1 + \frac{\cos(v)}{4(t+8)^4}$ and $\mathbf{A}(t) = -1$. Then we have,

$$\begin{aligned} |g(t,v) - g(t,\tilde{v})| &= \tfrac{1}{8(t+10)^2} |\tan(v) - \tan(\tilde{v})| \leq \tfrac{1}{8 \cdot 10^2} |v - \tilde{v}|, \\ |\mathfrak{F}(t,v) - \mathfrak{F}(t,\tilde{v})| &= |e^{-\frac{v}{10(t+5)^3}} - e^{-\frac{\tilde{v}}{10(t+5)^3}}| \leq \tfrac{1}{10 \cdot 5^3} |v - \tilde{v}|, \\ |\Gamma(t,v) - \Gamma(t,\tilde{v})| &= \tfrac{1}{4(t+8)^4} |\cos(v) - \cos(\tilde{v})| \leq \tfrac{1}{4 \cdot 8^4} |v - \tilde{v}|, \\ |\eta(v) - \eta(\tilde{v})| &= \tfrac{1}{30^2} |\sin(v) - \sin(\tilde{v})| \leq \tfrac{1}{30^2} |v - \tilde{v}|, \end{aligned}$$

and

$$L_g + M[L_\eta + L_G + L_g + L_g L_\eta + \|\mathbf{A}\| L_g T + L_{\mathfrak{F}} T] < 1.$$

Hence, the conditions **(L1)**-**(L4)** *are satisfied.*

3.2 Controllability

In this section, the approximate and exact controllability of system (1.2) is obtained. In the first case, we apply the technique developed by Bashirov & Ghahramanlou [14], which avoids the use of fixed point theorems. In the latter case, we transform the controllability problem to a fixed point one. Then, we apply Rothe's fixed point theorem to establish the desired result. In both cases, we assume that the associated linear system is controllable.

3.2.1 Approximate Controllability

Let us consider the following controlled neutral system with non-instantaneous impulses, non-local conditions, and infinite delay

$$\begin{aligned} \frac{d}{dt}[v(t) - g(t, v_t)] &= \mathbf{A}(t)v(t) + \mathbf{B}(t)u(t) + f(t, v_t, u(t)), \quad t \in \bigcup_{k=0}^{N} J_k^1, \\ v(t) &= \Gamma_k(t, v(t_k^-), u(t_k^-)), \quad t \in J_k^2, \; k = 1, \ldots, N, \\ v(s) + \zeta(v_{\lambda_1}, \ldots, v_{\lambda_q})(s) &= \phi(s) \quad s \in (-\infty, 0], \end{aligned} \tag{3.30}$$

where $u \in L^2([0,T]; \mathbb{R}^m)$. A detailed description of eq. (3.30) is given in (1.2). Assume that the nonlinear terms in (3.30) are smooth enough. Then, there exists a solution $v(\cdot)$ on $(-\infty, T]$ if, and only if, $v(\cdot)$ satisfies the following integral equation

$$v(t) = \begin{cases} W(t,0)\left[\phi(0) - \zeta(v_{\theta_1}, ..., v_{\theta_q})(0) - g(0, \phi - \zeta(v_{\theta_1}, ..., v_{\theta_q}))\right] \\ + \int_0^t W(t,s)\left[\mathbf{A}(s)g(s, v_s) + f(s, v_s, u(s))\right] ds + g(t, v_t) \\ + \int_0^t W(t,s)\mathbf{B}(s)u(s)ds, \quad t \in [0, t_1], \\ W(t, s_k)\left[\Gamma_k(s_k, v(t_k^-), u(t_k^-)) - g(s_k, v_{s_k})\right] \\ + \int_{s_k}^t W(t,s)\left[\mathbf{A}(s)g(s, v_s) + f(s, v_s, u(s))\right] ds + g(t, v_t) \\ + \int_{s_k}^t W(t,s)\mathbf{B}(s)u(s)ds, \quad t \in J_k^1 = (s_k, t_{k+1}], \quad k = 1, \ldots, N, \\ \Gamma_k(t, v(t_k^-), u(t_k^-)), \qquad t \in J_k^2 = (t_k, s_k], \quad k = 1, \ldots, N, \\ \phi(t) - \zeta(v_{\theta_1}, ..., v_{\theta_q})(t), \quad t \in (-\infty, 0]. \end{cases} \tag{3.31}$$

Definition 19. *We say that system (3.30) is approximate controllable on $[0,T]$ if for any $\phi \in \mathscr{C}_m, v_1 \in \mathbb{R}^n$ and $\varepsilon > 0$, there exists a control $u \in L^2([0,T]; \mathbb{R}^m)$ such that*

$$v(0) = -\zeta(v_{\lambda_1}, \ldots, v_{\lambda_q})(0) + \phi(0) \text{ and } \|v(T) - v_1\| < \varepsilon.$$

Before proving the approximate controllability of system (3.30), we recall some results of section 2.3. Let $\delta \in (0, T)$. Then, consider the non-autonomous controlled linear system with initial value $x_0 \in \mathbb{R}^n$ at $T - \delta$, namely,

$$\begin{aligned} x'(t) &= \mathbf{A}(t)x(t) + \mathbf{B}(t)x(t)u(t), \quad t \in [T-\delta, T], \\ x(T-\delta) &= x_0. \end{aligned} \tag{3.32}$$

A control w^δ steering the system (3.32) from the initial condition x_0 to the final state $x(T) = x_1$ is given by

$$w^\delta(t) = \mathbf{B}^*(t)W^*(T,t)(\Theta_{[T-\delta,T]})^{-1}(x_1 - W(T,T-\delta)x_0), \quad t \in [T-\delta, T]. \tag{3.33}$$

In order to prove the approximate controllability, we assume the following hypotheses:

(C1) The nonlinear terms g, f and $\Gamma, k = 1, ..., N$ are smooth enough so that the system (3.30) admits a unique solution given by (3.31).

(C2) The functions g and f satisfy

$$\|g(t,\varphi)\| \leq \alpha(\|\varphi(-s_N)\|) \text{ and } \|f(t,\varphi,\mu)\| \leq \beta\left(\|\varphi(-s_N)\|\right) \tag{3.34}$$

where $\alpha, \beta : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ are continuous functions.

(C3) For $\delta \in (0,T)$, the linear system (3.32) is assumed to be exact controllable on $[T-\delta, T]$.

Theorem 21. *Under the assumptions* **(C1)-(C3)**, *the semilinear neutral system with non-instantaneous impulses, non-local conditions and infinite delay* (3.30) *is approximate controllable on* $[0,T]$.

