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Negotiations, fairness and social welfare

Trabajo de integración curricular presentado como requisito para la obtención del título de Matemático.

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Dedication

"To God, my parents, my professors, and my friends."

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Resumen

Este trabajo estudia el proceso de las negociaciones, justicia y bienestar social en el problema de la asignación de recursos justa. Para ello, se analiza la literatura más relevante en materia de bienestar social. Además, herramientas de matemáticas discretas se utilizan para probar resultados relevantes previamente establecidos. Adicionalmente, los resultados ya conocidos en esta área se utilizaron para proponer y probar nuevos resultados en un contexto más general que la literatura previa. Se presenta además un análisis detallado de las relaciones, implicaciones y consecuencias de los teoremas. Este trabajo refuerza los fundamentos matemáticos sobre los que se construye la teoría del bienestar social y sus aplicaciones. Además, amplía el espectro de estudio de la teoría del bienestar social al proponer nuevos resultados usando criterios más generales que aquellos usados en trabajos previos.

Palabras Clave: Bienestar social, Justicia, Negociaciones.

Abstract

This work studies the process of negotiations, fairness and social welfare in the fair allocation of goods problem. For this, the most relevant literature regarding social welfare is analyzed. Also, mathematical tools from discrete mathematics are used to show relevant results that have already been stated. Additionally, already known results in this area were used to propose and prove new results in a more general context than previous literature. Also, a detailed analysis of relationships, implications and consequences of the theorems is presented. This work reinforces the mathematical foundations over which social welfare theory and its applications are constructed. Also, it widens the spectre of study of theory of social welfare by proposing new results using more general criteria than the ones used in previous works.

Keywords: Fairness, Negotiations, Social welfare.

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Chapter 1

Introduction

Fair allocation of goods is a problem that concerns rational living beings. This action, intrinsically present in the life of people may sometimes be ignored because of its frequent and natural use. From the beginning of history, different societies around the world, have been developing varied ways of facing the fair allocation of goods problem. Hence, it has played a major role in the progress or declining of whole civilizations.

In this chapter, the allocations of goods problem will be presented from a mathematical point of view. This will allow subjecting this process to the rigor of mathematics. The coming story, is a particular case that shows how complicated, expensive and relevant the allocation of goods problem could be.

It is April 1994, and Ecuador's most important business man Luis Noboa Naranjo lays on a bed with a complicated health condition. He is 78 years old, and his health has not being improving lately. Unfortunately, Mr Luis Noboa Naranjo passes away on 28th April 1994 and some of the most difficult and expensive inheritance trials of Ecuador begin. Mr Noboa Naranjo heirs are his 6 sons and daughters.

Among the assets that Mr Noboa heirs claimed as inheritance, there is the bananas exportation business in Ecuador, two Banks in Nassau and Miami, some apartments in Guayaquil and New York, one of the biggest fleet of refrigerated boats in the world, among many other businesses, real state properties and cash money.

This conflict starts a legal fight which is not only relevant for the Noboa family, but also for all Ecuadorians since the enterprises managed by this family bring jobs to around a hundred thousand families in the country and their wealth represents a considerable percentage of the Ecuador Gross Domestic Product (GDP). The lawsuits among Mr Noboa heirs carry on for years since not all of them are satisfied with the inheritance that they receive.

Here, we find a problem with the division of inheritance among people, which is pretty common for most of us, independently from our societies. In regard of this particular case, the question that naturally raises is how can we split the inherited assets among the heirs in such way that all of them are satisfied as much it is possible?

1.1 Background

As it has been shown in the last example, fair allocation of goods is not always an easy task. In these problems, agents usually will act in a rational way and try to obtain the best outcome possible. Hence, many researchers have been dealing with this problem from many years ago from different approaches like mathematics, computer science, economy, social sciences, among others [1, 2, 3, 4]. One of the most common approaches is through mathematical modeling. Here, authors have been working giving a mathematical characterization to actions like allocations of goods to agents, appreciations of agents over goods or tasks, negotiations, and also to principles like rationality, social well-being and fairness.

In the fair allocation problem, the main purpose is to efficiently allocate goods to agents. The accomplishment of this goal is measured using the social welfare value yielded by allocations. Also, it is equally relevant to obtain allocations with fairness properties. Regarding welfarism, researchers have presented different ways of studying social welfare. Some of them are the utilitarianism approach, the egalitarianism approach, and the Nash social welfarism. This last social welfare has been presented by John Nash in the latest fifties in [5].

The division of goods problem has even appeared in the bible in the book of Genesis. This is when Abraham and Lot divide a piece of Land. First, Abraham chooses where the dividing line should go and then lot chooses the area that he finds more convenient. In this case, the good (land) is divisible. This problem, in particular, considers goods that can be infinitely divided and there is a field of study of this problem called the "fair cake-cutting" problem. Many researchers have worked with this particular problem. For instance, for three divisible goods, Selfridge and John H. Conway independently have shown that there is an algorithm that reaches an envy-free allocation in a finite number of steps [6]. A similar result for four agents was proposed by Aziz et al. [7] and for any number of agents by the same author [8].

The other case that is also relevant to consider, is when goods are not divisible. That means goods that can not be split or cut without losing their value are considered. In this area, works [1] have focused on finding ways to negotiate allocations of goods for agents in such a way that the convergence to allocations with interesting properties is guaranteed. Sometimes, the goods that are split among agents are not always non-negatively valued by agents. That is, it is also important to consider goods that are valued negatively. For this, Aziz et al. [9] has also studied several procedures to allocate real-valued indivisible goods that reach some maximality and fairness properties.

Most of the current literature is focused in solving the fair allocation of goods problem consider the utilitarian social welfare and additive valuation functions [1, 3, 10, 11]. Hence, a great number of results are guaranteed under these suppositions.

In this work, negotiations, fairness criteria and social welfare criteria are studied for indivisible goods and considering more general social welfare and valuation functions. Here, some of the most relevant results regarding the existence of negotiations that reach allocations with strong properties are studied. Also, some of the studied results are presented for more general contexts than utilitarian social welfare or additive functions. One of the intentions of this work is to be as self contained as possible, that is why most of the relevant definitions, theorems, lemmas, corollaries, proofs and examples are illustrated in this same work.

1.2 Problem statement

Researches in many fields such as mathematics and computer science have been dealing with the fair allocation of goods problem for many years and they have came up with some ingenuous solutions to this problem. Nonetheless, most of the proposed solutions to this problem are guaranteed under the assumption of utilitarian social welfare functions and additive valuation functions. Hence, it is necessary to also study the fair allocation of goods problem in a more general context.

1.3 Objectives

1.3.1 General objective

Study the allocation of goods problem from a mathematical point of view and propose new results that enrich the social welfare theory literature. This, by considering more general criteria than the currently used ones.

1.3.2 Specific objectives

- Review the most relevant fair allocation of goods literature to understand the current results and limitations of this area.
- Propose theorems, lemmas, and corollaries based on current results found in the fair allocation of goods literature to enrich this knowledge area.
- Prove and illustrate previous results and new findings in the allocation of goods problem to evidence these results.

1.4 Contributions

The main contributions of this work are:

1. Proof of Lemma 2. A proof of the fact that under modular valuation functions such that agents value the empty set as zero, the valuation of a bundle is equal to the sum of valuations of goods in the given bundle has been proposed.
2. Lemma 3. It has been established and proved that under supermodular valuation functions such that agents value the empty set as zero, the valuation of a bundle is as a minimum the sum of valuations of goods in the given bundle.
3. Proof of Lemma 4. A proof of the fact that under submodular valuation functions such that agents value the empty set as zero, the valuation of a bundle is at most the sum of valuations of goods in the given bundle.

4. Proof of Lemma 5. A proof of the fact that under modular valuation functions such that agents value the empty set as zero, the valuation of a bundle of goods with exception of a second bundle of goods contained in the first bundle is equal to the valuation of the first bundle minus the valuation of the second bundle of goods. Additionally, it has been proved that if the valuation functions are also non-negative, then the function is monotonic.

5. Lemma 6. It has been proposed and shown that under supermodular valuation functions such that agents value the empty set as zero, the valuation of a bundle of goods with exception of a second bundle of goods contained in the first bundle is not always equal, less or greater than the valuation of the first bundle minus the valuation of the second bundle of goods. Additionally, it has been proved that if the valuation functions are also non-negative, then the function is monotonic.

6. Lemma 7. It has been established and shown that under submodular valuation functions such that agents value the empty set as zero, the valuation of a bundle of goods with exception of a second bundle of goods contained in the first bundle is not always equal, less or greater than the valuation of the first bundle minus the valuation of the second bundle of goods. Additionally, it has been proved that if the valuation functions are also non-negative, then the function is not always monotonic.

7. Proof of Theorem 1. A proof of the fact that under modular valuation functions such that agents value the empty set as zero, allocating goods to agents that maximize them will result in a maximal utilitarian allocation.

8. Lemma 10. It has been proposed and proved that if valuation functions are positive, then Nash maximal allocations are Pareto optimal allocations.

9. Lemma 11. It has been established and proved that under monotone valuation functions, proportionality implies proportionality up to one good.

10. Lemmas 12, 13 and Corollary 2. It has been stated and proved that under additive valuation functions, envy-free allocations are also envy-free up to the least positively valued good which in turn are also envy-free up to one good. This last implication is also true for monotone valuation functions. As a consequence, these results establish that envy-free allocations are also envy-free up to one good under additive valuation functions.

11. Corollary 3. It has been proposed and shown that if additive valuation functions are considered, then a proportional up to one good allocation does always exist.

12. Proofs of Lemmas 17 and 18. The proofs of Lemmas 17 and 18 have been presented to show that under additive valuation functions, there exists a set of maximal Nash allocations that are Pareto optimal and envy-free up to one good.

13. Corollary 4. It has been proposed and proved that Under additive valuation functions, there exists a set of maximal Nash allocations that are Pareto optimal and proportional up to one good.

14. Theorem 3. It has been shown that given an initial non-maximal allocation, it is possible to find a sequence of socially rational deals converging to a maximal allocation.

1.5 Document Distribution

This thesis is organized as follows: In Chapter 2, the preliminary concepts of the fair allocation of goods problem required to understand this work are introduced. In Chapter 3, the efficiency and fairness criteria are studied as desired properties of allocations. In Chapter 4, the deals and the negotiations are introduced as a way to define a process in which agents exchange their goods according to a rationality criterion. Also, the Knaster procedure is studied. Finally, in Chapter 5 the conclusions of this work and on going research are presented.

Chapter 2

Preliminaries

The allocation of goods problem may present different subjective appreciations among the people studying it. But, it is important to notice that there are some ingredients that are common in this problem independently of the context or people.

In this chapter, the main ingredients that are always present in the allocation of resources problem will be introduced using mathematical tools. Also, a numerical function to study the preferences of agents involved in the allocation of resources problem will be studied. This function, will later permit the study of the satisfaction of a society in regard of a particular way of allocating goods. Hence, being useful to evaluate and compare the success of the procedures.

2.1 Main ingredients

The allocation of goods problem considers three main ingredients: a set of agents, a set of goods, and the preferences that each agent establishes over the goods. The goods could be either divisible or indivisible, which means that they can or cannot be split respectively. Also, the allocations will be studied as functions that distribute the goods among the agents. In this subsection, the main ingredients and the relationship among them are studied.

2.1.1 Agents

The first ingredient is the set of *agents*. The nature of the agents present in a resource allocation problem is not known. The finite set of agents is denoted by

$$\mathcal{N} = \{a_1, a_2, ..., a_n\}.$$

To ease the notation, when only the information about the position and number of the agents is required, the following notation will be used,

$$\mathcal{N} = [n] = \{1, 2, ..., n\}.$$

For simplicity, this last notation will be preferred unless specified other. The size of the set $\mathcal{N}$ is n, with $n \in \mathbb{N}$. In what follows it is assumed that the set of agents $\mathcal{N}$ has at least two elements, i.e., $|\mathcal{N}| = n \geq 2$.

2.1.2 Goods

The second ingredient which will be constant along the problem is the set of *goods*. This finite set is denoted by

$$\mathcal{M} = \{g_1, g_2, ..., g_m\}.$$

When only the information about the position of a good is required, the set of goods will be denoted as

$$\mathcal{M} = [m] = \{1, 2, ..., m\}.$$

The size of the set $\mathcal{M}$ is m, with $m \in \mathbb{N}$. The set of goods $\mathcal{M}$ must contain at least one good, $m \geq 1$.

Goods could be of two types; *divisible goods* or *indivisible goods*. A good is said to be *divisible* if its value does not decrease when it is split. For instance, money, shares of a company, or a cake are divisible goods. While a good is said to be *indivisible* if it loses its value when it is divided. For example, a house, a pet, or an artwork are indivisible goods. The goods in the set $\mathcal{M}$ can also be grouped in bundles; hence, a *bundle* is a subset of $\mathcal{M}$. Notice also that a bundle may contain a single element or the empty set. The set of all possible bundles is denoted by $2^{\mathcal{M}}$.

In this work, only indivisible goods are considered. Despite this, some of the results or ideas of the divisible goods literature are also applied when the goods are indivisible. Some of these results are studied in Chapter 3, Section 3.2.

2.1.3 Preferences of agents over goods

The last main ingredient is the *preference* that each agent establish over some bundle of $\mathcal{M}$. In general, each preference is given by a total preorder over $2^{\mathcal{M}}$. A total preorder is a total and transitive relation.

Definition 1. *Let i be an agent in $\mathcal{N}$. A relation $\sqsupseteq_i$ over $2^{\mathcal{M}}$ is a preference relation of agent i if $\sqsupseteq_i$ is a total preorder over $2^{\mathcal{M}}$.*

If $\sqsupseteq_i$ is a preference relation of agent i, then given $S_1, S_2 \in 2^{\mathcal{M}}$, $S_1 \sqsupseteq_i S_2$, it is interpreted as: the bundle S_1 is *more or equally preferred* than the bundle S_2. The following example shows the establishment of a preference relation of an agent with respect to two goods.

Example 1. *Let $\mathcal{N} = \{1\}$ be a set of a single agent and $\mathcal{M} = \{g_1, g_2\}$ a set of two goods. Now, if agent 1 prefers good g_2 over or equally than good g_1 then,*

$$\{g_2\} \sqsupseteq_1 \{g_1\}.$$

One way to establish the preference of each agent is by a numerical function. This type of function is called valuation function and it is studied with more detail in the section 2.2.

2.1.4 Allocations

In this subsection, the task of distributing resources among agents is studied. In fact, it is easy to make a distribution of goods among agents. However, the relevance of the problem lies in that the distribution must be as successful as possible, in the sense that it must contain properties such as efficiency and fairness. These properties will be studied later in Chapter 3. There are two aspects that must be taken into account when allocating goods to agents: all goods must be distributed and each good must be allocated to a unique agent. This task is modeled by the *allocation function.* The definition of an allocation is given below.

Definition 2. *An allocation A is a function*

$$\begin{aligned} A : \mathcal{N} &\longrightarrow 2^{\mathcal{M}} \\ i &\longmapsto A(i) \end{aligned}$$

such that $A(i) \cap A(j) = \emptyset$*, and* $\bigcup_{i \in \mathcal{N}} A(i) = \mathcal{M}$.

Here, $A(i)$ denotes the bundle of goods that the function A allocates to the agent i. Notice that Definition 2 implies that the bundle that an agent i receives is different from the bundle that any other agent j gets, for different $i, j \in \mathcal{N}$. Definition 2 also implies that the union of all allocations is the set of goods $\mathcal{M}$, which means that all goods must be allocated by A.

In what follows, $\mathcal{N}^{\mathcal{M}}$ will denote the set of all possible allocations given a set of agents $\mathcal{N}$ and a set of goods $\mathcal{M}$. The size of $\mathcal{N}^{\mathcal{M}}$ is n^m.

Remark 1. *When defining an allocation as in Definition 2, the function A induces a partition of the set $\mathcal{M}$ of size n. Furthermore, a partition of size n of the set $\mathcal{M}$ induces an allocation of goods from the agents in $\mathcal{N}$ to the set of parts of $\mathcal{M}$. In fact, recall that a partition $\Pi_n = \{P_1, P_2, ..., P_n\}$ of size n of the set $\mathcal{M}$ is a grouping of the elements in $\mathcal{M}$ such that $P_i \cap P_j = \emptyset$ for any two different $i, j \leq n$ and also $\cup_{i \in \mathcal{N}} P_i = \mathcal{M}$. On the one hand, the two conditions of an allocation of goods for n agents, given by its definition, satisfy the two conditions of a partition of size n. On the other hand, by the definition of a n size partition, the two conditions of a partition also satisfy the two conditions of an allocation of goods for n agents. Hence, the definition of an allocation for n agents induces a partition of size n of $\mathcal{M}$. Also, a partition of size n of the set $\mathcal{M}$ induces an allocation for n agents.*

The following example shows all possible allocations that can be achieved given two agents and two goods.

Example 2. *Let $\mathcal{N} = \{1, 2\}$ be a set of two agents and let $\mathcal{M} = \{g_1, g_2\}$ be a set of two goods. For this allocation problem, there are $n^m = 4$ possible allocations. These allocations are described in Table 2.1.*

Note that each allocation is a partition of $\mathcal{M}$ of size $n = 2$.

	A_0	A_1	A_2	A_3
1	$\{g_1, g_2\}$	$\{g_1\}$	$\{g_2\}$	$\emptyset$
2	$\emptyset$	$\{g_2\}$	$\{g_1\}$	$\{g_1, g_2\}$

Table 2.1: All possible allocations for agents in $\mathcal{N}$.

Allocations are functions that assign goods to each agent participating in this problem. Some agents may not receive a good, in that case, it is said that the allocation of this agent is the empty set. In the following section, a function through which agents establish their preferences over the goods will be studied.

2.2 Valuation functions

It is common for each agent to establish their preference through numerical functions. These type of functions are called *valuation functions*. This section develops the basic notions of valuation functions and some properties that will be useful later on. As it has been said before, valuation functions express numerically the preferences or personal appreciations of the agents over some bundles of goods. A valuation function is defined from parts of $\mathcal{M}$ to the reals. Formally,

Definition 3. *Let $\mathcal{N}$ and $\mathcal{M}$ be sets of goods and agents respectively and let i be an agent in $\mathcal{N}$. A valuation function of agent i over $\mathcal{M}$, denoted by v_i, is any function v_i from $2^{\mathcal{M}}$ to $\mathbb{R}$,*

$$\begin{aligned} v_i : 2^{\mathcal{M}} &\longrightarrow \mathbb{R} \\ S &\longmapsto v_i(S) \end{aligned}$$

In order to ease notation, $v_i(g_k)$ will preferred instead of $v_i(\{g_k\})$ to express the valuation of agent i over the good g_k. In the following lemma, it is being shown that a valuation function induces a preference relation over the set $\mathcal{M}$.

Lemma 1. *Let i be in $\mathcal{N}$ and let $S_1, S_2 \in 2^{\mathcal{M}}$ be any two different bundles. Then, if v_i is a valuation function, v_i induces a preference relation of the agent i over S_2 and S_1.*

Proof. Suppose that $i \in \mathcal{N}$ and v_i is a valuation function for agent i. Let $\sqsupseteq_i$ be a relation over $2^{\mathcal{M}}$ given by: $\forall\, S_1, S_2 \in 2^{\mathcal{M}}$,

$$S_2 \sqsupseteq_i S_1 \Leftrightarrow v_i(S_2) \geq v_i(S_1). \tag{2.1}$$

As $\geq$ is the usual order over $\mathbb{R}$, then $\geq$ is transitive and a total preorder over $\mathbb{R}$. Thus, by (2.1), $\sqsupseteq_i$ is a total preorder over $2^{\mathcal{M}}$. □

The coming example shows how a numerical function induces a preference relation over some bundles of goods.

Example 3. *Let* $\mathcal{N} = \{1\}$ *be a set of a single agent and* $\mathcal{M} = \{g_1, g_2\}$ *a set of two goods. Then, agent* 1 *expresses its valuation over the goods in the set* $\mathcal{M}$ *by* $v_1(g_1) = 3$ *and* $v_1(g_2) = 5$*. This implies that* $v_1(g_2) \geq v_1(g_1)$ *and hence* $\{g_2\} \sqsupseteq_1 \{g_1\}$*.*

There are particular characterizations of the valuation function which will be useful to identify certain ways in which agents value goods. Nonetheless, when defining valuation functions in these ways, it is not necessary to highlight its relation with the agents. Hence, in these cases the notation $v(S)$ will be used instead of $v_i(S)$.

Definition 4. *Let* S_1 *and* S_2 *be bundles of* $\mathcal{M}$ *and let* v *be a valuation function. Then,* v *is called*

- *Non-negative if* $v(S_1) \geq 0$*.*
- *Dichotomous if* $v(S_1) = 0$ *or* $v(S_1) = 1$*.*
- *Monotonic if* $S_1 \subseteq S_2$*, implies that* $v(S_1) \leq v(S_2)$*.*
- *Modular if* $v(S_1 \cup S_2) = v(S_1) + v(S_2) - v(S_1 \cap S_2)$*.*
- *Supermodular if* $v(S_1 \cup S_2) \geq v(S_1) + v(S_2) - v(S_1 \cap S_2)$*.*
- *Submodular if* $v(S_1 \cup S_2) \leq v(S_1) + v(S_2) - v(S_1 \cap S_2)$*.*
- *0-1 if* v *is modular,* $v(\emptyset) = 0$ *and* $v(g) = 0$ *or* $v(g) = 1$ *for* $g \in \mathcal{M}$*.*

Notice that if v is a modular valuation function, then v is also a supermodular and submodular valuation function. There are other valuations functions which will not be considered in the present work. The following example shows some valuation functions that are non-negative, dichotomous, monotonic, modular, supermodular or submodular.

Example 4. *Let* $\mathcal{N} = \{1, 2, 3\}$ *be a set of three agents and* $\mathcal{M} = \{g_1, g_2\}$ *be a set of two goods. Then, some valuations of the agents over the bundles are given in Table 2.2.*

	$\{g_1\}$	$\{g_2\}$	$\{g_1, g_2\}$
v_1	5	2	4
v_2	0	1	1
v_3	2	3	7

Table 2.2: Valuations of the agents over all possible bundles for Example 4.

Here, notice that v_1 is non-negative since there are not negative valuations. v_1 is also submodular since the valuation of the union of goods g_1 and g_2 is smaller that the sum of the individual valuations.

Valuation v_2 is dichotomous since its valuations for any bundle of $\mathcal{M}$ is either 1 or 0. v_2 is also monotonic since the valuations of either g_1 and g_2 are smaller or equal that the valuation of the set $\{g_1, g_2\}$, which contains the previously mentioned goods. v_2 is also modular since the valuation of the union of the goods g_1 and g_2 is the same as the sum of their individual valuations. v_2 is supermodular and submodular since it is modular. v_2 is also non-negative. Finally, v_2 is also a 0-1 valuation function since it valuaes goods as 0 or 1 and it is modular.

In this example, notice also that v_3 is supermodular since the valuation of the union of goods g_1 and g_2 is greater or equal to the sum of the individual valuations of these goods. v_2 is also non-negative and monotonic.

All of the previously mentioned properties of the valuation functions are summarized in Table 2.3.

	Non-negative	Dichot.	Monotonic	Modular	Supermod.	Submod.	0-1
v_1	Yes					Yes	
v_2	Yes	Yes	Yes	Yes	Yes	Yes	Yes
v_3	Yes		Yes		Yes		

Table 2.3: Characterizations of valuations in Table 2.2.

