UNIVERSIDAD DE INVESTIGACIÓN DE TECNOLOGÍA EXPERIMENTAL YACHAY

Escuela de Ciencias Matemáticas y Computacionales

TÍTULO: ON THE INVERSE PROBLEM FOR EULER-POINCARÉ EQUATIONS

Trabajo de integración curricular presentado como requisito para la obtención del título de Matemática

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Urcuquí, marzo 2023

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Dedication

To my sister, Luciana.

I will forever be grateful for your support

and all the joy you brought into my life.

Samantha Maribel Naranjo Guevara

Acknowledgment

In the first speech I heard at Yachay Tech Ph.D. Ares Rosakis talked about how choosing a career is one of the most important things students have to do. He said a career must be “an extension of ourselves” and “something that we wake up every morning eager to pursue.” That feeling is exactly what I found studying mathematics. Being part of the Yachay and especially the Mathematical and Computational Sciences School was a lot of fun and it helped me discover who I am.

I am so grateful to all the remarkable and inspiring professors at the university, who were always passionate about science. I would like to thank in particular my advisers Cédric Campos and Eusebio Ariza for your all support even in difficult times. I also want to express my gratitude to professors Hugo Leiva, Juan Mayorga, Rafael Amaro, Antonio Acosta, Saba Infante, Freddy Cuenca, Raúl Manzanilla, Juan Carlos López, Hugo Campos, Franklin Camacho, Juan Lobos, and Andreas Griewank.

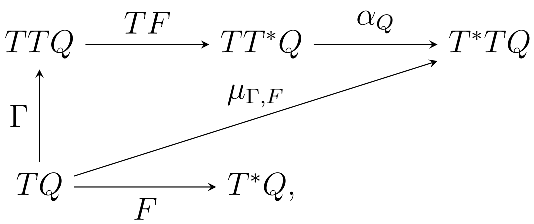
Besides lectures, I also got to spend a lot of time with wonderful people at the German Club. Elisabeth Griewank is one of them. I will forever be grateful for all the opportunities she and her husband offered me. Next, I want to thank my friends Carolina, Kerly, Mateo, Adrián, Eduardo and my classmates. I made some of my best memories with you. You will always have a special place in my heart. Finally, I could not have done this without my family: my parents Genoveva and Wilson, my sister Luciana, my grandparents Luis, Matilde. I appreciate your support.

Samantha Maribel Naranjo Guevara

Resumen

En este trabajo de titulación se estudian los problemas directos e inversos de las ecuaciones de Euler-Lagrange, ecuaciones discretas de Euler-Lagrange, ecuaciones de Euler-Poincaré y las ecuaciones discretas de Euler-Poincaré.

Primero, se estudia el problema inverso de las muy conocidas ecuaciones de Euler-Lagrange, el cual se puede estudiar de diferentes formas. Una de ellas, es la llamada nueva formulación geométrica, que se puede resumir en el siguiente diagrama.



Esta formulación dice que una Ecuación Diferencial de Segundo Orden, SODE (por sus siglas en inglés) Γ, de una variedad continua Q, es variacional (i.e., tiene solución) si y solo si existe un difeomorfismo local de modo que $Im(\mu_{\Gamma,F})$ sea una subvariedad Lagrangiana de (T^*TQ, ω_{TQ}).

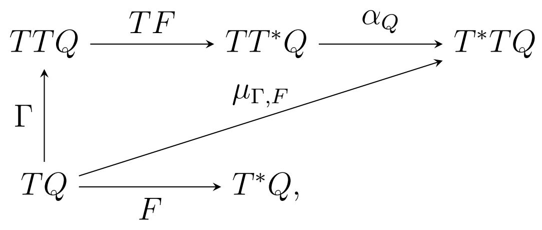
Los objetivos principales de este trabajo son encontrar una versión discreta de este diagrama y condiciones (si es posible) para que una ecuación diferencial de segundo orden en el tangente de una variedad discreta tenga solución.

Palabras Clave:

Ecuaciones de Euler-Lagrange, Ecuaciones de Euler-Poincaré, Ecuaciones Discretas de Euler-Lagrange, Ecuaciones Discretas de Euler-Poincaré, Grupo de Lie, Álgebra de Lie, Invarianza a Izquierda.

Abstract

In this bachelor thesis, we study the direct and inverse problems for the Euler-Lagrange, discrete Euler-Lagrange, Euler-Poincaré, and discrete Euler-Poincaré equations.
First, we study the inverse problem for the widely known Euler-Lagrange equations, which can be approached in different ways. The most important approach for this work is the so called new geometrical formulation, which can be summarized in the following diagram.



It states that a Second Order Differential Equation Γ in the tangent bundle of a continuous manifold Q is variational (i.e., it has a solution) if and only if there is a local diffeomorphism F such that $Im(\mu_{\Gamma,F})$ is a Lagrangian submanifold of (T^*TQ, ω_{TQ}).
The main objectives are to find a trivialized version of this theorem and conditions (if possible) such that a second order differential equation on the tangent bundle of a discrete manifold G is variational.

Keywords:
Euler-Lagrange Equations, Euler-Poincaré Equations, Discrete Euler-Lagrange Equations, Discrete Euler-Poincaré Equations, Lie Group, Lie Algebra, Left Invariance.

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Introduction

In classical mechanics the Euler-Lagrange equations (EL) are of great significance. These equations are be derived from a simple but interesting problem. Imagine we want to move a a toy car from the start to the finish line and we are given a track. We could easily find out how much energy it takes us to move this car, but what if we could find another track, which can minimize the energy we are spending? We now take this example to a more general space.

Given a smooth manifold Q, a smooth function $L : TQ \to \mathbb{R}$ and a path $\gamma : [0,1] \to Q$ from q_0 to q_1 in Q. How much energy does it take go through this path? The answer lies in the following formula

$$\mathcal{A}(\gamma) := \int_0^1 L(\gamma^i(t), \dot{\gamma}^i(t))dt.$$

Out of all the paths from q_0 to q_1, which path let us spend the less energy? We shall minimize $\mathcal{A}$. After minimizing $\mathcal{A}$ we get the equations well-known Euler-Lagrange equations

$$\frac{\partial L}{\partial q^i} - \frac{\mathrm{d}}{dt}\left(\frac{\partial L}{\partial v^i}\right) = 0.$$

Up till now we have only considered continuous spaces. If we let Q be a discrete space, and consider a discrete Lagrangian $\mathbb{L}_d : Q \times Q \to \mathbb{R}$, and a discrete path $\{q_k\}_{k=0,\ldots,n-1}$ we will have to minimize

$$\mathcal{A}_{\mathbb{L}_d}(\{q_k\}_{k=0,\ldots,n-1}) := \sum_{k=0}^{n-1} \mathbb{L}_d(q_k, q_{k+1}).$$

Then, we will get the discrete Euler-Lagrange equations (DEL)

$$D_1\mathbb{L}_d(q_0, q_{k+1}) + D_2\mathbb{L}_d(q_{k-1}, q_k) = 0,$$

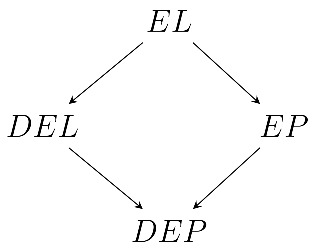
where D_1 is the derivative of $\mathbb{L}_d$ respect to the first component and D_2 is the derivative of $\mathbb{L}_d$ respect to the second component, which are discrete versions of the normal derivative. If we then consider another continuous abstract space, the Lie algebra of a Lie group and a Lagrangian function on that space, we can derive Euler-Poincaré equations (EP):

$$L_g^*(\partial_g l) + \frac{d}{dt}\frac{\partial l}{\partial \xi} + ad^*[\partial_\xi l] = 0.$$

And yet again, when we let the Lie algebra be discrete, and minimize a function, we get the discrete Euler-Poincaré equations (DEP):

$$D_1\mathbb{L}_q(q_k, q_{k+1}) + D_2\mathbb{L}_q(q_{k-1}, q_k) = 0. \tag{1.1}$$

These four equations and spaces are connected by symmetries and isomorphisms. So, solving the problem in one space is similar to solving it in the other three spaces (see the next diagram).



The inverse problem of the calculus of variations can be stated as follows. If we are given a set of differential equations, is it possible to find a set of multipliers such that the system admits a regular solution (a Lagrangian L).

The inverse problem of the calculus of variations has also been studied and it is well known. For instance, [1] gives a detailed solution for this problem, its discrete version and the inverse problem for Euler-Poincaré equations.

We aim to understand the four equation mentioned before hand and their inverse problems. Moreover, we want to find how those spaces are connected to each other.

Our main goal is to solve the inverse problem for the discrete Euler-Poincaré equations using similar methods to those used in the other three inverse problems.

This document is organized as follows: In Chapter 2, we provide the mathematical framework. We fist review some concepts on algebra that include the definitions of groups, algebras and isomorphisms. Then, we present some concepts on topology. Those are vital to understand the topological properties of manifolds and how they resemble Euclidean spaces. Finally, we introduce our main mathematical object: manifolds, smooth structures, tangent bundles, Lie groups and Lie algebras.

In Chapter 3, we present our results. We start with the Euler-Poincaré equation and point out a small mistake found in [2]. Then, we introduce our main result: the solution of the inverse problem for Euler-Poincaré equations.

In Chapter 4, we include some conclusions and give recommendations.

Chapter 2

Theoretical Framework

In this chapter we provide a brief overview on relevant topics for our work and it is structured as follows. First, we give a short review on definitions related to metric spaces, Euclidean spaces and algebras. Then, we cover definitions and theorems on manifolds and Lie groups. Finally, we introduce the Lagrangian formulation and the inverse problem of calculus of variation.

2.1 Some Topics on Topology and Algebra

In this section we review the most fundamental concepts that will help us understand subsequent definitions. We used [3], [4], [5], [6] and [7] as a reference. We begin this section with a quick review on algebra.

Definition 2.1.1 (Equivalence Relation)**.** *Consider two non-empty sets A and B.*

i) We say that r is a relation of A in B if $r \subseteq A \times B$. By definition, $a \in A$ is related to $b \in B$ (denoted $a \sim b$) if and only if $(a, b) \in r$.

ii) A relation r of A in A is said to be an equivalence relation if the following holds

- $\forall a \in A : a \sim a,$
- $\forall a_1, a_2, a_3 \in A : a_1 \sim a_2, a_2 \sim a_3 \implies a_1 \sim a_3,$
- $\forall a_1, a_2 \in A : a_1 \sim a_2 \implies a_2 \sim a_1.$

iii The equivalence class of $a \in A$ is defined as the set $[a] = \{b \in A : b \sim a\}$.

Elements of an equivalent class share the same properties and are said to be equivalent. Choosing a representative element of a class and proving that some property holds is enough to prove it in the whole class.

Definition 2.1.2 (Group)**.** *Given a non-void set G and a binary operation $(\cdot,\cdot) : G \times G \mapsto G$, we say that G is a group if and only if the following properties hold.*

$$\begin{aligned}
&\forall g, h, k \in G : g \cdot (h \cdot k) = (g \cdot h) \cdot k;\\
&\exists e \in G, \forall g \in G : g \cdot e = g \cdot e = g;\\
&\forall g \in G, \exists h \in G : g \cdot h = h \cdot g = e.
\end{aligned}$$

The element e is referred to as the unit or neutral element of G. Moreover, in the third property h is called the inverse of g. From now on We denote the inverse of g as g^{-1}. Additionally, if the group is commutative; i.e.,

$$\forall g, h \in G : g \cdot h = h \cdot g,$$

we say that it is an Abelian group.
Whenever we write $(G, \cdot, e)$ we a referring to the group G endowed with the binary operation $(\cdot,\cdot)$ and the unit element e.

Definition 2.1.3 (Isomorphism)**.** *Let $(G, \cdot, e)$ and $(H, \star, \hat{e})$ be two groups and a bijective function $f : G \mapsto H$. We say that f is an isomorphism if and only if*

$$\begin{aligned}
&f(e) = \hat{e};\\
&\forall g, h \in G : f(g \cdot h) = f(g) \star f(h).
\end{aligned}$$

If f is not bijective it is called a ***homomorphism****.*

In other words, an isomorphism is a function that preserves the structure of a group. One can also say that two isomorphic groups have the same properties.
If there is an isomorphism between two groups G and H, we say that they are isomorphic and it is denoted $G \cong H$. We will say that an isomorphism is canonical when the isomorphism is unique or it is the most natural and simple function between G and H. It is important to mention that since f is bijective, the isomorphism has an inverse.

Definition 2.1.4 (Ring)**.** *Let $(G, \oplus)$ be an Abelian group and $\cdot : G \times G \mapsto G$ be another binary operation on G. We say that $(G, \oplus, \cdot)$ is a ring if and only if*

- *The binary operation $\cdot$ is associative; i.e.,*

$$\forall g, h, k \in G : (g \cdot h) \cdot k = g \cdot (h \cdot k).$$

- *The binary operation $\cdot$ is distributive with respect to $\oplus$; i.e.,*

$$\forall g, h, k \in G : g \cdot (h \oplus k) = (g \cdot h) \oplus (g \cdot k);$$
$$\forall g, h, k \in G : (h \oplus k) \cdot g = (h \cdot g) \oplus (k \cdot g).$$

Depending on the properties a ring has, it can be classified into the following types.

- *Ring with unity if and only if*

$$\exists e \in G, \forall g \in G : g \cdot e = e \cdot g = g; \tag{2.1}$$

- *Commutative ring if and only if*

$$\forall g, h \in G : g \cdot h = h \cdot g; \tag{2.2}$$

- *An integral domain if and only if conditions (2.1), (2.2) hold, and*

$$\forall g \in G \backslash \{0\}, \forall h, k \in G : (g \cdot h = g \cdot k \vee h \cdot g = k \cdot g) \implies h = k; \tag{2.3}$$

- *A field if and only if conditions (2.1), (2.2) hold and*

$$\forall g \in G \backslash \{0\}, \exists k \in G : g \cdot k = k \cdot g = e. \tag{2.4}$$

Definition 2.1.5 (Linear Space)**.** *Given an Abelian group $(G, \odot)$, a field $\mathbb{K}$ and an external operation $\cdot : K \times G \mapsto G$, we say that $(G, \odot, \cdot)$ is a linear (or vector) space over $\mathbb{K}$ if and only if*

$$\exists e \in \mathbb{K}, \forall g \in G : e \cdot g = g;$$
$$\forall g \in G, \forall \alpha, \beta \in \mathbb{K} : (\alpha\beta) \cdot g = \alpha \cdot (\beta \cdot g),$$
$$\forall g \in G, \forall \alpha, \beta \in \mathbb{K} : (\alpha \odot \beta) \cdot g = (\alpha \cdot g) \odot (\beta \cdot g),$$
$$\forall \alpha \in \mathbb{K}, \forall g, k \in G : \alpha \cdot (g \odot k) = (\alpha \cdot g) \odot (\alpha \cdot k).$$

Given two linear spaces $(G, \odot, \cdot)$ and $(H, \oplus, +)$we say that $T : G \to H$ is a linear operator if and only if

$$\forall \alpha \in \mathbb{R}, \forall g_1 g_2 \in G : T(\alpha g_1 \odot g_2) = \alpha T(g_1) \oplus T(g_2).$$

We denote the set of linear operators from G to H as $L(G, H)$. Whenever $H = \mathbb{R}$, an element of $L(G, \mathbb{R})$ is known as a real-valued linear functional or a covector of G. Moreover, we denote $G^* = L(G, \mathbb{R})$, which is referred to as the dual space of G. Furthermore, for any $T\colon G \to H$ there is a canonical function T^* such that $T^* : H^* \to G^*$. Given two covectors T_1 and T_2, we have that $(T_1 \circ T_2)^* = T_2^* \circ T_1^*$, and $(T_1^{-1})^* = (T_1^*)^{-1}$.

Definition 2.1.6 (Inner Product)**.** *Consider a vector space X. An inner product is a function $\langle \cdot, \cdot \rangle : X \times X \mapsto \mathbb{R}$ that satisfies the following properties*

$$\begin{aligned}
&\forall x, y, z \in X : \langle x + y, z \rangle = \langle x, z \rangle + \langle y, z \rangle; \\
&\forall x, y \in X, \forall a \in \mathbb{R} : \langle ax, y \rangle = a \langle x, y \rangle; \\
&\forall x, y \in X : \langle x, y \rangle = \langle y, x \rangle; \\
&\forall x \in X : \langle x, x \rangle \geq 0.
\end{aligned}$$

Definition 2.1.7 (Euclidean Space)**.** *A Euclidean space is a vector space that is equipped with an inner product.*

In particular, we have that $\mathbb{R}$ is a Euclidean space whose inner product is

$$\begin{aligned}
\langle \cdot, \cdot \rangle \colon \mathbb{R}^n \times \mathbb{R}^n &\longrightarrow \mathbb{R} \\
(x, y) &\longmapsto \langle x, y \rangle = x \cdot y.
\end{aligned}$$

When $n = 1$, the inner product is the (scalar) multiplication.

