UNIVERSIDAD DE INVESTIGACIÓN DE TECNOLOGÍA EXPERIMENTAL YACHAY

Escuela de Ciencias Físicas y Nanotecnología

TÍTULO: Study of hydrogenic model of dyons with zero spin

Trabajo de integración curricular presentado como requisito para la obtención del título de Físico

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DEDICATION

"In loving memory of my father"

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"To all those who supported me to make this work successful. In particular, to my family for accompanying me in this process, and to my advisor for her help, patience and dedication".

Resumen

La existencia de monopolos magnéticos tiene importancia teórica en el Electromagnetismo. Estas nuevas fuentes de magnetismo generalizan las ecuaciones de Maxwell y las vuelve simétricas con respecto a los campos y sus fuentes. Además, bajo consideraciones de la Mecánica Cuántica, los monopolos permiten la cuantización de la carga eléctrica, un hecho experimental. Extendiendo el concepto de particula eléctrica, se introducen las partículas con carga eléctrica y magnética llamadas dyones.

En este proyecto de tesis se presenta la generalización de un átomo de hidrógeno de espin cero a un modelo relativista atómico de hidrógeno con dyones usando la ecuación de Klein-Gordon. Se muestra la derivación de la ecuación de Klein Gordon para la partícula de movimiento relativo. Además, se calculan las soluciones analíticas de la ecuación. Se obtiene el espectro discreto de la energía, y la densidad de carga de la partícula orbitante. Para un sistema de cargas magnéticas y eléctricas positivas en el núcleo y negativas para la partícula orbitante, y considerando los primeros valores permitidos de N y l, se encontró que el átomo de dyon actúa con mayor fuerza de interacción entre las cargas del núcleo y la partícula secundaria a comparación del átomo estándar . Esto se observó al comparar la distancia entre el núcleo y las concentraciones de densidad de carga del átomo de dyon con el átomo de hidrógeno relativista de espin cero.

Palabras Clave: ecuación de Klein-Gordon, ecuaciones de Maxwell, monopolos magnéticos, dyones.

Abstract

The existence of magnetic monopoles has theoretical importance in Electromagnetism. These new sources of magnetism generalize Maxwell's equations and make them symmetric with respect to fields and its sources. Also, under considerations of Quantum Mechanics, magnetic monopoles allow the quantization of electric charge, an experimental fact. Extending the concept of electric particle, electrically and magnetically charged particles called dyons are introduced.

This thesis project presents the generalization of a zero spin hydrogen atom to a relativistic atomic model of hydrogen with dyons using the Klein-Gordon equation. The derivation of the Klein Gordon equation for the particle of relative motion is shown. In addition, the analytical solutions of the equation are calculated. The discrete spectrum of the energy is obtained, and the charge density of the orbiting dyon. For a system of positive magnetic and electric charges in the nucleus and negative for the orbiting particle, and considering the first allowed values of N y l, it was found that the dyon atom acts with a greater force of interaction between the charges of the nucleus and the secondary particle compared to the standard atom. This was observed by comparing the distance between the nucleus and charge density concentrations from the dyon atom with the relativistic zero spin hydrogen atom.

Keywords: Klein-Gordon equation, Maxwell equations, magnetic monopoles, dyons.

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Chapter 1

Introduction

Electromagnetism is one of the most studied theories. Nevertheless, there are two issues that arise from this branch of Physics. The first one is the symmetry of Maxwell's equations in vacuum, and the loss of this one when electric charges are present. The second one consists in that electric charge appears in integer multiple of the elementary electric charge in materials, which has been measured experimentally.[7] In 1931, Dirac published an article in which he was seeking for an explanation of the quantization of electric charge using a quantum-mechanical argument. He founded that the electric charge can be quantized if magnetic monopoles exists.[10,21] In this way, both issues could be solved: symmetry of Maxwell's equations and charge quantization. However, the experimental detection of magnetic monopoles has not been successful.[19] Some feautures have been studied from the interaction between magnetic monopoles and charged particles.[8,9]

In 1969, Julian Schwinger proposed a new type of particle with both electric and magnetic charge without violating any of the results given by Dirac.[25] This dual charged particle is called dyon, and several features of this particle have been theoretically researched. In particular, the interaction between a dyon and another charged particle in unbound states such as scattering can be found in the literature.[2,4,26]

On the other hand, bound states have been used to explore other systems such as the hydrogen atom under relativistic and non-relativistic theoretical frameworks. In such systems, energy, probability or charge density, as well as angular momentum and other physical magnitudes are calculated and analyzed.[6,30] Also, another atomic systems have been studied in Nature. Specifically, exotic atoms are atoms in which one of the electrons has been replaced by a negatively charged heavy particle such as muon, K-meson, pion, Σ-hyperon, etc.[15] In this research area, the description of phenomena at high energies requires the investigation of relativistic wave equations. In particular, the Klein- Gordon equation correctly describes spinless relativistic particles like pions.

Dyon systems have been investigated about bound states to model hydrogenic atomic systems with spin[22] and using a non-relativistic theoretical frameworks.[23,29] The generalization of the spinless hydrogen atom to spinless relativistic hydrogenic atom with dual charged particles is presented in this thesis project.

1.1 Problem Statement

A new property of particles is founded in dyons. This new magnetic charge coupled to electric particles displays new features in physical systems. Several properties can be explored in two body problems. The present work shows the dynamics of binary systems of dyons under the relativistic quantum mechanics context.

1.2 General and Specific Objectives

The main objective of this thesis project is to analyze the energy spectrum and charge density of a hydrogenic atom composed of electric and magnetic charges with zero spin using the Klein-Gordon equation. In order to achieve this objetive, it is necessary to satisfy the following specific objectives:

- To derive the Klein Gordon equation for a system of two dyons that interact through electromagnetic fields.
- To obtain the bound states of the Klein Gordon equation for a system of two dyons.
- To explore the behavior of energy for nuclei of different electric and magnetic charges.
- To find the radial charge density using the radial solution of the Klein Gordon equation.
- To compare the energy and charge density of the system of two dyons with the standard relativistic hydrogenic atom.

Chapter 2

Methodology

2.1 Classical Description of Dyons

2.1.1 Extended Maxwell equations

Magnetic monopoles are proposed particles, which contain an isolated magnetic charge. Analogously to electric charge, the hypothetical magnetic charges has a single north pole or a single south pole. In order to explore the interaction between magnetic and electric charges, let us postulate the existence of magnetic charge and current densities ρ_m and $\mathbf{j}_m$, respectively. To consider these new sources of magnetic fields, Maxwell's equations must be modified. The extended Maxwell's equations are given by[28]

$$\nabla \cdot \mathbf{E} = \rho_e, \qquad -\nabla \times \mathbf{E} - \frac{\partial \mathbf{B}}{\partial t} = \mathbf{j_m}, \tag{2.1}$$
$$\nabla \cdot \mathbf{B} = \rho_m, \qquad \nabla \times \mathbf{B} - \frac{\partial \mathbf{E}}{\partial t} = \mathbf{j_e},$$

where the bold terms indicate vectors, the Heaviside-Lorentz units have been considered and the speed of light c as the unit for natural units. So that, the magnetic field is now generated not only by moving electric charges, but also by magnetic monopoles, whereas the electric field is now generated not only by electric charges, but also by moving magnetic monopoles.

The equations (2.1) can be written in covariant formulation, if we define the electromagnetic field tensor ($F^{\mu\nu}$) and the dual tensor ($\tilde{F}^{\mu\nu}$). The elements of the dual tensor are obtained from ($F^{\mu\nu}$) by putting $\mathbf{E} \to \mathbf{B}$ and $\mathbf{B} \to -\mathbf{E}$. As a result, the covariant form of the extended Maxwell equations are[17]

$$\partial_\mu F^{\mu\nu} = j_e^\nu, \tag{2.2}$$

$$\partial_\mu \tilde{F}^{\mu\nu} = j_m^\nu, \tag{2.3}$$

where $(j_e^\mu) = (\rho_e, \mathbf{j_e})$, $(j_m^\mu) = (\rho_m, \mathbf{j_m})$ are the electric and magnetic four-currents, respectively. The covariance of an equation means that its form does not change under Lorentz transformation.It implies that various quantities such as ρ, $\mathbf{j}$, $\mathbf{E}$, $\mathbf{B}$ that enter into the Maxwell equations or the Lorentz force transform in well-defined ways under Lorentz transformations.[16]

2.1.2 Energy- momentum tensor and Lorentz Force

For an isolated system, some observable quatities are preserved during the evolution of time. In particular, the conservation of energy and momentum are required in Electrodynamics. Then it is fundamental that, also in the presence of magnetic charges, exists a conservation of energy and momentum. These laws can be introduced into a generic relativistic theory through the conservation of the energy- momentum tensor ($T^{\mu\nu}$).[17] That is,

$$\partial_\mu T^{\mu\nu} = 0. \tag{2.4}$$

Since the standard energy momentum is invariant under a duality transformation, then the form of the standard energy momentum tensor can be taken for the generalized case of magnetic charges:[17]

$$T^{\mu\nu} = T^{\mu\nu}_{em} + T^{\mu\nu}_{p}, \tag{2.5}$$

where

$$T^{\mu\nu}_{em} = F^{\mu\alpha}F_{\alpha}^{\ \nu} + \frac{1}{4}\eta^{\mu\nu}F^{\alpha\beta}F_{\alpha\beta}, \qquad T^{\mu\nu}_{p} = \sum_n m_n \int u^\mu_n u^\nu_n \delta^4(x-y_n)ds_n.$$

Here, $\eta^{\mu\nu}$=diag$(1,-1,-1,-1)$ is the component of the Minkowski metric, u^μ_n is the four velocity component, ds_n is the proper time and y_n is the four position component for a particle n. The energy momentum tensor contains two contributions. The first one is because of the electromagnetic field and the second contribution is due to particles. To find out under wich conditions the energy momentum tensor remains conserved, the four divergence of $T^{\mu\nu}$ is computed [1†]:

$$\partial_\mu T^{\mu\nu} = \sum_n \int \left[\frac{dp^\mu_n}{ds_n} - (e_n F^{\mu\alpha} + g_n \tilde{F}^{\mu\alpha})u_{n\alpha}\right]\delta^4(x-y_n)ds_n = 0, \tag{2.6}$$

$$\frac{dp^\nu_n}{ds_n} - (e_n F^{\nu\alpha} + g_n\tilde{F}^{\nu\alpha})u_{n\alpha} = 0, \tag{2.7}$$

where the four-momentum $p^\mu_n = mu^\mu_n$ has been considered for the n particle. The equation (2.6) implies the new generalized Lorentz force (2.7). This equation can be written in the three-dimensional notation and, reinserting the speed of light c it gives (2.8). The new Lorentz force describes the interaction between external electromagnetic fields and a particle with electric charge e_n and magnetic charge g_n.

$$\frac{d\mathbf{p}_n}{dt} = e_n\left(\mathbf{E} + \frac{\mathbf{v}_n}{c}\times\mathbf{B}\right) + g_n\left(\mathbf{B} - \frac{\mathbf{v}_n}{c}\times\mathbf{E}\right). \tag{2.8}$$

