UNIVERSIDAD DE INVESTIGACIÓN DE TECNOLOGÍA EXPERIMENTAL YACHAY

Escuela de Ciencias Matemáticas y Computacionales

TÍTULO: 2-Dimensional Quaternionic Fourier Transform and Applications.

Trabajo de integración curricular presentado como requisito para la obtención del título de Matemático

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Urcuquí, 20 de Julio de 2022

Urcuquí, 21 de junio de 2022

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ACTA DE DEFENSA No. UITEY-ITE-2022-00013-AD

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Dedication

"Dedicated to my parents Lourdes and Walter
and my grandparents María and Nicolás."

Acknowledgements

First of all, I want to thank all the people that have been part of my life. Each one of them has contributed a lot to arrive until this point and building my character and personality.

I want to thank my lovely Yachay Tech University and all my professors, who shared all their knowledge and always motivated me to become a researcher. I am incredibly grateful to Professor Juan Mayorga, who showed me the beauty of math and inspired me to become a mathematician, and to my advisor, Professor Eusebio Ariza, who was the first professor who introduced me to the world of pure math, also by previewing my future as a mathematician with a most straightforward strong phrase, ”As we mature mathematically, one can better understand the theory reviewed above” and by being my support to develop this work.

Finally, I want to thank my parents, grandmother, and my friends in Yachay Tech, who have been important, especially Sebastián, Fernando, María, and Doménica, for being my support in my hardest times, and Daniela and Jeimmy, who helped me to grow, believe in myself and have the motivation to continue achieving new challenges.

Carlos Chipantiza.

Resumen

En este trabajo, definimos la Transformada de Fourier Cuaterniónica Bidimensional (izquierda) (**2D-QFT**) de $f \in L^1\left(\mathbb{R}^2; \mathbb{H}\right)$, la cual es la función $\mathcal{F}_q\{f\} : \mathbb{R}^2 \to \mathbb{H}$ definida por

$$\mathcal{F}_q\{f\}(\boldsymbol{\omega}) = \widehat{f}(\boldsymbol{\omega}) = \int_{\mathbb{R}^2} e^{-\mu\boldsymbol{\omega}\cdot\boldsymbol{x}} f(\boldsymbol{x}) d^2\boldsymbol{x},$$

donde $\boldsymbol{x} = x_1\boldsymbol{e}_1 + x_2\boldsymbol{e}_2$, $\boldsymbol{\omega} = \omega_1\boldsymbol{e}_1 + \omega_2\boldsymbol{e}_2$, con kernel de Fourier cuaterniónico $e^{-\mu\boldsymbol{\omega}\cdot\boldsymbol{x}}$ tal que $|\mu| = 1$.

Derivamos las propiedades de desplazamiento, modulación y convolución y establecemos el teorema de Plancherel y el teorema de derivación vectorial. Además, estudiaremos la aplicación de esta transformada de Fourier a la resolución de la ecuación del calor.

Palabras claves: Cuaternión, Transformada de Fourier Cuaterniónica Bidimensional (izquierda), ecuación de calor.

Abstract

In this work, we define the Two-dimensional (left) Quaternion Fourier Transform (**2D-QFT**) of $f \in L^1\left(\mathbb{R}^2;\mathbb{H}\right)$, which is the function $\mathcal{F}_q\{f\} : \mathbb{R}^2 \to \mathbb{H}$ defined by

$$\mathcal{F}_q\{f\}(\boldsymbol{\omega}) = \widehat{f}(\boldsymbol{\omega}) = \int_{\mathbb{R}^2} e^{-\mu\boldsymbol{\omega}\cdot\boldsymbol{x}} f(\boldsymbol{x}) d^2\boldsymbol{x},$$

where $\boldsymbol{x} = x_1\boldsymbol{e}_1 + x_2\boldsymbol{e}_2$, $\boldsymbol{\omega} = \omega_1\boldsymbol{e}_1 + \omega_2\boldsymbol{e}_2$, with quaternion Fourier kernel $e^{-\mu\boldsymbol{\omega}\cdot\boldsymbol{x}}$ such that $|\mu| = 1$.

We derive the shift, modulation, and convolution properties and establish the Plancherel and vector differential theorem. Furthermore, we will study the application of this Fourier transform to the resolution to the heat equation.

Keywords: Quaternion, Two-dimensional (left) Quaternion Fourier Transform, heat equation.

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Chapter 1

Introduction

The use of quaternion algebra to generalize the conventional Fourier transform (FT) and employ it in image analysis has recently become popular. For example, to calculate the FT of a color image without splitting it into three gray-scale images, suggested a single and holistic FT that treats a color image as a vector field [15]. Quaternions are used to represent color picture pixels in this system. The **quaternion Fourier Transform** (QFT) is the name given to these extensions [6].

This transformation is based on a unit pure quaternion μ. The value of μ is chosen to produce embedding spaces that are resilient and/or have perceptual qualities. In the proposed method, μ is a function of a block's mean color value, and a perceptual component [9]. A color picture pixel may be converted to a quaternion pixel by putting the three components (in the case of RGB images, the red, green, and blue components) into the three imaginary portions of the quaternion while leaving the real part zero [34, 15]. This decision is neither arbitrary nor coincidental. A complete quaternion can be thought of as the ratio of two vectors or the number that multiplies one vector to produce another. This is a valuable geometric interpretation that is used in color picture filter design [15].

Due to the noncommutative property of quaternion multiplication, there are at least three different types of 2D QFTs as follows (see [13],[24],[23],[8],[31],[33])

$$\begin{aligned}
\mathcal{F}_q^{\mathrm{I}}\{f\}(\boldsymbol{\omega}) &= \int_{\mathbb{R}^2} e^{-\mu_1 \boldsymbol{\omega}\cdot\boldsymbol{x}} f(\boldsymbol{x}) d^2\boldsymbol{x}, \quad \boldsymbol{\omega}\cdot\boldsymbol{x} = \omega_1 x_1 + \omega_2 x_2,\\
\mathcal{F}_q^{\mathrm{II}}\{f\}(\boldsymbol{\omega}), &= \int_{\mathbb{R}^2} f(\boldsymbol{x}) e^{-\mu_1 \boldsymbol{\omega}\cdot\boldsymbol{x}} d^2\boldsymbol{x}, \quad \boldsymbol{\omega}\cdot\boldsymbol{x} = \omega_1 x_1 + \omega_2 x_2,\\
\mathcal{F}_q^{\mathrm{III}}\{f\}(\boldsymbol{\omega}), &= \int_{\mathbb{R}^2} e^{-\mu_1 \omega_1 x_1} f(\boldsymbol{x}) e^{-\mu_2 \omega_2 x_2} d^2\boldsymbol{x},
\end{aligned}$$

where μ_1 and μ_2 are any two unit pure quaternions ($\mu_1^2 = \mu_2^2 = -1$) that are orthogonal to each other. These three QFTs are so-called **left-side**, **right-side** and **double-side**, or type I, II and III, respectively.

Assefa et al. [3] used QFTs of type II to establish the 2D quaternion Stockwell (QS) transform and then apply it for the analysis of local color image spectra. Recently, Guo and Zhu [20] introduced the quaternion Fourier–Mellin moments as a generalization of traditional Fourier–Mellin moments to quaternion algebra. Several properties of this generalization are investigated using QFTs of type II [6].

In this work, we concentrate on $2D - QFT$ of Type I with kernel $\mu_1 = \mu \in \mathbb{H}$. We derive the shift, modulation and convolution properties and also establish the Plancherel theorem. Furthermore, we will study the application of this Quaternion Fourier transform to the resolution of partial differential equations.

This document is organized as follows:

- In Chapter 2 we summarize some applications of the quaternions.
- In Chapter 3 we present the mathematical support for the Fourier Transform. The concept of convolution, the Fourier inversion theorem is present as such as the relation of this theory with the spaces L^1 and L^2. Furthermore, we review the extension to $\mathbb{R}^n$ and some applications.
- In Chapter 4 we review the mathematical support of some properties about quaternions.
- In Chapter 5 we introduce the two-dimensional (left) Quaternion Fourier Transform. Some important properties are studied. Finally, we will use the Quaternion Fourier Transform to solve a partial differential equation.
- In Chapter 6 we present our conclusions and recommendations.

Chapter 2

Quaternion's Applications

In the Chapter 4 we shall review with more details about quaternions. For now, we are just going to focus on some applications that can help the reader understand the importance of this theory and will be a guide for future works.

2.1 A brief summary of history

Quaternions were invented in 1843 by the Irish mathematician William Rowan Hamilton. He had worked since 1830 in the field of complex numbers and in 1833 obtained the result that complex numbers form an algebra of pairs of real numbers. W.R. Hamilton tried for more than ten years to extend this concept to triples of real numbers with one real and two imaginary units. He could himself well imagine suitable operations of addition and multiplication of triples (which he later defined as 'vectors'), but he was unable to find a method for the division of such vectors. W.R. Hamilton was often asked by his elder son during breakfast: "Father, have you learned how to divide vectors?" After the rejection of the commutativity law of multiplication, he could answer on October 16, 1843. "Yes, I have!" This introduction of a non-commutative structure into this algebra led to the dispute over more than twenty years with other famous mathematicians of his time [21].

He had the idea for quaternions while strolling down the Royal Canal on his way to an Irish Academy conference, and he was so thrilled with his discovery that he scratched the fundamental formula of quaternion algebra,

$$i^2 = j^2 = k^2 = ijk = -1,$$

into the stone of the Brougham bridge. Although the quaternions are not commutative, they are associative, and they constitute the quaternion group. Moreover, the quaternions are members of William Rowan Hamilton's non-commutative division algebra [38].

2.2 Previous works

2.2.1 Applications to Physics

Despite the fact that for many years after its conception, quaternions were considered a "solution in search of a problem," applications in classical mechanics and relativity theory were discovered in the early twentieth century. The capacity of quaternions to represent three-dimensional rotations around any axis led to the usage of quaternion algebra in rotational kinematics equations [29].

Applications to Estimation of Rotating Body Attitude

If the vehicle's angular velocity is known and there are further observations of one or more known stars, one must estimate the vehicle's attitude relative to some inertial reference frame. Of course, the differential equation of the vehicle's rotation can be used alone. However, measurement and computation mistakes make it difficult to arrive at a right solution, especially over a lengthy period of time. If one obtains extra measurements of some external items, the situation can be improved. We assume that these are known stars that are periodically monitored with the assistance of on-board measuring equipment. The equation of a rotating body motion is taken into consideration, as well as the equation of observations in quaternion form (see [1]).

2.2.2 Dual Quaternion and Applications to Neuroscience

Using quaternions, we can easily handle rotations, but not translations. However, since the sources of the reference frames linked with each body are rarely the same, we must deal with translations. The rotation center of the eye, for example, is not the same as the rotation center of the head. In addition to rotation, we will use dual quaternions' geometric algebra (see [14],[27]) to deal with translations. Dual quaternions may also be used to succinctly depict a screw motion, which is defined as a combination of rotation and translation along the rotation axis [27].

3D kinematics using dual quaternions

Much behavioral neuroscience research is planned in one or two spatial dimensions, but when scientists attempt to solve problems in three dimensions (3D), they typically face obstacles or extra challenges. Lower-dimensional findings aren't always three-dimensionally extendable. In motor planning of eye, gaze, or arm motions, or sensorimotor transformation difficulties, the 3D kinematics of external (stimuli) or internal (body parts) items must regularly be considered: how to describe the 3D location and orientation of these things and connect them together. The dual quaternions make it simple to represent rotations, translations, and screw movements, as well as combinations of these, by providing a simple way to define 3D kinematics for the position (point transformation) or coupled position and orientation (coupled position and orientation) (via line transformation) (for more detail see [27]).

2.2.3 Applications to computer science

Computer/robot vision, computer graphics, and animation all benefit from the analytical properties of quaternions. Special focus is paid to vision methods for 3D pose estimation, animation systems that incorporate viewpoint/object rotations, motion interpolation algorithms, and quaternion fractals. The benefit of using quaternions over other representations like Euler angles isn't restricted to singularity-free kinematics connections; we can also utilize quaternions to construct closed-form solutions for algebraic systems with unknown rotational parameters. Some of the exquisite mathematical features of quaternions in complex space, together with a set of useful equations, are offered for the benefit of the technically inclined [29].

In Section 4.2.1 we will give more details about rotation in $\mathbb{R}^3$. For now, we will just give a brief summary of this and its applications in science computer. Three-dimensional rotations are traditionally represented using a set of Euler angles ψ, ϕ, θ, which signify rotations about independent coordinate axes. A succession of rotations can therefore be used to generate any general rotation, as shown below.

$$\begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \cos\varphi & 0 & \sin\varphi \\ 0 & 1 & 0 \\ -\sin\varphi & 0 & \cos\varphi \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\psi & -\sin\psi \\ 0 & \sin\psi & \cos\psi \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}. \tag{2.1}$$

If $\boldsymbol{v} = (l, m, n)$ denote a vector in three-dimensional space, then θ denotes a rotational transformation about this vector

$$\begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} l^2(1-\cos\theta)+\cos\theta & lm(1-\cos\theta)-n\sin\theta & nl(1-\cos\theta)+m\sin\theta \\ lm(1-\cos\theta)+n\sin\theta & m^2(1-\cos\theta)+\cos\theta & mn(1-\cos\theta)-l\sin\theta \\ nl(1-\cos\theta)-m\sin\theta & mn(1-\cos\theta)+l\sin\theta & n^2(1-\cos\theta)+\cos\theta \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}.$$

Clearly, the shortest path between two object orientations is defined by a single rotation around an analogous axis. A quaternion product may be used to describe the aforementioned transformation:

$$\mathbf{w} = \mathbf{q}\mathbf{v}\mathbf{q}^* \tag{2.2}$$

where

$$\mathbf{w} = \left(0, x', y', z'\right), \quad \mathbf{v} = (0, x, y, z), \text{ and } \mathbf{q} = \left(\cos\frac{\theta}{2}, l\sin\frac{\theta}{2}, m\sin\frac{\theta}{2}, n\sin\frac{\theta}{2}\right)$$

The length of a vector is obviously unaffected by rotation, therefore $|\mathbf{w}| = |\mathbf{v}|$. As a result, for a proper rotation, $\mathbf{q}$ in (2.2) must be a unit vector. We also have the condition $\mathbf{q}^{-1} = \mathbf{q}*$ when $\mathbf{q}$ is a unit quaternion (for more detail see Section 4.1.1). We may therefore define

$$L_q(\mathbf{v}) = \mathbf{q}\mathbf{v}\mathbf{q}^*.$$

This operator is used widely in both classical mechanics and computer graphics as a three-dimensional space quaternion rotation operator that works on any vector $\mathbf{v}$. It is straightforward to show that the operator is linear, therefore

$$\mathrm{L_q}(k\mathbf{a} + \mathbf{b}) = k\,\mathrm{L_q}(\mathbf{a}) + \mathrm{L_q}(\mathbf{b})$$

for any constant k, and vector quaternions $\mathbf{a}$ and $\mathbf{b}$.

Quaternion Applications in Graphics and Animation

Let $\mathbf{q}_1$ and $\mathbf{q}_2$ be two unit quaternions that express three-dimensional rotations in any direction. The first rotation, followed by the second, has the following composition in quaternion notation:

$$\begin{aligned} \mathrm{L}_{q2}\left(\mathrm{L}_{q1}(\mathbf{v})\right) &= \mathbf{q}_2\left(\mathbf{q}_1\mathbf{v}\mathbf{q}_1{}^*\right)\mathbf{q}_2{}^* \\ &= \left(\mathbf{q}_2\mathbf{q}_1\right)\mathbf{v}\left(\mathbf{q}_2\mathbf{q}_1\right)^* \\ &= \mathrm{L}_{q2q1}(\mathbf{v}). \end{aligned}$$

As a result, the quaternion product $\mathbf{q}_2\mathbf{q}_1$ defines the quaternion rotation operator $\mathrm{L}q_2q_1$, which represents a series of operators $\mathrm{L}q_1$ followed by $\mathrm{L}q_2$. The composition of any number of rotations may be generalized using this characteristic.

The rotation of the frame when the point is inertially fixed is the dual operation of rotating a point within a fixed frame. The fixed point's coordinate transformation with respect to the rotating frame is given by [26]

$$\mathbf{v}' = \mathbf{q}^*\mathbf{v}\mathbf{q} = \mathrm{L}_q^{-1}(\mathbf{v}) = \mathrm{L}_{q^*}(\mathbf{v}).$$

When defining rotations, quaternions provide a variety of benefits over Euler angles. The angle and the axis (vector) are the sole parameters used to parameterize rotations using quaternions, whereas Euler angles describe a rotation as a composite of three distinct rotations about coordinate axes. Furthermore, when the second Euler angle approaches 90 degrees, the mutually independent feature of the Euler angle rotations breaks down, resulting in the loss of one degree of freedom. The gimbal lock [37] is the name given to this occurrence. Despite several Euler angle formulations in the rotation matrix (2.1), this singularity remains. In a graphic animation, the state of the gimbal lock might have negative consequences.

In comparison to a quaternion-based method, the parameterization of orientation using Euler angles necessitates more arithmetic operations. A few essential issues are summarized in Table 2.1

	Quaternions	**Euler Angles**
Representation of Rotations	4 Elements	9 Elements (16 Elements if homogeneous coordinates are used)
Composition of Rotations	16 Multiplications and 12 Additions	27 Multiplications and 18 Additions

Table 2.1: Computational Complexity: Quaternions vs. Euler Angles. Source [29]

Chapter 3

The Fourier Transform

In this chapter, we shall present some concepts about the Fourier Transform, which are important for our study. Our main reference is [16], in which the reader will be able to find all the proof with all details.

In the same manner that Fourier series are used to expand functions on a finite interval, the Fourier transform allows functions on the whole real line $\mathbb{R} = (-\infty, +\infty)$ to be expanded as (continuous) superpositions of the fundamental oscillatory functions $e^{i\xi x}(\xi \in \mathbb{R})$. Let us run a few formal calculations to offer some motivation.

Assume that f is a $\mathbb{R}$ function. We can expand f on the interval $[-l, l]$ in a Fourier series (see [16]) for every $l > 0$, and we want to investigate what happens when we allow $l \to +\infty$, To this goal, the Fourier expansion is written as follows: $x \in [-l, l]$,

$$f(x) = \frac{1}{2l} \sum_{-\infty}^{+\infty} c_{n,l} e^{i\pi nx/l}, \quad c_{n,l} = \int_{-l}^{l} f(y) e^{-ixny/l} dy.$$

Let $\Delta\xi = \pi/l$ and $\xi_n = n\Delta\xi = n\pi/l$; then these formulas become

$$f(x) = \frac{1}{2\pi} \sum_{-\infty}^{+\infty} c_{n,l} e^{i\xi_n x} \Delta\xi, \quad c_{n,l} = \int_{-l}^{l} f(y) e^{-i\xi_n y} dy.$$

Let us suppose that $f(x)$ vanishes rapidly as $x \to |+\infty|$; then $c_{n,l}$ will not change much if we extend the region of integration from $[-l, l]$ to $(-\infty, +\infty)$:

$$c_{n,l} \approx \int_{-\infty}^{+\infty} f(y) e^{-i\xi_n y} dy.$$

This last integral is a function only of ξ_n, which we call $\widehat{f}(\xi_n)$, and we now have

$$f(x) \approx \frac{1}{2\pi} \sum_{-\infty}^{+\infty} \widehat{f}(\xi_n) e^{-i\xi_n x} \Delta\xi \quad (|x| < l).$$

This looks very much like a Riemann sum. If we now let $l \to +\infty$, so that $\Delta\xi \to 0$, the $\approx$ should become $=$ and the sum should turn into an integral, thus:

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \widehat{f}(\xi) e^{i\xi x} d\xi, \tag{3.1}$$

where

$$\widehat{f}(\xi) = \int_{-\infty}^{+\infty} f(x)e^{-i\xi x}dx. \tag{3.2}$$

These limiting calculations are utterly nonrigorous as they stand; nonetheless, the final result is correct under suitable conditions on f. The function $\widehat{f}$ given by (3.2) is called the **Fourier transform** of f, and (3.1) is the **Fourier inversion theorem**.