Some of the valuation functions in Definition 4 are known to possess some interesting properties given more constrains. For instance, considering modular valuation functions such that agents value the empty set as zero, it is true that the valuation of a bundle is the sum of the respective valuations of each good in that bundle. Formally,

Lemma 2. *If v is modular and $v(\emptyset) = 0$, then for all $S \in 2^{\mathcal{M}}$ with $S \neq \emptyset$,*

$$v(S) = \sum_{s \in S} v(s). \tag{2.2}$$

Proof. Let $S = \{s_1, \cdots, s_k\}$ be a non-empty bundle. Mathematical induction over the number of elements in S is used for the proof. If $|S| = 1$, then $v(S) = \sum_{s \in S} v(s) = v(s_1)$. Let $k > 1$ be a non-negative integer. The induction hypothesis is: if $|S| < k$, then

$v(S) = \sum_{s \in S} v(s)$. Suppose that $|S| = k$ and let g^* be in S, then

$$\begin{aligned} v(S) = v(S \setminus \{g^*\} \cup \{g^*\}) &= v(S \setminus \{g^*\}) + v(\{g^*\}) - v(S \setminus \{g^*\} \cap \{g^*\}) \\ &= \sum_{s \in S \setminus \{g^*\}} v(\{s\}) + v(\{g^*\}) - v(\emptyset) \\ &= \sum_{s \in S \setminus \{g^*\}} v(\{s\}) + v(\{g^*\}) \\ &= \sum_{s \in S} v(\{s\}). \end{aligned}$$

Thus, for all $S \in 2^{\mathcal{M}}$ with $S \neq \emptyset$, the equation (2.2) is true. □

In the same context than Lemma 2, if supermodular valuation functions are considered such that the empty set is always valued as zero by all agents, then the valuation of a bundle is as a minimum the sum of the valuations of each good in that bundle. Formally,

Lemma 3. *If v is supermodular and $v(\emptyset) = 0$, then for all $S \in 2^{\mathcal{M}}$ with $S \neq \emptyset$,*

$$v(S) \geq \sum_{s \in S} v(s). \tag{2.3}$$

Proof. Similar to the proof when valuation functions are assumed to be modular, let $S = \{s_1, \cdots, s_k\}$ be a non-empty bundle. Mathematical induction over the number of elements in S is used for the proof. If $|S| = 1$, then $v(S) \geq \sum_{s \in S} v(\{s\}) = v(s_1)$ follows immediately. Let $k > 1$ be a non-negative integer. The induction hypothesis is: if $|S| < k$, then $v(S) \geq \sum_{s \in S} v(s)$. Now, suppose that $|S| = k$ and let g^* be in S, then

$$\begin{aligned} v(S) = v(S \setminus \{g^*\} \cup \{g^*\}) &\geq v(S \setminus \{g^*\}) + v(\{g^*\}) - v(S \setminus \{g^*\} \cap \{g^*\}) \\ &\geq \sum_{s \in S \setminus \{g^*\}} v(\{s\}) + v(\{g^*\}) - v(\emptyset) \\ &= \sum_{s \in S \setminus \{g^*\}} v(\{s\}) + v(\{g^*\}) \\ &= \sum_{s \in S} v(\{s\}). \end{aligned}$$

Thus, for all $S \in 2^{\mathcal{M}}$ with $S \neq \emptyset$, the equation (2.3) is true. □

Similarly to Lemmas 2 and 3, if submodular valuation functions are considered such that agents value the empty set as zero, then the valuation of a bundle is at most the sum of the valuations of each good in that bundle. Formally,

Lemma 4. *If v is submodular and $v(\emptyset) = 0$, then for all $S \in 2^{\mathcal{M}}$ with $S \neq \emptyset$,*

$$v(S) \leq \sum_{s \in S} v(s). \tag{2.4}$$

Proof. Similar to the proof when valuation functions are assumed to be modular, let $S = \{s_1, \cdots, s_k\}$ be a non-empty bundle. Mathematical induction over the number of elements in S is used for the proof. If $|S| = 1$, then $v(S) \leq \sum_{s \in S} v(\{s\}) = v(s_1)$ follows immediately. Let $k > 1$ be a non-negative integer. The induction hypothesis for this problem is: if $|S| < k$, then $v(S) \leq \sum_{s \in S} v(s)$. Now, suppose that $|S| = k$ and let g^* be in S, then

$$\begin{aligned} v(S) = v(S \setminus \{g^*\} \cup \{g^*\}) &\leq v(S \setminus \{g^*\}) + v(\{g^*\}) - v(S \setminus \{g^*\} \cap \{g^*\}) \\ &\leq \sum_{s \in S \setminus \{g^*\}} v(\{s\}) + v(\{g^*\}) - v(\emptyset) \\ &= \sum_{s \in S \setminus \{g^*\}} v(\{s\}) + v(\{g^*\}) \\ &= \sum_{s \in S} v(\{s\}). \end{aligned}$$

Thus, for all $S \in 2^{\mathcal{M}}$ with $S \neq \emptyset$, the equation (2.4) is true. □

So far, properties of valuation functions regarding valuations of bundles and the sum of goods in them have been studied. Now, additional facts will also be shown in relation to the disunion of sets and the monotony of valuation functions. For instance, the following lemma shows that under modular valuation functions such that agents value as zero the empty set, the valuation of a bundle of goods S_2 except another bundle of goods $S_1 \subseteq S_2$ is equal to the valuation of the first bundle S_2 minus the valuation of the second bundle S_1. Additionally, if non-negative valuation functions are also considered, then v is a monotonic valuation function. Formally,

Lemma 5. *Let v be a modular valuation function with $v(\emptyset) = 0$ and let S_1 and S_2 be bundles.*

1. *If $S_1, S_2 \in 2^{\mathcal{M}}$ with $S_1 \subseteq S_2$, then $v(S_2 \setminus S_1) = v(S_2) - v(S_1)$.*

2. *If v is non-negative, then v is monotonic.*

Proof. 1. Let $S_2 = \{s_1, ..., s_k\}$ be a non-empty set, and $S_1 \subseteq S_2$. Notice that if $S_1 = \emptyset$, then the result is trivial since

$$v(S_2) = v(S_2 \setminus \emptyset) + v(\emptyset) = v(S_2 \setminus S_1) + v(S_1).$$

Hence, suppose that S_1 is non-empty. Mathematical induction over the number of elements in S_2 is used for the proof. If $|S_2| = 1$, then $v(S_2) = v(S_1) = v(S_2 \setminus S_1) + v(S_1)$ follows since $S_1 = S_2$. Let $K > 1$ be a non-negative integer. The induction hypothesis is: if $|S_2| < k$, then $v(S_2) = v(S_2 \setminus S_1) + v(S_1)$. Now, suppose $|S_2| = k$ and let g^* be a new element in S_2. Then,

$$\begin{aligned} v(S_2) = v\left(S_2 \setminus \{g^*\} \cup \{g^*\}\right) &= v(S_2 \setminus \{g^*\}) + v(\{g^*\}) - v(S_2 \setminus \{g^*\} \cap \{g^*\}) \\ &= v\Big((S_2 \setminus \{g^*\}) \setminus S_1\Big) + v(S_1) + v(\{g^*\}) \\ &= \sum_{s \in S_2 \setminus \left(\{g^*\} \cup S_1\right)} v(\{s\}) + v\left(\{g^*\}\right) + v(S_1) \\ &= \sum_{s \in S_2 \setminus S_1} v(\{s\}) + v(S_1) \\ &= v\left(S_2 \setminus S_1\right) + v(S_1). \end{aligned}$$

Notice that since g^* is a new element in S_2 when $|S_2| = k$, then $g^* \notin S_1$ from the induction hypothesis. Hence, $\{g^*\} \cup S_1 \neq S_1$. Thus, if $S_1, S_2 \in 2^{\mathcal{M}}$ with $S_1 \subseteq S_2$, then $v(S_2 \setminus S_1) = v(S_2) - v(S_1)$.

2. Let $S_2 = \{s_1, ..., s_k\}$ be a non empty set, and $S_1 \subseteq S_2$. Let also v be a non-negative valuation function. Notice that if $S_1 = \emptyset$, then the result is trivial since $v(S_2) \geq v(\emptyset) = v(S_1)$ always holds. Hence, suppose that S_1 is non-empty. Mathematical induction over the number of elements in S_2 is used for the proof. If $|S_2| = 1$, then $v(S_2) = v(\{s_1\}) \geq v(\{s_1\}) = v(S_1)$ follows since $S_1 = S_2$. Let $K > 1$ be a non-negative integer. The induction hypothesis is: if $|S_2| < k$, then $v(S_2) \geq v(S_1)$. Now, suppose $|S_2| = k$ and let g^* be a new element in S_2. Then,

$$\begin{aligned} v(S_2) = v(S_2 \setminus \{g^*\} \cup \{g^*\}) &= v(S_2 \setminus \{g^*\}) + v(\{g^*\}) - v(S_2 \setminus \{g^*\} \cap \{g^*\}) \\ &\geq v(S_1) + v(\{g^*\}) \\ &\geq v(S_1). \end{aligned}$$

This implies that v is a monotonic valuation function. Notice that since g^* is a new element in the set S_2, then, $\{g^*\} \notin S_1$. Thus, if $S_1, S_2 \in 2^{\mathcal{M}}$ with $S_1 \subseteq S_2$, and valuation functions v are non-negative, then v is monotonic, that is, $v(S_2) \geq v(S_1)$. □

Similarly to Lemma 5, the following lemma shows that under supermodular valuation functions such that agents value as zero the empty set, the valuation of a bundle of goods S_2 except another bundle of goods $S_1 \subseteq S_2$ is not equal to the valuation of the first bundle S_2 minus the valuation of the second bundle S_1. Additionally, if non-negative valuation functions are also considered, then v is a monotonic valuation function. Formally,

Lemma 6. *Let v be a supermodular valuation function with $v(\emptyset) = 0$ and let S_1 and S_2 be bundles.*

1. *If $S_1, S_2 \in 2^{\mathcal{M}}$ with $S_1 \subseteq S_2$, then $v(S_2 \setminus S_1) = v(S_2) - v(S_1)$ does not hold.*

2. If v is non-negative, then v is monotonic.

The first statement of Lemma 6 will be evidenced by Example 5, while the second statement will be demonstrated as follows.

Proof. 2. Let $S_2 = \{s_1, ..., s_k\}$ be a non empty set, and $S_1 \subseteq S_2$. Let also v be a non-negative valuation function. Notice that if $S_1 = \emptyset$, then the result is trivial since $v(S_2) \geq v(\emptyset) = v(S_1)$ always holds. Hence, suppose that S_1 is non-empty. Mathematical induction over the number of elements in S_2 is used for the proof. If $|S_2| = 1$, then $v(S_2) = v(\{s_1\}) \geq v(\{s_1\}) = v(S_1)$ follows since $S_1 = S_2$. Let $K > 1$ be a non-negative integer. The induction hypothesis is: if $|S_2| < k$, then $v(S_2) \geq v(S_1)$. Now, suppose $|S_2| = k$ and let g^* be a new element in S_2. Then,

$$\begin{aligned} v(S_2) = v(S_2 \setminus \{g^*\} \cup \{g^*\}) &\geq v(S_2 \setminus \{g^*\}) + v(\{g^*\}) - v(S_2 \setminus \{g^*\} \cap \{g^*\}) \\ &\geq v(S_1) + v(\{g^*\}) \\ &\geq v(S_1). \end{aligned}$$

This implies that v is a monotonic valuation function. Notice that since g^* is a new element in the set S_2, then, $\{g^*\} \notin S_1$. Thus, if $S_1, S_2 \in 2^{\mathcal{M}}$ with $S_1 \subseteq S_2$, and valuation functions v are non-negative. Then v is monotonic, that is, $v(S_2) \geq v(S_1)$. □

The coming example shows that the first statement of Lemma 6 is true. That is, under supermodular valuation functions such that agents give a zero valuation to the empty set, the valuation of a bundle of goods S_2 except another bundle of goods $S_1 \subseteq S_2$ is not equal to the valuation of the first bundle S_2 minus the valuation of the second bundle.

Example 5. *Let $\mathcal{N} = \{1\}$ be a set of a single agent and let $\mathcal{M} = \{g_1, g_2, g_3\}$ be a set of three goods. Let also v be a supermodular valuation function of the agent over elements in $2^{\mathcal{M}}$ be given as in Table 2.4. Consider also $v_1(\emptyset) = 0$.*

	$\{g_1\}$	$\{g_2\}$	$\{g_3\}$	$\{g_1, g_2\}$	$\{g_1, g_3\}$	$\{g_2, g_3\}$	$\{g_1, g_2, g_3\}$
v_1	1	2	3	4	5	6	6

Table 2.4: Supermodular valuation of an agent over elements in $2^{\mathcal{M}}$.

First, let $S_2 = \{g_1, g_2, g_3\}$ and $S_1 = \{g_1\}$. Then,

$$v_1(S_2 \setminus S_1) = 6 \geq 5 = v_1(S_2) - v_1(S_1).$$

Now, let $S_2 = \{g_2, g_3\}$ and $S_1 = \{g_2\}$. Then,

$$v_1(S_2 \setminus S_1) = 3 \leq 4 = v_1(S_2) - v_1(S_1).$$

These two calculations show that for a supermodular valuation function where agents value the empty set as zero, it is not always possible that $v_1(S_2) = v_1(S_2) - v_1(S_1)$ whenever $S_1 \subseteq S_2$. Notice that in fact, neither $\leq$ nor $\geq$ hold in this context.

Analogously to Lemmas 5 and 6, the following lemma shows that under submodular valuation functions such that agents value as zero the empty set, the valuation of a bundle of goods S_2 except another bundle of goods $S_1 \subseteq S_2$ is not equal to the valuation of the first bundle S_2 minus the valuation of the second bundle S_1. Additionally, if non-negative valuation functions are also considered, then v is not a monotonic valuation function. Formally,

Lemma 7. *Let v be a submodular valuation function with $v(\emptyset) = 0$ and let S_1 and S_2 be bundles.*

1. *If $S_1, S_2 \in 2^{\mathcal{M}}$ with $S_1 \subseteq S_2$, then $v(S_2 \setminus S_1) = v(S_2) - v(S_1)$ does not hold.*
2. *If v is non-negative, then v is not always monotonic.*

The coming examples will evidence the truthfulness of Lemma 7. For the first statement, the following example shows that for submodular valuations functions, it is not true that $v(S_2 \setminus S_1) = v(S_2) - v(S_1)$.

Example 6. *Let $\mathcal{N} = \{1\}$ be a set of a single good and let $\mathcal{M} = \{g_1, g_2, g_3\}$ be a set of three goods. Let also v be a supermodular valuation function of this agent over elements of $2^{\mathcal{M}}$ be given as in Table 2.5. Consider also $v(\emptyset) = 0$.*

	$\{g_1\}$	$\{g_2\}$	$\{g_3\}$	$\{g_1, g_2\}$	$\{g_1, g_3\}$	$\{g_2, g_3\}$	$\{g_1, g_2, g_3\}$
v_1	1	2	3	2	3	4	6

Table 2.5: Submodular valuation of an agent over elements in $2^{\mathcal{M}}$ for Example 6.

First, let $S_2 = \{g_1, g_2, g_3\}$ and $S_1 = \{g_1\}$. Then,

$$v_1(S_2 \setminus S_1) = 4 \leq 5 = v_1(S_2) - v_1(S_1).$$

Now, let $S_2 = \{g_2, g_3\}$ and $S_1 = \{g_2\}$. Then,

$$v_1(S_2 \setminus S_1) = 3 \geq 2 = v_1(S_2) - v_1(S_1).$$

These two calculations show that for a submodular valuation function with $v_1(\emptyset) = 0$ it is not always possible that $v_1(S_2) = v_1(S_2) - v_1(S_1)$ whenever $S_1 \subseteq S_2$. Notice that in fact, neither $\leq$ nor $\geq$ hold in this context.

Finally, the following example shows that the second statement of Lemma 7 is true. That is, when $v(\emptyset) = 0$ and valuation functions are submodular and non-negative, then v is not monotonic.

Example 7. *Let $\mathcal{N} = \{1\}$ be a set of a single agent and let $\mathcal{M} = \{g_1, g_2\}$ be a set of three goods. Let also v be a submodular valuation function of the agent over subsets of the goods such that $v_1(\emptyset) = 0$ be given as in Table 2.6.*

	$\{g_1\}$	$\{g_2\}$	$\{g_1, g_2\}$
v_1	1	2	1

Table 2.6: Submodular valuation of an agent over elements in $2^{\mathcal{M}}$ for Example 7.

Let $S_2 = \{g_1, g_2\}$ and $S_1 = \{g_2\}$. Then,

$$v_1(S_1) = 2 \geq 1 = v_1(S_2).$$

This calculation shows that for a submodular valuation function with $v_1(\emptyset) = 0$ it is not always possible that $v_1(S_1) \leq v_1(S_2)$. That is, v_1 is not always monotonic in this context.

Up to now, the basic characterizations of the preferences of agents expressed over goods have been introduced. Some of these functions such as modular, supermodular, and submodular valuation functions have shown to possess interesting properties regarding valuations over bundles of goods and their elements. Further, some of these functions have shown that under additional properties such as the non-negativeness, they are also monotonic.

Now, in the next subsection, new valuation functions will be presented. These valuation functions are the *additive*, *superadditive* and *subadditive valuation functions* and they will be constructed from simpler functions studied so far. Some common criteria necessary to the construction of these functions are the assumption of non-negativeness and zero valuation of agents over the empty set.

2.2.1 Additive, superadditive, and subadditive valuations

Now that the basic characterizations of valuation functions have been introduced, it is timely to present functions with a more interesting structure. These new valuation functions, gather more than one characteristic contained in simpler functions. This is useful since many results are obtained using these newly created characterizations. In this sense, the following definition presents some of these mentioned valuation functions.

Definition 5. *Let v be a non-negative valuation function such that $v(\emptyset) = 0$. Then, it is said that v is*

- *Additive valuation if v is modular.*
- *Superadditive valuation if v is supermodular.*
- *Subadditive valuation if v is submodular.*

An additive valuation function is always modular; however, a modular valuation function is not always additive. The same is true for supermodular and submodular valuation functions. In the following example, this fact will be elucidated.

Example 8. *Let $\mathcal{N} = \{1, 2, 3, 4\}$ be a set of four agents and let $\mathcal{M} = \{g_1, g_2\}$ be a set of two goods. Let also $v_1, ..., v_4$ be valuation functions of agents over elements in $2^{\mathcal{M}}$ defined as in Table 2.7.*

	$\emptyset$	$\{g_1\}$	$\{g_2\}$	$\{g_1, g_2\}$
v_1	0	2	3	5
v_2	0	2	3	7
v_3	0	2	3	4
v_4	1	2	3	5

Table 2.7: Valuations of four agents over elements in $2^{\mathcal{M}}$.

Here, notice that v_1 is an additive valuation function since it is non-negative, modular and $v_1(\emptyset) = 0$. Also, v_2 is a superadditive valuation function since it is non-negative, supermodular and $v_2(\emptyset) = 0$. Similarly, v_3 is a subadditive valuation function since it is non-negative, submodular and $v_3(\emptyset) = 0$. Finally, v_4 is a modular valuation function. Note also that v_4 is neither an additive nor a superadditive nor a subadditive valuation function. This, since $v_4(\emptyset) \neq 0$.

Example 8 evidences the fact that additive, superadditive and subadditive valuation functions are also modular, supermodular and submodular respectively. Nonetheless, the inverse is not always true since modular, supermodular or submodular valuation functions are not all the time non-negative or $v(\emptyset) = 0$.

Now, as one of the first uses of functions in Definition 5, notice that by Lemma 2, if the valuation function of an agent over the singleton sets in $2^{\mathcal{M}}$ is known, then, the valuation of any non-empty bundle can also be known. This is true particularly when the valuation function is additive. Formally,

Corollary 1. *If v is an additive valuation function, then for all $S \in 2^{\mathcal{M}}$,*

$$v(S) = \sum_{s \in S} v(s) \tag{2.5}$$

Notice that this corollary restricts the functions in Lemma 2 to non-negative valuation functions. The coming example illustrates the convenience of knowing the valuations of the singleton sets in $2^{\mathcal{M}}$ to compute the valuations of other sets in $2^{\mathcal{M}}$ for additive valuation functions.

Example 9. *Let $\mathcal{N} = \{1\}$ be a set of a single agent and let $\mathcal{M} = \{g_1, g_2\}$ be a set of two goods. Also, let v be an additive valuation function defined as in Table 2.8*

	$\{g_1\}$	$\{g_2\}$
v_1	2	3

Table 2.8: Valuation of agent 1 over goods in $\mathcal{M}$.

Now, it is possible to know the valuation of the agents to bundles $S \in 2^{\mathcal{M}}$ as follows.

$$\begin{aligned} v_1(\{g_1\}) &= 2 \\ v_1(\{g_2\}) &= 3 \\ v_1(\{g_1, g_2\}) &= v_1(\{g_1\}) + v_1(\{g_2\}) = 5. \end{aligned}$$

Then, the valuation of the bundle $S = \{g_1, g_2\}$ was obtained using the valuations of the singleton sets in $2^{\mathcal{M}}$. However, when valuation functions are superadditive or subadditive, it is not enough to know the valuations that the agent gives to each element. For these cases, further information is required.

At this moment, a formal definition of a valuation function has been presented. Also, it has been demonstrated that these valuation functions induce a preference relation of agents over goods. Moreover, different characterizations of valuation functions have been proposed as they are useful in different contexts.

Notice that the fact that valuation functions induce a preference relation of agents in $\mathcal{N}$ over the bundles present in $2^{\mathcal{M}}$ gives us some extra information. Indeed, since there are usually bundles which are preferred over others, the allocation function must try to assign to each agent the bundle that they value the most. This task produces various different allocations. Hence, it is necessary to identify which of all possible allocations assigns the bundles more successfully than others. This is studied by the social welfare criterion. This criterion measures the success of a particular allocation by creating a preference relation over the allocations.

2.3 Social welfare

A common approach to measuring the success of an allocation is by using the *social welfare* principle (sw). Endriss et al. [1], affirms that the social welfare is *"a formal tool to assess how the distribution of resources amongst the members of a society affects the well-being of a society as a whole"*.

The social welfare, in general, is a binary relation over the set of all allocations $\mathcal{N}^{\mathcal{M}}$ that classifies allocations. Particularly, the set $\mathcal{N}^{\mathcal{M}}$ is endowed with a total preorder $\sqsupseteq$ and is denoted by $(\mathcal{N}^{\mathcal{M}}, \sqsupseteq)$. This total preorder is a preference relation that permits a classification of allocations according to its given preference.

This work considers the social welfare defined through numerical functions, called social welfare functions. Before giving its formal definition and some examples, the following section describes the relationship between valuation functions and allocations.

2.3.1 Valuations of allocations

Valuation functions have already been introduced as numerical functions that express the preferences of agents over bundles. Additionally, the allocation function has also been presented as a function that assigns goods to agents. Hence, it is natural to think that valuation functions and allocations are strongly related.

In fact, since allocations belong to the set $2^{\mathcal{M}}$, the valuation $v_i(A(i))$ is well defined. Here, $A(i)$ is the set of all goods allocated to agent i. The following example shows different valuations given by agents to some allocations.

Example 10. *Let $\mathcal{N} = \{1, 2\}$ be a set of two agents, $\mathcal{M} = \{g_1, g_2\}$ a set of two goods, and $v_1(\emptyset) = v_2(\emptyset) = 0$ for all agents. Then, the valuations of the agents over elements in $2^{\mathcal{M}}$ are given in Table 2.9.*

	$\{g_1\}$	$\{g_2\}$	$\{g_1, g_2\}$
v_1	5	2	7
v_2	3	1	4

Table 2.9: Valuations of agents over bundles for Example 10.

Then, all possible allocations are given in Table 2.10.

	A_0	A_1	A_2	A_3
1	$\{g_1, g_2\}$	$\{g_1\}$	$\{g_2\}$	$\emptyset$
2	$\emptyset$	$\{g_2\}$	$\{g_1\}$	$\{g_1, g_2\}$

Table 2.10: Possible allocations for agents in $\mathcal{N}$ for Example 10.

The valuations that agents give to their allocations are given in Table 2.11.

	A_0	A_1	A_2	A_3
v_1	7	5	2	0
v_2	0	1	3	4

Table 2.11: Valuations of agents over allocations in Table 2.10.

Here, the valuation that agent 1 gives to its bundle allocated by allocation A_0 is

$$v_1(A_0(1)) = v_1(\{g_1, g_2\}) = 7.$$

Also, the valuation that agent 2 gives to its bundle allocated by allocation A_0 is

$$v_2(A_0(2)) = v_2(\emptyset) = 0.$$

The same follows for allocations A_1, A_2, and A_3.

Notice that allocations in Example 10 are performed considering all possible ways in which the set $\mathcal{M}$ can be partitioned. This implies that when the number of elements in $\mathcal{N}$ or $\mathcal{M}$ increases, then $\mathcal{N}^{\mathcal{M}}$ grows exponentially. Thence, finding all possible allocations could become a costly task. Therefore, an immediate solution to this problem could be only looking for allocations with interesting properties. For instance, allocations assigning goods to the agents that value them the most are good candidates worth looking for. These particular type of allocations will be called the *auction winner allocations* and they are of special interest. This, since they present special properties that will be analyzed later in this same chapter.

Auction winner allocations

In everyday life, an auction is a public sale in which goods are sold to the person who pays the most for them. That is, the person who made a better offer over a good than its competitors ends up with the good. This person is called the auction winner.