Proof. Let $\phi \in \mathscr{C}_m, v_1 \in \mathbb{R}^n$ and $\varepsilon > 0$, we must find a control u such that the solution $v(t) = v(t,\phi,v_1,u)$ of the semilinear system (3.30) satisfies $\|v(T) - v_1\| < \varepsilon$. Indeed, consider any fixed control $u \in L^2([0,T];\mathbb{R}^m)$ and δ such that $0 < \delta < \min\{T - s_N, s_N, \varepsilon/MN\}$, where

$$M = \sup_{s\in[0,T]}\{\|W(T,s)\|\|\mathbf{A}(s)\|\}, \quad N = \sup_{s\in[0,T]}\{\alpha(\|v(s)\|) + \beta(\|v(s)\|)\}.$$

Now, define the control u^δ as follows

$$u^\delta(t) = \begin{cases} u(t), & \text{if } \ 0 \leq t \leq T-\delta, \\ w^\delta(t), & \text{if } \ T-\delta < t \leq T, \end{cases}$$

where

$$w^\delta(t) = \mathbf{B}^*(t)W^*(T,t)(\Theta_{[T-\delta,T]})^{-1}(v_1 - W(T,T-\delta)v_0)(t),$$

with v_0 to be defined later. Since $T - \delta > s_N$, the solution $v^\delta(t) = v(t,\phi,u^\delta)$ at time T can be written as

$$\begin{aligned} v^\delta(T) = W(T,s_N)&\left[\delta_N\left(s_N, v^\delta(t_N^-), u^\delta(t_N^-)\right) - g(s_N, v^\delta_{s_N})\right] + g(T, v^\delta_T) \\ &+ \int_{s_N}^T W(T,s)[\mathbf{A}(s)g(s,v^\delta_s) + f(s,v^\delta_s,u^\delta(s))]ds + \int_{s_N}^T W(T,s)\mathbf{B}(s)u^\delta(s)ds \end{aligned} \tag{3.35}$$

Adding $W(T,T-\delta)g(T-\delta, v^\delta_{T-\delta}) - W(T,T-\delta)g(T-\delta, v^\delta_{T-\delta}) = 0$ to the righ hand side of (3.35) and by the cocycle property $W(t,s)W(s,l) = W(t,l)$, we have that

$$\begin{aligned}
v^\delta(T) &= g(T, v^\delta_T) \\
&\quad + W(T,T-\delta)\Big\{W(T-\delta, s_N)[\delta_k(s_N, v^\delta(t_N^-), u^\delta(t_N^-)) - g(s_N, v^\delta_{s_N})] \\
&\quad + \int_{s_N}^{T-\delta} W(T-\delta, s)[\mathbf{A}(s)g(s, v^\delta_s) + f(s, v^\delta_s, u^\delta(s))]ds + g(T-\delta, v^\delta_{T-\delta}) \\
&\quad + \int_{T-\delta}^{T} W(T,s)\mathbf{B}(s)u^\delta(s)ds\Big\} \\
&\quad + \int_{T-\delta}^{T} W(T,s)[\mathbf{A}(s)g(s, v^\delta_s) + f(s, v^\delta_s, u^\delta(s))]ds \\
&\quad + \int_{T-\delta}^{T} W(T,s)\mathbf{B}(s)u^\delta(s)ds - W(T,T-\delta)g(T-\delta, v^\delta_{T-\delta})
\end{aligned}$$

Hence,

$$\begin{aligned}
v^\delta(T) &= g(T, v^\delta_T) + W(T,T-\delta)v^\delta(T-\delta) \\
&\quad + \int_{T-\delta}^{T} W(T,s)[\mathbf{A}(s)g(s, v^\delta_s) + f(s, v^\delta_s, w^\delta(s))]ds \\
&\quad + \int_{T-\delta}^{T} W(T,s)\mathbf{B}(s)w^\delta(s)ds - W(T,T-\delta)g(T-\delta, v^\delta_{T-\delta}).
\end{aligned}$$

The solution $x(t) = x(t, v_0, w^\delta)$ of the initial value problem (3.32) at time T, for the control w^δ and the initial condition v_0, is given by:

$$x(T) = W(T,T-\delta)v_0 + \int_{T-\delta}^{T} W(T,s)\mathbf{B}(s)w^\delta(s)ds.$$

Letting $v_0 = v^\delta(T-\delta) + W(T-\delta, T)g(T, v^\delta_T) - g(T-\delta, v^\delta_T)$ and $v_1 = x(T)$, we then have that

$$\begin{aligned}
\|v^\delta(T) - v_1\| &\le \int_{T-\delta}^{T} \|W(T,s)\| \left[\|\mathbf{A}(s)\|\|g(s, v^\delta_s)\| + \|f(s, v^\delta_s, w^\delta(s))\|\right] ds \\
&\le \int_{T-\delta}^{T} M \left[\alpha(\|v^\delta(s-s_N)\|) + \beta(\|v^\delta(s-s_N)\|)\right] ds
\end{aligned}$$

Since $0 < \delta < s_N$ and $T-\delta \le s \le T$, we have that $s - s_N \le T - s_N \le T - \delta$, consequently, $v^\delta(s-s_N) = v(s-s_N)$. Finally, we obtain

$$\|v^\delta(T) - v_1\| \le \int_{T-\delta}^{T} M\left[\alpha(\|v(s-s_N)\|) + \beta(\|v(s-s_N)\|)\right] ds \le \delta M N < \varepsilon.$$

□

Example 4. *In this section we propose the following example of a neutral differential system including non-instantaneous impulses, non-local conditions and infinite delay,*

$$\begin{aligned}
\frac{d}{dt}[v(t) - g(t, v_t)] &= \mathbf{A}(t)v(t) + \mathbf{B}(t)u(t) + f(t, v_t, u(t)), \quad t \in \bigcup_{k=0}^{N} J_k^1, \\
v(t) &= \Gamma_k(t, v(t_k^-), u(t_k^-)), \quad t \in J_k^2, k = 1, \dots, N, \\
v(s) + \zeta(v_{\lambda_1}, \dots, v_{\lambda_q})(s) &= \phi(s) \quad s \in (-\infty, 0].
\end{aligned} \tag{3.36}$$

where $\mathbf{A}(t) = a(t)\mathbf{A}, \mathbf{B}(t) = b(t)\mathbf{B}$, *with* $\mathbf{A}_{n\times n}$ *and* $\mathbf{B}_{n\times n}$ *being constant matrices. Moreover,* $a \in L^1([0,T]), b \in C([0,T])$ *and*

$$\int_0^T a(t)dt \neq 0, \quad b(t) \neq 0, \quad t \in [0,T]. \tag{3.37}$$

From [75], the linear system

$$v'(t) = \mathbf{A}(t)v(t) + \mathbf{B}(t)u(t), \quad t \in [0,T]$$

is exactly controllable in $[0,T]$ *if, and only if, Kalman's rank condition is satisfied, i.e.,*

$$\operatorname{rank}(\mathbf{B}|\mathbf{AB}|\ldots|\mathbf{A}^{n-1}\mathbf{B}) = n. \tag{3.38}$$

Here we assume that condition (3.38) *holds. The nonlinear terms* $f : [0,T]\times\mathscr{C}_m\times\mathbb{R}^m \longrightarrow \mathbb{R}^n$ *and* $g : [0,T]\times\mathscr{C}_m \longleftrightarrow \mathscr{C}_m$ *are given as follows*

$$f(t,\varphi,u) = \begin{pmatrix} \sqrt[3]{\sin\|u\|+1} & \cdot & \sqrt[3]{\varphi_1(-s_N)} \\ \sqrt[3]{\sin\|u\|+1} & \cdot & \sqrt[3]{\varphi_2(-s_N)} \\ \vdots & \cdot & \vdots \\ \sqrt[3]{\sin\|u\|+1} & \cdot & \sqrt[3]{\varphi_n(-s_N)} \end{pmatrix},$$

$$g(t,\varphi) = \begin{pmatrix} \sqrt[3]{\varphi_1(-s_N)} \\ \sqrt[3]{\varphi_2(-s_N)} \\ \vdots \\ \sqrt[3]{\varphi_n(-s_N)} \end{pmatrix}.$$