Definition 2.1.8 (Algebra)**.** *Given a linear space $(\mathcal{A}, \odot, \oplus)$ over a field $\mathbb{K}$ and an internal bilinear operation $\cdot\colon \mathcal{A} \times \mathcal{A} \to \mathcal{A}$, we say that $\mathcal{A}$ is a non-associative algebra if*

$$\begin{aligned}
&\forall a, b, c \in \mathcal{A} : a \cdot (b \odot c) = (a \odot b) \cdot (a \odot c). \\
&\forall a, b, c \in \mathcal{A} : (a \odot b) \cdot c = (a \odot c) \cdot (b \odot c). \\
&\forall a, b \in \mathcal{A}, \forall \lambda \in \mathbb{R} : \lambda(a \cdot b) = (\lambda a) \cdot b = a \cdot (\lambda b).
\end{aligned}$$

In addition, we call $\mathcal{A}$ an associative algebra if

$$\forall a, b, c \in \mathcal{A} : a \cdot (b \cdot c) = (a \cdot b) \cdot c;$$

an algebra with unity if

$$\exists e \in \mathcal{A}, \forall a \in \mathcal{A} : a \cdot e = e \cdot a = a;$$

and a commutative algebra if

$$\forall a, b \in \mathcal{A} : a \cdot b = b \cdot a;$$

The set of all $n \times n$ matrices endowed with the ordinary matrix multiplication as its internal operation is an associative algebra with unity. On the other hand, $\mathbb{R}^3$ endowed with the cross-product is a non-associative algebra with unity.

Definition 2.1.9 (Group of transformations)**.** *Given a set A, and let the composition of functions $(\circ)$ be an internal bilinear operation, we define the set*

$$Sym(A) = \{\eta : A \to A : \eta \text{ is bijective.}\}$$

When endowed with $\circ$, $(Sym(A), \circ)$ is a group known as the symmetry group.
A group of transformations is any subgroup of $Sym(A)$.

Next we will introduce the definition of topology and review important topological properties.

Definition 2.1.10 (Metric)**.** *Let X be a non empty set. A metric is a function $d : X \times X \mapsto \mathbb{R}$ such that the following conditions hold.*

$$\forall x, y \in X : d(x, y) = d(y, x);$$
$$\forall x, y \in X : d(x, y) = 0 \iff x = y;$$
$$\forall x, y, z \in X : d(x, z) \leq d(x, y) + d(y, z).$$

Definition 2.1.11 (Metric Space)**.** *Let X be a non-void set and d a metric on that set. We say that the pair (X, d) is a metric space.*

Given a metric space , $x_0 \in X$ and $r > 0$ we define an open ball with radius r centered at x_0 as the set

$$B(x_0, r) = \{x \in X : d(x, x_0) < r\}.$$

The sphere and closed ball with radius r centered at x_0 are given by

$$S(x_0, r) = \{x \in X : d(x, x_0) = r\},$$
$$\overline{B}(x_0, r) = \{x \in X : d(x, x_0) \leq r\},$$

respectively. In the particular case where $X = \mathbb{R}^n$, $n \in \mathbb{N}$ is endowed with the Euclidean metric d_n, we denote $B^n := B(\mathbf{0}, 1)$, $\overline{B}^n := \overline{B}(\mathbf{0}, 1)$, with $\mathbf{0} \in \mathbb{R}^n$, and the sphere as $S^n := S(\mathbf{0}, 1) = \{x \in \mathbb{R}^{n+1} : d(x, \mathbf{0}) = 1\}$, where $\mathbf{0} \in \mathbb{R}^{n+1}$.

Definition 2.1.12 (d-open). *Let (X, d) be a metric space and $A \subseteq X$. We say A is d-open if*

$$\forall x \in A, \exists r > 0 : B(x, r) \subseteq A.$$

Before we introduce the definition of topology, it is important to recall that given a set X, we denote by $\mathcal{P}(X)$ the set of all subsets of X. This set will be of use in subsequent definitions.

Definition 2.1.13 (Topology). *Given X be a non-void space, $\mathcal{P}(X)$ a partition of X and $\mathcal{T} \subset \mathcal{P}(X)$. $\mathcal{T}$ is said to be a topology on X if and only if*

$$\emptyset \in \mathcal{T} \wedge X \in \mathcal{T}; \tag{T1}$$
$$A, B \in \mathcal{T} \implies A \cap B \in \mathcal{T}; \tag{T2}$$
$$(A_{\lambda \in \Lambda}) \subset \mathcal{T} \implies \bigcup_{\lambda \in \Lambda} A_\lambda \in \mathcal{T}. \tag{T3}$$

The elements of $\mathcal{T}$ are known as open sets and the pair $(X, \mathcal{T})$ is known as a topological space. $(A_\lambda)_{\lambda \in \Lambda}$ denotes a family of sets of $\mathcal{T}$. The second condition strictly calls for a finite number of open sets while the third still holds for an infinite number of sets.

Proposition 2.1.1 (Metric Spaces Induce a Topology). *A metric space (X, d) is a topological space with the topology*

$$\mathcal{T}_d = \{A \subseteq X : A \text{ is } d\text{-open}\}.$$

$\mathcal{T}_d$ is known as the induced topology on X. The space $(X, \mathcal{T}_d)$ is also known as a metrizable space.

Proof. In order to prove T1, note that $\emptyset$ is d-open because there are no elements of this set that could be outside of a ball. On the other hand, given an arbitrary $x_0 \in X$ we know there is $r > 0$ such that $B(x_0, r) = \{x \in X : d(x, x_0) < r\}$. In addition, $B(x_0, r) \subseteq X$. Thus, X is d-open.
To prove T2 let $A, B \subseteq X$ be d-open; i.e.,

$$\forall x \in A \exists r_1 > 0 : B(x, r_1) \subseteq A,$$
$$\forall x \in B \exists r_2 > 0 : B(x, r_2) \subseteq B.$$

If we let $x \in A \cap B$ and choose $r = \min(r_1, r_2)$, we can see that

$$B(x, r) = B(x, r_1) \cap B(x, r_2) \subseteq A \cap B.$$

It remains to prove T3. Let $(A_\lambda)_{\lambda\in\Lambda}$ be a family of d-open sets and denote $A = \bigcup_{\lambda\in\Lambda} A_\lambda$. Given a generic $x \in A$, we choose $\lambda \in \Lambda$ such that

$$B(x, r_\lambda) \subseteq A_\lambda.$$

Then, we have

$$x \in B(x, r_\lambda) \subseteq A_\lambda \subseteq \bigcup_{\lambda\in\Lambda} A_\lambda = A.$$

Since x was chosen arbitrarily, we conclude that A is d-open. □

Definition 2.1.14 (Neighbourhood)**.** *Let $(X, \mathcal{T})$ be a topological space, $x \in X$ and $V \subseteq X$. V is a neighbourhood of x if and only if*

$$\exists U \in \mathcal{T} : x \in U \subset V.$$

The set of all neighbourhoods of x is denoted $\mathcal{N}(x)$.

Next, we will see some characterizations of open and closed sets.

Definition 2.1.15 (Interior. Adherence. Boundary.)**.** *Consider a topological space $(X, \mathcal{T})$ a set $A \subseteq X$, and $x \in X$. We say that x is*

- *an interior point of A if and only if $A \in \mathcal{N}(x)$;i.e., $\exists V \in \mathcal{T} : x \in V \subseteq A$;*

- *an adherent point of* A *if and only if* $\forall B \in \mathcal{N}(x) : A \cap B = \emptyset$*;*
- *a boundary point of* A *if and only if* $\forall B \in \mathcal{N}(x) : A \cap B = \emptyset \wedge A^{\mathsf{c}} \cap B = \emptyset$.

The sets of all interior, adherent and boundary points are denoted $int(A)$*,* $ad(A)$ *and* $bd(A)$*, respectively.*

Remark 2.1.1. *From Definition 2.1.15, it follows that for all* $A \subset X$

$$int(A) \subseteq A,$$
$$A \subseteq ad(A),$$
$$bd(A) = ad(A) \cap ad(A^{\mathsf{c}}).$$

Theorem 2.1.1. *Let* $(X, \mathcal{T})$ *be a topological space and* $A \subseteq X$*. Then,*

- $A \in \mathcal{T} \iff int(A) = A$*,*
- $ad(A)$ *is closed,*
- $bd(A)$ *is closed.*

A proof for this theorem can be found in [6] and [8].

Definition 2.1.16 (Hausdorff space)**.** *Given a topological space* $(X, \mathcal{T})$*, we say that it is a Hausdorff space if and only if*

$$\forall x, y \in X, x \neq y, \exists U \in \mathcal{N}(x), \exists V \in \mathcal{N}(y) : U \cap V = \emptyset.$$

The following theorem shows there is a relationship between metric and Hausdorff spaces.

Theorem 2.1.2. *Consider a metric space* (X, d)*. Then,* $(X, \mathcal{T}_d)$ *is a Hausdorff space.*

Proof. Let $A, B \subseteq (X, \mathcal{T}_d)$. It is enough to prove that

$$\forall x, y, x \neq y, \exists r_1, r_2 > 0 : B(x, r_1) \cap B(x, r_2) = \emptyset.$$

Let $x, y \in X$, $x \neq y$, generic and choose $r_1, r_2 \in (0, d(x, y)/3)$. Let's assume by the sake of contradiction that

$$B(x, r_1) \cap B(x, r_2) \neq \emptyset; \text{i.e.,}$$
$$\exists z \in B(x, r_1) \cap B(x, r_2) : d(x, z) < r_1 \wedge d(x, y) < r_2.$$

Using the triangle inequality we have that

$$d(x,y) \leq d(x,z) + d(y,z) < r_1 + r_2 < \frac{2}{3}d(x,y),$$

which is a contradiction. Since x and y were chosen arbitrarily, we have proved $(X, \mathcal{T}_d)$ is a Hausdorff space. □

Definition 2.1.17 (Topological Base). *Let $(X, \mathcal{T})$ be a topological space and $\mathcal{B} \subseteq \mathcal{T}$. We say that $\mathcal{B}$ is a base of $\mathcal{T}$ if and only if*

$$\forall U \in \mathcal{T}, \exists (\mathcal{B}_\lambda)_{\lambda \in \Lambda} \subseteq \mathcal{B} : U = \bigcup_{\lambda \in \Lambda} \mathcal{B}_\lambda.$$

That is, every open set can be written as the union of elements of $\mathcal{B}$.

Definition 2.1.18 (Topological Subspace). *Given a topological space $(X, \mathcal{T})$, a set $Y \subset \mathcal{T}$, the subspace topology (also known as induced topology) $\mathcal{T}_Y$ on Y is given by*

$$\mathcal{T}_Y = \{Y \cap U : U \in \mathcal{T}\}.$$

From definition 2.1.13 we can see that $Y \cap U, U \in \mathcal{T}$ is an open set. It follows that $\mathcal{T}_Y$ is a topology itself. We denote $\mathcal{N}_Y(y)$, $y \in Y$ the set of neighbourhoods of y in Y, and it is not to be confused with $\mathcal{N}(y)$ as defined in 2.1.14. We will see that the Hausdorff property is hereditary. In order to prove that we need the following proposition.

Proposition 2.1.2. *Let $(X, \mathcal{T})$ be a topological space, $Y \subset X$ a non-empty subset and $\mathcal{T}_Y$ its topology. Then,*

$$y \in Y : \hat{U} \in \mathcal{N}_Y(y) \iff \exists U \in \mathcal{N}(y) : \hat{U} = U \cap Y.$$

Proof. First, let's prove that

$$\hat{U} \in \mathcal{N}_Y(y) \implies \exists U \in \mathcal{N}(y) : \hat{U} = U \cap Y.$$

Assume that $\hat{U} \in \mathcal{N}_Y(y)$, i.e.,

$$\exists \hat{A} \in \mathcal{T}_Y : y \in \hat{A} \subset \hat{U}.$$

Then, by definition 2.1.18, we know that

$$\exists A \in \mathcal{T} : \hat{A} = A \cap Y.$$

Note that $A \cup \hat{U} \in \mathcal{T}$. If we take $U = A \cup \hat{U}$, then $U \cap H = (A \cup \hat{U}) \cap H = \hat{A} \cup \hat{U} = \hat{U}$. Now, we will prove that

$$\exists U \in \mathcal{N}(y) : \hat{U} = U \cap Y \implies \hat{U} \in \mathcal{N}_Y(y).$$

Suppose there is a neighbourhood $U \in \mathcal{N}(y)$ and $\hat{U} = U \cap Y$. Then, by definition 2.1.14, there is an open set $A \in \mathcal{T}$ such that $y \in A \subset U$. Therefore.

$$y \in A \cap Y \subset U \cap Y = \hat{U}.$$

Hence, $\hat{U} \in \mathcal{N}_Y(y)$. □

Theorem 2.1.3. *If X is a topological Hausdorff space and Y is a topological subspace of X, then Y is also a Hausdorff space.*

Proof. Assume $(X, \mathcal{T})$ is a Hausdorff space and Y is a subspace of X. We have to prove that

$$\forall x, y \in Y, x \neq y, \exists \hat{U} \in \mathcal{N}_Y(x), \exists \hat{V} \in \mathcal{N}_Y(y) : \hat{U} \cap \hat{V} = \emptyset.$$

Let $x, y \in Y, x \neq y$ be arbitrary. Since X is Hausdorff, we know there exist $U \in \mathcal{N}(x)$ and $V \in \mathcal{N}(y)$ such that $U \cap V \neq \emptyset$. By proposition 2.1.2 we take

$$\hat{U} = U \cap Y,$$
$$\hat{V} = V \cap Y.$$

Then,

$$\hat{U} \cap \hat{V} = (U \cap Y) \cap (V \cap Y) = (U \cap V) \cap Y = \emptyset.$$

We have proved that $(Y, \mathcal{T}_Y)$ is a Hausdorff space. □

Definition 2.1.19 (Second Countable Space)**.** *A topological space $(X, \mathcal{T})$ is a second countable space if and only if $\mathcal{T}$ has a countable base.*

Theorem 2.1.4. *A subspace of a second-countable space is also second countable.*

Proof. Let $(X, \mathcal{T})$ be a topological space and $(Y, \mathcal{T}_Y)$ a topological subspace. Assume $\mathcal{B} \subseteq \mathcal{T}$ is countable base of $\mathcal{T}$. Then $\mathcal{B}_Y = \{B \cap Y : B \in \mathcal{B}\}$ is a countable base for $(Y, \mathcal{T}_Y)$. □

Remark 2.1.2. *We claim that for every $n \in \mathbb{N}$, $(\mathbb{R}^n, \mathcal{T}_{d_n})$ is a second countable space.*

We will now present some concepts on compact sets. First, recall that given a topological space $(X, \mathcal{T})$ and a subset $A \subseteq X$, a family $(A_\lambda)_{\lambda\in\Lambda}$ is a cover of A if $A \subseteq \bigcup_{\lambda\in\Lambda} A_\lambda$. That is, a cover of a given set is a family whose union contains all the elements of that set. A cover is said to be open provided that for every $\lambda \in \Lambda$, $A_\lambda \in \mathcal{T}$.

Definition 2.1.20 (Compact Set)**.** *Let $(X, \mathcal{T})$ be a topological space. A subset $A \subseteq X$ is said to be compact if and only if for every open cover $(A_\lambda)_{\lambda\in\Lambda}$ of A, there is an finite set $I \subseteq \Lambda$ such that $(A_\iota)_{\iota\in I}$ is also an open cover of A. The family $(A_\iota)_{\iota\in I}$ is called an finite open subcover of A.*
The union or intersection of compact sets is also compact.

Definition 2.1.21 (Precompact Set)**.** *A set is precompact if and only if its adherence is compact. Precompact sets can also be called relatively compact.*

Remark 2.1.3. *A ball $B(x, r) \in \mathbb{R}^n, x \in \mathbb{R}^n, r > 0$ is precompact.*

Proposition 2.1.3. *Let $(X, \mathcal{T})$ be a topological second countable space and $\mathbb{A}$ an open cover for X. Then, $\mathbb{A}$ has a countable subcover.*

Proof. Assume that $\mathbb{B}$ is a countable base of X and let

$$\tilde{\mathbb{B}} = \{B \in \mathbb{B} : \exists A \in \mathbb{A}, B \subseteq A\}.$$

Since $\tilde{\mathbb{B}} \subseteq \mathbb{B}$ and $\mathbb{B}$ is a countable base of X, so is $\tilde{\mathbb{B}}$. If we take a generic $x \in X$, we can see that

$$\exists A \in \mathbb{A} : x \in A.$$

On the account of the fact that $\mathbb{B}$ is a base and $\mathbb{A}$ is a cover we have that

$$\exists B \in \mathbb{B} : x \in B \subset A.$$

Note that $B \in \tilde{\mathbb{B}}$. Since x was arbitrary, it follows that $\bigcup\limits_{B\in\tilde{\mathbb{B}}} B = X$. Let

$$\tilde{\mathbb{A}} = \{\tilde{A} \in \mathbb{A} : B \in \tilde{\mathbb{B}} \wedge B \subseteq A\}. \tag{2.5}$$

From 2.5, it is clear that $X \subseteq \bigcup\limits_{\tilde{A}\in\tilde{\mathbb{A}}} \tilde{A}$. Thus, $\tilde{\mathbb{A}}$ set is an open countable cover for X. □

The following statements will help us understand the relationship between closed sets, compactness and continuity.