2.1.3 Electromagnetic fields for dyons

The Liénard–Wiechert potentials describe the classical electromagnetic effect of a point electric charge in arbitrary motion in terms of a vector potential and a scalar potential. The expressions for the standar fields $\mathbf{E}$ and $\mathbf{B}$ can be derived from the potentials using the relations $\mathbf{E} = -\nabla A^0 - c^{-1}\partial\mathbf{A}/\partial t$ and $\mathbf{B} = \nabla\times\mathbf{A}$. Taking in account low velocities and a Taylor series expansion in powers of $1/c$, the fields are such that[12, 17]

$$\begin{aligned}\mathbf{E} &= \frac{e}{4\pi}\left\{\frac{\mathbf{R}}{R^3} - \frac{1}{2c^2R}\left[\mathbf{a} + (\hat{\mathbf{R}}\cdot\mathbf{a})\hat{\mathbf{R}} + \frac{[3(\hat{\mathbf{R}}\cdot\mathbf{v})^2 - v^2]\hat{\mathbf{R}}}{R}\right]\right\} + ...\\ \mathbf{B} &= \frac{e}{4\pi c}\mathbf{v}\times\frac{\mathbf{R}}{R^3} + ...\end{aligned} \tag{2.9}$$

[1]See Appendix B

Since the generalized Maxwell Equations are invariant under the duality transformation [2†], the fields generated by a dyon (approximation to first order $1/c$) can be derived using (2.9), the duality rules $\mathbf{E} \rightarrow \mathbf{B}$, $\mathbf{B} \rightarrow -\mathbf{E}$, $e_r \rightarrow g_r$, $g_r \rightarrow -e_r$, and the linearity of the generalized Maxwell Equations . Then, the fields take the form of[5,17]

$$\begin{aligned} \mathbf{E} &= \frac{e}{4\pi}\frac{\mathbf{R}}{R^3} - \frac{g}{4\pi}\frac{\mathbf{V}}{c}\times\frac{\mathbf{R}}{R^3}, \\ \mathbf{B} &= \frac{g}{4\pi}\frac{\mathbf{R}}{R^3} + \frac{e}{4\pi}\frac{\mathbf{V}}{c}\times\frac{\mathbf{R}}{R^3}. \end{aligned} \tag{2.10}$$

2.1.4 Two body system

The study of a binary system with no external forces of two interacting point particles can be reduced to a one body problem. The task to reduce the two body problem is reached, if the relative variables are introduced. Specifically, for two arbitrary dyons with masses m_1, m_2 and charges $(e_1, g_1), (e_2, g_2)$ respectively, the relative coordinate $\mathbf{r} = \mathbf{s_2} - \mathbf{s_1}$ is defined as shown in Figure 2.1.

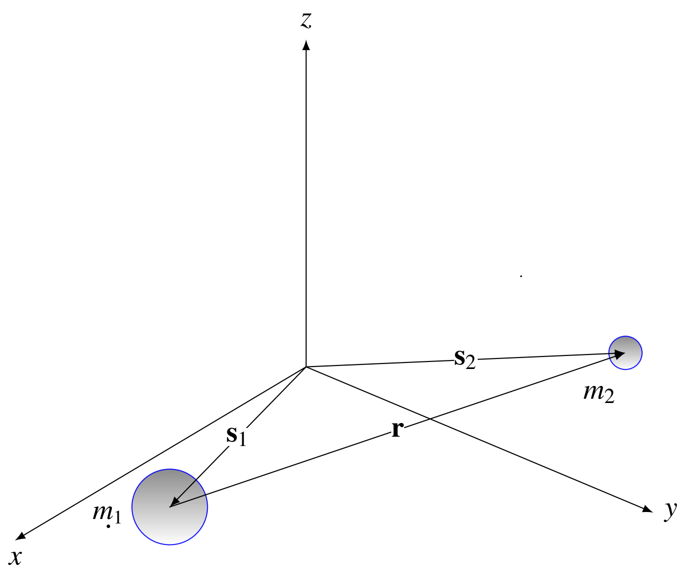


Figure 2.1: Frame of reference for two arbitrary dyons

The motion equations for a dyonic system are given by the generalized Lorentz force (2.8) for each particle. In addition, the specific form of the electric and magnetic fields are taken from (2.10).

$$\begin{aligned} \mathbf{F}_{21} = m_2\mathbf{a}_2 &= e_2\left(\mathbf{E}_1 + \frac{\mathbf{v}_2}{c}\times\mathbf{B}_1\right) + g_2\left(\mathbf{B}_1 - \frac{\mathbf{v}_2}{c}\times\mathbf{E}_1\right), \\ &= \frac{e_1e_2 + g_1g_2}{4\pi}\frac{\mathbf{r}}{r^3} + \frac{e_2g_1 - e_1g_2}{4\pi}\frac{\mathbf{v}}{c}\times\frac{\mathbf{r}}{r^3}, \\ \mathbf{F}_{12} = m_1\mathbf{a}_1 &= e_1\left(\mathbf{E}_2 + \frac{\mathbf{v}_1}{c}\times\mathbf{B}_2\right) + g_1\left(\mathbf{B}_2 - \frac{\mathbf{v}_1}{c}\times\mathbf{E}_2\right), \\ &= -\frac{e_1e_2 + g_1g_2}{4\pi}\frac{\mathbf{r}}{r^3} - \frac{e_2g_1 - e_1g_2}{4\pi}\frac{\mathbf{v}}{c}\times\frac{\mathbf{r}}{r^3}. \end{aligned} \tag{2.11}$$

[2] See Appendix A

2.1.4.1 Relative motion

The Newton's third law holds for the dyon interaction with the fields (2.10). That is, $\mathbf{F}_{21} = -\mathbf{F}_{12}$. The consequence is the uniform motion of the center of mass: $\mathbf{a}_{CM} = \mathbf{0}$. Thus, the kinetic energy of the center of mass is constant. The system of equations (2.11) under the relative coordinates describe the dynamics of a fictitious particle with mass m , such that the motion equation is given by[17]

$$m\frac{d\mathbf{v}}{dt} = \frac{e_1e_2 + g_1g_2}{4\pi}\frac{\mathbf{r}}{r^3} + \frac{e_2g_1 - e_1g_2}{4\pi}\frac{\mathbf{v}}{c} \times \frac{\mathbf{r}}{r^3}, \qquad m = \frac{m_1m_2}{m_1 + m_2}.$$

Or,

$$m\frac{d\mathbf{v}}{dt} = \frac{q}{4\pi}\frac{\mathbf{r}}{r^3} + \frac{g}{4\pi}\frac{\mathbf{v}}{c} \times \frac{\mathbf{r}}{r^3} = \mathbf{E} + \frac{\mathbf{v}}{c} \times \mathbf{B}, \tag{2.12}$$

where $q = e_1e_2 + g_1g_2$ and $g = e_2g_1 - e_1g_2$.

The equation (2.12) corresponds to the standard Lorentz force for a electric charge equal to 1. That is, the system can be thought as an eletric charge under the influence of the electric field $\mathbf{E}$ generated by a fictitious static particle with charge q, and $\mathbf{B}$ is the field created by the same imaginary particle with also magnetic charge g. This Lorentz force admits a Lagrange description with the so called generalized force in terms of the electric potential A^0 and the vector potential $\mathbf{A}$ such that $\mathbf{E} = \nabla A^0 - c^{-1}\partial\mathbf{A}/\partial t$ and $\mathbf{B} = \nabla \times \mathbf{A}$. In particular, the dynamics of the relative dyon ruled by (2.12) can be promoted to the Lagrange formalism with the force such that[11]

$$F_j = \frac{d}{dt}\left(\frac{\partial U}{\partial \dot{q}_j}\right) - \frac{\partial U}{\partial q_j}, \qquad U = A^0 - \frac{\mathbf{v}}{c} \cdot \mathbf{A}. \tag{2.13}$$

$$\mathcal{L} = T - U = \frac{1}{2}mv^2 - A^0 + \frac{\mathbf{v}}{c} \cdot \mathbf{A}. \tag{2.14}$$

From (2.14), the canonical momentum is obtained

$$p_i = \frac{\partial \mathcal{L}}{\partial \dot{q}_i} = mv_i + \frac{A_i}{c}.$$

2.1.5 Vector potential

A static magnetic monopole of charge g, located at the origin of coordinates, generates a radial field $\mathbf{B} = g\mathbf{x}/4\pi r^3$ which implies $\nabla\cdot\mathbf{B} = g\delta^3(\mathbf{x})$. The non zero divergence of $\mathbf{B}$ shows that there is no vector potential $\mathbf{A}$ such that $\mathbf{B} = \nabla \times \mathbf{A}$ with domain $\mathbb{R}^3$.

The task for searching a vector potential for the magnetic monopole can be done, if a magnetic dipole is considered. The system of the magnetic monopole is equivalent to a tightly wound solenoid that stretches off to infinity, or as a charge g at the end of a line of magnetic dipoles, as shown in Figure 2.2. Any of these equivalent configurations can be described considering the vector potential of a magnetic dipole $\mathbf{A}(\mathbf{x}) = [\mathbf{m} \times (\mathbf{x} - \mathbf{x}')] / 4\pi|\mathbf{x} - \mathbf{x}'|^3$. Here, $\mathbf{m}$ is the magnetic dipole moment, $\mathbf{x}$ is the field point, and $\mathbf{x}'$ marks the source point. So that, an element vector potential for a magnetic dipole element at $\mathbf{x}'$ is[16]

$$d\mathbf{A} = -d\mathbf{m} \times \nabla(1/4\pi|\mathbf{x} - \mathbf{x}'|), \qquad d\mathbf{m} = gd\mathbf{l}'. \tag{2.15}$$

Therefore for a string of dipoles or solenoid whose location is determined by the string L, the total vector potential is

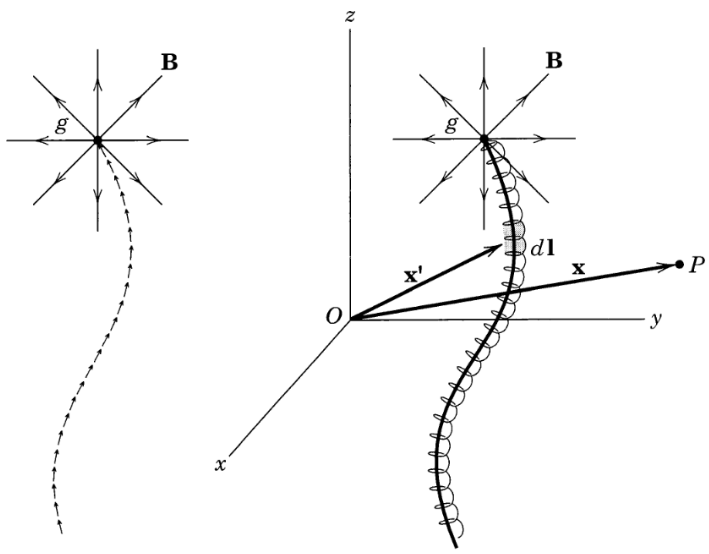


Figure 2.2: Two equivalent representations of magnetic monopoles, taken from [16].

$$\mathbf{A}_L = -\frac{g}{4\pi}\int_L d\mathbf{l}' \times \nabla\left(\frac{1}{|\mathbf{x}-\mathbf{x}'|}\right). \tag{2.16}$$

As a particular application, the case in which the thin solenoid lays along the negative z-axis and the magnetic monopole is at the origin, gives the line element $d\mathbf{l} = dz'\hat{\mathbf{z}}$, and the corresponding potential[14,27]

$$\mathbf{A}_L = \frac{g}{4\pi}\frac{1-\cos(\theta)}{r\sin(\theta)}\hat{\phi}, \tag{2.17}$$

where r=$|\mathbf{x}-\mathbf{x}'|$ and $\sin(\theta) \neq 0$. The curl of the vector potential (2.17) defines the magnetic field $\mathbf{B} = g\mathbf{r}/4\pi r^3$ in all space except at $r = 0$ and except along the negative semi-axis z. If $\sin(\theta) \to 0$ is considered, the curl of the vector potential (2.16) with the solenoid in the negative z axis is such that[14]

$$\nabla \times \mathbf{A}_L = \frac{g}{4\pi r^3}\mathbf{r} + g\delta(x)\delta(y)\Theta(-z)\hat{\mathbf{z}}, \tag{2.18}$$

where $\delta(x), \delta(y)$ are Dirac delta functions and $\Theta(-z)$ is the Heaviside step function. The singular term $\mathbf{B}_{\text{string}} = g\delta(x)\delta(y)\Theta(-z)\hat{\mathbf{z}}$ is known as a **Dirac string**, and it corresponds to infinite values along the negative semi-axis z. Then, the potential in (2.18) corresponds not to a single isolated magnetic pole, but rather to a semi-infinite and infinitely thin solenoid.The position of such a solenoid is not fixed and it can have any form.