The functions $\zeta : \mathscr{C}_m^q \longleftrightarrow \mathscr{C}_m$ *and* $\Gamma_k : (t_k, s_k]\times\mathbb{R}^n\times\mathbb{R}^m \longrightarrow \mathbb{R}^n, k = 1, ..., N$, *are defined by*

$$\zeta\left(\varphi_1,\varphi_2,\cdots,\varphi_q\right) = \sum_{i=1}^{q}\begin{pmatrix} \sin\left(\varphi_{i1}\right) \\ \sin\left(\varphi_{i2}\right) \\ \vdots \\ \sin\left(\varphi_{in}\right) \end{pmatrix},$$

$$\Gamma_k(t,v,u) = \cos\left(\sqrt{\|u\|+1}\right)\begin{pmatrix} \sin\left(v_1^k\right) \\ \sin\left(v_2^k\right) \\ \vdots \\ \sin\left(v_n^k\right) \end{pmatrix}.$$

Then

$$\begin{aligned}\|f(t,\varphi,u)\| &\leq \sqrt{n}\|\varphi(-s_N)\|^{2/3} + 2\sqrt{n}\sin\|u\|^{2/3} + 2\sqrt{n} \leq \sqrt{n}\|\varphi(-s_N)\|^{2/3} \\ &+ 3\sqrt{n} := \alpha(\|\varphi(-s_N)\|),\end{aligned}$$

$$\|g(t,\varphi)\| \leq \sqrt{n}\|\varphi(-s_N)\|^{2/3} := \beta(\|\varphi(-s_N)\|).$$

Since ζ *and* $\Gamma_k, k = 1, ..., N$ *are bounded, the conditions* **(C1)-(C3)** *are satisfied. Thus, the system* (3.36) *is approximate controllable in* $[0,T]$.

3.2.2 Exact Controllability

In this part, we obtain the exact controllability of the following neutral problem with non-instantaneous impulses, non-local conditions and infinite delay

$$\begin{aligned} \frac{d}{dt}[v(t)-g(t,v_t)] &= \mathbf{A}(t)v(t)+\mathbf{B}(t)u(t)+f(t,v_t,u(t)), \quad t\in\bigcup_{k=0}^{N} J_k^1,\\ v(t) &= \Gamma_k(t,v(t_k^-)), \quad t\in J_k^2, k=1,\dots,N,\\ v(s)+\zeta(v_{\lambda_1},\dots,v_{\lambda_q})(s) &= \phi(s) \quad s\in(-\infty,0]. \end{aligned} \tag{3.39}$$

The control u in (3.39) belongs to the space of controls $\mathcal{P}C_u := \mathcal{P}C_u((0,T];\mathbb{R}^m)$, given by

$$\mathcal{P}C_u = \left\{u:(0,T]\longrightarrow\mathbb{R}^m : u \text{ is bounded and } u\in C\left(\cup_{k=0}^{N} J_k^1;\mathbb{R}^m\right)\right\},$$

endowed with the norm

$$\|u\| = \sup_{t\in[0,T]}\|u(t)\|_{\mathbb{R}^m}.$$

Definition 20. *We say that system (3.39) is exactly controllable on $[0,T]$ if for any $\phi\in\mathscr{C}_m, v_1\in\mathbb{R}^n$, there exists a control $u\in\mathcal{P}C_u$ such that the solution v of (3.39) verifies*

$$v(0)=-\zeta(v_{\lambda_1},\dots,v_{\lambda_q})(0)+\phi(0) \text{ and } v(T)=v_1.$$

To establish our main result, we list the subsequent assumptions:

(D1) The nonlinear terms are globally Lipschitz, i.e.,

(*i*) $\|\zeta(v)-\zeta(y)\|_m \le L_0\|v-y\|_{mq}, \quad v,y\in\mathscr{C}_m^q,$

(*ii*) $\|g(t,\varphi)-g(t,\phi)\|_{\mathbb{R}^n}\le L_g\|\varphi-\phi\|_m, \quad \varphi,\phi\in\mathscr{C}_m,$

(*iii*) $\|f(t,\varphi,u)-f(t,\phi,w)\|_{\mathbb{R}^n}\le L_f\{\|\varphi-\phi\|+\|u-w\|\}, \quad \varphi,\phi\in\mathscr{C}_m, u,w\in\mathbb{R}^m,$

(*iv*) $\|\Gamma_k(t,v)-\Gamma_k(s,y)\|_{\mathbb{R}^n}\le L_k\{|t-s|+\|v-y\|_{\mathbb{R}^n}\}, k=1,...,N, t,s\in J_k^2, v,y\in\mathbb{R}^n.$

Given a bounded subset E of $\mathcal{P}C_m$, there exist continuous functions $\xi_1:[0,T]\longrightarrow\mathbb{R}_+$, $\xi_2:(-\infty,0]\longrightarrow\mathbb{R}_+$ depending on E such that $\xi_1(0)=\xi_2(0)=0$, and for any $v\in E$, the following holds

(*v*) $\|g(\tau_2,v_{\tau_2})-g(\tau_1,v_{\tau_1})\|\le\xi_1(|\tau_2-\tau_1|)\|v\|_{\mathcal{PC}_{mT}}, \quad \tau_2,\tau_1\in[0,T],$

(*vi*) $\|\zeta(\tilde v)(\tau_2)-\zeta(\tilde v)(\tau_1)\|\le\xi_2(|\tau_2-\tau_1|)\|\tilde v\|_{mq}, \quad \tau_2,\tau_1\in(-\infty,0].$

(D2) (*i*) $\|f(t,\varphi,u)\|\le\alpha_0\|\varphi\|^{a_0}+\beta_0\|u\|^{b_0}+c_0, \quad \varphi\in\mathscr{C}_m, u\in\mathbb{R}^m, t\in[0,T]$

(*ii*) $\|\Gamma_k(t,v)\|\le\alpha_k\|v\|^{a_k}+c_k, \quad k=1,...,N, v\in\mathbb{R}^n, t\in(t_k,s_k]$

(*iii*) $\|\zeta(\tilde v)\|\le\rho\|\tilde v\|^{d_0}, \quad \tilde v\in\mathscr{C}_m^q$

(*iii*) $\|g(t,\varphi)\|\le\|\varphi\|^{d_1}, \quad \varphi\in\mathscr{C}_m, t\in[0,T]$

where $0 \leq a_k < 1, k = 0, ..., N$, $0 \leq b_0 < 1$, $0 \leq d_0 < 1$, $0 \leq d_1 < 1$ and $\alpha_k, \beta_k, c_k, \rho$ are positive constants for $k = 0, ..., N$.

Assume that the nonlinear terms in (3.39) are smooth enough. Then there exists a solution $v(\cdot)$ on $(-\infty, T]$ if, and only if, $v(\cdot)$ satisfies the following integral equation

$$v(t) = \begin{cases} W(t,0)\left[\phi(0) - \zeta(v_{\lambda_1}, ..., v_{\lambda_q})(0) - g(0, \phi - \zeta(v_{\lambda_1}, ..., v_{\lambda_q}))\right] \\ + \int_0^t W(t,s)\left[\mathbf{A}(s)g(s, v_s) + f(s, v_s, u(s))\right] ds + g(t, v_t) \\ + \int_0^t W(t,s)\mathbf{B}(s)u(s)ds, \quad t \in [0, t_1], \\ W(t, s_k)\left[\Gamma_k(s_k, v(t_k^-)) - g(s_k, v_{s_k})\right] \\ + \int_{s_k}^t W(t,s)\left[\mathbf{A}(s)g(s, v_s) + f(s, v_s, u(s))\right] ds + g(t, v_t) \\ + \int_{s_k}^t W(t,s)\mathbf{B}(s)u(s)ds, \quad t \in J_k^1 = (s_k, t_{k+1}], k = 1, \ldots, N, \\ \Gamma_k(t, v(t_k^-)), \qquad t \in J_k^2 = (t_k, s_k], \qquad k = 1, \ldots, N, \\ \phi(t) - \zeta(v_{\lambda_1}, ..., v_{\lambda_q})(t), \quad t \in (-\infty, 0]. \end{cases} \tag{3.40}$$