Proposition 2.1.4. *Let $(X, \mathcal{T})$ be a topological space. If $A \subseteq X$ is compact, then A is a closed set.*

Proposition 2.1.5. *If $(X, \mathcal{T})$ is a Hausdorff space and $A \subseteq X$ is compact, then A is closed.*

Theorem 2.1.5. *Given two topological spaces $(X_1, \mathcal{T}_1)$, $(X_2, \mathcal{T}_2)$ and a continuous function $f : X_1 \to X_2$, if $A \subseteq X_1$ is compact, so is $f(A)$.*

This theorem is of great interest because it shows that continuous functions can transfer properties (compactness in this case).

Definition 2.1.22 (Homeomorphism)**.** *Given two topological spaces X, Y and a bijective function $f : X \mapsto Y$. We say that f is a homeomorphism if and only if both f and f^{-1} are continuous functions.*

Corollary 2.1.1. *Let $(X, \mathcal{T}_1)$ and $(Y, \mathcal{T}_2)$ be two homeomorphic topological spaces. Then, X is compact if and only if Y is compact.*

Definition 2.1.23 (Locally Compact Set)**.** *A topological space $(X, \mathcal{T})$ is locally compact if and only if for every $x \in X$, x has a compact neighbourhood.*

Definition 2.1.24 (Refinement)**.** *Let $(X, \mathcal{T})$ be a topological space, $\mathcal{A}$ and $\mathcal{B}$ two covers of X. Cover $\mathcal{B}$ is called a refinement of $\mathcal{A}$ if and only if*

$$\forall B \in \mathcal{B}, \exists A \in \mathcal{A} : B \subseteq A.$$

Remark 2.1.4. *Note that each subcover is a refinement of a cover.*

Definition 2.1.25 (Locally Finite Set)**.** *Let $(X, \mathcal{T})$ be topological space. A collection of sets $C \subseteq \mathcal{P}(X)$ is said to be locally finite if and only if for each element of X there is a neighbourhood that intersects at most finitely many sets of C.*

Definition 2.1.26 (Paracompact Set)**.** *A topological space $(X, \mathcal{T})$ is paracompact if and only if every open cover of X admits an open locally finite refinement.*

Proposition 2.1.6. *If $(X, \mathcal{T})$ is a compact topological space, then it is a paracompact space.*

Proof. Suppose$X$ is a compact space and let $\mathcal{A}$ be an arbitrary open cover of X. Then, $\mathcal{A}$ has a finite subcover $\mathcal{B}$. Since $\mathcal{B}$ is a locally finite refinement of $\mathcal{A}$ and $\mathcal{A}$ was taken arbitrarily, X is a paracompact space. □

Definition 2.1.27 (Partition). *Let $(X, \mathcal{T})$ be a topological space. We say that $\mathcal{P} \subseteq \mathcal{P}(X)$ is a partition of X if and only if*

- $\emptyset \notin \mathcal{P}$,
- $\mathcal{P}$ *is a cover of* X,
- $\forall P, Q \in \mathcal{P} : P \cap Q = \emptyset$ *($\mathcal{P}$ is pairwise disjoint).*

If $\mathcal{P} = X$ we say the partition is trivial. If all the elements of a partition are open, it is called an open partition.

Definition 2.1.28 (Connected Set). *A topological space $(X, \mathcal{T})$ is said to be disconnected if and only if there is a non trivial open partition $\mathcal{P} \subseteq \mathcal{P}(X)$. If no such partition exists, the space is referred to as connected.*

A more intuitive way to see if a space is disconnected is the following.

Proposition 2.1.7. *A topological space $(X, \mathcal{T})$ is said to be disconnected if and only if there is a binary open partition $\mathcal{P} = \{A, B\}$.*

Proof. Assume $(X, \mathcal{T})$ is disconnected and let $\mathcal{P} = \{P_i\}_{i \in I}$ be a non trivial partition of X. Fix $i_0 \in I$.

$$\{P_{i_0} \bigcup_{i \in I, i \neq i_0} P_i\}$$

is an open binary partition of X.
The converse is trivial. □

Remark 2.1.5. *The space $(\mathbb{R}^n, \mathcal{T}_{d_n})$ is a connected space. The interval $[0, 1]$ is also connected.*

Theorem 2.1.6. *Let $(X_1, \mathcal{T}_1)$, $(X_2, \mathcal{T}_2)$ be two topological spaces and $f : X_1 \to X_2$ a continuous function. If X_1 is connected, then $f(X)$ is connected.*

Connection is also a topological property.

Corollary 2.1.2. *Let $(X_1, \mathcal{T}_1)$ and $(X_2, \mathcal{T}_2)$ be two homeomorphic topological spaces. Then, X is connected if and only if Y is connected.*

Definition 2.1.29 (Convex Set). *A set $S \subseteq \mathbb{R}_n$ is convex if and only if*

$$\forall x, y \in S : \{(1 - t)x + ty : t : [0, 1]\} \subseteq S$$

Proposition 2.1.8. *If $S \subseteq \mathbb{R}_n$ is a convex set, then it is path-connected.*

Definition 2.1.30 (Path-connected Set)**.** *Let $(X, \mathcal{T})$ and $x, y \in X$. An path from x to y in X is a continuous function $\sigma : [0,1] \to X$ such that $\sigma(0) = x$ and $\sigma(1) = y$.*
A space is said to be path-connected if and only if for all $x, y \in X$ there is a path from x to y.

The empty set is not path-connected since there are no two points that can be connected by a path.

Proposition 2.1.9. *If $(X, \mathcal{T})$ is path-connected, then it is connected.*

Proof. By means of contradiction, assume $(X, \mathcal{T})$ is path-connected and disconnected. Then, there is an open partition $\{A, B\}$ of X. Since this space is path-connected, there are $x \in A$, $y \in B$ and a path $\sigma : [0,1] \to X$ from x to y.
Since $[0,1]$ is connected and σ is continuous, by 2.1.6, we have that $\sigma[0,1]$ is connected. Let $\mathcal{P} = \{A \cap \sigma[0,1], B \cap \sigma[0,1]\}$. Since $x \in A \cap \sigma[0,1]$ and $y \in B \cap \sigma[0,1]$, these sets are non-empty and $\mathcal{P}$ is a partition of $\sigma[0,1]$. Hence $\sigma[0,1]$ is disconnected, which is a contradiction. Therefore, X should be connected. □

Proposition 2.1.10. *Let $(X_1, \mathcal{T}_1)$, $(X_2, \mathcal{T}_2)$ be two topological spaces and $f : X_1 \to X_2$ a continuous function. If $A \subseteq X_1$ is path-connected, then $f(A)$ is path-connected.*

Definition 2.1.31 (Locally Path-connected Set)**.** *A topological space $(X, \mathcal{T})$ is said to be locally connected if and only if it has a base of path connected subsets.*

Proposition 2.1.11. *Let $(X, \mathcal{T})$ be a locally connected topological space. X is connected if and only if it is path-connected.*

We have made a quick review on topology. Now we will introduce homotopy and the fundamental group. The latter will be of great importance in the next section. The main reference for these concepts is [9].

Homotopy comes from the Greek words ”homo” and ”topos”, which mean similar place. Two spaces are homotopic if one can transform (deform) into another via a continuous function. Typical examples of homotopies include i) the mug that can be deformed into a torus and ii) two 2-dimensional curves that have the same endpoints. The following figure illustrates the latter example. The doted line represents a curve that can be continuously deformed into the non-dotted line.

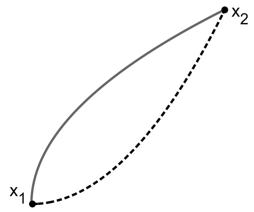


Figure 2.1: Two curves that can be continuously deformed into one another

Definition 2.1.32 (Homotopy). *Consider two topological spaces X and Y and continuous functions F_1 and F_2 from X to Y. An homotopy from F_1 to F_2 is a continuous map $H : X \times I \mapsto Y$, where $I \subset \mathbb{R}$, such that*

$$\forall x \in X : H(x,0) = F_1(x),$$
$$\forall x \in X : H(x,1) = F_2(x).$$

F_1 and F_2 are referred to as homotopic and it is denoted $F_1 \simeq F_2$. If, in addition,

$$\forall t \in \mathbb{R}, \forall x \in A \subseteq X : H(x,t) = F_1(x) = F_2(x),$$

F_1 and F_2 are called homotopic relative to A.
These homotopies are equivalence relations on the set of all continuous functions.

Definition 2.1.33 (Path-Homotopy). *Let X be a topological space and f_1, f_2 two paths on X. We say that f_1 and f_2 are path-homotopic if and only if they are homotopic relative to the set $\{0,1\}$.*
Path homotopy is an equivalence relation on the set of all paths and it is known a path-class.

Definition 2.1.34 (Loop). *Consider a topological space X and $q \in X$. We say that a loop with base on q is path from q to q.*
Loop is an equivalence relation. The set of all classes of loops based at q is denoted $\pi_1(X,q)$. This set is known as the fundamental group of X with base on q.

Definition 2.1.35 (Path Product). *Consider a topological space X and two paths $f_1, f_2 : I \mapsto X$ such that $f_1(1) = f_2(0)$. We define the path product as the function $f_1 * f_2 : I \mapsto X$ such that*

$$f_1 * f_2 = \begin{cases} f_1(2t), & 0 \leq t \leq 1/2 \\ f_2(2t-1), & 1/2 \leq t \leq 1. \end{cases}$$

Proposition 2.1.12. *Let $[f_1]$ and $[f_2]$ be path-classes. Then, their product $[f_1] \cdot [f_2]$ is the class of the path product $[f_1 * f_2]$.*

The fundamental group is indeed a group equipped with the product defined above. The identity of this group is the path-class of the constant path (denoted c_q) and the inverse of a path-class $[f]$ is the class of the reverse path $f(1-s)$. Note that the identity loop c_q is simply a loop that can be contracted to a single point.

Definition 2.1.36 (Simply Connected)**.** *Let X be a topological path-connected space and take an arbitrary $q \in X$. We say that X is simply connected if and only if the fundamental group $\pi_1(X,q) = \{[c_q]\}$.*

Simply connected spaces can be pictured as spaces that have no holes. For instance, the sphere S^2 is simply connected, while the torus is not simply connected.

In the next chapter, we will study the inverse problem of the Euler-Poincaré equations on Lie groups and Lie algebras (which are non-associative). Before we take a look at them, we have to provide some concepts on manifolds.

2.2 Manifolds

A manifold can be seen as a generalization of a surface that is not embedded in a Euclidean space although it locally resembles one. In fact, a manifold exists as an object in its own right [10]. Take a ball, for example. We can study the properties of this ball without considering the space it is in, where or how it is placed; all that matters is the object. Moreover, if we look at neighbourhood of any point in the ball, we will notice that it looks like the Euclidean space $\mathbb{R}^3$. In this section we will cover a few concepts on manifolds and it will be based on [11] and [12].

2.2.1 Smooth Manifolds

Definition 2.2.1 (Topological Manifold)**.** *Consider a topological space M of dimension n (also known as an n-manifold). We say that M is a manifold if and only if the following holds.*

- *M is a Hausdorff space,*
- *M is a second countable space; i.e., $\mathcal{T}$ (the topology in M) has a countable base,*

- *For each $p \in M$ there is a neighbourhood homeomorphic to an open set in $\mathbb{R}^n$; i.e.,*

$$\begin{aligned}&\forall p \in M, \exists U \in \mathcal{T} : p \in U,\\ &\forall p \in M, \exists \hat{U} \subseteq \mathbb{R}^n, \text{ and}\\ &\text{there is an homeomorphism } \varphi : U \to \hat{U}.\end{aligned}$$

The last statement states that every point on the manifold has a neighbourhood that is homeomorphic to an open set on $\mathbb{R}^n$. That is what we mean when we say a manifold resembles a Euclidean space. In order to illustrate what a manifold is, we will see a quick example.

Example 2.2.1 (The unit circle)**.** *First we need to prove the unit circle S^1 is a Hausdorff space. We know that $(\mathbb{R}^2, d_2)$ is a metric space. Thus, by proposition 2.1.1, $(\mathbb{R}^2, \mathcal{T}_{d_2})$ is a topological space. Moreover, by theorem 2.1.2, we know the Hausdorff condition holds. We now consider the subspace topology $(S^1, \mathcal{T}_{S^1})$. From theorem 2.1.3, it follows that the former space is also Hausdorff.*

Using theorem 2.1.4 and the fact that $(\mathbb{R}^2, \mathcal{T}_{d_2})$ is a second countable space, we can see that the subspace $(S^1, \mathcal{T}_{S^1})$ is also second countable.

It only remains to prove the third property. Let's denote the following sets

$$H_u = \left\{(x, y) \in \mathbb{R}^2 \colon y > 0\right\}, \qquad H_b = \left\{(x, y) \in \mathbb{R}^2 \colon y < 0\right\},$$
$$H_r = \left\{(x, y) \in \mathbb{R}^2 \colon x > 0\right\}, \qquad H_l = \left\{(x, y) \in \mathbb{R}^2 \colon x < 0\right\},$$

where u, b, r, l stand for up, bottom, right and left, respectively. On one hand, we choose $U_u = H_u \cap S^1$, $U_b = H_b \cap S^1$, $U_r = H_r \cap S^1$ and $U_l = H_l \cap S^1$. Notice that we need four sets to cover the entire circle. Notice that if we only used U_u and U_b, two points would be missing; (-1,0) and (1,0). Therefore, we also need the left and right halves to cover the entire circle. On the other hand, we take $\hat{U} = B^1 = (-1, 1)$. Now, consider the following function.

$$\begin{aligned} f \colon \hat{U} &\longrightarrow \mathbb{R}\\ x &\longmapsto \sqrt{1 - x^2}.\end{aligned}$$

Note that the graph of f gives us U_u and U_r while U_b and U_l are the graph of $-f$. Finally, *we define the homeomorphism*

$$\begin{aligned}\varphi_u \colon U_u &\longrightarrow \hat{U}\\ (x,y) &\longmapsto \varphi(x,y) = y.\end{aligned}$$

Similarly, we can define φ_b, φ_r and φ_l. This proves that the third condition holds. We conclude that S^1 is a smooth manifold of dimension one.

Example 2.2.2. *Another example is the space $\mathbb{R}^n$, $n \in \mathbb{N}$, itself since it is a second countable Hausdorff space and we can take the homeomorphism φ to be the identity function.*

Definition 2.2.2 (Coordinate Charts)**.** *Given a topological manifold M of dimension n, let $U \subseteq M$, $\hat{U} \subseteq \mathbb{R}^n$ be open sets, and $\varphi : U \to \hat{U}$ a homeomorphism. A coordinate chart on M is the pair (U, φ).*

Moreover,

- if $p \in U$ and $\varphi(p) = 0$, we say the chart φ is centered at p,
- if $\varphi(U)$ is an open ball, then U is referred to as a coordinate ball,
- the map φ is referred to as a local coordinate map and the component function $(x_1, x_2, \ldots, x_n)$ of φ;i.e., $\varphi(p) = (x_1(p), x_2(p), \ldots, x_n(p))$ is called a local coordinate on U.

In example 2.2.1 (U_u, φ_u), (U_b, φ_b), (U_r, φ_r) and (U_l, φ_l) are coordinate charts.

Since topological manifolds have such a strong connection with Euclidean spaces, they also have topological properties.

Lemma 2.2.1. *Consider a topological manifold M of dimension n. M has a base of precompact coordinate balls.*

Proof. We will prove this lemma in two steps. First, in the case where M has only one chart. Then, when M has more than one chart.

- Assume M has a single chart (M, φ) such that $\varphi \colon M \longrightarrow \hat{U} \subseteq \mathbb{R}^n$. As a consequence of the Archimedean property, we can define the set

$$B = \{B(x,r) \subset \mathbb{R}^n : r \in \mathbb{Q} \wedge x \in \mathbb{Q}^n\},$$

and take $B(x, r') \subseteq \hat{U}$ for some $r' > r$, which is a precompact ball. We can clearly see that B is a countable base for the topology of $\hat{U}$. Since φ is a homeomorphism (and thus a continuous function), we can see that each element of the set

$$\varphi^{-1}(B) = \{\varphi^{-1}(B(x, r)) : r \in \mathbb{Q} \wedge x \in \mathbb{Q}^n\}$$

is a precompact coordinate ball. Therefore, $\varphi^{-1}(B)$ is a countable basis for U of precompact coordinate balls.