In the context of social welfare theory, an auction winner allocation is defined as an allocation in which all agents received the good that they valued (offered) the most. Formally,

Definition 6. *Suppose that for each agent $i \in \mathcal{N}$, v_i is a valuation function. For all $g \in \mathcal{M}$, consider w_g the maximal valuation of good g,*

$$w_g = max\{v_i(g) : i \in \mathcal{N}\}.$$

Notice that w_g is well defined since $\mathcal{N}$ is finite and hence the maximum exists. There may be several agents that maximize g, let $\mathcal{N}_g$ be set of such agents,

$$\mathcal{N}_g = \{i \in \mathcal{N} : v_i(g) = w_g\}.$$

Notice that $|\mathcal{N}_g| \leq |\mathcal{N}| = n$, where $|\mathcal{N}_g|$ is the number of agents maximizing g. Now, let a vector of agents β be

$$\beta = (i_1, \ldots, i_m), \tag{2.6}$$

where i_k for $1 \leq k \leq m$ represents an agent i that maximizes the good g_k such that $i_k \in \mathcal{N}_{g_k}$. Notice that since there may more than an agent maximizing a good $g \in \mathcal{M}$, the total number of possibilities of choosing β is $|\mathcal{N}_1| \times \cdots \times |\mathcal{N}_m|$. The other case is also possible; that is, there may be agents that do not maximize a single good. This implies that

$$\mathcal{N} \setminus \bigcup_{k=1}^{m} \{i_k\} \neq \emptyset.$$

Now, let an auction winner allocation A^w be defined by: for β,

$$A^w = \langle A^w(1), A^w(2), ..., A^w(n)\rangle$$

where for $i \in \mathcal{N}$, $A^w(i)$ is obtained by

$$A^w(i) \ni g_k \Leftrightarrow i = i_k \quad or \quad A^w(i) = \emptyset \Leftrightarrow i \in \mathcal{N} \setminus \bigcup_{k=1}^{m} \{i_k\}. \tag{2.7}$$

Then, denote by $\mathcal{W}$ the set of all different allocations that assigns to the agents the goods they prefer the most, i.e.,

$$\mathcal{W} = \{A^w : A^w \text{ is defined for every } \beta\}. \tag{2.8}$$

It is possible to compute an auction winner allocation by the following algorithm proposed by Camacho et al. [12].

Definition 7 (Algorithm for finding an auction winner allocation)**.** *The algorithm requires three inputs.*
$\mathcal{N}$: *a set of agents.*
$\mathcal{M}$: *a set of goods.*
$\mathcal{V}$: *a set of valuation functions.*

Then,

1 *Let $V_0 = (0, \cdots, 0)$ be a vector of n valuations of agents over the last allocation. Let also $A^w = \langle\emptyset, \cdots, \emptyset\rangle$ be an initial allocation of size n.*

2 *For $k = 1$ to m*

   2.1 *Let the maximum valuation of g_k be $w_k = \max\{v_i(g_k) : i \in \mathcal{N}\}$.*

   2.2 *Let the set of agents that maximize g_k be $\mathcal{N}^w = \{i \in \mathcal{N} : v_i(g_k) = w_k\}$.*

   2.3 *Let the set of agents with a previously minimal valuation be $\mathcal{N}^s = \{i \in \mathcal{N}^w : (V_{k-1})_i = \min\{(V_{k-1})_j : j \in \mathcal{N}^w\}\}$.*

   2.4 *Since agents are represented as positions, let the leftmost agent in $\mathcal{N}^s$ be i_k=$\min\{i : i \in \mathcal{N}^s\}$.*

   2.5 *Let the allocation $A^w(i_k) = A^w(i_k) \cup \{g_k\}$.*

   2.6 *The new valuations are $(V_k)_i = (V_{k-1})_i + v_i(g_k)$ for $i = i_k$ and $(V_k)_i = (V_{k-1})_i$ for $i \neq i_k$.*

3 *End for*

Recall that given a set of agents, a set of goods, and valuation functions for each agent, the number of vectors β that could be obtained is at most n^m (in case all agents share the same valuation function; these are called identical valuation functions). This implies that the set $\mathcal{W}$ is at most the same size as $\mathcal{N}^{\mathcal{M}}$. This fact is established in the following lemma.

Lemma 8. *Suppose that for each agent i, v_i is a valuation function. If $\mathcal{W}$ is the set of all the allocations that assign goods to agents that prefer the most, then*

$$|\mathcal{W}| \leq |\mathcal{N}^{\mathcal{M}}| = n^m.$$

Moreover, if all agents maximize all goods, then $\mathcal{W} = \mathcal{N}^{\mathcal{M}}$.

Proof. Recall that the total number of possibilities of choosing β defined in equation (2.6) is $|\mathcal{N}_1| \times \cdots \times |\mathcal{N}_m|$. Then, $|\mathcal{N}_{g_k}|$ is at most n if all agents maximize the good g_k. Also, since there are m goods,

$$\begin{aligned} |\mathcal{W}| = |\mathcal{N}_1| \times \cdots \times |\mathcal{N}_m| &\leq n \times \cdots \times n \\ &= n^m \\ &= |\mathcal{N}^{\mathcal{M}}|. \end{aligned}$$

Which shows that $|\mathcal{W}| \leq |\mathcal{N}^{\mathcal{M}}|$. In particular, if all agents maximize all goods, then $|\mathcal{W}| = |\mathcal{N}^{\mathcal{M}}|$. Now, given that all allocations in $\mathcal{W}$ are different and since $\mathcal{N}^{\mathcal{M}}$ is the set of all possible allocations, this implies that $\mathcal{W} = \mathcal{N}^{\mathcal{M}}$. □

In the following example, the set of auction winner allocation is computed. Also, the size of this set is compared with the size of the whole set of allocations.

Example 11. *Let $\mathcal{N} = \{1, 2\}$ be a set of two agents and $\mathcal{M} = \{g_1, g_2\}$ be a set of two goods such that $v_1(\emptyset) = v_2(\emptyset) = 0$ for all agents. Then, the valuations that the agents give to the bundles are given in Table 2.12.*

	$\{g_1\}$	$\{g_2\}$	$\{g_1, g_2\}$
v_1	5	7	12
v_2	5	6	11

Table 2.12: Valuations of agents 1 and 2 for Example 11

Now, all the possible allocations of two goods in $\mathcal{M}$ among two agents in $\mathcal{N}$ are given in Table 2.13. Also, the valuations of these allocations are given in Table 2.14.

Here, notice that A_0 and A_2 are two auction winner allocations since both of them assign the goods to the agents that value the most. Hence, $\mathcal{W} = \{A_0, A_2\}$ since these

	A_0	A_1	A_2	A_3
1	$\{g_1, g_2\}$	$\{g_1\}$	$\{g_2\}$	$\emptyset$
2	$\emptyset$	$\{g_2\}$	$\{g_1\}$	$\{g_1, g_2\}$

Table 2.13: Possible allocations for agents in $\mathcal{N}$ for Example 11.

	A_0	A_1	A_2	A_3
v_1	12	5	7	0
v_2	0	6	5	11

Table 2.14: Valuations of allocations in Table 2.13.

allocations give the goods to the agents that maximize its valuation. Also, notice that $|\mathcal{W}| = 2 \leq 4 = |\mathcal{N}^{\mathcal{M}}|$.

Now, note that if $v_2(\{g_2\}) = 7$*, then any allocation would have been an auction winner allocation; hence, we would have had that* $|\mathcal{W}| = 4 = |\mathcal{N}^{\mathcal{M}}|$.

So far, the relationship between valuation functions and allocations has been clarified. In this sense, it has been shown that the idea of an agent valuating its allocation is the same as considering that an allocation for an agent is a bundle of goods. Hence, the valuation function in this context is well defined.

Now, since agents may prefer bundles of agents over others, some allocations of goods will be more successful than others in assigning the goods. Consequently, to evaluate the success of some allocations in comparison with others, it is necessary to consider the social welfare function. This function studies the performance of an allocation from a social point of view. That is, the social welfare function analyzes the success of an allocation focusing on the society rather than individually. This was explained in more detail at the beginning of Section 2.3.

2.3.2 Social welfare functions

Up to now, it is well understood that given a set of agents and a set of goods, it is possible to perform a finite number of different allocations. The total number of allocations is n^m, where n and m are the numbers of agents and goods respectively. This implies, that as the number of agents or goods increases, the number of allocations increases exponentially. Hence, it is pertinent to search for the "best" allocation among all possible allocations. For this, the social welfare function will be useful to model the social welfare criterion explained at the beginning of Section 2.3. Hence, the formal definition of a social welfare function will be given first as follows.

Definition 8 (Social welfare function)**.** *A social welfare function sw is any function from*

$\mathcal{N}^{\mathcal{M}}$ *to* $\mathbb{R}$,

$$\begin{aligned} sw : \mathcal{N}^{\mathcal{M}} &\longrightarrow \mathbb{R} \\ A &\longmapsto sw(A). \end{aligned}$$

If sw is a social welfare function, then it defines a total preorden $\sqsupseteq$ over $\mathcal{N}^{\mathcal{M}}$ given by: $\forall A, A' \in \mathcal{N}^{\mathcal{M}}$

$$A \sqsupseteq A' \Leftrightarrow sw(A) \geq sw(A').$$

Now, by the fact that the *social welfare* function induces a total preorder over a finite set of allocations, a maximum exists. That is, there exists an allocation (or allocations) whose maximum social welfare is greater or equal to other allocations. This allocation (or allocations) will be named the *maximal allocation* and is defined as follows.

Definition 9 (Maximal allocation)**.** *Let sw be a social welfare function and A be an allocation. Then, A is said to be maximal with respect to sw (or simply maximal) if there exists no other allocation A' such that $sw(A) < sw(A')$.*

Note that if sw is a social welfare function, then there is always a maximal allocation. Nonetheless, depending on the social welfare function, finding a maximum allocation can not always be an easy task.

The measurement of the well-being of a society is characterized in different ways by the social welfare function. For instance, a common approach to computing the social welfare is by adding up all the valuations that the agents give to the bundle of goods assigned by a particular allocation. This function is known as the *utilitarian social welfare function.*

2.3.3 Utilitarian social welfare

One way to model the social welfare of an allocation is by aggregating all the individual valuations that agents give to their allocated bundle. Hence, the *utilitarian social welfare function* is defined as follows.

Definition 10 (Utilitarian social welfare function)**.** *A utilitarian social welfare function sw_u is defined by*

$$\begin{aligned} sw_u : \mathcal{N}^{\mathcal{M}} &\longrightarrow \mathbb{R} \\ A &\longmapsto sw_u(A) = \sum_{i \in \mathcal{N}} v_i(A(i)). \end{aligned}$$

The set of all maximal allocations with respect to utilitarian social welfare is denoted by MSW_u. Notice that maximizing the social welfare criterion by means of a utilitarian social welfare function does not guarantee that all agents will receive at least a good. In fact, under modular valuation functions such that agents value the empty set as zero, allocating all goods to a single agent that maximizes all the goods will maximize the utilitarian social welfare.

Now, a new result comes as a consequence of the nature of auction winner allocations explained at the beginning of this section. In particular, the following theorem shows that for modular valuation functions such that agents value the empty set as zero, it is true that the set of auction winner allocations is the set of allocations maximizing the utilitarian social welfare. Formally,

Theorem 1. *If for all $i \in \mathcal{N}$, v_i is a modular valuation function with $v_i(\emptyset) = 0$, then*

$$\mathcal{W} = MSW_u, \tag{2.9}$$

where $\mathcal{W}$ is defined as (2.8) .

Proof. First, let us show that if $A \in \mathcal{W}$, then $A \in MSW_u$. Let A' be any allocation in $\mathcal{N}^{\mathcal{M}}$ and let $i \in \mathcal{N}$. Then,

$$A'(i) \;=\; \mathcal{M} \cap A'(i) = \left(\bigcup_{j \in \mathcal{N}} A(j) \right) \cap A'(i) = \bigcup_{j \in \mathcal{N}} (A(j) \cap A'(i)).$$

So,

$$A'(i) = \bigcup_{j \in \mathcal{N}} (A(j) \cap A'(i)). \tag{2.10}$$

Note that for all $j, k \in \mathcal{N}$ with $j \neq k$,

$$(A(j) \cap A'(i)) \cap (A(k) \cap A'(i)) = \emptyset.$$

Since v_i is modular and $v_i(\emptyset) = 0$, then by equation (2.10),

$$v_i(A'(i)) = \sum_{j \in \mathcal{N}} v_i(A(j) \cap A'(i)). \tag{2.11}$$

On the other hand, as $A \in \mathcal{W}$, by equation (2.7), if $g \in A(j) \cap A'(i)$ for any $j \in \mathcal{N}$, then $v_j(g) \geq v_i(g)$. So, by equation (2.11),

$$v_i(A'(i)) \leq \sum_{j \in \mathcal{N}} v_j(A(j) \cap A'(i))). \tag{2.12}$$

However,

$$\begin{aligned}
sw_u(A') &= \sum_{i \in \mathcal{N}} v_i(A'(i)) \\
&\leq \sum_{i \in \mathcal{N}} \sum_{j \in \mathcal{N}} v_j(A(j) \cap A'(i)) \\
&= \sum_{j \in \mathcal{N}} \sum_{i \in \mathcal{N}} v_j(A(j) \cap A'(i)) \\
&= \sum_{j \in \mathcal{N}} v_j(A(j) \cap (\bigcup_{i \in \mathcal{N}} A'(i))) \\
&= \sum_{j \in \mathcal{N}} v_j(A(j)) \\
&= sw_u(A).
\end{aligned}$$

So, for any $A' \in \mathcal{N}^{\mathcal{M}}$, $sw_u(A') \leq sw_u(A)$. Thus, $A \in MSW_u$. Hence, it has been demonstrated that $\mathcal{W} \subseteq MSW_u$.

Now, let us prove that $MSW_u \subseteq \mathcal{W}$. That is equivalent to show that $\mathcal{W}^c \subseteq MSW_u^c$. Suppose that $A \in \mathcal{W}^c$. Then, there exist $g \in \mathcal{M}$ and $i \in \mathcal{N}$ such that

$$g \in A(i) \text{ and } v_i(g) < w_g,$$

where $w_g = \max\left\{v_j(g) : j \in \mathcal{N}\right\}$. Let an agent $j \in \mathcal{N}$ such that $v_j(g) = w_g$ and note that

$$v_i(g) < v_j(g). \tag{2.13}$$

Now, consider the allocation G, given by

$$G(k) = \begin{cases} A(k) & if \quad k \neq i, j, \\ \\ A(i) \setminus \{g\} & if \quad k = i, \\ \\ A(j) \cup \{g\} & if \quad k = j. \end{cases}$$

Now,

$$\begin{aligned} sw_u(G) &= \sum_{k \in \mathcal{N}} v_k(G(k)) \\ &= \sum_{\substack{k \in \mathcal{N} \\ k \neq i,j}} v_k(G(k)) + v_i(G(i)) + v_i(G(j)) \\ &= \sum_{\substack{k \in \mathcal{N} \\ k \neq i,j}} v_k(A(k)) + v_i(A(i)) - v_i(g) + v_j(A(j)) + v_j(g) \\ &= \sum_{k \in \mathcal{N}} v_k(A(k)) + v_j(g) - v_i(g) \\ &= sw_u(A) + v_j(g) - v_i(g). \end{aligned}$$

Now, by inequality 2.13 it is true that $v_i(g) < v_j(g)$ and hence $v_j(g) - v_i(g) > 0$. So,

$$sw_u(G) > sw_u(A).$$

Therefore, $A \notin MSW_u$. Thus, $\mathcal{W}^c \subset MSW_u^c$. This allows us to conclude that $\mathcal{W} = MSW_u$ as desired. □

The following example shows that in fact, the set of auction winner allocations $\mathcal{W}$ is the set of allocations that maximize utilitarian social welfare MSW_u when valuation functions are modular and agents value as zero to the empty set for all agents.

	A_0	A_1	A_2	A_3
v_1	7	5	2	0
v_2	0	1	3	4
sw_u	7	6	5	4

Table 2.15: Utilitarian social welfare values for Example 12.

Example 12. *Let us consider modular valuations as in Example 10. Then, the utilitarian social welfare values for each allocation are given in Table 2.15.*

Here, $MSW_u = \{A_0\}$ *since allocation* A_0 *is the allocation with the maximum utilitarian social welfare. On the other hand, giving all the goods to the agent that maximizes its valuation is the same as giving both goods to the agent* 1 *as in allocation* A_0 *(Check Example 10). This implies that* $\mathcal{W} = \{A_0\}$*. Therefore, it is clear that* $MSW_u = \mathcal{W}$*.*

Theorem 1 does not hold when valuation functions are supermodular or submodular. This can be checked by the following two counter examples. First, the coming example shows that under supermodular valuation functions, the set of allocations maximizing the utilitarian social welfare, is not the same as the set of auction winner allocations.

Example 13. *Let* $\mathcal{N} = \{1, 2\}$ *be a set of agents, let* $\mathcal{M} = \{g_1, g_2\}$ *be a set of goods and let us consider the supermodular valuations such that* $v_1(\emptyset) = v_2(\emptyset) = 0$ *as given in Table 2.16.*

	$\{g_1\}$	$\{g_2\}$	$\{g_1, g_2\}$
v_1	5	2	7
v_2	3	1	10

Table 2.16: Valuations of agents 1 and 2 for Example 13.

Then, if allocations are performed as in Table 2.13 from Example 11, the valuations and the utilitarian social welfare of each allocation are given in Table 2.17.

	A_0	A_1	A_2	A_3
v_1	7	5	2	0
v_2	0	1	3	10
sw_u	7	6	5	10

Table 2.17: Utilitarian social welfare values for Example 13.

In the previous table, $MSW_u = \{A_3\}$ *since allocation* A_3 *possesses the greatest utilitarian social welfare. On the other hand, giving all the goods to the agent that maximizes its valuation is the same as giving both goods to agent* 1 *as in allocation* A_0*. This implies that* $\mathcal{W} = \{A_0\}$*. Therefore,* $MSW_u \neq \mathcal{W}$*.*

In the same context than Example 13, in the next exemplification, submodular valuation functions such that agents value the empty set as zero will be considered. Then, it will be shown that under the already mentioned conditions, the set of allocation maximizing the utilitarian social welfare is not the same as the set of auction winner allocations.

Example 14. *Let* $\mathcal{N} = \{1, 2\}$ *be a set of agents,* $\mathcal{M} = \{g_1, g_2\}$ *be a set of goods and let us consider the submodular valuations such that* $v_1(\emptyset) = v_2(\emptyset) = 0$ *given in Table 2.18.*

	$\{g_1\}$	$\{g_2\}$	$\{g_1, g_2\}$
v_1	5	2	3
v_2	3	1	4

Table 2.18: Valuations of agents 1 and 2 for Example 14.

Then, if allocations are performed as in Table 2.13 from Example 11. The valuations of the allocations and utilitarian social welfare values for each allocation are given in Table 2.19.

	A_0	A_1	A_2	A_3
v_1	3	5	2	0
v_2	0	1	3	4
sw_u	3	6	5	4

Table 2.19: Utilitarian social welfare values for Example 14.

Here, $MSW_u = \{A_1\}$ *since* A_1 *possesses the maximum utilitarian social welfare. On the other hand, giving all the goods to the agent that maximizes its valuation is the same as giving both goods to agent* 1 *as in allocation* A_0*. This implies that* $\mathcal{W} = \{A_0\}$*. Therefore,* $MSW_u \neq \mathcal{W}$

Another way to measure social well-being is by means of the *Nash social welfare function.* This social welfare function is similar to its utilitarian counterpart. While the utilitarian function adds up all the valuations of agents over the bundles allocated, the Nash social welfare function multiplies these valuations.

2.3.4 Nash social welfare

A social welfare function is a Nash social welfare function if it multiplies the valuations that each agent have over the bundles assigned by an allocation A. Formally,

Definition 11 (Nash social welfare)**.** *A Nash social welfare function sw_N is defined by*

$$\begin{aligned} sw_N : \mathcal{N}^{\mathcal{M}} &\longrightarrow \mathbb{R} \\ A &\longmapsto sw_N(A) = \prod_{i \in \mathcal{N}} v_i(A(i)). \end{aligned}$$

The set of all maximal allocations with respect to Nash social welfare is denoted by MSW_N. Notice that if the social welfare criterion is defined by means of the Nash social welfare function then maximal allocations guarantee that all agents value its allocations with a non-zero value. This, since if some agent values its allocation as zero, then $sw_N = 0$. The only exception is when at least an agent values all the bundles by zero. In that case, any allocation maximizes the Nash social welfare. The following example shows the computation of the Nash social welfare from some allocations.

Example 15. *Let us consider valuation functions as in Example 10. Then, the Nash social welfare values for each allocation are given in Table 2.20.*

	A_0	A_1	A_2	A_3
v_1	7	5	2	0
v_2	0	1	3	4
sw_u	0	5	6	0

Table 2.20: Nash social welfare values for Example 15.

Here, $MSW_N = \{A_2\}$ since allocation A_2 is the allocation with the maximum Nash social welfare. In this case, $sw_N(A_2) = 2 \times 3 = 6$ by definition of the Nash social welfare. A similar process is performed for allocations A_0, A_1 and A_3.

The Nash social welfare function is a relevant characterization of the social welfare criterion. Researches have shown that this social welfare possesses useful properties for the study of the allocations of goods problem [13]. Particularly, when goods are divisible, authors have shown that the Nash social welfare has interesting properties and they have also presented some interesting solutions to the allocation of goods problem in this setting [14]. Nonetheless, when studying indivisible goods, some results do not longer hold [13]. In fact, Lee [15] has shown that under additive valuation functions, finding a Nash maximal allocation is an APX-hard problem. Hence, the problem of allocating indivisible goods using the Nash social welfare characterization is still keeping researches busy.

The last way of measuring the social well-being that will be studied in this work is by considering how happy is the less fortunate member of a society. Saying in other words, the well-being of society will be measured according to the smallest valuation given by an agent. This is modeled by the *egalitarian social welfare* function.

2.3.5 Egalitarian social welfare

A social welfare function is an egalitarian social welfare function if it measures the well-being of a society by only considering the valuation of the agent with the smallest valuation. Formally,

Definition 12 (Egalitarian social welfare)**.** *An egalitarian social welfare function sw_e is defined by*

$$\begin{aligned} sw_e : \mathcal{N}^{\mathcal{M}} &\longrightarrow \mathbb{R} \\ A &\longmapsto sw_e(A) = \min\{v_i(A(i)) : i \in \mathcal{N}\}. \end{aligned}$$

The set of all allocations that maximize the egalitarian social welfare will be denoted as MSW_e. Notice that if the social welfare criterion is characterized by the egalitarian social welfare function, then if there exist at least an agent with a constant valuation that is the smallest among the other agent's valuation, then the social welfare will never increase.

The following example, shows the computation of the egalitarian social welfare for four different allocations.

Example 16. *Let us consider valuation functions as in Example 10. Then, the egalitarian social welfare values for each allocation are given in Table 2.21.*

	A_0	A_1	A_2	A_3
v_1	7	5	2	0
v_2	0	1	3	4
sw_e	0	1	2	0

Table 2.21: Egalitarian social welfare values

Here, $sw_e(A_0) = \min\{7, 0\} = 0$ by definition of egalitarian social welfare. A similar process is performed for allocations A_1, A_2 and A_3. In this case, $MSW_e = \{A_2\}$ since allocation A_2 is the allocation with the maximum egalitarian social welfare.

It is important to notice that not all allocations that maximize a particular social welfare will immediately maximize other social welfare functions. The following example shows how different allocations may maximize different social welfare functions.

Example 17. *Let $\mathcal{N} = \{1, 2\}$ be a set of agents, $\mathcal{M} = \{g_1, g_2\}$ be a set of goods and let us consider the supermodular valuations such that $v_1(\emptyset) = v_2(\emptyset) = 0$ given in Table 2.22.*

	$\{g_1\}$	$\{g_2\}$	$\{g_1, g_2\}$
v_1	5	2	3
v_2	3	1	4

Table 2.22: Valuations of agents 1 and 2 for Example 17.

	A_0	A_1	A_2	A_3
v_1	3	5	2	0
v_2	0	1	3	4

Table 2.23: Valuations of allocations for Example 17.

Consider the allocations as given in Table 2.13 from Example 11. Hence, the valuations of these allocations are given in Table 2.23.

Now, compute the values for the distinct social welfare characterizations. This new information is given in Table 2.24.

	A_0	A_1	A_2	A_3
sw_u	3	6	5	4
sw_N	0	5	6	0
sw_e	0	1	2	0

Table 2.24: Distinct social welfare values.

Notice that there is not necessarily a single allocation that maximizes the utilitarian, egalitarian and Nash social welfare characterizations at the same time. For instance, $A_1 \in MSW_u$ while $A_2 \in MSW_N \cup MSW_e$. This implies that it is not always true that

$$MSW_u = MSW_N = MSW_e.$$

The requirement for an allocation to be maximal is strong [1]. The maximality is a very desired property to be pursued when facing an allocation of goods problem. However, this is not the only relevant criterion to be considered. There are other requirements for an allocation in order to be considered as a successful. Particularly, allocations which satisfy agents from an individual point of view, as well as from a social point of view are also worth looking for.

2.4 Conclusion

To conclude, the three main ingredients of the allocation of goods problem have been introduced: the agents, the goods, and the valuation functions. The set of agents and goods will be constant along the problem, as well as the valuations of the agents. Other topics also analyzed in this chapter are the following.

- Particular types of functions such as the additive, superadditive and subadditive valuation functions are studied. These functions are special valuations that are structured using other simpler valuation functions.
- Different characterizations of the social welfare principle have been introduced. For instance, the utilitarian, Nash and egalitarian social welfare functions. These functions permit the classification of allocations according to a total preorder. The sets of allocations that maximize some of these social welfare functions are studied as well as the "auction winner allocation finder" algorithm that generates some members of these sets.
- It has been shown that under modular valuation functions such that agents value the empty set as zero, the set of allocation maximizing the utilitarian social welfare is the set of auction winner allocations. Similar results using supermodular and submodular valuation functions have been proven to be false using counterexamples.