Suppose for a moment that system (3.39) is controllable, that is, given any $\phi \in \mathscr{C}_m, v^{t_{k+1}} \in \mathbb{R}^n, k = 0, ..., N$, we can find a control $u \in \mathcal{PC}_u$ such that the solution $v(\cdot)$ satisfies

$$v(0) = \phi(0) - \zeta(v_{\lambda_1}, \ldots, v_{\lambda_q})(0) \text{ and } v(t_{k+1}) = v^{t_{k+1}}, \text{ where } v^{t_{N+1}} = v_1.$$

Then, for $t \in [0, t_1]$ we characterize the control

$$u(t) = \Upsilon_0 \mathfrak{S}_0(v, u) := \mathbf{B}^*(t) W^*(t_1, t)(\Theta_{[0,t_1]})^{-1} \mathfrak{S}_0(v, u),$$

where

$$\begin{aligned} \mathfrak{S}_0(v, u) = v^{t_1} - W(t_1, 0)[\phi(0) - \zeta(v_{\lambda_1}, ..., v_{\lambda_q})(0) - g(0, \phi - \zeta(v_{\lambda_1}, ..., v_{\lambda_q}))] \\ - \int_0^{t_1} W(t_1, s)[\mathbf{A}(s)g(s, v_s) + f(s, v_s, u(s))]ds - g(t_1, v_{t_1}). \end{aligned} \tag{3.41}$$

Similarly, for $t \in (s_k, t_{k+1}], k = 1, ..., N$, we sustain the control

$$u(t) = \Upsilon_k \mathfrak{S}_k(v, u) := \mathbf{B}^*(t) W^*(t_{k+1}, t)(\Theta_{[s_k, t_{k+1}]})^{-1} \mathfrak{S}_k(v, u),$$

where

$$\begin{aligned} \mathfrak{S}_k(v, u) = v^{t_{k+1}} - W(t_{k+1}, s_k)[\Gamma_k(s_k, v(t_k^-)) - g(s_k, v_{s_k})] \\ - \int_{s_k}^{t_{k+1}} W(t_{k+1}, s)[\mathbf{A}(s)g(s, v_s) + f(s, v_s, u(s))]ds - g(t_{k+1}, v_{t_{k+1}}). \end{aligned} \tag{3.42}$$

Now, we define the operator $\mathcal{Q} : \mathcal{PC}_m \times \mathcal{PC}_u \longrightarrow \mathcal{PC}_m \times \mathcal{PC}_u$ by the form

$$\mathcal{Q}(v, u) = (\mathcal{Q}_1(v, u), \mathcal{Q}_2(v, u)) = (y, w).$$

Operators $\mathcal{Q}_1 : \mathcal{P}C_m \times \mathcal{P}C_u \longrightarrow \mathcal{P}C_m$ and $\mathcal{Q}_2 : \mathcal{P}C_m \times \mathcal{P}C_u \longrightarrow \mathcal{P}C_u$ are given by

$$y(t) = \mathcal{Q}_1(v,u)(t) = \begin{cases} W(t,0)\left[\phi(0) - \zeta(v_{\lambda_1},...,v_{\lambda_q})(0) - g(0,\phi - \zeta(v_{\lambda_1},...,v_{\lambda_q}))\right] \\ + \int_0^t W(t,s)\left[\mathbf{A}(s)g(s,v_s) + f(s,v_s,u(s))\right]ds + g(t,v_t) \\ + \int_0^t W(t,s)\mathbf{B}(s)u(s)ds, \quad t \in [0,t_1], \\ W(t,s_k)\left[\Gamma_k(s_k, v(t_k^-)) - g(s_k, v_{s_k})\right] \\ + \int_{s_k}^t W(t,s)\left[\mathbf{A}(s)g(s,v_s) + f(s,v_s,u(s))\right]ds + g(t,v_t) \\ + \int_{s_k}^t W(t,s)\mathbf{B}(s)u(s)ds, \quad t \in J_k^1 = (s_k, t_{k+1}], k = 1,\dots,N, \\ \Gamma_k(t, v(t_k^-)), \qquad t \in J_k^2 = (t_k, s_k], \qquad k = 1,\dots,N, \\ \phi(t) - \zeta(v_{\lambda_1},...,v_{\lambda_q})(t), \quad t \in (-\infty, 0]. \end{cases} \tag{3.43}$$

and

$$w(t) = \mathcal{Q}_2(v,u)(t) = \begin{cases} \Upsilon_0 \mathfrak{S}_0(v,u), \quad t \in [0,t_1], \\ \Upsilon_k \mathfrak{S}_k(v,u), \quad t \in (s_k, t_{k+1}], k = 1,...,N, \\ 0, \quad t \in (t_k, s_k], k = 1,...,N. \end{cases} \tag{3.44}$$

Theorem 22. *Assume that conditions* **(D1)-(D2)** *are satisfied. Then the system* (3.39) *is controllable if and only if for all $\phi \in \mathscr{C}_m$ and $v_1 \in \mathbb{R}^n$, the operator $\mathcal{Q}$ has a fixed point, i.e.,*

$$\exists (v,u) \in \mathcal{P}C_m \times \mathcal{P}C_u : \quad \mathcal{Q}(v,u) = (v,u).$$

Now, we are in a position to state and prove the main theorem of this subsection.

Theorem 23. *Suppose conditions* **(D1)-(D2)** *hold and the linear system* (2.45) *is controllable in any interval $[\alpha,\beta]$, with $0 < \alpha < \beta \le T$. Then the semilinear neutral system* (3.39) *is controllable in $[0,T]$. Moreover, for any $\phi \in \mathscr{C}_m$ and $v^{t_{k+1}} \in \mathbb{R}^n, k = 0,...,N$, there exists $u \in \mathcal{P}C_u$ such that the corresponding solution $v(t) = v(t,\phi,u)$ of* (3.39) *fulfills*

$$v(0) = \phi(0) - \zeta(v_{\lambda_1},\dots,v_{\lambda_q})(0) \text{ and } v^{t_{k+1}} = v(t_{k+1}), k = 0,...,N,$$

with

$$v^{t_{N+1}} = v(T) = v_1,$$

and

$$u(t) = \begin{cases} \mathbf{B}^*(t)W^*(t_1,t)(\Theta_{[0,t_1]})^{-1}\mathfrak{S}_0(v,u), \quad t \in [0,t_1], \\ \mathbf{B}^*(t)W^*(t_{k+1},t)(\Theta_{[s_k,t_{k+1}]})^{-1}\mathfrak{S}_k(v,u), \quad t \in (s_k, t_{k+1}], k = 1,...,N, \\ 0, \quad t \in (t_k, s_k], k = 1,...,N, \end{cases}$$

where $\mathfrak{S}_0(v,u)$ and $\mathfrak{S}_0(v,u)$ are the same as defined in (3.41) *and* (3.42)*, respectively.*

Proof. For better readability, the proof of this Theorem will be divided in Statements.
Statement 1: The operator $\mathcal{Q}$ is continuous.