- From proposition 2.1.3 and since M is a second countable space, we know that M can be covered by a countable set of charts. Note that by definition each point in M is contained in a chart. Thus, their union is a countable base for the topology of M. Let (U, φ) be any of those charts. We already know that U has a countable base of precompact coordinate balls in U, but we still need a set of precompact balls in the manifold. Take a coordinate ball $V \subseteq U$. We know that $ad(V)$ is compact. By proposition 2.1.5 we have that V is closed in M. Therefore, $ad(V)$ is the same in U and M. This means that V is in fact precompact in M. Since V was arbitrary, we have proved that M has a base of precompact coordinate balls.

□

Proposition 2.2.1. *Topological manifolds are locally compact.*

This proposition is a consequence of lemma 2.2.1.

Theorem 2.2.1. *Topological manifolds are paracompact spaces.*

A proof for this theorem can be found in [11]. As we have already seen, second countability plays an important role in the topological properties of a manifold. The latter theorem might be the main consequence because paracompact manifolds are metrizable (the inverse is also true) and having a metric will allow us to do calculus on the manifold.

Proposition 2.2.2. *Let M be an n-manifold. The following propositions are true.*

i) M is locally path-connected.

ii) M is connected if and only if it is path-connected.

Proof. This proposition is an immediate consequence of the theorems on connected sets.

i) Recall that M has a base of precompact coordinate balls (2.2.1). Since balls in $\mathbb{R}^n$ are path-connected, by proposition 2.1.10 we can see that coordinate balls are also path-connected. Hence, M has a base of path-connected sets.

ii) By definition M is a topological manifold. This proof follows directly from proposition 2.1.11.

□

Now that we know what a manifold is, we would like to apply some ideas of calculus on manifolds. In order to do this, we will introduce smooth (or differentiable) structures.

Definition 2.2.3 (Diffeomorphism). *Let M, N be n-manifolds and $f : M \to N$ a bijective function. We say that f is a diffeomorphism if and only if f and f^{-1} are differentiable in the Euclidean sense.*

Diffeomorphisms are an important part of this work because they allow us to transfer ideas and properties from one manifold to another without altering those properties.

Definition 2.2.4 (Transition Map). *Let M be a topological n-manifold and (U, φ), (V, ψ) two charts. The composite function $\varphi \circ \psi^{-1} : \varphi(U \cap V) \to \psi(U \cap V)$ is called the transition map from φ to ψ.*
Two charts are said to be smoothly compatible if either $U \cap V = \emptyset$ or their transition map is a diffeomorphism.

Definition 2.2.5 (Atlas). *An atlas is a collection of charts that covers an n-dimensional manifold M. An atlas is said to be maximal if and only if it is not properly contained in any other atlas. An atlas is called smooth if and only if any two charts of the collection are smoothly compatible.*
A smooth structure on M is called a smooth maximal atlas.

Definition 2.2.6 (Smooth Manifold). *Let M be a topological n-manifold and $\mathcal{A}$ a maximal smooth atlas. A smooth manifold is defined as the pair $(M, \mathcal{A})$.*

Definition 2.2.7 (Open Submanifold). *Consider a manifold M and an open subset $U \subseteq M$. Let's denote A the set of all smooth charts of M and define the set*

$$\mathcal{A} = \{(V, \varphi) \in A : V \subseteq U\}.$$

Note that $\mathcal{A}$ is an atlas for U. Thus, U is an open submanifold of M.

In the next section we will work with smooth manifolds; unless otherwise stated.

Definition 2.2.8 (Smooth Function). *Given an n-manifold M and a function $f : M \to \mathbb{R}$, we say that f is a smooth function if and only for each $p \in M$ there is a chart $(U, \varphi : U \to$*

$\hat{U}$) such that $p \in U$ and the composite function $f \circ \varphi^{-1} : \hat{U} \to \mathbb{R}$ is smooth; i.e., infinitely differentiable. This composite function is known as the coordinate representation of f. The set of all smooth real-valued functions is denoted $C^\infty(M)$ and it is a vector space over $\mathbb{R}$ endowed with the sum and multiplication of functions.

Even though map and function are used as synonyms, to avoid misunderstandings we will use the word map exclusively for functions from one manifold into another.

Definition 2.2.9 (Smooth Map). *Let M and N be smooth manifolds and consider the map $F : M \to N$. F is said to be a smooth map if and only if*

i) $\forall p \in M, \exists (U, \varphi), p \in U, \exists (V, \psi), F(p) \in V : F(U) \subseteq V$, and

ii) $\hat{F} := \psi \circ F \circ \varphi^{-1} : \varphi(U) \to \psi(V)$ is smooth. $\hat{F}$ is known as the coordinate representation of F.

The last function is called the coordinate representation of F.

Proposition 2.2.3. *Let M and N be smooth manifolds. If the map $F : M \to N$ is smooth, then it is continuous.*

Proof. Assume that F is a smooth map. Then, condition i) and ii) from 2.2.9 hold. Let $p \in M$ be arbitrary. Recall that $\psi \circ F \circ \varphi^{-1}$ is differentiable and thus continuous. Since φ and ψ are homeomorphism, their inverse functions exist and are continuous. Therefore, we have that

$$F|_U = \psi^{-1} \circ (\psi \circ F \circ \varphi^{-1}) \circ \varphi : U \to V$$

is a composition of continuous functions. Hence, it is continuous. Since p was generic, we have proved that F is continuous in M. □

Proposition 2.2.4. *Consider smooth topological manifolds M, N and P. Then*

i) the identity map $Id : M \to M$ is a smooth map;

ii) every constant map $C : M \to N$ is smooth;

iii) if $U \subseteq M$ is an open submanifold of M, then the inclusion map $id : U \to M$ is smooth;

iv) if $F : M \to N$ and $G : N \to P$ are smooth maps, then so is $G \circ F : M \to P$.

The study of manifolds is extensive. However, we have recall only those definitions and theorems that are relevant for our work. Further concepts can be found in [11] and [10].

2.2.2 Tangent Bundle

In this section we will see a generalization of total derivative and derivative in the direction of a vector based at some point.

Definition 2.2.10 (Geometric Tangent Vector)**.** *Given a vector $a \in \mathbb{R}^n$, we say that the geometric tangent space to $\mathbb{R}^n$ at a is the set*

$$\mathbb{R}^n_a = \{a\} \times \mathbb{R}^n = \{(a, v) : v \in \mathbb{R}^n\}.$$

This can be understood as the set of arrows attached to a.
From now on, we will denote $v_a = (a, v)$. An element of $\mathbb{R}^n_a$ is called a geometric tangent vector and it is a real vector space whose basis are the unit vectors $e_i|_a, i = 1, \ldots, n$.

Definition 2.2.11 (Directional Derivative)**.** *Let $a \in \mathbb{R}^n$ and $v_a \in \mathbb{R}^n_a$ be a geometric tangent vector. The map $D_v|_a : C^\infty(\mathbb{R}^n) \to \mathbb{R}$ defined by*

$$D_v|_a f := D_v|_a f(a) = \left.\frac{d}{dt}\right|_{t=0} f(a + tv) \tag{2.6}$$

provides the derivative of f in the direction of v at a.

Remark 2.2.1. *This function satisfies the product rule*

$$D_v|_a(fg) = f(a)D_v|_a g + g(a)D_v|_a f. \tag{2.7}$$

Moreover, if we let $v_a = v^i e_i|_a$ and use the chain rule, we get

$$D_v|_a f = v^i \frac{\partial f}{\partial x^i}(a). \tag{2.8}$$

Note that we have used the Einstein summation convention.

Definition 2.2.12 (Derivation)**.** *Let $a \in \mathbb{R}^n$ and $\omega : C^\infty(\mathbb{R}^n) to \mathbb{R}$. Function ω is said to be a derivation at a provided it is linear over the real numbers and it satisfies the product rule*

$$\omega(fg) = f(a)\omega g + g(a)\omega f. \tag{2.9}$$

The set of all derivation of at a is denoted $T_a\mathbb{R}^n$ and it is a vector space under the sum of *derivations and scalar multiplication; i.e., for all* $\omega_1, \omega_2 \in T_a\mathbb{R}^n$, $f \in C^\infty(\mathbb{R}^n)$ *and* $c \in \mathbb{R}$

$$(\omega_1 + \omega_2)f = \omega_1 f + \omega_2 f \text{ and}$$
$$(c\omega_1)f = c(\omega_1 f).$$

$T_a\mathbb{R}^n$ *is also known as the tangent space of* $\mathbb{R}^n$ *at* a.

Proposition 2.2.5. *Let* $a \in \mathbb{R}^n$. *The map* $\mathbb{R}^n_a \ni v_a \mapsto D_v|_a \in T_a\mathbb{R}^n$ *is an isomorphism and thus* $\mathbb{R}^n_a \cong T_a\mathbb{R}^n$.

The proof of proposition 2.2.5 can be found in [11].

Corollary 2.2.1. *Given* $a \in \mathbb{R}^n$ *the derivatives*

$$D_{e_1}|_a = \left.\frac{\partial}{\partial x^1}\right|_a, \ldots, D_{e_n}|_a = \left.\frac{\partial}{\partial x^n}\right|_a \tag{2.10}$$

form a basis for $T_a\mathbb{R}^n$.

Note that derivations are not actually vectors at some point, but linear functions. However, thinking of them as vectors helps us understand their meaning. Moreover, we will see that properties of derivatives are similar to those of derivatives in the Euclidean space $\mathbb{R}^n$.

Since we will be working on manifolds, we would like to find a generalization of derivation and the set $T_a\mathbb{R}^n$. The definition of a derivation on a manifold is analogue to definition 2.2.12. What changes is the space where the derivation is defined.

Definition 2.2.13 (Tangent Vector on Manifolds)**.** *Given a manifold* M *and an element* $p \in M$, *we say that a function* $v : C^\infty(M) \to \mathbb{R}$ *is a derivation at* p *provided that*

$$\forall f \in C^\infty(M) : v(fg) = f(p)vg + g(p)vf. \tag{2.11}$$

Similarly, T_pM *is a vector space known as the tangent space to* M *at* p.

Lemma 2.2.2. *Let* M *be a manifold,* $p \in M$, $v \in T_pM$ *and* $f, g \in C^\infty(M)$. *Then, the following properties hold.*

i) If f *is a constant linear map, then* $vf = 0$.

ii) If $f(p) = g(p)$, *then* $v(fg) = 0$.

We can see that 2.2.2 also holds when $M = \mathbb{R}^n$. Recall that the total derivative of a map between Euclidean spaces is a linear map that is represented by its Jacobian matrix. In the more abstract setting of manifolds, this concept (total derivative) can be generalized as a linear map between tangent spaces.

Definition 2.2.14 (Differential)**.** *Consider two smooth n-manifolds M and N, $p \in M$ arbitrary, and a smooth map $F : M \to N$. The linear map*

$$dF_p \colon T_pM \to T_{F(p)}N$$

is said to be the differential of F at p. Moreover, for each $f \in C^\infty(N)$ and $v \in T_pM$, we have

$$dF_p(v)(f) = v(f \circ F).$$

Notice that $dF_p(v)$ is defined from $C^\infty(N)$ to $\mathbb{R}$ is a linear operator.

Remark 2.2.2. *Given $f, g \in C^\infty(N)$ we have that*

$$\begin{aligned} dF_p(v)(fg) &= v((fg) \circ F) \\ &= v((f \circ F)(g \circ F)) \\ &= (f \circ F)(p)v(g \circ F) + (g \circ F)(p)v(f \circ F) \\ &= f(F(p))dF_p(v)(g) + g(F(p))dF_p(v)(f). \end{aligned}$$

The following properties are essential and will be used frequently in the next chapter.

Proposition 2.2.6. *Given smooth manifolds M, N and P, $p \in M$, and two smooth maps $F : M \to N$, $G : N \to P$, the following statements hold*

1. *$dF_p : T_pM \to T_{F(p)}N$ is a linear map.*
2. *$d(G \circ F)_p : T_pM \to T_{G \circ F(p)}P$ and $d(G \circ F)_p = dG_{F(p)} \circ dF_p$.*
3. *$d(Id_M)_p = Id_{T_pM} : T_pM \to T_pM$ (the differential of the identity in M is the identity map in the tangent space T_pM).*
4. *If F is a diffeomorphism, then dF_p is an isomorphism and $(dF_p)^{-1} = d(F^{-1})_{F(p)}$.*

Another important result is that tangent spaces preserve the dimension of the manifold.

Proposition 2.2.7. *If M is a smooth n-manifold and $p \in M$, then T_pM has the same dimension as M.*

An analog result for 2.2.1 is the following.

Proposition 2.2.8. *Let M be a topological space, $p \in M$. Then, for each chart $(U, (x^i))$ that contains p the coordinate vectors*

$$\left.\frac{\partial}{\partial x^1}\right|_p, \ldots, \left.\frac{\partial}{\partial x^n}\right|_p$$

form a basis of T_pM.

Using the coordinate vector we can rewrite the differential and do computations easily. Let $U \subset \mathbb{R}^n$, $V \subset \mathbb{R}^m$ be open subsets, $p \in U$. We let $(x^1, \ldots, x^n)$ be the coordinates in $U$and $(y^1, \ldots, y^m)$ the coordinates in V. Consider the smooth map $F : U \to V$, the differential $dF_p : T_pM \to T_{F(p)}N$ and $f : C^\infty(\mathbb{R}^n)$. Then

$$\begin{aligned} dF_p\left(\left.\frac{\partial}{\partial x^i}\right|_p\right)f &= \left.\frac{\partial}{\partial x^i}\right|_p (f \circ F) \\ &= \frac{\partial f}{\partial y^j}(F(p))\frac{\partial F^j}{\partial x^i}(p) \\ &= \left(\frac{\partial F^j}{\partial x^i}(p)\left.\frac{\partial}{\partial y^j}\right|_{F(p)}\right)f. \end{aligned}$$

Therefore, we have that

$$dF_p\left(\left.\frac{\partial}{\partial x^i}\right|_p\right) = \frac{\partial F^j}{\partial x^i}(p)\left.\frac{\partial}{\partial y^j}\right|_{F(p)}. \tag{2.12}$$

Recall that we are using the summation notation and 2.12 is actually the Jacobian matrix of F at p; i.e.,

$$\begin{pmatrix} \frac{\partial F^1}{\partial x^1}(p) & \cdots & \frac{\partial F^1}{\partial x^n}(p) \\ \vdots & \ddots & \vdots \\ \frac{\partial F^m}{\partial x^1}(p) & \cdots & \frac{\partial F^m}{\partial x^n}(p) \end{pmatrix}. \tag{2.13}$$

In a similar way, consider two smooth manifolds M, N, a smooth map $F : M \to N$, a chart (U, φ) that contains p and a chart (V, ψ) that contains p. Recall that the coordinate representation (which was given in definition 2.2.9) is

$$\hat{F} : \psi \circ F \circ \varphi^{-1} : \varphi(U \cap F^{-1}(V)) \to \psi(V).$$

Thus, $d\hat{F}$ is represented by the Jacobian matrix above. Now, for the sake of simplicity, we let $\hat{p} := \varphi(p)$ and then compute dF_p and get

$$dF_p\left(\frac{\partial}{\partial x^i}\bigg|_p\right) = \frac{\partial \hat{F}^j}{\partial x^i}(\hat{p})\frac{\partial}{\partial y^j}\bigg|_{F(p)}.$$

The differential dF_p can also be denoted F_* and throughout this work we will work with both notations. Moreover, F_* will be referred to as the pushforward of F.

Definition 2.2.15 (Tangent Bundle). *Let Q be an n-dimensional smooth manifold. The tangent bundle of Q is defined as the disjoint union of all the tangent spaces; i.e.,*

$$TQ = \coprod_{p\in Q} T_pQ = \{(p,v) : p \in Q \text{ and } v \in T_pQ\}, \tag{2.14}$$

where T_pQ is the tangent space of Q at point p.

Now, we consider the natural projection $\pi : TQ \to Q$ such that $(p, v) \mapsto p$. Using this map we can define a natural topology and smooth structure on TQ by using charts (U, φ) of Q. We consider $(\pi^{-1}(U), \widetilde{\varphi})$, where $\widetilde{\varphi} : \pi^{-1}(U) \to \mathbb{R}^{2n}$ and

$$\widetilde{\varphi}\left(v^i\frac{\partial}{\partial x^i}\bigg|_p\right) := \Big(x^1(p),\dots,x^n(p),v^1,\dots,v^n\Big) = (x^i, v^i), \tag{2.15}$$

with $\varphi(p) = (x^1(p), \dots, x^n(p))$, which are called natural coordinates on TQ. This structure makes TQ a $2n$-dimensional manifold.