Other properties over allocations of goods are still required to consider these allocations as suitable solutions to the allocation of goods problem. The two most important properties that will be studied in the next chapter are *efficiency* and *fairness*. On the one hand, *efficiency* will be presented as a way to study the satisfaction of a society with respect to an allocation. While *fairness* will characterize how do agents feel about a particular allocation of goods from a very personal point of view. In some cases, these two properties show up at the same time but most of the time they will not.

Chapter 3

Efficiency and fairness

Finding allocations that maximize either utilitarian, Nash or egalitarian social welfare is a desired result when dealing with the problem of allocation of indivisible goods. The maximality property is a high level requirement for an allocation. Achieving this property for an allocation, does not always implies that the allocation is the best of all allocations. The maximality property considers the success of an allocation from an social point of view, while it ignores the well-being of each agent.

This chapter studies the *efficiency* and the *fairness* criteria, which will be useful to overcome some of the current limitations of the maximality property. The *efficiency* criterion is a manner to evaluate the satisfaction of a society with respect to an allocation [12]. Under the utilitarian social welfare, the efficiency is also considered as a relaxation of the maximality criterion [1]. On the other hand, the *fairness* criterion studies the success of an allocation from the personal point of view of the agents. This section is inspired in the work of Endriss et al. [1] and Caragiannis et al. [13].

3.1 Efficiency

The first new property of an allocation that will be discussed in this chapter is the *efficiency*. This criterion is studied through the *Pareto optimality* which is also known as the Pareto efficiency and evaluates the satisfaction of a society as a whole concerning an allocation. In some contexts, the existence of the Pareto optimality is a guaranteed property; for instance, under the utilitarian social welfare, maximal allocations are also Pareto optimal [1]. Nonetheless, the converse of this result is not true. For this reason, under the utilitarian social welfare, the Pareto optimality property is a relaxation of the maximality property. Also, when the Nash or egalitarian characterizations of the social welfare are considered, Pareto optimality is no longer assured. This implies that some allocations that maximize some social welfare are also Pareto optimal. Still, this does not mean that Pareto optimality only exists in maximal allocations. In fact, it is possible to find non-maximal allocations that are Pareto optimal.

3.1.1 Pareto optimality

The Pareto optimality criterion has been widely used in the allocation of goods literature as a property of allocations [1, 13, 9, 12]. In this sense, an allocation A is Pareto optimal if there exists no other allocation A' where at least an agent increases strictly its valuation while the other agents do not diminish their valuations [16, 13]. Formally,

Definition 13 (Pareto optimality)**.** *An allocation A is Pareto optimal (PO) if there exists no other allocation A' that satisfies the following conditions:*

- $v_i(A(i)) \leq v_i(A'(i))$ *for all* $i \in \mathcal{N}$, *and*
- $v_j(A(j)) < v_j(A'(j))$ *for some* $j \in \mathcal{N}$.

The following example shows some Pareto optimal allocations given sets of two agents and two goods.

Example 18. *Let* $\mathcal{N} = \{1, 2\}$ *and* $\mathcal{M} = \{g_1, g_2\}$ *be sets of agents and goods respectively. Also, let us consider the additive valuations of agents over the goods given in Table 3.1.*

	$\{g_1\}$	$\{g_2\}$
v_1	5	2
v_2	3	1

Table 3.1: Valuations of goods for Example 18.

Now, consider all possible allocations and its valuations as given in Tables 3.2 and 3.3 respectively.

	A_0	A_1	A_2	A_3
1	$\{g_1, g_2\}$	$\{g_1\}$	$\{g_2\}$	$\emptyset$
2	$\emptyset$	$\{g_2\}$	$\{g_1\}$	$\{g_1, g_2\}$

Table 3.2: Possible allocations for agents in $\mathcal{N}$ for Example 18.

	A_0	A_1	A_2	A_3
v_1	7	5	2	0
v_2	0	1	3	4

Table 3.3: Valuations of allocations in Table 3.2.

In the last table, A_0 is Pareto optimal. This happens since any other allocation distinct from A_0 will decrease the valuation of agent 1. Hence, there is no other allocation that will keep or increase the valuation of all agents and increase strictly the valuation of some agent without affecting agent 1. Therefore, A_0 is a Pareto optimal allocation. Notice that A_0 is not the only Pareto optimal allocation. In fact, in this example A_1, A_2 and A_3 are also Pareto optimal allocations for the same reason than A_0 is.

Next, it is important to talk about the relationship among the allocations that maximize the social welfare and the Pareto efficient allocations. Some results appear immediately as consequence of the definition of Pareto efficiency and maximality. For instance, it is true that an allocation that maximizes the utilitarian social welfare is also Pareto optimal. In some works, it has already been shown that this result holds if only additive valuation functions are considered [12]. This result is formalized in the following lemma.

Lemma 9. *An allocation A that maximizes the utilitarian social welfare is Pareto optimal.*

Proof. Let A be a utilitarian maximal allocation. For the sake of contradiction, suppose that A is not a Pareto optimal allocation. This last assertion implies that there exists other allocation A' such that $v_i(A(i)) \leq v_i(A'(i))$ for all $i \in \mathcal{N} \setminus \{j\}$ and $v_j(A(j)) < v_j(A'(j))$ for agent j. So,

$$\left(\sum_{i \in \mathcal{N} \setminus \{j\}} v_i(A(i)) \right) + v_j(A(j)) \leq \left(\sum_{i \in \mathcal{N} \setminus \{j\}} v_i(A'(i)) \right) + v_j(A'(j))$$

and hence $sw_u(A) < sw_u(A')$. This contradicts the fact that A is a maximal allocation. Hence, A must be Pareto optimal. □

It also important to notice that Lemma 9 does not hold for all social welfare characterizations. That is, an allocation maximizing either the Nash or the egalitarian social welfare, is not necessarily Pareto optimal. The next example shows how some allocations that posses the maximality property with respect to some social welfare characterization are not necessarily Pareto optimal.

Example 19. *Let $\mathcal{N} = \{1, 2\}$ and $\mathcal{M} = \{g_1, g_2\}$ be sets of agents and goods respectively. Also, let us consider the valuations of agents over the goods such that $v_1(\emptyset) = v_2(\emptyset) = 0$ as given in Table 3.4.*

	$\{g_1\}$	$\{g_2\}$	$\{g_1, g_2\}$
v_1	2	0	1
v_2	1	0	1

Table 3.4: Valuations of bundles for Example 19.

	A_0	A_1	A_2	A_3
v_1	1	2	0	0
v_2	0	0	1	1

Table 3.5: Valuations of allocations for Example 19.

Now, given all the possible allocations as given in Table 3.2 from Example 18, the valuations for these allocations are given in Table 3.5. Finally, the distinct values for the social welfare characterizations are given in Table 3.6

	A_0	A_1	A_2	A_3
sw_u	1	2	1	1
sw_N	0	0	0	0
sw_e	0	0	0	0

Table 3.6: Social welfare values for Table 3.5.

In the last table, allocation $A_1 \in MSW_u$ is Pareto optimal since no other allocation will increase the valuation of any agent without damaging the other agents. Furthermore, $MSW_N = MSW_e = \mathcal{N}^{\mathcal{M}}$. Notice also that not all allocations are Pareto optimal. Check for instance allocation A_0. It is clear that A_0 is a maximal Nash and a maximal egalitarian allocation. Nonetheless, A_0 is not Pareto optimal since allocation A_1 keeps the valuation of agent 2 while improves strictly the valuation of agent 1.

Notice that Lemma 9 holds for a general valuation function v. The converse of this lemma is not true. That is, not all Pareto optimal allocations maximize some social welfare. This can be easily seen by the nature of an allocation that maximizes a social welfare. If an allocation is Pareto optimal, it may be possible to find another allocation with a higher social welfare but may also diminish the valuation of some agent. Hence, a Pareto optimal allocation does not always maximize a social welfare. Consider for instance Example 18. In that example, allocation A_0 is known to be Pareto optimal. Nonetheless, A_0 is not a utilitarian maximal allocation. Similar examples can be constructed for Pareto allocation which are neither Nash nor egalitarian maximal allocations. Hence, not all Pareto optimal allocations are also maximal allocations.

Despite the fact that not all Nash and egalitarian maximal allocations are also Pareto optimal, it is possible to show that for a particular type of valuation function, Nash maximal allocations are Pareto optimal [13]. In particular, this result can be shown if positive valuation functions are considered. This result is formalized in the following lemma.

Lemma 10. *If valuation functions are positive, then an allocation A that maximizes Nash social welfare is Pareto optimal.*

Proof. Let A be a Nash maximal allocation. For the sake of contradiction, suppose that A is not a Pareto optimal allocation. This last assertion implies that there exists other allocation A' such that $v_i(A(i)) \leq v_i(A'(i))$ for all $i \in \mathcal{N} \setminus \{j\}$ and $v_j(A(j)) < v_j(A'(j))$ for agent j. Then, since valuations are positive, it is true that

$$\left(\prod_{i \in \mathcal{N} \setminus \{j\}} v_i(A(i))\right) v_j(A(j)) \leq \left(\prod_{i \in \mathcal{N} \setminus \{j\}} v_i(A'(i))\right) v_j(A'(j))$$

So, $sw_N(A) < sw_N(A')$. This contradicts the fact that A is a maximal allocation. Hence, A must be Pareto optimal. □

As it has been seen, the efficiency criterion is an interesting way to study how agents in a society feel about a particular allocation. This criterion is based in the idea that an allocation will be considered as convenient if any other allocations may affect some agents. This, despite the possibility that some agents can improve. This property has been demonstrated to always exist in allocations that maximize the utilitarian social welfare. Nonetheless, this result does not hold when allocations maximize Nash or egalitarian social welfare. The converse is also false. That is, allocations that are Pareto optimal are not guaranteed to maximize neither utilitarian, Nash nor egalitarian social welfare functions.

Now, a criterion to study the satisfaction an allocation from the personal point of view of the agents involved will be described. The criterion is called the *fairness* principle and plays a great role when agents consider if their allocation is convenient for them or not. Hence, the fairness principle influences on the personal decision of an agent about being satisfied with their allocated bundle or not. This criterion will be studied with more detail in Section 3.2.

3.2 Fairness

Additionally to the efficiency or maximality properties sought in an allocation, the performance of a solution is also measured using the *fairness criterion.* This criterion analyzes the performance of an allocation from the personal point of view of the agents. Hence, the best allocations are the fairest ones. In this sense, there are several characterizations of the fairness principle that model different approaches of understanding fairness. Recall that fairness is a subjective principle and there is not a common agreement over which is the right definition of fairness. This implies that what is fair for a society could be completely unfair for other. Some of the most common fairness criteria are the *proportionality* and the *envy-freeness.* These characterization will be studied with detail in this chapter.

As an initial characterization of the fairness principle, the idea of a proportional allocation of goods arises. That is, agents demand at least a proportional part of goods depending on the number of agents. For instance, if there are two goods to be allocated to two agents with additive valuations and both of them want the goods equally, then each agent will be satisfied with one of the two goods.

3.2.1 Proportionality

The *proportionality criterion* is the most natural way of interpreting fairness in an allocation. This criterion is relatively easy to achieve for divisible goods when agents have common additive valuations over a set of divisible goods. However, the problem is different when dealing with indivisible goods. In this regard, an allocation is proportionally fair if every agent receives at least one n-th part of the valuation that they give to the whole set of goods [3]. Formally,

Definition 14 (Proportional fairness)**.** *An allocation A is said to be proportionally fair (PROP) if for all $i \in \mathcal{N}$,*

$$v_i(A(i)) \geq v_i(\mathcal{M})/n$$

where n is the number of agents.

Proportional fairness does not always exist in allocations. Consider for example a situation where two agents demand a single similarly appreciated indivisible good. In the following example, this case will be illustrated.

Example 20. *Let $\mathcal{N} = \{1, 2\}$ and $\mathcal{M} = \{g\}$ be sets of agents and a good respectively. Consider also positive valuations v_1, v_2 such that $v_1(\emptyset) = v_2(\emptyset) = 0$. In this case, any allocation will assign the good to either of the agents. This will leave some agent with a zero valuation since they have not received the good. Hence, this allocation will be always unfair from a proportional fairness point of view.*

The previous example shows that even though proportional fairness is a good approach to reach a fair allocation, it is not always possible to find a completely proportional fair allocation. Thence, since proportional fairness is not always possible to attain, there are some relaxations of this criterion such as the *proportional fairness up to one good.* This relaxation has been proposed by Conitzer et al.[17] and the idea is that the proportional fairness property for an agent i will be reached when this agent receives an additional good. Formally,

Definition 15 (Proportional fairness up to one good)**.** *An allocation A is said to be proportionally fair up to one good (PROP1) if for all $i \in \mathcal{N}$ and for some $g \in \mathcal{M}$,*

$$v_i(A(i) \cup \{g\}) \geq v_i(\mathcal{M})/n$$

where n is the number of agents.

The next example shows how it is possible to find an allocation that is proportional fair up to one good.

Example 21. *Let $\mathcal{N} = \{1, 2\}$ be a set of agents, $\mathcal{M} = \{g_1, g_2\}$ be a set of goods. Let also consider the additive valuations of the agents over subsets of $\mathcal{M}$ given in Table 3.7.*

Then, consider allocations and its valuations as given in Tables 3.8. and 3.9.

	$\{g_1\}$	$\{g_2\}$	$\{g_1, g_2\}$
v_1	5	2	7
v_2	3	1	4

Table 3.7: Valuations of agents 1 and 2 for Example 21.

	A_0	A_1	A_2	A_3
1	$\{g_1, g_2\}$	$\{g_1\}$	$\{g_2\}$	$\emptyset$
2	$\emptyset$	$\{g_2\}$	$\{g_1\}$	$\{g_1, g_2\}$

Table 3.8: Possible allocations for agents in $\mathcal{N}$ for Example 21.

	A_0	A_1	A_2	A_3
v_1	7	5	2	0
v_2	0	1	3	4

Table 3.9: Valuations of allocations in Table 3.8.

Here, check that for allocation A_1, the following inequalities hold,

$$\begin{aligned} v_1(A_1(1)) &= 5 \geq 3.5 = v_1(\mathcal{M})/2 \\ v_2(A_1(2)) &= 1 \ngeq 2 = v_2(\mathcal{M})/2. \end{aligned}$$

Notice that A_1 is not a proportional allocation since the valuation of agent 2 to its allocated bundle is not at least as half of its total valuation over $\mathcal{M}$. Nonetheless, if the good g_1 were included in the allocation of agent 2, then

$$\begin{aligned} v_1(A_1(1)) &= 5 \geq 3.5 = v_1(\mathcal{M})/2 \\ v_2(A_1(2) \cup \{g_1\}) &= 4 \geq 2 = v_2(\mathcal{M})/2. \end{aligned}$$

Hence, after aggregating the good g_1 to agent 2's bundle, proportionality is reached. Thence, even though allocation A_1 is not proportional fair (PROP), it is proportional fair up to one good (PROP1).

The following lemma shows that under monotone valuations, a proportionally fair allocation is also proportionally fair up to one good.

Lemma 11. *For monotone valuations, proportionality implies proportionality up to one good.*

Proof. Let us suppose that A is a proportionally fair allocation. This implies that $v_i(A(i)) \geq v_i(\mathcal{M})/n$ for any agent $i \in \mathcal{N}$. Now, since the valuations are monotone, it is true that $v_i(A(i)) \leq v_i(A(i) \cup \{g\})$ for any $i \in \mathcal{N}$ and for some $g \in \mathcal{M}$. Hence $v_i(A(i) \cup \{g\}) \geq v_i(A(i)/n$ for any $i \in \mathcal{N}$ and for some $g \in \mathcal{M}$. This last, is the definition of proportional fairness up to one good. Hence, this implies that if monotone valuation functions are considered, a proportionally fair allocation A is also proportionally fair up to one good. □

Up to now, the first fairness criterion known as proportionality has been studied. This characterization is based on the idea that all agents must receive a proportion of the allocated goods that they consider fair. Also, a relaxation of this principle has been presented and the fact that under monotone valuation functions, proportionality implies proportionality up to one good has been proven.

The next characterization of the fairness principle is through the *envy-freeness*. This is, agents do not envy other agents. This new criterion is different from the proportionality criterion in that agents do care more about other agents. Hence, for the study of the second fairness criterion, it is necessary to define envy in the first place.

3.2.2 Envy-freeness

The idea of envy-freeness raises since in a completely fair society, envy does not exist among its members. In this sense, Endriss et al. [1] defines envy between agents i and j as if agent i values more the bundle allocated to agent j rather than its own assigned bundle. Formally,

Definition 16 (Envy)**.** *Let i, j be any pair of agents in $\mathcal{N}$ and let A be an allocation. It is said that agent i envies agent j if*

$$v_i(A(i)) < v_i(A(j)).$$

An allocation is said to be envy-free if there is no envy among agents. Saying it formally,

Definition 17 (Envy-freeness)**.** *An allocation A is said to be envy-free (EF) if for any pair of agents $i, j \in \mathcal{N}$,*

$$v_i(A(i)) \geq v_i(A(j)).$$

Obtaining a completely envy-free allocation cannot always be possible in general conditions; in fact, even checking if an envy-free allocation exists is a computationally intractable task; that is, there exist no efficient algorithms to solve this problem [10]. Indeed, Example 22 evidences that sometimes completely envy-free allocations do not exist.

Example 22. *Similarly to Example 20, let $\mathcal{N} = \{1, 2\}$ and $\mathcal{M} = \{g\}$ be sets of agents and a good respectively. Consider also positive valuation functions such that $v_1(\emptyset) = v_2(\emptyset) = 0$.*

In this case, any allocation will give the good to an arbitrary agent $i \in \mathcal{N}$. This will leave the other agent j with a zero valuation since they have not received the good. Hence, this allocation will always be unfair from an envy-freeness point of view.

In the last example, it has been shown that for some settings, the problem of finding an envy-free allocation is unattainable. Furthermore, if envy-free allocations exist, it can be computationally costly to find [18]. For this reason, researchers have weakened the envy-free principle with new characterizations known as the *envy-freeness up to the least positively valued good* [13, 19]. Regarding the first concept, an allocation is said to be envy-free up to the least positively valued good if envy-freeness is reached by removing any positively valued good to the envied agent's allocated bundle. Formally,

Definition 18 (Envy-freeness up to the least positively valued good)**.** *An allocation A is said to be envy-free up to the least positively valued good (EFx) if for any pair of agents $i, j \in \mathcal{N}$ and for any $g \in A(j)$ such that $v_i(g) > 0.$,*

$$v_i(A(i)) \geq v_i(A(j) \setminus \{g\}).$$

Then, it can be shown that under additive valuations, envy-freeness implies envy-freeness up to the least positively valued good. This result is illustrated in the following lemma.

Lemma 12. *Under additive valuation functions, envy-freeness implies envy-freeness up to the least positively valued good.*

Proof. Let us suppose that A is an envy-free allocation. This implies that $v_i(A(i)) \geq v_i(A(j))$ for any agents $i, j \in \mathcal{N}$. Since valuations are additive it is true that

$$v_i(A(j)) \geq v_i(A(j) \setminus \{g\})$$

for any $g \in \mathcal{M}$. Hence,

$$v_i(A(i)) \geq v_i(A(j) \setminus \{g\})$$

for any $g \in A(j)$ such that $v_i(g) > 0$. Therefore, if A is EF, it is also EFx. □

Up to now, the envy-freeness characterization has been relaxed to envy-freeness up to the least positively valued good. This allows finding fair allocations more easily than using the strong envy-freeness criterion. However, it is still not guaranteed that an envy-free up to the least positively valued good allocation can always be found. For instance, in the next example there are some allocations where envy persist even after removing a positively valued good. In this case, it is not true that these allocations reach envy-freeness after removing any positively valued good.

Example 23. *Let $\mathcal{N} = \{1, 2\}$ and $\mathcal{M} = \{g_1, g_2\}$ be sets of agents and goods respectively. Also, consider additive valuations of the agents over subsets of $\mathcal{M}$ such that $v_1(\emptyset) = v_2(\emptyset) = 0$ as given in Table 3.10.*

	$\{g_1\}$	$\{g_2\}$	$\{g_1, g_2\}$
v_1	5	2	7
v_2	3	1	4

Table 3.10: Valuations of agents 1 and 2 for Example 23.

	A_0	A_1	A_2	A_3
v_1	7	5	2	0
v_2	0	1	3	4

Table 3.11: Valuations of agents 1 and 2 over allocations for Example 23.

Then, consider allocations performed as in Table 3.8 from Example 21. Thence, the valuations of these allocations are given in Table 3.11.

Here, note that for allocation A_1 the following inequalities hold.

$$v_1(A_1(1)) = 5 \geq 2 = v_1(A_1(2))$$
$$v_2(A_1(2)) = 1 \not\geq 3 = v_2(A_1(1)).$$

Notice that A_1 is not an envy-free allocation by itself since agent 2 envies agent 1. Now, suppose that A_1 is an EFx allocation, then it will be enough to remove any good from agent 2's bundle to reach envy-freeness in A_1.

However, if good g_2 were removed from agent 1's bundle,

$$v_1(A_1(1)) = 5 \geq 2 = v_1(A_1(2))$$
$$v_2(A_1(2)) = 1 \not\geq 2 = v_2(A_1(1) \setminus \{g_2\}).$$

This implies that removing good g_2 from agent 1's bundle has not reached the purpose of eliminating envy among agents 2 and 1 as expected. This implies that sometimes, removing any positively valued good from the envied agent's bundle will not guarantee the envy-freeness property. On the other hand, it can be seen that removing the specific good g_2 from agent 1's bundle will in fact remove envy among agents.

By the previous example, notice that it is still necessary to relax even more the envy-free up to the least positively valued good concept to guarantee the existence of allocations close to having the envy-free property. Hence, Caragiannis et al. [13] has proposed an even more relaxed criterion than EFx called *envy-freeness up to one good.* This criterion, states that an envy-free allocation can be found after the removal of a particular good from the envied agent's bundle. The coming definition will formalize the *envy-freeness up to one good* criterion.

Definition 19 (Envy-freeness up to one good)**.** *An allocation A is said to be envy-free up to one good (EF1) if for any pair of agents $i, j \in \mathcal{N}$, there exist $g \in A(j)$ such that*

$$v_i(A(i)) \geq v_i(A(j) \setminus \{g\}).$$

As it has been explained before, the envy-freeness up to the least positively valued good property is not always reached in allocations. Nonetheless, the absence of this property will not avoid that the envy-freeness up to one good property will be present. Indeed, the following example shows that in a context where an envy-free up to the least positively valued good allocation does not exist, an envy-free up to one good allocation does exist.

Example 24. *Recall Example 23. Here, A_1 is not an envy-free allocation since agent 2 envies agent 1. But, it is enough to remove the good g_1 from $A_1(1)$ to reach envy-freeness.*

Now that a relaxation of the envy-freeness criterion has been introduced, it is natural to find new implications. In particular, the following lemma shows that for monotone valuation functions, an envy-free up to the least positively valued good allocation is also an envy-free up to one good allocation. Formally,

Lemma 13. *Under monotone valuation functions, envy-free up to the least positively valued good implies envy-free up to one good.*

Proof. Let us suppose that A is an envy-free allocation up to the least positively valued good. This implies that $v_i(A(i)) \geq v_i(A(j) \setminus \{g\})$ for any agents $i, j \in \mathcal{N}$ and for the least positively valued good g. Now since valuations are monotone it is true that

$$v_i(A(j) \setminus \{g\}) \geq v_i(A(j) \setminus \{g^*\})$$

for any agents $i, j \in \mathcal{N}$ and for some $g^* \in \mathcal{M}$. Hence,

$$v_i(A(i)) \geq v_i(A(j) \setminus \{g^*\}),$$

for any agents $i, j \in \mathcal{N}$ and for some $g^* \in \mathcal{M}$. This is the definition of a proportional up to one good allocation. Therefore, if A is an envy-free up to the least positively valued good allocation then A is also an envy-free up to one good allocation. □

Finally, there is also an important implication between envy-freeness and envy-freeness up to one good. Thence, in the following corollary, it is established that under additive valuation functions, an envy-free allocation is also an envy-free up to one good allocation.

Corollary 2. *Under additive valuations, envy-freeness implies envy-freeness up to the least positively valued good, and in turn this last property implies envy-freeness up to one good.*

Proof. Consider additive valuation functions. By Lemma 12, envy-freeness implies envy-freeness up to the least positively valued good and by Lemma 13, envy-freeness up to the least positively valued good implies envy-freeness up to one good. The desired result follows immediately by transitivity. □

The envy-freeness up to one good criterion is the last envy-freeness relaxation that will be studied. This, since the existence of an envy-freeness up to one good allocation is guaranteed under additive valuations [13]. This is accomplished with the use of a classic algorithm known as the *round-Robin algorithm*. The process is simple and follows the coming procedure. The first agent receives its most valued good. Then, the second agent receives its most valued good among the goods that are not allocated yet. This process continues until agent n receives its allocation. Once that every agent has received a good, the process starts again from agent 1 to allocate the remaining goods until all goods are allocated. This procedure is further explained in the following definition.