We shall prove that the operators $\mathcal{Q}_1$ and $\mathcal{Q}_2$ are continuous.
Let $t \in [0, t_1]$, then **(D1)**-(i), (ii), (iii) imply

$$\|\mathcal{Q}_1(v,u)(t) - \mathcal{Q}_1(y,w)(t)\| \leq C_1\|v-y\| + C_2\|u-w\|, \tag{3.45}$$

where constants C_1 and C_2 are such that

$$\begin{aligned} C_1 &= M(L_0 q + L_g L_0 q + t_1\|\mathbf{A}\|L_g + t_1 L_f + L_g),\\ C_2 &= M t_1(L_f + \|\mathbf{B}\|). \end{aligned}$$

Let $t \in (s_k, t_{k+1}], k = 1, ..., N$, then **(D1)**-($ii$), ($iii$), ($iv$) yield

$$\|\mathcal{Q}_1(v,u)(t) - \mathcal{Q}_1(y,w)(t)\| \leq D_k\|v-y\| + C_3\|u-w\|, \tag{3.46}$$

where

$$\begin{aligned} D_k &= M(L_k + L_g + t_{k+1}\|\mathbf{A}\|L_g + t_{k+1}L_f),\\ C_3 &= MT(L_f + \|\mathbf{B}\|). \end{aligned}$$

Let $t \in (t_k, s_k], k = 1, ..., N$, from **(D1)**-($iv$), we obtain

$$\|\mathcal{Q}_1(v,u)(t) - \mathcal{Q}_1(y,w)(t)\| \leq L_k\|v-y\|. \tag{3.47}$$

Let $t \in (-\infty, 0]$, then **(D1)**-(i) implies that

$$\|\mathcal{Q}_1(v,u)(t) - \mathcal{Q}_1(y,w)(t)\| \leq L_0 q\|v-y\|. \tag{3.48}$$

The inequalities (3.45),(3.46),(3.47) and (3.48) imply the continuity of $\mathcal{Q}_1$. Likewise, the continuity of $\mathcal{Q}_2$ comes from the continuity of $\mathbf{B}, W, \Theta_{[0,t_1]}, \Theta_{[s_k,t_{k+1}]}, k = 1, ..., N, \mathfrak{S}_0$ and $\mathfrak{S}_k$.

Statement 2: The operator $\mathcal{Q}$ maps bounded sets into equicontinuous sets.

Let $E \subset \mathcal{PC}_m \times \mathcal{PC}_u$ be bounded and consider the sequel:
Let $\tau_1, \tau_2 \in (0, t_1]$ with $0 < \tau_1 < \tau_2 \leq t_1$, then from **(D1)**-($v$), we have that

$$\begin{aligned} \|\mathcal{Q}_1(v,u)(\tau_2) - \mathcal{Q}_1(v,u)(\tau_1)\| \leq& \|W(\tau_2,0) - W(\tau_1,0)\|\Big[\|\phi(0)\| + \Big\|\zeta(v_{\lambda_1},...,v_{\lambda_q})\Big\|\\ &+ \Big\|g(0, \phi - \zeta(v_{\lambda_1},...,v_{\lambda_q}))\Big\|\Big]\\ &+ \int_0^{\tau_1}\|W(\tau_2,s) - W(\tau_1,s)\|\|\mathbf{B}(s)\|\|u(s)\|ds\\ &+ \int_{\tau_1}^{\tau_2}\|W(\tau_2,s)\|\|\mathbf{B}(s)\|\|u(s)\|ds\\ &+ \xi_1(|\tau_2 - \tau_1|)\|v\| + \|W(\tau_2,s) - W(\tau_1,s)\|\\ &\times\left(\int_0^{\tau_1}\|\mathbf{A}(s)g(s,v_s) + f(s,v_s,u(s))\|ds\right)\\ &+ \int_{\tau_1}^{\tau_2}\|W(\tau_2,s)\|\|\mathbf{A}(s)g(s,v_s) + f(s,v_s,u(s))\|ds, \end{aligned}$$

and

$$\begin{aligned} \|\mathcal{Q}_2(v,u)(\tau_2) - \mathcal{Q}_2(v,u)(\tau_1)\| \leq& \Big\|(\Theta_{[0,t_1]})^{-1}\mathfrak{S}_0(v,u)\Big\|\\ &\times\|\mathbf{B}^*(\tau_2)W^*(t_1,\tau_2) - \mathbf{B}^*(\tau_1)W^*(t_1,\tau_1)\|. \end{aligned}$$

Let $s_k < \tau_1 < \tau_2 \leq t_{k+1}, k = 1, ..., N$, then **(D1)**-$(v)$ yields

$$\begin{aligned}\|\mathcal{Q}_1(v,u)(\tau_2) - \mathcal{Q}_1(v,u)(\tau_1)\| \leq & \|W(\tau_2, s_k) - W(\tau_1, s_k)\| \left\|\Gamma_k(s_k, v(t_k^-)) - g(s_k, v_{s_k})\right\| \\ & + \int_{s_k}^{\tau_1} \|W(\tau_2, s) - W(\tau_1, s)\| \|\mathbf{B}(s)\| \|u(s)\| ds \\ & + \int_{\tau_1}^{\tau_2} \|W(\tau_2, s)\| \|\mathbf{B}(s)\| \|u(s)\| ds \\ & + \xi_1(|\tau_2 - \tau_1|)\|v\| + \|W(\tau_2, s) - W(\tau_1, s)\| \\ & \times \left(\int_{s_k}^{\tau_1} \|\mathbf{A}(s)g(s, v_s) + f(s, v_s, u(s))\| ds \right) \\ & + \int_{\tau_1}^{\tau_2} \|W(\tau_2, s)\| \|\mathbf{A}(s)g(s, v_s) + f(s, v_s, u(s))\| ds,\end{aligned}$$

and

$$\begin{aligned}\|\mathcal{Q}_2(v,u)(\tau_2) - \mathcal{Q}_2(v,u)(\tau_2)\| \leq & \left\|(\Theta_{[s_k, t_{k+1}]})^{-1} \mathfrak{S}_k(v,u)\right\| \\ & \times \|\mathbf{B}^*(\tau_2)W^*(t_{k+1}, \tau_2) - \mathbf{B}^*(\tau_1)W^*(t_{k+1}, \tau_1)\|.\end{aligned}$$

Let $t_k < \tau_1 < \tau_2 \leq s_k, k = 1, ..., N$, then

$$\|\mathcal{Q}_1(v,u)(\tau_2) - \mathcal{Q}_1(v,u)(\tau_1)\| = \left\|\Gamma_k(\tau_2, v(t_k^-)) - \Gamma_k(\tau_1, v(t_k^-))\right\| \leq L_k|\tau_2 - \tau_1|.$$

Let $-\infty < \tau_1 < \tau_2 \leq 0$, then from **(D1)**-$(vi)$, we obtain

$$\begin{aligned}\|\mathcal{Q}_1(v,u)(\tau_2) - \mathcal{Q}_1(v,u)(\tau_1)\| \leq & \|\phi(\tau_2) - \phi(\tau_1)\| \\ & + \left\|\zeta(v_{\lambda_1,...,\lambda_q})(\tau_2) - \zeta(v_{\lambda_1,...,\lambda_q})(\tau_1)\right\| \\ \leq & \|\phi(\tau_2) - \phi(\tau_1)\| + \xi_2(|\tau_2 - \tau_1|)\|v\|.\end{aligned}$$

Since $|\tau_2 - \tau_1| \to 0, \xi_1(|\tau_2 - \tau_1|) \to 0, \xi_1(|\tau_2 - \tau_1|) \to 0, \|W(\tau_2, s) - W(\tau_1, s)\| \to 0$ and $\|W(\tau_2, s_k) - W(\tau_1, s_k)\| \to 0$ for all $k = 1, ..., N$ as $\tau_1 \to \tau_2$, the foregoing inequalities imply that $\mathcal{Q}_1(E)$ is equicontinuous. Analogously, since $\|\mathbf{B}^*(\tau_2)W^*(t_1, \tau_2) - \mathbf{B}^*(\tau_1)W^*(t_1, \tau_1)\| \to 0, \|\mathbf{B}^*(\tau_2)W^*(t_{k+1}, \tau_2) - \mathbf{B}^*(\tau_1)W^*(t_{k+1}, \tau_1)\| \to 0$ for all $k = 1, ..., N$ as $\tau_1 \to \tau_2$ and $\mathfrak{S}_0, \mathfrak{S}_k$ are bounded in E, we obtain that $\mathcal{Q}_2(E)$ is equicontinuous.
Statement 3: $\mathcal{Q}(E)$ is relatively compact.

