UNIVERSIDAD DE INVESTIGACIÓN DE TECNOLOGÍA EXPERIMENTAL YACHAY

Escuela de Ciencias Matemáticas y Computacionales

TÍTULO: Controllability for time varying Semi Linear Systems with Impulses, Infinite Delay, and Non local Conditions

Trabajo de integración curricular presentado como requisito para la obtención
del título de Matemático

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Dedication

"To my eternal grandmother María,
my beloved parents, Carmen and Oswaldo,
my loved brother, Anthony,
my dear girlfriend Joselyn,
and to my second house Yachay Tech"

Acknowledgment

First of all, I would like to thank God and my eternal grandmother Maria who have guided me and given me the strength from heaven to fulfill each of the dreams I have set for myself. I thank my parents, Carmen and Oswaldo, who constantly supported my decisions, and gave me the best education I could have had. They have never stopped me from doing what I wanted to do, in terms of academic decisions, and I could not be more grateful for that. I would also like to thank my brother Anthony with whom I have shared my entire life and we have supported each other to get ahead to make our parents proud. During my college career, I also met amazing people and one of them is my girlfriend Joselyn Vizuete, who became my best friend, confidant and a fundamental part of my achievements. Everything I have achieved is thanks to them, to their infinite love, support and trust.

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Thank you all, thank you Yachay Tech.

Resumen

En este trabajo, en primer lugar, se estudia la existencia y unicidad de soluciones para ecuaciones retardadas bajo ciertas condiciones tales como condiciones no locales, impulsos y retardo infinito. En segundo lugar, bajo algunas condiciones sobre el término no lineal, se consigue la controlabilidad aproximada. Además, probamos la controlabilidad exacta, demostrando la conjetura de que la controlabilidad exacta se preserva bajo la influencia de condiciones no locales, impulsos y retardo infinito si se asumen algunas condiciones. Por último, como aplicación de nuestro resultado, presentamos un ejemplo en el que se verifican todas las condiciones asumidas se verifican.
Palabras Clave: existencia de soluciones, controllabilidad Aproximada, controlabilidad exacta, impulsos, retardo infinito, condiciones no locales, teorema de punto fijo de Rothe, teorema de punto fijo de Karakostas.

Abstract

In this work, first of all, we study the existence and uniqueness of solutions for retarded equations under certain conditions such as nonlocal conditions, impulses, and infinite delay. Secondly, under some conditions on the nonlinear term, the approximate controllability is achieved. Moreover, we prove the exact controllability, proving the conjecture that the exact controllability is preserve under the influence of nonlocal conditions, impulses, and infinite delay if some conditions are assumed. Finally, as an application of our result, we present an example were all the conditions assumed are verified.
Keywords: existence of solutions, approximate controllability, exact controllability impulses, infinite delay, non-local conditions, Rothe's fixed point theorem, Karakostas's fixed point theorem.

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Chapter 1

Introduction

1.1 Background

One of the most interesting tools currently available for analyzing and predicting real-life problems is mathematical modeling. The first step to describe these problems mathematically is to find a model that represents them. For this reason, control systems arise to describe behaviors from simple cases, such as heat conduction in electrical or electronic systems, to more complex cases such as population growth planning, crop treatment, and general mathematical modeling. This is why control theory has gained great importance as a discipline in recent years. There is an extensive literature on control theory for continuous systems, as examples we have the most important works used in our research which are Curtain and Pritchard [1], Curtain and Zwart [2]. On the other hand, the literature for discrete systems is less extensive and in many researches they are visualized in an introductory way.

A control problem can be applied to phenomena involving external factors such as impulses, finite or infinite delays, non-local conditions or random variables, such as noise, which improve the accuracy of the mathematical model. These perturbations occur naturally in real-life problems and help us to describe the model in a better way. In this work we shall study the controllability for systems on finite dimensional Banach spaces. For this purpose, we consider non-autonomous semi-linear retarded equation, under the influence of infinite delay, impulses, and nonlocal conditions. In this sense, we will first prove the existence of solutions of semilinear systems and, consequently, concepts and characterizations of exact controllability and approximate controllability for linear and semilinear systems in finite dimensional spaces will be presented.

There are many works on existence and uniqueness of solutions for semilinear systems with evolution equations. Some of them are [3, 4, 5, 6, 7, 8, 9, 10]. In particular, in [3], the existence of periodic mild solution of evolution equations have been studied by implementing Sadovki's fixed point theorem and Kuratowski's measure of non-compactness. In this sense, we will define a particular phase space $\mathfrak{B}$ using ideas from [11, 12, 13] and satisfying Hale-Kato axiomatic theory. Since our problem is under the influence of nonlocal condition, impulses, and unbounded delay, it suggests us to use a new method to prove the existence result by applying Karakosta's Fixed Point Theorem (see [6, 7]).

Adittionally, there are some papers on the controllability of the linear system (4.1.1), it may be possible to see [14], [15] and [16].In particular, we can use the information from the work carried out by Lukes in [17], J.C. Coron in [18].

In the same way, Vidyasager in [19] proved the controllability by using the nonlinear term $\mathcal{H}$ does not depend $u \in \mathbb{R}^m$ and Schauder Fixed Point Theorem. Dauer in [20] identify conditions on the non linear term $\mathcal{H}$ and prove the controllability of the semilinear system without nonlocal conditions and impulses. But, V. N. Do [21] weaker the conditions on the non linear term $\mathcal{H}$ to prove the controllability of the system (1.2.2) without non local conditions, delays, and impulses. It is worth remembering that all these conditions are highly dependent on the linear system (4.1.1); especially, on the fundamental matrix $\Phi(t)$ of the linear system (2.3.5).

On the other hand, there are other notions of controllability, for example, local controllability, such as [14], [22], [23], [24], [25], [26],[27],[28], and [29], but, without non local conditions and impulses. For evolution equations there are works in [30] by Zhi-Qing Zhu and Qing-Wen Lin, and in [31] by J.J. Nieto and C.C. Tiesdell. Also, in [32] and [33], we can find the Rothe's fixed point theorem, which is the key point for this work. In addition, in [9] S. Selvi and M. Malika Arjunan studied the controllability of differential systems with finite delay and impulses by using Monch's Fixed Point Theorem and measures of non-compactness.

In the case of infinite-dimensional Banach spaces, we are clear that some of the ideas presented here can be used to solve the controllability of evolution equations with nonlocal conditions, delays, and impulses. Recall that the nonlinear term involving all variables, time, state and control. Additionally, there exist some results from [34],[32], [35] and [36], that can be applied in our work.

1.2 Problem statement

The non-autonomous semilinear system under study, which contains these three disturbances: non-local conditions, infinite delay and impulses, is represented as follows.

$$\begin{cases} z'(t) = \mathcal{A}(t)z(t) + \mathcal{H}(t, z_t), \quad t \neq t_k, t > 0, \\ z(s) + \mathcal{D}(z_{\tau_1}, z_{\tau_2}, \cdots, z_{\tau_q})(s) = \varphi(s), \quad s \in \mathbb{R}_- = (-\infty, 0], \\ z(t_k^+) = z(t_k^-) + \mathcal{J}_k(t_k, z(t_k)), \quad k = 1, 2, \cdots, p. \end{cases} \tag{1.2.1}$$

where $0 < t_1 < t_2 < \cdots < t_p$, $0 < \tau_1 < \tau_2 < \cdots < \tau_q$.
z_t, $\mathcal{H}$,$\mathcal{J}_k$ and $\mathcal{D}, \varphi$ are continuous and suitable functions, related with the delays, the non-linear part of the system, the impulses, and the nonlocal conditions, respectively.
$\mathcal{A}(t)$ is a continuous matrix of dimension n × n; $\varphi \in \mathfrak{B}$, with $\mathfrak{B}$ being the phase space satisfying the axiomatic theory proposed by Hale and Kato , $z_t(s) = z(t+s)$, $z_t \in \mathfrak{B}$, and finally $\mathcal{H} : (-\infty, \tau] \times \mathfrak{B} \to \mathbb{R}^n$, $\mathcal{J}_k : (-\infty, \tau] \times \mathbb{R}^n \to \mathbb{R}^n$ and $\mathcal{D} : (\mathfrak{B})^q \to \mathfrak{B}$.
z_t shows the evolution of the state during time t, and allows to remember the historical past of φ, moving part of the present to the past. Moreover, it is fundamental to remark that in the system (1.2.1) we are going to prove the existence, uniqueness and prolongation of its solutions.
According to the literature, the natural space to study this type of problems is the Banach

space $\mathcal{CA}_{d\tau}$ endowed with the supremum norm and given by

$$\mathcal{CA}_{d\tau} = \{z : (-\infty, \tau] \to \mathbb{R}^n : z\Big|_{\mathbb{R}_-} \in \mathfrak{B} \text{ and } z\Big|_{(0,\tau]} \text{ is a continuous except at } t_k,$$
$$k = 1, 2, ..., p, \text{ where side limits } z(t_k^+), z(t_k^-) \text{ exist and } z(t_k^+) = z(t_k)\}.$$

To analyze the controllability, we shall consider the following control problem belonging to the semilinear system (1.2.1) and which is represented as follows.

$$\begin{cases} z'(t) = \mathcal{A}(t)z(t) + \mathcal{B}(t)u(t) + h(t, z_t, u(t)), \quad t \neq t_k, \quad t > 0 \\ z(s) + \mathcal{D}(z_{\tau_1}, z_{\tau_2}, \cdots, z_{\tau_q})(s) = \varphi(s), \quad s \in \mathbb{R}_- = (-\infty, 0] \\ z(t_k^+) = z(t_k^-) + J_k(t_k, z(t_k)), \quad k = 1, 2, \cdots, p, \end{cases} \tag{1.2.2}$$

where $0 < t_1 < t_2 < \cdots < t_p < \tau$, $0 < \tau_1 < \tau_2 < \cdots < \tau_q < \tau$, are fixed real numbers. The non-linear part $\mathcal{H}$ in this control problem is represented as follows:

$$\mathcal{H}(t, z_t) = \mathcal{B}(t)u(t) + h(t, z_t, u(t))$$

where $h : [0, \tau] \times \mathfrak{B} \times \mathbb{R}^m \to \mathbb{R}^n$ is an smooth enough function.
The matrix $\mathcal{B}(t)$ of dimension n $\times$ m, is continuous. Moreover, u $\in L^2([0, \tau]; \mathbb{R}^m)$ is the control function and is continuous.
Finally, recall that since we want to prove the controllability of the system on a finite interval [0, τ], we shall consider a finite number of impulses.

Chapter 2

Theoretical Framework

[Theoretical framework text here]

In this chapter, we shall introduce the spaces in which our problem will be studied and review some definitions, lemmas, and theorems used in this work.

2.1 Definitions, Lemmas, and Theorems

First of all, let us define the space of normalized piecewise continuous function, represented by $\mathcal{CA}_p = \mathcal{CA}_p((-\infty, 0]; \mathbb{R}^n)$. This space contains functions such that their restriction to any interval of the form $[a, 0]$ is a piecewise continuous function. i.e.,

$$\mathcal{CA}_p = \left\{ \varphi : \; (-\infty, 0] \longrightarrow \mathbb{R}^n : \varphi\Big|_{[a,0]} \text{is a piecewise continuous function}, \forall a < 0 \right\}$$

Applying ideas from [13, 11, 12], the function $d : \mathbb{R} \to \mathbb{R}_+$ is considered in such a way that

1. d is decreasing,

2. $d(0) = 1$,

3. $d(-\infty) = +\infty$.

Then, we describe the function space $\mathcal{C}_{dp}$

$$\mathcal{C}_{dp} = \left\{ z \in \mathcal{CA}_p : \sup_{s \leq 0} \frac{\|z(s)\|}{d(s)} < \infty \right\}.$$

In the following references [3, 13, 12, 11], this space is used as a Banach space. A representation of the proof that $\mathcal{C}_{dp}$ is a Banach space can be seen in [37].

Lemma 1 (See [37]). *$\mathcal{C}_{dp}$ is a Banach space equipped with the following norm*

$$\|z\|_{dp} = \sup_{s \leq 0} \frac{\|z(s)\|}{d(s)}, \quad \text{with } z \in \mathcal{C}_{dp}$$

Furthermore, we consider the larger space of functions as defined in the introduction of this work

$$\mathcal{CA}_{d\tau} = \{z : (-\infty, \tau] \to \mathbb{R}^n : z\Big|_{\mathbb{R}_-} \in \mathfrak{B} \text{ and } z\Big|_{(0,\tau]} \text{ is a continuous except at } t_k,$$
$$k = 1, 2, ..., p, \text{ where side limits } z(t_k^+), z(t_k^-) \text{ exist and } z(t_k^+) = z(t_k)\}.$$

As a result of Lemma 1, the following lemma can be deduced

Lemma 2. *$\mathcal{CA}_{d\tau}$ is a Banach space equipped with the following norm*

$$\|z\| = \left\| z|_{\mathbb{R}_-} \right\|_{\mathfrak{B}} + \|z|_I\|_\infty$$

where $\|z|_I\|_\infty = \sup_{t \in I = (0,\tau]} \|z(t)\|$.

Therefore, the phase space will be represented as follows

$$\mathfrak{B} := \mathcal{C}_{dp},$$

equipped with the norm

$$\|z\|_{\mathfrak{B}} = \|z\|_{dp}.$$

It is not hard to verify that phase space $\mathfrak{B}$ satisfies the following Hale and Kato axiomatic theory for the phase space of retarded differential equations with infinite delay:

$A1)$ If $z \in \mathcal{CA}_{d\tau}$, then for every $t \in [0, \tau]$ the following conditions hold:

(i) z_t is in $\mathfrak{B}$;

(ii) $\|z(t)\| \leq H\|z_t\|_{\mathfrak{B}}$;

(iii) $\|z_t\|_{\mathfrak{B}} \leq K(t)\sup\{\|z(s)\| : 0 \leq s \leq t\} + M(t)\|z_t\|_{\mathfrak{B}}$, where $H \geq 0$ is a constant, $K,\ M : [0,\infty) \to [0,\infty)$, K is continuous and M is locally bounded, and $H,\ K,\ M$ are independent of $z(t)$.

$A2$) For the function $z(\cdot)$ in A1), z_t is a $\mathfrak{B}-$valued continuous function on $[0,\tau]$.

$A3$) The space $\mathfrak{B}$ is complete.

For more details about this axiomatization, one can see [11, 13, 12].

In order to prove our existence theorem, we consider the following lemma. Details of the proof of this lemma can be found in [37].