2.2 Generalized Dirac Quantization Condition

2.2.1 Path integral approach

The dynamics of a quantum system is described by the Schrodinger equation. It admits a unique solution $\psi(t,\mathbf{x})$ once the initial wavefunction is known. As the equation is linear and for a time-independent Hamiltonian, the solution can have an integral representation

$$\psi(t,\mathbf{x}) = \int K(t,\mathbf{z},\mathbf{x})\psi(0,\mathbf{z})d^3z, \tag{2.19}$$

where $K(t, \mathbf{z}, \mathbf{x})$ is a complex scalar function called Schrodinger Kernel . This expression can be expressed in terms of a path integral given by the Feynman's formula

$$K(T, \mathbf{z}, \mathbf{x}) = \int_{\mathbf{z}}^{\mathbf{x}} D\mathbf{r}(t) \exp\left(\frac{i}{\hbar} \int_0^T \mathcal{L} dt\right). \tag{2.20}$$

In this expression, $D\mathbf{r}$ denotes a functional measure on the space of all paths, and the exponential term contains the classical action involving the Lagrangian of the system. For the dyon case, it is possible to consider different Dirac strings with different observers. One observer can adopte a description with a string γ' with an associated term $K'(T, \mathbf{z}, \mathbf{x})$, and a wavefunction ψ'.

The relation between the Schrodinger kernels K and K' can be found by defining a surface Σ such that the boundary is $\partial\Sigma = \gamma \cup \gamma'$,that is Σ corresponds to an infinitely extended half-plane. In the domain $\mathbb{R}^3 \backslash \Sigma$, the vector potentials and the Lagrangians differ, respectively, by the factors

$$\mathbf{A}_{\gamma'} = \mathbf{A}_{\gamma} + \nabla\Lambda, \tag{2.21}$$

$$\mathcal{L}' - \mathcal{L} = \frac{1}{c}\mathbf{v} \cdot (\mathbf{A}_{\gamma'} - \mathbf{A}_{\gamma}) = \frac{1}{c}\mathbf{v} \cdot \nabla\Lambda = \frac{1}{c}\frac{d\Lambda}{dt}. \tag{2.22}$$

The integration along an arbitrary path from the positions $\mathbf{r}(0) = \mathbf{z}$ to $\mathbf{r}(T) = \mathbf{x}$ over the Lagrangians (2.22) gives the classical actions. For paths $\{\mathbf{r}(t)_i\}$ wich do not intersect the surface Σ, the actions appearing in the exponents of the Schrodinger kernels differ by

$$\int_0^T \mathcal{L}' dt - \int_0^T \mathcal{L} dt = \frac{[\Lambda(\mathbf{r}(T)) - \Lambda(\mathbf{r}(0))]}{c} = \frac{[\Lambda(\mathbf{x}) - \Lambda(\mathbf{z})]}{c}. \tag{2.23}$$

The difference is independet of the path $\mathbf{r}(t)_i$, then the corresponding exponentials in (2.20) can be taken out of the path integral.Therefore, the kernels K and K' are related in such a way

$$K'(T, \mathbf{z}, \mathbf{x}) = \exp\left[-\frac{\Lambda(\mathbf{x}) - \Lambda(\mathbf{z})}{i\hbar c}\right] K(T, \mathbf{z}, \mathbf{x}). \tag{2.24}$$

This kernel relation allows to write the solution for the Schrodinger equation as the time-evolution formula of the original observer:

$$\psi'(\mathbf{x}) = e^{i\Lambda(\mathbf{x})/\hbar c}\psi(\mathbf{x}) = U\psi(\mathbf{x}). \tag{2.25}$$

For paths wich intersects the surface Σ at the points $\{\mathbf{r}(t)_i\}$, the integral of $\mathcal{L}' - \mathcal{L}$ acquieres an additional contribution of g/c due to the discontinuity of the function Λ. Then,

$$\int_0^T \mathcal{L}' dt - \int_0^T \mathcal{L} dt = \frac{\Lambda(\mathbf{x}) - \Lambda(\mathbf{z}) - Ng}{c},$$

where N is the number of intersections in Σ, and the kernel is such that

$$K'(T, \mathbf{z}, \mathbf{x}) = \exp\left[-\frac{\Lambda(\mathbf{x}) - \Lambda(\mathbf{z})}{i\hbar c}\right] \int_{\mathbf{z}}^{\mathbf{x}} D\mathbf{x} \exp\left(-\frac{iNg}{\hbar c}\right) \exp\left(\frac{i}{\hbar} \int_0^T \mathcal{L} dt\right). \tag{2.26}$$

The phase $e^{-iNg/\hbar c}$ changes with the function $\mathbf{r}$(t),and hence it can not be taken out of the path integral. The kernel K' is related to K by the transformation law (2.25), if it is imposed the equality $e^{-iNg/\hbar c} = 1$ for every integer N. This equations leads to the generalized Dirac quantization condition:

$$-\frac{g}{\hbar c} = \frac{e_1 g_2 - e_2 g_1}{\hbar c} = 2\pi n, \qquad n \in \mathbb{Z}. \tag{2.27}$$

This condition ensures the unobservability of the Dirac String in Quantum Mechanics. Another approaches have been derived to find the Dirac quantization condition for a magnetic charge and an electric charge giving the value of $qg = 2\pi\hbar c n$[14] . The relation (2.27) is the generalization of this condition to the case of dyons.[17]

2.3 The Klein Gordon Equation

The equation that describes relativistic spin-zero particles is the Klein Gordon equation. That is, it is a relativistic quantum mechanical wave equation for particles with no internal degrees of freedom for spin. It can be obtained using the dispersion relation for a relativistic free particle: $E^2 = p^2c^2 + m^2c^4$. According to the principle of correspondence, the energy and momentum operators are respectively[30]

$$\hat{E} = i\hbar\frac{\partial}{\partial t}, \qquad \hat{\mathbf{p}} = -i\hbar\nabla. \tag{2.28}$$

The operators (2.28) can be introduced into the four momentum $\hat{p}^\mu = (\hat{E}/c, \hat{\mathbf{p}})$. Then, using the relativistic energy in invariant form, the Klein Gordon equation is obtained for free particles:

$$\begin{aligned} p^\mu p_\mu &= \frac{E^2}{c^2} - \mathbf{p}\cdot\mathbf{p} = m^2c^2, \\ \left(\frac{1}{c^2}\frac{\partial^2}{\partial^2 t} - \nabla^2 + \frac{m^2c^2}{\hbar^2}\right)\psi &= 0. \end{aligned} \tag{2.29}$$

If we introduce the d'Alembert operator $\Box = \eta^{\mu\nu}\partial_\nu\partial_\mu = \partial^2/(c^2\partial t^2) - \nabla^2$, the Klein-Gordon equation can be written as $(\Box + m^2c^2/\hbar^2)\psi = 0$.The equation (2.29) is manifestly covariant because the wave operator is a scalar, and if the mass m and the wave function ψ are also scalars, the equation and the wave function have the same form in all inertial reference frames.

2.3.1 Density function

In analogy to the Schrödinger equation, a law of conservation is founded in the Klein Gordon case. Multiplying from the left the Klein Gordon equation for free particles by ψ^*, and considering the difference with the complex conjugate of the equation with the factor ψ in the left hand side, then it yields[6]

$$\begin{aligned} &\psi^*\left(\hat{p}_\mu\hat{p}^\mu - m^2c^2\right)\psi - \psi\left(\hat{p}_\mu\hat{p}^\mu - m^2c^2\right)\psi^* = 0, \\ &\nabla_\mu\cdot(\psi^*\nabla^\mu\psi - \psi\nabla^\mu\psi^*) = 0, \\ &\frac{\partial}{\partial t}\left(\psi^*\frac{\partial\psi}{\partial t} - \psi\frac{\partial\psi^*}{\partial t}\right) - \nabla\cdot(\psi^*\nabla\psi - \psi\nabla\psi^*) = 0. \end{aligned} \tag{2.30}$$

The last equation in (2.30) can be written as a continuity equation, if the term $(j_\mu) = (c\rho, -\mathbf{j}) = (\psi^*\nabla_\mu\psi - \psi\nabla_\mu\psi^*)i\hbar/2m$ is considered. Therefore, it is possible to express the last equations as the well known continuity equation: $\partial\rho/\partial t + \nabla\cdot\mathbf{j} = 0$. So, j_μ can be taken as the four current density definition for a free particle. The relation between the flux of $\mathbf{j}$ through a closed surface and the divergence of the current in the volume enclosed is given by divergence theorem or Gauss's theorem. So that, an integration over the entire configuration yields

$$\iiint_V \frac{\partial \rho}{\partial t} dV = \frac{\partial}{\partial t} \iiint_V \rho dV = -\iiint_V \nabla \cdot \mathbf{j} dV = -\iint_{\partial V} \mathbf{j} \cdot d\mathbf{S} = 0. \tag{2.31}$$

The meaning of the previous result is that $\iiint_V \rho dV$ is constant with respect to time, and the natural guess is to interpret ρ as a probability density. Nevertheless, the probability interpretation is not applicable since at a given time t both ψ and $\partial\psi/\partial t$ may have arbitrary values. Then, ρ may be either positive or negative, which is in contradiction with the definition of proability.

The alternative interpretation is the four current density of charge by multiplication of the current density with the elementary charge e to give

$$J_\mu = \pm \frac{ie\hbar}{2m} \left(\psi^* \nabla_\mu \psi - \psi \nabla_\mu \psi^* \right) = (c\mathcal{P}, -\mathcal{J}), \tag{2.32}$$

where $\mathcal{J}$ denotes the charge current density and the charge density $\mathcal{P}$ is allowed to be positive, negative or zero. This equates with the existence of particles and antiparticles, and it leds to charge degrees of freedom of particles.[6,20]

2.3.2 Energy solutions

Some physical aspects can be observed in the free Klein Gordon equation. In particular, the spectrum of energy is a continuous set of positive and negative energies. The interpretation of the two parts of the spectrum is that the Klein Gordon equation indicates the existence of both particles with energy $E > 0$ and charge density $\rho > 0$, and particles with $-E < 0$ or $E > 0$ and charge density $\rho < 0$. These solutions are connected with the existence of antiparticles. Particles and antiparticles have the same mass, the same positive energy, opposite charge and oppositive current. Then, there exists two possible solutions for a given momentum $\mathbf{p}$:[20]

$$\psi_{(\pm)} = A_{(\pm)} \exp\left[\frac{i}{\hbar} (\mathbf{p}.\mathbf{x} \mp |E_p| t) \right], \qquad E_p = \pm c\sqrt{p^2 + m^2 c^2},$$

where $A_{(\pm)}$ are normalizations constants. The general solution of the wave equation is always a linear combination of both types of functions. $\psi_{(+)}$ specifies particles with charge $+e$; $\psi_{(-)}$ specifies particles with the same mass, but with charge $-e$.