Definition 20 (Round-Robin algorithm)**.** *This algorithm requires three inputs.*
$\mathcal{N}$: *a set of agents.*
$\mathcal{M}$: *a set of goods.*
$\mathcal{V}$: *a set of valuation functions. Then,*

1. *Let A be the allocation that starts assigning $\emptyset$ to each agent, $A(i) \leftarrow \emptyset$, for $i \in \mathcal{N}$.*
2. *Let $\mathcal{M} \leftarrow \{g_1, g_2, \ldots, g_m\}$ be the set of goods to allocate.*
3. *Let $i \leftarrow 1$ be the agent identifier that starts with the first agent from $\mathcal{N}$.*
4. *While $\mathcal{M} \neq \emptyset$, there are goods to allocate.*
   (a) *Let $r \leftarrow \arg\max_{g \in \mathcal{M}}\{v_i(g)\}$ be the good in $\mathcal{M}$ most valued by agent i.*
   (b) *Allocates r to i, $A(i) \leftarrow A(i) \cup \{r\}$*
   (c) *Remove r from $\mathcal{M}$, $\mathcal{M} \leftarrow \mathcal{M} \setminus \{r\}$*
   (d) *If $i = n$, the last agent received a good.*
      - *$i \leftarrow 1$, start again with the first agent.*
   (e) *Else*
      - *$i \leftarrow i + 1$, assign to next agent.*

Up to now, two main characterizations of the fairness principle and its relaxations have been studied. These two characterizations allow us to find important results regarding successful allocations. In particular, in the following section, some consequences regarding fairness and efficiency will be analyzed.

3.3 Results on efficiency and fairness

Allocations which only possess either the maximality or the fairness property are not always the most convenient solutions to the allocation of goods problem. In this sense, allocations with both maximality and fairness criteria are specially desired. However, it is not always possible to accomplish both properties at the same time within the same allocation. For instance, utilitarian maximal allocations are known to be unfair since they only focus in increasing the sum of all valuations. Also, Nash maximal allocations are known to posses strong fairness criteria since they tend to allocate goods to all agents in a more fair way.

There are special relationship among allocations with fairness attributes. For instance, it has been shown that under submodular valuations such that agents value the empty set as zero, allocations that are envy-free are also proportional [10, 9]. This result is formalized in the next lemma.

Lemma 14. *Under submodular valuation functions such that $v(\emptyset) = 0$, an envy-free allocation A is also a proportionally fair allocation.*

Proof. Let $\mathcal{N} = \{1, 2, ..., n\}$ be a set of n agents and let A be an allocation. Suppose that v_i is a submodular valuation functions such that $v_i(\emptyset) = 0$, for $i \in \mathcal{N}$. Also, suppose that A is an envy-free allocation. This implies that for any agent i,

$$\begin{aligned} v_i(A(i)) &\geq v_i(A(1)) \\ v_i(A(i)) &\geq v_i(A(2)) \\ &\vdots \\ v_i(A(i)) &\geq v_i(A(n)). \end{aligned}$$

Hence, by adding up the left and right hand side of these inequalities,

$$n(v_i(A(i))) \geq v_i(A(1)) + v_i(A(2)) + \cdots + v_i(A(n)),$$

which is,

$$v_i(A(i)) \geq v_i(\mathcal{M})/n,$$

by the submodularity of valuations and since $v_i(\emptyset) = 0$ for $i \in \mathcal{N}$. Hence, A is also a proportionally fair allocation. □

In a similar fashion than Lemma 14, it is also true that under additive valuation functions such that agents value as zero the empty set, an envy-free up to one good allocation is also proportional up to one good [9]. Formally,

Lemma 15. *Under additive valuation functions, an envy-free up to one good allocation A is also a proportionally fair up to one good allocation.*

Proof. Let $\mathcal{N} = \{1, 2, ..., n\}$ be a set of n agents and let A be an allocation. Suppose that v_i is an additive valuation function for $i \in \mathcal{N}$. Also, suppose that A is an envy-free allocation up to one good. This implies that there are agents envying other agents if a good is not removed from the envied agent's bundle. Hence, let the set of agents j being envied by an agent i be denoted by $\mathcal{N}^* = \{j \in \mathcal{N} : i \text{ envies } j\}$. Then, for agents in $\mathcal{N}^*$ it is true that,

$$v_i(A(i)) \geq v_i(A(j) \setminus \{g_j\}),$$

for some $g_j \in A(j)$. This implies that,

$$v_i(A(i) \cup \{g_j\}) \geq v_i(A(j)). \tag{3.1}$$

On the other hand, for agents $k \in \mathcal{N} \setminus \mathcal{N}^*$ and for any good $g \in \mathcal{M}$ it is always true that,

$$v_i(A(i) \cup \{g\}) \geq v_i(A(k)). \tag{3.2}$$

Now, let us denote the set of goods that when removed from agent j's bundle, allocation A is envy-free be denoted by $\mathcal{M}^* = \{g_j \in \mathcal{M} : v_i(A(i)) \geq v_i(A(j) \setminus \{g_j\})\}$. Then, let us define g^* such that $v_i(g^*) \geq v_i(g_j)$ for $g_j \in \mathcal{M}^*$. Thence, by adding up valuations of all agents over the bundle $A(i) \cup \{g^*\}$, and by equations (3.1) and (3.2), the following inequalities hold.

$$\begin{aligned} n(v_i(A(i) \cup \{g^*\})) &\geq \sum_{j \in \mathcal{N}^*} v_i(A(i) \cup \{g_j\}) + \sum_{j \in \mathcal{N} \setminus \mathcal{N}^*} v_i(A(i) \cup \{g_j\}) \\ &\geq \sum_{j \in \mathcal{N}^*} v_i(A(j)) + \sum_{k \in \mathcal{N} \setminus \mathcal{N}^*} v_i(A(k)) \\ &= \sum_{j \in \mathcal{N}} v_i(A(j)) \\ &= v_i(\mathcal{M}). \end{aligned}$$

This, finally implies that,

$$v_i(A(i) \cup \{g^*\}) \geq v_i(\mathcal{M})/n.$$

Hence, A is also a proportionally fair up to one good allocation. □

The previous lemma is relevant since shows that under particular conditions, allocations that have the envy-freeness property are also proportionally fair up to one good. Hence there exist allocations that posses more than a fairness property.

As Caragiannis et al. [13] proposed and proved, under additive valuation functions, the existence of an envy-free up to one good allocation can be guaranteed. In fact, the following lemma, states that under additive valuations, it is possible to find an envy-free up to one good allocation by using the round-Robin algorithm detailed in Definition 20 [13].

Lemma 16. *Given additive valuation functions, it is always possible to find an envy-free up to one good allocation using the round-Robin allocation strategy.*

Proof. Suppose that the allocation of goods process follows the round-Robin algorithm. Hence, there are m steps. Consider an agent $i \in \mathcal{N}$. Now, separate the allocation process in phases each one starting when agent i receives a good and terminating before they receive the new good. In each phase, each agent receives a single good. Now, in the first phase each agent receives the good that they value the most and was not previously allocated. This implies that it is possible that agent i envies other agent j who previously received its most valued good g_k. Hence, removing g_k from j will make the allocation *EF*. Now, in the following phases, there will be no other goods with equal or greater valuation than g_k that could also cause envy. This, since the round-Robin process allocates goods with greater or equal valuation first. Therefore, it is enough to remove the good g_k from agent j's bundle to make the allocation EF. Hence, by following this allocation procedure, it is guaranteed that an $EF1$ allocation will be found. □

The following corollary expresses that if additive valuations are considered, proportional up to one good allocations are guaranteed to always exist. This result comes as consequence of Lemmas 15 and 16.

Corollary 3. *Under additive valuation functions, proportional up to one good allocations do always exist.*

Proof. Notice that by Lemma 16, under additive valuation functions, an $EF1$ allocation does always exist. Now, by Lemma 15, under the same valuation functions, an $EF1$ allocation is also $PROP1$. Hence, $PROP1$ allocations do always exist under the previously mentioned conditions. □

Lemma 16 has shown that under additive valuation functions, envy-free up to one good allocations do always exists. Nonetheless, this result does not guarantee that these allocation posses other properties like Pareto optimality. Indeed, it is not true that an envy-free up to one good allocation is also Pareto optimal. The following example illustrates this result.

Example 25. *Let $\mathcal{N} = \{1, 2\}$ be a set of two agents and $\mathcal{M} = \{g_1, g_2\}$ be a set of two goods. Consider also additive valuations of the agents over subsets of $\mathcal{M}$ given as in Table 3.12. Then, consider allocations as in Table 3.13 and its valuations given as in Table 3.14.*

Here, notice that for allocation A_2, the following inequalities hold.

$$v_1(A_2(1)) = 0 \geq 0 = v_1(A_2(2) \setminus \{g_1\}),$$
$$v_2(A_2(2)) = 0 \geq 0 = v_2(A_2(1) \setminus \{g_2\}).$$

	$\{g_1\}$	$\{g_2\}$	$\{g_1, g_2\}$
v_1	1	0	1
v_2	0	1	1

Table 3.12: Valuations of agents 1 and 2 for Example 25.

	A_0	A_1	A_2	A_3
1	$\{g_1, g_2\}$	$\{g_1\}$	$\{g_2\}$	$\emptyset$
2	$\emptyset$	$\{g_2\}$	$\{g_1\}$	$\{g_1, g_2\}$

Table 3.13: Possible allocations for agents in $\mathcal{N}$ for Example 25.

	A_0	A_1	A_2	A_3
v_1	1	1	0	0
v_2	0	1	0	1

Table 3.14: Valuations of allocations in Table 3.13.

Hence, A_2 is an envy-free up to one good allocation. However, A_2 is not a Pareto optimal allocation since any other allocation will strictly increase the valuation of at least an agent without diminishing the valuation of the other agents. Hence, not all $EF1$ allocations are also Pareto optimal.

As an important result, Caragiannis et al. [13] has shown that under additive valuation functions, some allocations that maximize Nash social welfare possess two properties: envy-freeness up to one good and Pareto optimality. In order to prove this result, Lemmas 17 and 18 will be studied. First, the coming lemma shows that given modular valuation functions such that the empty set is valued as zero by all agents, the ratio between valuations of any two agents j, i over a certain good is at most equal to the ratio among the sums of all goods allocated to agents j, i respectively. In turn, this last ratio is equal to the ratio between valuations of agents j, i over allocations to agent j. Formally,

Lemma 17. *Let $\mathcal{N}$ and $\mathcal{M}$ be sets of agents and goods respectively and consider modular valuation functions such that $v_i(\emptyset) = 0$ for agents $i \in \mathcal{N}$. Let also A be an allocation of goods function. Then, for agents $i, j \in \mathcal{N}$, suppose that for at least a good $g \in A(j)$, $v_i(g) > 0$. Hence, if a good g_k is defined as*

$$g_k = \underset{g \in A(j), v_i(g) > 0}{\arg\min} \frac{v_j(g)}{v_i(g)},$$

then the following relation holds [1]

$$\frac{v_j(g_k)}{v_i(g_k)} \leq \frac{\sum_{g\in A(j)} v_j(g)}{\sum_{g\in A(j)} v_i(g)} = \frac{v_j(A(j))}{v_i(A(j))}. \tag{3.3}$$

Proof. Consider modular valuation functions such that $v_i(\emptyset) = 0$, for $i \in \mathcal{N}$. Let also g_k be defined as $g_k = \arg\min_{g\in A(j), v_i(g)>0} \frac{v_j(g)}{v_i(g)}$. Hence, since g_k minimizes the ratio $\frac{v_j(g)}{v_i(g)}$ for $g \in A(j)$ and $v(g) > 0$, then it is true that

$$v_j(g_k)v_i(g_l) \leq v_i(g_k)v_j(g_l)$$

for any other good $g_l \in A(j)$ different from g_k. Then, let $\mathcal{M}^{A(j)} = \{\hat{g}_1, ..., \hat{g}_p\}$ denote the set of goods allocated to agent j by allocation A, where $p = |A(j)|$. Hence, for all goods $\hat{g}$ in $\mathcal{M}^{A(j)}$ it is also true that

$$\begin{aligned} v_j(g_k)v_i(\hat{g}_1) &\leq v_i(g_k)v_j(\hat{g}_1)\\ v_j(g_k)v_i(\hat{g}_2) &\leq v_i(g_k)v_j(\hat{g}_2)\\ &\vdots\\ v_j(g_k)v_i(\hat{g}_p) &\leq v_i(g_k)v_j(\hat{g}_p). \end{aligned}$$

Then by adding up the values in the left and right side of the previous inequalities, gives

$$v_j(g_k)\left(\sum_{\hat{g}\in\mathcal{M}^{A(j)}} v_i(\hat{g})\right) \leq v_i(g_k)\left(\sum_{\hat{g}\in\mathcal{M}^{A(j)}} v_j(\hat{g})\right),$$

which as a consequence is

$$\frac{v_j(g_k)}{v_i(g_k)} \leq \frac{\sum_{g\in A(j)} v_j(g)}{\sum_{g\in A(j)} v_i(g)}.$$

Finally, the equality in equation (3.3) holds since valuations v_i are modular valuation functions such that $v_i(\emptyset) = 0$, for $i \in \mathcal{N}$. □

The next lemma that will be used to prove Theorem 2 will now be presented. This lemma states that implication 3.4 holds.

[1]In Lemma 17, the function $\arg\min(f(x))$ is the preimage of the minimum of the function $f(x)$. For example, consider a function $f(x)$ defined as $f(x) = x^2$ on a discrete domain $\{1, 2, 3\}$. Then, $\arg\min\{f(x)\} = \arg\min\{f(1) = 1, f(2) = 4, f(3) = 9\} = \arg\{f(1)\} = 1$. Hence, the value 1 minimizes the function $f(x)$ in the given domain.

Lemma 18. *Let $\mathcal{N}$ and $\mathcal{M}$ be sets of agents and goods respectively and consider modular valuation functions such that $v_i(\emptyset) = 0$ for agents $i \in \mathcal{N}$. If A is an allocation such that for some pair $i, j \in \mathcal{N}$ there exist at least a good $g \in A(j)$ such that $v_i(g) > 0$ and there exist at least a good $g' \in A(j)$ such that $v_j(g') > 0$. Then, it is true that*

$$\frac{v_j(g_k)}{v_i(g_k)}\left(v_i(A(i)) + v_i(g_k)\right) < v_j(A(j)) \implies \left[1 - \frac{v_j(g_k)}{v_j(A(j))}\right]\left[1 + \frac{v_i(g_k)}{v_j(A(i))}\right] > 1. \tag{3.4}$$

Proof. Consider modular valuation functions such that $v(\emptyset) = 0$ for agents $i \in \mathcal{N}$. Then, starting from the left hand side of Equation 3.4,

$$\frac{v_j(g_k)}{v_i(g_k)}(v_i(A(i)) + v(g_k)) < v_j(A(j)),$$

$\implies$

$$v_j(g_k)v_i(A(i)) + v_i(g_k)v_j(g_k) < v_j(A(j))v_i(g_k),$$

$\implies$

$$v_j(A(j))v_i(A(i)) < v_j(A(j))v_i(A(i)) + v_j(A(j))v_i(g_k) - v_j(g_k)v_i(A(i)) - v_i(g_k)v_j(g_k),$$

$\implies$

$$v_j(A(j))v_i(A(i)) < (v_j(A(j)) - v_j(g_k))(v_i(A(i)) + v_i(g_k)),$$

$\implies$

$$\frac{v_i(A(i))}{(v_i(A(i)) + v_i(g_k))} < \frac{(v_j(A(j)) - v_j(g_k))}{v_j(A(j))},$$

$\implies$

$$\frac{v_i(A(i))}{(v_i(A(i)) + v_i(g_k))} < \left[1 - \frac{v_j(g_k)}{v_j(A(j))}\right],$$

$\implies$

$$1 < \left[1 - \frac{v_j(g_k)}{v_j(A(j))}\right]\left[1 + \frac{v_i(g_k)}{v_j(A(i))}\right],$$

which shows that Equation 3.4 holds.

□

The following algorithm, stated by [13], helps us find a particular kind of Nash maximal allocation A^*. First, a maximum set of agents which possess positive valuations is considered. Then, an allocation A^* maximizing Nash social welfare among agents in S will be sought; the set of all of those allocations A^* with this property will de denoted as $\mathcal{N}^{\mathcal{M}^S}$.

Definition 21 (Algorithm for finding a Nash Maximal Allocation A^*)**.** *The algorithm requires three inputs.*

$\mathcal{N}$: *a set of agents.*

$\mathcal{M}$: *a set of goods.*

$\mathcal{V}$: *a set of valuation functions.*

Then,

1 Let $S \in \arg\max_{\mathcal{N}^* \subseteq \mathcal{N} : A \in \mathcal{N}^{\mathcal{M}} \text{ s.t. } v_i(A(i))>0, \forall i \in \mathcal{N}^*} |\mathcal{N}^*|$.

2 Let the allocation maximizing Nash social welfare be $A' \leftarrow \arg\max_{A \in \Pi_{|S|}} \Pi_{i \in S} v_i(A(i))$.

3 $A_i^* \leftarrow A_i'$*, for* $i \in S$.

4 $A_i^* \leftarrow \emptyset$*, for* $i \in \mathcal{N} \setminus S$.

This algorithm generates an important set of allocations $\mathcal{N}^{\mathcal{M}^S}$ that maximize Nash social welfare using the largest set of agents that provide positive valuations to allocation. The set of allocations $\mathcal{N}^{\mathcal{M}^S}$ will now be shown to posses some important properties such as maximality and Pareto optimality.

Example 25 has shown that some Nash maximal allocations that are envy-free up to one good are not necessarily Pareto optimal. For this reason, Lemmas 17 and 18 have stated some inequalities and implications that will be used in the following theorem. This theorem, proposed by [13] states that under additive valuation functions there exists a set of allocations maximizing Nash social welfare that are also Pareto optimal.

Theorem 2. *Under additive valuation functions, there exists a set of allocations in* MSW_N *that are Pareto optimal and envy-free up to one good.*

Proof. Consider the set $\mathcal{N}^{\mathcal{M}^S}$ defined before Definition 21. Let A^* be an allocation in $\mathcal{N}^{\mathcal{M}^S}$. Then, by Lemma 10, A^* is a Pareto optimal allocation. Now, in order to show that A^* is also $EF1$, let us suppose that A^* is not an $EF1$ allocation. This implies that some agent $i \in \mathcal{N}$ envies other agent $j \in \mathcal{N}$ even after removing a particular good g_k from agent j's bundle. For this, let us choose

$$g_k = \underset{g \in A(j), v_i(g)>0}{\arg\min} \frac{v_j(g)}{v_i(g)}.$$

Notice that $v_i(g) \neq 0$ since agent i envies agent j for at least a good g. Also, let us denote as A' to the new allocation that removes good g_k from j's bundle and gives to agent i. Notice that the valuations of agents $\mathcal{N} \setminus \{i, j\}$ are the same in both allocations A^* and A' since their bundle do not change. The purpose of this proof, is to show that $sw_N(A') > sw_N(A^*)$ which would be a contradiction to the assumption that A^* is a Nash maximal allocation. First, since valuations are positive, $sw_N(A^*) > 0$. Now, by the g_k election and by Lemma 17 it is true that,

$$\frac{v_j(g_k)}{v_i(g_k)} \leq \frac{\sum_{g \in A(j)} v_j(g)}{\sum_{g \in A(j)} v_i(g)} = \frac{v_j(A(j))}{v_i(A(j))}. \tag{3.5}$$

Then, by the supposition that agent i envies agent j even after removing the good g_k, it is clear that $v_i(A(i)) < v_i(A(j) \setminus \{g_k\})$ which in fact is,

$$v_i(A(i)) + v_i(g_k) < v_i(A(j)), \tag{3.6}$$

since all valuations are assumed to be additive. Now, by multiplying equations (3.5) and (3.6),

$$\frac{v_j(g_k)}{v_i(g_k)}(v_i(A(i)) + v_i(g_k)) < v_j(A(j)).$$

Now, by the definition of Nash social welfare and by Lemma 18 the following implications arise,

$$\frac{v_j(g_k)}{v_i(g_k)}(v_i(A(i)) + v_i(g_k)) < v_j(A(j))$$

$$\implies \left[1 - \frac{v_j(g_k)}{v_j(A(j))}\right]\left[1 + \frac{v_i(g_k)}{v_j(A(i))}\right] > 1$$

$$\implies \frac{sw_N(A')}{sw_N(A^*)} > 1.$$

By the last implication, it is true that $sw_N(A') > sw_N(A^*)$ which is a contradiction since it is assumed that A^* is a Nash maximal allocation. Hence, A^* is an $EF1$ allocation. □

At this moment, the last lemma has shown that agents that belong to the set S do not envy each other up to one good given an allocation A. Now, it is also important to show that agents outside S do not envy agents in S. Formally.

Lemma 19. *Under additive valuation functions, allocations in $\mathcal{N}^{\mathcal{M}^S}$ are envy-free up to the least positively valued good and Pareto optimal even if agents with zero valuations are considered.*

Proof. Consider the set $\mathcal{N}^{\mathcal{M}^S}$ defined before Definition 21. Let A^* be an allocation in $\mathcal{N}^{\mathcal{M}^S}$ that also consider agents with zero valuation. This implies that $sw_N(A^*) = 0$. This allocation is Pareto optimal since agents in S already provide a maximal allocation. Also, since agents valuating its allocation with zero will not increase the social welfare. Additionally, by the proof of Lemma 2, A^* is $EF1$ over the agents in S. Hence, now it is necessary to show that A^* is also $EF1$ with agents in $\mathcal{N} \setminus S$. Therefore, it is essential to show that all agents i in $\mathcal{N} \setminus S$ do not envy agent $j \in S$ up to one good. Hence, by contradiction, suppose that there is an agent $i \in \mathcal{N} \setminus S$ that envies other agent j in S. Therefore, let us choose some good $g_j \in A(j)$ such that $v_j(g_j) > 0$. Since agent i envies agent j even after removing the good g_j, it is true that $v_i(A(i)) < v_i(A(j) \setminus \{g_j\})$. Hence, there exists another good $g_i \in A(j) \setminus \{g_j\}$ such that $v_i(g_i) > 0$. But this is a contradiction since $i \in \mathcal{N} \setminus S$. Hence, A^* must be $EF1$ even if agents with zero valuation are considered.

Therefore, an allocation that maximizes Nash social welfare is $EF1$ and PO under additive valuations. □

As an additional it is now possible to show that under additive valuation functions, there exists a set of Nash maximal allocations that are Pareto optimal and proportionally fair up to one good. This corollary is also given the fact that an envy-free up to one good allocation is a proportionally fair up to one good allocation.

Corollary 4. *Under additive valuation functions, there exists a set of allocations in MSW_N that are Pareto optimal and proportional up to one good.*

Proof. This result is a consequence of Lemma 15 and Theorem 2. □

3.4 Conclusion

In this chapter, two important criteria to study the success of an allocation were studied. First, the Pareto optimality criterion comes as a manner to study social satisfaction of allocations. Also, the fairness principle appears as a way of analyzing an allocation from a more personal point of view of the agents. In this sense, there are some characterizations of these two criteria that formalize notions such as proportionality and envy-freeness. Allocations with these properties are regarded as the best solutions to the allocations of goods problem.

Nonetheless, achieving both of efficiency and envy-freeness could be difficult. For instance, it can be easy to get a utilitarian maximal allocation but it does not always posses fairness criteria. The converse occurs with Nash maximal allocations. In particular, the main results of this chapter are the following.

- The most important characterizations of efficiency and fairness have been analyzed. For instance, the Pareto optimality, the proportional fairness and envy-freeness have been studied, as well as some of its relaxations.
- The existence of envy-free up to one good and proportionally fair up to one good allocations under modular valuations such that $v(\emptyset) = 0$ and using the round-Robin algorithm has been proved.
- It has been shown that under additive valuations, there exists a set of allocations maximizing Nash social welfare that are Pareto optimal and envy-free up to one good. This result also guarantees the existence of proportionally fair up to one good allocations too.

Thence, notice that up to now allocations with maximality and fairness criteria are obtained by looking among all the possible allocations. But the difficulty of this process increases exponentially as the number of agents and goods involved increases. Then, a new way to find allocations with desired properties is required.

For this, in the next chapter a process called *negotiation* will be studied. By this strategy, allocations with particular properties are achieved by following a determined procedure and in a finite number of steps.

Chapter 4

Negotiations

In previous chapters, the ingredients of the problem of allocation of resources were presented. Some of these ingredients, usually have more than a single characterization. This is since these concepts try to represent the most common ways of thinking in a society, which of course is not standard among all humans. Hence, phenomena like preferences and fairness need to have more than one definition.

One of the tools that reunite all of the ingredients at the same time allowing its study, is the allocation function. These functions do not simply allocate goods to agents, but also allows the analysis of the quality of the executed actions. Hence, some results will raise regarding the quality of certain allocations. Therefore, it is remarkably important to try to get the best allocations out of all the possible allocations.