Lemma 3 (See [37])**.** *The following estimate holds for all $s \in [0,\tau]$:*

$$\|z_s\|_{\mathfrak{B}} \leq \|z\|_{\mathcal{CA}_{d\tau}}, \quad \textit{with } z \in \mathcal{CA}_{d\tau}.$$

In the same way, we shall consider the following space

$$\mathfrak{B}^q = \mathfrak{B} \times \mathfrak{B} \times \cdots \times \mathfrak{B} = \prod_{i=1}^{q} \mathfrak{B}$$

equipped with the following norm

$$\|z\|_q = \sum_{i=1}^{q} \|z_i\|_{\mathfrak{B}}$$

Additionally, we shall consider the following Banach space

$$\mathcal{CA}_{d\tau}((-\infty,\tau];\mathbb{R}^n) \times C([0,\tau];\mathbb{R}^m),$$

equipped with the norm

$$|||(z,u)||| = \|z\|_{\mathcal{CA}_{d\tau}} + \|u\|_0.$$

Also, the following norm is considered in $\mathbb{R}^n \times \mathbb{R}^m$

$$\|(z,u)\|_1 = \|z\|_{\mathbb{R}^n} + \|u\|_{\mathbb{R}^m} = \|z\| + \|u\|, \quad \forall (z,u) \in \mathbb{R}^n \times \mathbb{R}^m.$$

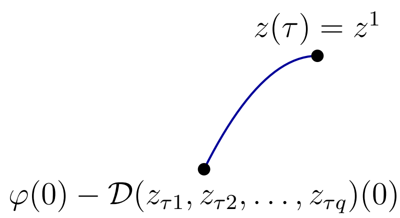
If we have an arbitrary $(z,u) \in \mathcal{CA}_{d\tau}((-\infty,\tau];\mathbb{R}^n) \times C([0,\tau];\mathbb{R}^m)$, the following quantity is also considered:

$$\|h(\cdot,z,u)\|_0 = \sup_{t\in[0,\tau]} \|h(t,z_t,u(t))\|_{\mathbb{R}^n}.$$

Now, we give the most important definition of this work, to study the controllability of the semilinear system (1.2.2):

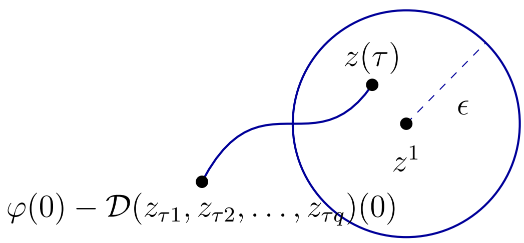
Definition 1. (**Controllability**) *The system (1.2.2) is said to be controllable on $[0,\tau]$ if for every $\varphi \in \mathfrak{B}$, $z^1 \in \mathbb{R}^n$, there exists $u \in L^2([0,\tau];\mathbb{R}^m)$ such that the solution $z(t)$ of (1.2.2) verifies:*

$$z(0) + \mathcal{D}(z_{\tau_1}, z_{\tau_2}, \dots, z_{\tau_q})(0) = \varphi(0) \quad \textit{and} \quad z(\tau) = z^1.$$



Definition 2. **(Approximate Controllability)** *The system* (1.2.2) *is said to be approximately controllable on* $[0,\tau]$ *if for every* $\varphi \in \mathfrak{B}$*,* $z^1 \in \mathbb{R}^n$ *and* $\epsilon > 0$*, there exists* $u \in L^2([0,\tau];\mathbb{R}^m)$ *such that the solution* $z(t)$ *of* (1.2.2) *corresponding to* u *verifies:*

$$z(0) + \mathcal{D}(z_{\tau_1}, \dots, z_{\tau_q})(0) = \varphi(0), \quad \textit{and} \quad \left\| z(\tau) - z^1 \right\|_{\mathbb{R}^n} < \epsilon.$$



Definition 3. **(Equicontractivity)** *Let Z be a Banach space and $\{T_n\}_{n\in I}$ be a family of operators $T_n : Z \to Z$. The family $\{T_n\}_{n\in\mathbb{N}}$ is said to be equicontractive, if there exists $0 < L < 1$ such that:*

$$\|T_n z_1 - T_n z_2\| \leq L\|z_1 - z_2\|, \quad z_1, z_2 \in Z, \quad n \in I = \mathbb{N}$$

Lemma 4. **(Generalized Gronwall-Bellman)** *Let a non negative function $z \in \mathcal{CA}_{d\tau}$ satisfy, for $t \geq t_0$, the inequality*

$$z(t) \leq C + \int_{t_0}^{t} v(s)z(s)ds + \sum_{t_0<t_k<t} \beta_k u(t_k),$$

where $C \geq 0$, $\beta_k \geq 0$, $v(s) \geq 0$, and t_k 's are the discontinuity points of first type for the function z. Then we have

$$z(t) \leq C \prod_{t_0<t_k<t} (1 + \beta_k) e^{\int_{t_0}^{t} v(s)ds}$$

Theorem 1. *(See [38]) (G.L. Karakostas Fixed Point Theorem) Let Z and Y be Banach spaces and D be a closed convex subset of Z, and let $\mathcal{C} : D \to Y$ be a continuous operator such that $\mathcal{C}(D)$ is a relatively compact subset of Y, and*

$$\mathcal{T} : D \times \overline{\mathcal{C}(D)} \to D$$

is a continuous operator such that the family $\{\mathcal{T}(\cdot, y) : y \in \overline{\mathcal{C}(D)}\}$ is equicontractive. Then, the operator equation

$$\mathcal{T}(z, \mathcal{C}(z)) = z$$

admits a solution on D.

Theorem 2. *(Rothe's Fixed Theorem, [39],[40], [41]) Let E be a Banach space. Let $B \subset E$ be a closed convex subset such that the zero of E is contained in the interior of B. Let $\Psi : B \to E$ be a continuous mapping with $\Psi(B)$ relatively compact in E and $\Psi(\partial B) \subset B$. Then there is a point $x^* \in B$ such that $\Psi(x^*) = x^*$.*

2.2 Characterization of Dense Range Operators

Theorem 3. *(Curtain & Pritchard's lectures notes [1] and Curtain & Zwart [2]). Let W and Z be Hilbert spaces, $\mathscr{G} \in L(W, Z)$ and $\mathscr{G}^* \in L(Z, W)$. Then, the following statements* *hold:*

a) Ran($\mathscr{G}$) = Z if, and only if, there exist α such that $\|\mathscr{G}^* z\|_W \geq \alpha \|z\|_Z, \quad z \in Z$

b) $\overline{Ran(\mathscr{G})}$ = Z if, and only if, ker($\mathscr{G}^*$) ={0}.

Proof. Let´s prove a).

⇒) Assume that Ran($\mathscr{G}$) = Z

In order to prove the first direction, the proof will be carried out in steps:

Step 1 Assume that $\mathscr{G}$ is one-to-one map. Since Ran($\mathscr{G}$) = Z, we have that $\mathscr{G}$ is a bijection and from a result of the open mapping theorem, it follows that $\mathscr{G}^{-1} \in L(Z, W)$. And, since

$$\left(\mathscr{G}^{-1}\right)^* = (\mathscr{G}^*)^{-1} \in L(W, Z)$$

there exists a constant $\beta > 0$ such that the following inequality is satisfied:

$$\| \left(\mathscr{G}^*\right)^{-1} w\|_Z \leq \beta \|w\|_W, \quad \forall w \in W.$$

Then, doing a change of variable

$$z = (\mathscr{G}^*)^{-1} w \iff w = \mathscr{G}^* z,$$

we get that

$$\|\mathscr{G}^* z\|_W \leq \alpha \|z\|_Z, \quad \forall z \in Z$$

with

$$\alpha = \frac{1}{\beta}.$$

Step 2 For the general case, we consider the Hilbert space W as a direct sum:

$$W = X \oplus ker(\mathscr{G}), \qquad \text{with} \quad X = [Ker(\mathscr{G})]^{\perp}$$

Since X is a closed sublinear space, then X is also a Hilbert space with the same norm

$$\|w\|_W = \|w\|_X, \quad \forall w \in X$$

Now, we define the following linear operator

$$\begin{aligned}\hat{\mathscr{G}} :& X \to Z\\ & w \mapsto \hat{\mathscr{G}}w = \mathscr{G}w.\end{aligned}$$

Trivially, $\hat{\mathscr{G}}$ is one-to-one. Therefore, $\hat{\mathscr{G}}$ is bijective and, analogous to step 1, we get that

$$\|\hat{\mathscr{G}}^* z\|_X \geq \alpha \|z\|_Z, \quad \forall z \in Z,$$

with

$$\alpha = \frac{1}{\beta}$$

Additionally, from Riesz representation theorem, we know that

$$\begin{aligned}\|\hat{\mathscr{G}}^* z\|_X &= \sup_{\substack{w\in X\\ \|w\|_X \leq 1}} \left| \left\langle w, \hat{\mathscr{G}}^* z \right\rangle_W \right| \\ &= \sup_{\substack{w\in X\\ \|w\|_X \leq 1}} \left| \left\langle \hat{\mathscr{G}} w, z \right\rangle_W \right| \\ &\leq \sup_{\substack{w\in W\\ \|w\|_W \leq 1}} \left| \langle \mathscr{G} w, z \rangle_W \right| \\ &= \sup_{\substack{w\in W\\ \|w\|_W \leq 1}} \left| \langle w, \mathscr{G}^* z \rangle_W \right| \\ &= \|\mathscr{G}^* z\|_W.\end{aligned}$$

Hence,

$$\|\mathscr{G}^* z\|_W \geq \frac{1}{\beta} \|z\|_Z, \quad \forall z \in Z$$

$\Leftarrow$) Let's prove the reverse direction. Assume that

$$\|\mathscr{G}^* z\|_W \geq \alpha \|z\|_Z, \quad \forall z \in Z,$$

and consider the following inequalities

$$\begin{aligned}\|\mathscr{G}^* z\|_W^2 &\geq \alpha^2 \|z\|_Z^2, \quad \forall z \in Z.\\ \langle \mathscr{G}\mathscr{G}^* z, z\rangle &\geq \alpha^2 \|z\|_Z^2, \quad \forall z \in Z.\end{aligned}$$

Applying Cauchy-Schwartz inequality, we get that

$$\|\mathscr{G}\mathscr{G}^*z\|_Z \geq \alpha^2\|z\|_Z, \quad \forall z \in Z. \tag{2.2.1}$$

Now, let´s show that Ran$(\mathscr{G}\mathscr{G}^*) = Z$. First, we shall see that Ran$(\mathscr{G}\mathscr{G}^*)$ is closed. In fact, let $\tilde{z}$ be an accumulation point of Ran$(\mathscr{G}\mathscr{G}^*)$. Then, there exist a sequence $\{z_n\}_{n=1}^{\infty} \subset Z$ such that

$$\lim_{n\to\infty} \mathscr{G}\mathscr{G}^*z_n = \tilde{z}. \tag{2.2.2}$$

From (2.2.1) , we get that

$$\|z\|_Z \leq \frac{1}{\alpha^2}\|\mathscr{G}\mathscr{G}^*z\|_Z.$$

So,

$$\|z_n - z_m\|_Z \leq \frac{1}{\alpha^2}\|\mathscr{G}\mathscr{G}^*z_n - \mathscr{G}\mathscr{G}^*z_m\|_Z.$$

Since $\{\mathscr{G}\mathscr{G}^*z_n\}_{n=1}^{\infty}$ is a Cauchy sequence, $\|z_n - z_m\| \to 0$ as n, m $\to \infty$. Hence, $\{z_n\}_{n=1}^{\infty}$ is a Cauchy sequence as well. Thus,

$$\lim_{n\to\infty} z_n = z.$$

Passing to the limit in (2.2.2), we get that

$$\mathscr{G}\mathscr{G}^*z = \tilde{z},$$

i.e., $\tilde{z} \in Ran(\mathscr{G}\mathscr{G}^*)$. Now, suppose that Ran$(\mathscr{G}\mathscr{G}^*) \subsetneq$ Z. Then, by the perpendicular theorem, there exist z_0 with $\|z_0\|_Z = 1$ such that

$$z_0 \perp Ran(\mathscr{G}\mathscr{G}^*),$$

i.e.,

$$\langle \mathscr{G}\mathscr{G}^*z, z_0\rangle_Z = 0, \quad \forall z \in Z.$$

In particular,

$$\langle \mathscr{G}\mathscr{G}^*z_0, z_0\rangle_Z = 0.$$

From, (2.2.1), we get that $\|z_0\|^2 = 0$, i.e., $z_0 = 0$, which is a contradiction. So,

$$Ran(\mathscr{G}\mathscr{G}^*) = Z.$$

Clearly, we have that

$$Ran(\mathscr{G}\mathscr{G}^*) \subset Ran(\mathscr{G}).$$

Hence,

$$Ran(\mathscr{G}) = Z.$$

Let's prove b)

$\Rightarrow$) Assume that $\overline{Ran(\mathscr{G})} = Z$ and $\mathscr{G}^*$ is not one-to-one. Then, there exists $z_0 \neq 0$ such that $\mathscr{G}^* z_0 = 0$, i.e.,

$$\begin{aligned} 0 &= \langle w, \mathscr{G}^* z_0 \rangle_W, \quad \forall w \in W \\ &= \langle \mathscr{G} w, z_0 \rangle_Z. \end{aligned}$$

Since we assume that $\overline{Ran(\mathscr{G})} = Z$, there exists a sequence $\mathscr{G} w_n \to z_0$. So that,

$$0 = \langle \mathscr{G} w_n, z_0 \rangle_Z, \quad \forall n \in \mathbb{N}$$

Passing to the limit, as n $\to \infty$, we get that

$$\begin{aligned} 0 &= \langle z_0, z_0 \rangle_Z \\ &= \|z_0\|_Z^2 \end{aligned}$$

i.e., $z_0 = 0$, which is a contradiction.
Hence, $\mathscr{G}^*$ is one-to-one, i.e., $\ker(\mathscr{G}^*) = \{0\}$.

$\Leftarrow$) Assume that $\mathscr{G}^*$ is one-to-one and $\overline{Ran(\mathscr{G})} \subset Z$.
From the perpendicular theorem, we know that there exists $z_0 \neq 0$, $z_0 \in Z$ such that

$$\langle \mathscr{G} w, z_0 \rangle_Z = 0 \Longrightarrow \langle w, \mathscr{G}^* z \rangle_W = 0 \quad \forall w \in W.$$

Putting $w = \mathscr{G}^* z_0 \in W$, we get that

$$\begin{aligned} \|\mathscr{G}^* z_0\|_W^2 &= \langle \mathscr{G}^* z_0, \mathscr{G}^* z_0 \rangle \\ &= 0, \end{aligned}$$

i.e., $\mathscr{G}^* z_0 = 0$ and $z_0 \neq 0$, which is a contraddiction. Hence, $\overline{Ran(\mathscr{G})} = Z$.