Coordinates (x^i, v^i) are called the natural coordinates on TQ. Let M and N be smooth manifolds and F a smooth map. The differential $dF : TM \to TN$ is called the global differential and is given by

$$dF(x^1,\dots,x^n) = \left(F^1(x),\dots,F^n(x),\frac{\partial F^1}{\partial x^i}(x)v^i,\dots,\frac{\partial F^n}{\partial x^i}(x)v^i\right).$$

The properties of this map are a generalization of properties 2, 3 and 4 in 2.2.6. However, there is a slight difference from 4. In fact, if F is a diffeomorphism, then so is it differential.

Definition 2.2.16 (Vector Field). *A smooth vector field on Q is a smooth map $X : Q \to TQ$ such that for all $p \in Q$ we have $X_p = X(p) \in T_pQ$*

The set of smooth vector fields is denoted by $\mathfrak{X}(M)$. If $(U, (x^i))$ are local coordinates of Q, for any $p \in U$ we can write $X_p \in T_pM$ in terms of the coordinate base:

$$X_p = X^i(p) \frac{\partial}{\partial x^i}\bigg|_p, \tag{2.16}$$

where each $X^i : U \to \mathbb{R}$ is $C^\infty(\mathbb{R})$ and called the component functions of X. Since each X^i is smooth, we can write (2.16) as an equation between vector fields.

$$X = X^i \frac{\partial}{\partial x^i} \tag{2.17}$$

Definition 2.2.17 (Integral Curves)**.** *Let V be a vector field on Q and $\gamma : J \to M$ be a differentiable curve. Then γ is called an integral curve of V if*

$$\forall t \in J : \gamma'(t) = V_{\gamma(t)} \tag{2.18}$$

Definition 2.2.18 (Flow Domain)**.** *A open subset $\mathcal{D}$ of $\mathbb{R} \times Q$ is a flow domain if*

$$\forall p \in Q : \mathcal{D}^{(p)} = \{t \in \mathbb{R} : (t, p) \in \mathcal{D}\}$$

contains the zero and is an open interval.

Definition 2.2.19 (Smooth Flow)**.** *A smooth flow is a smooth map $\theta : \mathcal{D} \to Q$, with $\mathcal{D}$ being a flow domain with the following properties*

- $\forall s \in \mathcal{D}^{(p)}, \forall t \in \mathcal{D}^{(\theta(s,p))}, s + t \in \mathcal{D}^{(P)} : \theta(t, \theta(s, p)) = \theta(t + s, p),$
- $\forall p \in Q : \theta(0, p) = p.$

For a smooth flow we define the *infinitesimal generator of* θ as the vector field V such that

$$\forall p \in Q : V_p = \theta^{(p)\prime}(0),$$

where $\theta^{(p)} : \mathbb{R} \to Q$ is such that $\theta^{(p)}(t) = \theta(t, p)$.

The **Fundamental Theorem on Flows** (see [11]) states that for any $V \in \mathfrak{X}$ there exits a unique smooth maximal flow $\theta : \mathcal{D} \to Q$ such that its infinitesimal generator is V. This flow is called the *flow of V* .

2.2.3 Lie Group

We consider a new kind of manifolds, called Lie groups. These manifolds are the group of symmetries of another manifolds.

Definition 2.2.20 (Lie Group)**.** *A Lie group G is an n-dimensional smooth manifold which has a group structure where the multiplication and inverse operations are both smooth.*

For any element g of a Lie group G we define the map $L_g : G \to G$ as follows,

$$L_g(h) = gh. \tag{2.19}$$

This map is called *left translation*. It can easily be shown that L_g is a diffeomorphism on G.

As an example of a Lie group we consider the general linear group $GL(n, R)$, which is the set of $n \times n$ matrices with real entries, with the usual matrix multiplication as the multiplication operation. The smoothness of this operation comes from the fact that entries of matrix product is given by polynomial functions. The same apply for the inverse.

Since Lie groups are in particular groups, we can define homomorphisms between them.

Definition 2.2.21 (Lie Group Homomorphism)**.** *Let G, H be Lie groups. A Lie group homomorphism is a map $F : G \to H$ that is smooth and a homomorphism in the algebraic sense.*

Lie groups act on other manifolds. Let M a smooth manifold and G a Lie group. The **left action** of G on M is a map

$$\theta : G \times M \to M$$
$$(g, p) \mapsto gp$$

If this map is smooth, we called it a **smooth action**. We will see that left actions arise naturally in the next chapter.

By fixing $g \in G$ we define $\theta_g : M \to M$ given by $\theta_g(p) = gp$. Hence, for any $g_1, g_2 \in G$ we have the following properties

$$\theta_{g_1} \circ \theta_{g_2} = \theta_{g_1 g_2},$$

$$\theta_e = Id_M$$

Since $\theta_{g^{-1}}$ is the inverse of θ_g and it is smooth, θ_g is a diffeomorphism.

From the algebraic point of view we know that an orbit of $p \in M$ is given by the set $G \cdot p := \{g \cdot p : g \in G\}$ and a stabilizer is $G_p := \{g \in G : g \cdot p = p\}$.

The action is *transitive* if the only orbit is the whole M, while it is called *free* if the stabilizer is trivial; i.e., $G_p = \{e\}$.

Let G be a Lie group. Let us consider the left action $g_1 \mapsto g_2 g_1^{-1}$. When we consider a Lie group acting on itself we get some particular properties. If g_1 and g_2 are generic elements of G, we take $g = g_2 g_1^{-1} \in G$. Then since $g g_1 = g_2$ we get that G acts transitively on itself. Also, it can be proved that the action is free.

Definition 2.2.22 (F-related)**.** *Let $X \in \mathfrak{X}(M)$, M, N smooth manifolds, and $F : M \to N$ a smooth map. Let us assume that there exists a smooth vector field Y on N such that*

$$\forall p \in M : dF_p(X_p) = Y_{F(p)}$$

Then we say that X and Y are F-related.

Let Q be a manifold, $X \in \mathfrak{X}(Q)$ and $f \in \mathcal{C}^\infty(Q)$. We define $Xf : Q \to \mathbb{R}$ such that $p \mapsto X_p f$. Moreover, we have that for all $f, g \in C^\infty(Q))$

$$X(fg) = fXg + gXf.$$

2.2.4 Lie Algebra

Definition 2.2.23 (Lie Brackets)**.** *Let X and Y be two smooth vector fields on Q. The Lie bracket of X and Y is a new smooth vector field defined by*

$$[X, Y] = XYf - YXf, \ \forall f \in \mathcal{C}^\infty(Q), \tag{2.20}$$

where $(XYf) = X(Yf)$.

By writing $X = X^i \partial/\partial x^i$ and $Y = Y^j \partial/\partial x^j$ respect to some smooth local coordinates (x^i) for Q, we can write the coordinate expression of Lie bracket

$$[X, Y] = \left(X^i \frac{\partial Y^j}{\partial x^i} - Y^i \frac{\partial X^j}{\partial x^i} \right) \frac{\partial}{\partial x^j}. \tag{2.21}$$

The Lie bracket has the following properties.

1. Bilinearity: For all $a, b \in \mathbb{R}$

$$[aX + bY, Z] = a[X, Z] + b[Y, Z], \tag{2.22}$$

$$[Z, aX + bY] = a[Z, X] + b[Z, Y]. \tag{2.23}$$

2. Antisymmetry:

$$[X, Y] = -[Y, X].$$

3. Jacobi Identity:

$$[X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]] = 0.$$

4. For $f, g \in \mathcal{C}^\infty(Q)$

$$[fX, gY] = fg[X, Y] + (fXg)Y - (gYf)X.$$

Now we consider a definition that describes a way to calculate directional derivatives of vector fields.

Definition 2.2.24 (Lie Derivative)**.** *Let p be a element of Q, $V, W \in \mathfrak{X}(Q)$, and θ the flow of V. We define the Lie derivative of W with respect to V by*

$$(\mathcal{L}_V W)_p = \lim_{t \to 0} \frac{d(\theta_{-t})_{\theta_t(p)}(W_{\theta_t(p)}) - W_p}{t}, \tag{2.24}$$

which is well defined since both vectors in the numerator belong to T_pQ.

The Lie derivative can be expressed independently of the flow. In fact, we have that

$$\mathcal{L}_V W = [V, W]. \tag{2.25}$$

Definition 2.2.25 (Lie Algebra)**.** *A Lie algebra is a real vector space $(\mathfrak{g}, [\ ])$ where the internal operation called bracket satisfies the bilinearity, anti-symmetry and Jacobi identity properties for any $X, Y.Z \in \mathfrak{g}$.*

The set $\mathfrak{X}(Q)$ endowed with the Lie bracket operation is a Lie algebra. Now suppose we have a Lie group G. A vector space X is called *left-invariant* if

$$d\left(L_g\right)_{g'}\left(X_{g'}\right) = X_{gg'}, \quad \text{for all } g, g' \in G.$$

For any X, Y smooth left-invariant vector fields on G, $[X, Y]$ is also left-invariant. Thus we define the **Lie algebra of G** as the set of smooth left-invariant vector fields and denote it $Lie(G)$.

In addition, it can be shown that the map $\varepsilon : Lie(G) \to T_eG$ defined by $\varepsilon(X) = X_e$ is an isomorphism, i.e. $Lie(G) \cong T_eG$. Thanks to this relation, we can identify unambiguously a left-invariant vector field on G with a vector in T_eG. Thus, G, T_eG and $Lie(G)$ have the same dimension.

For Lie algebras $\mathfrak{g}$ and $\mathfrak{h}$ a map $K : \mathfrak{g} \to \mathfrak{h}$ is called a *Lie algebra homomorphism* if

$$\forall X, Y \in \mathfrak{g} : K[A, Y] = [K(A), K(Y)] \tag{2.26}$$

If a map K satisfies (2.26) we say that it preserves brackets.

From a Lie group homomorphism we can get a Lie algebra homomorphism. For Lie groups G and H and their respective Lie algebras $\mathfrak{g}$ and $\mathfrak{h}$, consider a Lie group homomorphism $F : G \to H$. Then we have that,

$$\forall X \in \mathfrak{g}, \exists! Y \in \mathfrak{h} : X \text{ and } Y \text{ are } F\text{-related.}$$

Usually the unique vector field Y is denoted as F_*X. One can show that the map

$$F_* : \mathfrak{g} \to \mathfrak{h} \tag{2.27}$$

defines a Lie algebra homomorphism.

Definition 2.2.26 (One parameter Subgroup)**.** *A one-parameter subgroup is a Lie group homomorphism $\gamma : \mathbb{R} \to G$, with $\mathbb{R}$ the additive group.*

The following statement gives us an important tool to do computations on Lie algebras.

Corollary 2.2.2. *Given smooth manifolds, a diffeomorphism $F : M \to N$ and two vector fields $X, Y \in \mathfrak{X}(M)$, we have that the pushforward of the Lie bracket is the Lie bracket of the pushforward; i.e.,*

$$F_*[X, Y] = [F_*X, F_*Y].$$

2.2.5 Cotangent Bundle

We will give a brief review on tangent covectors. We have seen that covectors are elements of a dual space. Let V be a finite vector space. The dimension of of V is the same as the dimension of V^*. In order to avoid misunderstandings, we will use upper indices when it comes to a basis of the V^* and lower indices for the basis of V.
Tangent covectors are a generalization of covectors.

Definition 2.2.27 (Cotangent Space)**.** *Let M be a smooth manifold and $p \in M$. We denote $T_p^*M := (T_pM)^*$ the cotangent space of M at p.*
The coordinate representation of a covector ω is

$$\omega_i = \omega\left(\frac{\partial}{\partial x^i}\bigg|_p\right).$$

Definition 2.2.28 (Cotangent Bundle)**.** *Let M be an n-dimensional smooth manifold. The tangent bundle of M is defined as the disjoint union of all the cotangent spaces; i.e.,*

$$T^*M = \coprod_{p\in Q} T_p^*M. \tag{2.28}$$

A section of the dual space T^*M is called a covector field. The component functions of a covector field ω are characterized by

$$\omega_i(p) = \omega_p\left(\frac{\partial}{\partial x^i}\bigg|_p\right).$$

Given two smooth manifolds M, N, $p \in M$, a smooth map F between them, we can see that the differential $dF_p : T_pM \to T_{F(p)}N$ has a dual function given by

$$(dF_p)^* := dF_p^* : T_{F(p)}^*N \to T_p^*M.$$

This function is known as the **pullback by F at** p. If, in addition, we are given a covector field ω, then the pullback of ω by F is defined by

$$(F^*\omega)_p = dF_p^*(\omega_{F(p)})$$

and for every vector $v \in T_pM$ we have that $(F^*\omega)_p(v) = \omega_{F(p)}(dF_p(v))$.

2.2.6 The exponential Map

Definition 2.2.29 (Exponential Map)**.** *Let G be a Lie group. The exponential map* exp *from its Lie algebra $Lie(G)$ to G is defined as,*

$$\exp(X) = \gamma(1) \tag{2.29}$$

where γ is the one parameter subgroup generated by X, which coincides with the integral curve γ of X such that $\gamma(0) = e$.

For any $X \in Lie(G)$ its integral curve can be expressed as

$$\gamma(s) = \exp(sX). \tag{2.30}$$

From (2.30), for $g \in G$, $\eta \in Lie(G)$ we get

$$\lim_{\varepsilon \to 0} g \exp(\varepsilon\eta) = g. \tag{2.31}$$

2.2.7 Adjoint representation

Let G be a Lie group, $\mathfrak{g}$ its Lie algebr and $g \in G$. The Lie homomorphism

$$\begin{aligned} C_g : G &\to G \\ h &\mapsto ghg^{-1}, \end{aligned}$$

called the conjugation map, has interesting properties. From this map we define $Ad(g)$ as its induced Lie algebra homomorphism, i.e., $Ad(g) := (C_g)_* : \mathfrak{g} \to \mathfrak{g}$ given as in (2.27). We have that, $(C_g)_* \in GL(\mathfrak{g})$, the general linear group of the vector space $\mathfrak{g}$. From the previous development, we get a new map called the *Adjoint representation of G*, $Ad : G \to GL(\mathfrak{g})$, which is a Lie group representation [13]. In the same way, we define the map $\mathsf{ad} : \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g})$, called the *adjoint representation of* $\mathfrak{g}$ defined by using the Lie brackets,

$$\forall X \in \mathfrak{g} : \mathsf{ad}[X](Y) = \mathsf{ad}_X(Y) := [X, Y]$$

It can be shown that ad is a Lie algebra homomorphism. Moreover we have that $Ad_* = \mathsf{ad}$. From ad we also consider its dual maps ad^* from $\mathfrak{g}^*$ to itself with the following property.

$$\forall X, \eta \in \mathfrak{g}, \omega \in \mathfrak{g} : \langle\, \mathsf{ad}^*_X(\omega), \eta\rangle = \langle\, \omega, \mathsf{ad}_X(\eta)\rangle \tag{2.32}$$

2.2.8 Differential Forms and Tensors

In the inverse problem of calculus of variation we have to deal with tensors and differential forms. We will introduce those concepts and their properties. Our main guide for this section is [14]. Let V be a vector space. Recall that $L(V, \mathbb{R})$ is the set of all real linear functions on V. Moreover, $L(V, \mathbb{R}) = V^*$.

Given $r, s \in \mathbb{N} \cup \{0\}$, we denote by $T^r_s(V) := L^{r+s}(V^*, \overset{r}{\ldots}, V^*, V, \overset{s}{\ldots}, V; \mathbb{R})$ the set of all real multilinear functions on $V^* \overset{r}{\times} V^* \times V \overset{r}{\times} V$. The elements of $T^r_s(V)$ are called **tensors** of type (r, s) or (r, s)-tensors. It is easy to see that elements of $L(V, \mathbb{R}) = V^*$ and $L(V^*, \mathbb{R}) = V^{**} = V$ are $(0, 1)$-tensors and $(1, 0)$-tensors respectively.

Let M be a manifold. The definition of a tensor field is analogous to that of vector field. In fact, a tensor field is a section of the set $T^r_s(M)$;i.e, a vector field is an element of $\mathfrak{X}(T^r_s(M))$.

A differential one form is an element of $\mathfrak{X}(T^0_1(M))$.

In the preceding sections we have cover the necessary concepts to understand the Lagrangian formulation and the inverse problem of the calculus of variations along with its variations.

2.3 Classical Mechanics

We will introduce the Euler-Lagrange equations considering two different approaches. First, we will use calculus of variations an than we will take a look at the geometrical formulation.