When the concept the "best allocation" arises, it refers to some particular definition. An allocation is deemed to be the best if it possesses more and stronger properties of efficiency and fairness than the rest of the allocations. In this sense, Chapter 3 has shown that there are some allocations having some properties that also imply others. By this way, it is necessary to obtain these "best" allocations to be successful in solving the fair allocation of goods problem.

Given the demand for obtaining allocations having more than one strong as possible property, the idea of a process to do this emerges conveniently. In fact, this chapter will be devoted to the study of interactions among agents called the *negotiations*. These interactions will allow agents to trade goods looking for an improvement in their outcome. Some of the results of this chapter are inspired by the work of Chevaleyre et al. [10] and Caragiannis et al. [13].

4.1 Deals

Up to now, it is known that there are some allocations with the maximality property that also possess some fairness properties under particular conditions. Hence, to reach these results, a trading process among agents will be studied. This process will allow agents to exchange their goods among them looking for an improvement from a personal or social point of view. This process is called the *negotiation* and is structured by a sequence of *deals*. Here, a *deal* is defined as the agreement among agents to exchange their current allocated bundles. Saying it formally,

Definition 22. *A deal* $\delta = (A, A')$*, where* A *and* A' *are two different allocations, is an agreement among agents to exchange their current allocated bundles.*

There are different characterizations of this concept which will be useful later. This, since sometimes only a single traded good among agents will be enough to solve particular problems. While, in other settings, the whole bundle of goods will be required to be traded. Hence, some of the most common deals found in the literature are the following.

Definition 23. *Given two different allocations* A *and* A'*, a deal* $\delta = (A, A')$ *is called,*

- *1-deal, if only a single item is traded.*
- *Swap deal, if two agents exchange a single good each.*
- *Cluster deal, if a bundle of goods is traded.*
- *Combined deal, if it is a swap and a cluster deal.*
- *Equitable deal, if the condition* $\min\{v_i(A(i))\} < \min\{v_i(A'(i))\}$ *holds for* i *such that* $v_i(A(i)) \neq v_i(A'(i))$.

Note that if in a cluster deal, the bundle contains a single good, then the deal is also a 1-deal. Notice also that an equitable deal could also be a 1-deal, swap deal or a cluster deal as long as it holds the required condition. Some other similar relations are found among deals in Definition 23. There are other characterizations of a deal which will not be studied in this work.

In the same sense, when more than a deal is performed, it is necessary to define a *negotiation.* Formally,

Definition 24. *A negotiation is a sequence of deals*

$$(\delta_i)_{i=1}^{r} = \{\delta_1, \delta_2, ..., \delta_r\},$$

such that $\delta_i = (A_{i-1}, A_i)$.

The coming example illustrates a 1-deal, a swap deal, a cluster deal and a combined deal as defined in Definition 23.

Example 26. *Let* $\mathcal{N} = \{1, 2\}$ *be a set of two agents and let* $\mathcal{M} = \{g_1, g_2, g_3\}$ *be a set of three goods. Consider also the allocations given as in Table 4.1*

Here, let us define the deals $\delta_1 = (A_0, A_1)$*,* $\delta_2 = (A_1, A_2)$ *and* $\delta_3 = (A_2, A_3)$*. Then,* δ_1 *is a 1-deal since only the good* g_2 *is traded from agent* 1 *to agent* 2*; this deal is also a cluster-deal since the bundle traded is the singleton set* $\{g_2\}$*. Likewise, the deal* δ_2 *is a swap deal since the goods* g_1 *and* g_2 *are exchanged among agents* 1 *and* 2*. Finally, the deal* δ_3 *is a combined deal since two bundles are exchanged among agents* 1 *and* 2*.*

	A_0	A_1	A_2	A_3
1	$\{g_1, g_2\}$	$\{g_1\}$	$\{g_2\}$	$\{g_1, g_3\}$
2	$\{g_3\}$	$\{g_2, g_3\}$	$\{g_1, g_3\}$	$\{g_2\}$

Table 4.1: Distinct allocations of three goods for Example 26.

A 1-deal, a swap deal, a cluster deal and a combined deal do not take into account the valuations of the agents over bundles of goods. On the other hand, an equitable deal requires the valuation of the agents to be studied. The following example shows an equitable deal as defined in Definition 23.

Example 27. *Let $\mathcal{N} = \{1, 2, 3\}$ be a set of three agents and let A_0, A_1 and A_2 be three different allocations of goods. Let also the valuations of agents over these allocations be given by Table 4.2*

	A_0	A_1	A_2
v_1	8	8	8
v_2	6	7	6
v_3	3	5	4

Table 4.2: Valuations of agents over three allocations for Example 26.

Here, let us define the deals $\delta_1 = (A_0, A_1)$ and $\delta_2 = (A_1, A_2)$. Hence, the deal δ_1 is an equitable deal since given that the agents 2 and 3 are involved in the deal, the minimum of its valuation increases from 3 *to* 5*. On the contrary, the deal δ_2 is not equitable since given that the agents 2 and 3 are involved in the deal, the minimum of its valuation decreases from* 5 *to* 4*.*

This last example is of special interest since it has depicted the fact that some deals may result in a better valuation for the agents involved while other deals may bring equal or even worse valuations than in previous allocations. Hence, since most of the decisions that humans take are based on a rational way of thinking, it is expected that the deals also express that rational way of reasoning. That is, to model the rational human decision-making process, it is necessary to introduce a characterization of this quality in the allocation of goods problem. In particular, since rationality is related to a decision taking process, it will be defined with a strong relation with deals.

4.1.1 Rational deals

The necessity of characterizing rationality, is also reinforced by the fact that agents establish preference relations over the goods by means of valuation functions (this topic was analyzed in detail in Chapter 2). To generalize the idea of rational decisions among agents, it is assumed that all agents will act in a rational way.

In this context, the payments (monetary compensations) appear as a tool to define rationality and in order to facilitate the negotiation process. This, particularly when the circumstances do not allow a deal to be performed naturally. Formally,

Definition 25. *A payment* p *associated to a deal* δ *is a function*

$$\begin{aligned} p_\delta : \mathcal{N} &\longrightarrow \mathbb{R} \\ i &\longmapsto p_\delta(i), \end{aligned}$$

such that $\sum_{i \in \mathcal{N}} p_\delta(i) = 0.$

Here, $p_\delta(i)$ denotes the payment of agent i. If $p_\delta(i) > 0$, then agent i pays the quantity of $p_\delta(i)$. On the contrary, if $p_\delta(i) < 0$, then agent i will receive the quantity of $p_\delta(i)$. The definition of payment is strongly related to rationality. This, since a payment can be established as a monetary compensation for agents losing its valuations given a particular deal; also, for agents increasing its valuation, a payment could be interpreted as a tax imposed to the gained value. Hence, a deal is said to be rational for an agent if there exists a payment that is superior to the lose of valuation or inferior to the gain of valuation. Formally

Definition 26. *Given an agent* $i \in \mathcal{N}$*, a deal* $\delta = (A, A')$ *is said to be individually rational* (IR) *if there exist a payment function* p_δ *such that* $v_i(A'(i)) - v_i(A(i)) > p_\delta(i)$.

This last definition states that a deal is individually rational if the agent i must pay less money than what they earn with the deal $\delta = (A, A')$; or in the case that the new valuation in A' is smaller than what they already have, the deal is said to be individually rational if the agent i will be paid more money than what they loose with the deal $\delta = (A, A')$.

The coming example, shows the relevance of a payment function to define individual rationality. Note that showing the set of goods is not essential to explain the example. Nonetheless, the set of goods is intrinsically present in the problem setting.

Example 28. *Let* $\mathcal{N} = \{1, 2\}$ *be a set of two agents and let* A_0, A_1 *be two different allocations of goods. Let also the valuations of agents over these allocations be given in Table 4.3*

	A_0	A_1
v_1	8	7
v_2	6	11

Table 4.3: Valuations of agents over two allocations for Example 28.

Here, let us define the deal $\delta_1 = (A_0, A_1)$*. Now, notice that the deal* δ_1 *is convenient for agent* 2 *since its valuation will increase from* 6 *to* 11*. However, if agent* 1 *accepts the deal* δ_1*, they will decrease their valuation from* 8 *to* 7*. This implies that agents will not agree on the deal* δ_1 *since each agent is not pleased with the new allocation.*

Hence, if a payment function associated to the deal δ_1 *is introduced, the process will be the following: Let us define the payments* $p_{\delta_1}(1) = -4$ *and* $p_{\delta_1}(2) = 4$*. Then, agent* 1 *will be payed* 4 *and agent* 2 *will have to pay* 4*. This new scenario is convenient for both agents since both of them are gaining and therefore,* δ_1 *is individually rational.*

To conclude this example, note that the deal $\delta_1 = (A_0, A_1)$ *would not have been possible without the help of a payment function. For this reason, payments are used to facilitate deals in a society with rational thinking agents.*

Although payments will always help the performance of a deal, it is not always true that these payments exist. The following example shows that it is not always possible to obtain a payment function associated with a deal.

Example 29. *Let* $\mathcal{N} = \{1, 2\}$ *be a set of two agents and let* A_0, A_1 *be two different allocations of goods. Let also the valuations of agents over these allocations be given by Table 4.4,*

	A_0	A_1
v_1	8	7
v_2	7	8

Table 4.4: Valuation of agents over two allocations for Example 29.

Here, let us define the deal $\delta_1 = (A_0, A_1)$*. Now, similarly to previous examples, notice that the deal* $\delta_1 = (A_0, A_1)$ *is convenient for agent* 2 *but not for agent* 1*. Hence, to find a payment function and make the deal* $\delta_1 = (A_0, A_1)$ *individually rational, it is necessary to solve the following system of equations.*

$$\begin{cases} p_{\delta_1}(1) < -1 \\ p_{\delta_1}(2) < 1 \\ p_{\delta_1}(1) + p_{\delta_1}(2) = 0 \end{cases} . \tag{4.1}$$

However, the system 4.1 has not solution. Therefore, there exists no payment function p_{δ_1} *that makes the deal* $\delta_1 = (A_0, A_1)$ *individually rational.*

Since the criterion of individual rationality is studied from the personal point of view of the agents, it is now natural to require an analysis of this criterion from a more general point of view. Additionally, since allocations usually affect all members of a society, a *social rationality* criterion is also relevant. Hence, it is said that for two different allocations A and A', the deal $\delta = (A, A')$ is *socially rational* if it increases the social welfare. Formally,

Definition 27. *A deal $\delta = (A, A')$ is said to be socially rational if $sw(A) < sw(A')$.*

A particular social rationality characterization is the utilitarian social rationality. Thence, under the utilitarian social welfare, it is said that a deal is socially rational if $sw_u(A) < sw_u(A')$. The following example illustrates a socially rational deal given a utilitarian characterization of the social welfare.

Example 30. *Let $\mathcal{N} = \{1, 2\}$ be a set of two agents and let A_0, A_1 be two different allocations of goods. Let also the valuations of agents over these allocations and the utilitarian social welfare be given by Table 4.5*

	A_0	A_1
v_1	8	7
v_2	6	11
sw_u	14	18

Table 4.5: Valuations of agents and utilitarian social welfare for Example 30.

Here, let us define the deal $\delta_1 = (A_0, A_1)$. Then, the deal δ_1 is socially rational since $sw_u(A) = 14 \leq 18 = sw_u(A')$. That is, δ_1 is socially rational since the social welfare has increased from 14 to 18.

The following lemma stated and proved by Endriss et al. [1] shows that a deal is both individually and socially rational if the utilitarian social welfare is considered. Formally,

Lemma 20. *Under the utilitarian social welfare function, a deal $\delta = (A, A')$ is individually rational for all $i \in \mathcal{N}$, if and only if $\delta = (A, A')$ is socially rational.*

Proof. ($\Longrightarrow$) Suppose that the deal $\delta = (A, A')$ is individually rational. Then, for all $i \in \mathcal{N}$ there are payments $p_\delta(i)$ such that $v_i(A'(i)) - v_i(A(i)) > p_\delta(i)$. Hence,

$$\begin{aligned} v_1(A'(1)) - v_1(A(1)) &> p_\delta(1), \\ v_2(A'(2)) - v_2(A(2)) &> p_\delta(2), \\ &\vdots \\ v_n(A'(n)) - v_n(A(n)) &> p_\delta(n). \end{aligned}$$

Then, by adding these inequalities on the left and right hand side and since p_δ is a payment function it follows that,

$$sw_u(A') > sw_u(A).$$

This, implies that the deal $\delta = (A, A')$ is utilitarian socially rational.

($\Longleftarrow$) Suppose that the deal $\delta = (A, A')$ is utilitarian socially rational. Now, let us define a payment function p_δ for each agent $i \in \mathcal{N}$ as,

$$p_\delta(i) = v_i(A'(i)) - v_i(A(i)) - \frac{sw_u(A') - sw_u(A)}{n},$$

where $n = |\mathcal{N}|$. In fact, p_δ is a payment function since,

$$\begin{aligned}
\sum_{i\in\mathcal{N}} p_\delta(i) &= \sum_{i\in\mathcal{N}} \left(v_i(A'(i)) - v_i(A(i)) - \frac{sw_u(A') - sw_u(A)}{n} \right) \\
&= sw_u(A') - sw_u(A) - n\left(\frac{sw_u(A') - sw_u(A)}{n} \right) \\
&= sw_u(A') - sw_u(A) - sw_u(A') + sw_u(A) \\
&= 0.
\end{aligned}$$

Now, notice that since $\delta = (A, A')$ is a socially rational deal, then $sw_u(A') - sw_u(A) > 0$ and hence

$$v_i(A'(i)) - v_i(A(i)) > v_i(A'(i)) - v_i(A(i)) - \frac{sw_u(A') - sw_u(A)}{n} = p_\delta(i).$$

This implies that there exists a payment function $p_\delta(i)$ such that $v_i(A'(i)) - v_i(A(i)) > p_\delta(i)$ for all $i \in \mathcal{N}$ and hence $\delta = (A, A')$ is an individually rational deal as desired.

□

The next example, shows that an individually rational deal is a socially rational deal. The fact that the other way around is also true is illustrated as well.

Example 31. *Let $\mathcal{N} = \{1, 2\}$ be a set of two agents and let A_0, A_1 be two different allocations of goods. Let also the valuations of agents over these allocations be given in Table 4.6*

Here, let us define the deal $\delta_1 = (A_0, A_1)$. By Example 28, it has already been shown that δ_1 is individually rational. Additionally, by Example 30, it has also been shown that δ_1 is socially rational. Hence, δ_1 is both individual and socially rational.

In the same context, a different rationality criterion for the agents will be defined. Unlike individual rationality criterion defined in Definitions 26 and 27, this new criterion called the *cooperative rationality* will not be focused in the existence of payments or the

	A_0	A_1
v_1	8	7
v_2	6	11

Table 4.6: Valuation of agents over two allocations for Example 31.

increase of the social welfare to consider a deal as rational. Instead, this characterization states that a deal is cooperatively rational if at least an agent increases its valuation without damaging other agents [1]. Formally,

Definition 28. *A deal* $\delta = (A, A')$ *is said to be cooperatively rational* (CR) *if* $v_i(A'(i)) \geq v_i(A(i))$ *for all* $i \in \mathcal{N}$ *and* $v_j(A'(j)) > v_j(A(j))$ *for at least an agent* $j \in \mathcal{N}$*.*

The following example shows how cooperative rational deals are performed using sets of two agents and two goods.

Example 32. *Let* $\mathcal{N} = \{1, 2\}$ *be a set of two agents and let* $\mathcal{M} = \{g_1, g_2\}$ *be a set of two goods. Consider also the valuation functions as given in Table 4.7.*

	$\{g_1\}$	$\{g_2\}$	$\{g_1, g_2\}$
v_1	1	2	1
v_2	1	1	2

Table 4.7: Valuations of agents 1 and 2 for Example 32.

Then, consider the allocations of bundles of goods as in Table 4.8.

	A_0	A_1	A_2	A_3
1	$\{g_1, g_2\}$	$\{g_1\}$	$\{g_2\}$	$\emptyset$
2	$\emptyset$	$\{g_2\}$	$\{g_1\}$	$\{g_1, g_2\}$

Table 4.8: Possible allocations for agents in $\mathcal{N}$ for Example 32.

Hence, the valuations and the utilitarian social welfare values of these allocations are given in Table 4.9.

Consider the deal $\delta_1 = (A_0, A_1)$*. This is a cooperatively rational deal since the valuations of all agents are either maintained or increased. For this same reason, the deal* $\delta_2 = (A_1, A_2)$ *is cooperatively rational. Finally, the deal* $\delta_2 = (A_2, A_3)$ *is not cooperatively rational since the valuation of agent* 1 *diminishes.*

	A_0	A_1	A_2	A_3
v_1	1	1	2	0
v_2	0	1	1	2

Table 4.9: Valuations of allocations for agents in $\mathcal{N}$ for Example 32.

Similarly to Lemma 20, it is also true that under the utilitarian social welfare, a cooperatively rational deal will increase the utilitarian social welfare [1]. Hence, the deal is also socially rational. This happens since cooperative rational deals will increase the valuation of at least an agent without decreasing the valuation of the other agents. This implies that the utilitarian social welfare of the new allocation reached by this deal is larger than the utilitarian social welfare of the previous allocation. Formally,

Lemma 21. *Under the utilitarian social welfare, a cooperatively rational deal is a socially rational deal.*

Proof. Let $\delta = (A, A')$ be a cooperatively rational deal. This implies that there exists an agent $j \in \mathcal{N}$ such that $v_j(A(j)) < v_j(A'(j))$ while for other agents it is true that $v_i(A(i)) \leq v_i(A'(i))$ for all $i \in \mathcal{N}$. Suppose that the only agent increasing its valuation is j. Hence,

$$sw_u(A) = \sum_{i \in \mathcal{N}} v_i(A(i)) \leq \sum_{i \in \mathcal{N} \setminus \{j\}} v_i(A'(i)) + v_j(A'(j)) = sw_u(A').$$

Thence, $\delta = (A, A')$ is a socially rational deal. □

This last result does not hold under Nash or Egalitarian social welfare. That is, some cooperatively rational deals are not socially rational if Nash or egalitarian social welfares are considered. A similar negative result is obtained when considering the other way of Lemma 21. That is, under utilitarian, Nash and egalitarian social welfare a socially rational deal is not always a cooperatively rational deal.

4.1.2 Results

The following results will use payments, deals and previous lemmas and definitions to find new allocations with some interesting properties. For instance, the coming result shows that it is possible to find a negotiation based on socially rational deals that finds a maximal allocation without considering individual deals. The process on how to find allocations that maximize social welfare considering also individual deals will be also studied in this chapter.

Theorem 3. *Given an initial allocation A_0, if it is not a maximal allocation, then there exists a sequence of socially rational deals starting from A_0 that converges to a maximal allocation A^*.*

Proof. Let A_0 be an initial non-maximal allocation. Let also $\mathcal{N}^{\mathcal{M}^j} \subseteq \mathcal{N}^{\mathcal{M}}$ denote the set of all allocations with a social welfare greater than $sw(A_j)$. That is,

$$\mathcal{N}^{\mathcal{M}^j} = \{A_i : sw(A_j) < sw(A_i)\}.$$

Then, the set of allocations with a greater social welfare than $sw(A_0)$ is $\mathcal{N}^{\mathcal{M}^0} = \{A_i : sw(A_0) < sw(A_i)\}$. Now, perform the deal $\delta_1 = (A_0, A_1)$, where $A_1 \in \mathcal{N}^{\mathcal{M}^0}$. If A_1 is not a maximal allocation, it implies that there is still a maximal allocation in $\mathcal{N}^{\mathcal{M}}$. Thence, perform the deal $\delta_2 = (A_1, A_2)$, where $A_2 \in \mathcal{N}^{\mathcal{M}^1}$. This process continues until $\mathcal{N}^{\mathcal{M}^r} = \emptyset$ for some $A_r \in \mathcal{N}^{\mathcal{M}}$ and generates a sequence of deals $(\delta_i)_{i=1}^r = \{\delta_1, \delta_2, ..., \delta_r\}$, where $\delta_i = (A_{i-1}, A_i)$; this means that there is not an allocation with a greater social welfare than $sw(A_r)$. Therefore, A_r is a maximal allocation.

Now, notice that each deal performed is a socially rational deal since making a deal from an allocation A_j to any allocation $A_i \in \mathcal{N}^{\mathcal{M}^j}$ makes $sw(A_j) < sw(A_i)$. Therefore, the sequence of deals $(\delta_i)_{i=1}^r = \{\delta_1, \delta_2, ..., \delta_r\}$ is a sequence of socially rational deals converging to a maximal allocation A_r. □

Note that Theorem 3 holds for a general social welfare. Hence, an special case of this theorem will be studied related with utilitarian social welfare [10]. Thence, the following corollary states that Theorem 3 holds in particular from an utilitarian social welfare point of view.

Corollary 5. *Under the utilitarian social welfare, given an initial allocation A_0, if it is not maximal, then there exists a sequence of socially rational deals starting from A_0 that converges to $A^* \in MSW_u$.*

Proof. By Theorem 3, there must exist a sequence of utilitarian socially rational deals starting from A_0 that converges to A^* and maximizes the utilitarian social welfare.

□

The following example shows that considering the utilitarian social welfare and given a non-maximal allocation there exist a sequence of deals converging to a maximal allocation. These deals are socially rational in the sense that increase the utilitarian social welfare.

Example 33. *Let $\mathcal{N} = \{1, 2\}$ be a set of two agents, and let $\mathcal{M} = \{g_1, g_2\}$ be a set of two goods. Consider the valuations given in Table 4.10.*

Then, consider the allocations of bundles of goods as in Table 4.11.

Hence, the valuations and utilitarian social welfare of these allocations are given in Table 4.12,

Here, let A_0 be an initial allocation and consider the deals $\delta_1 = (A_0, A_1)$ and $\delta_2 = (A_1, A_2)$. Hence, the sequence of deals $(\delta_i)_{i=1}^2 = \{\delta_1, \delta_2\}$ is a sequence of socially rational deals since both deals increase the utilitarian social welfare. Additionally, this sequence converges to the allocation $A_2 \in MSW_u$.

	$\{g_1\}$	$\{g_2\}$	$\{g_1, g_2\}$
v_1	5	2	7
v_2	3	1	5

Table 4.10: Valuations of agents 1 and 2 for Example 33.

	A_0	A_1	A_2	A_3
1	$\{g_2\}$	$\{g_1\}$	$\{g_1, g_2\}$	$\emptyset$
2	$\{g_1\}$	$\{g_2\}$	$\emptyset$	$\{g_1, g_2\}$

Table 4.11: Possible allocations for agents in $\mathcal{N}$ for Example 33.

	A_0	A_1	A_2	A_3
v_1	2	5	7	0
v_2	3	1	0	5
sw_u	5	6	7	5

Table 4.12: Valuations of allocations in Table 4.14 for agents in $\mathcal{N}$.

Studying the context when deals are socially rational and valuation functions are of any sort is useful to understand the generality of these results. However, in the following subsection, too general results may not be useful. That is, it is enough to consider particular characterizations of the social welfare and valuation functions that also hold for previous theorems, lemmas, and corollaries. For instance, Endriss et al. [1], has shown that considering the utilitarian social welfare and under additive valuation functions, there exists a sequence of socially rational deals starting from a utilitarian non-maximal allocation that converges to a utilitarian maximal allocation. In this sense, the following corollary is a particularization of Theorem 3.

Corollary 6. *Under the utilitarian social welfare let A_0 be an initial allocation. Consider also additive valuation functions. Then, if A_0 is not maximal, there exist a sequence of socially rational 1-deals starting from A_0 that converges to $A^* \in MSW_u$.*

Proof. Let A_0 be an initial allocation which is not a maximal. Let also $\mathcal{N}^{\mathcal{M}^j} \subseteq \mathcal{N}^{\mathcal{M}}$ denote the set of all allocations with a greater utilitarian social welfare than $sw_u(A_j)$ such that only a good $g^* \in \mathcal{M}$ is allocated in different ways in A_j and A_i. That is,

$$\mathcal{N}^{\mathcal{M}^j} = \{A_i : sw_u(A_j) < sw_u(A_i) \,|\, \exists!\, g^* \in \mathcal{M} \text{ differently allocated in } A_j \text{ and } A_i\}.$$

Then, the set of allocations with greater social welfare than $sw(A_0)$ such that only a good

$g^* \in \mathcal{M}$ is allocated in different ways in A_0 and A_i, is

$$\mathcal{N}^{\mathcal{M}^0} = \{A_i : sw_u(A_0) < sw_u(A_i) \,|\, \exists! g^* \in \mathcal{M} \text{ differently allocated in } A_0 \text{ and } A_i\}.$$

Now, perform the deal $\delta_1 = (A_0, A_1)$, where $A_1 \in \mathcal{N}^{\mathcal{M}^0}$. If A_1 is not a maximal allocation, it implies that there is still a maximal allocation in $\mathcal{N}^{\mathcal{M}}$. Thence, perform the deal $\delta_2 = (A_1, A_2)$, where $A_2 \in \mathcal{N}^{\mathcal{M}^1}$. This process continues until $\mathcal{N}^{\mathcal{M}^r} = \emptyset$ for some $A_r \in \mathcal{N}^{\mathcal{M}}$ and generates a sequence of deals $(\delta_i)_{i=1}^r = \{\delta_1, \delta_2, ..., \delta_r\}$, where $\delta_i = (A_{i-1}, A_i)$; this means that there is not an allocation with a greater social welfare than $sw_u(A_r)$. Therefore, A_r is a maximal allocation.