We'll start by establishing a few notational conventions. We'll be working with real-line functions, and the majority of our integrals will be definite integrals over the entire line. As a result, we'll agree that an integral sign without specified bounds denotes an integral over $\mathbb{R}$ (rather than an indefinite integral):

$$\int f(x)dx = \int_{-\infty}^{+\infty} f(x)dx.$$

Moreover, L^2 will mean $L^2(\mathbb{R})$, the space of square-integrable functions on $\mathbb{R}$. We also introduce the space $L^1 = L^1(\mathbb{R})$ of (absolutely) integrable functions on $\mathbb{R}$:

$$L^1 = \left\{ f : \int |f(x)|dx < +\infty \right\}.$$

Remark 1. *We note that L^1 is not a subset of L^2, and that L^2 is not a subset of L^1.*

Remark 2. *The singularities of a function in L^1 (that is, places where the function's values tend to $+\infty$) can be worse than those of a function in L^2, because squaring a large number makes it larger; on the other hand, functions in L^2 do not have to decay as quickly at infinity as those in L^1, because squaring a small number makes it smaller.*

We have the following useful facts:

(i) If $f \in L^1$ and f is bounded, then $f \in L^2$. Indeed,

$$|f| \leq M \Longrightarrow \quad |f|^2 \leq M|f| \Longrightarrow \int |f(x)|^2dx \leq M \int |f(x)|dx < +\infty.$$

(ii) If $f \in L^2$ and f vanishes outside a finite interval $[a, b]$, then $f \in L^1$. This follows from the Cauchy-Schwarz inequality:

$$\int |f(x)|dx = \int_a^b 1 \cdot |f(x)|dx \leq (b-a)^{1/2} \left(\int_a^b |f(x)|^2 dx \right)^{1/2} < +\infty.$$

3.1 Convolution

Definition 1 (Convolution). *If f and g are functions on $\mathbb{R}$, their convolution is the function $f * g$ defined by*

$$f * g(x) = \int f(x-y)g(y)dy, \tag{3.3}$$

provided that the integral exists.

Various conditions can be imposed on f and g to ensure that the integral will be absolutely convergent for all $x \in \mathbb{R}$, for example:

(i) If $f \in L^1$ and g is bounded (say $|g| \leq M$), then

$$\int |f(x-y)g(y)|dy \leq M \int |f(x-y)|dy = M \int |f(y)\,|\,dy < +\infty.$$

(ii) If f is bounded (say $|f| \leq M$) and $g \in L^1$, then

$$\int |f(x-y)g(y)|dy \leq M \int |g(y)|dy < +\infty.$$

(iii) If f and g are both in L^2, then by the Cauchy-Schwartz inequality,

$$\int |f(x-y)g(y)|dy \leq \sqrt{\int \left[f(x-y)\right|^2 dy} \sqrt{\int |g(y)|^2 dy} = \|f\|\|g\| < +\infty.$$

(iv) If f is piecewise continuous and g is bounded and vanishes outside a finite interval $[a,b]$, then $f * g(x)$ exists for all x, since the function $y \to f(x-y)$ is bounded on $[a,b]$ for any x.

(v) It can be shown that if f and g are both in L^1, then $f * g(x)$ exists for "almost every" x, i.e., for all x except for some set having Lebesgue measure zero; moreover, $f * g \in L^1$.

This list could go on and on. We assume implicitly in what follows that the functions we specify meet the requirements for all integrals in question to be absolutely convergent.

We now investigate the basic algebraic and analytic properties of convolutions

Theorem 1. *Convolution obeys the same algebraic laws as ordinary multiplication:*

(i) $f * (ag + bh) = a(f * g) + b(f * h)$ *for any constants* a, b*;*

(ii) $f * g = g * f$*;*

(iii) $f * (g * h) = (f * g) * h$*.*

Theorem 2. *Suppose that* f *is differentiable and the convolutions* $f * g$ *and* $f' * g$ *are well-defined. Then* $f * g$ *is differentiable and* $(f * g)' = f' * g$*. Likewise, if* g *is differentiable, then* $(f * g)' = f * g'$*.*

We emphasize that in Theorem 2 one can throw the derivative in $(f * g)'$ onto either factor. Thus $f * g$ is at least as smooth as either f or g, even when the other factor has no smoothness properties.

Let us take a moment to make a few observations that may help to clarify the meaning of convolutions. To begin with, consider the convolution integral as a limit of Riemann sums.

$$\int f(x-y)g(y)dy \approx \sum f\left(x-y_j\right) g\left(y_j\right) \Delta y_j.$$

The function $f_j(x) = f\left(x-y_j\right)$ is the function f translated along the x -axis by the amount y_j, so the sum on the right is a linear combination of translates of f with coefficients $g\left(y_j\right) \Delta y_j$. We can therefore think of $f * g$ as a continuous superposition of translates of f; and since $f * g = g * f$, it is also a continuous superposition of translates of g.

Second, convolutions may be interpreted as "**moving weighted averages**."

Remark 3. *The **average value** of a function f on the interval $[a, b]$ is defined to be*

$$(b-a)^{-1} \int_a^b f(y)dy.$$

As a generalization of the previous Remark, we have the following definition

Definition 2 (Weighted average of f). *The weighted average of f on $[a, b]$ with respect to a non-negative weight function w is*

$$\frac{\int_a^b f(y)w(y)dy}{\int_a^b w(y)dy}.$$

Suppose now that g is non-negative and $\int g(y)dy = 1$. If we write $f * g(x)$ as $\int f(y)g(x-y)dy$, we see that $f * g(x)$ is the weighted average of f (on the whole line) with respect to the **weight function** $w(y) = g(x-y)$.

Observation 1. *If $g(x) = 0$ for $|x| > a$ then $g(x-y) = 0$ for $|x-y| > a$, so $f * g(x)$ is a weighted average of f on the interval $[x-a, x+a]$. In particular, if*

$$g(x) = \begin{cases} (2a)^{-1} & \text{if } -a < x < a, \\ 0 & \text{otherwise }, \end{cases}$$

then

$$f * g(x) = \frac{1}{2a} \int_{x-a}^{x+a} f(y)dy,$$

which is the (ordinary) average of f on the interval $[x-a, x+a]$.

Remark 4. *One respect in which convolution does not resemble ordinary multiplication is that whereas $f * 1 = f$ for all f, there is no function g such that $f * g = f$ for all f. (The Dirac " δ -function" does the job, but it is not a genuine function). However, we can easily find sequences $\{g_n\}$ such that $f * g_n$ converges to f as $n \to +\infty$. In fact, If $g(x)$ vanishes (or at least is negligibly small) outside an interval $|x| < a$, then $f * g(x)$ will be a weighted average of the values of f on the interval $[x-a, x+a]$, and if a is very small this should be approximately $f(x)$.*

As an excellent example to visualize Remark 4, we have the following example

Example 1. *Suppose that $g \in L^1$, and for $\epsilon > 0$ let*

$$g_\epsilon(x) = \frac{1}{\epsilon} g\left(\frac{x}{\epsilon}\right). \tag{3.4}$$

That is, g_ϵ is obtained from g by compressing the graph in the x -direction by a factor of ϵ and simultaneously stretching it in the y direction by a factor of $1/\epsilon$ (We are thinking of the case $\epsilon < 1$; if $\epsilon > 1$ the words compressing and stretching should be interchanged. See Figure 3.1) As $\epsilon \to 0$ the graph of g_ϵ becomes a sharp spike at $x = 0$, but the area under the graph remains constant:

$$\int g_\epsilon(x)dx = \int g\left(\frac{x}{\epsilon}\right) d\left(\frac{x}{\epsilon}\right) = \int g(y)dy.$$

More generally, the substitution $x = \epsilon y$ yields

$$\int_a^b g_\epsilon(x)dx = \int_{a/\epsilon}^{b/\epsilon} g(y)dy. \tag{3.5}$$

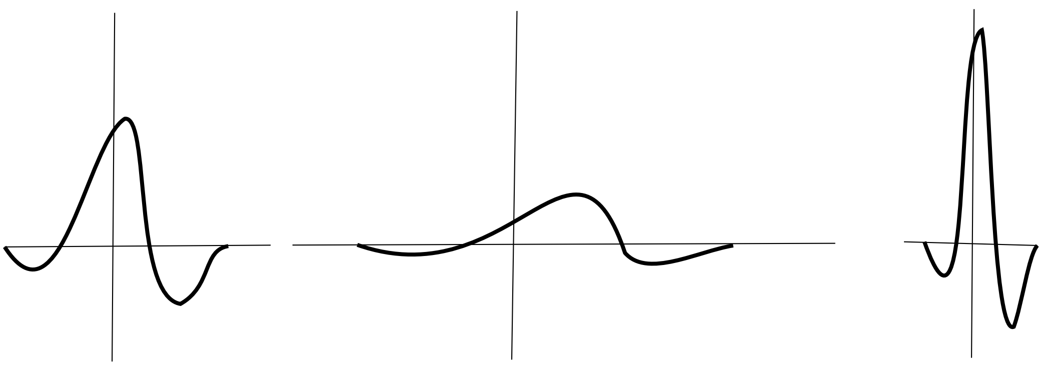

Figure 3.1: A function $g(x)$ (left) and its dilates $g_2(x) = \frac{1}{2} g\left(\frac{1}{2}x\right)$ (middle) and $g_{1/2}(x) = 2g(2x)$ (right).

With this in mind, we can state a precise theorem.

Theorem 3. *Let g be an L^1 function such that $\int_{-\infty}^{+\infty} g(y)dy = 1$, and let $\alpha = \int_{-\infty}^{0} g(y)dy$ and $\beta = \int_0^{+\infty} g(y)dy$. (Note that $\alpha + \beta = 1$ and that $\alpha = \beta = \frac{1}{2}$ if g is even.) Suppose that f is piecewise continuous on $\mathbb{R}$, and suppose either that f is bounded or that g vanishes outside a finite interval so that $f * g(x)$ is well-defined for all x. If g_ϵ is defined by (3.4), then*

$$\lim_{\epsilon \to 0} f * g_\epsilon(x) = \alpha f(x+) + \beta f(x-) \quad \text{for all } x.$$

In particular, if f is continuous at x, then

$$\lim_{\epsilon \to 0} f * g_{\epsilon}(x) = f(x). \tag{3.6}$$

Moreover, if f is continuous at every point in the bounded interval $[a, b]$, the convergence in (3.6) is uniform on $[a, b]$.

There are several variants of Theorem 3, which say that $f * g_{\epsilon} \to f$ in some sense or other as $\epsilon \to 0$ under suitable hypotheses on f and g. We shall content ourselves with stating a result for norm convergence of L^2 functions.

Theorem 4. *Suppose $g \in L^1$ is bounded and satisfies $\int g(y)dy = 1$, If $f \in L^2$ then $f * g(x)$ is well-defined for all x, and if g_{ϵ}, is defined as in (3.4), $f * g_{\epsilon}$ converges to f in norm as $\epsilon \to 0$.*

The family $\{g_k\}$ in Theorems 3 and 4 is called an **approximate identity**, since the operation of convolution with g_{ϵ} tends to the identity operator as $\epsilon \to 0$ (see Figure 3.2).

One of the functions g that is most often used in this context is the **Gaussian**

$$G(y) = \pi^{-1/2} e^{-y^2}. \tag{3.7}$$

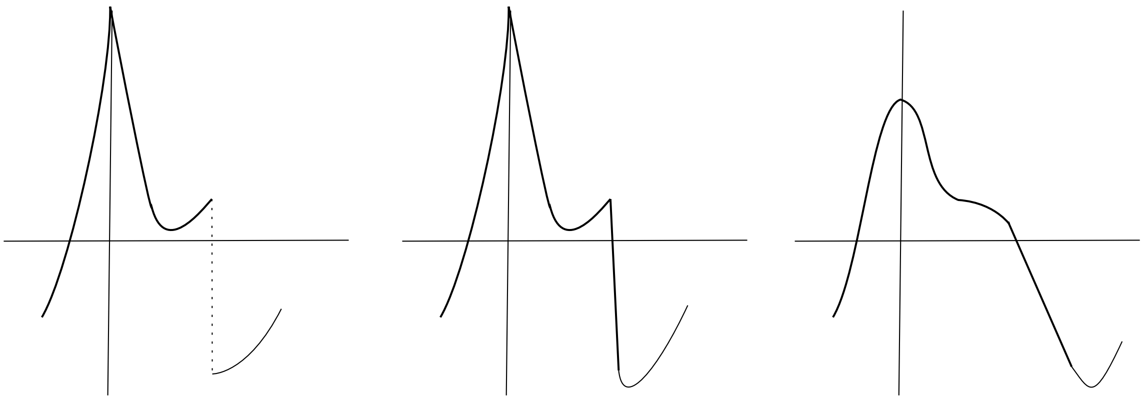

Figure 3.2: A function f with an infinite singularity and a jump discontinuity (left), $f * G_{0.1}$ (middle), and $f * G_{0.3}$ (right), where G is the Gaussian (3.7).

It satisfies $\int G(y)dy = 1$ because

$$\int_{-\infty}^{+\infty} e^{-y^2} dy = 2\int_{0}^{+\infty} e^{-y^2} dy = \int_{0}^{+\infty} e^{-t} t^{-1/2} dt = \Gamma\left(\frac{1}{2}\right) = \pi^{1/2}. \tag{3.8}$$

G is even, so that when it is used as the g in Theorem 3 we have $\alpha = \beta = \frac{1}{2}$. G and its dilated versions G_{ϵ} have the property that all their derivatives are bounded integrable functions. Indeed, it is easily established by induction that $G^{(k)}(y) = P_k(y)e^{-y^2}$

where P_k is a polynomial of degree k, and it follows that $\left|G^{(k)}(y)\right| \leq C_k e^{-y^2}$, with similar estimates (involving some powers of ϵ) for G_e. Hence we can apply Theorems 3 and 4: If f is (say) bounded and piecewise continuous, then $f * G_\epsilon$ is of class $C^{(\infty)}$, and it approximates f when ϵ is small. These convolutions may be regarded as "smeared out" or "smoothed out" versions of f. We've established a method for approximating generic functions with smooth ones, which can be beneficial in various scenarios. It proves the following fundamental result in particular.

Theorem 5 (**The Weierstrass Approximation Theorem**). *If f is a continuous function on $[a, b]$ $(-\infty < a < b < +\infty)$, then f is the uniform limit of polynomials on $[a, b]$ That is for any $\delta > 0$ there is a polynomial P such that*

$$\sup_{a \leq x \leq b} |f(x) - P(x)| < \delta.$$

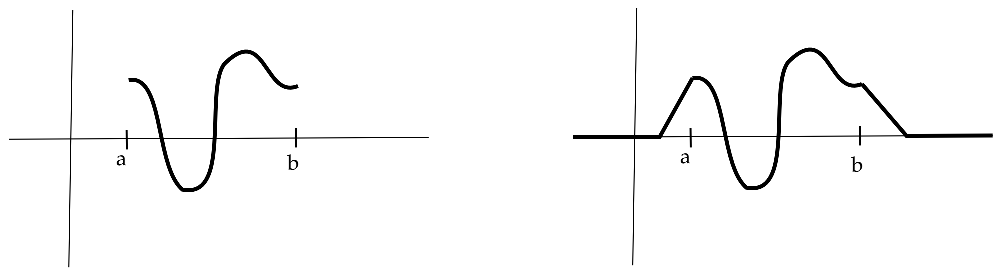

Figure 3.3: A continuous function f on $[a, b]$ (left) and a continuous extension of f to $\mathbb{R}$(right).

The Gaussian is not the only commonly used approximate identity. Another one is given by

$$H(y) = \frac{1}{\pi\left(1 + y^2\right)},$$

which, arises in the solution of the Dirichlet problem for a half-plane. It has the same properties as G in terms of being even and having bounded integrable functions as derivatives of all orders; as a result, it may approximate generic bounded functions smoothly. Another approximation identity with comparable qualities and an extra aspect that makes it particularly useful in certain situations is given by

$$K(y) = \begin{cases} C^{-1} e^{-1/\left(1-y^2\right)} & \text{for } |y| < 1 \\ 0 & \text{for } |y| \geq 1 \end{cases} \quad C = \int_{-1}^{1} e^{-1/\left(1-y^2\right)} dy. \tag{3.9}$$

K possesses derivatives of all orders, even at $y = \pm 1$ (because $e^{-1/\left(1-y^2\right)}$ vanishes to infinite order as y approaches 1 from the left or -1 from the right), and it vanishes outside the bounded set $|y| \leq 1$. Hence the convolutions $f * K_\epsilon$ are well-defined for any piecewise continuous f, bounded or not, and they provide smooth approximations to all such f.

3.2 The Fourier Transform

Definition 3 (Fourier transform). *If f is an integrable function on $\mathbb{R}$, its Fourier transform is the function $\widehat{f}$ on $\mathbb{R}$ defined by*

$$\widehat{f}(\xi) = \int e^{-i\xi x} f(x)dx.$$

We shall also occasionally write

$$\mathcal{F}[f(x)] = \widehat{f}(\xi)$$

for the Fourier transform of f. (This involves an ungrammatical use of the symbols x and ξ but is sometimes the clearest way of expressing things.)

Since $e^{-i\xi x}$ has absolute value 1, the integral converges absolutely for all ξ and defines a bounded function of ξ :

$$|\widehat{f}(\xi)| \leq \int |f(x)|dx. \tag{3.10}$$

Moreover, since $|e^{-i\eta x} f(x) - e^{-i\xi x} f(x)\Big| \leq 2|f(x)|$, the dominated convergence theorem implies that $\widehat{f}(\eta) - \widehat{f}(\xi) \to 0$ whens $\eta \to \xi$, that is, $\widehat{f}$ is continuous. The following theorem summarizes some of the other basic properties of the Fourier transform.

Theorem 6. *Suppose $f \in L^1$*

(a) For any $a \in \mathbb{R}$

$$\mathcal{F}[f(x-a)] = e^{-ia\xi}\widehat{f}(\xi) \text{ and } \mathcal{F}\left[e^{iax} f(x)\right]) = \widehat{f}(\xi - a).$$

(b) If $\delta > 0$ and $f_\delta(x) = \delta^{-1} f(x/\delta)$ as in 3.4, then

$$[f_\delta]\widehat{\ }(\xi) = \widehat{f}(\delta\xi) \quad \text{and} \quad \mathcal{F}[f(\delta x)] = [\widehat{f}]_\delta(\xi).$$

(c) If f is continuous and piecewise smooth and $f' \in L^1$, then

$$[f']\widehat{\ }(\xi) = i\xi\widehat{f}(\xi).$$

On the other hand, if $xf(x)$ is integrable, then

$$\mathcal{F}[xf(x)] = i[\widehat{f}]'(\xi).$$

(d) If also $g \in L^1$, then

$$(f * g)\widehat{\ } = \widehat{f}\,\widehat{g}.$$

Parts (a), (b), and (c) exhibit a remarkable set of correspondences between functions and their Fourier transforms.

i) When a function is translated, its Fourier transform is multiplied by an exponential, and vice versa.;

ii) When a function is dilated by the factor δ, its Fourier transform is dilated by the factor $1/\delta$, and vice versa.

iii) When a function is differentiated, its Fourier transform is multiplied by the coordinate variable, and vice versa. (Of course, this formulation is a little sloppy; there are $-1, i,$, and δ factors to consider.)

iv) Part (d) continues the symmetry between f and $\widehat{f}$: From (d) and the Fourier inversion formula below, it follows that

$$\widehat{f} * \widehat{g} = 2\pi(fg)\widehat{\ }. \tag{3.11}$$

One other basic property of Fourier transforms of L^1 functions should be mentioned here. We observed earlier that if $f \in L^1$, then $\widehat{f}$ is a bounded, continuous function on $\mathbb{R}$; but something more is true. We have the following Lemma

Lemma 1 (The Riemann-Lebesgue Lemma). *If $f \in L^1$, then $\widehat{f}(\xi) \to 0$ as $\xi \to |+\infty|$.*

3.3 The Fourier inversion theorem

We now turn to the Fourier inversion formula, that is, the procedure for recovering f from $\widehat{f}$. The heuristic arguments in the introduction to this chapter led us to the formula

$$f(x) = \frac{1}{2\pi}\int e^{i\xi x}\widehat{f}(\xi)d\xi. \tag{3.12}$$

Observation 2. *Note that this is the same as the formula that gives $\widehat{f}$ in terms of f, except for the plus sign in the exponent and the factor of 2π. This accounts for the symmetry between f and $\widehat{f}$ in Theorem 6 .*

Our mission now is to determine the correctness of (3.12). This is not as clear as the question of whether the Fourier series of a periodic function f converges to f. The first issue is that $\widehat{f}$ may not be in L^1, as $\widehat{\chi}_a(\xi)$ indicates (see [16]), therefore the integral in (3.12) is not absolutely convergent in this situation. Even if it is, one cannot obtain (3.12) by simply inserting $\widehat{f}$ in the defining formula,

$$\int e^{i\xi x}\widehat{f}(\xi)d\xi = \iint e^{i\xi(x-y)}f(y)dyd\xi,$$

and interchanging the order of integration, because the integral $\int e^{i\xi(x-y)}d\xi$ is divergent. The easiest solution to each of these issues is to multiply $\widehat{f}$ by a "**cutoff**

function" to make the integrals converge and then to pass to the limit as the cutoff is removed.