Since the functions g, f, ζ and Γ_k are smooth enough, there are positive constants such that for all $(v,u) \in E$ it pursue that

$$\begin{aligned}& \|g(t, v_t)\| \leq M_0, \|f(t, v_t, u(t))\| \leq M_1, \|\zeta(v)\| \leq M_2, \\ & \left\|(\Theta_{[s_k, t_{k+1}]})^{-1}\mathfrak{S}_k\right\| \leq M_{k+3}, k = 0, ..., N, \\ & \left\|\Gamma_k(t, v(t_k^-))\right\| \leq M_{N+k+3}, k = 1, ..., N.\end{aligned}$$

Thus, $\mathcal{Q}(E)$ is bounded.
Now, let $\{\psi_i = (y_i, w_i) : i \in \mathbb{N}\}$ be a sequence in $\mathcal{Q}(E) \subset \mathcal{PC}_m \times \mathcal{PC}_u$. Subsequently, $(w_i)_{i\in\mathbb{N}}$ is an uniformly and equicontinuous family on $[0, t_1]$. By the Arzelà Ascoli theorem, there is a convergent subsequence $(w_i^1)_{i\in\mathbb{N}} \subseteq (w_i)_{i\in\mathbb{N}}$ on $[0, t_1]$. Since $(w_i^1)_{i\in\mathbb{N}}$ is uniformly

bounded and equicontinuous on $[t_1, t_2]$, then there is a subsequence $(w_i^2)_{i\in\mathbb{N}} \subseteq (w_i^1)_{i\in\mathbb{N}}$ which is convergent on $[t_1, t_2]$. Maintaining this process, the subsequence $(w_i^{N+1})_{i\in\mathbb{N}}$ converges uniformly on each interval $[0, t_1], [t_1, t_2] \cdots [t_N, T]$. On the other hand, since $(y_i)_{i\in\mathbb{N}}$ is contained in $\mathcal{Q}_1(E) \subset \mathcal{PC}_m$, then $y_i\Big|_{(-\infty,-\lambda_q]} = \phi - \zeta(\phi_{\lambda_1}, \phi_{\lambda_2}, ..., \phi_{\lambda_q})$ for $i \in \mathbb{N}$, and $(y_i)_{i\in\mathbb{N}}$ is bounded and equicontinuous on $[-\lambda_q, t_1]$. Arzelà Ascoli theorem implies that there is a subsequence $(y_i^1)_{i\in\mathbb{N}} \subseteq (y_i)_{i\in\mathbb{N}}$ which is convergent on $(-\infty, t_1]$. Moreover, $(y_i^1)_{i\in\mathbb{N}}$ has a convergent subsequence $(y_i^2)_{i\in\mathbb{N}}$ on $[t_1, t_2]$. Continuining with this process, the subsequence $(y_i^{N+1})_{i\in\mathbb{N}}$ converges uniformly on each interval $(-\infty, t_1], [t_1, t_2], \cdots [t_N, T]$. Therefore, the subsequence $\{\psi_i^{N+1} = (y_i^{N+1}, w_i^{N+1})\}$ of $(\psi_i)_{i\in\mathbb{N}}$ is uniformly convergent. Thus, $\mathcal{Q}(E)$ is relatively compact.

Statement 4:The following limit holds

$$\lim_{|||(v,u)|||\to\infty} \frac{|||\mathcal{Q}(v,u)|||}{|||(v,u)|||} = 0$$

where $|||(v,u)||| = \|v\| + \|u\|$ is the norm of the product space $\mathcal{PC}_m \times \mathcal{PC}_u$.

For $\mathfrak{S}_0$ and $\mathfrak{S}_k, k = 1, ..., N$, we have the following

$$\begin{aligned}\|\mathfrak{S}_0(v,u)\| \leq& \|v^{t_1}\| + \|W(t_1,0)\|\|\phi(0) - \zeta(\tilde{v})(0) - g(0, \phi - \zeta(\tilde{v}))\| + \|g(t_1, v_{t_1})\| \\ &+ \int_0^{t_1} \|W(t_1,s)\|\|\mathbf{A}(s)g(s,v_s) + f(s,v_s,u(s))\|ds\end{aligned}$$

Hypotheses **(D2)**-*(i)*, *(iii)*, *(iv)* yield

$$\begin{aligned}\|\mathfrak{S}_0(v,u)\| \leq& \|v^{t_1}\| + M\|\phi(0)\| + M[\rho\|\tilde{v}\|^{d_0} + \|\phi - \zeta(\tilde{v})\|^{d_1}] + \|v_{t_1}\|^{d_1} \\ &+ Mt_1[\|\mathbf{A}\|\|v_s\|^{d_1} + \alpha_0\|v_s\|^{a_0} + \beta_0\|u\|^{b_0} + c_0] \\ \leq& \Lambda_0 + M[\rho q^{d_0}\|v\|^{d_0} + 2^{d_1}\rho^{d_1}q^{d_0 d_1}\|v\|^{d_0 d_1}] + \|v\|^{d_1} \\ &+ Mt_1[\|A\|\|v\|^{d_1} + \alpha_0\|v\|^{a_0} + \beta_0\|u\|^{b_0}],\end{aligned}$$

where $\Lambda_0 = \|v^{t_1}\| + M[\|\phi(0)\| + 2^{d_1}\|\phi\|^{d_1} + t_1 c_0]$.

$$\begin{aligned}\|\mathfrak{S}_k(v,u)\| \leq& \|v^{t_{k+1}}\| + \|W(t_{k+1}, s_k)\|\|\Gamma_k(s_k, v(t_k^-)) - g(s_k, v_{s_k})\| + \|g(t_{k+1}, v_{t_{k+1}})\| \\ &+ \int_{s_k}^{t_{k+1}} \|W(t_{k+1}, s)\|\|\mathbf{A}(s)g(s,v_s) + f(s,v_s,u(s))\|ds.\end{aligned}$$

(D2)-*(i)*, *(ii)*, *(iv)* imply

$$\begin{aligned}\|\mathfrak{S}_k(v,u)\| \leq& \|v^{t_{k+1}}\| + M[\alpha_k\|v(t_k^-)\|^{a_k} + c_k + \|v_{s_k}\|^{d_1}] + \|v_{t_{k+1}}\|^{d_1} \\ &+ M(t_{k+1} - s_k)[\|\mathbf{A}\|\|v_s\|^{d_1} + \alpha_0\|v_s\|^{a_0} + \beta_0\|u\|^{b_0} + c_0] \\ \leq& \Lambda_1 + M[\alpha_k\|v\|^{a_k} + \|v\|^{d_1}] + \|v\|^{d_1} + MT[\|\mathbf{A}\|\|v\|^{d_1} + \alpha_0\|v\|^{a_0} + \beta_0\|u\|^{b_0}],\end{aligned}$$

where $\Lambda_1 = \|v^{t_{k+1}}\| + M[c_k + c_0 T]$.