Corollary 1. *If $dim(Z) < \infty$, then the following statements hold:*

a.- Ran($\mathscr{G}$) = Z,

b.- There exist α such that $\|\mathscr{G}z\|_z \geq \alpha\|z\|_Z$, z∈ Z,

c.- $\ker(\mathscr{G}^*) = \{0\}$.

Proof. Remember that all finite dimensional linear space is closed, then $\mathrm{Ran}(\mathscr{G}) = \overline{Ran(\mathscr{G})}$. Then, applying the theorem 3, we obtain the result. □

Corollary 2. *Let W, Z be the Hilbert spaces and $\mathscr{G} \in L(W, Z)$ such that $Ran(\mathscr{G}) = Z$. Then, $w_z = \mathscr{G}^*(\mathscr{G}\mathscr{G}^*)^{-1}z$ is the solution of the equation*

$$\mathscr{G}w = z,$$

with minimum norm, i.e.,

$$\|w_z\|_W = \inf\{w \in W : \mathscr{G}w = z\}.$$

Proof. By using the open mapping theorem and Theorem 3 is clear that there exist $(\mathscr{G}\mathscr{G}^*)^{-1}$ and $w_z = \mathscr{G}^*(\mathscr{G}\mathscr{G}^*)^{-1}z$ is the solution of $\mathscr{G}w = z$. □

Remark 1. *Under the conditions of Corollary 2 the operator $\Upsilon : Z \to W$ defined by:*

$$\Upsilon z = \mathscr{G}^*(\mathscr{G}\mathscr{G}^*)^{-1}z,$$

is a right inverse to $\mathscr{G}$, in the sense that

$$\mathscr{G}\Upsilon = I.$$

The result developed in thi section will be used in the next section where we shall present the controllability of the linear system corresponding to our mathematical model. In addition to presenting the controllability for the case on we are focused, non-autonomous, we shall present in Section 3 an algebraic characterization of the controllability for the autonomous case. □

2.3 Controllability of Linear Systems in Finite Dimensional Spaces

2.3.1 Non-Autonomous Systems

Corresponding to system (1.2.2), we shall consider the linear initial value problem

$$\begin{cases} z'(t) = \mathcal{A}(t)z(t) + \mathcal{B}(t)u(t), \quad z \in \mathbb{R}^n, \quad t \in [a,b], \\ z(a) = z^0, \end{cases} \tag{2.3.3}$$

where $u \in L^2([a,b];\mathbb{R}^m)$ and $z^0 \in \mathbb{R}^n$. From [42] we know that (2.3.3) admits only one solution given by the next expression:

$$z(t) = \mathscr{U}(t,a)z^0 + \int_a^t \mathscr{U}(t,s)\mathcal{B}(s)u(s)ds, \;\; t \in [a,b], \tag{2.3.4}$$

where $\mathscr{U}(t,s) = \Phi(t)\Phi^{-1}(s)$ is the expression for the evolution operator with $s,t \in \mathbb{R}$ and $\Phi(t)$ is the fundamental matrix of the uncontrolled linear system

$$z'(t) = \mathcal{A}(t)z(t). \tag{2.3.5}$$

Therefore, the matrix $\Phi(t)$ satisfies:

$$\begin{cases} \Phi'(t) = \mathcal{A}(t)\Phi(t), \\ \Phi(0) = I_{\mathbb{R}^n}, \end{cases} \tag{2.3.6}$$

where $I_{\mathbb{R}^n}$ is the identity matrix of dimension n $\times$ n.

Proposition 1. *$\mathscr{U}$ (t,s) satisfies the following properties for all t,r,s in $\mathbb{R}$:*

a) $\mathscr{U}$(t,s) is continuous;

b) $\mathscr{U}^{-1}(s,t) = \mathscr{U}(t,s)$;

c) $\mathscr{U}$(t,t) $= I_{\mathbb{R}^n}$;

d) $\mathscr{U}$(t,r) $\mathscr{U}$(r,s) $=$ $\mathscr{U}$(t,s);

e) $\mathscr{U}' = \frac{\partial}{\partial t}\mathscr{U}(\text{t,s}) = \mathcal{A}(t)\mathscr{U}(\text{t,s})$;

f) There exist constants $M > 0$ and $w > 0$ such that:

$$\|\mathscr{U}(t,s)\| \leq Me^{w\mathcal{A}(t-s)}, \qquad a \leq s \leq t \leq b; \tag{2.3.7}$$

To characterize the controllability of the linear system (2.3.3) in terms of linear operators, we define the following operators:

The controllability maps: $\mathscr{G} : L^2([a,b];\mathbb{R}^m) \longrightarrow \mathbb{R}^n$ is given by

$$\mathscr{G}u = \int_a^b \mathscr{U}(\tau,s)\mathcal{B}(s)u(s)ds. \tag{2.3.8}$$

And,the adjoint operators $\mathscr{G}^* : \mathbb{R}^n \longrightarrow L^2([a,b];\mathbb{R}^m)$ of the operator $\mathscr{G}$ is given by

$$(\mathscr{G}^*z)(s) = \mathcal{B}^*(s)\mathscr{U}^*(\tau,s)z, \quad \forall s \in [a,b], \quad \forall z \in \mathbb{R}^n.$$

The controllability Gramian: $\mathscr{W} : \mathbb{R}^n \to \mathbb{R}^n$ is given by

$$\mathscr{W}z = \mathscr{G}\mathscr{G}^*z = \int_a^b \mathscr{U}(\tau,s)\mathcal{B}(s)\mathcal{B}^*(s)\mathscr{U}^*(\tau,s)zds. \tag{2.3.9}$$

Proposition 2. *The systems (2.3.3) is controllable on* $[a,b]$ *if, and only if,* $\text{Ran}(\mathscr{G}) = \mathbb{R}^n$.

Since the controllability of the linear system is characterized by surjectivity of the operator $\mathscr{G}$, the result obtained in Section 2.1 can be applied to prove the following lemma.

Lemma 5. *(see [43]) The following statements are equivalent*

a) $\text{Ran}(\mathscr{G}) = \mathbb{R}^n$.

b) $\ker(\mathscr{G}^*) = \{0\}$.

c) $\exists \gamma > 0 \;/\; \langle \mathscr{G}\mathscr{G}^*z, z\rangle > \gamma\|z\|^2,\ z \neq 0$ *in* $\mathbb{R}^n$.

d) $\exists (\mathscr{W})^{-1} \in L(\mathbb{R}^n)$.

e) $\mathcal{B}^*(s)\mathscr{U}^*(\tau,s)z = 0, \quad \forall s \in [a,b] \Rightarrow z = 0$.

Therefore, the operators $\Upsilon : \mathbb{R}^n \to L^2([a,b];\mathbb{R}^m)$ defined by

$$\Upsilon z = \mathcal{B}^*(\cdot)\mathscr{U}^*(b,\cdot)\mathscr{W}^{-1}z = \mathscr{G}^*(\mathscr{G}\mathscr{G}^*)^{-1}z, \tag{2.3.10}$$

is called the steering operator and it is a right inverse of $\mathscr{G}$, in the sense that

$$\mathscr{G}\Upsilon = I.$$

Moreover,

$$\|\mathscr{W}^{-1}z\|^2_{\mathbb{R}^n} = \|(\mathscr{G}\mathscr{G}^*)^{-1}z\|^2_{\mathbb{R}^n} \leq \gamma^{-1}\|z\|^2_{\mathbb{R}^n}, \quad z \in \mathbb{R}^n,$$

and a control steering the system (2.3.3) from initial state z_0 to a final state z_1 at time $b > 0$ is given by

$$u(t) = \mathcal{B}^*(t)\mathscr{U}^*(b,t)\mathscr{W}^{-1}(z^1 - \mathscr{U}(b,a)z^0) = \Upsilon(z^1 - \mathscr{U}(b,a)z^0)(t), \quad t \in [a,b]. \tag{2.3.11}$$

In order to apply the controllability results obtained in this section to our mathematical model, we will use the following lemma.

Lemma 6. *(see [44]) The system (2.3.3) is controllable with control $u \in L^2([a,b];\mathbb{R}^m)$ if, and only if, it is controllable with control $u \in S$ with S be any dense subspace of $L^2([a,b];\mathbb{R}^m)$, i.e.,*

$$\text{Ran}(\mathscr{G}) = \mathbb{R}^n \quad \Longleftrightarrow \quad \text{Ran}(\mathscr{G}|_S) = \mathbb{R}^n,$$

2.3.2 Autonomous Systems

Here we shall present a known algebraic characterization of controllability for the autonomous case, also known as Kalman rank condition for controllability. This results was one of the first results obtained in the Control Theory by Rudolf E. Kalman in 1963 and this marked the beginning of a new branch of studies, controllability of linear dynamical systems (see [45]).

Considering that the matrices $\mathcal{A}$(t) and $\mathcal{B}$(t) are constant and from the controllability point of view, we can represent the autonomous linear system as the pair ($\mathcal{A}$, $\mathcal{B}$).

And, the fundamental matrix is given by

$$\Phi(t) = e^{\mathcal{A}t}$$

Moreover, the solution for the autonomous system is given as (2.3.4), where the evolution operator is given by the following expression:

$$\mathscr{U}(t,s) = e^{\mathcal{A}(t-s)}$$

and the controllability maps by

$$\mathscr{G}u = \int_a^b e^{\mathcal{A}(t-s)}\mathcal{B}(s)u(s)ds.$$

Algebraic Characterization of Controllability for Systems ($\mathcal{A}$, $\mathcal{B}$)

Theorem 4. *(Kalman Condition) The system ($\mathcal{A}$, $\mathcal{B}$) is controllable on [a,b] if, and only if,*

$$\text{rank}[\mathcal{B}|\mathcal{A}\mathcal{B}|\cdots|\mathcal{A}^{n-1}\mathcal{B}] = n$$

where, let S be a matrix, the rank[S] is the maximum number of columns linearly independent.

Chapter 3

Existence of Solutions for Retarded Equations with Infinite Delay, Impulses, and Nonlocal Conditions

In this section we show that the non-autonomous differential equation (1.2.1) with impulses, infinite delay and non-local conditions has a solution on $(-\infty, \tau]$, for $\tau > 0$. First, we show a characterization of the solution for the semilinear system as the solution of an integral equation. Second, we shall apply Karakostas Fixed Point Theorem to get solutions to the system (1.2.1) and finally, under certain hypotheses, the uniqueness of the solutions will be proved.

3.1 Integral formula of the solution

Proposition 3 (See [37]). *Problem (1.2.1) admits a solution $z(\cdot)$ on $(-\infty, \tau]$ if, and only if, $z(\cdot)$ satisfies the following expression,*

$$\begin{aligned} z(t) &= \mathscr{U}(t,0)[\varphi(0) - \mathcal{D}(z_{\tau_1}...z_{\tau_q})(0)] + \int_0^t \mathscr{U}(t,s)\mathcal{H}(s,z_s)ds \\ &+ \sum_{0<t_k<t} \mathscr{U}(t,t_k)\mathcal{J}_k(t_k, z(t_k)), \quad t \in (0,\tau], \\ z(t) &= -\mathcal{D}\left(z_{\tau_1}, z_{\tau_2}, z_{\tau_3}, \dots, z_{\tau_q}\right)(t) + \varphi(t), \quad t \in (-\infty, 0]. \end{aligned}$$

3.2 Hypotheses

In this subsection, we shall assume the hypotheses that will allow us to prove the first existence theorem. The constant M is defined in (2.3.7)

(H1) Let $\mathcal{K} : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ and $\tilde{\psi} : \mathbb{R}_+ \to \mathbb{R}_+$ be increasing and continuous functions. The following conditions are satified for the non linear term $\mathcal{H} : [0,\tau] \times \mathfrak{B} \to \mathbb{R}^n$:

i) $\|\mathcal{H}(t,\eta_1) - \mathcal{H}(t,\eta_2)\|_{\mathcal{R}^n} \leq \mathcal{K}(\|\eta_1\|_{\mathfrak{B}}, \|\eta_2\|_{\mathfrak{B}}) \|\eta_1 - \eta_2\|_{\mathfrak{B}}, \quad \forall \eta_1, \eta_2 \in \mathfrak{B}$, $\forall t \in I = (0,\tau]$,

ii) $\|\mathcal{H}(t,\eta)\|_{\mathbb{R}^n} \leq \tilde{\psi}(\|\eta\|_{\mathfrak{B}}), \quad \forall \eta \in \mathfrak{B}$.

(H2) There exist positive constants $L_q, r_k, k = 1, 2, ..., p$ such that $\forall y, z \in \mathbb{R}^n, t \in I$:

i) $\|\mathcal{J}_k(t,y) - \mathcal{J}_k(t,z)\|_{\mathbb{R}^n}, \leq r_k \|y - z\|_{\mathbb{R}^n}$ and $ML_q q < M \sum_{k=1}^{p} r_k < \frac{1}{4}$.

ii) $\mathcal{D}(0) = 0$, and

$$\|\mathcal{D}(\tilde{y}) - \mathcal{D}(\tilde{z})\|_{\mathfrak{B}} \leq L_q \sum_{i=1}^{q} \|\tilde{z}_i - \tilde{y}_i\|_{\mathfrak{B}}, \quad \forall \tilde{y}, \tilde{z} \in (\mathfrak{B})^q.$$

(H3) For such τ there exist positive constant α such that

$$\left(ML_q q + M \sum_{k=1}^{p} r_k \right) (\|\tilde{\varphi}\| + \alpha) + \tau M \tilde{\psi}(\|\tilde{\varphi}\| + \alpha) \leq \frac{\alpha}{2},$$

where $\tilde{\varphi} \in \mathcal{CA}_{d\tau}$ is such that

$$\tilde{\varphi} = \begin{cases} \mathscr{U}(t,0)\varphi(0), & t \in (0,\tau], \\ \varphi(t), & t \in \mathbb{R}_-. \end{cases} \tag{3.2.1}$$

(H4) For α as in (H3) we have the following inequality

$$\tau M \mathcal{K}(\|\tilde{\varphi}\| + \alpha, \|\tilde{\varphi}\| + \alpha) + M \sum_{k=1}^{p} r_k < \frac{1}{2}.$$

Theorem 5. *The system (1.2.1) has at least one solution on $(-\infty, \tau]$ under the hypothesis (H1) - (H3)*

3.3 Existence of solutions

In order to apply Karakosta's fixed point theorem, we consider the following operators:

$$\begin{aligned} \mathcal{T} &: \mathcal{CA}_{d\tau} \times \mathcal{CA}_{d\tau} \longrightarrow \mathcal{CA}_{d\tau}, \\ \mathcal{C} &: \mathcal{CA}_{d\tau} \longrightarrow \mathcal{CA}_{d\tau}, \end{aligned}$$

where

$$\mathcal{T}(z,y)(t) = \begin{cases} y(t) + \sum_{0<t_k<t} \mathscr{U}(t,t_k)\mathcal{J}_k(t_k, z(t_k)), & t \in (0,\tau], \\ -\mathcal{D}(z_{\tau_1}, ..., z_{\tau_q})(t) + \varphi(t), & t \in \mathbb{R}_-, \end{cases}$$

and

$$\mathcal{C}(z)(t) = \begin{cases} \mathscr{U}(t,0)[\varphi(0) - \mathcal{D}(z_{\tau_1}, ..., z_{\tau_q})(0)] + \int_0^t \mathscr{U}(t,s)\mathcal{H}(s, z_s)ds, & t \in (0,\tau], \\ \varphi(t), & t \in \mathbb{R}_-. \end{cases}$$

Now, we define the the following closed and convex set

$$D = D(\alpha, \tau, \varphi) = \{y \in \mathcal{CA}_{d\tau} : \|y - \tilde{\varphi}\| \leq \alpha\}. \tag{3.3.2}$$

Consequently, the problem of finding solutions to the system (1.2.1) is reduced to find solutions of the next operator equation

$$\mathcal{T}(z, \mathcal{C}(z)) = z.$$

We shall use Karakostas Fixed Point Theorem to find solutions of such equation. In order to apply the theorem, we shall verify that $\mathcal{C}$ is continuous, $\mathcal{C}(D)$ is a relatively compact set, $\{\mathcal{T}(\cdot, y) : y \in \overline{\mathcal{C}(D)}\}$ is equicontractive and finally we will check that $\mathcal{T}(\cdot, \mathcal{C}(\cdot))(D) \subseteq D$. We divide the proof in the following Affirmations:

Affirmation 1: $\mathcal{C}$ is continuous.