The minimun energy of a positive energy particle is mc^2, whereas the maximun energy of a negative energy particle is $-mc^2$. Both positive and negative energy particles can take on any value of momentum between $+\infty$ and $-\infty$. The energies are separated by a forbidden region of energy defined by $-mc^2 < E_{\text{forbidden}} < mc^2$. As a consequence, a promotion of energy greater than $2mc^2$ can excite a negative energy particle up into the positive energy states.[13,20]

2.3.3 Interaction with electromagnetic fields

The coupling of a charged particle to an electromagnetic field into the Klein Gordon equation can be achieved using the so called minimal coupling. The electromagnetic field is described by the four potential vector $(A^\mu) = (A^0, \mathbf{A})$ where A^0 is the electric scalar potential and $\mathbf{A}$ is the magnetic 3-vector potential. Then, the minimal coupling is given by the substitutions[6]

$$\hat{E} \to i\hbar \frac{\partial}{\partial t} - eA_0, \qquad \mathbf{p} \to -i\hbar\nabla - \frac{e}{c}\mathbf{A}. \tag{2.33}$$

The above quantum operators are compressed to the four-dimensional and covariant form as $\hat{p}^{\mu} \rightarrow \hat{p}^{\mu} - A^{\mu}e/c$. The replacement of the four momentum into the invariant $\hat{p}^{\mu}\hat{p}_{\mu}$ promotes the equation (2.29) to the Klein Gordon equation with electromagnetic interaction:

$$\begin{aligned}
\left(\hat{p}^{\mu} - \frac{e}{c}A^{\mu}\right)\left(\hat{p}_{\mu} - \frac{e}{c}A_{\mu}\right)\psi &= m^2c^2\psi, \\
\frac{1}{c^2}\left(i\hbar\frac{\partial}{\partial t} - eA_0\right)^2\psi &= \left[\sum_{k=1}^{3}\left(i\hbar\frac{\partial}{\partial x^k} + \frac{e}{c}A^k\right)^2 + m^2c^2\right]\psi, \\
\frac{1}{c^2}\left(i\hbar\frac{\partial}{\partial t} - eA_0\right)^2\psi &= \left[\left(i\hbar\nabla + \frac{e\mathbf{A}}{c}\right)^2 + m^2c^2\right]\psi.
\end{aligned} \tag{2.34}$$

2.3.4 Pair creation in a high potential

Under strong fields, some fraction of the positive charge density is always mixed with the negative part. That is ,charge density is not positive definite.[6,13] This new feature can be seen in high Coulomb potentials such as quadratic or exponential potentials. In this case, the electromagnetic vector potential is $\mathbf{A} = \mathbf{0}$ and $eA_0(r) = Ze^2V(r)$ such that $V(r) \rightarrow 1/r$ for large r. Then, for a stationay state, the charge density is given by[6]

$$\mathcal{P}(\mathbf{r}) = e\frac{[\epsilon - eA_0(\mathbf{r})]}{mc^2}\psi\psi^*(\mathbf{r}) = e\frac{[\epsilon + Ze^2V(r)]}{mc^2}\psi\psi^*(\mathbf{r}), \tag{2.35}$$

$$\epsilon > eA_0 \Rightarrow \mathcal{P} > 0, \qquad \epsilon < eA_0 \Rightarrow \mathcal{P} < 0.$$

In the first case the charge density has the same sign as the charge e of the particle ; in the second it is the other way round. The charge density has the opposite sign to the particle charge e, whenever the potential energy has values so that $\epsilon < eA_0$. The physical meaning of this change of sign of $\mathcal{P}$ in strong fields can only be understood within the frame of the field theory where the number of particles becomes variable. The interpretation is that in the areas of strong fields, particles-antiparticles pairs will be produced.

Chapter 3

Results & Discussion

3.1 Klein Gordon equation for the hydrogenic model of dyons

Recalling the classical two body system analized in the section 2.1.4, the motion equations are completely decoupled into the relative motion equation and the center of mass equation. Since the center of mass of the system performs a uniform linear motion, and by a suitable choice of ineartial reference frame, $\mathbf{r_{CM}}(t) = \mathbf{0}$, then it is sufficient to consider the dynamics of the relative dyon so that

$$\begin{aligned} \mathcal{L} &= \mathcal{L}_{CM} + \mathcal{L}_{relative}, \\ \mathcal{L} &= \mathcal{L}_{relative}, \end{aligned} \tag{3.1}$$

where the relative Lagrangian is given by the relation (2.14). To promote the classical system to a quantum relativistic system, the minimal coupling is necessary. As the Lagrangian (3.1) describes a fictitious particle with charge equal to 1, then the minimal coupling showed in the relation (2.33) is such that

$$p^{\nu} \to p^{\nu} - \frac{A^{\nu}}{c}, \tag{3.2}$$

where $(A^{\mu}) = (A^0, \mathbf{A})$ with $A^0 = q/4\pi r$, and $\mathbf{A}$ is the vector potential showed in (2.17) in the restricted domain $D = \mathbb{R}^3 \backslash \gamma$.

In the quantum regime, the corresponding operators for the energy and momentum are

$$\hat{E} \to i\hbar\frac{\partial}{\partial t} - A_0, \qquad \hat{\mathbf{p}} \to -i\hbar\nabla - \frac{\mathbf{A}}{c}. \tag{3.3}$$

Then, the Klein Gordon equation for the relative dyon is obtained by substituting the four-momentum operator in the Einstein relation in the covariant form $\hat{p}^{\mu}\hat{p}_{\mu}$, as was shown in the section 2.3.3, and it is equivalent to (2.34) with $e = 1$:

$$\frac{1}{c^2}\left(i\hbar\frac{\partial}{\partial t} - A_0\right)^2 \psi = \left[\left(i\hbar\nabla + \frac{\mathbf{A}}{c}\right)^2 + m^2c^2\right]\psi. \tag{3.4}$$

3.2 Decoupling of the equation

To find a stationary solution for the equation, we consider the method of separation of variables in spherical coordinates so that the wave function is $\psi = R(r)Y(\theta,\phi)e^{-iEt}$. Also, natural units will be considered: $\hbar = c = 1$. Then, (3.4) is given by

$$(E - A_0)^2 \psi = \left(-\nabla^2 + i\mathbf{A}.\nabla + i\nabla.\mathbf{A} + A^2 + m^2\right)\psi, \tag{3.5}$$

$$\left[r^2(E - A_0)^2 + \left(\frac{r^2}{R}R'' + \frac{2r}{R}R'\right) - m^2r^2\right] = \frac{Q}{Y}, \tag{3.6}$$

where $A = |\mathbf{A}|$ and

$$Q = YA^2 + 2iA\csc(\theta)\frac{\partial Y}{\partial\phi} - \frac{\partial^2 Y}{\partial\theta^2} - \cot(\theta)\frac{\partial Y}{\partial\theta} - \csc^2(\theta)\frac{\partial^2 Y}{\partial\phi^2}. \tag{3.7}$$

The equation (3.6) contains only radial terms and (3.7) solely angular terms. So, the equations must be related by a constant λ such that $Q = \lambda Y$.

In the interest of finding a explicit form of the angular solution, let's consider the angular momentum for the system of dyons. It obeys the quantum relation [1†]

$$\hat{\mathbf{L}} = -i\mathbf{e}_\phi\frac{\partial}{\partial\theta} + \mathbf{e}_\theta\left(\frac{i}{\sin(\theta)}\frac{\partial}{\partial\phi} + A\right) - \frac{g}{4\pi}\mathbf{e}_r, \tag{3.8}$$

$$L_z = L_3 = -i\frac{\partial}{\partial\phi} - \frac{g}{4\pi}. \tag{3.9}$$

The angular momentum relations (3.8) and (3.9) are identical to the standard angular momentum except for the terms A, $g/4\pi$,$\mathbf{e}_r$. Also, since A does not depend of ϕ, and the commutator of a constant is zero then

$$[\mathbf{L}^2, L_3] = \left[2i\frac{A}{\sin(\theta)}\frac{\partial}{\partial\phi}, L_3\right] + [A^2, L_3] + \left[\frac{g^2}{16\pi^2}, L_3\right] = 0.$$

So it holds that $\mathbf{L}^2, L_3$ obey the standard commutation relation $[\mathbf{L}^2, L_3] = 0$. Since $\mathbf{L}^2$ commutes with L_3, it is possible to find a common basis of eigenfunctions for the two operators. Let be $Y(\theta,\phi)$ the common eigenvector. The eigenvectors for the L_z operator are such that

$$L_3Y = \left(-i\frac{\partial}{\partial\phi} - \mu\right)Y = KY, \qquad \mu = g/4\pi, \tag{3.10}$$

$$Y = \Theta(\theta)e^{i(\mu+K)\phi} = \Theta(\theta)e^{ik\phi}, \qquad k = K + \mu. \tag{3.11}$$

This fact allows to consider the eigenvectors $Y = \Theta(\theta)e^{ik\phi}$ for the operator $\mathbf{L}^2$. This result will be useful in the following section.

3.3 Radial equation

Considering the previous result, then $\psi = R(r)\Theta(\theta)e^{i(k\phi-Et)}$. To find a familiar differential equation, consider the variable change $R = \Lambda(r)/r$. So, the radial equation (3.6) is such that

[1]See Appendix C

$$\Lambda'' + \left[E^2 - m^2 - \frac{2qE}{4\pi r} + \left(\frac{q^2}{16\pi^2} - \lambda\right)\frac{1}{r^2}\right]\Lambda = 0. \tag{3.12}$$

The constant λ in the equation $Q = \lambda Y$ can be found if the squared angular momentum for dyons is considered. First, it is necessary to note that the operator

$$\hat{\mathbf{L}}^2 = -\cot(\theta)\frac{\partial}{\partial\theta} - \frac{\partial^2}{\partial\theta^2} - \csc^2(\theta)\frac{\partial^2}{\partial\phi^2} + 2i\frac{A}{\sin(\theta)}\frac{\partial}{\partial\phi} + A^2 + \frac{g^2}{16\pi^2}, \tag{3.13}$$

is identical to the operator in the angular equation (3.7) except for a constant $g^2/16\pi^2 = \mu^2$. By adding the constant μ^2 into the equation (3.7), then the eigenvalue problem for this operator can be written as

$$\mathbf{L}^2 Y = (\lambda + \mu^2)Y. \tag{3.14}$$

Since the angular momentum operator obeys the standard commutation relations, the eignvalues of $\hat{\mathbf{L}}^2$ can be written as $(\lambda + \mu^2) = l(l+1)$, and they must satisfy the constraints $l = |\mu|, |\mu|+1, |\mu|+2, ...$,and$-l \leq K \leq l$ simultaneously.[3]

On the other hand, the binding energy is defined as $E_{\text{binding}} = |E| - mc^2 = |E| - m$. It is necessary to search for energy solutions in the range $-m < E < m$ to get bound states in the system. Then, the variable changes $b = 2\sqrt{m^2 - E^2}$, $z = br$,$\nu = \sqrt{(l+1/2)^2 - \mu^2 - q^2/16\pi^2}$, $\xi = -2qE/4\pi b$ are well defined, and they give the following differential equation

$$\frac{d^2\Lambda}{dz^2} + \left[-\frac{1}{4} + \frac{\xi}{z} - \frac{\nu^2 - 1/4}{z^2}\right]\Lambda = 0. \tag{3.15}$$

The general solution for the Whittaker's equation (3.15) is expressed in terms of the linear independent functions $M_{\xi,\nu}(z)$ and $M_{\xi,-\nu}(z)$,[18] where

$$M_{\xi,\nu}(z) = e^{-z/2}z^{(1/2)+\nu}F_1(\nu + 1/2 - \xi, 2\nu + 1, z),$$

$$M_{\xi,-\nu}(z) = e^{-z/2}z^{(1/2)-\nu}F_1(1/2 - \nu - \xi, 1 - 2\nu, z), \quad F_1(a, c, z) = \sum_{n'=0}^{\infty}\frac{(a)_{n'}}{(c)_{n'}}\frac{z^{n'}}{n'!}. \tag{3.16}$$