2.3.1 Lagrangian Formulation

We let $L : TQ \to \mathbb{R}$ be a Lagrangian on a smooth manifold Q. This space will be referred to as the configuration space. The local coordinates of Q are denoted (q^i), while the local coordinates on TQ are (q^i, v^i), where

$$v = v^i \frac{\partial}{\partial q^i} \tag{2.33}$$

The tangent bundle TQ describes the states of our system, see [15], which are position and velocity.

Let us consider the following set of curves which map real values to Q:

$$\mathcal{C}^2\left(\left[t_0, t_1\right], Q, q_0, q_1\right) = \mathcal{C}^2(q_0, q_1) = \left\{\gamma \in \mathcal{C}^2\left(\left[t_0, t_1\right], Q\right) : \gamma\left(t_i\right) = q_i, i = 0, 1\right\}. \tag{2.34}$$

For these function we consider their lifts, described in the next definition.

Definition 2.3.1 (Lift of a curve). *For a curve in (2.34) we define its lift as the smooth curve $\tilde{\gamma} : [t_0, t_1] \to TQ$ such that*

$$\forall f \in C^\infty(Q) : \left(\tilde{\gamma}\left(t_0\right)\right)(f) = \frac{\mathrm{d}}{\mathrm{d}t}(f \circ \gamma)\bigg|_{t=t_0}. \tag{2.35}$$

In the local coordinates on TQ we can write

$$\tilde{\gamma}(t) = \left(\gamma^i(t), \dot{\gamma}^i(t)\right). \tag{2.36}$$

In order to use the variational approach we have to add perturbations to the curves, but it is important that the endpoints remain untouched. Otherwise, we would end up with a curve with different start and end points.

Definition 2.3.2. *Let $\gamma \in \mathcal{C}^2\left(q_0, q_1\right)$ and take a small $\epsilon > 0$. We say that the curve*

$$\gamma_s : [-\epsilon, \epsilon] \to \mathcal{C}^2\left(q_0, q_1\right)$$

is a variation of γ if and only if γ_0 is equivalent to γ.

Now consider the vector field $\delta\gamma$ (over the curve γ) such that for a fixed t in $[t_0, t_1]$

$$\delta\gamma(t) = \frac{\mathrm{d}\gamma_s(t)}{\mathrm{d}s}\bigg|_{s=0}. \tag{2.37}$$

If $\delta\gamma(t_0) = \delta\gamma(t_1) = 0$ (in other words, the variations vanish at the boundary points), we say that this vector field is an **infinitesimal variation** of γ.

Recall that $\mathcal{C}^2(q_0, q_1)$ is an infinite dimensional manifold. Its tangent space manifold at γ is given by

$$T_\gamma(\mathcal{C}^2(q_0, q_1)) = \left\{\delta\gamma \in \mathcal{C}^1([t_0, t_1], TQ) : \tau_Q \circ \delta\gamma \equiv \gamma, \delta\gamma(t_i) = 0, i = 0, 1\right\}. \tag{2.38}$$

Definition 2.3.3 (Critical Point). *Given a function $\mathcal{F} \in \mathcal{C}^1(\mathcal{C}^2(q_0, q_1), \mathbb{R})$, we say that a curve $\gamma \in \mathcal{C}^2(q_0, q_1)$ is a critical point of $\mathcal{F}$ provided*

$$\forall \gamma_s : \left.\frac{\mathrm{d}(\mathcal{F} \circ \gamma_s)}{\mathrm{d}s}\right|_{s=0} = 0. \tag{2.39}$$

Note that this is a generalization of the critical point of a function in a Euclidean space.

The sum action $\mathcal{A}_L : \mathcal{C}^2([t_0, t_1], Q, q_0, q_1) \to \mathbb{R}$ given by

$$\mathcal{A}_L(\gamma) = \int_{t_0}^{t_1} L(\tilde{\gamma}(t))\mathrm{d}t = \int_{t_0}^{t_1} L\left(\gamma^i(t), \dot{\gamma}^i(t)\right)\mathrm{d}t \tag{2.40}$$

returns the total cost to go from q_0 to q_1 on a curve $\gamma \in \mathcal{C}^2([t_0, t_1], Q, q_0, q_1)$.

We would like to find a curve γ that minimizes this cost. The following theorem can be found in [15].

Theorem 2.3.1 (Euler-Lagrange equations.). *Let Q be a smooth manifold, $q_0, q_1 \in Q$ and consider a Lagrangian $L : TQ \to \mathbb{R}$. A curve $\gamma \in \mathcal{C}^2([t_0, t_1], Q, q_0, q_1)$ is a critical point of the sum action $\mathcal{A}_L$ if and only if the lift $\tilde{\gamma}$ of γ satisfies the differential equations*

$$\frac{\partial L}{\partial q^i} \circ \tilde{\gamma} - \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial v^i} \circ \tilde{\gamma}\right) = 0, \tag{2.41}$$

where (q^i, v^i) are coordinates in a neighbourhood of $\tilde{\gamma}$.

The differential equations

$$\frac{\partial L}{\partial q^i} - \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial v^i}\right) = 0, \tag{EL}$$

are known as the Euler-Lagrange equations.

Proof. First, we need to find the derivative $\mathcal{A}_L'$ of the integral action $\mathcal{A}_L$ given in (2.40). We know that

$$\mathcal{A}_L'(\gamma) = \lim_{\varepsilon \to 0} \frac{\mathcal{A}_L(\gamma + \varepsilon\delta\gamma) - \mathcal{A}_L(\gamma)}{\varepsilon}.$$

From (2.40) we get that

$$\begin{aligned}
\mathcal{A}'_L(\gamma) &= \lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\int_{t_0}^{t_1} L\left(\gamma^i(t)+\varepsilon\delta\gamma^i(t),\dot{\gamma}^i(t)+\varepsilon\delta\dot{\gamma}^i(t)\right)\mathrm{d}t-\int_{t_0}^{t_1} L\left(\gamma^i(t),\dot{\gamma}^i(t)\right)\mathrm{d}t\\
&=\int_{t_0}^{t_1}\lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\left[L\left(\gamma^i(t)+\varepsilon\delta\gamma^i(t),\dot{\gamma}^i(t)+\varepsilon\delta\dot{\gamma}^i(t)\right)-L\left(\gamma^i(t),\dot{\gamma}^i(t)\right)\right]\mathrm{d}t\\
&=\int_{t_0}^{t_1}\left[\frac{\partial L}{\partial q^i}(\gamma^i(t),\dot{\gamma}^i(t))\ \delta\gamma^i+\frac{\partial L}{\partial v^i}(\gamma^i(t),\dot{\gamma}^i(t))\ \delta\dot{\gamma^i}\right]\mathrm{d}t\\
&=\int_{t_0}^{t_1}\frac{\partial L}{\partial q^i}(\gamma^i(t),\dot{\gamma}^i(t))\ \delta\gamma^i\mathrm{d}t-\int_{t_0}^{t_1}\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial v^i}(\gamma^i(t),\dot{\gamma}^i(t))\ \delta\gamma^i\mathrm{d}t+\frac{\partial L}{\partial v^i}(\gamma^i(t),\dot{\gamma}^i(t))\ \delta\gamma^i\bigg|_{t_0}^{t_1}\\
&=\int_{t_0}^{t_1}\left[\frac{\partial L}{\partial q^i}(\gamma^i(t),\dot{\gamma}^i(t))-\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial v^i}(\gamma^i(t),\dot{\gamma}^i(t))\right]\delta\gamma^i\mathrm{d}t.
\end{aligned}$$

In order to get minimize $\mathcal{A}_L$, we let $\mathcal{A}'_L(\gamma)=0$. Hence,

$$\frac{\partial L}{\partial q^i}\circ\tilde{\gamma}-\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial v^i}\circ\tilde{\gamma}\right)=0.$$

Since the latter is 0 for every δ, we get the Euler-Lagrange equations

$$\frac{\partial L}{\partial q^i}-\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial v^i}\right)=0.$$

□

We have cover fundamental concepts from algebra to topology and introduced the definition of manifold. Moreover, we have scratched the surface of classical mechanics. This concludes this chapter.

Chapter 3

Results

3.1 Preliminary Results

In this section we present and prove important results that will yield the main result.

3.1.1 Discrete Euler-Lagrange Equations

We consider a discrete Lagrangian $\mathbb{L}_d$ on a discrete space Q.

Remark 3.1.1. *$Q \times Q$ is the analogue discrete version of TQ.*

Smooth curves on a continuous Q are changed by a sequence of points on the discrete space Q , which we denoted as follows

$$\mathcal{C}_d(Q) = \{q_k : \{k\}_{k=0}^{n} \to Q\}. \tag{3.1}$$

Also, we consider steps of time $h = t_k - t_{k-1}$ for $k = 1, \ldots, n$. Then the discrete Lagrangian is given by

$$\mathbb{L}_d : Q \times Q \to \mathbb{R} \tag{3.2}$$

$$(q_{k-1}, q_k) \mapsto L\left(\frac{q_{k-1} + q_k}{h}, \frac{q_k - q_{k-1}}{h}\right). \tag{3.3}$$

Before we jump to the minimization problem, we have to introduce discrete derivatives, see [16].

Definition 3.1.1. *Given $q_1, q_2 \in Q$, we say that the discrete derivative D_1 with respect to the first component of $\mathbb{L}_d$ as a function in $T^*_{q_1}Q$ and the $D_2 \in T^*_{q_2}Q$ such that they correspond to $d(\mathbb{L}_d)(q_1, q_2)$.*

By using (3.2) the integral action can be approximated in the following way

$$\begin{aligned}
\mathcal{A}_L(\mathcal{C}_d(Q)) &= \int_0^{Nh} L\left(q^i(t), v^i(t)\right) \mathrm{d}t && (3.4)\\
&\approx \sum_{k=1}^{N} \int_{(k-1)h}^{kh} L(q^i(t), \dot{q}^i(t))dt && (3.5)\\
&\approx \sum_{k=1}^{N} hL\left(\frac{q_{k-1}+q_k}{h}, \frac{q_k - q_{k-1}}{h}\right) && (3.6)\\
&= h\sum_{k=1}^{N} \mathbb{L}_d(q_{k-1}, q_k) && (3.7)\\
&=: \mathcal{A}_{\mathbb{L}_d}(\{q_0, \dots, q_n\}) && (3.8)
\end{aligned}$$

Assume that $\delta q_0 = \delta q_N = 0$. To get the critical points we calculate the derivative of the discrete integral action,

$$\begin{aligned}
\mathcal{A}'_{\mathbb{L}_d}(\{q_0, \dots, q_n\}) &= \lim_{\varepsilon \to 0} \frac{\mathcal{A}_{\mathbb{L}_d}(\{q_k + \varepsilon \delta q_k\}_{k=0}^N) - \mathcal{A}_{\mathbb{L}_d}(\{q_k\}_{k=0}^N)}{\varepsilon}\\
&= h \lim_{\varepsilon \to 0} \frac{\sum_{k=0}^{N-1} (\mathbb{L}_d(q_k + \varepsilon\delta q_k, q_{k+1} + \varepsilon \delta q_{k+1}) - \mathbb{L}_d(q_k, q_{k+1}))}{\varepsilon}\\
&= h \sum_{k=0}^{N-1} D_1\mathbb{L}_d(q_k, q_{k+1})\delta q_k + D_2\mathbb{L}_d(q_k, q_{k+1})\delta q_{k+1}\\
&= h \sum_{k=1}^{N-1} D_1\mathbb{L}_d(q_k, q_{k+1})\delta q_k + \sum_{k=0}^{N-2} D_2\mathbb{L}_d(q_k, q_{k+1})\delta q_{k+1}\\
&= h \sum_{k=1}^{N-1} (D_1\mathbb{L}_d(q_k, q_{k+1}) + D_2\mathbb{L}_d(q_{k-1}, q_k))\delta q_k = 0,
\end{aligned}$$

for any δq_k. Then, by using this discretization, we get the discrete Euler-Lagrange equations

$$D_1\mathbb{L}_d(q_k, q_{k+1}) + D_2\mathbb{L}_d(q_{k-1}, q_k) = 0. \qquad \text{(DEL)}$$

Therefore, by knowing the value q_0, q_1 we can obtain the whole set $\mathcal{C}_d(Q)$; i.e., a discrete curve that minimizes the value of the discrete integral sum.

3.1.2 Euler-Poincaré Equations

Now we consider the continuous Lagrangian system (G, L) where G is a Lie group and $L \in \mathcal{C}^2(TG)$ is a Lagrangian. In general we cannot identify the tangent bundle TQ as a cartesian product $Q \times Q$. However, when we consider a Lie group and its Lie algebra, we can find an identification.

Theorem 3.1.1. *Given a Lie group G and its Lie algebra $\mathfrak{g}$, we have that $TG \cong G \times \mathfrak{g}$.*

Proof. Take $g \in G$ and consider the left action $L_g : G \to G$. Recall that $\mathfrak{g} = T_eG \cong Lie(G)$, with T_eG the tangent space at the identity element e. The isomorphism is obtained through the left traslations $L_g : h \in G \mapsto gh \in G$. Their differential at e are $d(L_g)_e = T_eG \to T_gG$. Then we get the isomorphism that we wanted.
The inverse of this isomorphism is given by $d(L_{(\cdot)})_{g^{-1}} = T_gG \to T_eG$. □

Now we consider a new lagrangian $L = l(g, \xi)$ where $g(t) \in G$ and $\xi(t) \in \mathfrak{g}$. Our goal is to find the critical points of sum action (2.40) for our redefined Lagrangian

$$l : G \times \mathfrak{g} \to \mathbb{R}$$

when $\delta A_l(g, \xi) = 0$. We will do this by working with the exponential map.

By using (2.31), we define infinitesimal variations δg of $g \in G$, for any $\eta \in \mathfrak{g}$ by

$$\delta g = g\eta := \frac{d}{d\varepsilon}(g \exp(\varepsilon\eta))|_{\varepsilon=0} = d(L_g)(\eta), \tag{3.9}$$

where $\eta(0) = \eta(T) = e$. For the case of $\xi \in Lie(G)$ we consider its infinitesimal variation as $\delta\eta = \dot{\eta} + [\xi, \eta]$.

With this variations we consider the associated integral action of l, which is now reformulated as

$$A_l \left(g \exp(\varepsilon\eta); \xi + \varepsilon \left(\dot{\eta} + [\xi, \eta] \right) \right),$$

where g, η and ξ are functions of time. Then, for the critical points we have

$$\begin{aligned}\frac{\delta A_l}{\delta(g,\xi)} &= \lim_{\varepsilon\to 0}\frac{1}{\varepsilon}A_l\left(g\exp\left(\varepsilon\eta\right);\xi+\varepsilon(\dot{\eta}+[\xi,\eta])\right)\\ &= \lim_{\varepsilon\to 0^+}\frac{1}{\varepsilon}\int_0^T l(g\exp(\varepsilon\eta);\xi+\varepsilon(\dot{\eta}+[\xi,\eta]))dt\\ &= \int_0^T \frac{d}{d\varepsilon}l\left(g\exp(\varepsilon\eta);\xi+\varepsilon(\eta+[\xi,\eta])\right)\Big|_{\varepsilon} = 0.\end{aligned}$$

Since g, η, ξ depend on t we have

$$\begin{aligned}&= \int_0^T \frac{d}{d\varepsilon}l\left(g\exp(\varepsilon\eta);\varepsilon+\varepsilon(\dot{\eta}+[\varepsilon,\eta]\right)dt\\ &= \int_0^T \frac{\partial l}{\partial g}\Big|_{(g,\xi)}\frac{d}{d\varepsilon}(g+\exp(\varepsilon\eta))\Big|_{\varepsilon=0}+\frac{\partial l}{\partial\xi}\Big|_{(g,\xi)}(\dot{\eta}+[\xi,\eta])\quad dt\\ &= \int_0^T \frac{\partial l}{\partial g}\Big|_{(g,\xi)}d(L_g)_e(\eta)+\frac{\partial l}{\partial\xi}\Big|_{(g,\xi)}(\dot{\eta}+[\xi,\eta])dt\\ &= \int_0^T \frac{\partial l}{\partial g}\Big|_{(g,\xi)}d(L_g)_e(\eta)+\frac{\partial l}{\partial\xi}\Big|_{(g,\xi)}(\dot{\eta}+[\xi,\eta])\,dt\\ &= \int_0^T L_g^*\left(\partial_g l\right)(\eta)+\frac{\partial l}{\partial\xi}\Big|_{(g,\xi)}\dot{\eta}+\frac{\partial l}{\partial\xi}\Big|_{(g,\xi)}[\xi,\eta]dt.\end{aligned}$$

Since $ad_\xi(\eta) = [\xi,\eta]$,

$$= \int_0^T L_g^*(\partial_g l)(\eta)+\frac{\partial l}{\partial\xi}\Big|_{(g,\xi)}\dot{\eta}+\frac{dl}{d\varepsilon}\Big|_{(y,\xi)}ad_\xi(\eta)dt$$

By definition of ad^* we have that

$$ad^*[\partial_\xi l](\eta) = \frac{dl}{d\xi}\Big|_{(g,\xi)}ad_\xi(\eta).$$

Then,

$$= \int_0^T \left(L_g^*(\partial_g l)\eta + \left.\frac{\partial l}{\partial \xi}\right|_{(g,\xi)} \dot{\eta} + ad^*[\partial_\xi l]\eta \right) dt. \tag{3.10}$$

Also, by using integration by parts we get,

$$\int_0^T \left.\frac{\partial l}{\partial \xi}\right|_{(g,\xi)} \dot{\eta} = \partial_\xi l \eta\Big|_0^T - \int_0^T \frac{d}{dt}\frac{\partial l}{\partial \xi}\eta dt.$$

Since $\eta(0) = \eta(T) = 0$, the first term in the right hand side is zero. By replacing in (3.10), we get

$$\int_0^T L_g^*(\partial_g l) + \frac{d}{dt}\frac{\partial l}{\partial \xi} + ad^*[\partial_\xi l])\eta dt = 0$$

for all $\eta : [0, T] \to \mathfrak{g}$ with $\eta(0) = \eta(T) = 0$. Thus, we get the following condition to find critical points.

$$L_g^*(\partial_g l) + \frac{d}{dt}\frac{\partial l}{\partial \xi} + ad^*[\partial_\xi l] = 0. \tag{EP}$$

These equations are known as the Euler-Poincaré equations.