We shall use Lemma 3 and the hypothesis (H1)-i),(H2)-ii) to obtain the following estimations for $z, y \in \mathcal{CA}_{d\tau}$.

Considering $t \in (0, \tau]$, we have that,

$$\begin{aligned}
&\|\mathcal{C}(z)(t) - \mathcal{C}(y)(t)\|_{\mathbb{R}^n} \qquad (3.3.3)\\
&\leq ML_q \sum_{i=1}^{q} \|y_{\tau_i}(0) - z_{\tau_i}(0)\|_{\mathbb{R}^n} + \int_0^t \|\mathscr{U}(t,s)\| \, \|\mathcal{H}(s, z_s) - \mathcal{H}(s, y_s)\|_{\mathbb{R}^n} \, ds\\
&\leq ML_q \sum_{i=1}^{q} \sup_{t\in(0,\tau]} \|z(t) - y(t)\|_{\mathbb{R}^n} + \int_0^t M \, \|\mathcal{H}(s, z_s) - \mathcal{H}(s, y_s)\|_{\mathbb{R}^n} \, ds\\
&\leq ML_q q \| \left.(z-y)\right|_I \|_\infty + M \int_0^t \mathcal{K}\left(\|z_s\|_{\mathfrak{B}}, \|y_s\|_{\mathfrak{B}}\right) \|z_s - y_s\|_{\mathfrak{B}} \, ds\\
&\leq ML_q q \| \left.(z-y)\right|_I \|_\infty + M \int_0^t \mathcal{K}\left(\|z\|, \|y\|\right) \|z - y\| \, ds\\
&= ML_q q \| \left.(z-y)\right|_I \|_\infty + M\mathcal{K}\left(\|z\|, \|y\|\right) \|z - y\| \int_0^t ds\\
&= ML_q q \| \left.(z-y)\right|_I \|_\infty + M\mathcal{K}\left(\|z\|, \|y\|\right) \|z - y\| (t-0)\\
&\leq ML_q q \| \left.(z-y)\right|_I \|_\infty + M\mathcal{K}(\|z\|, \|y\|)\|z - y\|\tau\\
&\leq ML_q q\|z - y\| + M\tau\mathcal{K}(\|z\|, \|y\|)\|z - y\|. \qquad (3.3.4)
\end{aligned}$$

And, if $t \in (-\infty, 0]$. Then,

$$\|\mathcal{C}(z)(t) - \mathcal{C}(y)(t)\|_{\mathbb{R}^n} = \|\varphi(t) - \varphi(t)\|_{\mathbb{R}^n} = 0,$$

that is, $\left\| \left.(\mathcal{C}(z) - \mathcal{C}(y))\right|_{\mathbb{R}_-} \right\|_{\mathfrak{B}} = 0$.

Hence, by taking the supremum on I, by (3.3.3), we get that

$$\begin{aligned}
\|\mathcal{C}(z) - \mathcal{C}(y)\| &= \| \left.(C(z) - C(y))\right|_{\mathbb{R}_-} \|_{\mathfrak{B}} + \| \left.(C(z) - C(y))\right|_I \|_\infty\\
&= \| \left.(\mathcal{C}(z) - \mathcal{C}(y))\right|_I \|_\infty\\
&\leq \left(ML_q + M\tau\mathcal{K}(\|z\|, \|y\|)\right) \|z - y\|.
\end{aligned}$$

Therefore, we conclude that $\mathcal{C}$ is locally Lipschitz. Hence, it implies that $\mathcal{C}$ is continuous.

Affirmation 2: $\mathcal{C}$ maps bounded sets of $\mathcal{CA}_{d\tau}$ into bounded sets $\mathcal{CA}_{d\tau}$

It is enough to prove that for any $R > 0$, $\exists r > 0$ s.t. for each $y \in B_R = \{z \in \mathcal{CA}_{d\tau} : \|z\| \leq R\}$ we have that $\|\mathcal{C}(y)\| \leq r$. Let's take $y \in B_R$ and baring in mind (H1)-ii), (H2)-ii), and Lemma 3, the following estimates hold

Considering $t \in (0, \tau]$, we have that,

$$\begin{aligned}
\|\mathcal{C}(y)(t)\|_{\mathbb{R}^n} &\leq \left\| \mathscr{U}(t,0) \left\{ \varphi(0) - \mathcal{D}\left(y_{\tau_1}, y_{\tau_2}, \ldots, y_{\tau_q}\right)(0) \right\} \right\| \\
&\quad + \int_0^t \|\mathscr{U}(t,s)\mathcal{H}(s, y_s)\|_{\mathbb{R}^n} ds \\
&\leq M \left\{ \|\varphi(0)\|_{\mathbb{R}^n} + L_q q \|y\| \right\} + M\tilde{\psi}(\|y\|) \int_0^t ds \\
&= M \left\{ \|\varphi(0)\|_{\mathbb{R}^n} + L_q q \|y\| \right\} + M\tilde{\psi}(\|y\|)(t-0) \\
&\leq M \left\{ \|\varphi(0)\|_{\mathbb{R}^n} + L_q q R \right\} + \tau M \tilde{\psi}(R) = l.
\end{aligned}$$

And, for t $\in (-\infty, 0]$, we have that,

$$\|\mathcal{C}(y)(t)\|_{\mathbb{R}^n} = \|\varphi(t)\|_{\mathbb{R}^n}, \quad \forall t \in (-\infty, 0],$$

from which follows that,

$$\begin{aligned}
\left\| (C(y))|_{\mathbb{R}_-} \right\|_{\mathfrak{B}} &= \sup_{t \leq 0} \frac{\|C(y)(t)\|_{\mathbb{R}^n}}{d(t)} \\
&= \sup_{t \leq 0} \frac{\|\varphi(t)\|_{\mathbb{R}^n}}{d(t)} \\
&= \|\varphi\|_{\mathfrak{B}},
\end{aligned}$$

Taking supremum on $t \in [0, \tau]$ and putting $r = \|\varphi\|_{\mathfrak{B}} + l$, we have that

$$\|\mathcal{C}(y)\| = \| \left.(C(y))\right|_{\mathbb{R}_-} \|_{\mathfrak{B}} + \| \left.(C(y))\right|_I \|_\infty \leq r.$$

Hence, Affirmation 2 holds.

Affirmation 3: $\mathcal{C}$ maps bounded sets of $\mathcal{CA}_{d\tau}$ into equicontinuous sets of $\mathcal{CA}_{d\tau}$.

Let's consider B_R as in Affirmation 2. We shall prove that $\mathcal{C}(B_R)$, on the interval is $(-\infty, \tau]$

is equicontinuous. From definition of $\mathcal{C}$, it is enough to show this on $(0,\tau]$.
Let's take $y \in B_R$. Baring in mind (H1)-ii), (H2)-ii) and Lemma 3, we have the following estimates,

$$
\begin{aligned}
&\|\mathcal{C}(y)(t_2) - \mathcal{C}(y)(t_1)\|_{\mathbb{R}^n} \\
\leq & \Big\| \mathscr{U}(t_2,0)\left\{\varphi(0) - \mathcal{D}\left(y_{\tau_1}, y_{\tau_2}, \ldots, y_{\tau_q}\right)(0)\right\} + \int_0^{t_2} \mathscr{U}(t_2,s)\,\mathcal{H}(s,y_s)\,ds \\
& - \mathscr{U}(t_1,0)\left\{\varphi(0) - \mathcal{D}\left(y_{\tau_1}, y_{\tau_2}, \ldots, y_{\tau_q}\right)(0)\right\} - \int_0^{t_1} \mathscr{U}(t_1,s)\,\mathcal{H}(s,y_s)\,ds \Big\|_{\mathbb{R}^n} \\
\leq & \|[\mathscr{U}(t_2,0) - \mathscr{U}(t_1,0)]\left\{\varphi(0) - \mathcal{D}\left(y_{\tau_1}, y_{\tau_2}, \ldots, y_{\tau_q}\right)(0)\right\}\|_{\mathbb{R}^n} \\
& + \Big\| \int_0^{t_1} \mathscr{U}(t_2,s)\,\mathcal{H}(s,y_s)\,ds + \int_{t_1}^{t_2} \mathscr{U}(t_2,s)\,\mathcal{H}(s,y_s)\,ds \\
& - \int_0^{t_1} \mathscr{U}(t_1,s)\,\mathcal{H}(s,y_s)\,ds \Big\|_{\mathbb{R}^n} \\
\leq & \|\mathscr{U}(t_2,0) - \mathscr{U}(t_1,0)\| \left(\left\|\varphi(0) - \mathcal{D}\left(y_{\tau_1}, y_{\tau_2}, \ldots, y_{\tau_q}\right)(0)\right\|_{\mathbb{R}^n} \right) \\
& + \int_0^{t_1} \left\| [\mathscr{U}(t_2,s) - \mathscr{U}(t_1,s)]\,\mathcal{H}(s,y_s) \right\|_{\mathbb{R}^n} ds + \int_{t_1}^{t_2} \|\mathscr{U}(t_2,s)\,\mathcal{H}(s,y_s)\|_{\mathbb{R}^n}\,ds \\
\leq & \|\mathscr{U}(t_2,0) - \mathscr{U}(t_1,0)\| \left(\|\varphi(0)\|_{\mathbb{R}^n} + L_q \sum_{i=1}^{q} \|y_i(t)\|_{\mathbb{R}^n} \right) \\
& + \tilde{\psi}(\|y\|) \int_0^{t_1} \|\mathscr{U}(t_2,s) - \mathscr{U}(t_1,s)\|\,ds + M\tilde{\psi}(\|y\|) \int_{t_1}^{t_2} ds \\
\leq & \|\mathscr{U}(t_2,0) - \mathscr{U}(t_1,0)\| \left(\|\varphi(0)\|_{\mathbb{R}^n} + L_q q \|y\|_{\mathbb{R}^n} \right) \\
& + \tilde{\psi}(R) \int_0^{t_1} \|\mathscr{U}(t_2,s) - \mathscr{U}(t_1,s)\|\,ds + M\tilde{\psi}(R)(t_2 - t_1) \\
\leq & \|\mathscr{U}(t_2,0) - \mathscr{U}(t_1,0)\| \left(\|\varphi(0)\|_{\mathbb{R}^n} + L_q q R \right) \\
& + \tilde{\psi}(R) \int_0^{t_1} \left\| (\mathscr{U}(t_2,s) - \mathscr{U}(t_1,s)) \right\| ds + M\tilde{\psi}(R)(t_2 - t_1).
\end{aligned}
$$

From the continuity of the evolution operator $\mathscr{U}(t,s)$, we obtain that

$$\|\mathcal{C}(y)(t_2) - \mathcal{C}(y)(t_1)\|_{\mathbb{R}^n} \to 0 \quad \text{as} \quad t_2 \to t_1,$$

independently on $y \in B_R$.

Affirmation 4: The subset $\mathcal{C}(D)$ is relatively compact in $\mathcal{CA}_{d\tau}$.

Let D be the bounded subset of $\mathcal{CA}_{d\tau}$ defined in (3.3.2). By Affirmations 2 and 3, $\mathcal{C}(D)$ is bounded and equicontinuous in $\mathcal{CA}_{d\tau}$. Let $\{y_n\}_{n\in\mathbb{N}} \subseteq \mathcal{C}(D)$; then

$$y_n\Big|_{\mathbb{R}_-} = \varphi, \quad \forall n \in \mathbb{N}.$$

Hence, $y_n\Big|_{\mathbb{R}_-}$ converges uniformly on $\mathbb{R}_-$.

Now, putting $\phi_n = y_n\Big|_{[0,\tau]}$, we get that $\{\phi_n\}_{n\in\mathbb{N}} \subseteq \mathcal{CA}_{d\tau}$, endowed with the sumpremum norm. Let us put $t_0 = 0$ and $t_{p+1} = \tau$. Then, applying Arzela-Ascoli Theorem, the sequence $\{\phi_n\}_{n\in\mathbb{N}}$ contains a subsequence $\{\phi_n^1\}_{n\in\mathbb{N}}$ that converges in the interval $[t_0, t_1]$. Now, applying again Arzela-Ascoli Theorem, the sequence $\{\phi_n^1\}_{n\in\mathbb{N}}$ contains a subsequence $\{\phi_n^2\}_{n\in\mathbb{N}}$ that converges in the interval $[t_1, t_2]$.

Continuing with this process we find a subsequence $\{\phi_n^{p+1}\}_{n\in\mathbb{N}}$ of $\{\phi_n\}_{n\in\mathbb{N}}$ that converges in each interval $[t_k, t_{k+1}]$, with $k = 0, 1, 2, \dots, p$. Therefore,

$$\phi_n^{p+1} = y_n^{p+1}\Big|_{[0,\tau]}, \quad \text{converges on} \quad [0, \tau].$$

Consequently, $\{y_n^{p+1}\}_{n\in\mathbb{N}}$ converges uniformly on $(-\infty, \tau]$. Thus, $\mathcal{C}(D)$ is relatively compact, and the proof of Affirmation 4 is completed.