One convenient cutoff function is $e^{-\epsilon^2\xi^2/2}$: For any fixed $\epsilon > 0$ it decreases rapidly as $\xi \to |+\infty|$, and to remove it we simply let $\epsilon \to 0$. Accordingly, instead of (3.12) for $f \in L^1$ we consider

$$\frac{1}{2\pi}\int e^{i\xi x}e^{-\epsilon^2\xi^2/2}\widehat{f}(\xi)d\xi = \frac{1}{2\pi}\iint e^{i\xi(x-y)}e^{-\epsilon^2\xi^2/2}f(y)dyd\xi.$$

Now the double integral is absolutely convergent and it is permissible to interchange the order of integration. The ξ -integral is evaluated as follows

$$\begin{aligned}\int e^{i\xi(x-y)}e^{-\epsilon^2\xi^2/2}d\xi &= \mathcal{F}\left[e^{-\epsilon^2\xi^2/2}\right](y-x)\\ &= \frac{\sqrt{2\pi}}{\epsilon}e^{-(x-y)^2/2\epsilon^2}.\end{aligned}$$

On the other hand,

$$\begin{aligned}\frac{1}{2\pi}\int e^{i\xi x}e^{-\epsilon^2\xi^2/2}\widehat{f}(\xi)d\xi &= \frac{1}{\epsilon\sqrt{2\pi}}\int f(y)e^{-(x-y)^2/2\epsilon^2}dy\\ &= f * \phi_\epsilon(x),\end{aligned}$$

where

$$\phi(x) = \frac{1}{\sqrt{2\pi}}e^{-x^2/2}, \quad \phi_\epsilon(x) = \frac{1}{\epsilon}\phi\left(\frac{x}{\epsilon}\right) = \frac{1}{\epsilon\sqrt{2\pi}}e^{-x^2/2\epsilon^2}.$$

But this is exactly the case with Theorem 3 and Example (3.7) (with ϵ replaced by $\epsilon\sqrt{2}$), and we conclude that if f is piecewise continuous,

$$\lim_{\epsilon\to 0}\frac{1}{2\pi}\int e^{i\xi x}e^{-\epsilon^2\xi^2/2}\widehat{f}(\xi)d\xi = \frac{1}{2}[f(x-)+f(x+)]$$

for all x. Hence, our primary outcome has been reached.

Theorem 7 (The Fourier Inversion Theorem). *Suppose f is integrable and piecewise continuous on $\mathbb{R}$, defined at its points of discontinuity so as to satisfy $f(x) = \frac{1}{2}[f(x-)+f(x+)]$ for all x. Then*

$$f(x) = \lim_{\epsilon\to 0}\frac{1}{2\pi}\int e^{i\xi x}e^{-\epsilon^2\xi^2/2}\widehat{f}(\xi)d\xi, \quad x \in \mathbb{R}. \tag{3.13}$$

Moreover, if $\widehat{f} \in L^1$, then f is continuous and

$$f(x) = \frac{1}{2\pi}\int e^{i\xi x}\widehat{f}(\xi)d\xi, \quad x \in \mathbb{R}. \tag{3.14}$$

Observation 3. *The inversion formula (3.14) or its alternative (3.13) describes a general function f as a continuous superposition of the exponential functions $e^{i\xi x}$.*

In this sense, the Fourier series expansion of periodic functions gives an equivalent for nonperiodic functions.

Corollary 1. *If $\widehat{f} = \widehat{g}$, then $f = g$.*

If ϕ is the Fourier transform of $f \in L^1$, we say that f is the inverse Fourier transform of ϕ and write $f = \mathcal{F}^{-1}\{\phi\}$, The operation $\mathcal{F}^{-1}$ is well-defined by Corollary 1.

Remark 5. *Functions f such that f and $\widehat{f}$ are both in L^1 exist in great abundance, one needs only a little smoothness of f to ensure the necessary decay of $\widehat{f}$ at infinity. For example, if the functions f and $\widehat{f}$, are both bounded, continuous, and integrable, and so f and $\widehat{f}$ are also in L^2.*

A number of variations on the Fourier inversion theorem are possible. For one thing, a version of (3.13) is true for functions $f \in L^1$ that are not piecewise continuous; namely, if $f \in L^1$, we have

$$f(x) = \lim_{\epsilon \to 0} \frac{1}{2\pi} \int e^{i\xi x} e^{-\epsilon^2 \xi^2/2} \widehat{f}(\xi) d\xi$$

for "almost every" $x \in \mathbb{R}$, in the sense of Lebesgue measure. For another, one can replace the cutoff function $e^{-\epsilon^2 \xi^2 t^2/2}$ in (3.13) by any of a large number of other functions with similar properties. (See Folland [17]. Theorem 8.31.) On the more naive level, one can ask whether the integral in (3.12) can be interpreted simply as a (principal value) improper integral, that is, whether

$$f(x) = \lim_{r \to \infty} \frac{1}{2\pi} \int_{-r}^{r} e^{i\xi x} \widehat{f}(\xi) d\xi.$$

This amounts to using the cutoff function that equals 1 on $[-r, r]$ and 0 elsewhere, and letting $r \to +\infty$; it is the obvious analogue of evaluating a Fourier series as the limit of its symmetric partial sums. Just as in that case, piecewise continuity of f does not suffice, but piecewise smoothness does.

Theorem 8. *If f is integrable and piecewise smooth on $\mathbb{R}$, then*

$$\lim_{r \to +\infty} \frac{1}{2\pi} \int_{-r}^{r} e^{i\xi x} \widehat{f}(\xi) d\xi = \frac{1}{2}[f(x-) + f(x+)] \tag{3.15}$$

for every $x \in \mathbb{R}$.

3.4 The Fourier transform on L^2

We built the Fourier transform in the space L^1, but our experience with Fourier series implies that the space L^2 should play an important role as well. This is indeed the case, There is an initial difficulty to be overcome, in that the integral $\int e^{-i\xi x} f(x) dx$

may not converge if f is in L^2 but not in L^1, but there is a way around this problem. The key observation is that the analogue of Parseval's formula holds for the Fourier transform. Namely, suppose that f and g are L^1 functions such that $\widehat{f}$ and $\widehat{g}$ are in L^1. Then f, g, $\widehat{f}$, and $\widehat{g}$ are also in L^2, and by (3.14) we have

$$\begin{aligned} 2\pi\langle f,g\rangle &= 2\pi\int f(x)\overline{g(x)}dx \\ &= \iint f(x)\overline{e^{i\xi x}\widehat{g}(\xi)}d\xi dx \\ &= \iint f(x)e^{-i\xi x}\overline{\widehat{g}(\xi)}dxd\xi \\ &= \int \widehat{f}(\xi)\overline{\widehat{g}(\xi)}d\xi \\ &= \langle \widehat{f},\widehat{g}\rangle. \end{aligned}$$

The Fourier transform, in other words, maintains inner products up to a factor of 2π. In particular, taking $g = f$, we obtain

$$\|\widehat{f}\|^2 = 2\pi\|f\|^2,$$

which is the "**Parseval formula**" for the Fourier transform.

Now, if f is an arbitrary L^2 function, we can find a sequence $\{f_n\}$ such that f_n and $\widehat{f_n}$ are in L^1 and $f_n \to f$ in the L^2 norm. Then

$$\|\widehat{f_n} - \widehat{f_m}\|^2 = 2\pi\,\|f_n - f_m\|^2 \to 0 \quad \text{as} \quad m,n \to +\infty,$$

so $\{\widehat{f_n}\}$ is a Cauchy sequence in L^2. Since L^2 is complete, it has a limit that can be demonstrated to depend only on f and not on the approximation sequence $\{f_n\}$.

The domain of the Fourier transform is therefore extended to include all of L^2, and a simple limiting argument demonstrates that this extended Fourier transform still retains the norm and inner product up to a factor of 2π, as well as the features of Theorem 6. In summary, we have the following outcome.

Theorem 9. (**The Plancherel Theorem**). *The Fourier transform, defined originally on $L^1 \cap L^2$ extends uniquely to a map from L^2 to itself that satisfies*

$$\langle \widehat{f},\widehat{g}\rangle = 2\pi\langle f,g\rangle \quad \textit{and} \quad \|\widehat{f}\|^2 = 2\pi\|f\|^2 \quad \textit{for all} \;\; f,g \in L^2.$$

Moreover, the formulas of Theorem 6 still hold for L^2 functions.

If f is in L^2 but not in L^1, the integral $\int f(x)e^{-i\xi x}dx$ defining $\widehat{f}$ may not converge pointwise, but it may be interpreted by a limiting process like the one we used in the inversion formula (3.13). That is, if $f \in L^2$, as $\epsilon \to 0$ the functions g^ϵ defined by

$$g^\epsilon(\xi) = \int e^{-i\xi x}e^{-\epsilon^2x^2/2}f(x)dx,$$

converge in the L^2 norm, and pointwise almost everywhere, to $\widehat{f}$. Likewise, the functions f^ϵ defined by

$$f^\epsilon(x) = \frac{1}{2\pi}\int e^{i\xi x}e^{-\epsilon^2\xi^2/2}\widehat{f}(\xi)d\xi,$$

converge in the L^2 norm, and pointwise almost everywhere, to f.
The Fourier inversion theorem is also a useful device for computing Fourier transforms. Indeed, upon setting $\phi = \widehat{f}$ the inversion formula (3.14) can be restated as

$$\phi = \widehat{f} \implies f(x) = (2\pi)^{-1}\widehat{\phi}(-x).$$

(The original formula (3.14) is valid when ϕ and $\widehat{\phi}$ are in L^1, but in the present form it continues to hold for any $\phi \in L^2$,) But this means that if ϕ is the Fourier transform of a known function f, we can immediately write down the Fourier transform of ϕ by setting $\xi = -x$:

$$\phi = \widehat{f} \quad \implies \quad \widehat{\phi}(\xi) = 2\pi f(-\xi).$$

N°	Funtion	Fourier transform
1.	$f(x)$	$\widehat{f}(\xi)$
2.	$f(x-c)$	$e^{-ic\xi}\widehat{f}(\xi)$
3.	$e^{icx}f(x)$	$\widehat{f}(\xi-c)$
4.	$f(ax)$	$a^{-1}\widehat{f}(a^{-1}\xi)$
5.	f'	$i\xi\widehat{f}(\xi)$
6.	$f(x)$	$i\widehat{f}'(\xi)$
7.	$(f*g)(x)$	$\widehat{f}\widehat{g}$
8.	$f(x)g(x)$	$(2\pi)^{-1}(\widehat{f}*\widehat{g})(\xi)$
9.	$e^{-ax^2/2}$	$\sqrt{2\pi/a}e^{-\xi^2/2a}$
10.	$(x^2+a^2)^{-1}$	$(\pi/a)e^{-a\lvert\xi\rvert}$
11.	$e^{-a\lvert x\rvert}$	$2a(\xi^2+a^2)^{-1}$
12.	$\chi_a(x) = \begin{cases} 1 & (\lvert x\rvert < a) \\ 0 & (\lvert x\rvert > a) \end{cases}$	$2\xi^{-1}\sin a\xi$
13.	$x^{-1}\sin ax$	$\pi\chi_a(\xi) = \begin{cases} 1 & (\lvert\xi\rvert < a) \\ 0 & (\lvert\xi\rvert > a) \end{cases}$

Table 3.1: SOME BASIC FOURIER TRANSFORMS

Table 3.1 contains a brief list of basic Fourier transform formulas that we have derived in this section. Much more extensive tables of Fourier transforms are available - for example, Erdelyi et al. [10].

3.5 Multivariable convolutions and Fourier transforms

In this section we consider functions of n real variables, that is, functions on the space $\mathbb{R}^n$ of n -tuples of real numbers. The notation for points in $\mathbb{R}^n$ will be $\boldsymbol{x} = (x_1, \ldots, x_n)$. We denote by $\boldsymbol{x} \cdot \boldsymbol{y}$ and $|\boldsymbol{x}|$ the usual dot product and norm on $\mathbb{R}^n$:

$$\begin{aligned} \boldsymbol{x} \cdot \boldsymbol{y} &= x_1 y_1 + x_2 y_2 + \cdots + x_n y_n, \\ |\boldsymbol{x}| &= (\boldsymbol{x} \cdot \boldsymbol{x})^{1/2} = \left(x_1^2 + \cdots + x_n^2\right)^{1/2}. \end{aligned}$$

Also, an integral sign with no limits will denote integration over all of n -space:

$$\int f(\boldsymbol{x}) d\boldsymbol{x} \text{ means } \int_{-\infty}^{+\infty} \ldots \int_{-\infty}^{+\infty} f(x_1, \ldots, x_n)\, dx_1 \ldots dx_n.$$

Most of the ideas of sections 3.1-3.2 can be generalized in a straightforward way to functions of several variables[1]. We then sketch out the expansions of our prior results to functions on $\mathbb{R}^n$, adding details only where new concepts are needed. As previously, convolutions are defined as follows:

$$f * g(\boldsymbol{x}) = \int f(\boldsymbol{y}) g(\boldsymbol{x} - y) d\boldsymbol{y}.$$

The basic algebraic properties of convolution stated in Theorem 1 and the differentiation property of Theorem 2 (with ordinary derivatives replaced by partial derivatives) still remain valid in the n-variable case, with the same proofs:

(i) $f * (ag + bh) = a(f * g) + b(f * h)$;

(ii) $f * g = g * f$;

(iii) $(f * g) * h = f * (g * h)$;

(iv) $\partial_j (f * g) = (\partial_, f) * g = f * \left(\partial_j g\right) \quad \left(\partial_j = \partial / \partial x_j\right)$.

By other hand, since

$$d(r\boldsymbol{x}) = (r dx_1) \cdots (r dx_n) = r^n d\boldsymbol{x} \quad (r > 0),$$

[1]One exception: For functions with more than one variable, the concepts of one-sided limits, piecewise continuity, and piecewise smoothness have no clear analogs. We will not be concerned with minimal smoothness assumptions for functions with several variables, which necessitate a more complex theory.

the appropriate analogue of the dilation formula (3.4) is

$$g_\epsilon(\boldsymbol{x}) = \epsilon^{-n} g\left(\epsilon^{-1}\boldsymbol{x}\right). \tag{3.16}$$

The factor ϵ^{-n} is the right one to ensure that the integral of g_ϵ is independent of ϵ :

$$\int g_\epsilon(\boldsymbol{x}) d\boldsymbol{x} = \int g\left(\epsilon^{-1}\boldsymbol{x}\right) d\left(\epsilon^{-1}\boldsymbol{x}\right) = \int g(\boldsymbol{y}) d\boldsymbol{y}.$$

The notion of one-sided limits does not make sense for functions of several variables, but we still have the following analogue of Theorems 3 and 4.

Theorem 10. *Suppose $g \in L^1$ and $\int g(x)dx = 1$, and let g_ϵ be defined by (3.16).*

(a) Suppose that either f is bounded or g vanishes outside a bounded set, so that $f * g$ is well-defined. If f is continuous at $\boldsymbol{x}$, then

$$\lim_{x \to 0} f * g_\epsilon(\boldsymbol{x}) = f(\boldsymbol{x}).$$

If f is continuous on a closed, bounded set D. the convergence is uniform on D

(b) If $f \in L^2$, then

$$\lim_{\delta \to 0} \|f * g_\epsilon - f\| = 0.$$

The Fourier transform of an integrable function f on $\mathbb{R}^n$ is defined by

$$\widehat{f}(\boldsymbol{\xi}) = \mathcal{F}[f(\boldsymbol{x})] = \int e^{-i\boldsymbol{\xi}\cdot\boldsymbol{x}} f(\boldsymbol{x}) d\boldsymbol{x}.$$

The estimate $|\widehat{f}(\boldsymbol{\xi})| \leq \int |f(\boldsymbol{x})| d\boldsymbol{x}$ and the fact that $\widehat{f}(\boldsymbol{\xi})$ is continuous and tends to zero as $|\boldsymbol{\xi}| \to +\infty$ are still valid. The basic transformational properties of the n-dimensional Fourier transform are just as in Theorem 6, with one new feature.

Theorem 11. *Suppose $f \in L^1(\mathbb{R}^n)$.*

a) For any $\boldsymbol{a} \in \mathbb{R}^n$

$$\mathcal{F}[f(\boldsymbol{x}-\boldsymbol{a})] = e^{-i\boldsymbol{a}\cdot\boldsymbol{\xi}} \widehat{f}(\boldsymbol{\xi}) \quad \text{and} \quad \mathcal{F}\left[e^{i\boldsymbol{a}\cdot\boldsymbol{x}} f(\boldsymbol{x})\right] = \widehat{f}(\boldsymbol{\xi}-\boldsymbol{a}).$$

b) If $\delta > 0$ and $f_\delta(\boldsymbol{x}) = \delta^{-n} f\left(\delta^{-1}\boldsymbol{x}\right)$, then

$$[f_\delta]\widehat{\ }(\boldsymbol{\xi}) = \widehat{f}(\delta\boldsymbol{\xi}) \quad \text{and} \quad \mathcal{F}[f(\delta\boldsymbol{x})] = (\widehat{f})_\delta(\boldsymbol{\xi}).$$

c) If $\partial f/\partial x_j$, exists and is in L^1, then

$$\left[\partial f/\partial x_j\right]\widehat{\ }(\boldsymbol{\xi}) = i\xi_j \widehat{f}(\boldsymbol{\xi}),$$

whereas if $x_j f(x)$ is integrable, then

$$\mathcal{F}\left[x_j f(\boldsymbol{x})\right] = i\partial\widehat{f}/\partial\xi_j.$$

d) If $g \in L^1$ and $f * g \in L^1$, then

$$(f * g)\widehat{} = \widehat{f}\widehat{g}.$$

e) The Fourier transform commutes with rorations: If R is a rotation of $\mathbb{R}^n$, then

$$\mathcal{F}\left[f(R\boldsymbol{x})\right] = \widehat{f}(R\boldsymbol{\xi}).$$

The following is another useful and elementary fact. Suppose f is a product of functions of the individual variables:

$$f(\boldsymbol{x}) = f_1(x_1) f_2(x_2) \cdots f_n(x_n).$$

Then the Fourier transform of f is the corresponding product of one-dimensional Fourier transforms:

$$\widehat{f}(\boldsymbol{\xi}) = \widehat{f}_1(\xi_1) \widehat{f}_2(\xi_2) \cdots \widehat{f}_n(\xi_n).$$

This is so because $e^{-i\boldsymbol{\xi}\cdot\boldsymbol{x}} = \left(e^{-i\xi_1 x_1}\right) \ldots \left(e^{-i\xi_n} x_n\right)$, so the n-dimensional integral defining $\widehat{f}$ decomposes into a product of one-dimensional integrals. An important example is the Gaussian $f(\boldsymbol{x}) = e^{-a|\boldsymbol{x}|^2/2}(a > 0)$, which is the product of the functions $f_j\left(x_j\right) = e^{-ax_j^2/2}$. As in one dimension, we have the following important result

$$\mathcal{F}\left[e^{-a|\boldsymbol{x}|^2/2}\right] = \left(\frac{2\pi}{a}\right)^{n/2} e^{-|\boldsymbol{\xi}|^2/2a}, \quad a > 0. \tag{3.17}$$

With this knowledge, the arguments that lead to the formulas (3.13) and (3.14) result in the n-dimensional Fourier inversion formula:

Theorem 12 (The Fourier Inversion Theorem). *If f is integrable and continuous an $\mathbb{R}^n$, then*

$$f(\boldsymbol{x}) = \lim_{\epsilon \to 0} \frac{1}{(2\pi)^n} \int e^{i\boldsymbol{\xi}\cdot\boldsymbol{x}} e^{-\epsilon^2|\boldsymbol{\xi}|^2} \widehat{f}(\boldsymbol{\xi}) d\boldsymbol{\xi}, \quad \boldsymbol{x} \in \mathbb{R}^n. \tag{3.18}$$

If also $\widehat{f}$ is integrable, then

$$f(\boldsymbol{x}) = \frac{1}{(2\pi)^n} \int e^{i\boldsymbol{\xi}\cdot\boldsymbol{x}} \widehat{f}(\boldsymbol{\xi}) d\boldsymbol{\xi}. \tag{3.19}$$

If f is only in L^1, formula (3.18) holds for virtually every x, and the cutoff function $e^{-\epsilon^2\xi^2/2}$ may be substituted by a variety of alternatives, just as it does in the one-dimensional case. However, the n-dimensional analogs of Theorem 7.6, in which the Fourier inversion integral is interpreted as the pointwise limit of integrals over a family of bounded regions Ω_r, which expand to fill out $\mathbb{R}^n$ as $r \to +\infty$, are rather delicate and not particularly useful, and we will not attempt to discuss them here.