In order to search a more precise form of the radial solution, the asymptotic behavior $z \to \infty$ of (3.15) will be considered. In the infinite limit, the terms z^{-1}, z^{-2} vanish. Then, the resulting expression is

$$\left(\frac{d^2}{dz^2} - \frac{1}{4}\right)\Lambda = 0, \tag{3.17}$$

$$\Lambda(z) = C_1 e^{-z/2} + C_2 e^{z/2}. \tag{3.18}$$

Because of the normalization condition, then $C_2 = 0$. On the other hand, in the limit $z \to 0$ the quantities $1/4, z^{-1}, z^{-2}$ satisfy the inequality $1/4 << z^{-1} << z^{-2}$. That is, z^{-2} is the dominant term as can be seen in the limit $z^{-2}/z^{-1} \to \infty$ when $z \to 0$. Therefore, it is necessary to consider only the z^{-2} term in the equation (3.15) to explore the behavior of the radial solution in the neighborhood of $z \to 0$. Then, the equation (3.15) is reduced to

$$\left(\frac{d^2}{dz^2} - \frac{\nu^2 - 1/4}{z^2}\right)\Lambda = 0, \tag{3.19}$$

$$\Lambda(z) = D_1 z^{1/2+\nu} + D_2 z^{1/2-\nu}. \tag{3.20}$$

Since ν can take positive values, and the wave function must not have any nonintegrable divergence at the origin then $D_2 = 0$. An additional condition can be found in the asymptotic behavior of F_1. For $z \to \infty$, it holds that[1]

$$F_1(a, c, z) = \frac{\Gamma(c)}{\Gamma(a)} e^z z^{a-c}[1 + O(|z|^{-1})]. \tag{3.21}$$

The hypergeometric function diverges, but the normalization condition implies that the series of F_1 must breaks off at some $n' = N$. Then, the coefficient $(a)_{N+1}$ must satisfy

$$(a)_{N+1} = a(a + 1)(a + 2)...(a + N) = 0 \Rightarrow a + N = 0, \tag{3.22}$$

$$\nu + 1/2 - \xi = -N. \tag{3.23}$$

From the above results, the function F_1 is a polynomial of degree N and the radial function takes the form of

$$R = \frac{\Lambda}{r} = \frac{C}{r}(br)^{\nu+1/2} e^{-br/2} F_1(-N, 2\nu + 1, br). \tag{3.24}$$

3.4 Angular equation

The angular equation (3.7) and the angular wave function $Y = \Theta(\theta)e^{i(K+\mu)\phi}$ with the variable changes $x = \cos(\theta)$,$\Theta = (1 - x)^{-(a+b)/2}(1 + x)^{-(b-a)/2}\omega$ lead to the differential equation

$$\left(x^2 - 1\right)\omega''(x) - 2[a + (b - 1)x]\omega'(x) + (-b - \lambda)\omega(x) = 0, \tag{3.25}$$

where $a = k - g/4\pi$ and $b = g/4\pi$. The additional changes $z = (1 + x)/2$, $\alpha + \beta + 1 = 2(1 - b), \alpha\beta = -b - \lambda, \gamma = a - b + 1$ give the hypergeometric differential equation

$$z(1 - z)\frac{d^2\omega}{dz^2} + [\gamma - (\alpha + \beta + 1)z]\frac{d\omega}{dz} - \alpha\beta\omega = 0. \tag{3.26}$$

The general solution of (3.26) is given in terms of hypergeometric functions F of second order:

$$\omega(z) = C_1 F(\alpha, \beta, \gamma, z) + C_2 z^{-\gamma+1} F(\alpha - \gamma + 1, \beta - \gamma + 1, 2 - \gamma, z), \tag{3.27}$$

where

$$F(\alpha, \beta, \gamma, z) = \sum_{n=0}^{\infty} \frac{(\alpha)_n(\beta)_n}{(\gamma)_n} \frac{z^n}{n!}.$$

The function F is finite if it is reduced to a finite polynomial. Then to get physically acceptable solutions, the series must terminate at a certain power $n = 0, 1, 2, ...$ The condition is achieved, if α or β is a negative integer. Let be $\alpha = -n$ and returnig some variable changes, it holds that

$$\alpha + \beta + 1 = 2(1 - b),$$

$$\lambda + \mu^2 = n^2 - \mu + n - 2n\mu + \mu^2, \tag{3.28}$$

$$l(l+1) = n^2 - \mu + n - 2n\mu + \mu^2 = (n-\mu)(n-\mu+1). \tag{3.29}$$

The relation (3.29) implies a restriction in the quantum number l. It shows that $l = n - \mu$. Using the relation (3.28) into the equation (3.25) and the variables changes $-2b = \alpha + \beta, 2a = \beta - \alpha$ lead to the following equation

$$(1-x^2)\omega'' + [\beta - \alpha - (\alpha + \beta + 2)x]\omega' + n(\alpha + \beta + n + 1)\omega = 0, \tag{3.30}$$

which is satisfied by the Jacobi polynomials $\omega = P_n^{\alpha,\beta}(x)$. Then, the angular function is such that

$$Y_{\mu l K}(\theta, \phi) = (1-x)^{-(\alpha+\mu)/2}(1+x)^{-(\mu-\alpha)/2} P_n^{\alpha,\beta}(x) e^{i(K+\mu)\phi}. \tag{3.31}$$

3.5 Energy

Notice that the relative mass showed in the section 2.1.4.1 tends to m_2 due to the masses relation $m_1 \gg m_2$ for the hydrogen like atom. Also, this masses relation locates the reference frame near from the mass m_1,. From now, we will consider the approximation of $m \approx m_2$, and the reference frame attached to the mass m_1. So that, the wave function ψ corresponds to the orbiting dyon. Considering the previous idea, the energy spectrum for the orbiting dyon is reached if the changes of variables are returned in the expression (3.23)

$$\sqrt{\left(l + \frac{1}{2}\right)^2 - \mu^2 + \frac{q^2}{16\pi^2}} + \frac{1}{2} + \frac{2qE}{4\pi 2\sqrt{m^2 - E^2}} + N = 0,$$

By clearing algebraically the term of the energy, it holds that

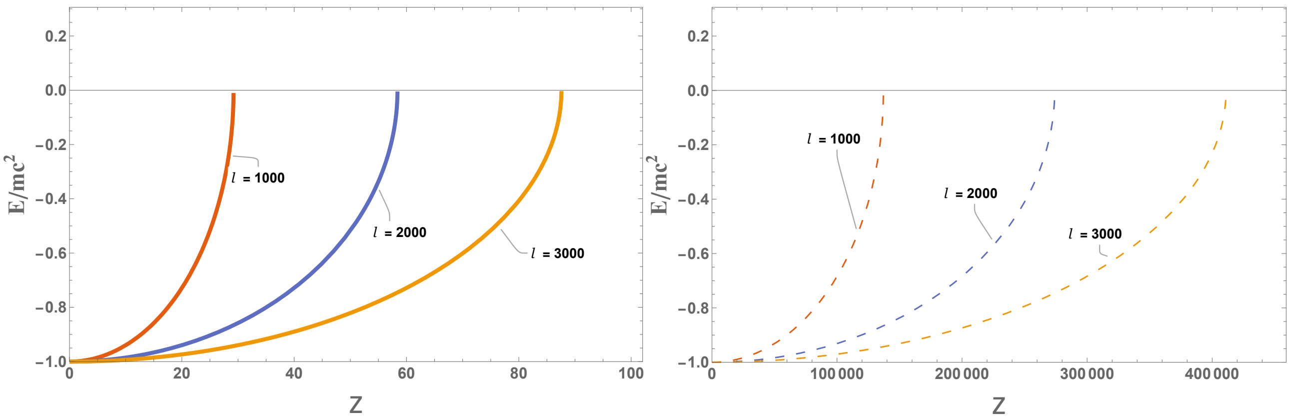
$$E = E_{Nl\mu} = \pm m \left[1 - \frac{q^2}{q^2 + 16\pi^2 \left(\sqrt{(l+1/2)^2 - \mu^2 - q^2/16\pi^2} + 1/2 + N \right)^2} \right]^{1/2}. \tag{3.32}$$

In order to find the allowed values for the magnetic and electric charges in the hydrogen atom with dyons, the quantization conditions must be considered. The existence of the elementary electric charge $e = \pm e_0$ and the Dirac's quantization condition implies the relation

$$g = \frac{2\pi}{e}\Omega', \qquad \Omega' \in \mathbb{Z}. \tag{3.33}$$

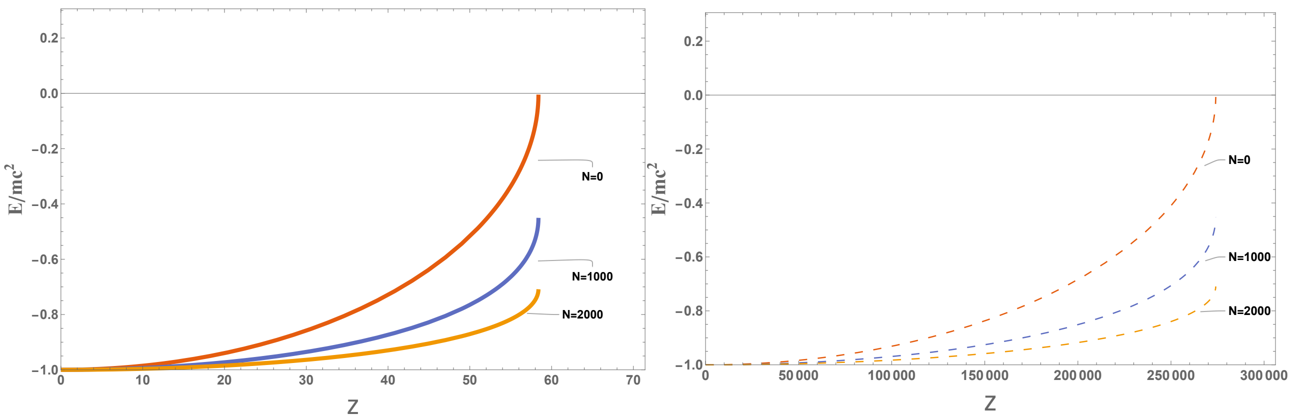
Taking $\Omega' = \pm 1$ for the minimal value of the magnetic charge, then $g_0 = \pm 2\pi/e_0$. Let be a system with a point nucleus that contains the charge $(e_1, g_1) = (Ze_0, Z2\pi/e_0)$, and the second particle is such that $(e_2, g_2) = (-e_0, -2\pi/e_0)$. Then, $g = 0$ satisfies the quantization condition (2.27). Since the square root in the energy expression (3.32), the allowed energies for the system start from $l = 34$. In Figure 3.1, for $N = 0$ some energy curves are showed with increasing Z and fixed values $l = 1000, 2000, 3000$. Here, the fine- structure constant $\alpha = e^2/4\pi$ and the relation (3.33) were considered to do the calculations.

For the dyon system with a level $l = 1000$, there are no solutions for $Z > 30$ and similarly for the greater l levels when $Z > 59, 88$, respectively. The negative energy corresponds to the negative



(a) (b)

Figure 3.1: Energy curves for different l values and fixed $N = 0$ for the dyon-dyon system (a), and the electron-proton system (b). The energy is given in units of the rest energy mc^2



(a) (b)

Figure 3.2: Energy curves for fixed $l = 2000$ and several N values for the dyon-dyon system (a), and the electron-proton system (b) . The energy is given in units of the rest energy mc^2 .

charge densities related to the negative charges $(-e_0, -g_0)$. As can be seen in the Figure 3.1 (a), it is necessary large values of the quantum number l as well as big N to show notable changes in the energy. This is due to the magnetic charge contribution in the effective charge q. In the figures 3.1 and 3.2, the behavior of the energy for the dyon system is similar to the standard hydrogen atom, but with a much more restrictive range of allowed nuclear charges.