Consequently,

$$\begin{aligned}\|\mathcal{Q}_2(v,u)(t)\| \leq& \|\mathbf{B}^*(t)\|\|W^*(t_1,t)\|\|(\Theta_{[0,t_1]})^{-1}\mathfrak{S}_0(v,u)\| \leq \|\mathbf{B}(t)\|\|W(t_1,t)\|\gamma^{-1}\|\mathfrak{S}_0(v,u)\| \\ \leq& \|\mathbf{B}\|M\gamma^{-1}\Lambda_0 + \|\mathbf{B}\|M^2\gamma^{-1}[\rho q^{d_0}\|v\|^{d_0} + 2^{d_1}\rho^{d_1}q^{d_0 d_1}\|v\|^{d_0 d_1}] \\ &+ \|\mathbf{B}\|M\gamma^{-1}\|v\|^{d_1} + \|\mathbf{B}\|M^2\gamma^{-1}t_1[\|\mathbf{A}\|\|v\|^{d_1} + \alpha_0\|v\|^{a_0} + \beta_0\|u\|^{b_0}], \\ & t \in [0, t_1]. \end{aligned} \tag{3.49}$$

$$\begin{aligned}\|\mathcal{Q}_2(v,u)(t)\| \leq& \|\mathbf{B}^*(t)\|\|W^*(t_{k+1},t)\|\|(\Theta_{[s_k,t_{k+1}]})^{-1}\mathfrak{S}(v,u)\| \leq \|\mathbf{B}\|M\gamma_k^{-1}\|\mathfrak{S}_k(v,u)\| \\ \leq& \|\mathbf{B}\|M\gamma_k^{-1}\Lambda_1 + \|\mathbf{B}\|M^2\gamma_k^{-1}[\alpha_k\|v\|^{a_k} + \|v\|^{d_1}] + \|\mathbf{B}\|M\gamma_k^{-1}\|v\|^{d_1} \\ &+ \|\mathbf{B}\|M^2\gamma_k^{-1}T[\|\mathbf{A}\|\|v\|^{d_1} + \alpha_0\|v\|^{a_0} + \beta_0\|u\|^{b_0}], \\ & t \in (s_k, t_{k+1}], k = 1, ..., N. \end{aligned} \tag{3.50}$$

$$\begin{aligned}\|\mathcal{Q}_1(v,u)(t)\| \leq& M\|\phi(0)\| + M[\rho q^{d_0}\|v\|^{d_0} + 2^{d_1}\|\phi\|^{d_1} + 2^{d_1}\rho^{d_1}q^{d_0d_1}\|v\|^{d_0d_1}] + \|v\|^{d_1} \\ &+ Mt_1[\|\mathbf{A}\|\|v\|^{d_1} + \alpha_0\|v\|^{a_0} + \beta_0\|u\|^{b_0} + c_0] + M^2\|\mathbf{B}\|^2t_1\gamma^{-1}\|\mathfrak{S}_0(v,u)\| \\ \leq& \Lambda_2 + \Lambda_3\Big(M\|\phi(0)\| + M[\rho q^{d_0}\|v\|^{d_0} + 2^{d_1}\|\phi\|^{d_1} + 2^{d_1}\rho^{d_1}q^{d_0d_1}\|v\|^{d_0d_1}] + \|v\|^{d_1} \\ &+ Mt_1[\|\mathbf{A}\|\|v\|^{d_1} + \alpha_0\|v\|^{a_0} + \beta_0\|u\|^{b_0} + c_0]\Big), \quad t \in [0, t_1], \end{aligned} \tag{3.51}$$

where $\Lambda_2 = M^2\|\mathbf{B}\|^2t_1\gamma^{-1}\|v^{t_1}\|$ and $\Lambda_3 = M^2\|\mathbf{B}\|^2t_1\gamma^{-1} + 1$.

$$\begin{aligned}\|\mathcal{Q}_1(v,u)(t)\| \leq& M[\alpha_k\|v\|^{a_k} + c_k + \|v\|^{d_1}] + \|v\|^{d_1} \\ &+ MT[\|\mathbf{A}\|\|v\|^{d_1} + \alpha_0\|v\|^{a_0} + \beta_0\|u\|^{b_0} + c_0] + M^2\|\mathbf{B}\|^2T\gamma_k^{-1}\|\mathfrak{S}_k(v,u)\| \\ \leq& \Lambda_4 + \Lambda_5\Big(M[\alpha_k\|v\|^{a_k} + c_k + \|v\|^{d_1}] + \|v\|^{d_1} \\ &+ MT[\|\mathbf{A}\|\|v\|^{d_1} + \alpha_0\|v\|^{a_0} + \beta_0\|u\|^{b_0} + c_0]\Big), \quad t \in (s_k, t_{k+1}], \end{aligned} \tag{3.52}$$

where $\Lambda_4 = M^2\|\mathbf{B}\|^2T\gamma_k^{-1}\|v^{t_{k+1}}\|$ and $\Lambda_5 = M^2\|\mathbf{B}\|^2T\gamma_k^{-1} + 1$.

$$\|\mathcal{Q}_1(v,u)(t)\| \leq \alpha_k\|v\|^{a_k} + c_k, \quad t \in (t_k, s_k]. \tag{3.53}$$

Let $K_1 = \Lambda_3 + \|\mathbf{B}\|M\gamma^{-1}$. Therefore, from (3.49) and (3.51), we sustain

$$\begin{aligned}|||\mathcal{Q}(v,u)||| =& \|\mathcal{Q}_1(v,u)\| + \|\mathcal{Q}_2(v,u)\| \\ \leq& K_2 + K_3\|v\|^{d_1} + K_4\|v\|^{d_0d_1} + K_5\|v\|^{d_0} + K_6\|v\|^{a_0} + K_7\|u\|^{b_0},\end{aligned}$$

where

$$\begin{aligned}&K_2 = \Lambda_2 + M\left[\Lambda_3(\|\phi(0)\| + 2^{d_1}\|\phi\|^{d_1} + t_1c_0) + \|\mathbf{B}\|\gamma^{-1}\Lambda_0\right], \\ &K_3 = \Lambda_3 + \Lambda_3Mt_1\|\mathbf{A}\| + \|\mathbf{B}\|M\gamma^{-1} + \|\mathbf{B}\|M^2\gamma^{-1}t_1\|\mathbf{A}\|, K_4 = M2^{d_1}\rho^{d_1}q^{d_0d_1}K_1, \\ &K_5 = M\rho d^{d_0}K_1, K_6 = Mt_1\alpha_0K_1, K_7 = Mt_1\beta_0K_1.\end{aligned}$$

Let $\overline{K_1} = \Lambda_5 + \|\mathbf{B}\|M\gamma_k^{-1}$. From (3.50) and (3.52), we obtain

$$\begin{aligned}|||\mathcal{Q}(v,u)||| =& \|\mathcal{Q}_1(v,u)\| + \|\mathcal{Q}_2(v,u)\| \\ \leq& \overline{K_2} + \overline{K_3}\|v\|^{d_1} + \overline{K_4}\|v\|^{a_k} + \overline{K_5}\|v\|^{a_0} + \overline{K_6}\|u\|^{b_0},\end{aligned}$$

where

$$\begin{aligned}&\overline{K_2} = \Lambda_4 + M[\Lambda_5(c_k + Tc_0) + \|\mathbf{B}\|\gamma_k^{-1}\Lambda_1], \overline{K_3} = (M + 1 + MT\|\mathbf{A}\|)\overline{K_1}, \overline{K_4} = M\alpha_k\overline{K_1}, \\ &\overline{K_5} = MT\alpha_0\overline{K_1}, \overline{K_6} = MT\beta_0\overline{K_1}.\end{aligned}$$