The Plancherel theorem also remains true in the n -dimensional setting, with the same proof, except that the factor of 2π must be replaced by $(2\pi)^n$

$$\langle \widehat{f}, \widehat{g} \rangle = (2\pi)^2 \langle f, g \rangle, \quad \|\widehat{f}\|^2 = (2\pi)^n \|f\|^2.$$

Limiting processes like (3.18) can again be used to construct the Fourier transform and inverse Fourier transform of L^2 functions.

3.5.1 Applications

In any number of dimensions, the many discussions of linear operators that commute with translations are valid. The Fourier transform, in particular, turns any constant-coefficient differential operator into polynomial multiplication. For example, if

$$L[u] = au + \sum_{j=1}^{n} b_j \frac{\partial u}{\partial x_j} + \sum_{j,k=1}^{n} c_{jk} \frac{\partial^2 u}{\partial x_j \partial x_k},$$

so,

$$\big(L[u]\big)\widehat{\ }(\boldsymbol{\xi}) = P(\boldsymbol{\xi})\widehat{u}(\boldsymbol{\xi}) \quad \text{where } P(\boldsymbol{\xi}) = a + i\sum_{j=1}^{n} b_j \xi_j - \sum_{jx=1}^{n} c_{jk}\xi_j\xi_k.$$

Formally, then, we can solve the non homogeneous equation $L(u) = f$ on $\mathbb{R}^n$ by taking $\widehat{u} = \widehat{f}/P$. If $P(\boldsymbol{\xi})$ is never zero, this leads to a Fourier integral formula for u,

$$u(\boldsymbol{x}) = \frac{1}{(2\pi)^n} \int \frac{\widehat{f}(\boldsymbol{\xi})}{P(\boldsymbol{\xi})} e^{\mathcal{Z}\cdot \boldsymbol{x}} d\boldsymbol{\xi}, \tag{3.20}$$

or to the convolution formula $u = f * \mathbf{K}$ where $\mathbf{K}$ is the inverse Fourier transform of $1/P$. If P has zeros, the situation is technically more complicated since once has to worry about the integrability of $\widehat{f}/P$, but the same ideas work in principle.

We can solve the initial value problem for the heat equation in $\mathbb{R}^n$,

$$u_t = k\nabla^2 u, \quad u(\boldsymbol{x}, 0) = f(\boldsymbol{x}),$$

by taking the Fourier transform in $\boldsymbol{x}$ just as before. In view of (3.17), the result is

$$u(\boldsymbol{x}, t) = f * \mathbf{K}_t(\boldsymbol{x}), \quad \mathbf{K}_t(\boldsymbol{x}) = (4\pi k t)^{-n/2} e^{-|\boldsymbol{x}|^2/4kt},$$

which has the same physical interpretation as before. Similarly, to solve the Dirichlet problem in a half-space

$$H = \big\{(x_1 \ldots\ldots, x_{n+1}) : x_{n+1} > 0\big\}$$

in $\mathbb{R}^{n+1}$, we adopt the notation $y = x_{n+1}, \boldsymbol{x} = (x_1, \ldots, x_n)$, so that the problem is

$$\nabla^2 u = \nabla^2_{\boldsymbol{x}} u + u_{yy} = 0 \text{ in } H, \quad u(\boldsymbol{x}, 0) = f(\boldsymbol{x}).$$

Taking the Fourier transform in $\boldsymbol{x}$ and imposing the requirement of boundedness, we obtain $\widehat{u}(\boldsymbol{\xi}, t) = \widehat{f}(\boldsymbol{\xi})e^{-|\boldsymbol{\xi}|y}$ as before, and hence $u(\boldsymbol{x}, y) = f * \boldsymbol{P}_y(\boldsymbol{x})$ where $\widehat{\boldsymbol{P}}_y(\boldsymbol{\xi}) = e^{-|\boldsymbol{\xi}|y}$. The calculation of the inverse Fourier transform of $e^{-|\boldsymbol{\xi}|y}$ is trickier than in the 1-dimensional case. The result is the n-dimensional Poisson integral formulas:

$$u(\boldsymbol{x}, y) = f * \boldsymbol{P}_y(\boldsymbol{x}), \quad \boldsymbol{P}_y(\boldsymbol{x}) = \frac{\Gamma((n+1)/2)}{\pi^{(n+1)/2}} \frac{y}{\big(|\boldsymbol{x}|^2 + y^2\big)^{(n+1)/2}}.$$

Chapter 4

Quaternions

In this chapter we shall present a number of concepts of quaternions which are important for our study. Our main reference is [21].

4.1 Real and complex quaternions

4.1.1 Elementary properties of real quaternions

In this section we assemble a number of well known properties of real quaternions, which we found in different classic books in this field.

Let $e_0 = 1, e_1, e_2, e_3$ the basic elements in $\mathbb{R}^4$ where the vector e_k is to identify with this 4-tupel which has at the $(k+1)$-th component the number one and is zero otherwise. An arbitrary element $x \in \mathbb{R}^4$ now has the representation $x = x_0e_0 + x_1e_1 + x_2e_2 + x_3e_3$. We will call the part $x_0e_0 =: Sc\ x$ the **scalar part** of x and $\underline{x} = x - x_0e_0$, written by $Vec\ x$ is the **vector part** of x. Let $x, y \in \mathbb{R}^4$. Now we define a product in $\mathbb{R}^4$ which fulfils the following conditions:

i) $e_1^2 = e_2^2 = e_3^2 = -1$,

ii) $e_1e_2 = e_3, e_2e_3 = e_1, e_3e_1 = e_2$,

iii) $e_ie_j + e_je_i = 0\ (i, j = 1, 2, 3; i \neq j)$.

If we write for short $(\underline{x}, \underline{y}) := x_1y_1 + x_2y_2 + x_3y_3$ and $\underline{x} \times \underline{y} := \left(x_2y_3 - x_3y_2\right) e_1 + (x_3y_1 - x_1y_3)e_2 + (x_1y_2 - x_2y_1)e_3$, then by using of the algebraic rules (i),(ii),(iii) the **quaternionc product** xy is given by

$$xy = x_0y_0 - (\underline{x}, \underline{y}) + x_0\underline{y} + y_0\underline{x} + \underline{x} \times \underline{y}.$$

After introduction of this product, the tuple $\left(\mathbb{R}^4, .\right)$ is called **algebra of real quaternions**. In honour of W.R. HAMILTON we signify $(\mathbb{R}^4, .)$ by $\mathbb{H}$. If $x = \underline{x}$ then x is called a **pure quaternion**. The subset of all pure quaternions is denoted by $Vec\ \mathbb{H}$, while the subset of all scalars will be signified by $Sc\ \mathbb{H}$. For $x \in Vec\ \mathbb{H}$, we identify $x = \underline{x}$. The quaternion $\bar{x} = x_0 - \underline{x}$ is called the **conjugate** to x. The mapping $x \longmapsto \bar{x}$ is called **conjugation**. The number $|x|$ defined by $x\bar{x} = |x|^2$ is named **norm** or **absolute value** of x. If $|x| = 1$ then a quaternion is said to be a **unit quaternion**.

Example 2. *Let $x = 2 + e_1 + 3e_2 + e_3$ then $\bar{x} = 2 - e_1 - 3e_2 - e_3$. Sc $x = 2$, $|x|^2 = 4 + 1 + 9 + 1 = 15$. Let $y = e_1 - 2e_3$ be another quaternion then*

$$\begin{aligned} xy &= 2 - (-1) + 2e_1 - 2e_3 + (-6)e_1 + 4e_2 - 3e_3 \\ &= 3 - 4e_1 + 4e_1 - 5e_3. \end{aligned}$$

Lemma 2. *xy is an $\mathbb{R}$ -bilinear and associative product, but it is not commutative.*

Remark 6. *For $x, y \in \mathbb{H}$ and $\mu \in \mathbb{R}$ one has the relations*

(i) $\overline{x+y} = \overline{x} + \overline{y}$,

(ii) $\overline{\mu x} = \mu \overline{x}$,

(iii) $\overline{\overline{x}} = x$,

(iv) $\overline{xy} = \overline{y}\,\overline{x}$ *which is named **anti-involution**.[6]*

Corollary 2. *Let $x, y \in Vec\ \ \mathbb{H}$, then*

(i) $x \times y = Vec\ \ (xy) = \frac{1}{2}(xy - yx)$,

(ii) $(x, y) = Sc\ \ (\bar{x}y) = \frac{1}{2}(\bar{x}y + y\bar{x})$.

Corollary 3. *From $x^2 = y^2$ it does not follow that $x = \pm y$.*

Proof. Take for instance e_1 and e_2. □

Lemma 3. *A quaternion x is a vector if and only if $x^2 < 0$ or $x = 0$.*

Proof. ($\Rightarrow$) Let $x = \underline{x}$, then

$$\begin{aligned} x^2 &= -x\bar{x} \\ &= -(x, x) \\ &= -\left(x_1^2 + x_2^2 + x_3^2\right). \end{aligned}$$

($\Leftarrow$) In the opposite direction we have $x^2 = x_0^2 - |\underline{x}|^2 + 2x_0\underline{x}$. Now only two cases are possible:

(i) If $x_0 = 0$ and then we are ready.

(ii) If $\underline{x} = 0$ and $x_0 \neq 0$ then x^2 would be positive contrary to the assumption.

□

Lemma 4. *A quaternion x is real if and only if for each other quaternion y holds $yx = xy$.*

Proof. It is necessary to show only that from the commutativity relation with an arbitrary quaternion y it follows that x has to be a real number. Taking for $y = e_1$ we get

$$\begin{aligned} xe_1 &= -x_1 + x_0e_1 - x_2e_3 + x_3e_2 \\ &= -x_1 + x_0e_1 + x_2e_3 - x_3e_2 \\ &= e_1x. \end{aligned}$$

Hence $0 = 2(x_2e_3 - x_3e_2)$. It immediately follows that $x_2 = x_3 = 0$. After repeating this procedure with the quaternion $y = e_2$ we obtain that x_1 also has to be zero. □

Proposition 1. *For all $x, y \in \mathbb{H}$ the identity $|xy| = |x||y|$ holds. If $x \neq 0$ then x^{-1} exists and we have*

$$x^{-1} = \bar{x}|x|^{-2}. \tag{4.1}$$

From (4.1) we can see that $\mathbb{H}$ is a normed division algebra. [6] Furthermore, we get

$$|x|^{-1} = \left|x^{-1}\right|.$$

Lemma 5. *Let $\underline{y}$ be an invertible vector. Then for any vector $\underline{x}$ the expression $\underline{y}\underline{x}\underline{y}^{-1}$ is also a vector.*

Proof. As $\underline{x}$ is a vector, $\underline{x}^2$ is real and non-positive. Using Lemma 4 we have $\left(\underline{y}\underline{x}\underline{y}^{-1}\right)^2 = \underline{x}^2$ and so $\underline{y}\underline{x}\underline{y}^{-1}$ has to be a vector. □

Corollary 4. *The map $\rho_y : \mathbb{R}^3 \ni \underline{x} \to -\underline{y}\underline{x}\underline{y}^{-1} \in \mathbb{R}^3$ is a reflection in the plane which lies orthogonal to the vector $\underline{y}$.*

Proof. The map ρ_y is linear and $\rho_y\underline{y} = -y$, while for any vector $r \perp y$ it follows that

$$\begin{aligned} \rho_y(r) &= -\underline{y}r\underline{y}^{-1} \\ &= r\underline{y}\underline{y}^{-1} \\ &= r. \end{aligned}$$

□

Corollary 5. *Each rotation in $\mathbb{R}^3$ has for some non-zero $\underline{y}$ the form $-\rho_y$. Conversely, any such map can be seen as rotation in $\mathbb{R}^3$.*

Proof. We already know that the product of two plane reflections is just a rotation and vice versa. □

Lemma 6. *Let y be a quaternion. Then there exists a vector $\underline{a} \neq 0$, such that $y\underline{a}$ is also a vector.*

Proof. Let $\underline{a}$ be a vector in $\mathbb{R}^3$ orthogonal to the vector part of y. Then $y\underline{a} = y_0\underline{a} + \underline{y} \times \underline{a} \in \mathbb{R}^3$. □

Corollary 6. *Each quaternion can be described as a product of two vectors.*

Lemma 7. *An arbitrary unit quaternion can be represented as the product $y = y^{-1}x^{-1}$, where $x = \underline{x}$ and $y = \underline{y}$ are non-zero vectors.*

Proof. We know from Lemma 6 that for any unit quaternion e a non-zero vector $\underline{x}$ exists such that $e\underline{x}$ is a vector. Because of $|e| = 1$ we have $|e\underline{x}| = |\underline{x}|$. In this way $e\underline{x}$ has to be a rotation. Then there exists a vector $\underline{y} \neq 0$ with $e\underline{x} = \underline{y}\underline{x}\underline{y}^{-1}$ and so the statement follows. □

4.1.2 Representation of real quaternions

We will group here some of the most important properties of the representation of real quaternions.

Theorem 13. *An arbitrary quaternion $x \in \mathbb{H}$, $\underline{x} \neq 0$ permits the representation*

$$x = |x|(\cos\phi + \omega(x)\sin\phi)$$

where $\phi = \operatorname{arc\,cot}\left(x_0/|\underline{x}|\right)$ *and* $\omega(x) = \underline{x}/|\underline{x}| \in S^3$.

Proof. It is well known that

$$\sin\phi = \frac{1}{\sqrt{(1+\cot^2\phi)}} \quad \text{and} \quad \cos\phi = \frac{\cot\phi}{\sqrt{(1+\cot^2\phi)}}.$$

We obtain under our assumption

$$\sin\operatorname{arc\,cot}\frac{x_0}{|\underline{x}|} = \frac{1}{\sqrt{1+\left(\frac{x_0}{|\underline{x}|}\right)^2}} \quad \text{and} \quad \cos\operatorname{arc\,cot}\frac{x_0}{|\underline{x}|} = \frac{\frac{x_0}{|\underline{x}|}}{\sqrt{1+\left(\frac{x_0}{|\underline{x}|}\right)^2}}$$

Then we get by a straightforward calculation

$$\cos\phi + (\underline{x}/|\underline{x}|)\sin\phi = x/|x|,$$

which verifies our theorem. □

Remark 7. *For $x_0 = 0$, we get the decomposition mentioned in Proposition 1.*

Example 3. *Let $x = 3 + 2e_1 + 2e_2 + e_3$ then $|x| = 3\sqrt{2}$, $|\underline{x}| = 3$, $\phi = 45°$. Thus we obtain the representation*

$$x = 3\sqrt{2}\left[\cos 45° + \frac{(2c_1 + 2c_2 + \epsilon_3)}{3}\sin 45°\right].$$

Corollary 7 (MOIVRE's formula.). *Let $x \in \mathbb{H}$, $\underline{x} \neq 0, n \in \mathbb{N}$, then the following formula is valid:*

$$(\cos \rho + \omega(x) \sin \phi)^n = \cos n\phi + \omega(x) \sin n\phi.$$

Lemma 8. *The $\mathbb{R}$-linear hull of the set $\{1, x\}$, where x is not real, forms a subalgebra which is isomorphic to $\mathbb{C}$.*

Theorem 14. *The algebra of real quaternions* H *can be represented by the matrices*

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix}, \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix}.$$

Therefore each quaternion x permits the representation

$$x = \begin{pmatrix} \bar{z}_1 & -\bar{z}_2 \\ z_2 & z_1 \end{pmatrix},$$

with $z_1 := x_0 + ix_1$ and $z_2 := x_2 + ix_3$.

Corollary 8. *The algebra of real quaternions $\mathbb{H}$ can be generated by the real matrices*

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix},$$

$$\begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}.$$

Remark 8. *In [1] the following good relations between quaternions and matrices are given. For this reason we introduce a vector $u = (u_0, u_1, u_2, u_3)^T \in \mathbb{R}^4$ and a corresponding matrix*

$$T^{\pm}(u) = \begin{pmatrix} u_0 & -\underline{u}^T \\ \underline{u} & u_0 E_3 \pm K(\underline{u}) \end{pmatrix},$$

with

$$k(\underline{u}) = \begin{pmatrix} 0 & -u_3 & u_2 \\ u_3 & 0 & -u_1 \\ -u_2 & u_1 & 0 \end{pmatrix} \quad \text{and} \quad E_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

It is clear that $K(\underline{v})\underline{w} = \underline{v} \times \underline{w}$, where $\underline{v}, \underline{w} \in \mathbb{R}^3$.

Lemma 9. *Let $u, v, w \in \mathbb{R}^4$. Then we have*

i) $vw = T^+(v)w = T^-(w)v.$

ii) $uvw = T^+(u)T^+(v)w = T^+(u)T^-(w)v = T^-(w)T^-(v)u.$

4.1.3 Complex quaternions

More than real quaternions, complex quaternions play an important role in theoretical physics. Let us now discuss the fundamental properties of quaternions with complex-valued coefficients. We will use the so-called **PAULI matrices**:

$$\sigma_0 = \begin{pmatrix} 1 & 0 \\ (1 & 1 \end{pmatrix}, \quad \sigma_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

PAULI matrices form an **algebra P**. Then we get the representation of an arbitrary element $x \in P$ in the form

$$x = x_0\sigma_0 + x_1\sigma_1 + x_2\sigma_2 + x_3\sigma_3 + x_4\sigma_2\sigma_3 + x_5\sigma_3\sigma_1 + x_6\sigma_1\sigma_2 + x_7\sigma_1\sigma_2\sigma_3$$

We have to distinguish four classes of elements, namely complex linear combinations of scalars σ_0, vectors $\sigma_1, \sigma_2, \sigma_3$, bivectors $\sigma_2\sigma_3, \sigma_3\sigma_1, \sigma_1\sigma_2$ and pseudoscalars $\sigma_1\sigma_2\sigma_3$. PAULI matrices satisfy the conditions $\sigma_i\sigma_j + \sigma_j\sigma_i = 2\delta_{ij}\sigma_0$, $i, j = 1, 2, 3$, where δ_{ij} denotes KRONECKER's symbol. On account of $(\sigma_1\sigma_2\sigma_3)^2 = -\sigma_0$ we obtain that the linear space generated by $\{\sigma_0, \sigma_1\sigma_2\sigma_3\}$ is isomorphic to the field of complex numbers $\mathbb{C}$.