If we introduce the principal quantum number $n_p = N + l + 1$, the first levels $1s, 2p, 3s, 3p$,etc are forbidden states for the dyon-dyon system because the minimum allowed level starts for $l > 33$. Also, the system can not show a transition from positive energy to a negative one under an increasing nuclear charges as can be noticed in the limit $Z \to \infty$ in the energy expression (3.32).

Since $\mu = 0$, the energy is reduced to $E_{n_p l \mu} = E_{n_p l}$. So that for a level l and a given n_p, the system has different energies. However, the $2l + 1$ degeneracy related with the quantum number K remains.

3.6 Charge density

To find the charge density expression for the relative dyon, the Klein Gordon equation (3.4) can be used in natural units. Following the procedure showed in,[6] the equation is multiplied by ψ^* from the left hand side and subtracted the complex conjugate so that

$$\begin{aligned}
&\eta^{\mu\nu}\left(i\frac{\partial}{\partial x^{\nu}} - A_{\nu}\right)\left(i\frac{\partial}{\partial x^{\mu}} - A_{\mu}\right)\psi = m^2\psi, \\
&0 = \psi^*\left[-\eta^{\mu\nu}\left(\frac{\partial}{\partial x^{\nu}} + iA_{\nu}\right)\left(\frac{\partial}{\partial x^{\mu}} + iA_{\mu}\right)\right]\psi \\
&\quad - \psi\left[-\eta^{\mu\nu}\left(\frac{\partial}{\partial x^{\nu}} - iA_{\nu}\right)\left(\frac{\partial}{\partial x^{\mu}} - iA_{\mu}\right)\right]\psi^*, \\
&0 = \eta^{\mu\nu}\left[\frac{\partial}{\partial x^{\mu}}\left(\psi\frac{\partial}{\partial x^{\nu}}\psi^* - \psi^*\frac{\partial}{\partial x^{\nu}}\psi\right) - 2\frac{\partial}{\partial x^{\mu}}\left(\psi iA_{\nu}\psi^*\right)\right].
\end{aligned} \tag{3.34}$$

Denoting J_ν as

$$J_{\nu} = \frac{ie}{2m}\left(\psi^*\frac{\partial}{\partial x^{\nu}}\psi - \psi\frac{\partial}{\partial x^{\nu}}\psi^*\right) - \frac{e}{m}A_{\nu}\psi\psi^*,$$

then the relation (3.34) shows the continuity equation $\partial_\mu J^\mu = 0$. Notice that in the section 2.3.1, the term J_ν is interpreted as the four-current density of electric charge. Since the dyon carries both magnetic and electric charges at the same time, then the four- current associated to the magnetic charge is the same but with the replacement $e_0 \to g_0$.

The component $J_0 = \mathcal{P}$ gives the charge density. Taking the wave function for the stationary state $\psi = R(r)Y(\theta, \phi)e^{-iEt} = \psi(\mathbf{r})e^{-iEt}$, then

$$\mathcal{P} = \frac{ie}{2m}\left[-iE\psi(\mathbf{r})e^{iEt}\psi(\mathbf{r})e^{-iEt} - iE\psi(\mathbf{r})e^{-iEt}\psi(\mathbf{r})e^{iEt}\right] - \frac{e}{m}A_0\psi(\mathbf{r})\psi(\mathbf{r}),$$

$$\mathcal{P}_e = \pm e_0\left(\frac{E - A_0}{m}\right)\psi(\mathbf{r})\psi(\mathbf{r}), \qquad \mathcal{P}_g = \pm g_0\left(\frac{E - A_0}{m}\right)\psi(\mathbf{r})\psi(\mathbf{r}). \tag{3.35}$$

To explore the charge density for the system, let's consider the radial charge density divided by the charge $\mathcal{P}_r/(e, g)$. That is, $(E - A_0)R^2r^2/m$.

The figures 3.3 and 3.4 shows the curves of $\mathcal{P}_r/(e, g)$ for a system with mass $m = 139.577$ MeV, a point nucleus with charge (e_0, g_0) and a second particle with charge $(-e_0, -g_0)$. The plots indicates that the charge density concentration moves away from the nucleus when N increases, but the density is gradually distributed to regions surrounding the nucleus. Similarly when l increases, the concentration shifts to the right. This is the same behavior than the standard relativistic hydrogen atom with a heavy electron mass (pion) $m = 139.577$ MeV, as can be seen in the figures 3.5 and 3.6.

For the relativistic standard hydrogen atom, the electric charge density is shown in the figure 3.5 for the first allowed levels of N and the minimum fixed value of l. For the levels $N = 2, 3$, the electric charge density concentration lies between the range of (2000 − 4000) fm and (4000 − 6000) fm respectively. For $N = 0, 1$, the concentrations of charge density do not exceed the 2000 fm. Analogously for the case of the dyon, the figure 3.3 shows the charge density curves for several first levels of N with the minimum fixed value of l. It can be seen that the charge density concentration for the initial values of $N = 0, 4$ and for greater values such as $N = 8, 12$, the concentrations do not reach the 60 fm. Therefore, for these levels, the charge density concentration is closest to the nucleus for the dyon, and farthest from the nucleus for the standard hydrogen atom. In these levels, it indicates that this dyon system acts like a standard hydrogen atom with a stronger interaction between the proton and electron.

The displacements of the charge can be seen in a general way by noting that the density function $\mathcal{P}_r$ can be written in terms of the variable $z = br$, such that $\mathcal{P}_r = \mathcal{P}_r(br)$, where b is given by the relation $b = 2\sqrt{m^2 - E^2}$. That is,

$$\mathcal{P}_r = \mathcal{P}_r(br) = \frac{e(E - A_0)R^2(br)^2}{b^2 m}, \tag{3.36}$$

where

$$R = \frac{bC}{br}(br)^{\nu+1/2}e^{-br/2}F_1(-N, 2\nu + 1, br), \qquad A_0 = \frac{bq}{4\pi br}.$$

Furthermore, as l or N increases, the energy value E approaches m^2, and as a consequence b tends to zero. Since $0 < b < 1$ and $\mathcal{P}_r = \mathcal{P}_r(br)$, then the effect on the density function is a horizontal dilation, that is, the charge density moves to regions further away from the nucleus.

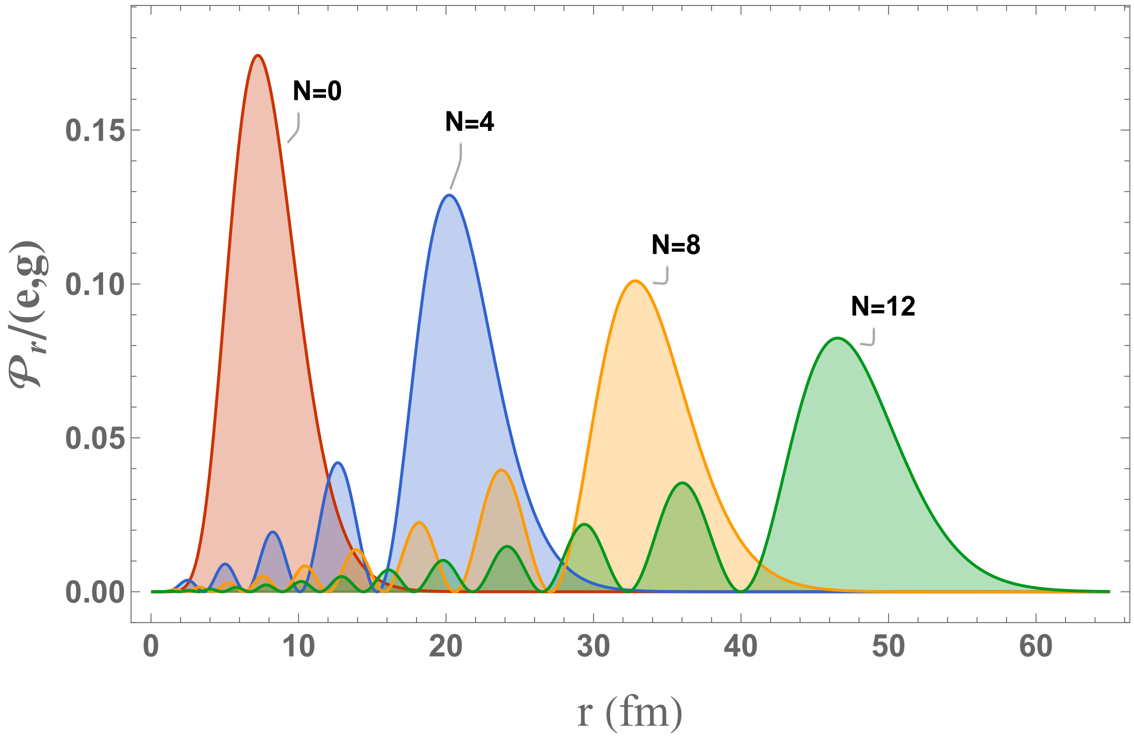


Figure 3.3: Curves of $\mathcal{P}_r/(e, g)$ for $l = 34$ and $N = 0, N = 4, N = 8, N = 12$.

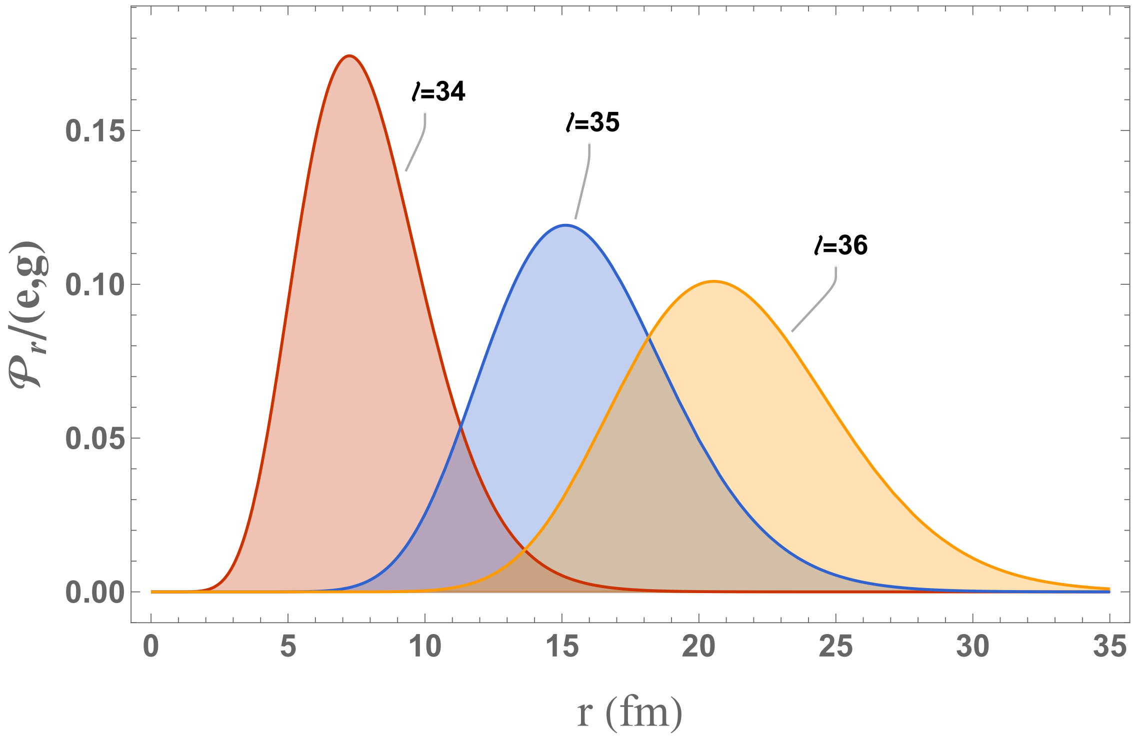


Figure 3.4: Curves of $\mathcal{P}_r/(e, g)$ for $N = 0$ and $l = 34, l = 35, l = 36$.