Affirmation 5: The family $\{\mathcal{T}(\cdot, y) : y \in \overline{\mathcal{C}(D)}\}$ is equicontractive.

Let's take $z, x \in CA_{d\tau}$, $y \in \overline{\mathcal{C}(D)}$, and $t \in (0, \tau]$. From the hypothesis (H2)-i), we get:

$$\begin{aligned}
&\|\mathcal{T}(z, C(y))(t) - \mathcal{T}(x, C(y))(t)\|_{\mathbb{R}^n} \\
&\leq \left\| \sum_{0<t_k<t} \mathscr{U}(t, t_k)\, \mathcal{J}_k(t_k, z(t_k)) - \sum_{0<t_k<t} \mathscr{U}(t, t_k)\, \mathcal{J}_k(t_k, x(t_k)) \right\|_{\mathbb{R}^n} \\
&\leq \sum_{0<t_k<t} \left\| \mathscr{U}(t, t_k)\left(\mathcal{J}_k(t_k, z(t_k)) - \mathcal{J}_k(t_k, x(t_k))\right) \right\|_{\mathbb{R}^n} \\
&\leq M \sum_{k=1}^{p} \left\| \left(\mathcal{J}_k(t_k, z(t_k)) - \mathcal{J}_k(t_k, x(t_k))\right) \right\|_{\mathbb{R}^n} \\
&\leq M \sum_{k=1}^{p} r_k \left\| z(t_k) - x(t_k) \right\|_{\mathbb{R}^n} \\
&\leq \left(M \sum_{k=1}^{p} r_k \right) \|(z - x)|_I\|_0 \\
&\leq \left(M \sum_{k=1}^{p} r_k \right) \|z - x\|.
\end{aligned}$$

Hence,

$$\|\mathcal{T}(z, C(y)) - \mathcal{T}(x, C(y))\| \leq \left(M \sum_{k=1}^{p} r_k \right) \|z - x\|. \tag{3.3.5}$$

For $t \in (-\infty, 0]$ and by using ii) of (H_2), we obtain

$$\begin{aligned}
&\frac{1}{d(t)} \|\mathcal{T}(z, C(y))(t) - \mathcal{T}(x, C(y))(t)\|_{\mathbb{R}^n} \\
&\leq \frac{1}{d(t)} \left\| \mathcal{D}\left(z_{\tau_1}, z_{\tau_2}, \ldots, z_{\tau_q}\right)(t) - \mathcal{D}\left(x_{\tau_1}, x_{\tau_2}, \ldots, x_{\tau_q}\right)(t) \right\|_{\mathbb{R}^n} \\
&\leq \left\| \mathcal{D}\left(z_{\tau_1}, z_{\tau_2}, \ldots, z_{\tau_q}\right) - \mathcal{D}\left(x_{\tau_1}, x_{\tau_2}, \ldots, x_{\tau_q}\right) \right\|_{\mathfrak{B}} \\
&\leq L_q q \left\| (z - x)|_{\mathbb{R}_-} \right\|_{\mathfrak{B}} \\
&\leq M L_q q \|z - x\|.
\end{aligned}$$

By taking the supremum in t, we have that,

$$\left\| \left(\mathcal{T}(z, C(y)) - \mathcal{T}(x, C(y))\right)|_{\mathbb{R}_-} \right\|_{\mathfrak{B}} \leq M L_q q \|z - x\|. \tag{3.3.6}$$

Therefore, from (3.3.5) and (3.3.6), we obtain that

$$\|\mathcal{T}(z, C(y)) - \mathcal{T}(x, C(y))\| \leq \frac{1}{2}\|z - x\|.$$

which is a contraction independently of $y \in \overline{C(D)}$. Hence, the family $\{\mathcal{T}(\cdot, y) : y \in \overline{C(D)}\}$ is equicontractive.

Affirmation 6: Finally, we shall prove that

$$\mathcal{T}(\cdot, C(\cdot))(D(\alpha, \tau, \varphi)) \subseteq D(\alpha, \tau, \varphi)$$

Let us take $z \in D(\alpha, \tau, \varphi)$ and consider $t \in (-\infty, 0]$. Then, for $t \in (0, \tau]$, and baring in mind (H1)-ii),(H2)-ii), (H3) and Lemma 3, we have that

$$\begin{aligned}
\|\mathcal{T}(z, C(z))(t) - \tilde{\varphi}(t)\|_{\mathbb{R}^n} \leq & M \left\|\mathcal{D}\left(z_{\tau_1}, z_{\tau_2}, \ldots, z_{\tau_q}\right)(0)\right\|_{\mathbb{R}^n} + \int_0^t \|\mathscr{U}(t, s)\mathcal{H}(s, z_s)\|_{\mathbb{R}^n} ds \\
& + \sum_{0<t_k<t} \|\mathscr{U}(t, t_k)\mathcal{J}_k(t_k, z(t_k))\|_{\mathbb{R}^n} \\
\leq & ML_q q\|z\| + M\tau\tilde{\psi}(\|z\|) + M\sum_{k=1}^{p} r_k\|z\| \\
\leq & ML_q q(\|\tilde{\varphi}\| + \alpha) + M\tau\tilde{\psi}(\|\tilde{\varphi}\| + \alpha) + \left(M\sum_{k=1}^{p} r_k\right)(\|\tilde{\varphi}\| + \alpha) \\
= & \left(ML_q q + M\sum_{k=1}^{p} r_k\right)(\|\tilde{\varphi}\| + \alpha) + M\tau\tilde{\psi}(\|\tilde{\varphi}\| + \alpha) \\
\leq & \alpha/2.
\end{aligned}$$

Moreover, for $t \in (-\infty, 0]$, from (H2)-ii) and (H3)-ii), we obtain

$$\begin{aligned}
\frac{1}{d(t)}\|\mathcal{T}(z, C(z))(t) - \tilde{\varphi}(t)\|_{\mathbb{R}^n} &\leq \|\mathcal{D}\left(z_{\tau_1}, z_{\tau_2}, \ldots, z_{\tau_q}\right)(t)\|_{\mathbb{R}^n} \\
&\leq L_q q\|z\| \leq ML_q q\|z\| \leq ML_q q(\|\tilde{\varphi}\| + \alpha) \leq \alpha/2.
\end{aligned}$$

Hence, $\mathcal{T}(\cdot, B(\cdot))D(\alpha, \tau, \varphi) \subseteq D(\alpha, \tau, \varphi)$. Since Affirmation 1, Affirmation 4 and Affirmation 5 hold, the conditions of Karakostas Fixed Point Theorem are satisfied for the closed and convex set given in (3.3.2), and the proof of Theorem 5 immediately follows by applying Theorem 1.

3.4 Uniqueness and prolongation of solutions

Theorem 6. *Assume that (H4) and the conditions of Theorem 1 holds. Then, the problem (1.2.1) has only one solution on the interval* $(-\infty, \tau]$.

Proof. Let z^1 and z^2 are two solutions for problem (1.2.1). Considering $t \in [0, \tau]$, we have that:

$$\begin{aligned}\|z^1(t) - z^2(t)\|_{\mathbb{R}^n} &\leq \|(\mathcal{D}(z^1_{\tau_1}, z^1_{\tau_2}, \dots, z^1_{\tau_q}))(0) - (\mathcal{D}(z^2_{\tau_1}, z^2_{\tau_2}, \dots, z^2_{\tau_q}))(0)\|_{\mathbb{R}^n} \\ &\quad + \int_0^t \|\mathscr{U}(t,s)(\mathcal{H}(s, z^1_s) - \mathcal{H}(s, z^2_s))\| ds \\ &\quad + \sum_{0<t_k<t} \|\mathscr{U}(t, t_k)(\mathcal{J}_k(t_k, z^1(t_k)) - \mathcal{J}_k(t_k, z^2(t_k)))\| \\ &\leq \left(L_q q + M\tau\mathcal{K}(\|\tilde{\varphi}\| + \alpha), \|\tilde{\varphi}\| + \alpha) + M \sum_{k=1}^{p} r_k \right) \|z^1 - z^2\|.\end{aligned}$$

Note that, from (H1)-(H4), we know that:

$$\tau M\mathcal{K}(\|\varphi\| + \alpha), \|\varphi\| + \alpha) + M \sum_{k=1}^{p} r_k + L_q q < 1,$$

Thus,

$$z^1\Big|_I = z^2\Big|_I$$

.

In the same way, we have the following estimation for t $\in \mathbb{R}_-$

$$\|(z^1 - z^2)|_{\mathbb{R}_-}\|_{\mathfrak{B}} \leq L_q q \|(z^1 - z^2)|_{\mathbb{R}_-}\|_{\mathfrak{B}},$$

Thus,

$$z^1\Big|_{\mathbb{R}_-} = z^2\Big|_{\mathbb{R}_-}.$$

Hence, $z^1 = z^2$. □

Now, we have to consider the following subset $\tilde{D}$ of $\mathbb{R}^N$:

$$\tilde{D} = \{y \in \mathbb{R}^n : \|y\|_{\mathbb{R}^n} \leq R\}, \quad \text{with} \quad R = \|\tilde{\phi}\| + \alpha. \tag{3.4.7}$$

Therefore, for all $z \in D$ we have $z(t) \in \tilde{D}$ for t $(-\infty, \tau]$.

We shall say that $(-\infty, s_1)$ is a maximal interval of existence of the solution $z(\cdot)$ of problem (1.2.1) if there is not solution of the (1.2.1) on $(-\infty, s_2)$ with $s_2 > s_1$.

Theorem 7. *Suppose that the conditions of Theorem 6 hold. If z is a solution of problem (1.2.1) on $(-\infty, s_1)$ and s_1 is maximal, then either $s_1 = +\infty$ or there exists a sequence $\tau_n \to s_1$ as $n \to \infty$ s.t. $z(\tau_n) \to \partial\tilde{D}$.*

Proof. Suppose, for the purpose of contradiction, that there exist a neighborhood N of $\partial\tilde{D}$ such that $z(t)$ does not enter in it, for $0 < s_2 \le t < s_1$. We can take $N = \tilde{D} \backslash B$, where B is a closed subset of $\tilde{D}$, then $z(t) \in B$ for $0 < t_p < s_2 \le t < s_1$. We need to prove that $\lim_{t \to s_1^-} z(t) = z_1 \in B$. Indeed, if we consider $0 < t_p < s_2 \le \ell < t < s_1$, then:

$$\begin{aligned}
\|z(t) - z(\ell)\|_{\mathbb{R}^n} \le\ & \|\mathscr{U}(t,0) - \mathscr{U}(\ell,0)\|\|\varphi(0)\|_{\mathbb{R}^n} \\
&+\|\mathscr{U}(t,0) - \mathscr{U}(\ell,0)\|\|\mathcal{D}(z_{\tau_1}, z_{\tau_2}, \dots, z_{\tau_q})(0)\|_{\mathbb{R}^n} \\
&+\int_0^\ell \|\mathscr{U}(t,s) - \mathscr{U}(\ell,s)\|\|\mathcal{H}(s,z_s)\|ds + \int_\ell^t \|(\mathscr{U}(t,s)\|\|\mathcal{H}(s,z_s)\|ds \\
&+\|\sum_{0<t_k<t} \mathscr{U}(t,t_k)\mathcal{J}_k(t_k, z(t_k)) - \sum_{0<t_k<t} \mathscr{U}(\ell,t_k)\mathcal{J}_k(t_k, z(t_k))\| \\
\le\ & (\|\mathscr{U}(t,0) - \mathscr{U}(\ell,0)\|)\left(\|\varphi(0)\|_{\mathbb{R}^n} + L_q q\right) \\
&+\left(\int_0^\ell \|\mathscr{U}(t,s) - \mathscr{U}(\ell,s)\|ds + \int_\ell^t \|(\mathscr{U}(t,s)\|ds\right)\Psi(R) \\
&+\|\mathscr{U}(t,\ell) - I_{\mathbb{R}^n}\| \sum_{k=1}^q \|\mathscr{U}(\ell,t_k)\|\|\mathcal{J}_k(t_k, z(t_k))\| \\
\le\ & (\|\mathscr{U}(t,0) - \mathscr{U}(\ell,0)\|)\left(\|\varphi(0)\|_{\mathbb{R}^n} + L_q q\right) \\
&+\left(\int_0^\ell \|\mathscr{U}(t,s) - \mathscr{U}(\ell,s)\|ds + \int_\ell^t \|(\mathscr{U}(t,s)\|ds\right)\Psi(R) \\
&+\|\mathscr{U}(t,\ell) - I_{\mathbb{R}^n}\| MR \sum_{k=1}^q r_k
\end{aligned}$$

Since $\mathscr{U}(t,s)$ is uniformly continuous for $t \ge 0$, then $\|z(t) - z(l)\|_{\mathbb{R}^n}$ goes to zero as $l < \ell \to s_1$. Therefore, $\lim_{t \to s_1} z(t) = z_1$ exists in $\mathbb{R}^n$, and since B is closed, z_1 belongs to B. This completes the proof.

□

Corollary 3. *In the conditions of Theorem 6, if the second part of hypothesis (H1) has* *changed to*

$$\|\mathcal{H}(t,\varphi)\| \leq \mu(t)(1+\|\varphi(0)\|_{\mathbb{R}^n}), \quad \varphi \in \mathfrak{B}, \quad t \in \mathbb{R},$$

where $\mu(\cdot)$ *is a continuous function on* $(-\infty,\infty)$, *then the problem (1.2.1) have a unique solution on* $(-\infty,\infty)$.

Proof.

$$\begin{aligned}
\|z(t)\|_{\mathbb{R}^n} &\leq M\left(\|z(0)\|_{\mathbb{R}^n}\right) + M\int_0^t \|\mathcal{H}(s,z_s)\|ds \\
&\quad + M\sum_{0<t_k<t} \|\mathcal{J}_k(t_k,z(t_k))\| \\
&\leq M\|z(0)\|_{\mathbb{R}^n} + \int_0^\tau M\mu(s)(1+\|z(s)\|_{\mathbb{R}^n})ds \\
&\quad + M\sum_{k=1}^{p} r_k\|z(t_k))\|_{\mathbb{R}} \\
&\leq M\left(\|z(0)\|_{\mathbb{R}^N} + \int_0^\tau \mu(s)ds\right) + \int_0^t M\mu(s)\|z(s)\|_{\mathbb{R}^n}ds \\
&\quad + \sum_{k=1}^{p} Mr_k\|z(t_k))\|_{\mathbb{R}^n}.
\end{aligned}$$

Applying Gronwall Inequality for impulsive differential equations (see [46, 47, 9, 8]), we obtain that

$$\|z(t)\|_{\mathbb{R}^n} \leq M\left(\|z(0)\|_{\mathbb{R}^n} + \int_0^\tau \mu(s)ds\right)\prod_{t_0<t_k<t}(1+Mr_k)e^{\int_0^\tau M\mu(s)ds},$$

This implies that $\|z(t)\|_{\mathbb{R}^n}$ stays bounded as $t \to s_1$ and we apply the Theorem 7 we get the result. $\square$

Chapter 4

Approximate Controllability of Time-Dependent Impulsive Semilinear Retarded Differential Equations with Infinite Delay and Nonlocal Conditions

In this section, the approximate controllability of a nonautonomous semilinear retarded system with infinite delay, impulses,nonlocal is proved without using fixed point technique, rather than this, we use a technique avoiding fixed point theorems employed by A.E. Bashirov et al.[48, 49, 50]. In this case we have to impose some conditions on the nonlinear term depending of the last impulse time t_p so that we can prove the approximate controllability of this system by living the impulses behind on a fixed curve in a short time interval, and from this position, we are able to reach a neighborhood of the final state on time τ, by assuming that the corresponding linear control system is exactly controllable on any interval $[t_0, \tau]$, $0 < t_0 < \tau$.