Remark 9. *The centre of P is $\mathbb{C}$.*

Lemma 10. *With $\varepsilon := i\sigma_0$ we get*

$$\begin{aligned} -\varepsilon\sigma_0 &= \sigma_1\sigma_2\sigma_3, \\ -\varepsilon\sigma_1 &= \sigma_2\sigma_3, \\ -\varepsilon\sigma_2 &= \sigma_3\sigma_1, \\ -\varepsilon\sigma_3 &= \sigma_1\sigma_2, \\ \varepsilon\sigma_1\sigma_2 &= +\sigma_3, \\ \varepsilon\sigma_2\sigma_3 &= +\sigma_1, \\ \varepsilon\sigma_3\sigma_1 &= +\sigma_2, \\ \varepsilon\sigma_1\sigma_2\sigma_3 &= +\sigma_0, \end{aligned}$$

i.e. the multiplication by ε transforms scalars into pseudoscalars, vectors into bivectors, bivectors into vector, and pseudoscalars into scalars.

Remark 10. *The operator of multiplication by ε is often called HODGE map and marked by $*$.*

Definition 4. *Let $x = x_1\sigma_1 + x_2\sigma_2 + x_3\sigma_3$ and $y = y_1\sigma_1 + y_2\sigma_2 + y_3\sigma_3$ be vectors in P. The formal determinant*

$$\det \begin{bmatrix} \sigma_1 & \sigma_2 & \sigma_3 \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{bmatrix}$$

*is called **cross product** and denoted by $x \times y$.*

Lemma 11. *The algebra P contains zero divisors.*

Proof. One has to take only the complex quaternions $(1-\sigma_1)$ and $(1+\sigma_1)$. The product of both is zero. □

We find for each element $x = p + q\varepsilon$ of the PAULI algebra with $p, q \in \mathbb{H}$ the conjugation (do not confuse with the notions of the complex-conjugation) acts as follows $x = \bar{p} + \bar{q}\varepsilon$.

Definition 5. *Let* $x = p + q\varepsilon, \quad p = p_0 + \underline{p} \quad q = q_0 + \underline{q}$. *Then we call the complex magnitude* $\|x\|_{\mathbb{C}}$ *of the element x defined by*

$$\|x\|_{\mathbb{C}}^2 = x\overline{x} = |p|^2 - |q|^2 + 2\varepsilon[p_0q_0 + (\underline{p}, \underline{q})]$$

complex-valued norm in P where $|p|^2 = p\underline{p}$ *and* $|q|^2 = q\bar{q}$.

Lemma 12. *There exists a unique real-valued norm* $\|x\|$ *of* $x \in P$ *which satisfies the conditions*

(i) $\|xy\| = \|x\|\|y\|$,

(ii) $\|\mu\sigma_0\| = |u|(\mu \in \mathbb{R})$

This norm is given by

$$\|x\|^4 = \left|\|x\|_{\mathbb{C}}^2\right|^2.$$

4.2 Rotations

As is well known, the matrices

$$R(\phi) = \begin{pmatrix} \cos\theta & -\sin\phi \\ \sin\phi & \cos\phi \end{pmatrix},$$

with $\phi \in \mathbb{R}$, form a group. It is easy to verify that for $\phi, \psi \in \mathbb{R}$ the relation

$$R(\phi + \psi) = R(\phi)R(\psi)$$

holds. A matrix of this type may be used to describe any rotation in the plane. **special orthogonal group** $SO(2)$ is the corresponding group of all rotations in the plane. The unit circle S^1 in the complex plane, with complex multiplication as the group action, is another manifestation of this group. The complex number $e^{i\phi}$ can be used to identify the matrix $R(\phi)$.

4.2.1 Rotations in $\mathbb{R}^3$

In $\mathbb{R}^3$, each rotation $R(\phi)$ is defined by an axis a and a rotation angle ϕ around this axis. $SO(3)$ denotes the group of all rotations. Linear transformations with matching matrices $R(\phi) = R$ that maintain the scalar product

$$(x, y) = (Rx, Ry)$$

and $\det R = 1$, can be used to realize this group. Rotations around the coordinate axis can be expressed by the matrices

$$R_1(a) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\alpha & -\sin a \\ 0 & \sin a & \cos a \end{pmatrix},$$

$$R_2(\beta) = \begin{pmatrix} \cos\beta & 0 & \sin\beta \\ 0 & 1 & 0 \\ -\sin\beta & 0 & \cos\beta \end{pmatrix},$$

$$R_3(\gamma) = \begin{pmatrix} \cos\gamma & -\sin\gamma & 0 \\ \sin\gamma & \cos\gamma & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

The angles a, β, γ are called **EULER angles**. Now, let us explain the role of quaternions for the representation of rotations in the space. We already know from Section 4.1.2 that a real quaternion x permits the binary representation

$$x = |x| \left(\frac{x_0}{|x|} + \omega \frac{|\underline{x}|}{|x|} \right),$$

with

$$\omega^2 = \left(\frac{\underline{x}}{|\underline{x}|} \right)^2 = -1.$$

Definition 6. *Let $\omega \in \operatorname{Vec}\mathbb{H}$, $\omega^2 = -1$. The exponential function $e^{\omega\theta}$ is defined by the formal expansion*

$$e^{\omega\theta} := 1 + \omega\theta - \frac{1}{2!}\theta^2 - \frac{1}{3!}\omega\theta^3 + \dots,$$

where we have uniform convergence of the expansion

$$e^{\theta} = 1 + \theta + \frac{1}{2!}\theta^2 + \dots, \theta \in \mathbb{R}$$

and the estimation

$$\left| e^{\omega\theta} \right| \leq 1 + |\theta| + \frac{1}{2!}|\theta|^2 + \dots = e^{|\theta|}.$$

Proposition 2. *Let $\omega \in Vec\ \mathbb{H}$ with $\omega^2 = -1$. Then*

(i) $e^{\omega\theta} = \cos\theta + \omega \sin\theta$.

(ii) $\left|e^{\omega\theta}\right| = 1$.

(iii) In the binary representation we have

$$\cos\theta = \frac{x_0}{|x|}, \quad \sin\theta = \frac{|\underline{x}|}{|x|}.$$

Proof. It is sufficient to compare separately the vector parts and scalar parts in the definition of $e^{\omega\theta}$. □

Corollary 9. *Let $x, y \in Vec\ \mathbb{H}$. Then*

(i) $xy = yx \longrightarrow e^x e^y = e^{x+y}$.

(ii) From $e^x e^y = e^{x+y}$ if does not follow that $xy = yx$.

Proof. We have only to verify (ii). For this reason we take $x := 3\pi e_1$ and $y := 4\pi e_2$. Then we obtain $e^{3\pi e_1} = -1, e^{4\pi e_2} = 1$ and $e^{3\pi e_1 + 4\pi e_2} = e^{((3/5)e_1 + (4/5)3_2)5\pi}$. The latter identity is minus one, because of $((3/5)e_1 + (4/5)e_2)^2 = -1$. On the other band, $4\pi e_2 \cdot 3\pi e_1 = -3\pi e_1 \cdot 4\pi e_2 \neq 3\pi \cdot e_1 4\pi e_2$. □

A similar definition can also be given in the algebra P.

Definition 7. *Let $\omega \in \text{Vec}\, P, \theta \in R, \omega^2 = 1$ then the exponential function can be defined by the series*

$$e^{\phi_\omega} = 1 + \omega\theta + \frac{1}{2!}\theta^2 + \omega\frac{1}{3!}\theta^3 + \cdots$$

The convergence of this series is ensured in definition 6.

Proposition 3. *Let $\omega \in Vec\ P$, $\omega^2 = 1$ then we have*

(i) $e^{\theta\omega} = \cosh\theta + \omega \sinh\theta$,

(ii) $e^{\theta(j\omega)} = \cos\theta + (j\omega)\sin\theta, j^2 = -1$,

where $j\omega$ is a unit vector in P.

Proof. It follows from the complex analysis in the plane. We have only to consider the relations $\cosh\theta = \cos j\theta$ and $\sin j\theta = -j \sinh\theta$. □

Remark 11. *In physic from the mapping $x \to e^{\theta e_1 e_2 e_3} x$ $(\theta \in \mathbb{R})$ with $(e_1 e_2 e_3)^2 = -1$, where $\{e_1, e_2, e_3\}$ is a basis in $\mathbb{R}^3$, is called the **LARMORREINICH transformation**.*

Corollary 10 (EULER's formula.). *For the rotation of a vector $x \in \text{Vec}\,\mathbb{H}$ we have the formula*

$$x' = x\cos\theta + (\cos\theta - 1)(x, \omega) + (\omega \times x)\sin\theta,$$

with $\omega^2 = -1$ and ω stands for the rotation axis.

Proof. We start with the decomposition $x = u + v$, with $u \times \omega = 0$ and $(v, \omega) = 0$. We do the same for $x' = u' + v'$ with $u' \times \omega = 0$ and $(v', \omega) = 0$ Since ω commutates with u, we get

$$u' = u = -(\omega, x)\omega.$$

Furthermore

$$v' = e^{\frac{1}{2}\theta\omega} v e^{-\frac{1}{2}\theta\omega} = v e^{-\theta\omega}.$$

Hence.

$$v' = v(\cos\theta - \omega \sin\theta) = v\cos\theta - (v \times \omega)\sin\theta$$

and

$$\begin{aligned} x' = u' + v' &= -(\omega, x)\omega + v\cos\theta - (v \times \omega)\sin\theta \\ &= -(\omega, x)\omega + x\cos\theta - u\cos\theta + (\omega \times v)\sin\theta \\ &= -(\omega, x)\omega - u\cos\theta + x\cos\theta + (\omega \times x)\sin\theta \\ &= (\cos\theta - 1)(\omega, x)\omega + x\cos\theta + (\omega \times x)\sin\theta. \end{aligned}$$

□

Corollary 11. *The coordinates of $x \in Vec\,\mathbb{H}$ transform as follows:*

$$\begin{aligned} x_1' &= x_1\cos\theta + \omega_1(\cos\theta - 1)\left[-(x_1\omega_1 + x_2\omega_2 + x_3\omega_3)\right] + (\omega_2 x_2 - \omega_3 x_2)\sin\theta, \\ x_2' &= x_2\cos\theta + \omega_2(\cos\theta - 1)\left[-(x_1\omega_1 + x_2\omega_2 + x_3\omega_3)\right] + (\omega_3 x_1 - \omega_1 x_3)\sin\theta, \\ x_3' &= x_3\cos\theta + \omega_3(\cos\theta - 1)\left[-(x_1\omega_1 + x_2\omega_2 + x_3\omega_3)\right] + (\omega_1 x_2 - \omega_2 x_1)\sin\theta. \end{aligned}$$

Remark 12. *For $\omega = e_1$ we get*

$$\begin{aligned} x_1' &= x_1\cos\theta + x_1(1 - \cos\theta), \\ x_2' &= x_2\cos\theta - x_3\sin\theta, \\ x_3' &= x_3\cos\theta + x_2\sin\theta. \end{aligned}$$

This is equivalent to

$$x' = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = R_1(\theta)x.$$

Similarly, we achieve $x' = R_2(\theta)x$ for $\omega = e_2$ and $x' = R_3(\theta)x$ for $\omega = e_3$.

Definition 8. *Let $u \in \mathbb{R}^3$. Each vector $x \in \mathbb{R}^3$ permits the decomposition $x = v + \lambda u$, with $v \in \{z : (z, u) = 0\} = u^{\perp}$. Then the map*

$$R_u x = v - \lambda u$$

*is called **reflection along** u, i.e., R_u is the identity onto $u^{\perp}$ and minus the identity on the fine through u.*

Corollary 12. *Any reflection has the representation*

$$R_u x = x - 2\frac{(x,u)}{(u,u)}u.$$

Proof. Let $x = v + \lambda u$ then $R_u x = v - \lambda u$. Obviously, $\lambda = \frac{(x,u)}{(u,u)}$, such that

$$R_u x = x - 2\lambda u.$$

□

Theorem 15 (CARTAN-DIEUDONNE)**.** *Each orthogonal transformation $R \in O\left(\mathbb{R}^3\right)$ can be represented as a product of a finite number s of reflections along the vectors $\omega_j, (\omega_j, \omega_j) \neq 0 (j = 1, \ldots, s)$, i.e., $R = R_{\omega 1}, R_{\omega 2} \ldots R_{\omega s}$.*

Remark 13. *Theorem 15 can be generalized to vector spaces with signature p, q and dimension n (for more detail see [22]).*

Chapter 5

Two-dimensional (left) Quaternion Fourier Transform (2D-QFT)

In this chapter, we introduce the Two-dimensional (left) Quaternion Fourier Transform ($2D - QFT$). Some important properties are studied. Moreover, we applied this transform to solve the heat equation in $\mathbb{R}^2$.

5.1 Basics

From Theorem 13, we can can written any quaternion q as

$$q = |q|e^{\mu\phi}, \tag{5.1}$$

where where $\theta = \arctan |\operatorname{Sc}(q)| / \operatorname{Vec}(q), 0 \leq \theta \leq \pi$, is the eigenangle or phase of q. When $|q| = 1, q$ is a unit quaternion. Euler's and De Moivre's formulas still hold in quaternion space, that is, for a pure unit quaternion the following holds:

$$\begin{aligned} e^{\mu\theta} &= \cos\phi + \mu \sin\phi, \\ e^{\mu n\phi} &= \left(\cos\phi + \mu\sin\phi\right)^n = \cos n\phi + \mu \sin n\phi. \end{aligned}$$

As in the algebra of complex numbers, we can define three nontrivial algebra involutions for quaternions

$$\begin{aligned} \alpha(q) &= -iqi = -i\left(q_0 + iq_1 + jq_2 + kq_3\right) i = q_0 + iq_1 - jq_2 - kq_3, \\ \beta(q) &= -jqj = -j\left(q_0 + iq_1 + jq_2 + kq_3\right) j = q_0 - iq_1 + jq_2 - kq_3, \\ \gamma(q) &= -kqk = -k\left(q_0 + iq_1 + jq_2 + kq_3\right) k = q_0 - iq_1 - jq_2 + kq_3. \end{aligned} \tag{5.2}$$

For our purposes, it is convenient to introduce the inner product of two quaternion-valued functions, $f, g : \mathbb{R}^2 \to \mathbb{H}$, as follows:

$$(f, g)_{L^2(\mathbb{R}^2;\mathbb{H})} = \int_{\mathbb{R}^2} \overline{g(\boldsymbol{x})} f(\boldsymbol{x}) d^2\boldsymbol{x}. \tag{5.3}$$

In particular, if $f = g$, then we obtain the associated norm:

$$\|f\|^2_{L^2(\mathbb{R}^2;\mathbb{H})} = \int_{\mathbb{R}^2} |f(\boldsymbol{x})|^2 d^2\boldsymbol{x}. \tag{5.4}$$

5.2 Basic Properties of 2D-QFT

We list some of their basic properties with proofs. Most of them are essentially straightforward extensions of the Two-dimensional Fourier Transform (2D-FT) properties. Denote by $\{\boldsymbol{e}_1, \boldsymbol{e}_2\}$ the standard basis of $\mathbb{R}^2$.

Definition 9. *Let $f \in L^1\left(\mathbb{R}^2; \mathbb{H}\right)$. We define the* ***Two-dimensional (left) Quaternion Fourier Transform*** *of f (**2D-QFT** for short) is the function $\mathcal{F}_q\{f\} : \mathbb{R}^2 \to \mathbb{H}$ defined by*

$$\mathcal{F}_q\{f\}(\boldsymbol{\omega}) = \widehat{f}(\boldsymbol{\omega}) = \int_{\mathbb{R}^2} e^{-\mu\boldsymbol{\omega}\cdot\boldsymbol{x}} f(\boldsymbol{x}) d^2\boldsymbol{x}, \tag{5.5}$$

where $\mu \in \mathbb{H}$ is a pure unit quaternion, $\boldsymbol{x} = x_1\boldsymbol{e}_1 + x_2\boldsymbol{e}_2$, $\boldsymbol{\omega} = \omega_1\boldsymbol{e}_1 + \omega_2\boldsymbol{e}_2$ and $e^{-\mu\boldsymbol{\omega}\cdot\boldsymbol{x}}$ is called the **quaternion Fourier kernel**.

Previous to present the Inversion formula for the $2D - QFT$, we present some important results (for more details see [2]).

Definition 10. *Let $f \in L^1(\mathbb{R}; \mathbb{H})$. We define the* ***One-dimensional (left) Quaternion Fourier Transform*** *of f (**1D−QFT** for short), by*

$$\mathcal{F}\{f(x)\}(t) = \widehat{f}(t) = \int_{\mathbb{R}} e^{-\mu t x} f(x) dx.$$

Theorem 16 (Fourier Inversion Formula for the 1D−QFT [2]). *Let f be an integrable and piecewise continuous on $\mathbb{R}$, with values in $\mathbb{H}$, defined at its points of discontinuity as to satisfy $f(x) = \frac{1}{2}[f(x^-) + f(x^+)]$ for all x. Then*

$$f(x) = \lim_{\varepsilon \to 0} \frac{1}{2\pi} \int e^{\mu t x} e^{-\varepsilon^2 t^2/2} \widehat{f}(t) dt, \quad x \in \mathbb{R}.$$

Moreover, if $\widehat{f} \in L^1(\mathbb{R}; \mathbb{H})$, then f is continuous and

$$f(x) = \frac{1}{2\pi} \int e^{\mu t x} \widehat{f}(t) dt, \quad x \in \mathbb{R}.$$

Theorem 17 (Inversion formula for the 2D-QFT). *Suppose that $f \in L^2\left(\mathbb{R}^2; \mathbb{H}\right)$ and $\mathcal{F}_q\{f\} \in L^1\left(\mathbb{R}^2; \mathbb{H}\right)$. Then the $2D - QFT$ is invertible with inverse*

$$\mathcal{F}_q^{-1}\left[\mathcal{F}_q\{f\}\right](\boldsymbol{x}) = f(\boldsymbol{x}) = \frac{1}{(2\pi)^2} \int_{\mathbb{R}^2} e^{\mu\boldsymbol{\omega}\cdot\boldsymbol{x}} \mathcal{F}_q\{f\}(\boldsymbol{\omega}) d^2\boldsymbol{\omega}. \tag{5.6}$$

Proof. Applying Definition 10 with respect to x_k, for $k = 1, 2$, by $\mathcal{F}\{f(\boldsymbol{x})\}$, we have

$$\mathcal{F}_{x_1}\{f(\boldsymbol{x})\}(\omega_1, x_2) = \int_{\mathbb{R}} e^{-\mu\omega_1 x_1} f(\boldsymbol{x}) dx_1$$

and

$$\mathcal{F}_{x_2}\{f(\boldsymbol{x})\}(x_1,\omega_2) = \int_{\mathbb{R}} e^{-\mu\omega_2 x_2} f(\boldsymbol{x})\, dx_2.$$

Then we are able to write

$$\begin{aligned}
\mathcal{F}_q\{f(x_1,x_2)\}(\boldsymbol{\omega}) &= \int_{\mathbb{R}^2} e^{-\mu\boldsymbol{\omega}\cdot\boldsymbol{x}} f(\boldsymbol{x}) d^2\boldsymbol{x} \\
&= \int_{\mathbb{R}} e^{-\mu\omega_1 x_1} \left(\int_{\mathbb{R}} e^{-\mu\omega_2 x_2} f(x_1,x_2) dx_2 \right) dx_1 \\
&= \int_{\mathbb{R}} e^{-\mu\omega_1 x_1} \mathcal{F}_{x_2}\{f\}(x_1,x_2) dx_1 \\
&= \mathcal{F}_{x_1 x_2}\{f(x_1,x_2)\}(\boldsymbol{\omega}).
\end{aligned}$$

By Theorem 16, we have

$$\begin{aligned}
f(x_1,x_2) &= \frac{1}{2\pi} \int_{\mathbb{R}} e^{\mu\omega_1 x_1} \mathcal{F}_{x_1}\{f\}(\omega_1,\omega_2) d\omega_1 \\
&= \frac{1}{2\pi} \int_{\mathbb{R}} e^{\mu\omega_1 x_1} \left(\frac{1}{2\pi} \int_{\mathbb{R}} \mathcal{F}_{x_2 x_1}\{f\}(\omega_1,\omega_2) d\omega_2 \right) d\omega_1 \\
&= \frac{1}{(2\pi)^2} \int_{\mathbb{R}^2} e^{\mu\omega_1 x_1} e^{\mu\omega_2 x_2} \mathcal{F}_{x_2 x_1}\{f\}(\omega_1,\omega_2) d\omega_2 d\omega_1.
\end{aligned}$$

On the other hand, since

$$e^{\mu\alpha} e^{\mu\beta} = e^{\mu\beta} e^{\mu\alpha}, \quad \forall \alpha,\beta \in \mathbb{R},$$

we have

$$\mathcal{F}_{x_1 x_2}\{f(x_1,x_2)\}(\omega_1,\omega_2) = \mathcal{F}_{x_2 x_1}\{f(x_1,x_2)\}(\omega_2,\omega_1).$$

Then,

$$f(x_1,x_2) = \frac{1}{2\pi} \int_{\mathbb{R}^2} e^{\mu\boldsymbol{\omega}\cdot\boldsymbol{x}} \mathcal{F}_q\{f\}(\omega_1,\omega_2) d^2\boldsymbol{\omega}.$$

□

5.2.1 Right linearity

It is easy to show the following lemma.