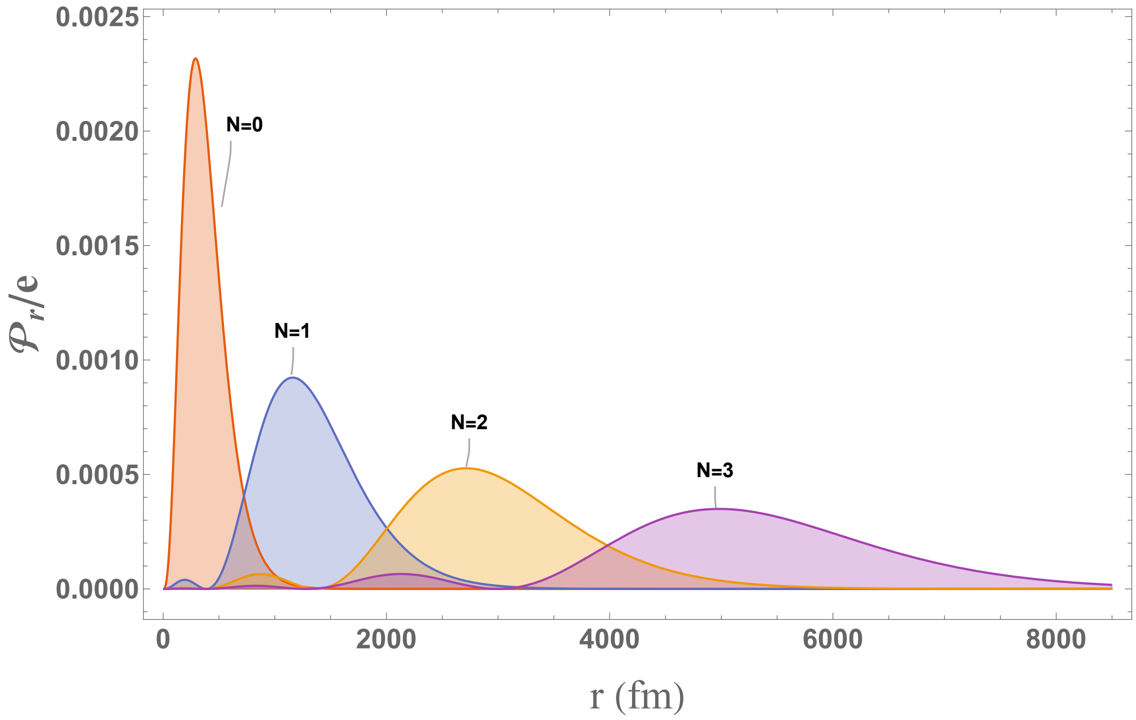


Figure 3.5: Curves of $\mathcal{P}_r/e$ for the relativistic standard hydrogen atom with $l = 0$ and $N = 0, N = 1, N = 2, N = 3$.

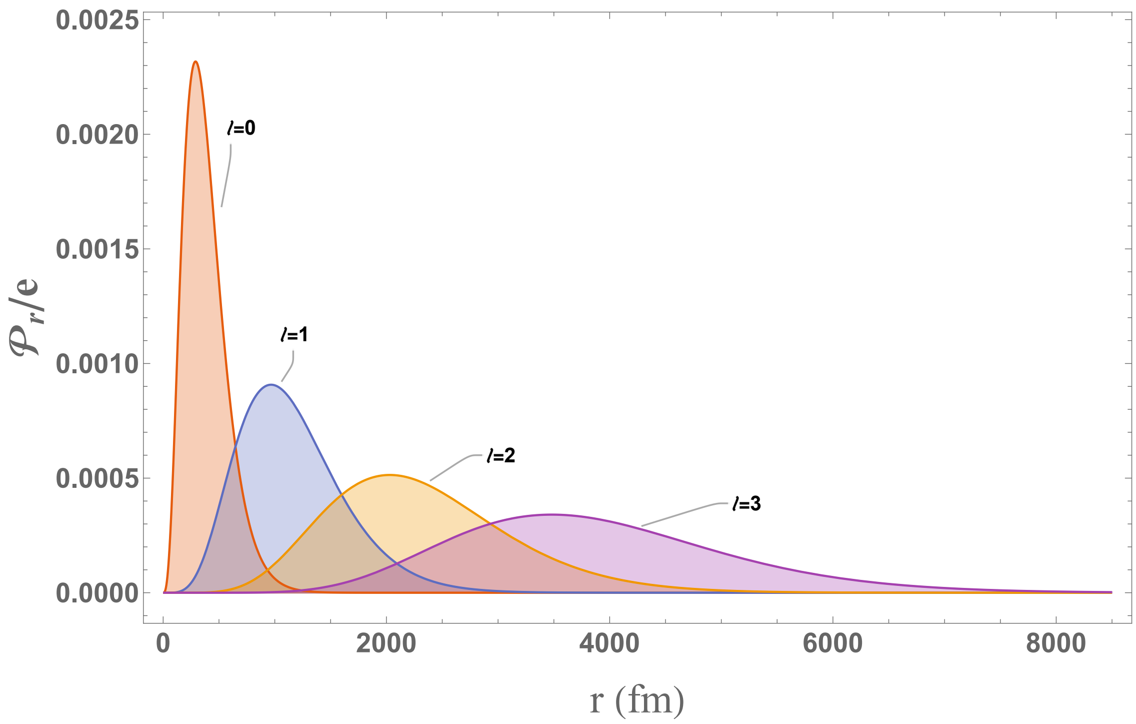


Figure 3.6: Curves of $\mathcal{P}_r/e$ for the relativistic standard hydrogen atom with $N = 0$ and $l = 0, l = 1, l = 2, l = 3$.

Chapter 4

Conclussions & Outlook

In this work, we have shown a study of an hydrogenic atomic model composed by two dyons. We have used the classical description of a two body system of dyons under the Lagrangian formalism, and the canonical quantization into the Einstein energy relation to promote the system to a relativistic quantum approach. The resulting equation was restricted to the domain $\mathbb{R}^3\backslash\gamma$ in order to get a consistent Klein Gordon equation with Quantum Mechanics.

It was considered to find solutions for stationary states of the system using the method of separation of variables. Two completely decoupled equations were obtained for the angular and radial variables. Furthermore, it was shown that the angular part of the equation can be related to an eigenvalue problem using the angular momentum operator associated with the system of dyons. The general solutions of the decoupled equations were found in terms of hypergeometric functions. By imposing conditions of normalization and integrability, the functions were reduced to finite polynomials. From this imposition, the energy expression was found in terms of the degree of the polynomial N, the μ term and the quantum number l associated with the angular momentum.

It was shown the energy spectrum is discrete and depends on three quantum numbers. To analyze the energy, it was considered an atomic system with a nucleus of variable electric and magnetic charge. The Dirac quantization condition was used to find the allowed value for elemental magnetic charge. Likewise, it was verified that the system satisfies the generalized quantization condition. The energy for this system was found to have a much more restricted range of allowed nuclei than the standard relativistic quantum hydrogen atom. In addition, the system did not show a transition of energy sign under an increasing potential energy by setting Z charges in the nucleus. Another important result was that in the system with $Z = 1$ the first levels $1s, 2p, 3s, 3p$, etc. are forbidden states because the minimum allowed value of the quantum number l is 34. With respect to the degeneracy of the energy, the system with $Z = 1$ exhibits $2l + 1$

The expression for the electric and magnetic charge density of the system was derived. For a system of positive elementary charges in the nucleus and negative for the orbiting particle, it was found that the radial charge density concentration moves away from the nucleus when N or l increases. It was also found that the charge density is located in regions much closer to the nucleus than in the standard hydrogen atom for the first allowed levels of N and l. In these levels, the results showed that this dyon system acts like a standard hydrogen atom with a stronger interaction between the proton and electron.

In this work, the nucleus was considered as a point particle. However, hydrogen-like-atoms contain a nucleus's finite extent. Then, this suggests improving the model of the dyon atom considering the size of the nucleus using a modified scalar or vector potential. This consideration could show bound states for larger Z charges and a smaller allowed quantum number l than the values founded in this work.

Further research can be done from the results obtained in this project. Since the Klein Gordon equation was derived using arbitrary charges, it is possible to explore other configurations for the system. In particular, a system with equal magnetic charge but different electric charges can be studied. In addition, different potentials can be used to model different interaction between electric charges. Specifically, interesting dynamics could be observed for an oscillator potential for the electric interaction and the vector potential for the magnetic charges.

Appendix A

Electromagnetic Duality Transformation

Let's define the rotation matrix $R(\theta)$ as

$$R(\theta) = \begin{pmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{pmatrix}. \tag{A.1}$$

We will explore the behavior of generalized Maxwell's equations under a rotation of the fields and sources. Consider the prime terms

$$\begin{pmatrix} \mathbf{E}' \\ \mathbf{B}' \end{pmatrix} = \begin{pmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{pmatrix} \begin{pmatrix} \mathbf{E} \\ \mathbf{B} \end{pmatrix}, \qquad \begin{pmatrix} j_e^{'\mu} \\ j_m^{'\mu} \end{pmatrix} = \begin{pmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{pmatrix} \begin{pmatrix} j_e^{\mu} \\ j_m^{\mu} \end{pmatrix}. \tag{A.2}$$

Then, it holds that

$$\begin{aligned} \nabla \cdot \begin{pmatrix} \mathbf{E}' \\ \mathbf{B}' \end{pmatrix} &= \nabla \cdot \begin{pmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{pmatrix} \begin{pmatrix} \mathbf{E} \\ \mathbf{B} \end{pmatrix} = \begin{pmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{pmatrix} \nabla \cdot \begin{pmatrix} \mathbf{E} \\ \mathbf{B} \end{pmatrix}, \\ \nabla \cdot \begin{pmatrix} \mathbf{E}' \\ \mathbf{B}' \end{pmatrix} &= \begin{pmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{pmatrix} \begin{pmatrix} j_e^{0} \\ j_m^{0} \end{pmatrix} = \begin{pmatrix} j_e^{0'} \\ j_m^{0'} \end{pmatrix}. \end{aligned} \tag{A.3}$$

$$\begin{aligned} \nabla \times \begin{pmatrix} \mathbf{E}' \\ \mathbf{B}' \end{pmatrix} &= \nabla \times \begin{pmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{pmatrix} \begin{pmatrix} \mathbf{E} \\ \mathbf{B} \end{pmatrix} = \begin{pmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{pmatrix} \nabla \times \begin{pmatrix} \mathbf{E} \\ \mathbf{B} \end{pmatrix}, \\ \nabla \times \begin{pmatrix} \mathbf{E}' \\ \mathbf{B}' \end{pmatrix} &= \begin{pmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{pmatrix} \begin{pmatrix} -\mathbf{j_m} - \partial \mathbf{B}/\partial t \\ \mathbf{j_e} + \partial \mathbf{E}/\partial t \end{pmatrix} = \begin{pmatrix} -\mathbf{j'_m} - \partial \mathbf{B}'/\partial t \\ \mathbf{j'_e} + \partial \mathbf{E}'/\partial t \end{pmatrix}. \end{aligned} \tag{A.4}$$

As can be seen in the relations $(A.3)$ and $(A.4)$, the Maxwell's equations are invariant under a continuos rotation transformation $R(\theta)$ of the fields and sources. So that, if $\mathbf{E}, \mathbf{B}, j_e^{\mu}, j_m^{\mu}$ are solutions, then the prime terms are also solutions. As a consequence of this property, if we know the sources and fields produced by a configuration of only electric charges, then the sources and fields produced by a system of only magnetic charges can be found inmediately by setting $\theta = \pi/2$, which implies $\mathbf{E} \to \mathbf{B}, \quad \mathbf{B} \to -\mathbf{E}, e_r \to g_r, \quad g_r \to -e_r$.