3.1.3 Inverse Problem for Euler-Lagrange Equation

As we have seen in subsection (2.3.1) for a given Lagrangian function we get a system of Second Order Differential Equations corresponding to the Euler-Lagrange equation. The inverse problem states that if a given system of second order differential equations has a related Lagrangian, i.e. if the given system is equivalent to some Euler-Lagrange equations. There are two approaches to this problem. In this section we study both. We have used [1] and [17] as a reference.

Variational Approach

Definition 3.1.2 (Liouville Vector Field)**.** *The Liouville vector field is a vector field* $\Delta \in \mathfrak{X}(TQ)$ *defined by*

$$\begin{aligned} \Delta\colon\ TQ &\longrightarrow TTQ \\ v_q &\longmapsto \left.\tfrac{d}{t}\right|_{t=0}(e^t v_q). \end{aligned}$$

Definition 3.1.3 (Vertical Endomorphism)**.** *The vertical endomorphism is a* $(1,1)$ *tensor* $\Delta \in \mathfrak{X}(TQ)$ *defined by*

$$\begin{array}{rccc} S\colon & TTQ & \longrightarrow & TTQ \\ & w_{v_q} & \longmapsto & \frac{d}{t}\Big|_{t=0}(v_q + tT\tau_Q(w_{v_q})) \end{array}$$

Definition 3.1.4 (SODE)**.** *A second order differential equation is called a SODE on the tangent bundle if and only if it is a vector field* $\Gamma \in \mathfrak{X}(TQ)$ *such that* $S(\Gamma) = \Delta$*, which can be written as*

$$\Gamma = \dot{q}^i \frac{\partial}{\partial \dot{q}^i} + \Gamma^i(q, \dot{q}) \frac{\partial}{\partial \dot{q}^i}$$

in coordinates. Here $S(\Gamma)$ *means the composition of* Γ *and* S*.We denote* $\ddot{q}^i := \Gamma^i(q, \dot{q})$

If we consider a time dependent system, we have that

$$\Gamma = \frac{\partial}{\partial t} + \dot{q}^i \frac{\partial}{\partial \dot{q}^i} + \Gamma^i(q, \dot{q}) \frac{\partial}{\partial \dot{q}^i}. \tag{3.11}$$

In order to carry out the derivatives in the Euler-Lagrange equations (EL) we consider a path $q : t \in I \to Q \ni q(t)$. Then, we get

$$\ddot{q} \frac{\partial^2 L}{\partial \dot{q}^k \partial \dot{q}^i} + \dot{q} \frac{\partial^2 L}{\partial q^j \partial \dot{q}^i} - \frac{\partial L}{\partial \dot{q}^i} = 0. \tag{3.12}$$

The first term contains the Hessian matrix, whose entries will be denoted g_{ki}. If the inverse of this matrix exists and we denote its components g^{ki}, we can rewrite (3.12) as

$$\ddot{q}^k = g^{ki} \left(-\dot{q} \frac{\partial^2 L}{\partial q^j \partial \dot{q}^i} + \frac{\partial L}{\partial \dot{q}^i} \right), \tag{3.13}$$

which is in $\mathfrak{X}(TQ)$. The Lagrangian L is said to be regular when matrix q_{ki} has an inverse [18]. The following theorems can be found in [1] and summarize how to find solution for inverse problem.

Theorem 3.1.2 (Inverse Problem of Variational Calculus)**.** *Let* $F_i(t, q, \dot{q}, \ddot{q}) = 0$*, with* $i = 1, \ldots, n$*, be a system of second order differential equation (not necessarily a SODE). There is a regular Lagrangian* $L : \mathbb{R} \times Q \to \mathbb{R}$ *such that for each* i

$$F_i(t, q, \dot{q}, \ddot{q}) = \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}^i} \right) - \frac{\partial L}{\partial q^i} \tag{3.14}$$

if and only if F_i satisfies the covariant Helmholtz conditions

$$\frac{\partial F_i}{\partial \dddot{q}^j} - \frac{\partial F_j}{\partial \dddot{q}^i} = 0, \tag{3.15}$$

$$\frac{\partial F_i}{\partial q^j} - \frac{\partial F_j}{\partial q^i} - \frac{1}{2}\frac{d}{dt}\left(\frac{\partial F_i}{\partial \dot{q}^j} - \frac{\partial F_j}{\partial \dot{q}^i}\right) = 0, \tag{3.16}$$

$$\frac{\partial F_i}{\partial \dot{q}^j} + \frac{\partial F_j}{\partial \dot{q}^i} - \frac{d}{dt}\left(\frac{\partial F_i}{\partial \ddot{q}^j} - \frac{\partial F_j}{\partial \ddot{q}^i}\right) = 0. \tag{3.17}$$

For a complete proof we refer the reader to [17].

Theorem 3.1.3 (Inverse Problem of Variational Calculus (Explicit Form))**.** *Consider a system of SODEs $\ddot{q} = \Gamma(t, q, \dot{q})$, $i = 1, \dots, n$. There is a regular Lagrangian $L : \mathbb{R} \times Q \to \mathbb{R}$ such that for each i*

$$g_{ij}\left(\ddot{q}^j - \Gamma^j(t, q, \dot{q})\right) = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}^i}\right) - \frac{\partial L}{\partial q^i}, \tag{3.18}$$

where g_{ij} is defined as in 3.13, if and only if the multipliers g_{ij} satisfy the Helmholtz conditions

$$\det(g_{ij}) \neq 0, \tag{3.19}$$

$$g_{ij} = g_{ji}, \tag{3.20}$$

$$\frac{\partial g_{ij}}{\partial \dot{q}^k} = \frac{\partial g_{ik}}{\partial \dot{q}^j}, \tag{3.21}$$

$$\Gamma(g_{ij}) - \nabla^k_j g_{ik} - \nabla^k_i g_{kj} = 0, \tag{3.22}$$

$$g_{ik}\Phi^k_j = g_{jk}\Phi^k_i, \tag{3.23}$$

with $\nabla^i_j = -\frac{1}{2}\frac{\partial \Gamma^i}{\partial \dot{q}^j}$ and $\Phi^k_j = \Gamma\left(\frac{\partial \Gamma^k}{\partial \dot{q}^j}\right) - 2\frac{\partial \Gamma^k}{\partial q^j} - -\frac{1}{2}\frac{\partial \Gamma^j}{\partial \dot{q}^j}\frac{\partial \Gamma^k}{\partial \dot{q}^i}$.

In this case, **finding a regular Lagrangian is equivalent to find multipliers that satisfy the Helmholtz conditions**. Moreover, solutions to the SODE are the same solutions for the Euler-Lagrange equations and the system of SODEs is called **variational**.

Note that Helmholtz conditions in 3.1.3 and 3.1.2 show there are symmetries and antisymmetries, which come from the Jacobian matrix.

Geometric Formulation

We consider a smooth manifold Q and define the canonical projections

$$\begin{array}{rccc} \tau_Q\colon & TQ & \to & Q \\ & (q^i,\dot{q}^i) & \mapsto & (q^i) \end{array} \quad \text{and} \quad \begin{array}{rccc} \pi_Q\colon & T^*Q & \to & Q \\ & (q^i,f^i) & \mapsto & (q^i). \end{array}$$

Let $L\colon TQ \to \mathbb{R}$ be a Lagrangian. We define $dL\colon T(TQ) \to \mathbb{R}$, which is given in coordinates by

$$dL = \frac{\partial L}{\partial q^i}dq^i + \frac{\partial L}{\partial \dot{q}^i}d\dot{q}^i. \tag{3.24}$$

The geometrical formulation allows us to study the inverse problem in a more general and abstract way. First, we will introduce the following functions.

Definition 3.1.5 (Poincaré-Cartan one-form)**.** *The Poincaré-Cartan one form Θ_L is the pullback of dL by the vertical endomorphism S; i.e., $\Theta_L = S^*(dL)$.*

Definition 3.1.6 (Poincaré-Cartan two-form)**.** *We define the Poincaré-Cartan two form Ω_L as $\Omega_L = -d\Theta_L$.*

Definition 3.1.7 (Energy Function)**.** *We define the energy function $E_L : TQ \to \mathbb{R}$ by $E_L = \Delta(L) - L$.*

Definition 3.1.8. *Given a two form β, we define the inclusion map i such that for each $v, w \in \mathfrak{X}(TQ)$ $i_v\beta = \beta(v,\cdot)$ and $i_w(i_v\beta) = \beta(v,w) \in \mathbb{R}$.*

Using these definitions we can prove that

$$i_\Gamma \Omega_L = dE_L \tag{3.25}$$

and

$$\mathcal{L}_\Gamma \Theta_L = dL, \tag{3.26}$$

for some SODE Γ. Equation (3.25) is the symplectic reformulation of the inverse problem, while equation (3.26) is known as the geometric formulation of the inverse problem. In fact, both equations are equivalent and solving the SODE Γ gives us the solution of the Euler-Lagrange equations (2.3.1). Furthermore, we have the following theorem.

Theorem 3.1.4. *A SODE $\Gamma \in \mathfrak{X}(TQ)$ is said to be variational if and only if there is a two-form Ω such that the following statements hold.*

$$d\Omega = 0, \tag{3.27}$$

$$\mathcal{L}_\Gamma \Omega = 0, \tag{3.28}$$

$$\Omega(v_1, v_2) = 0. \tag{3.29}$$

where v_1, v_2 are arbitrary vertical subspaces of the tangent bundle.

Recall that a two-form is said to be of maximal rank if and only if the Hessian matrix define by g_{ij} in (3.13) is invertible. see [18] for a proof of this theorem. The next approach (called the new geometrical approach in [1]) is the most important in this work. It will give us an insight on how to develop the remaining inverse problems. This approach is based on the existence of a local diffeomorphism (which behaves like a Legendre transformation of the Lagrangian) to prove that a SODE is derivable from a regular Lagrangian.

Definition 3.1.9 (Legendre Transformation)**.** *Let Q be a smooth manifold and $L : TQ \to \mathbb{R}$ a Lagrangian. The Legendre transformation of L is defined as*

$$\begin{array}{rccc} Leg_L\colon & TQ & \longrightarrow & T^*Q \\ & v_q & \longmapsto & Leg(v_q) \\ & (q^i, \dot{q}^i) & \longmapsto & (q^i, \frac{\partial L}{\partial \dot{q}^j}) \end{array}$$

Definition 3.1.10 (Tulczyjew Isomorphism)**.** *The Tulczyjew Isomorphism is an involution of the spaces TT^*Q and T^*TQ defined in local coordinates as*

$$\begin{array}{rccc} \alpha_Q\colon & TT^*Q & \longrightarrow & T^*TQ \\ & (q^i, p_i, \dot{q}^i, \dot{p}_i) & \longmapsto & (q^i, \dot{q}^i, p_i, \dot{p}_i). \end{array}$$

For more information on the Tulczyjew triple, we refer the reader to [19], [20] and [21]. We now let $F : TQ \to T^*Q$ be any local diffeomorphism, denote $TF : TTQ \to TT^*Q$ its tangent and define the one form $\mu_{\Gamma,F} := \alpha_Q \circ TF \circ \Gamma$. Note that $Im(\mu_{\Gamma,F})$ is a submanifold.

Remark 3.1.2. *The tangent maps of π_M can we written in coordinates as*

$$\begin{array}{rccc} T_{\pi_Q}\colon & TT^*Q & \longrightarrow & TQ \\ & (q^i, p_i, \dot{q}^i, \dot{p}_i) & \longmapsto & (q^i, \dot{q}^i). \end{array}$$

Remark 3.1.3. *The canonical projection map of the cotangent bundle is given in coordi-*

nates as

$$\begin{array}{rccc} \tau_{T^*Q}: & TT^*Q & \longrightarrow & T^*Q \\ & (q^i, p_i, \dot{q}^i, \dot{p}_i) & \longmapsto & (q^i, p_i). \end{array}$$

Remark 3.1.4. *The canonical Liouville form (see 3.1.2) can be written as*

$$\begin{array}{rccc} \theta_Q: & TT^*Q & \longrightarrow & \mathbb{R} \\ & v & \longmapsto & \langle \tau_{T^*Q}(v), T_{\pi_Q(v)} \rangle, \end{array}$$

which can be expressed in local coordinates as $\theta_Q(v) = p_i \dot{q}^i$.

Definition 3.1.11. *We denote* $\omega_Q = d\theta_Q = dp_i \wedge dq^i$ *the differential of the Liouville form, which is a symplectic form.*

Given the natural structure of the cotangent bundle, the pair (T^*Q, ω_Q) is a symplectic manifold. Using the Tulczyjew triple 3.1.10 as developed in [19], we will see the symplectic form for TQ is $w_{TQ} = dp_i \wedge dq^i + d\dot{p}_j \wedge d\dot{q}^j$, and the pair (T^*TQ, ω_{TQ}) is a symplectic manifold.

Remark 3.1.5. *The set* $Im(\mu_{\Gamma,F})$ *is a Lagrangian submanifold of* (T^*TQ, ω_{TQ}) *if and only if* $\mu_{\Gamma,F}$ *is closed; i.e.,* $d\mu_{\Gamma,F} = 0$.

Remark 3.1.6. *Assume that* $\mu_{\Gamma,F}$ *is closed. We know that*

$$\mu_{\Gamma,F} = \left(\frac{\partial F_i}{\partial q^j} \dot{q}^j + \frac{\partial F_i}{\partial \dot{q}^j} \Gamma^j \right) dq^i + F_i d\dot{q}^i.$$

Therefore,

$$\begin{aligned} d\mu_{\Gamma,F} =& \frac{\partial}{\partial q^j} \left(\left(\frac{\partial F_i}{\partial q^j} \dot{q}^j + \frac{\partial F_i}{\partial \dot{q}^j} \Gamma^j \right) dq^i + F_i d\dot{q}^i \right) dq^j \\ &+ \frac{\partial}{\partial \dot{q}^j} \left(\left(\frac{\partial F_i}{\partial q^j} \dot{q}^j + \frac{\partial F_i}{\partial \dot{q}^j} \Gamma^j \right) dq^i + F_i d\dot{q}^i \right) d\dot{q}^j \\ =& \frac{\partial^2 F_i}{\partial q^k \partial q^j} \dot{q}^j d\dot{q}^i \wedge dq^k + \frac{\partial^2 F_i}{\partial q^k \partial \dot{q}^j} \Gamma^j d\dot{q}^i \wedge dq^k + \frac{\partial F_i}{\partial q^j} \frac{\partial \Gamma^j}{\partial q^k} d\dot{q}^i \wedge dq^k \\ &+ \frac{\partial F_i}{\partial q^k} d\dot{q}^i \wedge dq^k + \frac{\partial^2 F_i}{\partial \dot{q}^k \partial q^j} \dot{q}^j d\dot{q}^i \wedge d\dot{q}^k + \frac{\partial F_i}{\partial q^k} d\dot{q}^i \wedge d\dot{q}^k \\ &+ \frac{\partial^2 F_i}{\partial \dot{q}^k \partial \dot{q}^j} \Gamma^j d\dot{q}^i \wedge d\dot{q}^k + \frac{\partial F_i}{\partial q^j} \frac{\partial \Gamma^j}{\partial \dot{q}^k} d\dot{q}^i \wedge d\dot{q}^k + \frac{\partial F_i}{\partial \dot{q}^k} d\dot{q}^i \wedge d\dot{q}^k = 0. \end{aligned}$$

Hence, the following conditions hold.

$$
\begin{aligned}
\frac{\partial F_i}{\partial \dot{q}^k} &= \frac{\partial F_k}{\partial \dot{q}^i}, \\
\frac{\partial^2 F_i}{\partial q^k \partial q^j} + \frac{\partial^2 F_i}{\partial q^k \partial \dot{q}^j} + \frac{\partial^2 F_i}{\partial q^k \partial q^j} &= \frac{\partial^2 F_k}{\partial q^i \partial q^j} + \frac{\partial^2 F_k}{\partial q^i \partial \dot{q}^j} + \frac{\partial^2 F_k}{\partial q^i \partial q^j}, \\
\frac{\partial F_i}{\partial \dot{q}^k} - \frac{\partial F_k}{\partial \dot{q}^i} &= \frac{\partial^2 F_i}{\partial \dot{q}^k \partial q^j} + \frac{\partial^2 F_i}{\partial \dot{q}^k \partial \dot{q}^j} + \frac{\partial^2 F_i}{\partial \dot{q}^k \partial q^j}.
\end{aligned}
$$

These conditions are similar to the covariant Helmholtz conditions. Remember that we are using the summation convention.