Lemma 13. *Let $f_1, f_2 \in L^2\left(\mathbb{R}^2; \mathbb{H}\right)$. The $2D-QFT$ is right $\mathbb{H}-$linear, that is,*

$$\mathcal{F}_q\left\{f_1\alpha + f_2\beta\right\}(\boldsymbol{\omega}) = \mathcal{F}_q\left\{f_1\right\}(\boldsymbol{\omega})\alpha + \mathcal{F}_q\left\{f_2\right\}(\boldsymbol{\omega})\beta, \quad \alpha,\beta \in \mathbb{H}. \tag{5.7}$$

Proof. By (5.5), we have that

$$\begin{aligned}
\mathcal{F}_q\left\{f_1\alpha + f_2\beta\right\}(\boldsymbol{\omega}) &= \int_{\mathbb{R}^2} e^{-\mu\boldsymbol{\omega}\cdot\boldsymbol{x}} \left\{f_1(\boldsymbol{x})\alpha + f_2(\boldsymbol{x})\beta\right\} d^2\boldsymbol{x} \\
&= \int_{\mathbb{R}^2} \left(e^{-\mu\boldsymbol{\omega}\cdot\boldsymbol{x}} f_1(\boldsymbol{x})\alpha + e^{-\mu\boldsymbol{\omega}\cdot\boldsymbol{x}} f_2(\boldsymbol{x})\beta \right) d^2\boldsymbol{x} \\
&= \mathcal{F}_q\left\{f_1\right\}(\boldsymbol{\omega})\alpha + \mathcal{F}_q\left\{f_2\right\}(\boldsymbol{\omega})\beta.
\end{aligned}$$

Note that left linearity is not valid for the $2D - QFT$. However, we can write

$$\begin{aligned}\mathcal{F}_q\left\{\alpha f_1 + \beta\, f_2\right\}(\boldsymbol{\omega}) &= \int_{\mathbb{R}^2} e^{-\mu\boldsymbol{\omega}\cdot\boldsymbol{x}}\left\{\alpha f_1(\boldsymbol{x}) + \beta f_2(\boldsymbol{x})\right\} d^2\boldsymbol{x} \\ &= \int_{\mathbb{R}^2}\left(e^{-\mu\boldsymbol{\omega}\cdot\boldsymbol{x}}\alpha f_1(\boldsymbol{x}) + e^{-\mu\boldsymbol{\omega}\cdot\boldsymbol{x}}\beta f_2(\boldsymbol{x})\right) d^2\boldsymbol{x} \\ &= \mathcal{F}_q\left\{\alpha f_1\right\}(\boldsymbol{\omega}) + \mathcal{F}_q\left\{\beta f_2\right\}(\boldsymbol{\omega}).\end{aligned}$$

□

5.2.2 Shift property

Lemma 14. *The $2D - QFT$ of a shifted function is given by*

$$\mathcal{F}_q\{f(\boldsymbol{x}-\boldsymbol{b})\}(\boldsymbol{\omega}) = e^{-\mu\boldsymbol{\omega}\cdot\boldsymbol{b}}\mathcal{F}_q\{f\}(\boldsymbol{\omega}). \tag{5.8}$$

Proof. From (5.5), we have

$$\mathcal{F}_q\{f(\boldsymbol{x}-\boldsymbol{b})\}(\boldsymbol{\omega}) = \int_{\mathbb{R}^2} e^{-\mu\boldsymbol{\omega}\cdot\boldsymbol{x}} f(\boldsymbol{x}-\boldsymbol{b}) d^2\boldsymbol{x}.$$

Substituting $\boldsymbol{t} = \boldsymbol{x} - \boldsymbol{b}$ in the above expression, we have $\boldsymbol{x} = \boldsymbol{t} + \boldsymbol{b}$ and $d^2\boldsymbol{x} = d^2\boldsymbol{t}$ Hence,

$$\begin{aligned}\mathcal{F}_q\{f(\boldsymbol{x}-\boldsymbol{b})\}(\boldsymbol{\omega}) &= \int_{\mathbb{R}^2} e^{-\mu\boldsymbol{\omega}\cdot(\boldsymbol{t}+\boldsymbol{b})} f(\boldsymbol{t}) d^2\boldsymbol{t} \\ &= \int_{\mathbb{R}^2} e^{-\mu\boldsymbol{\omega}\cdot(\boldsymbol{b}+\boldsymbol{t})} f(\boldsymbol{t}) d^2\boldsymbol{t} \\ &= \int_{\mathbb{R}^2} e^{-\mu\boldsymbol{\omega}\cdot\boldsymbol{b}} e^{-\mu\boldsymbol{t}\cdot\boldsymbol{\omega}} f(\boldsymbol{t}) d^2\boldsymbol{t} \\ &= e^{-\mu\boldsymbol{\omega}\cdot\boldsymbol{b}}\mathcal{F}_q\{f\}(\boldsymbol{\omega}).\end{aligned}$$

Here, we use the fact that: let $q_1,\ q_2 \in \mathbb{H}$, then $e^{q_1+q_2} = e^{q_1}e^{q_2}$ if and only if $q_1q_2 = q_2q_1$. This proves (5.8). □

5.2.3 Scaling property

Lemma 15. *Let $a \in \mathbb{R}\backslash\{0\}$. The $2D - QFT$ of the scaled function $f_a(\boldsymbol{x}) = f(a\boldsymbol{x})$ is given by*

$$\mathcal{F}_q\left\{f_a\right\}(\boldsymbol{\omega}) = \frac{1}{|a|^2}\mathcal{F}_q\{f\}\left(\frac{\boldsymbol{\omega}}{a}\right).$$

Proof. We first assume that $a > 0$. By Definition 9, we have

$$\mathcal{F}_q\left\{f_a\right\}(\boldsymbol{\omega}) = \int_{\mathbb{R}^2} e^{-\mu\boldsymbol{\omega}\cdot\boldsymbol{x}} f(a\boldsymbol{x}) d^2\boldsymbol{x}.$$

Substituting $\boldsymbol{u}$ for $a\boldsymbol{x}$, we have

$$\begin{aligned}\mathcal{F}_q\left\{f_a\right\}(\boldsymbol{\omega}) &= \frac{1}{a^2}\int_{\mathbb{R}^2} e^{-\mu\frac{\boldsymbol{\omega}}{a}\cdot\boldsymbol{u}} f(\boldsymbol{u})d^2\boldsymbol{u} \\ &= \frac{1}{a^2}\mathcal{F}_q\{f\}\left(\frac{\boldsymbol{\omega}}{a}\right).\end{aligned}$$

For $a < 0$, we have

$$\mathcal{F}_q\left\{f_a\right\}(\boldsymbol{\omega}) = \frac{1}{(-a)^2}\mathcal{F}_q\{f\}\left(\frac{\boldsymbol{\omega}}{a}\right),$$

which completes the proof.

5.2.4 Modulation property

Lemma 16. *Let* $\boldsymbol{\omega}_0 \in \mathbb{R}^2$ *and* $F_0(\boldsymbol{x}) = e^{\mu\boldsymbol{\omega}_0\cdot\boldsymbol{x}} f(\boldsymbol{x})$. *Then, we have*

$$\mathcal{F}_q\left\{F_0\right\}(\boldsymbol{\omega}) = \mathcal{F}_q\{f\}\left(\boldsymbol{\omega} - \boldsymbol{\omega}_0\right).$$

Proof. Using Definition 9 and simplifying it, we obtain

$$\begin{aligned}\mathcal{F}_q\left\{F_0\right\}(\boldsymbol{\omega}) &= \int_{\mathbb{R}^2} e^{-\mu\boldsymbol{\omega}\cdot\boldsymbol{x}} e^{\mu\boldsymbol{\omega}_0\cdot\boldsymbol{x}} f(\boldsymbol{x})d^2\boldsymbol{x} \\ &= \int_{\mathbb{R}^2} e^{-\mu(\boldsymbol{\omega}-\boldsymbol{\omega}_0)\cdot\boldsymbol{x}} f(\boldsymbol{x})d^2\boldsymbol{x} \\ &= \mathcal{F}_q\{f\}\left(\boldsymbol{\omega} - \boldsymbol{\omega}_0\right).\end{aligned}$$

□

Remark 14. *Note that this property is different from the usual modulation property of the* 2D *FT. If the modulation term is multiplied from the right, that is*

$$F_0(\boldsymbol{x}) = f(\boldsymbol{x}) \cdot e^{\mu\boldsymbol{\omega}_0\boldsymbol{x}}$$

then the modulation property holds only for

$$f(\boldsymbol{x}) = f_0(\boldsymbol{x}) + \mu f_1(\boldsymbol{x}), \quad f_0(\boldsymbol{x}), f_1(\boldsymbol{x}) \in \mathbb{R}, \mu \in \mathbb{H}.$$

In fact, by Corollary 9 and (5.1), we have

$$\mu e^x = e^x \mu.$$

Then

$$\begin{aligned}\mathcal{F}_q\left\{F_0\right\}(\boldsymbol{\omega}) &= \int_{\mathbb{R}^2} e^{-\mu\boldsymbol{\omega}\cdot\boldsymbol{x}} f(\boldsymbol{x}) e^{\mu\boldsymbol{\omega}_0\cdot\boldsymbol{x}} d^2\boldsymbol{x} \\ &= \int_{\mathbb{R}^2} e^{-\mu\boldsymbol{\omega}\cdot\boldsymbol{x}} \left(f_0(\boldsymbol{x}) + \mu f_1(\boldsymbol{x})\right) e^{\mu\boldsymbol{\omega}_0\cdot\boldsymbol{x}} d^2\boldsymbol{x} \\ &= \int_{\mathbb{R}^2} \left(e^{-\mu(\boldsymbol{\omega}-\boldsymbol{\omega}_0)\cdot\boldsymbol{x}} f_0(\boldsymbol{x}) + e^{-\mu(\boldsymbol{\omega}-\boldsymbol{\omega}_0)\cdot\boldsymbol{x}} \mu f_1(\boldsymbol{x})\right) d^2\boldsymbol{x} \\ &= \int_{\mathbb{R}^2} e^{-\mu(\boldsymbol{\omega}-\boldsymbol{\omega}_0)\cdot\boldsymbol{x}} f(\boldsymbol{x})d^2\boldsymbol{x} \\ &= \mathcal{F}_q\{f\}\left(\boldsymbol{\omega} - \boldsymbol{\omega}_0\right).\end{aligned}$$

5.3 Main Properties of the 2D-QFT

This section describes important properties of the $2D - QFT$, such as the Plancherel and convolution theorems. First, we establish the Plancherel theorem.

Theorem 18. ***(2D-QFT Plancherel).*** *Let $f, g \in L^2\left(\mathbb{R}^2; \mathbb{H}\right)$. Then, we have*

$$(f,g)_{L^2(\mathbb{R}^2;\mathbb{H})} = \frac{1}{(2\pi)^2}\left(\mathcal{F}_q\{f\}, \mathcal{F}_q\{g\}\right)_{L^2(\mathbb{R}^2;\mathbb{H})}. \tag{5.9}$$

In particular, with $f = g$, we have the Parseval theorem:

$$\|f\|_{L^2(\mathbb{R}^2;\mathbb{H})} = \frac{1}{2\pi}\left\|\mathcal{F}_q\{f\}\right\|_{L^2(\mathbb{R}^2;\mathbb{H})}. \tag{5.10}$$

Proof. Equation (5.9) follows from

$$\begin{aligned}
(f,g)_{L^2(\mathbb{R}^2;\mathbb{H})} &= \int_{\mathbb{R}^2} \overline{g(\boldsymbol{x})} f(\boldsymbol{x}) d^2\boldsymbol{x} \\
&= \frac{1}{(2\pi)^2}\int_{\mathbb{R}^2} \overline{g(\boldsymbol{x})}\left[\int_{\mathbb{R}^2} e^{\mu\boldsymbol{\omega}\cdot\boldsymbol{x}} \mathcal{F}_q\{f\}(\boldsymbol{\omega}) d^2\boldsymbol{\omega}\right] d^2\boldsymbol{x} \\
&= \frac{1}{(2\pi)^2}\int_{\mathbb{R}^2}\left[\int_{\mathbb{R}^2} \overline{g(\boldsymbol{x})} e^{\mu\boldsymbol{\omega}\cdot\boldsymbol{x}} d^2\boldsymbol{x}\right] \mathcal{F}_q\{f\}(\boldsymbol{\omega}) d^2\boldsymbol{\omega} \\
&= \frac{1}{(2\pi)^2}\int_{\mathbb{R}^2}\overline{\left[\int_{\mathbb{R}^2} \overline{e^{\mu\boldsymbol{\omega}\cdot\boldsymbol{x}}} g(\boldsymbol{x}) d^2\boldsymbol{x}\right]} \mathcal{F}_q\{f\}(\boldsymbol{\omega}) d^2\boldsymbol{\omega} \\
&= \frac{1}{(2\pi)^2}\int_{\mathbb{R}^2}\overline{\left[\int_{\mathbb{R}^2} e^{-\mu\boldsymbol{\omega}\cdot\boldsymbol{x}} g(\boldsymbol{x}) d^2\boldsymbol{x}\right]} \mathcal{F}_q\{f\}(\boldsymbol{\omega}) d^2\boldsymbol{\omega} \\
&= \frac{1}{(2\pi)^2}\int_{\mathbb{R}^2}\overline{\mathcal{F}_q\{g\}(\boldsymbol{\omega})} \mathcal{F}_q\{f\}(\boldsymbol{\omega}) d^2\boldsymbol{\omega} \\
&= \frac{1}{(2\pi)^2}\left(\mathcal{F}_q\{f\}, \mathcal{F}_q\{g\}\right)_{L^2(\mathbb{R}^2;\mathbb{H})}.
\end{aligned}$$

Which concludes (5.9). Now, if $f = g$ and by (5.4), we arrive to (5.10), which completes the proof of Theorem 18.

□

The most important property of the $2D - QFT$ for applications in signal processing is the convolution theorem [5]. Due to the non-commutativity of quaternion multiplication, we obtain the following definition.

Definition 11. *Let $f, g \in L^2\left(\mathbb{R}^2; \mathbb{H}\right)$. The convolution $f \star g$ is defined by*

$$(f \star g)(\boldsymbol{x}) = \int_{\mathbb{R}^2} f(\boldsymbol{x} - \boldsymbol{y}) g(\boldsymbol{y}) d^2\boldsymbol{y}. \tag{5.11}$$

Remark 15. *In general, $f \star g \neq g \star f$ because the quaternion multiplication is not commutative: $f(\boldsymbol{x} - \boldsymbol{y}) g(\boldsymbol{y}) \neq g(\boldsymbol{y}) f(\boldsymbol{x} - \boldsymbol{y})$.*

Theorem 19. *Let $f, g \in L^2\left(\mathbb{R}^2; \mathbb{H}\right)$ be two quaternion-valued functions. If g has the following representation*

$$g(\boldsymbol{x}) = g_0(\boldsymbol{x}) + \boldsymbol{i} g_1(\boldsymbol{x}) + \boldsymbol{j} g_2(\boldsymbol{x}) + \boldsymbol{k} g_3(\boldsymbol{x}),$$

then, the $2D - QFT$ of the convolution $f \star g$ is given by

$$\begin{aligned} \mathcal{F}_q\{f \star g\}(\boldsymbol{\omega}) =& F_q\{g_0\}(\boldsymbol{\omega}) F_q\left\{f\right\}(\boldsymbol{\omega}) + F_q\{g_1\}(\boldsymbol{\omega}) F_q\left\{f\right\}(\boldsymbol{\omega})\boldsymbol{i} \\ &+ F_q\{g_2\}(\boldsymbol{\omega}) F_q\left\{f\right\}(\boldsymbol{\omega})\boldsymbol{j} + F_q\{g_3\}(\boldsymbol{\omega}) F_q\left\{f\right\}(\boldsymbol{\omega})\boldsymbol{k}. \end{aligned} \tag{5.12}$$

Note that, denoting

$$e_0 := 1, \quad e_1 := i, \quad e_2 := j, \quad , e_3 := k;$$

we can rewritten (5.12) as follows

$$\mathcal{F}_q\{f \star g\}(\boldsymbol{\omega}) = \sum_{m=0}^{3} \mathcal{F}_q\{g_m\}(\boldsymbol{\omega}) \cdot \mathcal{F}_q\{f_m\}(\boldsymbol{\omega}) e_m. \tag{5.13}$$

Proof. Applying the definition of the $2D - QFT$ and by (5.11) we have

$$\begin{aligned} \mathcal{F}_q\{f \star g\}(\boldsymbol{\omega}) &= \int_{\mathbb{R}^2} e^{-\mu \boldsymbol{\omega} \cdot \boldsymbol{x}} (f \star g)(\boldsymbol{x}) d^2\boldsymbol{x} \\ &= \int_{\mathbb{R}^2} e^{-\mu \boldsymbol{\omega} \cdot \boldsymbol{x}} \left[\int_{\mathbb{R}^2} f(\boldsymbol{x} - \boldsymbol{y}) g(\boldsymbol{y}) d^2\boldsymbol{y} \right] d^2\boldsymbol{x} \\ &= \int_{\mathbb{R}^2} \left[\int_{\mathbb{R}^2} e^{-\mu \boldsymbol{\omega} \cdot \boldsymbol{x}} f(\boldsymbol{x} - \boldsymbol{y}) g(\boldsymbol{y}) d^2\boldsymbol{y} \right] d^2\boldsymbol{x} \\ &= \int_{\mathbb{R}^2} \left[\int_{\mathbb{R}^2} e^{-\mu \boldsymbol{\omega} \cdot \boldsymbol{x}} f(\boldsymbol{x} - \boldsymbol{y}) d^2\boldsymbol{x} \right] g(\boldsymbol{y}) d^2\boldsymbol{y}. \end{aligned} \tag{5.14}$$