4.1 Controlability of linear system

Associated with the semilinear system (1.2.1), we consider also the linear system

$$\begin{cases} z'(t) = \mathcal{A}(t)z(t) + \mathcal{B}(t)u(t), \quad t \in (t_0, \tau], \\ z(t_0) = z^0. \end{cases} \tag{4.1.1}$$

In this subsection, we shall present some known characterization of the controllability of the linear system (4.1.1) without impulses, infinite delays and nonlocal conditions. To this end, we note that for all $z_0 \in \mathbb{R}^n$ and $u \in L_2(0, \tau; \mathbb{R}^m)$ the initial value problem

$$\begin{cases} y' = \mathcal{A}(t)y(t) + \mathcal{B}(t)u(t), \quad y \in \mathbb{R}^n, \quad t \in [\tau - \delta, \tau], \\ y(\tau - \delta) = z^0, \end{cases} \tag{4.1.2}$$

admits only one solution given by

$$y(t) = \mathscr{U}(t, \tau - \delta)z^0 + \int_{\tau-\delta}^{t} \mathscr{U}(t, s)\mathcal{B}(s)u(s)ds, \ \ t \in [\tau - \delta, \tau], \tag{4.1.3}$$

Definition 4. *Corresponding with (4.1.2), we define the following matrix: The Gramian controllability matrix by:*

$$\mathscr{W}_{\tau\delta} = \int_{\tau-\delta}^{\tau} \mathscr{U}(\tau, t)\mathcal{B}(s)\mathcal{B}^*(s)\mathscr{U}^*(\tau, t)dt. \tag{4.1.4}$$

Proposition 4. *(See [51, 52]) The system (4.1.2) is controllable on $[\tau - \delta, \tau]$ if, and only if, the matrix $\mathscr{W}$ is invertible.*
Moreover, a control steering the system (4.1.2) from initial state z_0 to a final state z^1 on the interval $[\tau - \delta, \tau]$ is given by

$$v^{\delta}(t) = \mathcal{B}^*(t)\mathscr{U}^*(\tau, t)\mathscr{W}_{\tau\delta}^{-1}(z^1 - \mathscr{U}(\tau, \tau - \delta)z^0), \quad t \in [\tau - \delta, \tau]. \tag{4.1.5}$$

i.e.,
The corresponding solution $y^{\delta}(t)$ of the linear system (4.1.2) satisfies the boundary condition:

$$y^{\delta}(\tau - \delta) = z^0 \quad \textit{and} \quad y^{\delta}(\tau) = z^1.$$

4.2 Hypotheses

In order to study the approximate controllability of system (1.2.2), with techniques that evade the use of fixed point Theorems, We will assume the following hypothesis:

(H1) The nonlinear term h satisfies the following estimate

$$|h(t,\varphi,u)| \leq \zeta(\|\varphi(-t_p)\|), \qquad u \in \mathbb{R}^m, \varphi \in \mathcal{B}, \tag{4.2.6}$$

where $\zeta : \mathbb{R}_+ \to [0,\infty)$ is a continuous function. In particular, $\zeta(\xi) = a(\xi)^\beta + b$, with $\beta \geq 1$.

(H2) The linear control system (4.1.1) is exactly controllable on any interval $[\tau-\delta,\tau]$, for all δ with $0 < \delta < \tau$.

The hypothesis H2) is satisfied in the case that $\mathcal{A}(t) = \mathcal{A}$ and $\mathcal{B}(t) = \mathcal{B}$ are constant matrices since the algebraic Kalman's condition(see [15]) for exact controllability of linear autonomous systems does not depend on the time interval.

$$\text{rank}[\mathcal{B}|\mathcal{A}\mathcal{B}|\cdots|\mathcal{A}^{n-1}\mathcal{B}] = n$$

4.3 Approximate Controllability

In this section, we shall prove the main results of this work, the approximate controllability of the semilinear retarded system (1.2.2) with infinite delay, impulses, and nonlocal conditions. In this regard, according to (CITA), for all $\phi \in \mathfrak{B}$ and $u \in L^2(0,\tau;\mathbb{R}^m)$ the problem (1.2.2) admits only one solution $z \in \mathcal{CA}_{d\tau}$ given by

$$\begin{aligned} z(t) &= \mathscr{U}(t,0)\varphi(0) - \mathscr{U}(t,0)[(\mathcal{D}(z_{\tau_1},\ldots,z_{\tau_q}))(0)] + \int_0^t \mathscr{U}(t,s)\mathcal{B}(s)u(s)ds \\ &+ \int_0^t \mathscr{U}(t,s)h(s,z(s-t_p),u(t))ds + \sum_{0<t_k<t} \mathscr{U}(t,t_k)\mathcal{J}_k(t_k,z(t_k)u(t_k)), \quad t \in [0,\tau], \\ z(t) &+ (\mathcal{D}(z_{\tau_1},\ldots,z_{\tau_q}))(t) = \varphi(t), \quad t \in (-\infty,0]. \end{aligned} \tag{4.3.7}$$

Proof. Given $\varphi \in \mathfrak{B}$, a final state z^1 and $\varepsilon > 0$, we want to find a control $u^\delta \in L^2(0,\tau;\mathbb{R}^m)$ steering the system to a neighborhood of z^1 on $[0,\tau]$. Precisely, for $0 < \delta < min\{\tau - t_p, t_p\}$ small enough, there exists control $u^\delta \in L^2(0,\tau;\mathbb{R}^m)$ such that corresponding of solutions z^δ of (1.2.2) satisfies

$$\left\| z^\delta(\tau) - z^1 \right\| < \epsilon.$$

In indeed, we consider any fixed control $u \in L^2(0,\tau;\mathbb{R}^m)$ and the corresponding solution $z(t) = z(t, 0, \varphi, u)$ of the problem (1.2.2). For $0 < \delta < min\{\tau - t_p, t_p\}$ small enough, we define the control $u^\delta \in L^2(0,\tau;\mathbb{R}^m)$ as follows

$$u^\delta(t) = \begin{cases} u(t), & \text{if} \quad 0 \le t \le \tau - \delta, \\ v^\delta(t), & \text{if} \quad \tau - \delta < t \le \tau. \end{cases}$$

where

$$v^\delta(t) = \mathcal{B}^*(t)\mathscr{U}^*(\tau,t)(\mathscr{W}_{\tau\delta})^{-1}(z^1 - \mathscr{U}(\tau,\tau-\delta)z(\tau-\delta)), \quad \tau - \delta < t \le \tau.$$

Since $0 < \delta < \tau - t_p$, then $\tau - \delta > t_p$; and using the cocycle property $\mathscr{U}(t,l)\mathscr{U}(l,s) = \mathscr{U}(t,s)$, the corresponding solution $z^\delta(t) = z(t,0,\varphi,u^\delta)$ of the inpulsive, nonlocal and retarded system (1.2.2) at time τ can be written as follows:

$$\begin{aligned}
z^\delta(\tau) &= \mathscr{U}(\tau,0)\varphi(0) - \mathscr{U}(\tau,0)[(\mathcal{D}(z_{\tau_1},\dots,z_{\tau_q})(0)] + \int_0^\tau \mathscr{U}(\tau,s)\mathcal{B}(s)u^\delta(s)ds \\
&+ \int_0^\tau \mathscr{U}(\tau,s)h(s,z_s^\delta,u^\delta(s))ds + \sum_{0<t_k<\tau} \mathscr{U}(\tau,t_k)\mathcal{J}_k(t_k,z(t_k),u^\delta(t_k)) \\
&= \mathscr{U}(\tau,\tau-\delta)\Big\{ \mathscr{U}(\tau-\delta,0)\varphi(0) - \mathscr{U}(\tau-\delta,0)[(\mathcal{D}(z_{\tau_1},\dots,z_{\tau_1}))(0)] \\
&+ \int_0^{\tau-\delta} \mathscr{U}(\tau-\delta,s)\mathcal{B}(s)(s)u^\delta(s)ds \\
&+ \int_0^{\tau-\delta} \mathscr{U}(\tau-\delta,s)h(s,z_s^\delta,u^\delta(s))ds \\
&+ \sum_{0<t_k<\tau-\delta} \mathscr{U}(\tau-\delta,t_k)\mathcal{J}_k^e(t_k,z^\delta(t_k),u^\delta(t_k)) \Big\} \\
&+ \int_{\tau-\delta}^\tau \mathscr{U}(\tau,s)\mathcal{B}(s)u^\delta(s)ds + \int_{\tau-\delta}^\tau \mathscr{U}(\tau,s)h(s,z_s^\delta,u^\delta(s))ds \\
&= \mathscr{U}(\tau,\tau-\delta)z(\tau-\delta) + \int_{\tau-\delta}^\tau \mathscr{U}(\tau,s)\mathcal{B}(s)v^\delta(s))ds \\
&+ \int_{\tau-\delta}^\tau \mathscr{U}(\tau,s)h(s,z_s^\delta,v^\delta(s))ds.
\end{aligned}$$

So,

$$\begin{aligned} z^\delta(\tau) &= \mathscr{U}(\tau,\tau-\delta)z(\tau-\delta)+\int_{\tau-\delta}^{\tau}\mathscr{U}(\tau,s)\mathcal{B}(s)v^\delta(s)ds \\ &+ \int_{\tau-\delta}^{\tau}\mathscr{U}(\tau,s)h(s,z_s^\delta,v^\delta(s))ds. \end{aligned}$$

The corresponding solution $y^\delta(t) = y(t,\tau-\delta,z(\tau-\delta),v^\delta)$ of the initial value problem (4.1.2) at time τ, for the control v^δ and the initial condition $z_0 = z(\tau-\delta)$, is given by:

$$y^\delta(\tau) = \mathscr{U}(\tau,\tau-\delta)z(\tau-\delta)+\int_{\tau-\delta}^{\tau}\mathscr{U}(\tau,s)\mathcal{B}(s)v^\delta(s)ds,$$

and from Proposition 4, we get that

$$y^\delta(\tau) = z^1.$$

Therefore,

$$\left\|z^\delta(\tau)-z^1)\right\| \leq \int_{\tau-\delta}^{\tau}\|\mathscr{U}(\tau,s)\|\left\|h(s,z_s^\delta,u^\delta(s))\right\|ds.$$

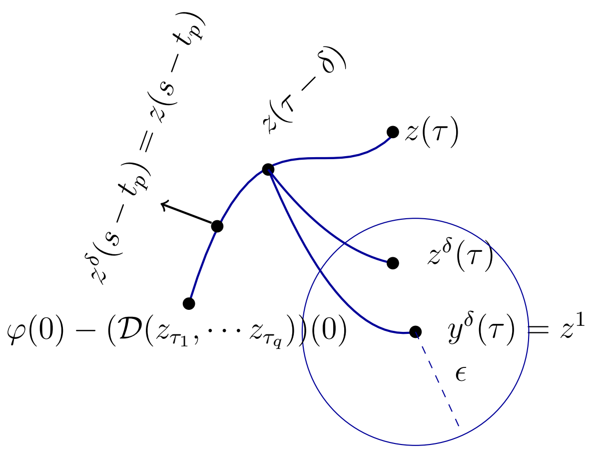
Now, since $0<\delta<t_p$ and $\tau-\delta\leq s\leq\tau$, then $s-t_p\leq\tau-t_p<\tau-\delta$ and

$$z^\delta(s-t_p) = z(s-t_p).$$

Hence, there exists δ small enough such that $0<\delta<\min\{t_p,\tau-t_p\}$ and

$$\begin{aligned} \left\|z^\delta(\tau)-z^1\right\| &\leq \int_{\tau-\delta}^{\tau}\|\mathscr{U}(\tau,s)\|\left\|h(s,z_s^\delta,v^\delta(s))\right\|ds \\ &\leq \int_{\tau-\delta}^{\tau}\|\mathscr{U}(\tau,s)\|\,\zeta(\|z(s-t_p\|)ds < \varepsilon. \end{aligned}$$

In the following figure one can see how the proof of this theorem goes.



With this, the proof of the theorem is completed. □

Chapter 5

Results

5.1 Exact Controllability of Retarded Semilinear Equations with Infinite Delay, Impulses and Nonlocal Conditions

The main objective of this section is to use Rothe's fixed point to prove that the system in (1.2.2) is exactly controllable. In order to do that, we shall consider the following hypotheses:

5.2 Hypotheses

(H1) The linear system associated to (1.2.2) is controllable on $[0, \tau]$,

(H2) The following statements hold:

$$\|h(t,\varphi,u)\|_{\mathbb{R}^n} \leq a_0\|\varphi(-t_p)\|_{\mathbb{R}^n}^{\alpha_0} + b_0\|u\|_{\mathbb{R}^m}^{\beta_0} + c_0, \quad u \in \mathbb{R}^m, \varphi \in \mathfrak{B}, \quad t \in [0,\tau], \tag{5.2.1}$$

$$\|\mathcal{J}_k(t_k,z,u)\|_{\mathbb{R}^n} \leq a_k\|z\|_{\mathbb{R}^n}^{\alpha_k} + b_k\|u\|_{\mathbb{R}^m}^{\beta_k} + c_k, \quad k = 1,2,3,\dots,p, u \in \mathbb{R}^m, \quad z \in \mathbb{R}^n, \tag{5.2.2}$$

$$\|\mathcal{D}(z)\|_C \leq e\|z\|_{\mathfrak{B}^q}^{\eta}, \quad z \in \mathfrak{B}^q((-\infty,0];(\mathbb{R}^n)^q)), \tag{5.2.3}$$

$$\|\mathcal{D}(z) - \mathcal{D}(w)\|_C \leq K\|z-w\|_{\mathfrak{B}^q}, \quad z,w \in \mathfrak{B}^q((-\infty,0];(\mathbb{R}^n)^q)), \tag{5.2.4}$$

with $0 < \eta \leq 1$, $0 < \alpha_k \leq 1$, $0 < \beta_k \leq 1$, $\quad k = 0, 1, 2, 3, \ldots, p$, and

$$z(t_k) = z(t_k^+) = \lim_{t \to t_k^+} z(t), \quad z(t_k^-) = \lim_{t \to t_k^-} z(t).$$

In addition to the conditions imposed to the operators and functions involving the system, it is also necessary to define some operators that help us to prove the controllability of the system (1.2.2).