Let be a electric charge e with velocity $\mathbf{v}$, then the fields produced at $\mathbf{E}_e, \mathbf{B_e}$ (approximation to first order $1/c$) are given by

$$\mathbf{E}_e = \frac{e}{4\pi} \frac{\mathbf{R}}{R^3}, \qquad \mathbf{B}_e = \frac{e}{4\pi c} \mathbf{v} \times \frac{\mathbf{R}}{R^3}. \tag{A.5}$$

Therefore, for a magnetic monopole with charge g the fields are such that

$$\mathbf{B}_m = \frac{g}{4\pi}\frac{\mathbf{R}}{R^3}, \qquad \mathbf{E}_m = -\frac{g}{4\pi c}\mathbf{v}\times\frac{\mathbf{R}}{R^3}. \tag{A.6}$$

Since the generalized Maxwell's are linear, then the electromagnetic fields generated by a dyon particle with charge (e, g) are the contribution of the fields $(A.5)$ and $(A, 6)$. So, one obtains the relation (2.10). That is,

$$\begin{aligned} \mathbf{E}_{dyon} &= \mathbf{E}_e + \mathbf{E}_m = \frac{e}{4\pi}\frac{\mathbf{R}}{R^3} - \frac{g}{4\pi}\frac{\mathbf{V}}{c}\times\frac{\mathbf{R}}{R^3}, \\ \mathbf{B}_{dyon} &= \mathbf{B}_e + \mathbf{B}_m = \frac{g}{4\pi}\frac{\mathbf{R}}{R^3} + \frac{e}{4\pi}\frac{\mathbf{V}}{c}\times\frac{\mathbf{R}}{R^3}. \end{aligned} \tag{A.7}$$

Appendix B

Four divergence of the energy- momentum tensor

The components for the energy- momentum can be founded if the Laws of Conservation for Electromagnetism are considered. The force per unit volume is given by

$$\mathbf{f} = \rho\mathbf{E} + \mathbf{j} \times \mathbf{B} = \nabla \cdot \mathbf{M} - \frac{\partial \mathbf{S}}{\partial t}, \tag{B.1}$$

where **S** is the Poynting vector and the components of **M** are such that

$$M_{ij} = E_i E_j + B_i B_j - \frac{1}{2}\delta_{ij}\left(E^2 + B^2\right).$$

For a region that is filled with no charges but only electromagnetic fields, then $\mathbf{f} = \mathbf{0}$ wich means the mechanical momentum is not changing. So, the continuity equation for the local momentum of the electromagnetic fields is given by

$$\frac{\partial \mathbf{S}}{\partial t} - \nabla \cdot \mathbf{M} = 0. \tag{B.2}$$

Also, for a region of space with no charges and using the Poynting's theorem, then dW/dt and the conservation equation for energy of fields is founded.

$$\frac{1}{2}\frac{\partial}{\partial t}\int \left(B^2 + E^2\right) dV + \int \mathbf{S}.d\mathbf{a} = 0,$$

$$\frac{1}{2}\frac{\partial}{\partial t}\left(B^2 + E^2\right) + \nabla \cdot \mathbf{S} = 0. \tag{B.3}$$

We introduce the Faraday tensor $F^{\mu\beta}$ and its dual tensor $\tilde{F}^{\alpha\beta}$ as:

$$F^{\alpha\beta} = \begin{pmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & -B_z & B_y \\ E_y & B_z & 0 & -B_x \\ E_z & -B_y & B_x & 0 \end{pmatrix}, \qquad \tilde{F}^{\alpha\beta} = \frac{1}{2}\epsilon^{\alpha\beta\gamma\delta}F_{\gamma\delta} = \begin{pmatrix} 0 & -B_x & -B_y & -B_z \\ B_x & 0 & E_z & -E_y \\ B_y & -E_z & 0 & E_x \\ B_z & E_y & -E_x & 0 \end{pmatrix}.$$

Therefore, the contribution of the electromagnetic energy and momentum can be expressed as follows

$$T^{\mu\nu}_{em} = F^{\mu\alpha}F_{\alpha}^{\ \nu} + \frac{1}{4}\eta^{\mu\nu}F^{\alpha\beta}F_{\alpha\beta}. \tag{B.4}$$

So that, $T_{em}^{00} = (E^2+B^2)/2, T_{em}^{i} = (\mathbf{E}\times\mathbf{B})^i = S^i$, etc. On the other hand, the particle contribution is related to the relativistic energy and momentum of the electric charges. Denote u_n^μ as the four velocity component, ds_n is the proper time and y_n is the four position component for the n particle, then the contribution of energy and momentum from particles can be expressed as

$$T_p^{\mu\nu} = \sum_n m_n \int u_n^\mu u_n^\nu \delta^4(x-y_n)ds_n. \tag{B.5}$$

The duality invariance allows to express the energy momentum tensor for dyons with the same form. The divergence of the two contributions is given by[17]

$$\begin{aligned}
\partial_\mu T_{em}^{\mu\nu} &= j_e^\alpha F_\alpha{}^\nu + F^{\mu\alpha}\partial_\mu F_\alpha{}^\nu + \frac{1}{2}F^{\alpha\beta}\partial^\nu F_{\alpha\beta}, \\
&= -F^{\nu\alpha} j_{e\alpha} + \frac{1}{2}F_{\alpha\beta}(\partial^\alpha F^{\beta\nu} + \partial^\beta F^{\nu\alpha} + \partial^\nu F^{\alpha\beta}), \\
&= -F^{\nu\alpha} j_{e\alpha} - \frac{1}{2}F_{\alpha\beta}\epsilon^{\alpha\beta\nu\mu} j_{m\mu}, \\
&= -F^{\nu\alpha} j_{e\alpha} - \tilde{F}^{\nu\alpha} j_{m\alpha}, \\
&= -\sum_n \int (e_n F^{\nu\alpha} + g_n \tilde{F}^{\nu\alpha}) u_{n\alpha}\delta^4(x-y_n)ds_n.
\end{aligned} \tag{B.6}$$

$$\begin{aligned}
\partial_\mu T_p^{\mu\nu} &= \sum_n \int \frac{dp_n^\nu}{ds_n}\delta^4(x-y_n)ds_n, \\
&= \sum_n \int \left(\frac{dp_n^\nu}{ds_n} - (e_n F^{\nu\alpha} + g_n \tilde{F}^{\nu\alpha})u_{n\alpha}\right)\delta^4(x-y_n)ds_n.
\end{aligned} \tag{B.7}$$

Appendix C

Angular momentum

The Consevation Laws requiere the contribution of electromagnetic fields to be sustained. In particular, the angular momentum associated to the fields must be considered since charges and fields exchange momentum, and only the total is conserved. The expression to calculate the angular momentum for dyons has the same form of the standar electrodynamics since the energy momentum tensor is conserved and it also mantains the same form.

$$\mathbf{L} = \mathbf{L}_p + \mathbf{L}_{em}. \tag{C.1}$$

For a two body system of dyons with charges (e_1, g_1) and (e_2, g_2) the field in the position $\mathbf{x}$ can be calculated using the fields $(A.7)$ to find the total fields of the system $\mathbf{E} = \mathbf{E}_{dyon1} + \mathbf{E}_{dyon2}$ and $\mathbf{B}$.

$$\mathbf{L}_{em} = \frac{1}{c}\iiint \mathbf{x}\times(\mathbf{E}\times\mathbf{B})d^3x = \frac{e_1g_2 - e_2g_1}{(4\pi)^2 c}\iiint \mathbf{x}\times\left(\frac{\mathbf{r}_1\times\mathbf{r}_2}{r_1^3 r_2^3}\right)d^3x, \tag{C.2}$$

$$\mathbf{L}_{em} = \frac{e_1g_2 - e_2g_1}{4\pi c}\frac{\mathbf{r}}{r}, \tag{C.3}$$

$$\mathbf{L}_p = \mathbf{s}_1\times m_1\mathbf{v}_1 + \mathbf{s}_2\times\mathbf{v}_2 = \mathbf{r}\times m\mathbf{v} + \mathbf{r}_{CM}\times(m_1+m_2)\mathbf{v}_{CM}. \tag{C.4}$$

If the coordinate system is located in the center of mass, then

$$\mathbf{L} = \mathbf{r}\times m\mathbf{v} + \frac{e_1g_2 - e_2g_1}{4\pi c}\frac{\mathbf{r}}{r}. \tag{C.5}$$

From the classical angular momentum, the quantum case can be obtained taking into account the canonical momentum relation shown in (2.14) for the relative motion, and the canonical commutation rule $[\hat{A}, \hat{B}] \to i\hbar\{A, B\}$ given in the section (3.1).

Therefore, the angular momentum operator is,

$$\begin{aligned}\hat{\mathbf{L}} &= \hat{\mathbf{r}}\times\left(-i\hbar\nabla - \frac{\mathbf{A}}{c}\right) + \frac{e_1g_2 - e_2g_1}{4\pi c}\mathbf{e}_r,\\ \hat{\mathbf{L}} &= -i\hbar\mathbf{e}_\phi\frac{\partial}{\partial\theta} + \mathbf{e}_\theta\left(\frac{i\hbar}{\sin(\theta)}\frac{\partial}{\partial\phi} + \frac{A}{c}\right) - \frac{g}{4\pi c}\mathbf{e}_r.\end{aligned} \tag{C.6}$$

So that, in natural units the quantum operators L_z and $\mathbf{L}^2$ are

$$L_z = L_3 = -i\frac{\partial}{\partial\phi} - \frac{g}{4\pi}, \tag{C.7}$$

$$\mathbf{L}^2 = -\cot(\theta)\frac{\partial}{\partial\theta} - \frac{\partial^2}{\partial\theta^2} - \csc^2(\theta)\frac{\partial^2}{\partial\phi^2} + 2i\frac{A}{\sin(\theta)}\frac{\partial}{\partial\phi} + A^2 + \frac{g^2}{16\pi^2}. \tag{C.8}$$

Appendix D

Semiclassical approach

From the result (C.2), the conservation of angular momentum implies

$$\mathbf{L}_1 = \mathbf{L}_2,$$

$$\Delta\mathbf{L}_p = -\Delta\mathbf{L}_{em} = \frac{e_2 g_1 - e_1 g_2}{4\pi c}\left(\frac{\mathbf{r_2}}{r_2} - \frac{\mathbf{r_1}}{r_1}\right). \tag{D.1}$$

Since the particles always move at a large distance from each other, they move practically with uniform linear motions. So, also the relative dyon do practically a uniform linear motion. This linear motion can be parameterized along the z direction. Then, $\mathbf{r_2}/r_2 \to \mathbf{e_z}$.

$$\Delta\mathbf{L}_p = \frac{e_2 g_1 - e_1 g_2}{2\pi c}\mathbf{e_z}. \tag{D.2}$$

According to Quantum Mechanics, the z component of the angular momentun is quantized $\Delta L_z = n\hbar$.Therefore,

$$\Delta\mathbf{L}_p = e_2 g_1 - e_1 g_2 = 2n\pi\hbar c. \tag{D.3}$$

The relation (D.3) was founded by Schwinger.[24]

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