Putting $\boldsymbol{z} = \boldsymbol{x} - \boldsymbol{y}$ and applying again the definition of the $2D - QFT$, we can rewrite (5.14) as

$$\begin{aligned}
\mathcal{F}_q\{f \star g\}(\boldsymbol{\omega}) &= \int_{\mathbb{R}^2} \left[\int_{\mathbb{R}^2} e^{-\mu\boldsymbol{\omega}\cdot(\boldsymbol{z}+\boldsymbol{y})} f(\boldsymbol{z}) d^2\boldsymbol{z} \right] g(\boldsymbol{y}) d^2\boldsymbol{y} \\
&= \int_{\mathbb{R}^2} \left[e^{-\mu\boldsymbol{\omega}\cdot\boldsymbol{y}} \int_{\mathbb{R}^2} e^{-\mu\boldsymbol{\omega}\cdot\boldsymbol{z}} f(\boldsymbol{z}) d^2\boldsymbol{z} \right] g(\boldsymbol{y}) d^2\boldsymbol{y} \\
&= \int_{\mathbb{R}^2} e^{-\mu\boldsymbol{\omega}\cdot\boldsymbol{y}} \mathcal{F}_q\{f\}(\boldsymbol{\omega}) \left[g_0(\boldsymbol{y}) + \boldsymbol{i} g_1(\boldsymbol{y}) + \boldsymbol{j} g_2(\boldsymbol{y}) + \boldsymbol{k} g_3(\boldsymbol{y}) \right] d^2\boldsymbol{y} \\
&= \left[\int_{\mathbb{R}^2} e^{-\mu\boldsymbol{\omega}\cdot\boldsymbol{y}} g_0(\boldsymbol{y}) d^2\boldsymbol{y} \right] \mathcal{F}_q\{f\}(\boldsymbol{\omega}) + \left[\int_{\mathbb{R}^2} e^{-\mu\boldsymbol{\omega}\cdot\boldsymbol{y}} g_1(\boldsymbol{y}) d^2\boldsymbol{y} \right] \mathcal{F}_q\{f\}(\boldsymbol{\omega}) \boldsymbol{i} \\
&\quad + \left[\int_{\mathbb{R}^2} e^{-\mu\boldsymbol{\omega}\cdot\boldsymbol{y}} g_2(\boldsymbol{y}) d^2\boldsymbol{y} \right] \mathcal{F}_q\{f\}(\boldsymbol{\omega}) \boldsymbol{j} + \left[\int_{\mathbb{R}^2} e^{-\mu\boldsymbol{\omega}\cdot\boldsymbol{y}} g_3(\boldsymbol{y}) d^2\boldsymbol{y} \right] \mathcal{F}_q\{f\}(\boldsymbol{\omega}) \boldsymbol{k} \\
&= \mathcal{F}_q\{g_0\}(\boldsymbol{\omega}) \mathcal{F}_q\{f\}(\boldsymbol{\omega}) + \mathcal{F}_q\{g_1\}(\boldsymbol{\omega}) \mathcal{F}_q\{f\}(\boldsymbol{\omega}) \boldsymbol{i} + \mathcal{F}_q\{g_2\}(\boldsymbol{\omega}) \mathcal{F}_q\{f\}(\boldsymbol{\omega}) \boldsymbol{j} \\
&\quad + \mathcal{F}_q\{g_3\}(\boldsymbol{\omega}) \mathcal{F}_q\{f\}(\boldsymbol{\omega}) \boldsymbol{k},
\end{aligned}$$

which finishes the proof. □

As a special case of Theorem 19, we have the following corollary.

Corollary 13. *(i) If $\mathcal{F}_q\{f\}(\boldsymbol{\omega}) \in \mathbb{R}$, then*

$$\mathcal{F}_q\{f \star g\}(\boldsymbol{\omega}) = F_q\{f\}(\boldsymbol{\omega}) F_q\{g\}(\boldsymbol{\omega}). \tag{5.15}$$

(ii) If $g(x) \in \mathbb{R}$, then

$$\mathcal{F}_q\{f \star g\}(\boldsymbol{\omega}) = F_q\{f\}(\boldsymbol{\omega}) F_q\{g\}(\boldsymbol{\omega}). \tag{5.16}$$

Note that the convolution of two Gaussian functions is again a Gaussian function by Corollary 13. In fact note that if $f, g \in L^2\left(\mathbb{R}^2; \mathbb{H}\right)$, are given by

$$f(\boldsymbol{x}) = e^{-a\boldsymbol{x}^2}; \quad g(\boldsymbol{x}) = e^{-b\boldsymbol{x}^2}, \quad a, b > 0.$$

Then, by Definition 11, we have

$$\begin{aligned}
(f \star g)(\boldsymbol{x}) &= \int_{\mathbb{R}^2} e^{-a(\boldsymbol{x}-\boldsymbol{y})^2} e^{-b\boldsymbol{y}^2} d\boldsymbol{y} \\
&= \int_{\mathbb{R}^2} e^{-a\boldsymbol{x}^2+2a\boldsymbol{x}\boldsymbol{y}-a\boldsymbol{y}^2-b\boldsymbol{y}^2} d\boldsymbol{y} \\
&= \exp\left(-a\boldsymbol{x}^2 + \frac{a^2\boldsymbol{x}^2}{a+b}\right) \int_{\mathbb{R}^2} \exp\left(-(a+b)\left(\boldsymbol{y} - \frac{a\boldsymbol{x}}{a+b}\right)^2\right) d\boldsymbol{y} \\
&= \exp\left(\frac{-ab\boldsymbol{x}^2}{a+b}\right) \int_{\mathbb{R}^2} e^{-(a+b)\boldsymbol{z}^2} d\boldsymbol{z} \\
&= \frac{\pi}{a+b} \exp\left(\frac{-ab\boldsymbol{x}^2}{a+b}\right)
\end{aligned}$$

in which the change of variable $\boldsymbol{z} = \boldsymbol{y} - \dfrac{a\boldsymbol{x}}{a+b}$ is used.

Applying the inverse $2D - QFT$ to the left-hand side of (5.15), we have the following Corollary, which is important for solving partial differential equations in quaternion algebra.

Corollary 14. *Assume that $\mathcal{F}_q\{f\}(\boldsymbol{\omega}) \in \mathbb{R}$. Then, we have*

$$\mathcal{F}_q^{-1}\left[\mathcal{F}_q\{f\}\mathcal{F}_q\{g\}\right](\boldsymbol{x}) = (f \star g)(\boldsymbol{x}). \tag{5.17}$$

Proof. By the $2D - QFT$ inversion, we have

$$\begin{aligned}
\mathcal{F}_q^{-1}\left[\mathcal{F}_q\{f\}\mathcal{F}_q\{g\}\right](\boldsymbol{x}) =& \frac{1}{(2\pi)^2} \int_{\mathbb{R}^2} e^{\mu\boldsymbol{\omega}\cdot\boldsymbol{y}} \left(F_q\{f\}(\boldsymbol{\omega})F_q\{g\}(\boldsymbol{\omega})\right) d^2\boldsymbol{\omega} \\
=& \frac{1}{(2\pi)^2} \int_{\mathbb{R}^2} e^{\mu\boldsymbol{\omega}\cdot\boldsymbol{y}} F_q\{f\}(\boldsymbol{\omega}) \left(\int_{\mathbb{R}^2} e^{-\mu\boldsymbol{\omega}\cdot\boldsymbol{y}} g(\boldsymbol{y}) d^2\boldsymbol{y}\right) d^2\boldsymbol{\omega} \\
=& \frac{1}{(2\pi)^2} \int_{\mathbb{R}^2} \left(\int_{\mathbb{R}^2} e^{\mu\boldsymbol{\omega}\cdot\boldsymbol{y}} F_q\{f\}(\boldsymbol{\omega}) e^{-\mu\boldsymbol{\omega}\cdot\boldsymbol{y}} d^2\boldsymbol{\omega}\right) g(\boldsymbol{y}) d^2\boldsymbol{y} \\
=& \int_{\mathbb{R}^2} \left(\frac{1}{(2\pi)^2} \int_{\mathbb{R}^2} e^{\mu\boldsymbol{\omega}\cdot(\boldsymbol{x}-\boldsymbol{y})} F_q\{f\}(\boldsymbol{\omega}) d^2\boldsymbol{\omega}\right) g(\boldsymbol{y}) d^2\boldsymbol{y} \\
=& \int_{\mathbb{R}^2} f(\boldsymbol{x}-\boldsymbol{y}) g(\boldsymbol{y}) d^2\boldsymbol{y} \\
=& (f \star g)(\boldsymbol{x}).
\end{aligned}$$

□

5.4 Differentiation of 2D-QFT

Differentiation properties of $2D - FT$ can be generalized to the proposed $2D - QFT$ without changing their invariant expressions (independence of coordinates) in terms of vector differentials and vector derivatives. First, we define the properties of vector differentials (compare to Hitzer and Bahri, [7] Bahri and Hitzer [25]). Let $(\mathbb{R}^n, Q) = V$ be a real vector space with basis $\{\widetilde{e}_1, \ldots, \widetilde{e}_n\}$ and nondegenerate quadratic form $Q : V \mapsto \mathbb{R}$:

$$Q(\boldsymbol{x}) = x_1^2 + x_2^2 + \cdots + x_p^2 - x_{p+1}^2 - \cdots - x_{p+q}^2, \quad p + q = n, \quad p, q \in \mathbb{Z}_+,$$

where $x = \sum_{i=1}^n x_i \widetilde{e}_i$ represents an arbitrary element of V and $\mathbb{Z}_+ = \{n \in \mathbb{Z} \mid n > 0\}$. It is required that the following basic multiplication rules hold:

$$\begin{array}{ll} \widetilde{e}_i^2 = 1, & 1 \leq i \leq p \\ \widetilde{e}_i^2 = -1, & p+1 \leq i \leq n \\ \widetilde{e}_i \widetilde{e}_j + \widetilde{e}_j \widetilde{e}_i = 0, & i \neq j, \quad 1 \leq i \leq n, \quad 1 \leq j \leq n. \end{array}$$

Sylvester's theorem guarantees that p and q do not depend on the choice of the basis $\{\widetilde{e}_1, \ldots, \widetilde{e}_n\}$. We use the abbreviation $\mathbb{R}^{p,q}$ for $(\mathbb{R}^n, Q)$. The pair p, q is called the signature of $\mathbb{R}^{p,q}$. The spaces $\mathbb{R}^{n,0}$ are called Euclidean spaces while all spaces of type $\mathbb{R}^{0,n}$ are called anti-Euclidean spaces [21].

Remark 16. $x \in \mathbb{R}^{0,2}$ *is of the form*

$$x = x_1 \widetilde{e}_1 + x_2 \widetilde{e}_2,$$

where

$$\widetilde{e}_1^2 = \widetilde{e}_2^2 = -1 \quad \text{and} \quad \widetilde{e}_1 \widetilde{e}_2 = -\widetilde{e}_2 \widetilde{e}_1.$$

Definition 12. *Let* $a \in \mathbb{R}^{0,2}$. *The vector differential* $a \cdot \nabla$ *along the direction* a *is defined by*

$$\boldsymbol{a} \cdot \nabla = a_1 \partial_1 + a_2 \partial_2, \tag{5.18}$$

where $\nabla = \widetilde{e}_1 \partial_1 + \widetilde{e}_2 \partial_2, a_k = a \cdot \widetilde{e}_k$ *and* $\partial_k = \frac{\partial}{\partial x_k}, k = 1, 2$. *Applying the vector derivative* ∇ *twice, we have*

$$\nabla^2 = \nabla\nabla = \left(\tilde{e}_1 \frac{\partial}{\partial x_1} + \tilde{e}_2 \frac{\partial}{\partial x_2}\right)\left(\tilde{e}_1 \frac{\partial}{\partial x_1} + \tilde{e}_2 \frac{\partial}{\partial x_2}\right) = -\left(\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2}\right), \tag{5.19}$$

which is $-\Delta$, *i.e., minus the Laplacian.*

Remark 17. *A function* $f \in L^2\left(\mathbb{R}^{0,2}, \mathbb{H}\right)$ *can be seen as a function* $f \in L^2\left(\mathbb{R}^2, \mathbb{H}\right)$. *In fact, if* $f : \mathbb{R}^{0,2} \to \mathbb{H}$, *we can write*

$$f(x_1 \widetilde{e}_1 + x_2 \widetilde{e}_2) = f(x_1, x_2).$$

Remark 17, implies that, for a given function $f \in L^2\left(\mathbb{R}^{0,2}, \mathbb{H}\right)$, the inverse formula can be used as (5.6).

Theorem 20. ***(Vector differential).*** *The $2D-QFT$ of the vector differential of $f \in L^2\left(\mathbb{R}^{0,2}; \mathbb{H}\right)$ is given by*

$$\mathcal{F}_q\{\boldsymbol{a}\cdot\nabla f(\boldsymbol{x})\}(\boldsymbol{\omega}) = \boldsymbol{a}\cdot\boldsymbol{\omega}\mu\mathcal{F}_q\{f\}(\boldsymbol{\omega}). \tag{5.20}$$

Proof. Applying the vector differential $a \cdot \nabla$ to the inversion formula (5.6), we have

$$\boldsymbol{a}\cdot\nabla f(\boldsymbol{x}) = \boldsymbol{a}\cdot\nabla\frac{1}{(2\pi)^2}\int_{\mathbb{R}^2} e^{\mu\boldsymbol{\omega}\cdot\boldsymbol{x}}\mathcal{F}_q\{f\}(\boldsymbol{\omega})d^2\boldsymbol{\omega}.$$

Since

$$\begin{aligned}\boldsymbol{a}\cdot\nabla\left(e^{\mu\boldsymbol{\omega}\cdot\boldsymbol{x}}\right) &= (a_1\partial_1 + a_2\partial_2)\left(e^{\mu(\omega_1 x_1+\omega_2 x_2)}\right)\\ &= (a_1\mu\omega_1 + a_2\mu\omega_2)e^{\mu\boldsymbol{\omega}\cdot\boldsymbol{x}}\\ &= (\boldsymbol{a}\cdot\boldsymbol{\omega})\mu e^{\mu\boldsymbol{\omega}\cdot\boldsymbol{x}}.\end{aligned}$$

Then

$$\begin{aligned}\boldsymbol{a}\cdot\nabla f(\boldsymbol{x}) &= \frac{1}{(2\pi)^2}\int_{\mathbb{R}^2}(\boldsymbol{a}\cdot\boldsymbol{\omega})\,\mu e^{\mu\boldsymbol{\omega}\cdot\boldsymbol{x}}\mathcal{F}_q\{f\}(\boldsymbol{\omega})d^2\boldsymbol{\omega}\\ &= \mathcal{F}_q^{-1}\left[\boldsymbol{a}\cdot\boldsymbol{\omega}\mu\mathcal{F}_q\{f\}(\boldsymbol{\omega})\right](\boldsymbol{x}),\end{aligned}$$

so,

$$\mathcal{F}_q\{\boldsymbol{a}\cdot\nabla f(\boldsymbol{x})\}(\boldsymbol{\omega}) = \boldsymbol{a}\cdot\boldsymbol{\omega}\mu\mathcal{F}_q\{f\}(\boldsymbol{\omega}).$$

□

As a special case of Theorem 20, we have the following Corollary.

Corollary 15. *Considering $a = \tilde{e}_k$, for $k = 1, 2$, one has*

$$\mathcal{F}_q\{\partial_k f(\boldsymbol{x})\}(\boldsymbol{\omega}) = \omega_k\mu\mathcal{F}_q\{f\}(\boldsymbol{\omega}). \tag{5.21}$$

Theorem 21. *Let $a, b \in \mathbb{R}^{0,2}$. Then, we have*

$$\mathcal{F}_q\{(\boldsymbol{a}\cdot\nabla)(\boldsymbol{b}\cdot\nabla)f\}(\boldsymbol{\omega}) = -(\boldsymbol{a}\cdot\boldsymbol{\omega})(\boldsymbol{b}\cdot\boldsymbol{\omega})\mathcal{F}_q\{f\}(\boldsymbol{\omega}). \tag{5.22}$$

Proof. Let $a, b \in \mathbb{R}^{0,2}$ and $f \in L^2(\mathbb{R}^{0,2}, \mathbb{H})$. Define $g(\boldsymbol{x}) = (\boldsymbol{b}\cdot\Delta)f(\boldsymbol{x})$. Then, we have

$$\begin{aligned}\mathcal{F}_q\{\boldsymbol{a}\cdot\Delta g\}(\boldsymbol{\omega}) &= \boldsymbol{a}\cdot\boldsymbol{\omega}\mu\mathcal{F}_q\{g\}(\boldsymbol{\omega})\\ &= \boldsymbol{a}\cdot\boldsymbol{\omega}\mu\mathcal{F}_q\{\boldsymbol{b}\cdot\Delta f\}(\boldsymbol{\omega})\\ &= (\boldsymbol{a}\cdot\boldsymbol{\omega})\mu(\boldsymbol{b}\cdot\boldsymbol{\omega})\mu\mathcal{F}_q\{f\}(\boldsymbol{\omega})\\ &= (\boldsymbol{a}\cdot\boldsymbol{\omega})(\boldsymbol{b}\cdot\boldsymbol{\omega})\mu^2\mathcal{F}_q\{f\}(\boldsymbol{\omega})\\ &= -(\boldsymbol{a}\cdot\boldsymbol{\omega})(\boldsymbol{b}\cdot\boldsymbol{\omega})\mathcal{F}_q\{f\}(\boldsymbol{\omega}).\end{aligned}$$

□

As a special case of Theorem 21, we have the following Corollary.

Corollary 16. *Considering $a = \widetilde{e}_k$, for $k = 1, 2$ and $b = \widetilde{e}_l$, for $l = 1, 2$, we have*

$$\mathcal{F}_q\{\partial_k \partial_l f(\boldsymbol{x})\}(\boldsymbol{\omega}) = -\omega_1 \omega_2 \mathcal{F}_q\{f\}(\boldsymbol{\omega}). \tag{5.23}$$

Theorem 22. *Let $x \in \mathbb{R}^{0,2}$. Then, the $2D - QFT$ of the mth vector moment is given by*

$$\mathcal{F}_q\left\{\boldsymbol{x}^m f(\boldsymbol{x})\right\}(\boldsymbol{\omega}) = \mu^m \nabla_{\boldsymbol{\omega}}^m \mathcal{F}_q\{f\}(\boldsymbol{\omega}), \quad m \in \mathbb{N}. \tag{5.24}$$

where ∇_ω is the vector derivative with respect to the vector variable index ω, i.e.

$$\nabla_{\boldsymbol{\omega}} = \tilde{\boldsymbol{e}}_1 \partial_{\omega_1} + \tilde{\boldsymbol{e}}_2 \partial_{\omega_2}. \tag{5.25}$$

Proof. A direct calculation of the first vector moment $(m = 1)$ gives

$$\begin{aligned}
\mathcal{F}_q\{\boldsymbol{x} f(\boldsymbol{x})\}(\boldsymbol{\omega}) &= \int_{\mathbb{R}^2} e^{-\mu \boldsymbol{\omega} \cdot \boldsymbol{x}} \boldsymbol{x} f(\boldsymbol{x}) d^2\boldsymbol{x} \\
&= \int_{\mathbb{R}^2} e^{-\mu \boldsymbol{\omega} \cdot \boldsymbol{x}} \mu \cdot \nabla_{\boldsymbol{\omega}} f(\boldsymbol{x}) d^2\boldsymbol{x} \\
&= \int_{\mathbb{R}^2} \mu \cdot \nabla_{\boldsymbol{\omega}} e^{-\mu \boldsymbol{\omega} \cdot \boldsymbol{x}} f(\boldsymbol{x}) d^2\boldsymbol{x} \\
&= \mu \cdot \nabla_{\boldsymbol{\omega}} \int_{\mathbb{R}^2} e^{-\mu \boldsymbol{\omega} \cdot \boldsymbol{x}} f(\boldsymbol{x}) d^2\boldsymbol{x} \\
&= \mu \cdot \nabla_{\boldsymbol{\omega}} \mathcal{F}\{f\}(\boldsymbol{\omega}).
\end{aligned}$$

In the second equality, we used the Corollary 3.18 from [25]. Also we are used the fact of $a \cdot \nabla$ is a scalar operator, therefore the left and the right vector differentials agree. Repeating this procedures $m - 1$ times, we have

$$\mathcal{F}_q\left\{\boldsymbol{x}^m f(\boldsymbol{x})\right\}(\boldsymbol{\omega}) = \mu^m \nabla_{\boldsymbol{\omega}}^m \mathcal{F}_q\{f\}(\boldsymbol{\omega}), \quad m \in \mathbb{N},$$

which completes the proof.