Now, we define the operator

$$\Theta : \mathcal{CA}_{d\tau}((-\infty, \tau]; \mathbb{R}^n) \times C([0, \tau]; \mathbb{R}^m) \to \mathcal{CA}_{d\tau}((-\infty, \tau]; \mathbb{R}^n) \times C([0, \tau]; \mathbb{R}^m)$$

by the following formula:

$$(y, v) = (\Theta_1(z, u), \Theta S_2(z, u)) = \Theta(z, u),$$

where Θ_1 and Θ_2 are defined as follow:

$$\Theta_1 : \mathcal{CA}_{d\tau}((-\infty, \tau]; \mathbb{R}^n) \times C([0, \tau]; \mathbb{R}^m) \to \mathcal{CA}_{d\tau}((-\infty, \tau]; \mathbb{R}^n),$$

and

$$\Theta_2 : \mathcal{CA}_{d\tau}((-\infty, \tau]; \mathbb{R}^n) \times C([0, \tau]; \mathbb{R}^m) \to C([0, \tau]; \mathbb{R}^m),$$

such that:

$$\begin{aligned} y(t) &= \Theta_1(z, u)(t) \\ &= \mathscr{U}(t, 0)\{\varphi(0) - \mathcal{D}(z_{\tau_1}, z_{\tau_2}, \ldots, z_{\tau_q})(0)\} \\ &\quad + \int_0^t \mathscr{U}(t, s)\mathcal{B}(s)(\Upsilon\mathfrak{L}(z, u))(s)ds \\ &\quad + \int_0^t \mathscr{U}(t, s)h(s, z_s, u(s))ds + \sum_{0<t_k<t} \mathscr{U}(t, t_k)\mathcal{J}_k(z(t_k), u(t_k)), \quad t \in [0, \tau], \\ y(t) &= \varphi(t), \qquad t \in (-\infty, 0], \end{aligned} \tag{5.2.5}$$

and

$$v(t) = \Theta_2(z, u)(t) = (\Upsilon\mathfrak{L}(z, u))(t) = \mathcal{B}^*(t)\mathscr{U}^*(\tau, t)\mathcal{W}^{-1}\mathfrak{L}(z, u), \tag{5.2.6}$$

with

$$\begin{aligned}\mathfrak{L}(z,u) = z^1 - \mathscr{U}(\tau,0)\{\varphi(0) - \mathcal{D}(z_{\tau_1}, z_{\tau_2},\dots,z_{\tau_q}))(0)\} \\ - \int_0^\tau \mathscr{U}(\tau,s)h(s,z_s,u(s))ds \\ - \sum_{0<t_k<\tau} \mathscr{U}(\tau,t_k)\mathcal{J}_k(z(t_k),u(t_k)).\end{aligned} \tag{5.2.7}$$

The following proposition follows trivially from the definition of the operator Θ.

Proposition 5. *The Semilinear System (1.2.2) with impulses, infinite delay, and nonlocal conditions is controllable if, and only if, for all initial state $\varphi \in \mathfrak{B}$ and a final state z^1 the operator Θ given by (5.2.5)-(5.2.7) has a fixed point. i.e.,*

$$\exists (z,u) \in \mathrm{Dom}(\Theta) \quad \textit{such that} \quad \Theta(z,u) = (z,u).$$

Theorem 8. *Suppose that conditions (5.2.1)-(5.2.4) hold and the linear system (4.1.1) is controllable on $[0,\tau]$. If $0 \le \alpha_k < 1$, $0 \le \beta_k < 1$, $k = 0,1,2,3,\dots,p$, $0 \le \eta < 1$, then the nonlinear system (1.2.2) is controllable on $[0,\tau]$. Moreover, exists a control $u \in C([0,\tau];\mathbb{R}^m)$ such that for a given $\varphi \in \mathfrak{B}, z^1 \in \mathbb{R}^n$ the corresponding solution $z^u(\cdot)$ of (1.2.2) satisfies:*

$$\begin{aligned} z_1 &= \mathscr{U}(\tau,0)\{\varphi(0) - \mathcal{D}(z_{\tau_1}, z_{\tau_2},\dots,z_{\tau_q})(0)\} + \int_0^\tau \mathscr{U}(\tau,s)\mathcal{B}(s)u(s)ds \\ &\quad + \int_0^\tau \mathscr{U}(\tau,s)h(s,z_s^u,u(s))ds + \sum_{0<t_k<\tau} \mathscr{U}(\tau,t_k)\mathcal{J}_k(t_k,z(t_k),u(t_k)),\end{aligned}$$

and

$$u(t) = \mathcal{B}^*(t)\mathscr{U}^*(\tau,t)\mathscr{W}^{-1}\mathscr{L}(z,u),$$

with

$$\begin{aligned}\mathscr{L}(z,u) &= z^1 - \mathscr{U}(\tau,0)\{\varphi(0) - \mathcal{D}(z_{\tau_1}, z_{\tau_2},\dots,z_{\tau_q})(0)\} - \int_0^\tau \mathscr{U}(\tau,s)h(s,z_s,u(s))ds \\ &\quad - \sum_{0<t_k<\tau} \mathscr{U}(\tau,t_k)\mathcal{J}_k(z(t_k),u(t_k)).\end{aligned}$$

Proof We shall prove this theorem by claims.

Statement 1. The operator Θ is continuous. In fact, to prove the continuity of Θ, it is enough to prove the continuity of the operators Θ_1 and Θ_2 defined above.

The continuity of Θ_1 follows from the continuity of the nonlinear functions $h(t, z_s, u)$, $\mathcal{J}_k(z, u)$, $\mathcal{D}(z)$ and the following estimate

$$\begin{aligned}\|\Theta_1(z,u) - \Theta_1(w,v)\| &\leq K_1\|z-w\| \\ &+K_2 \sup_{s\in J}\|h(s,z_s,u(s)) - h(s,w_s,v(s))\| \\ &+K_3 \sum_{0<t_k<t}\|\mathcal{J}J_k(t_k,z(t_k),u(t_k)) - \mathcal{J}_k(t_k,w(t_k),v(t_k))\|,\end{aligned}$$

where,

$$K_1 = K_4\widehat{K}, \quad K_2 = \frac{M}{w}\widehat{K}, \quad K_3 = M_3\widehat{K}, \quad with \quad \widehat{K} = 1 + \frac{M^2}{\omega}\|\mathcal{B}\|^2\|\mathscr{W}^{-1}\|,$$

and $\quad K_4 = M_3K.$

The continuity of the operator Θ_2 follows from the continuity of the operators $\mathfrak{L}$ and Υ define above.

Statement 2. Θ maps bounded sets of $\mathcal{CA}_{d\tau}((-\infty,\tau];\mathbb{R}^n) \times C([0,\tau];\mathbb{R}^m)$ into equicontinuous sets of $\mathcal{CA}_{d\tau}((-\infty,\tau];\mathbb{R}^n) \times C([0,\tau];\mathbb{R}^m)$.

Consider the following equality

$$\begin{aligned}\|\Theta(z,u)(t_2) - \Theta(z,u)(t_1)\|_1 &= \|\Theta_1(z,u)(t_2) - \Theta_1(z,u)(t_1)\| \\ &+\|\Theta_2(z,u)(t_2) - \Theta_2(z,u)(t_1)\|\end{aligned}$$

Let $D \subset \mathcal{CA}_{d\tau}((-\infty,\tau];\mathbb{R}^n) \times C([0,\tau];\mathbb{R}^m)$ be a bounded set. The equicontinuity for $\Theta(D)$ is given by the equicontinuity of each one of its components $\Theta_1(D)$, $\Theta_2(D)$, which are obtained from the continuity of $\mathscr{U}(t,s)$ and the following estimates $\forall(z,u) \in D$

$$\|\Theta_2(z,u)(t_2) - \Theta_2(z,u)(t_1)\| \leq \|\mathcal{B}^*(t_2)\mathscr{U}^*(\tau,t_2) - \mathcal{B}^*(t_1)\mathscr{U}^*(\tau,t_1)\|\|\mathscr{W}^{-1}\mathfrak{L}(z,u)\|,$$

Since $\mathscr{U}(t,s)$ is continuous $\|\mathscr{U}(t_2,s)-\mathscr{U}(t_1,s)\|$ goes to zero as $t_2 \to t_1$ and so does the sum and the integral from t_1 to t_2, which implies that $\Theta_1(D)$ is equicontinuous. Moreover, the equicontinuity of $\Theta_2(D)$ follows from the continuity of the evolution operator $\mathscr{U}(t,s)$. Hence, Θ maps bounded sets into equicontinuous sets.

Statement 3. The set $\Theta(D)$ is relatively compact. Indeed, let D be a bounded subset of $\mathcal{CA}_{d\tau}((-\infty,\tau];\mathbb{R}^n) \times C([0,\tau];\mathbb{R}^m)$. By the continuity of h, $\mathfrak{L}$, and $\mathcal{J}_k$, for $\forall (z,u) \in D$ it follows that

$$\|h(\cdot,z,u)\|_0 \leq M_5, \quad \|\mathscr{W}^{-1}\mathscr{L}(z,u)\| \leq M_6, \quad \|\mathcal{J}_k(z,u)\| \leq \mathcal{T}_k, \quad k=1,2,\dots,p,$$

where $\|\mathcal{J}_k(z,u)\| = \sup_{t\in[0,\tau]}\{\|\mathcal{J}_k(z(t,u(t))\|_{\mathbb{R}^n}\}$ $M_5,\ M_6,\ \mathcal{T}_1,\mathcal{T}_2,\dots,\mathcal{T}_k, \in \mathbb{R}$. Therefore, $\Theta(D)$ is uniformly bounded. Now, we consider a sequence a $\{\psi_i = (\psi_{1i},\psi_{2i}) : i=1,2,\dots\}$ in $\Theta(D)$. Since $\{\psi_{2i} : i=1,2,\dots\}$ is contained in $\Theta_2(D) \subset C([0,\tau];\mathbb{R}^m)$ and $\Theta_2(D)$ is an uniformly bounded and equicontinuous family, by Arzelà-Ascoli Theorem we can assume, without loss of generality, that $\{\psi_{2i} : i=1,2,\dots\}$ converges. On the other hand, since $\{\psi_{1i} : i=1,2,\dots\}$ is contained in $\Theta_1(D) \subset \mathcal{CA}_{d\tau}((-\infty,\tau];\mathbb{R}^n)$, then $\psi_{1i}\Big|_{(-\infty,-\tau_q]} = \varphi - \mathcal{D}(\varphi_{\tau_1},\varphi_{\tau_2},\dots,\varphi_{\tau_q})$, $i=1,2,\dots$. Taking into account that $\{\psi_{1i} : i=1,2,\dots\}$ is bounded and equicontinuous in $[0,t_1]$, we can apply Arzelà-Ascoli Theorem to ensure the existence of a subsequence $\{\psi_{1i}^1 : i=1,2,\dots\}$ of $\{\psi_{1i} : i=1,2,\dots\}$, which is uniformly convergent on $[0,t_1]$. Now, consider the sequence $\{\varphi_{1i}^1 : i=1,2,\dots\}$ on the interval $[t_1,t_2]$. On this interval the sequence $\{\psi_{1i}^1 : i=1,2,\dots\}$ is uniformly bounded and equicontinuous, and for the same reason, it has a subsequence $\{\psi_{1i}^2\}$ uniformly convergent on $[0,t_2]$. In this way, for the intervals $[t_2,t_3]$, $[t_3,t_4]$, $\dots$, $[t_p,\tau]$, we see that the sequence $\{\varphi_{1i}^{p+1} : i=1,2,\dots\}$ converges uniformly on the interval $[0,\tau]$.
Besides, in the interval $[-\tau_q,0]$ the function ψ_{1i} is piecewise continuous, then repeating the foregoing process we can assume that the subsequence $\{\psi_i^{p+1} = (\psi_{1i}^{p+1},\psi_{2i}^{p+1}) : i=1,2,\dots\}$ converges in $\Theta(D)$. This means that $\overline{\Theta(D)}$ is compact, i.e., $\Theta(D)$ is relatively compact.