Now, notice that each deal performed is socially rational since making a deal from an allocation A_j to any allocation $A_i \in \mathcal{N}^{\mathcal{M}^j}$ makes $sw_u(A_j) < sw_u(A_i)$. Additionally, each deal is a 1-deal since only a good is traded. Therefore, the sequence of deals $(\delta_i)_{i=1}^r = \{\delta_1, \delta_2, ..., \delta_r\}$ is a sequence of socially rational 1-deals converging to a utilitarian maximal allocation A_r. □

There is a special way of proving Corollary 6 and can be found in [1]. The following example shows that when considering additive valuation functions, the utilitarian social welfare and given an initial non-maximal allocation A_0, it is possible to find a sequence of deals converging to a maximal allocation. Each deal in this sequence is socially rational in the sense of increasing the utilitarian social welfare.

Example 34. *Consider the utilitarian social welfare. Also, let $\mathcal{N} = \{1, 2\}$ be a set of two agents, and let $\mathcal{M} = \{g_1, g_2\}$ be a set of two goods. Consider also additive valuations as given in Table 4.13.*

	$\{g_1\}$	$\{g_2\}$	$\{g_1, g_2\}$
v_1	5	2	7
v_2	3	1	4

Table 4.13: Valuations of agents 1 and 2 for Example 34.

Then, consider the allocations of bundles of goods as in Table 4.14.

	A_0	A_1	A_2	A_3
1	$\emptyset$	$\{g_1\}$	$\{g_1, g_2\}$	$\{g_2\}$
2	$\{g_1, g_2\}$	$\{g_2\}$	$\emptyset$	$\{g_1\}$

Table 4.14: Possible allocations for agents in $\mathcal{N}$ for Example 34.

	A_0	A_1	A_2	A_3
v_1	0	5	7	2
v_2	4	1	0	3
sw_u	4	6	7	5

Table 4.15: Valuations of allocations in Table 4.14 for agents in $\mathcal{N}$.

Hence, the additive valuations and utilitarian social welfare of these allocations are given in Table 4.15,

Here, let us set A_0 as the initial allocation and consider the two deals $\delta_1 = (A_0, A_1)$ and $\delta_2 = (A_1, A_2)$. Hence, the sequence of deals $(\delta_i)_{i=1}^2 = \{\delta_1, \delta_2\}$ is a sequence of socially rational deals since both deals increase utilitarian social welfare. Additionally, this is a sequence of 1-deals since in each deal, a single good is traded. Finally, this sequence converges to allocation $A_2 \in MSW_u$.

Using the Pareto optimality criterion, [1] has shown that any sequence of deals that are cooperatively rational will result in a Pareto optimal allocation of goods.

Lemma 22. *Given an initial allocation A_0, if it is not a Pareto optimal allocation, then there exists a sequence of cooperatively rational deals starting from A_0 that converges to a Pareto optimal allocation A^*.*

Proof. Let A_0 be an initial allocation which is not a maximal. Let also $\mathcal{N}^{\mathcal{M}^j} \subseteq \mathcal{N}^{\mathcal{M}}$ denote the set of all allocations A_k such that $v_i(A_k(i)) \geq v_i(A_l(i)$ for all $i \in \mathcal{N}$ and $v_k(A'(j)) > v_j(A_l(j))$ for at least an agent $j \in \mathcal{N}$ and for some allocation A_l. That is,

$$\mathcal{N}^{\mathcal{M}^l} = \{A_k : v_i(A_k(i)) \geq v_i(A_l(i)), \forall i \in \mathcal{N} \text{ and } \exists j \in \mathcal{N} : v_j(A_k(j)) > v_j(A_l(j))\}.$$

Then, the set of allocations meeting the previously mentioned conditions for A_0 is,

$$\mathcal{N}^{\mathcal{M}^0} = \{A_k : v_i(A_k(i)) \geq v_i(A_0(i)), \forall i \in \mathcal{N} \text{ and } \exists j \in \mathcal{N} : v_j(A_k(j)) > v_j(A_0(j))\}.$$

Now, perform the deal $\delta_1 = (A_0, A_1)$, where $A_1 \in \mathcal{N}^{\mathcal{M}^0}$. If A_1 is not a Pareto optimal allocation, it implies that there is still a Pareto optimal allocation in $\mathcal{N}^{\mathcal{M}}$. Thence, perform the deal $\delta_2 = (A_1, A_2)$, where $A_2 \in \mathcal{N}^{\mathcal{M}^1}$. This process continues until $\mathcal{N}^{\mathcal{M}^r} = \emptyset$ for some $A_r \in \mathcal{N}^{\mathcal{M}}$ and generates a sequence $(\delta_i)_{i=1}^r = \{\delta_1, \delta_2, ..., \delta_r\}$ where $\delta_i = (A_{i-1}, A_i)$; this means that there is not an allocation such that $v_i(A_k(i)) \geq v_i(A_r(i))$ for all $i \in \mathcal{N}$ and $v_i(A_k(j)) > v_j(A_r(j))$ for at least an agent $j \in \mathcal{N}$. Therefore, A_r is Pareto optimal.

Now, notice that each deal performed is cooperatively rational since making a deal from an allocation A_l to any other allocation $A_k \in \mathcal{N}^{\mathcal{M}^l}$ increases or keeps the valuation of all agents and increases strictly the valuation for at least an agent. Therefore, the sequence of deals $(\delta_i)_{i=1}^r = \{\delta_1, \delta_2, ..., \delta_r\}$ is a sequence of cooperatively rational deals converging to a Pareto optimal allocation A_r. □

The following example shows that given an initial allocation A_0 that is not Pareto optimal, it is possible to find a sequence of deals converging to a Pareto optimal allocation. In particular, these deals are cooperatively rational.

Example 35. *Let $\mathcal{N} = \{1, 2\}$ be a set of two agents, and let $\mathcal{M} = \{g_1, g_2\}$ be a set of two goods. Consider also additive valuation functions as given in Table 4.16.*

	$\{g_1\}$	$\{g_2\}$	$\{g_1, g_2\}$
v_1	1	1	2
v_2	3	2	2

Table 4.16: Valuations of agents 1 and 2 for Example 35.

Then, consider the allocations of bundles of goods as in Table 4.17.

	A_0	A_1	A_2	A_3
1	$\emptyset$	$\{g_1\}$	$\{g_2\}$	$\{g_1, g_2\}$
2	$\{g_1, g_2\}$	$\{g_2\}$	$\{g_1\}$	$\emptyset$

Table 4.17: Possible allocations for agents in $\mathcal{N}$ for Example 35.

Hence, the additive valuations and utilitarian social welfare of these allocations are given in Table 4.18,

	A_0	A_1	A_2	A_3
v_1	0	1	1	2
v_2	2	2	3	0

Table 4.18: Valuations of allocations in Table 4.17 for agents in $\mathcal{N}$.

Here, let us set A_0 as the initial allocation. Notice that this allocation is not Pareto optimal since other allocations are improving the valuations of all agents and improving strictly the valuation of at least one agent. Consider the two deals $\delta_1 = (A_0, A_1)$ and $\delta_2 = (A_1, A_2)$. Hence, the sequence of deals $(\delta_i)_{i=1}^2 = \{\delta_1, \delta_2\}$ is a sequence of cooperatively rational deals since both deals keep the valuations of all agents and increases strictly the *valuation of at least an agent participating in the deal. Finally, this sequence converges to allocation A_2 which is in fact a Pareto optimal allocation.*

In the same context of sequences of cooperatively rational deals, it is possible to show that there exists a sequence of these deals converging to a maximal utilitarian allocation using 0-1 valuation functions [1].

Lemma 23. *Under 0-1 valuation functions and considering the utilitarian social welfare, let A_0 be an initial allocation. Then, if $A_0 \notin MSW_u$ then there exists a sequence of cooperatively rational 1-deals starting from A_0 that converges to $A^* \in MSW_u$.*

Proof. Consider the utilitarian social welfare. Let A_0 be an initial allocation which is not a maximal. Let also $\mathcal{N}^{\mathcal{M}^l} \subseteq \mathcal{N}^{\mathcal{M}}$ denote the set of all allocations A_k such that the deal $\delta = (A_l, A_k)$ is cooperatively rational an only a goof $g^* \in \mathcal{M}$ is allocated differently in A_l and A_k. That is,

$$\mathcal{N}^{\mathcal{M}^l} = \{A_k : \delta = (A_l, A_k) \text{ is } CR \text{ and } \exists! g^* \in \mathcal{M} \text{ differently allocated in } A_l \text{ and } A_k\}.$$

Then, the set of all cooperatively rational allocations such that only a good $g^* \in \mathcal{M}$ is allocated in different ways in A_0 and A_k, is

$$\mathcal{N}^{\mathcal{M}^0} = \{A_k : \delta = (A_0, A_k) \text{ is } CR \text{ and } \exists! g^* \in \mathcal{M} \text{ differently allocated in } A_0 \text{ and } A_k\}.$$

Now, perform the deal $\delta_1 = (A_0, A_1)$, where $A_1 \in \mathcal{N}^{\mathcal{M}^0}$. If A_1 is not a maximal allocation, it implies that there is still a maximal allocation in $\mathcal{N}^{\mathcal{M}}$. Thence, perform the deal $\delta_2 = (A_1, A_2)$, where $A_2 \in \mathcal{N}^{\mathcal{M}^1}$. This process continues until $\mathcal{N}^{\mathcal{M}^r} = \emptyset$ for some $A_r \in \mathcal{N}^{\mathcal{M}}$ and generates a sequence of deals $(\delta_i)_{i=1}^r = \{\delta_1, \delta_2, ..., \delta_r\}$ where $\delta_i = (A_{i-1}, A_i)$; this means that there is not an allocation that allows to perform a cooperatively rational deal and only the allocation of a single good changes. This implies that there is not a good whose reallocation will improve the valuation of some agent without affecting other agent. Also, notice that the utilitarian social welfare reached is maximum. This, since moving a good will make an agent lose the valuation of 1 but will be recovered by other agent. Therefore, A_r is a maximal allocation.

Now, notice that each deal performed is cooperatively rational by definition of the set $\mathcal{N}^{\mathcal{M}^l}$. Additionally, each deal is a 1-deal since only a good is traded. Therefore, the sequence of deals $(\delta_i)_{i=1}^r = \{\delta_1, \delta_2, ..., \delta_r\}$ is a sequence of cooperatively rational 1-deals converging to a utilitarian maximal allocation A_r. □

Another efficiency result comes with equitable deals and egalitarian social welfare. In this setting, Endriss et al. [1], has shown that any sequence of equitable deals will result in an allocation with maximal sw_e.

Lemma 24. *Under the egalitarian social welfare, given an initial allocation A_0, if it is not maximal, then there exists a sequence of equitable deals starting from A_0 that converges to $A^* \in MSW_e$.*

Proof. Consider the egalitarian social welfare. Let A_0 be an initial non-maximal allocation. Also, given the allocation A_j, let $\mathcal{N}^{\mathcal{M}^j} \subseteq \mathcal{N}^{\mathcal{M}}$ denote the set of all allocations A_k such that $\min\{v_i(A_j(i))\} < \min\{v_i(A_k(i))\}$ for all agents involved in the deal $\delta = (A_j, A_k)$. That is,

$$\mathcal{N}^{\mathcal{M}^j} = \{A_k : \min\{v_i(A_j(i))\} < \min\{v_i(A_k(i))\} \text{ for } i \in \mathcal{N} \mid v_i(A_j(i)) \neq v_i(A_k(i))\}.$$

Then, given the allocation A_0, the set of allocations meeting the previously mentioned condition is

$$\mathcal{N}^{\mathcal{M}^0} = \{A_k : \min\{v_i(A_0(i))\} < \min\{v_i(A_k(i))\} \text{ for } i \in \mathcal{N} \mid v_i(A_0(i)) \neq v_i(A_k(i))\}.$$

Now, perform the deal $\delta_1 = (A_0, A_1)$, where $A_1 \in \mathcal{N}^{\mathcal{M}^0}$. If A_1 is not a maximal allocation, it implies that there is still a maximal allocation in $\mathcal{N}^{\mathcal{M}}$. Thence, perform the deal $\delta_2 = (A_1, A_2)$, where $A_2 \in \mathcal{N}^{\mathcal{M}^1}$. This process continues until $\mathcal{N}^{\mathcal{M}^r} = \emptyset$ for some $A_r \in \mathcal{N}^{\mathcal{M}}$ and generates a sequence of deals $(\delta_i)_{i=1}^r = \{\delta_1, \delta_2, ..., \delta_r\}$, where $\delta_i = (A_{i-1}, A_i)$. The last allocation A_r is an egalitarian maximal allocation since there are not other allocations that will increase the valuation of agents participating in the negotiation. The only possibility of existing an agent with a smaller valuation is if an agent has not participated in at least the last negotiation $\delta_r = (A_{r-1}, A_r)$. Nonetheless, if that happens, it means that at least these two last allocations are maximal.

Now, notice that each deal performed increases the valuation of the agent with the minimum valuation in the set. Hence, each deal is an equitable deal. Therefore, the sequence of deals $(\delta_i)_{i=1}^r = \{\delta_1, \delta_2, ..., \delta_r\}$ is a sequence of equitable deals converging to a maximal allocation A_r. □

As it has been shown earlier, another criteria to measure the performance of an allocation is to consider how fair the allocation is. In this sense, [10] has shown that under the Knaster payment scheme, any sequence of deals which strictly increase the sw_u of an allocation, will result in an allocation which is efficient and Chevaleyre proportional. These new definitions and results will be analyzed in the coming section.

4.2 Knaster Procedure

Now, the Knaster procedure, initially stated by the mathematician Bronislaw Knaster in the forties and adapted to indivisible goods by Chevaleyre et al. [10] states that it is possible to reach proportionally fairness in a new context given a maximal allocation. This result is achieved by using *compensation functions* added to the valuations of agents.

The *compensation functions* are functions of the form $\alpha_j : 2^{\mathcal{M}} \longrightarrow \mathbb{R}$. By adding the compensation functions to the valuations, each agent will also consider the compensating value to its utility over bundles of $2^{\mathcal{M}}$. The final utility function that will consider the valuation of each agent together with the compensation function is called the *compensated valuation function.* Formally,

Definition 29 (Compensated valuation function). *The function $u_i : 2^{\mathcal{M}} \longrightarrow \mathbb{R}$ given by*

$$u_i(S, \alpha_j) = v_i(S) - \alpha_j(S),$$

is known as the compensated valuation function. Here, v is a valuation function and α is a function of the form $\alpha_j : 2^{\mathcal{M}} \longrightarrow \mathbb{R}$ known as the compensation function.[1]

The *Chevaleyre proportionally fairness criterion* is a particular characterization of the fairness principle based on the classical proportional fairness criterion. The difference with the classical characterization of *proportional fairness* is that Chevaleyre proportional fairness considers the compensation values together with the valuations of the agents.

Definition 30 (Chevaleyre proportional fairness). *An allocation A is said to be Chevaleyre proportional fair if*

$$u_i(A(i), \alpha_i) \geq v_i(\mathcal{M})/n,$$

for all $i \in \mathcal{N}$ where n is the number of agents.

Thence, the Knaster procedure computes the *proportionality compensation function α_i* that will determine the amount of money that each agent will receive or will lose to obtain a *Chevaleyre proportionally fair allocation.*

Definition 31 (Knaster procedure). *Given an allocation A, compute the excess $ex_i(A(i))$ for each agent $i \in \mathcal{N}$. That is, determine*

$$ex_i(A(i)) = v_i(A(i)) - v_i(\mathcal{M})/n,$$

where $n = |\mathcal{N}|$. Now, compute $Ex(A)$ as the sum of all excesses of each agent $i \in \mathcal{N}$. That is,

$$Ex(A) = \sum_{i \in \mathcal{N}} ex_i(A(i)).$$

Finally, define the <u>proportionality compensation function</u> α_i as

$$\alpha_i(A(i)) = ex_i(A(i)) - \frac{Ex(A)}{n}.$$

[1]Notice that the agent associated to valuation v_i is denoted by i while agent associated to function α is denoted by j. This emphasises the fact that u_i may depend also from other agent j, although most of the time $i = j$.

Proposition 1. *Notice that*

$$\sum_{i\in\mathcal{N}} \alpha_i(A(i)) = 0.$$

As it has been stated before, the compensation functions will be considered in the utility of an agent with respect to a good. In this sense, the compensated valuation functions are defined. Using these utility functions, it is possible to show that under additive valuation functions and considering the utilitarian social welfare, a maximal allocation is also Chevaleyre proportional [1]. This result is reached by using the proportionality compensation functions and is stated formally as follows.

Theorem 4. *Under the utilitarian social welfare, consider additive valuation functions. Also, let A^* be a maximal allocation. If valuations of agents over bundles allocated by A^* are compensated by the proportionality compensation functions, then A^* is a Chevaleyre proportionally fair allocation.*

Proof. First notice that since A^* is a maximal allocation, then $Ex(A^*) \geq 0$. This is true since

$$\begin{aligned} Ex(A^*) &= \sum_{i\in\mathcal{N}} ex_i(A^*(i)) \\ &= \sum_{i\in\mathcal{N}} \left(v_i(A^*(i)) - \frac{v_i(\mathcal{M})}{n} \right) \\ &= sw_u(A^*) - n^{-1} \sum_{i\in\mathcal{N}} v_i(\mathcal{M}) \\ &\geq 0. \end{aligned}$$

The last line holds since $sw_u(A^*) \geq v_i(\mathcal{M})$ for any agent $i \in \mathcal{N}$. Thence,

$$\begin{aligned} u_i(A^*(i)) &= v_i(A^*(i)) - \left(ex_i(A^*(i)) - \frac{Ex(A^*)}{n} \right) \\ &= v_i(A^*(i)) - \left(v_i(A^*(i)) - \frac{v_i(\mathcal{M})}{n} - \frac{Ex(A^*)}{n} \right) \\ &= \frac{v_i(\mathcal{M})}{n} + \frac{Ex(A^*)}{n} \\ &\geq \frac{v_i(\mathcal{M})}{n}. \end{aligned}$$

The last line in this proof holds since $Ex(A^*) \geq 0$. Hence $u_i(A^*(i)) \geq v_i(\mathcal{M})/n$ as required. □

The following example shows the computation of the proportionality compensation function using the Knaster procedure. Then, considering additive valuation functions and the utilitarian social welfare, a maximal allocation is shown to also be Chevaleyre proportional fair.

	$\{g_1\}$	$\{g_2\}$	$\{g_3\}$
v_1	2	4	2
v_2	1	2	3
v_3	5	2	6

Table 4.19: Valuation of agents for Example 36.

Example 36. *Let $\mathcal{N} = \{1, 2, 3\}$ be a set of three agents and $\mathcal{M} = \{g_1, g_2, g_3\}$ be a set of three goods. Then, consider additive valuation functions as in Table 4.19.*

Now, choose the utilitarian maximal allocation A^* as $A^* = \langle\{g_2\}, \varnothing, \{g_1, g_3\}\rangle$ with utilitarian social welfare 15. This allocation is not proportionally fair in the usual sense since $v_2(\emptyset) = 0 < 2 = v_2(\mathcal{M})/n$. Hence, it is necessary to compute the Knaster compensation value to find an allocation which is Chevaleyre proportional.

First, compute the value $ex_i(A^*(i)) = v_i(A^*(i)) - v_i(\mathcal{M})/n$ for each agent $i \in \mathcal{N}$. That is,

$$\begin{aligned}
ex_1(A^*(1)) &= v_1(A^*(1)) - v_1(\mathcal{M})/n = 4 - 8/3 = 1.33,\\
ex_2(A^*(2)) &= v_2(A^*(2)) - v_2(\mathcal{M})/n = 0 - 6/3 = -2,\\
ex_3(A^*(3)) &= v_3(A^*(3)) - v_3(\mathcal{M})/n = 11 - 13/3 = 6.67.
\end{aligned}$$

Now, compute the sum of all of the excesses

$$\begin{aligned}
Ex(A^*) &= \sum_{i \in \mathcal{N}} ex_i(A^*(i))\\
&= ex_1(A^*(1)) + ex_2(A^*(2)) + ex_3(A^*(3))\\
&= 1.33 - 2 + 6.67\\
&= 6.
\end{aligned}$$

Then, the proportionality compensation values α_i are given by

$$\begin{aligned}
\alpha_1(A^*(1)) &= ex_1(A^*(1)) - Ex(A^*)/n = 1.33 - 6/3 = -0.67,\\
\alpha_2(A^*(2)) &= ex_2(A^*(2)) - Ex(A^*)/n = -2 - 6/3 = -4,\\
\alpha_3(A^*(3)) &= ex_3(A^*(3)) - Ex(A^*)/n = 6.67 - 6/3 = 4.67.
\end{aligned}$$

Now, the new compensated valuations are given for each agent by,

$$
\begin{aligned}
u_1(A^*(1), \alpha_1) &= v_1(A^*(1)) - \alpha_1(A^*(1)) = 4 + 0.67 = 4.67,\\
u_2(A^*(2), \alpha_2) &= v_2(A^*(2)) - \alpha_2(A^*(2)) = 0 + 4 = 4,\\
u_3(A^*(3), \alpha_3) &= v_3(A^*(3)) - \alpha_3(A^*(3)) = 11 - 4.67 = 6.33.
\end{aligned}
$$

Which in fact meet the criteria to make A^* a Chevaleyre proportionally fair allocation.

Hence, starting from a maximal allocation A^* and defining valuations of agents over bundles allocated by A^*, the allocation is Chevaleyre proportional. Nonetheless, this strategy does not always make an allocation Chevaleyre proportionally fair. If a given allocation A is not a maximal allocation, then the existence of a Chevaleyre proportionally fair allocation using the proportionality compensation function is not guaranteed.

4.2.1 Knaster payment scheme

The proportionality compensation functions are computed given a set of agents, a set of goods, an allocation and valuation functions. These compensation values have shown to provide a Chevaleyre proportionally fair allocation under additive valuation functions and considering an utilitarian approach to the social welfare. Then, the idea of obtaining a Chevaleyre proportionally fair allocation through a negotiations arises.

Thence, in order to achieve a Chevaleyre proportional allocation, it is necessary to obtain a sequence of deals converging to a maximal allocation in the first place. It has already been showed that these sequences of deals exist by Theorem 3 and its corollaries. Now, since the maximal allocation has been obtained, Theorem 4 has shown that under additive valuation functions, it is possible to obtain a maximal and Chevaleyre proportional allocation.

Hence, given a negotiation, the performed deals need to follow up the Knaster strategy to obtain compensation functions for each allocation. This ensures that the maximal allocation is Chevaleyre proportionally fair. Thence, a payment scheme will be defined as follows.

Definition 32 (Knaster payment scheme)**.** *Given any deal δ_j in a negotiation $(\delta_i)_{i=1}^r$, the payment associated to δ_j is,*

$$p_{\delta_j}(i) = \alpha_i(A_j(i)) - \alpha_i(A_{j-1}(i)),$$

for every $i \in \mathcal{N}$.

Hence, in a utilitarian context, given an initial allocation it is possible to find a sequence of socially rational deals converging to a maximal allocation that is Chevaleyre proportional. Here, payments follow the Knaster payment scheme. Formally,

Lemma 25. *Under additive valuation functions, consider the utilitarian social welfare and an initial allocation A_0. If A_0 is not maximal, then, there exist a sequence of socially rational deals converging to a maximal allocation that is Chevaleyre proportional.*

Proof. By Corollary 6, it is clear that under the utilitarian social welfare there exists a sequence of socially rational deals converging to a maximal allocation. Now, since this allocation is maximal, Theorem 4 guarantees that this allocation is Chevaleyre proportional as required. □

The following example shows that considering the utilitarian social welfare and under additive valuation functions, it is possible to reach a Chevaleyre proportional allocation through socially rational deals. The initial allocation of this sequence is a non-maximal allocation A_0 and the final allocation is also maximal.

Example 37. *Let $\mathcal{N} = \{1, 2, 3\}$ be a set of three agents and let $\mathcal{M} = \{g_1, g_2, g_3\}$ be a set of three goods. Consider also additive valuation functions as given in Table 4.20 and allocations and its valuations given as as in Tables 4.21 and 4.22 respectively.*

	$\{g_1\}$	$\{g_2\}$	$\{g_3\}$
v_1	2	4	2
v_2	1	2	3
v_3	5	2	6

Table 4.20: Valuation of agents for Example 37.