□

Theorem 23. *The $2D - QFT$ of the mth vector derivative is given by*

$$\mathcal{F}_q\left\{\nabla^m f\right\}(\boldsymbol{\omega}) = (\boldsymbol{\omega}\mu)^m \mathcal{F}_q\{f\}(\boldsymbol{\omega}), \quad m \in \mathbb{N}. \tag{5.26}$$

In particular, the case of the Laplacian $\Delta = \nabla^2$ is

$$\mathcal{F}_q\{\Delta f\}(\boldsymbol{\omega}) = -\boldsymbol{\omega}^2 \mathcal{F}_q\{f\}(\boldsymbol{\omega}). \tag{5.27}$$

Proof. A simple computation gives

$$\begin{aligned}
\nabla f(\boldsymbol{x}) &= \nabla \frac{1}{(2\pi)^2} \int_{\mathbb{R}^2} e^{\mu \boldsymbol{\omega} \cdot \boldsymbol{x}} \mathcal{F}_q\{f\}(\boldsymbol{\omega}) d^2\boldsymbol{\omega} \\
&= \frac{1}{(2\pi)^2} \int_{\mathbb{R}^2} \nabla e^{\mu \boldsymbol{\omega} \cdot \boldsymbol{x}} \mathcal{F}_q\{f\}(\boldsymbol{\omega}) d^2\boldsymbol{\omega} \\
&= \frac{1}{(2\pi)^2} \int_{\mathbb{R}^2} \boldsymbol{\omega} \mu e^{\mu \boldsymbol{\omega} \cdot \boldsymbol{x}} \mathcal{F}_q\{f\}(\boldsymbol{\omega}) d^2\boldsymbol{\omega} \\
&= \boldsymbol{\omega} \mu \frac{1}{(2\pi)^2} \int_{\mathbb{R}^2} e^{\mu \boldsymbol{\omega} \cdot \boldsymbol{x}} \mathcal{F}_q\{f\}(\boldsymbol{\omega}) d^2\boldsymbol{\omega} \\
&= \mathcal{F}_q^{-1}\left[\boldsymbol{\omega} \mu \mathcal{F}_q\{f\}(\boldsymbol{\omega})\right](\boldsymbol{x}).
\end{aligned} \tag{5.28}$$

Therefore, we have

$$\mathcal{F}_q\left\{\nabla f\right\}(\omega) = (\boldsymbol{\omega}\mu)\mathcal{F}_q\{f\}(\omega), \quad m \in \mathbb{N}.$$

Applying the vector differential, ∇, to (5.28) once more, we have

$$\begin{aligned}\mathcal{F}_q\left\{\nabla^2 f\right\} &= \mathcal{F}_q\{\nabla(\nabla f)\} \\ &= \boldsymbol{\omega}\mu\mathcal{F}_q\{\nabla f\}(\boldsymbol{\omega}) \\ &= (\boldsymbol{\omega}\mu)^2\,\mathcal{F}_q\{f\}(\boldsymbol{\omega}) \\ &= -\boldsymbol{\omega}^2\mathcal{F}_q\{f\}(\boldsymbol{\omega}).\end{aligned}$$

Then, applying mathematical induction, we have

$$\mathcal{F}_q\left\{\nabla^m f\right\}(\omega) = (\boldsymbol{\omega}\mu)^m\mathcal{F}_q\{f\}(\omega), \quad m \in \mathbb{N}, \tag{5.29}$$

which completes the proof. □

Remark 18. *Notice that $\boldsymbol{\omega}^2 = \boldsymbol{\omega}\cdot\boldsymbol{\omega} + \boldsymbol{\omega}\wedge\boldsymbol{\omega} = \boldsymbol{\omega}\cdot\boldsymbol{\omega}$. This means that $\boldsymbol{\omega}^m$ is a scalar if m is even and a vector if m is odd.*

5.5 Application of the 2D-QFT

In this section, we apply the $2D - QFT$ to partial differential equations in quaternion algebra (compare to Obolashvili [30] and Bahri [5]). We consider the following initial value problem:

$$\frac{\partial u}{\partial t} - \nabla^2 u = 0, \quad \text{on } \mathbb{R}^{0,2}\times(0,+\infty) \tag{5.30}$$

and

$$u(x,0) = f(x), \quad f \in \mathcal{S}\left(\mathbb{R}^{0,2};\mathbb{H}\right), \tag{5.31}$$

where $\mathcal{S}\left(\mathbb{R}^{0,2};\mathbb{H}\right)$ is the quaternion Schwartz space, that is, the set of rapidly decreasing functions from $\mathbb{R}^{0,2}$ to $\mathbb{H}$. Applying the $2D - QFT$ to both sides of (5.30) with respect to $\boldsymbol{x}$ and using (5.27) , we have

$$\begin{aligned}\mathcal{F}_q\left\{u_t\right\}(\omega) &= -\boldsymbol{\omega}^2\mathcal{F}_q\{u\}(\boldsymbol{\omega}) \\ &= -\left(\omega_1^2+\omega_2^2\right)\mathcal{F}_q\{u\}(\boldsymbol{\omega}).\end{aligned} \tag{5.32}$$

Assume that $u(\boldsymbol{x},t)$ is sufficiently nice to allow the interchange of differentiation with respect to t and the $2D - QFT$, that is,

$$\mathcal{F}_q\left\{\frac{\partial u}{\partial t}\right\} = \frac{\partial}{\partial t}\mathcal{F}_q\{u\}.$$

Then, the general solution of (5.32) is given by

$$\mathcal{F}_q\{u\}(\boldsymbol{\omega}, t) = Ce^{-\left(\omega_1^2+\omega_2^2\right)t}, \tag{5.33}$$

where C is a quaternion constant. We impose the initial condition

$$\mathcal{F}_q\{u\}(\boldsymbol{\omega}, 0) = \mathcal{F}_q\{f\}(\boldsymbol{\omega})$$

to obtain

$$\mathcal{F}_q\{u\}(\boldsymbol{\omega}, t) = e^{-\left(\omega_1^2+\omega_2^2\right)t}\mathcal{F}_q\{f\}(\boldsymbol{\omega}). \tag{5.34}$$

Note that the $2D-QFT$ of a Gaussian quaternion function is also a Gaussian quaternion function (compare to Bahri, Hitzer, Hayashi and Ashino [8]). In fact, consider a Gaussian quaternion function of the form

$$f(\boldsymbol{x}) = C_0 e^{-(a_1x_1^2+a_2x_2^2)},$$

where $C_0 = C_{00} + \boldsymbol{i}C_{01} + \boldsymbol{j}C_{02} + \boldsymbol{k}C_{03} \in \mathbb{H}$ is a quaternion constant and $a_1, a_2 \in \mathbb{R}$ are positive real constants. Then the $2D-QFT$ of f is given by

$$\begin{aligned}
\mathcal{F}_q\{f\}(\boldsymbol{\omega}) &= \frac{1}{(2\pi)^2}\int_{\mathbb{R}}\int_{\mathbb{R}} e^{-\mu(w_1x_1+\omega_2x_2)}C_0e^{-(a_1x_1+a_2x_2)}dx_1dx_2 \\
&= \frac{1}{(2\pi)^2}\int_{\mathbb{R}}\int_{\mathbb{R}} e^{-\mu\omega_1x_1}e^{-\mu w_2x_2}e^{-a_1x_1}e^{-a_2x_2}dx_1dx_2C_0 \\
&= \frac{1}{(2\pi)^2}\int_{\mathbb{R}}\int_{\mathbb{R}} e^{-\mu\omega_1x_1}e^{-\mu\omega_2x_2}e^{-a_1x_1}e^{-a_2x_2}dx_1dx_2C_0 \\
&= \frac{1}{2\pi}\int_{\mathbb{R}} e^{-\mu\omega_1x_1}\left[\frac{1}{2\pi}\int_{\mathbb{R}} e^{-\mu\omega_2x_2}e^{-a_2x_2^2}dx_2\right]e^{-ax_1}dx_1C_0.
\end{aligned}$$

Since

$$\begin{aligned}
\mathcal{F}\{f\}(\omega_1) &= \frac{1}{2\pi}\int_{\mathbb{R}} e^{-\mu\omega_1x_1}f(x_1)dx_1 \\
&= \frac{1}{2\pi}\int_{\mathbb{R}} e^{-\mu\omega_1x_1}e^{-a_1x_1^2}dx_1 \\
&= \frac{1}{2\pi}\int_{\mathbb{R}} e^{-\mu\omega_1x_1-a_1x_1^2}dx_1 \\
&= \frac{1}{2\pi}\int_{\mathbb{R}} \exp\left[-a_1\left(x_1+\frac{\mu\omega_1}{2a_1}\right)^2-\frac{\omega_1^2}{4a_1}\right]dx_1 \\
&= \frac{1}{2\pi}\exp\left(-\frac{\omega_1^2}{4a_1}\right)\int_{\mathbb{R}} e^{-a_1y_1^2}dy_1 \\
&= \frac{1}{2\pi}\exp\left(-\frac{\omega_1^2}{4a_1}\right)\sqrt{\frac{\pi}{a_1}} \\
&= \sqrt{\frac{1}{4\pi a_1}}e^{-\left(\frac{\omega_1^2}{4a_1}\right)},
\end{aligned}$$

in which the change of variable $y_1 = x_1 + \frac{\mu\omega_1}{2a}$ is used. In the same way, we have

$$\mathcal{F}\{f\}(\omega_2) = \sqrt{\frac{1}{4\pi a_2}} e^{-\left(\frac{\omega_2^2}{4a_2}\right)}.$$

Then

$$\begin{aligned}
\mathcal{F}_q\{f\}(\boldsymbol{\omega}) &= \frac{1}{2\pi}\int_{\mathbb{R}} e^{-\mu\omega_1 x_1}\left[\sqrt{\frac{1}{4\pi a_2}} e^{-\left(\frac{\omega_2^2}{4a_2}\right)}\right] e^{-a_1 x_1} dx_2 C_0 \\
&= \left[\sqrt{\frac{1}{4\pi a_2}} e^{-\left(\frac{w_1^2}{4a_1}\right)}\right] \frac{1}{2\pi}\int_{\mathbb{R}} e^{-\mu\omega_1 x_1} e^{-a_1 x_1} dx_1 C_0 \\
&= \left[\sqrt{\frac{1}{4\pi a_2}} e^{-\left(\frac{w_2^2}{4a_2}\right)}\right]\left[\sqrt{\frac{1}{4\pi a_1}} e^{-\left(\frac{\omega_1^2}{4a_1}\right)}\right] C_0 \\
&= \frac{1}{4\pi\sqrt{a_1 a_2}} e^{-\left(\frac{w_1^2}{4a_1}+\frac{w_2^2}{4a_2}\right)} C_0.
\end{aligned}$$

This shows that the $2D - QFT$ of the Gaussian quaternion function is another Gaussian quaternion function. More precisely, taking $a_1 = a_2$, we have

$$\begin{aligned}
\mathcal{F}_q\{f\}(\boldsymbol{\omega}) &= \frac{1}{4\pi\sqrt{a_1^2}} e^{-\left(\frac{w_1^2+w_2^2}{4a_1}\right)} C_0 \\
&= \frac{1}{4\pi|a_1|} e^{-\left(\frac{w_1^2+w_2^2}{4a_1}\right)} C_0.
\end{aligned}$$

By $t = a_1 > 0$, we have

$$\mathcal{F}_q\{f\}(\omega) = \frac{1}{4\pi t} e^{-\left(\frac{w_1^2+w_2^2}{4t}\right)} C_0,$$

so,

$$\frac{1}{4\pi t}\mathcal{F}_q\left\{e^{-\left(x_1^2+x_2^2\right)/(4t)}\right\} = e^{-\left(\omega_1^2+\omega_2^2\right)t}. \tag{5.35}$$

Applying the inverse $2D - QFT$, we have

$$\begin{aligned} u(\boldsymbol{x},t) &= \mathcal{F}_q^{-1}\left[e^{-\left(\omega_1^2+\omega_2^2\right)t}\mathcal{F}_q\{f\}\right](\boldsymbol{x}) \\ &= \mathcal{F}_q^{-1}\left[\frac{1}{4\pi t}\mathcal{F}_q\left\{e^{-\left(x_1^2+x_2^2\right)/(4t)}\right\}\mathcal{F}_q\{f\}\right](\boldsymbol{x}) \\ &= \mathcal{F}_q^{-1}\left[\mathcal{F}_q\left\{\frac{1}{4\pi t}e^{-\left(x_1^2+x_2^2\right)/(4t)}\right\}\mathcal{F}_q\{f\}\right](\boldsymbol{x}). \end{aligned}$$

Since

$$\mathcal{F}_q\left\{e^{-\left(x_1^2+x_2^2\right)/(4t)}\right\}(\boldsymbol{\omega}) = 4\pi t e^{-\left(\omega_1^2+\omega_2^2\right)t} \in \mathbb{R},$$

then we can apply the Convolution Corollary (5.15) and Corollary 14 to have

$$\begin{aligned} u(\boldsymbol{x},t) &= \mathcal{F}_q^{-1}\left[\mathcal{F}_q\left\{\frac{1}{4\pi t}e^{-\left(x_1^2+x_2^2\right)/(4t)}\right\}\mathcal{F}_q\{f\}\right](\boldsymbol{x}) \\ &= \mathcal{F}_q^{-1}\left[\mathcal{F}_q\left\{\frac{1}{4\pi t}e^{-\left(x_1^2+x_2^2\right)/(4t)}\star f\right\}\right](\boldsymbol{x}) \\ &= K_t(\boldsymbol{x})\star f, \end{aligned} \tag{5.36}$$

where $K_t(\boldsymbol{x}) = \frac{1}{4\pi t}e^{-\left(x_1^2+x_2^2\right)/(4t)}$. If we decompose $f = f_0 + \boldsymbol{i}f_1 + \boldsymbol{j}f_2 + \boldsymbol{k}f_3$, then (5.36) reduces to

$$u(\boldsymbol{x},t) = K_t(\boldsymbol{x})\star f_0 + \boldsymbol{i}K_t(\boldsymbol{x})\star f_1 + \boldsymbol{j}K_t(\boldsymbol{x})\star f_2 + \boldsymbol{k}K_t(\boldsymbol{x})\star f_3, \tag{5.37}$$

where $f_i \in \mathbb{R}, i = 0,1,2,3$. By Definition 11 of convolution, (5.11) gives

$$\begin{aligned} u(\boldsymbol{x},t) =& \frac{1}{4\pi t}\int_{\mathbb{R}^{0,2}} e^{-(\boldsymbol{y})^2/(4t)} f_0(\boldsymbol{x}-\boldsymbol{y})d^2\boldsymbol{y} + \frac{\boldsymbol{i}}{4\pi t}\int_{\mathbb{R}^{0,2}} e^{-(\boldsymbol{y})^2/(4t)} f_1(\boldsymbol{x}-\boldsymbol{y})d^2\boldsymbol{y} \\ &+ \frac{\boldsymbol{j}}{4\pi t}\int_{\mathbb{R}^{0,2}} e^{-(\boldsymbol{y})^2/(4t)} f_2(\boldsymbol{x}-\boldsymbol{y})d^2\boldsymbol{y} + \frac{\boldsymbol{k}}{4\pi t}\int_{\mathbb{R}^{0,2}} e^{-(\boldsymbol{y})^2/(4t)} f_3(\boldsymbol{x}-\boldsymbol{y})d^2\boldsymbol{y} \end{aligned} \tag{5.38}$$

by changing of variable, $\boldsymbol{z} = \boldsymbol{x} - \boldsymbol{y}$, (5.38) can be written as follow

$$\begin{aligned} u(\boldsymbol{x},t) =& -\frac{1}{4\pi t}\int_{\mathbb{R}^{0,2}} e^{-(\boldsymbol{x}-\boldsymbol{z})^2/(4t)} f_0(\boldsymbol{z})d^2\boldsymbol{z} - \frac{\boldsymbol{i}}{4\pi t}\int_{\mathbb{R}^{0,2}} e^{-(\boldsymbol{x}-\boldsymbol{z})^2/(4t)} f_1(\boldsymbol{z})d^2\boldsymbol{z} \\ &- \frac{\boldsymbol{j}}{4\pi t}\int_{\mathbb{R}^{0,2}} e^{-(\boldsymbol{x}-\boldsymbol{z})^2/(4t)} f_2(\boldsymbol{z})d^2\boldsymbol{z} - \frac{\boldsymbol{k}}{4\pi t}\int_{\mathbb{R}^{0,2}} e^{-(\boldsymbol{x}-\boldsymbol{z})^2/(4t)} f_3(\boldsymbol{z})d^2\boldsymbol{z}. \end{aligned} \tag{5.39}$$

Thus, we have the following theorem.

Theorem 24. *Let $f_i, i = 0,1,2,3$, belong to $L^p\left(\mathbb{R}^{0,2};\mathbb{R}\right), 1 \leq p \leq +\infty$. Put $u_i\ (\boldsymbol{x},t) = K_t(\boldsymbol{x})\star f_i$. Then, each $u_i(\boldsymbol{x},t), i = 0,1,2,3$, is a solution of*

$$\frac{\partial u_i}{\partial t} - \nabla^2 u_i = 0 \quad \text{on } \mathbb{R}^{0,2}\times(0,+\infty)$$

and

$$u(\boldsymbol{x},t) = u_0(\boldsymbol{x},t) + \boldsymbol{i}u_1(\boldsymbol{x},t) + \boldsymbol{j}u_2(\boldsymbol{x},t) + \boldsymbol{k}u_3(\boldsymbol{x},t)$$

is a solution of (5.30).

Proof. It is well-known that each $u_i(x,t) = K_t(x) \star f_i$ satisfies the heat equation [18]. Using the superposition principle, we have that

$$K_t(\boldsymbol{x}) \star f_0 + \boldsymbol{i}K_t(\boldsymbol{x}) \star f_1 + \boldsymbol{j}K_t(\boldsymbol{x}) \star f_2 + \boldsymbol{k}K_t(\boldsymbol{x}) \star f_3$$

is a solution of the quaternion heat equation (5.30). □

A similar argument gives the following theorem.

Theorem 25. *Let $f_i, i = 0,1,2,3$, be bounded and continuous. Then, we have the following:*

(i) Each $u_i(\boldsymbol{x},t) = K_t(\boldsymbol{x}) \star f_i, i = 0,1,2,3$, is continuous on $\mathbb{R}^{0,2} \times (0,+\infty)$ and satisfies $u_i(x,0) = f_i(x)$.

(ii) The solution u is continuous on $\mathbb{R}^{0,2} \times (0,+\infty)$ and satisfies $u(\boldsymbol{x},0) = f(\boldsymbol{x})$.

Chapter 6

Conclusions and Recommendations

6.1 Conclusions

Using basic quaternion properties, we introduced the two-dimensional (left) Quaternion Transform ($2D - QFT$). Because of the non-commutativity of the multiplication in the quaternion space $\mathbb{H}$, several important properties of the classical FT, such as modulation and convolution, must be modified. We also studied a simple application of the $2D - QFT$ to partial differential equations.

6.2 Recommendations

As a extension of this work, using the kernel of the $2D - QFT$ is possible to develop a $2D$ continuous quaternion wavelet transform ($2D - CQWT$) and prove some of the most important properties of this new transform, extending the theory of the classical continuous Wavelet Transform (WT).

It's also possible to develop the same properties established in this work in the 3-dimensional case.

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