Addittionally, from (3.53), it follows

$$|||\mathcal{Q}(v,u)||| = \|\mathcal{Q}_1(v,u)\| + \|\mathcal{Q}_2(v,u)\| \leq \alpha_k\|v\|^{a_k} + c_k.$$

Hence,

$$\begin{aligned}
\frac{|||\mathcal{Q}(v,u)|||}{\|(v,u)\|} \leq & \frac{K_2}{\|v\|+\|u\|} + K_3\|v\|^{d_1-1} + K_4\|v\|^{d_0 d_1-1} + K_5\|v\|^{d_0-1} + K_6\|v\|^{a_0-1} \\
& + K_7\|u\|^{b_0-1}; \\
\frac{|||\mathcal{Q}(v,u)|||}{\|(v,u)\|} \leq & \frac{\overline{K_2}}{\|v\|+\|u\|} + \overline{K_3}\|v\|^{d_1-1} + \overline{K_4}\|v\|^{a_k-1} + \overline{K_5}\|v\|^{a_0-1} + \overline{K_6}\|u\|^{b_0-1}; \\
\frac{|||\mathcal{Q}(v,u)|||}{\|(v,u)\|} \leq & \alpha_k\|v\|^{a_k-1} + \frac{c_k}{\|v\|+\|u\|}.
\end{aligned}$$

Therefore, we have

$$\lim_{|||(v,u)|||\to\infty} \frac{|||\mathcal{Q}(v,u)|||}{|||(v,u)|||} = 0.$$

Statement 5:The operator $\mathcal{Q}$ has at least one fixed point.

Presently, using the previous statement, for $0 < \varepsilon < 1$, there exists $R > 0$ such that

$$\frac{|||\mathcal{Q}(v,u)|||}{|||(v,u)|||} < \varepsilon \text{ if } |||(v,u)||| \geq R.$$

Thus, if $|||(v,u)||| = R$, then $|||\mathcal{Q}(v,u)||| \leq \varepsilon R < R$. Consequently,

$$\mathcal{Q}(\partial B(0,R)) \subset B(0,R), \tag{3.54}$$

where $B(0,R)$ is the closed ball of radius R centered at zero. Thus, by eq. (3.54), Statement 1, 2, 3, 4, and Theorem 14, we conclude that the operator $\mathcal{Q}$ has a fixed point, i.e.,

$$\exists (v,u) \in \mathcal{PC}_m \times \mathcal{PC}_u : \quad \mathcal{Q}(v,u) = (v,u),$$

which by Theorem 22, implies the controllability of system (3.39) on $[0,T]$. Furthermore,

$$u(t) = \begin{cases} \Upsilon_0 \mathfrak{S}_0(v,u), & t \in [0,t_1], \\ \Upsilon_k \mathfrak{S}_k(v,u), & t \in (s_k, t_{k+1}], k = 1, ..., N, \\ 0, & t \in (t_k, s_k], k = 1, ..., N. \end{cases} \tag{3.55}$$

such that for a given $\phi \in \mathcal{PC}_m$, and arbitrary points $v^{t_{k+1}}$ for $k = 0, ..., N$, the solution $v(t) = v(t,u)$ of (3.39) satisfies:

$$\begin{aligned}
v^{t_1} = \ & v(t_1) = W(t_1,0)\left[\phi(0) - \zeta(v_{\lambda_1}, ..., v_{\lambda_q})(0) - g(0, \phi - \zeta(v_{\lambda_1}, ..., v_{\lambda_q}))\right] \\
& + \int_0^{t_1} W(t_1,s)\left[\mathbf{A}(s)g(s,v_s) + f(s,v_s,u(s))\right]ds + g(t_1, v_{t_1}) \\
& + \int_0^{t_1} W(t_1,s)\mathbf{B}(s)u(s)ds; \\
v^{t_{k+1}} = \ & v(t_{k+1}) = W(t_{k+1},s_k)\left[\Gamma_k(s_k, v(t_k^-)) - g(s_k, v_{s_k})\right] \\
& + \int_{s_k}^{t_{k+1}} W(t_{k+1},s)\left[\mathbf{A}(s)g(s,v_s) + f(s,v_s,u(s))\right]ds + g(t_{k+1}, v_{t_{k+1}}) \\
& + \int_{s_k}^{t_{k+1}} W(t_{k+1},s)\mathbf{B}(s)u(s)ds, \quad k = 1, ..., N,
\end{aligned}$$

where $t_{N+1} = T$, and $v^{t_{N+1}} = v(T) = v^1$. This completes the proof. □

Chapter 4

Conclusions and Final Remarks

In this manuscript, the existence of solutions of the semilinear neutral equation with non-instantaneous impulses, non-local conditions, and unbounded delay

$$\begin{aligned}\frac{d}{dt}[v(t)-g(t,v_t)] &= \mathbf{A}(t)v(t)+\mathfrak{F}(t,v_t), \quad t\in J_k^1, k=0,1,\dots,\\ v(t) &= \Gamma_k(t,v(t_k^-)), \quad t\in J_k^2, k=1,\dots,\\ v(s)+\zeta(v_{\lambda_1},v_{\lambda_2},\dots,v_{\lambda_q})(s) &= \phi(s), \quad s\in\mathbb{R}_-=(-\infty,0],\end{aligned}$$

was proved. To this end, Karakostas's fixed point theorem was used. The phase space that we choose fulfills the axioms proposed by Hale & Kato for retarded equations with unbounded delay. However, in this case, our phase space is a subspace of piecewise continuous functions due to the impulses and non-local conditions. The uniqueness of solutions was obtained by imposing some conditions on the nonlinear terms.

Additionally, the approximate and exact controllability of the control neutral problem

$$\begin{aligned}\frac{d}{dt}[v(t)-g(t,v_t)] &= \mathbf{A}(t)v(t)+\mathbf{B}(t)u(t)+f(t,v_t,u(t)), \quad t\in\bigcup_{k=0}^{N} J_k^1,\\ v(t) &= \Gamma_k(t,v(t_k^-)), \quad t\in J_k^2, k=1,\dots,N,\\ v(s)+\zeta(v_{\lambda_1},\dots,v_{\lambda_q})(s) &= \phi(s) \quad s\in(-\infty,0].\end{aligned}$$

was obtained. To prove the approximate controllability, we assumed that the associated linear system is controllable and applied the technique employed by Bashirov & Ghahramanlou. On the other hand, to prove the exact controllability, we impose sublinear conditions on the nonlinear terms and assume the controllability of the linear system in any interval $[\alpha,\beta]\subset(0,T]$. Then, Rothe's fixed point theorem was applied.

The development of this work required the study of non-curricular topics such that DDEs, NDEs, DEs with non-local conditions, and Control Theory. Most of the theory and tools needed within this thesis were covered inside a research group of Yachay Tech students, alumni and colleagues from other countries.

As a future research on this subject, we will study the existence of solutions, controllability, and stability of the same problem in infinite dimension. Furthermore, the extension of our results to semilinear perturbed systems in time scales is a research area of our interest.

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