	A_0	A_1	A_2
1	$\{g_2\}$	$\{g_2\}$	$\{g_2\}$
2	$\{g_1, g_3\}$	$\{g_3\}$	$\varnothing$
3	$\varnothing$	$\{g_1\}$	$\{g_1, g_3\}$

Table 4.21: Allocations of goods for Example 37.

Now, following the Knaster procedure, compute the compensation functions for each agent and for each allocation as in Table 4.23.

Next, compute the compensated valuation functions u_i, for each agent and for each allocation as in Table 4.24

Then, the sequence of deals $(\delta_i)_{i=1}^2 = \{\delta_1, \delta_2\}$ where payments follow the Knaster payment scheme is Chevaleyre proportional. In fact, payments are given in Table 4.25

	A_0	A_1	A_2
v_1	4	4	4
v_2	4	3	0
v_3	0	5	11
sw_u	8	12	15

Table 4.22: Valuations of allocations in Table 4.21

	A_0	A_1	A_2
α_1	1.67	0.33	−0.67
α_2	2.33	0	−4
α_3	−4	−0.33	4.67

Table 4.23: Proportionality compensation functions

	A_0	A_1	A_2
u_1	2.33	3.67	4.67
u_2	1.67	3	4
u_3	4	5.33	6.33

Table 4.24: Compensated valuations using compensations from Table 4.23.

	δ_1	δ_2
$p(1)$	−1.33	−1
$p(2)$	−2.33	−4
$p(3)$	3.66	5

Table 4.25: Payments of three agents for Example 37.

Now, it is possible to establish similar results to Lemma 25 with particular types of valuation functions and deals. In the following corollary, it has been shown that under additive valuation functions, there exists a sequence of socially rational 1-deals converging to a maximal Chevaleyre proportional allocation [10].

Corollary 7. *Under the utilitarian social welfare, let A_0 be an initial allocation. Consider also additive valuation functions. Then, if A_0 is not maximal, there exists a sequence of socially rational 1-deals converging to a maximal allocation that is Chevaleyre proportional.*

Proof. By Corollary 6 it is known that considering the utilitarian social welfare and under additive valuation functions, it is possible to find a sequence of socially rational 1-deals converging to a maximal allocation. Moreover, since this allocation is maximal, Theorem 4 guarantees that this allocation is also Chevaleyre proportional as required. □

4.2.2 Path length

Previous theorems have stated that there exist sequences of deals converging to allocations with maximality and fairness criteria. The question that now arises is: how many of such deals as maximum will be required to reach those allocations? This question is answered by simply considering the set of all possible allocations $\mathcal{N}^{\mathcal{M}}$. In fact, there are at most n^m possible allocations and hence there are at most $n^m - 1$ possible deals.

Moreover, if social rationality is also required for deals, Chevaleyre et al. [10] has shown that it is possible to find valuation functions and an initial allocation that guarantees the existence of a sequence of socially rational deals converging to a maximal Chevaleyre proportional allocation in exactly $n^m - 1$ steps.

Lemma 26. *Considering the utilitarian social welfare, it is possible to find additive valuation functions for each agent and a sequence of deals $(\delta_i)_{i=1}^r$ meeting the following properties*

- *Deals in $(\delta_i)_{i=1}^r$ are socially rational.*
- *$A_r \in MSW_u$.*
- *$r = n^m - 1$.*
- *A_r is Chevaleyre proportional.*

Proof. First of all, let us construct the additive valuation functions of each agent as

$$v_i(S) = \left(\sum_{g_k \in S} v_1(g_k) \right) * (2^m)^{i-1}, \text{ where } v_1(g_k) = 2^{k-1} \tag{4.2}$$

Particularly, notice that valuation functions defined in this way give us different valuations for the sets containing only a single good. This happens since $v_i(g_k) = v_1(g_k) * (2^m)^{i-1}$ expressed as a binary number shows to which agent i the good g_k has been allocated. This is possible since a number 1 in the position $m(i-1)+k$ of the binary number (counting from right to the left) expresses that agent i received the good k. Hence, since there are a finite number of agents and goods there are n^m possible different allocations with different binary numbers expressing its social welfare, This implies that allocations will present different social welfare values. Thence, it is possible to arrange allocations linearly according to their social welfare values. Hence, let A_0 be the allocation with the smallest social welfare, let A_1 be the allocation with the following smallest social welfare, and so on. Thence, all these allocations are the following.

$$A_0, A_1, ..., A_r, \text{ where } r = n^m. \tag{4.3}$$

Now, consider the initial allocation A_0. This allocation is not maximal since it possesses the smallest social welfare. Hence, by Theorem 3, it is possible to find a sequence of socially rational deals converging to the maximal allocation A_r. In fact, let us define the sequence of deals $(\delta_i)_{i=1}^r = \{\delta_1, \delta_2, ..., \delta_r\}$ where $\delta_i = (A_{i-1}, A_i)$ given the allocations in 4.3. Each deal in this sequence is socially rational since the social welfare is strictly increased. This sequence of deals converges to the allocation $A_r \in MSW_u$ since this is the allocation with the greatest social welfare.

Now, given that all allocations in 4.3 have different social welfare values and all of them were considered in the sequence $(\delta_i)_{i=1}^r$ it is clear that $r = n^m$. Finally, by Lemma 25, the sequence $(\delta_i)_{i=1}^r$ converges to an allocation A_r that is also Chevaleyre proportional as desired.

□

The following example shows that it is possible to find additive valuation functions for each agent and a sequence of socially rational deals converging to an allocation that is maximal. Additionally, this allocation is Chevaleyre proportional and the number of deals performed is exactly $n^m - 1$.

Example 38. *Consider the utilitarian social welfare. Also, let $\mathcal{N} = \{1, 2\}$ be a set of two agents, and let $\mathcal{M} = \{g_1, g_2, g_3\}$ be a set of three goods. Consider also valuation functions given by Equation 4.2 and illustrated in Table 4.26*

	$\{g_1\}$	$\{g_2\}$	$\{g_3\}$
v_1	1	2	4
v_2	8	16	32

Table 4.26: Valuations of agents 1 and 2 for Example 38.

Then, all the possible valuations given by all possible ways of allocating three goods for two agents are expressed in Table 4.27.

Thence, all the possible partitions of goods for two agents (allocations) are given in Table

4.28.

Hence, the valuations of the partitions are given in Table 4.29.

g_1	g_1	g_2
1	2	4
1	2	32
1	16	4
1	16	32
8	2	4
8	2	32
8	16	4
8	16	32

Table 4.27: Different ways of allocating goods for agents in $\mathcal{N}$.

	A_0	A_1	A_2	A_3	A_4	A_5	A_6	A_7
1	$\{g_1, g_2, g_3\}$	$\{g_2, g_3\}$	$\{g_1, g_3\}$	$\{g_3\}$	$\{g_1, g_2\}$	$\{g_2\}$	$\{g_1\}$	$\emptyset$
2	$\emptyset$	$\{g_1\}$	$\{g_2\}$	$\{g_1, g_2\}$	$\{g_3\}$	$\{g_1, g_3\}$	$\{g_2, g_3\}$	$\{g_1, g_2, g_3\}$

Table 4.28: Possible allocations for agents in $\mathcal{N}$ for Example 38.

	A_0	A_1	A_2	A_3	A_4	A_5	A_6	A_7
v_1	$1+2+4$	$2+4$	$1+4$	4	$1+2$	2	1	0
v_2	0	8	16	$8+16$	32	$8+32$	$16+32$	$8+16+32$
sw_u	7	14	21	28	35	42	49	56

Table 4.29: Valuations of allocations in Table 4.28 for agents in $\mathcal{N}$.

Then, given the initial allocation A_0, the sequence of deals $(\delta_i)_{i=1}^{r} = \{\delta_1, \delta_2, ..., \delta_r\}$, where $r = n^m$ and $\delta_i = (A_{i-1}, A_i)$ visits all allocations and converges to a maximal Chevaleyre proportional allocation; this is achieved by Theorem 3 and Lemma 25.

Now that the proportionality criterion has been analyzed considering compensation functions, it is also relevant to study envy-freeness in this same context. In this sense, [10] has studied the envy-freeness fairness criterion considering compensation values. Hence, first, it is necessary to define envy-freeness considering compensation functions. Formally,

Definition 33 (Chevaleyre envy-freeness)**.** *An allocation A is called Chevaleyre envy-free if*

$$u_i(A(i), \alpha_i) \geq u_i(A(j), \alpha_j),$$

for $i, j \in \mathcal{N}$. Where u_i is a compensated valuation function for agent i.

Similar to the proportionality compensation function obtained through the Knaster process in Definition 31, a new compensation function will be defined. This new function considers the utilitarian social welfare of an allocation and also the valuations of every agent and will be called the *envy-freeness compensation function*

Definition 34. *Let the envy-freeness compensation function* $\alpha_i : 2^{\mathcal{M}} \longrightarrow \mathbb{R}$ *for an allocation* A *be defined as*

$$\alpha_i(A(i)) = v_i(A(i)) - \frac{sw_u(A)}{n}.$$

Proposition 2. *Notice that*

$$\sum_{i \in \mathcal{N}} \alpha_i(A(i)) = 0,$$

where α *is the envy-freeness compensation function.*

Similar to Theorem 4, now consider the envy-freeness compensation functions. In this case, if these valuation functions are considered, it is true that under supermodular valuation functions, a utilitarian maximal allocation is Chevaleyre envy-free. Formally,

Theorem 5. *Under the utilitarian social welfare, consider supermodular valuation functions such that* $v_i(\emptyset) = 0$ *for* $i \in \mathcal{N}$. *Also, let* A^* *be a maximal allocation. If valuations of agents over bundles allocated by* A^* *are compensated by the envy-freeness compensation functions, then* A^* *is a Chevaleyre envy-free allocation. That is,*

$$u_i(A^*(i), \alpha_i) \geq u_i(A^*(j), \alpha_j),$$

for any $i, j \in \mathcal{N}$.

Proof. First, let us show that under supermodular valuation functions, given a maximal allocation A^* it is true that

$$v_j(A^*(j)) \geq v_i(A^*(j)).$$

For the seek of contradiction, let us suppose that $v_j(A^*(j)) < v_i(A^*(j))$ holds. This means that giving the bundle $A^*(j)$ to agent i is a better option than assigning $A^*(j)$ to j. However, since A is maximal, this implies that if agent i receives $A^*(j)$, its valuation will decrease. Hence, $v_i(A^*(i) \cup A^*(j)) < v_i(A^*(i)) + v_i(A^*(j))$. But this last assertion contradicts the fact that v_i is a supermodular valuation function. Hence, it is true that $v_j(A^*(j)) \geq v_i(A^*(j))$. Now, given the envy-freeness compensation function α,

$$\begin{aligned}
u_i(A^*(i), \alpha_i) &= v_i(A^*(i)) - \alpha_i \\
&= v_i(A^*(i)) - v_i(A^*(i)) + sw_u(A^*)/n \\
&= v_j(A^*(j)) - v_j(A^*(j)) + sw_u(A^*)/n \\
&\geq v_i(A^*(j)) - v_j(A^*(j)) + sw_u(A^*)/n \\
&= v_i(A^*(j) - \alpha_j \\
&= u_i(A^*(j), \alpha_j).
\end{aligned}$$

Hence, it is true that $u_i(A^*(i), \alpha_i) \geq u_i(A^*(j), \alpha_j)$.

□

The following example shows that under supermodular valuation functions such that $v_i(\emptyset) = 0$ for $i \in \mathcal{N}$ and considering the utilitarian social welfare, a maximal allocation is also Chevaleyre envy-free.

Example 39. *Let $\mathcal{N} = \{1, 2, 3\}$ be a set of three agents and $\mathcal{M} = \{g_1, g_2\}$ be a set of two goods. Then, consider supermodular valuation functions such that $v_i(\emptyset) = 0$ for $i \in \mathcal{N}$ as given in Table 4.30.*

	$\{g_1\}$	$\{g_2\}$	$\{g_1, g_2\}$
v_1	2	4	7
v_2	1	2	3
v_3	5	2	11

Table 4.30: Supermodular valuations of agents for Example 39.

Now, choose the utilitarian maximal allocation A^* as $A^* = \langle \varnothing, \varnothing, \{g_1, g_2\} \rangle$. The utilitarian social welfare of A^* is 11. This allocation is not envy-free in the usual sense since $v_2(A^*(2)) = 0 \ngeq 3 = v_2(A^*(3))$. Hence, it is necessary to find compensation functions in such way that A^* is a Chevaleyre envy-free allocation.

Hence, by computing the envy-free compensation functions α_i as explained before,

$$\alpha_1(A^*(1)) = v_1(A^*(1)) - sw_u(A^*)/n = -11/3,$$
$$\alpha_2(A^*(2)) = v_2(A^*(2)) - sw_u(A^*)/n = -11/3,$$
$$\alpha_3(A^*(3)) = v_3(A^*(3)) - sw_u(A^*)/n = 11 - 11/3.$$

Now, the compensated valuation functions for each agent are given by,

$$u_1(A^*(1), \alpha_1) = v_1(A^*(1)) - \alpha_1(A^*(1)) = 11/3,$$
$$u_2(A^*(2), \alpha_2) = v_2(A^*(2)) - \alpha_2(A^*(2)) = 11/3,$$
$$u_3(A^*(3), \alpha_3) = v_3(A^*(3)) - \alpha_3(A^*(3)) = 11/3.$$

These utility functions meet the criteria to make A^* a maximal and Chevaleyre envy-free allocation. That is $u_i(A^*(i), \alpha_i) \geq u_i(A^*(j), \alpha_j)$ for any $i, j \in \mathcal{N}$. In fact $u_i(A^*(j), \alpha_j)$ is at most 11/3.

The globally uniform payment function (GUPF) appears in the definition of a payment scheme using the envy-freeness compensation function. This payment function has been studied by previous authors such as [20, 10].

Definition 35 (The *GUPF* payment scheme)**.** *Given any deal δ_k in a negotiation $(\delta_i)_{i=1}^r$, the payment associated to δ_k is,*

$$p_{\delta_k}(i) = \alpha_i(A_k(i)) - \alpha_i(A_{k-1}(i))$$

for every $i \in \mathcal{N}$. Here, α_i is the envy-freeness compensation function.

The fact that under supermodular valuations and considering the utilitarian social welfare, a maximal allocation will be Chevaleyre envy-free is interesting. Moreover, this result can be generalized and studied in the context of negotiations. Hence, the following lemma will show that it is possible to find a sequence of socially rational deals converging to a maximal Chevaleyre envy-free allocation [10].

Lemma 27. *Under the utilitarian social welfare, consider supermodular valuation functions such that $v_i(\emptyset) = 0$ for $i \in \mathcal{N}$. Then, there exists a sequence of socially rational deals converging to a maximal and Chevaleyre envy-free allocation.*

Proof. By Corollary 5, it is clear that under the utilitarian social welfare there exists a sequence of socially rational deals converging to a maximal allocation A^*. Now, notice that supermodular valuations are considered such that $v_i(\emptyset) = 0$ for agents $i \in \mathcal{N}$. Also, since A^* is maximal, Theorem 5 guarantees that this allocation is Chevaleyre envy-free as required. □

The following example shows that under the utilitarian social welfare and considering supermodular valuation functions such that $v_i(\emptyset) = 0$ for agents $i \in \mathcal{N}$, there exist a sequence of socially rational deals converging to a Chevaleyre envy-free allocation. This allocation is also maximal.

Example 40. *Let $\mathcal{N} = \{1, 2, 3\}$ be a set of three agents and $\mathcal{M} = \{g_1, g_2\}$ be a set of two goods. Then, consider supermodular valuation functions such that $v_i(\emptyset) = 0$ for agents $i \in \mathcal{N}$ as in Table 4.30 from Example 39. Consider also the allocations as in Table 4.31 and its valuations given in Table 4.32. The utilitarian social welfare values are also included in this last table.*

Now, compute the envy-free compensation functions as in Table 4.33

Next, compute the compensated valuation functions u_i, as in Table 4.34.

Consider the deals $\delta_1 = (A_0, A_1)$ and $\delta_2 = (A_1, A_2)$. Then, the sequence of deals $(\delta_i)_{i=1}^2 = \{\delta_1, \delta_2\}$ where payments follow the GUPF payment scheme reach an allocation that is maximal and is Chevaleyre envy-free. In fact, payments are given in Table 4.35. Notice that the last allocation in this sequence, has been proven to be Chevaleyre envy-free in Example 39.

	A_0	A_1	A_2
1	$\{g_1, g_2\}$	$\{g_2\}$	$\varnothing$
2	$\varnothing$	$\varnothing$	$\varnothing$
3	$\varnothing$	$\{g_1\}$	$\{g_1, g_2\}$

Table 4.31: Allocations of goods for Example 40.

	A_0	A_1	A_2
v_1	7	4	0
v_2	0	0	0
v_3	0	5	11
sw_u	7	9	11

Table 4.32: Valuations of allocations in Table 4.31 and utilitarian social welfare.

	A_0	A_1	A_2
α_1	$7 - 7/3$	$4 - 9/3$	$-11/3$
α_2	$-7/3$	$-9/3$	$-11/3$
α_3	$-7/3$	$5 - 9/3$	$11 - 11/3$

Table 4.33: Envy-freeness compensation functions.

	A_0	A_1	A_2
u_1	$7/3$	3	$11/3$
u_2	$7/3$	3	$11/3$
u_3	$7/3$	3	$11/3$

Table 4.34: Compensated valuation functions for Example 40.

	δ_1	δ_2
$p(1)$	$-11/3$	$-14/3$
$p(2)$	$-2/3$	$-2/3$
$p(3)$	$13/3$	$16/3$

Table 4.35: Payments of three agents for Example 40.

In this subsection, the Knaster procedure has been studied. By this methodology, the proportionality compensation function is found. This compensation function will permit us to find proportionally fair allocations under a particular context presented by the Chevaleyre proportionality criterion. Moreover, these allocations are also relevant since they are utilitarian maximal allocations. Additionally, under the same context in which Chevaleyre proportional allocations are studied, it has been shown that there is a sequence with at most $n^m - 1$ deals converging to a maximal Chevaleyre proportional allocation. In fact, additive valuation functions and socially rational deals were constructed to prove the existence of such a sequence of deals.

Finally, similarly than with the proportionality criteria, a new fairness criterion has been studied. This new criterion considers the envy-freeness characterization of fairness studied in Chapter 3. Then, it has also been showed that it is possible to find a sequence of socially rational deals converging to a Chevaleyre envy-free allocation. As an important note, the existence of such sequence of deals can only be guaranteed if valuation functions are supermodular. Similarly to the results given for the Chevaleyre proportional criterion, the utilitarian characterization of the social welfare is considered for the results based on the Chevaleyre envy-free criterion.

4.3 Conclusions

In this chapter, a procedure to exchange goods has been presented. This process is called the *deal* and is relevant due to its applicability to the real life. In this mechanism, it will be assumed that agents act rationally. That is, the behaviour of agents is commanded by the seek of a personal or a common improvement. Also, compensated valuation functions were presented and their usefulness to reach allocations with some fairness criteria has been demonstrated. Among the most important results presented in this chapter, the following are highlighted.

- Deals, such as 1-deals, swap deals, cluster deals, combined deals and equitable deals together with negotiations have been studied.
- Different criteria of rationality of agents such as social, individual and cooperative rationality have been presented. Also, the implications among them were proposed and proved.
- It has been shown that given an initial allocation A_0, if it is not maximal, then there exist a sequence of socially rational deals converging to a maximal allocation. Also, some lemmas and corollaries based on this result were analyzed.
- The Knaster procedure was studied as a way of finding compensation functions. These functions guarantee the existence of a sequence of socially rational deals converging to a maximal allocation that is Chevaleyre proportionally fair. Also, the upper bound $n^m - 1$ for the number of deals in the previously mentioned sequence is established.

- Moreover, another existence result has been studied using a different criterion of fairness, the Chevaleyre envy-freeness characterization. In this context, the envy-freeness compensation function has been used to guarantee the existence of a sequence of socially rational deals converging to a maximal Chevaleyre envy-free allocation.

The deals, negotiations, and new characterizations of the fairness principle were presented and illustrated using examples. Also, the theorems, lemmas, and corollaries that guarantee the properties of some allocations and the existence of particular sequences of deals were proved and also exemplified.

Chapter 5

Conclusions

Fair allocation of goods problem is relevant due to the immense applicability in real-life situations. Researchers have been studying it from different approaches such as social sciences, economy, computer science, and mathematics. In this work, the fair allocation of goods problem is studied from a mathematical point of view and focusing on indivisible goods. Given that most of the current research is focused on results concerning additive valuation functions, this work proposes similar findings using also superadditive and subadditive valuation functions. Some of the most relevant contributions of this work are the following.

1. Proof of Lemma 2. A proof of the fact that under modular valuation functions such that agents value the empty set as zero, the valuation of a bundle is equal to the sum of valuations of goods in the given bundle has been proposed. This allows us to use this result in the coming theorems and lemmas with a strong foundation.

2. Lemma 3. It has been established and proved that under supermodular valuation functions such that agents value the empty set as zero, the valuation of a bundle is as a minimum the sum of valuations of goods in the given bundle. This result allows us to understand the relationship between the valuation of a bundle and the sum of the goods in the bundle under supermodular valuation function.

3. Proof of Lemma 4. A proof of the fact that under submodular valuation functions such that agents value the empty set as zero, the valuation of a bundle is at most the sum of valuations of goods in the given bundle. This allows us to use this result in the coming lemmas with a strong foundation.

4. Proof of Lemma 5. A proof of the fact that under modular valuation functions such that agents value the empty set as zero, the valuation of a bundle of goods with exception of a second bundle of goods contained in the first bundle is equal to the valuation of the first bundle minus the valuation of the second bundle of goods. Additionally, it has been proved that if the valuation functions are also non-negative, then the function is monotonic. This allows us to use these results in the coming lemmas with a strong foundation.

5. Lemma 6. It has been proposed and shown that under supermodular valuation functions such that agents value the empty set as zero, the valuation of a bundle of goods with exception of a second bundle of goods contained in the first bundle is not always equal, less or greater than the valuation of the first bundle minus the valuation of the second bundle of goods. Additionally, it has been proved that if the valuation functions are also non-negative, then the function is monotonic. This allows us to use these results in the coming lemmas with a strong foundation.

6. Lemma 7. It has been established and shown that under submodular valuation functions such that agents value the empty set as zero, the valuation of a bundle of goods with exception of a second bundle of goods contained in the first bundle is not always equal, less or greater than the valuation of the first bundle minus the valuation of the second bundle of goods. Additionally, it has been proved that if the valuation functions are also non-negative, then the function is not always monotonic. These results were proven using counterexamples and allow us to use them in the coming lemmas with a strong foundation.

7. Proof of Theorem 1. A proof of the fact that under modular valuation functions such that agents value the empty set as zero, allocating goods to agents that maximize them will result in a maximal utilitarian allocation. This result is relevant since there already exist polynomial-time algorithms that allocate goods to agents that maximize them.

8. Lemma 10. It has been proposed and proven that if valuation functions are positive, then Nash maximal allocations are Pareto optimal allocations. This result is relevant since it establishes that similarly to utilitarian maximal allocations, Nash maximal allocations can also be Pareto optimal under the positiveness condition of valuation functions.

9. Lemma 11. It has been established and proven that under monotone valuation functions, proportionality implies proportionality up to one good. This result is relevant since considers the monotone valuation function, which is a more general function than the additive functions frequently used in the literature.

10. Lemmas 12, 13 and Corollary 2. It has been stated and proven that under additive valuation functions, envy-free allocations are also envy-free up to the least positively valued good which in turn are also envy-free up to one good. This last implication is also true for monotone valuation functions. As a consequence, these results establish that envy-free allocations are also envy-free up to one good under additive valuation functions. These results are relevant since they consider the monotone valuation function, which is a more general function than the additive functions frequently used in the literature.

11. Corollary 3. It has been proposed and shown that if additive valuation functions are considered, then proportional up to one good allocations do always exist. This result is important since it shows that under certain conditions, a fairness criterion is guaranteed to exist.

12. Proofs of Lemmas 17 and 18. The proofs of Lemmas 17 and 18 have been presented. The proofs of these lemmas are necessary to understand that under additive valuation functions, there exists a set of maximal Nash allocations that are Pareto optimal and envy-free up to one good.

13. Corollary 4. It has been proposed and proven that under additive valuation functions, there exists a set of maximal Nash allocations that are Pareto optimal and proportional up to one good. This result is relevant since shows that under particular conditions, Nash maximal allocations are not only envy-free up to one good but also proportionally fair up to one good.

14. Theorem 3. It has been shown that given an initial non-maximal allocation, it is possible to find a sequence of socially rational deals converging to a maximal allocation. This result is relevant since it is a generalization of a similar theorem which only considers the utilitarian social welfare.

In future work, it would be interesting to find specific procedures to find allocations with fairness and efficiency properties. Up to now, only the existence of sequences of deals converging to allocations with fairness property such as Chevaleyre envy-freeness and Chevaleyre proportionality are guaranteed to exist. Nonetheless, a protocol according to which agents reach such allocations is still an interesting problem to think about.

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