Statement 4. for $0 < \alpha_k < 1$, $0 < \beta_k < 1$, $k = 0, 1, 2, 3, \ldots, p$, $0 < \eta < 1$, the following limit holds.

$$\lim_{|||(z,u)|||\to\infty} \frac{|||\Theta(z,u)|||}{|||(z,u)|||} = 0,$$

where $|||(z,u)||| = \|z\|_0 + \|u\|_0$ is the norm in the space $\mathcal{CA}_{d\tau}((-\infty,\tau];\mathbb{R}^n) \times C([0,\tau];\mathbb{R}^m)$. Using the conditions (5.2.1)-(5.2.4), we get that

$$\|\mathfrak{L}(z,u)\| \leq$$
$$M_1 + M_2\{e\|z\|^{\eta} + a_0\|z\|^{\alpha_0} + b_0\|u\|^{\beta_0} + c_0\} + M_3 \sum_{0<t_k<\tau} \{a_k\|z\|^{\alpha_k} + b_k\|u\|^{\beta_k} + c_k\},$$

where

$$M_1 = \|z^1\| + M_3\|\varphi(0)\|, \quad M_2 = M_3 + \frac{M}{\omega} \quad \text{and} \quad M_3 = Me^{\omega\tau}.$$

$$\begin{aligned} \|\Theta_2(z,u)\| \;&\leq\; \|\mathcal{B}\|M_3M_1\|\mathscr{W}^{-1}\| + \|\mathcal{B}\|M_3M_2\|\mathscr{W}^{-1}\|\{e\|z\|^{\eta} + a_0\|z\|^{\alpha_0} + b_0\|u\|^{\beta_0} + c_0\} \\ &\quad + \|\mathcal{B}\|M_3^2\|\mathscr{W}^{-1}\| \sum_{0<t_k<\tau} \{a_k\|z\|^{\alpha_k} + b_k\|u\|^{\beta_k} + c_k\}. \end{aligned}$$

and

$$\begin{aligned} \|\Theta_1(z,u)\| \;&\leq\; M_3\|\varphi(0)\| + \frac{M^2}{\omega}\|\mathcal{B}\|^2\|\mathscr{W}^{-1}\|M_1 \\ &\quad + M_2\widehat{K}\{e\|z\|^{\eta} + a_0\|z\|^{\alpha_0} + b_0\|u\|^{\beta_0} + c_0\} \\ &\quad + M_3\widehat{K} \sum_{0<t_k<\tau} \{a_k\|z\|^{\alpha_k} + b_k\|u\|^{\beta_k} + c_k\}. \end{aligned}$$

Therefore,

$$\begin{aligned} |||\Theta(z,u)||| \;&=\; \|\Theta_1(z,u)\| + \|\Theta_2(z,u)\| \\ &\leq\; M_4 + \{\|\mathcal{B}\|M_3M_2\|\mathscr{W}^{-1}\| + M_2\widehat{K}\}\{e\|z\|^{\eta} + a_0\|z\|^{\alpha_0} + b_0\|u\|^{\beta_0} + c_0\} \\ &\quad + \{\|\mathcal{B}\|M_3^2\|\mathscr{W}^{-1}\| + M_3\widehat{K}\} \sum_{0<t_k<\tau} \{a_k\|z\|^{\alpha_k} + b_k\|u\|^{\beta_k} + c_k\}, \end{aligned}$$

where M_4 is given by:

$$M_4 = M_3\|\varphi(0)\| + \|\mathcal{B}\|\|\mathscr{W}^{-1}\|M_1\{M_3 + \frac{M^2}{\omega}\|B\|\}.$$

Hence,

$$\begin{aligned}
\frac{|||\Theta(z,u)|||}{|||(z,u)|||} \quad \leq \quad & \frac{M_4}{\|z\|+\|u\|} \\
& +\{\|\mathcal{B}\|M_3M_2\|\mathscr{W}^{-1}\| + M_2\widehat{K}\} \\
& \times\{e\|z\|^{\eta-1} + a_0\|z\|^{\alpha_0-1} + b_0\|u\|^{\beta_0-1} + \frac{c_0}{\|z\|+\|u\|}\} \\
& +\{\|\mathcal{B}\|M_3^2\|\mathscr{W}^{-1}\| + M_3\widehat{K}\} \times \\
& \sum_{0<t_k<\tau}\{a_k\|z\|^{\alpha_k-1} + b_k\|u\|^{\beta_k-1} + \frac{c_k}{\|z\|+\|u\|}\},
\end{aligned}$$

Consequently,

$$\lim_{|||(z,u)|||\to\infty} \frac{|||\Theta(z,u)|||}{|||(z,u)|||} = 0.$$

Statement 5. The operator Θ has a fixed point. In fact, by Claim 4, we know that for a fixed $0 < \rho < 1$ there exists $R > 0$ big enough such that

$$|||\Theta(z,u)||| \leq \rho|||(z,u)|||, \quad |||(z,u)||| = R.$$

Hence, if we denote by $B(0,R)$ the closed ball of center zero and radius $R > 0$, we get that $\Theta(\partial B(0,R)) \subset B(0,R)$. Since Θ is a compact operator, $\Theta(B)$ is relatively compact in $\mathcal{CA}_{d\tau}((-\infty,\tau];\mathbb{R}^n) \times C([0,\tau];\mathbb{R}^m)$, and maps the sphere $\partial B(0,R)$ into the interior of the ball $B(0,R)$, we can apply Rothe's fixed point theorem 2 to ensure the existence of a fixed point $(z,u) \in B(0,R) \subset \mathcal{CA}_{d\tau}((-\infty,\tau];\mathbb{R}^n) \times C([0,\tau];\mathbb{R}^m)$ such that

$$\Theta(z,u) = (z,u).$$

Hence, applying the Proposition 5, we get that the nonlinear system (1.2.2) is controllable on $[0, \tau]$. Moreover,

$$u = \Upsilon\mathfrak{L}(z,u) = \mathcal{B}^*(\cdot)\mathscr{U}^*(\tau,\cdot)\mathscr{W}^{-1}\mathfrak{L}(z,u),$$

such that for a given $\varphi \in \mathfrak{B}$, $z^1 \in \mathbb{R}^n$ the corresponding solution $z(t) = z(t,u)$ of (1.2.2) satisfies:

$$\begin{aligned} z^1 &= \mathscr{U}(\tau,0)\{\varphi(0) - \mathcal{D}(z_{\tau_1}, z_{\tau_2}, \dots, z_{\tau_q}))(0)\} + \int_0^\tau \mathscr{U}(\tau,s)\mathcal{B}(s)u(s)ds \\ &+ \int_0^\tau \mathscr{U}(\tau,s)h(s,z_s,u(s))ds + \sum_{0<t_k<\tau} \mathscr{U}(\tau,t_k)\mathcal{J}_k(t_k, z(t_k), u(t_k)). \end{aligned}$$

It completes the proof. □

Now, we present another version of the previous theorem, which follows from the estimates considered in Claim 4.

Theorem 9. *Suppose the linear system (4.1.1) is controllable on* $[0,\tau]$. *Then the nonlinear system (1.2.2) is controllable if one of the following statement holds:*

a) $\alpha_0 = 1$, $\max\{\alpha_k : k = 1,2,\dots,p\} < 1$, $\max\{\beta_k : k = 0,1,2,\dots,p\} < 1$, $\quad \eta < 1$ *and*

$$\{\|\mathcal{B}\| M_3 M_2 \|\mathscr{W}^{-1}\| + M_2\widehat{K}\}a_0 < 1.$$

b) $\beta_0 = 1$, $\max\{\beta_k : k = 1,2,\dots,p\} < 1$, $\max\{\alpha_k : k = 0,1,2,\dots,p\} < 1 \quad \eta < 1$ *and*

$$\{\|\mathcal{B}\| M_3 M_2 \|\mathscr{W}^{-1}\| + M_2\widehat{K}\}b_0 < 1.$$

c) $\beta_0 = \alpha_0 = 1$, $\max\{\beta_k : k = 1,2,\dots,p\} < 1$, $\max\{\alpha_k : k = 1,2,\dots,p\} < 1 \quad \eta < 1$ *and*

$$\{\|\mathcal{B}\| M_3 M_2 \|\mathscr{W}^{-1}\| + M_2\widehat{K}\}(a_0 + b_0) < 1.$$

d) $\beta_0 = \alpha_0 = \eta = 1$, $\max\{\beta_k : k = 1, 2, \ldots, p\} < 1$, $\max\{\alpha_k : k = 1, 2, \ldots, p\} < 1$ *and*

$$\{\|\mathcal{B}\| M_3 M_2 \|\mathscr{W}^{-1}\| + M_2 \widehat{K}\}(e + a_0 + b_0) < 1.$$

e) $\beta_0 < 1, \alpha_0 < 1$, $\max\{\beta_k : k = 1, 2, \ldots, p\} < 1$, $\max\{\alpha_k : k = 1, 2, \ldots, p\} = 1 \quad \eta < 1$ *and*

$$\{\|\mathcal{B}\| M_3^2 \|\mathscr{W}^{-1}\| + M_3 \widehat{K}\} \sum_{k \in S_\alpha} a_k < 1,$$

where $S_\alpha = \{k : \alpha_k = 1\}$.

f) $\beta_0 < 1, \alpha_0 < 1$, $\max\{\beta_k : k = 1, 2, \ldots, p\} = 1$, $\max\{\alpha_k : k = 1, 2, \ldots, p\} < 1 \quad \eta < 1$ *and*

$$\{\|\mathcal{B}\| M_3^2 \|\mathscr{W}^{-1}\| + M_3 \widehat{K}\} \sum_{k \in S_\beta} b_k < 1,$$

where $S_\beta = \{k : \beta_k = 1\}$.

g) $\beta_0 < 1, \alpha_0 < 1$, $\max\{\beta_k : k = 1, 2, \ldots, p\} = 1$, $\max\{\alpha_k : k = 1, 2, \ldots, p\} = 1 \quad \eta < 1$ *and*

$$\{\|\mathcal{B}\| M_3^2 \|\mathscr{W}^{-1}\| + M_3 \widehat{K}\}\Big(\sum_{k \in S_\alpha} a_k + \sum_{k \in S_\beta} b_k\Big) < 1,$$

where

$$M_2 = M_3 + \frac{M}{\omega}, \quad \widehat{K} = 1 + \frac{M^2}{\omega}\|\mathcal{B}\|^2 \|\mathscr{W}^{-1}\|, \textit{and} \quad M_3 = M e^{\omega \tau}.$$

Proof Let us consider any of the conditions $a) - g)$. Then, from the estimates obtained in Claim 4, we get that

$$\lim_{|||(z,u)||| \to \infty} \frac{|||\Theta(z, u)|||}{|||(z, u)|||} < \rho < 1.$$

Hence, there exists $R > 0$ such that

$$|||\Theta(z, u)||| \leq \rho |||(z, u)|||, \quad |||(z, u)||| = R.$$

Then, analogously to the previous theorem the proof of Theorem 9 immediately follows by applying Proposition 5.

□

5.3 An Example

In this section, we present an example to illustrate our results. In this regard, we will apply Theorem 8 to the semilinear time dependent control system with impulses, delay and nonlocal conditions given by

$$\begin{cases} z'(t) = \mathcal{A}(t)z(t) + \mathcal{B}(t)u(t) + h(t, z_t, u(t)), & t \in (0, \tau], t \neq t_k \\ z(s) + \mathcal{D}(z_{\tau_1}, z_{\tau_2}, \dots, z_{\tau_q})(s) = \varphi(s), & s \in [-t_p, 0], \\ z(t_k^+) = z(t_k^-) + \mathcal{J}_k(z(t_k), u(t_k)), & k = 1, 2, 3, \dots, p, \end{cases} \tag{5.3.8}$$

where $\mathcal{A}(t) = \mathfrak{a}(t)\mathcal{A}$, $\mathcal{B}(t) = \mathfrak{b}(t)\mathcal{B}$ with $\mathcal{A}$ and $\mathcal{B}$ $n \times n$ and $n \times m$ constant matrices, respectively.

$\mathfrak{a} \in L^1[0, \tau]$, $\mathfrak{b} \in C[0, \tau]$ and

$$\int_0^\tau \mathfrak{a}(s)ds \neq 0, \qquad \mathfrak{b}(t) \neq 0, \qquad t \in [0, \tau]$$

From [53] , if the Kalman's rank condition holds true

$$Rank[\mathcal{B}; \mathcal{A}\mathcal{B}; \cdots ; \mathcal{A}^{n-1}\mathcal{B}] = n,$$

then the following time dependent linear system

$$z'(t) = \mathcal{A}(t)z(t) + \mathcal{B}(t)u(t), \qquad t \in [0, \tau]$$

with $\mathcal{A}(t) = \mathfrak{a}(t)\mathcal{A}$, $\mathcal{B}(t) = \mathfrak{b}(t)\mathcal{B}$, is exactly controllable on $[0, \tau]$(see [53]). Here, the nonlinear terms and the impulsive functions are given as follows

$h : [0, \tau] \times \mathfrak{B} \times \mathbb{R}^m \to \mathbb{R}^n$,

$$h(t, \phi, u) = \begin{pmatrix} \sqrt[3]{\|u\| + 1} & \cdot & \sqrt[3]{\phi_1(-t_p)} \\ \sqrt[3]{\|u\| + 1} & \cdot & \sqrt[3]{\phi_2(-t_p)} \\ \vdots & \cdot & \vdots \\ \sqrt[3]{\|u\| + 1} & \cdot & \sqrt[3]{\phi_n(-t_p)} \end{pmatrix},$$

$\mathcal{D} : \mathfrak{B}^q \to \mathfrak{B}$,

given by

$$\mathcal{D}(\varphi_1, \varphi_2, \cdots, \varphi_1) = \sum_{i=1}^{q} \begin{pmatrix} sin(\varphi_{i1}) \\ sin(\varphi_{i2}) \\ \vdots \\ sin(\varphi_{in}) \end{pmatrix},$$

$\mathcal{J}_k : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$, $k = 1, 2, \cdots, p$,

given by

$$\mathcal{J}_k(z, u) = cos(\sqrt{\|u\| + 1}) \begin{pmatrix} sin(z_1^k) \\ sin(z_2^k) \\ \vdots \\ sin(z_n^k) \end{pmatrix},$$

Then

$$\|h(t, \varphi, u)\| \leq 2\sqrt{n}\|\varphi(-t_p)\| + 2\sqrt{n}\|u\| + 2\sqrt{n},$$

and since $\mathcal{D}$ and J_k, $k = 1, 2, \cdots, p$ are bounded, the conditions (5.2.1)-(5.2.4) are satisfied. Hence, the system (5.3.8) is exactly controllable on $[0, \tau]$.

Chapter 6

Conclusions and Final Remarks

In this work we prove the existence and uniqueness of solutions of retarded equations with infinite delay, impulses, and nonlocal conditions by applying Karakostas's fixed point theorem; showing that the phase space that we choose satisfies the axioms proposed by Hale and Kato to study retarded equations with unbounded delay, but in this case, our phase space is a subspace of the piecewise continuous functions due to impulses and non-local conditions.

Additionally, we prove the apprxoimate controllability of a control system governed by a retarded equation with infinite delay, impulses, and non-local conditions assuming that the non-linear term is bounded by a special class of function and the linear control system without impulses, infinite delay, and non-local conditions is exactly controllable on any interval $[\tau - \delta, \tau]$ for all δ with $0 < \delta < \tau$. In this part of the work, the condition imposed on the last impulse time helps us to go back to a previously chosen solution curve, from which we can steer the system to a neighborhood of the final state on time τ, thus proving the approximate controllability of the system.

Finally, we prove that the exact controllability is robust in the presence of such perturbations. In this case, we apply the Rothe's fixed point theorem to obtain the main result and assume that the linear system associated to the control system is exact controllable on $[0,\tau]$ and the nonlinear terms are sublinear, i.e., they satisfy a sublinearity inequality. This happens for many real life control systems where impulses, delays, and nonlocal conditions are intrinsic phenomena of the system. Moreover, in several papers it has been shown that the influence of impulses do not destroy the controllability of some known systems like the heat equation, the wave equation and the strongly damped wave equation([34, 33, 35, 36]). Therefore, the same ideas presented in this work to prove the exact controllability can be used to prove the controllability of infinite dimensional systems in Hilbert spaces where the dynamical is given by the infinitesimal generator A of a compact semigroup $\{T(t)\}_{t\geq 0}$, in this case we only get approximate controllability of the system.

Our future research will focus on studying the same results with non-instantaneous impulses, and infinite-dimensional Banach. Those are existence of bounded solutions, uniqueness, stability, approximate and exact controllability, as well as, other aspects of dynamical systems.

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