Next we have an important theorem.

Theorem 3.1.5. *We say that a SODE* Γ *is variational if and only if there is a local diffeomorphism* F *such that* $Im(\mu_{\Gamma,F})$ *is a Lagrangian submanifold of* (T^*TQ, ω_{TQ}).

This theorem can be summarized in the next diagram,

$$
\begin{array}{ccccc}
TTQ & \xrightarrow{TF} & TT^*Q & \xrightarrow{\alpha_Q} & T^*TQ \\
\uparrow{\scriptstyle \Gamma} & & & \nearrow{\scriptstyle \mu_{\Gamma,F}} & \\
TQ & \xrightarrow[F]{} & T^*Q, & &
\end{array}
$$

which can be expressed in local coordinates as

$$
\begin{array}{ccccc}
(q^i, \dot{q}^i, \dot{q}^i, \Gamma^j(q,\dot{q})) & \xrightarrow{TF} & \left(q^i, F_i, \dot{q}^i, \frac{\partial F_i}{\partial q^j}\dot{q}^j + \frac{\partial F_i}{\partial \dot{q}^j}\Gamma^j\right) & \xrightarrow{\alpha_Q} & \left(q^i, F_i, \frac{\partial F_i}{\partial q^j}\dot{q}^j + \frac{\partial F_i}{\partial \dot{q}^j}\Gamma^j, \dot{q}^i\right) \\
\uparrow{\scriptstyle \Gamma} & & & \nearrow{\scriptstyle \mu_{\Gamma,F}} & \\
(q^i, \dot{q}^i) & \xrightarrow{F} & (q^i, F_i). & &
\end{array}
$$

Proof. First we prove the necessary condition.

$\Leftarrow$ Let F be a local diffeomorphism. Suppose that $Im(\mu_{\Gamma,F})$ is a Lagrangian submanifold and define the two-form $\Omega := -d(F^*\theta_Q)$. We have to prove the conditions in theorem 3.1.4 are satisfied. Using properties of the exterior differentiation, we have that

$$d\left(-d(F^*\theta_Q)\right) = -d\left(d(F^*\theta_Q)\right) = 0. \tag{3.30}$$

That is, Ω is a closed form.
The proof of the third condition is trivial and comes directly from the definition of Lagrangian submanifold.
It remains to prove that $\mathcal{L}_\Gamma \Omega = 0$.
Using properties of the exterior differential and Lie derivative, and 3.1.5, we have that

$$\begin{aligned}
\mathcal{L}_\Gamma \Omega &= \mathcal{L}_\Gamma \left(-d(F^*\theta_Q) \right) \\
&= -d\mathcal{L}_\Gamma (F^*\theta_Q) \\
&= -d\left(\Gamma(F_i dq^i) + F_i d\dot{q}^i \right) \\
&= -d\left(\left(\frac{\partial F_i}{\partial q^j}\dot{q}^j + \frac{\partial F_i}{\partial \dot{q}^j}\Gamma^j \right) dq^i + F_i d\dot{q}^i \right) \\
&= -(d\mu_{\Gamma,F}) = 0.
\end{aligned}$$

We have proved that Γ is variational.

$\Rightarrow$ We assume that Γ is variational and there is a two form that satisfies conditions in theorem 3.1.4. We want to prove that $Im(\mu_{\Gamma,F})$ is a Lagrangian submanifold of (T^*TQ, ω_{TQ}). From theorem 3.1.4 we know that there is a two form Ω such that $d\Omega = 0$. Therefore, there is a one form Θ in an open set $U \subseteq TQ$ such that $\Omega = d\Theta$. From the third condition in theorem 3.1.4 it follows that for all v vertical subspace of the tangent bundle, there is a function $f : U \to \mathbb{R}$ such that

$$\Theta(v) = \langle df, v \rangle = df(v).$$

Now we define a one form $\widetilde{\Theta} = \Theta - df$. It is clear that $\widetilde{\Theta}$ satisfies condition three in theorem 3.1.4 and since $d\widetilde{\Theta} = d(\Theta - df)$, we have that $d\widetilde{\Theta} = \Omega$.
We choose

$$F : U \to T^*Q$$

define by

$$\langle F(v_q), w_q \rangle = \langle \widetilde{\Theta}(v_q), W_q \rangle \tag{3.31}$$

Here, $W_q \in TTQ$ and $T\tau_Q(W_q) = w_q$. We still have to prove that F is a diffeomorphism.
Then, we get that $\widetilde{\Theta} = F^*\theta_Q$. Since $\mu_{\Gamma,F} = d\mathcal{L}_\Gamma\Theta$ and $\mathcal{L}_\Gamma\Omega$, it follows that $d\mu_{\Gamma,F} = 0$. Thus, by remark 3.1.5, we get that $Im(\mu_{\Gamma,F})$ is a Lagrangian submanifold.

We have proved the necessary and sufficient conditions for this theorem. □

3.1.4 Inverse Problem for the Discrete Euler-Lagrange Equations

The inverse problem for the discrete Euler-Lagrange equations DEL is addressed in a similar way to the continuous case. In this case we will deal with second order difference equations (SOdE), which are a discrete version of a SODEs. Given a discrete space Q, we claim that $Q \times Q$ and $Q \times Q \times Q \times Q$ are discretizations of TQ and TTQ, respectively. We define the canonical projections

$$\begin{array}{rccc} pr_1 : & Q \times Q & \to & Q \\ & (q_1, q_2) & \mapsto & (q_1) \end{array} \quad \text{and} \quad \begin{array}{rccc} pr_2 : & Q \times Q & \to & Q \\ & (q_1, q_2) & \mapsto & (q_2). \end{array}$$

Recall that $\mathcal{C}_d(Q) = \{q_d : \{k\}_{k=0}^n \to Q\}$ is a discrete curve in Q.

Definition 3.1.12. *We say that $\Gamma_d : Q \times Q \to (Q \times Q) \times (Q \times Q)$ is a SOdE if and only if $pr_1 \circ \Gamma = Id$.*
Moreover, a system of SOdEs $q_{k+1} = \Gamma_d(q_{k-1}, q_k)$ is said to be variational if and only if there is a discrete Lagrangian $\mathbb{L}$ such that

$$D_1\mathbb{L}_d(q_k, q_{k+1}) + D_2\mathbb{L}_d(q_{k-1}, q_k) = 0. \tag{3.32}$$

and

$$q_{k+1} = \Gamma_d(q_{k-1}, q_k) \tag{3.33}$$

have the same solutions.

We now let $F_d : Q \times Q \to T^*Q$ be any local diffeomorphism, denote it tangent map $TF_d : (Q \times Q) \times (Q \times Q) \to T^*Q \times T^*Q$ and define the one form $\mu_{\Gamma_d, F_d} := TF_d \circ \Gamma_d$. Note that $Im(\mu_{\Gamma_d, F_d})$ is a submanifold.

Theorem 3.1.6. *A SOdE is variational if and only if there is a local diffeomorphism $F_d : Q \times Q \to T^*Q$ and $Im(\mu_{\Gamma_d, F_d})$ is a Lagrangian submanifold of $(T^*Q \times T^*Q, \Omega_Q)$.*

This theorem can be pictured in the following diagram.

$$
\begin{array}{ccc}
Q \times Q \times Q \times Q & \xrightarrow{TF_d} & T^*Q \times T^*Q \\
\Gamma_d \Big\uparrow & \overset{\mu_{\Gamma_d, F_d}}{\nearrow} & \\
Q \times Q & \xrightarrow[F_d]{} & T^*Q,
\end{array}
$$

which locally can be written as

$$
\begin{array}{ccc}
(q_{k-1}, q_k, q_k, \Gamma_d(q_{k-1}, q_k)) & \xrightarrow{TF_d} & (q_{k-1}, F_d(q_{k-1}, q_k), q_k, F_d(q_k, \Gamma_d(q_{k-1}, q_k))) \\
\Gamma_d \Big\uparrow & \overset{\mu_{\Gamma_d, F_d}}{\nearrow} & \\
(q_{k-1}, q_k) & \xrightarrow[F_d]{} & (q_{k-1}, F_d(q_{k-1}, q_k)).
\end{array}
$$

3.2 Main Results

In this section we present our main results.

3.2.1 Trivialization

In the subsequent subsections, we will work on discrete manifolds and its tangent and cotangent spaces. Our goal is to find trivializations of the manifolds introduced in the diagram of theorem 3.1.5. We will let G be a discrete Lie group.

Theorem 3.2.1. $TG \cong G \times \mathfrak{g}$

Proof. We have to find an isomorphism between TG and $G \times \mathfrak{g}$. First, consider the map $D_h L_g : T_g G \to T_g G$

$$D_h L_g(h) := \frac{d}{dt}\bigg|_{t=0} L_g(h(t)) = g\dot{h}. \tag{3.34}$$

It is easy to see this map is an isomorphism. We already know that $\mathfrak{g} \cong T_e G$. Then, we can define the isomorphism

$$
\begin{array}{rccc}
D_e L_g \colon & G \times \mathfrak{g} & \longrightarrow & TG \\
& (g, \xi) & \longmapsto & (g, D_e L_g \xi).
\end{array}
$$

The inverse of this function is defined by

$$
\begin{array}{rccc}
D_g L_{g^{-1}} \colon & TG & \longrightarrow & G \times \mathfrak{g} \\
& (g, v) & \longmapsto & (g, D_g L_{g^{-1}} v).
\end{array}
$$

□

From now on, we will denote $g^{-1}v := D_g L_{g^{-1}} v$ and $g\xi := D_e L_g \xi$, which shall not be confused with the element $g \in G$.

Theorem 3.2.2. *$T(G \times \mathfrak{g}) \cong TG \times T\mathfrak{g}$.*

Proof. First, consider two manifolds M, N and a path

$$\begin{aligned} \gamma\colon \quad I &\longrightarrow M \times N \\ t &\longmapsto (\gamma_1(t), \gamma_2(t)). \end{aligned}$$

Then, for any $f : M \times N \to \mathbb{R}$,

$$V_{p,q} = \left.\frac{d}{dt}\right|_{t=0} f(\gamma(t)) = \left.\frac{d}{dt}\right|_{t=0} f \circ \gamma(t).$$

Let $\gamma_1 : I \to M$ and $\gamma_2 : I \to N$ and denote $g := f(\cdot, q)$, $h := f(p, \cdot)$. Then, we can define

$$\begin{aligned} V_p &= \left.\frac{d}{dt}\right|_{t=0} g \circ \gamma_1(t), \\ V_q &= \left.\frac{d}{dt}\right|_{t=0} h \circ \gamma_2(t). \end{aligned}$$

Note that $V_{p,q} = [\gamma]$ is an equivalence class so that

$$\begin{aligned} T(M \times N) &\longrightarrow TM \times TN \\ V_{p,q} &\longmapsto (V_p, V_q). \end{aligned}$$

Therefore, if we take a path

$$\begin{aligned} \gamma\colon \quad I &\longrightarrow G \times \mathfrak{g} \\ t &\longmapsto (g(t), \xi(t)), \end{aligned}$$

we have that

$$\begin{aligned} T(G \times \mathfrak{g}) &\cong TG \times T\mathfrak{g} \\ (g, \xi, \dot{g}, \dot{\xi}) &\longmapsto (g, \dot{g}, \xi, \dot{\xi}). \end{aligned}$$

□

Theorem 3.2.3. *$TTG \cong T(G \times \mathfrak{g})$*

Proof. Given a $(g, \dot{g}, v, \dot{v}) \in TTG$, we know that

$$\frac{d}{ds}(\widetilde{g}(s), \dot{g}(s))\Big|_{s=0} =: (v, \dot{v}),$$

with $\widetilde{g}(0) = g$. From theorem 3.2.1, we have that

$$(\widetilde{g}(s), \widetilde{g}(s)^{-1}\dot{g}(s)) \in G \times \mathfrak{g}.$$

Note that

$$\begin{aligned}
\frac{d}{ds}(\widetilde{g}(s), \widetilde{g}(s)^{-1}\dot{g}(s))\big|_{s=0} &= \left(v, \frac{d}{ds}\widetilde{g}^{-1}(s)\Big|_{s=0} \dot{g} + \widetilde{g}^{-1}\frac{d}{ds}\dot{g}(s)\Big|_{s=0}\right) \\
&= \left(v, \frac{d}{ds}\widetilde{g}^{-1}(s)\Big|_{s=0} \dot{g} + g^{-1}\dot{v}\right) \\
&= \left((v, -g^{-1}vg^{-1}\dot{g} + g^{-1}\dot{v}\right) = \left((v, -g^{-1}(-v - g^{-1}\dot{g} + \dot{v})).\right)
\end{aligned}$$

Thus, we have that

$$\begin{array}{ccc}
TTG & \longrightarrow & T(G \times \mathfrak{g}) \\
(g, \dot{g}, v, \dot{v}) & \longmapsto & \left((v, -g^{-1}(-v - g^{-1}\dot{g} + \dot{v})).\right).
\end{array}$$

The inverse can be found easily. In fact, we have

$$\begin{array}{ccc}
T(G \times \mathfrak{g}) & \longrightarrow & TTG \\
(g, \xi, v, \eta) & \longmapsto & ((g, g\xi, v, v\xi + \xi\eta)).
\end{array}$$

□

Theorem 3.2.4. $TG \times T\mathfrak{g} \cong (G \times \mathfrak{g}) \times (G \times \mathfrak{g})$.

Proof. Since $\mathfrak{g}$ is a vector space, it follows that $T\mathfrak{g} = \mathfrak{g} \times \mathfrak{g}$. From theorem 3.2.1 it is easy to see that

$$\begin{array}{ccc}
TG \times T\mathfrak{g} & & (G \times \mathfrak{g}) \times (\mathfrak{g} \times \mathfrak{g}) \\
(g, v, \xi, \eta) & \longmapsto & (g, g^{-1}v, \xi, \eta).
\end{array}$$

□

Theorem 3.2.5. $(G \times \mathfrak{g}) \times (\mathfrak{g} \times \mathfrak{g}) \cong G \times \mathfrak{g} \times \mathfrak{g} \times \mathfrak{g}$.

Proof. The proof of this theorem is trivial. □

Corollary 3.2.1. $TTG \cong G \times \mathfrak{g} \times \mathfrak{g} \times \mathfrak{g}$

The proof of this corollary follows directly from theorem 3.2.1 to 3.2.5. In fact, we have that

$$\begin{array}{ccc} TTG & & G \times \mathfrak{g} \times \mathfrak{g} \times \mathfrak{g} \\ (g, \dot{g}, v, \dot{v}) & \longmapsto & (g, g^{-1}\dot{g}, g^{-1}v, g^{-1}\dot{v} - g^{-1}vg^{-1}\dot{g}), \end{array}$$

and

$$\begin{array}{ccc} G \times \mathfrak{g} \times \mathfrak{g} \times \mathfrak{g} & & TTG \\ (g, \xi, \mu, \eta) & \longmapsto & (g, g\xi, g\mu, g\eta - g\mu\xi). \end{array}$$

We denote the dual space of $G \times \mathfrak{g}$ by $G \times \mathfrak{g}^*$.

Theorem 3.2.6. *$T^*G \cong G \times \mathfrak{g}^*$.*

Proof. We construct the dual function using theorem 3.2.1. Let us consider the map $D_gL_{g^{-1}} : T_gG \to T_eG$. Then, we can define

$$\begin{array}{rccc} (D_gL_{g^{-1}})^* \colon & T_e^*G & \longrightarrow & T_g^*G \\ & \omega & \longmapsto & (D_gL_{g^{-1}})^*(\omega) = \eta, \end{array}$$

where $\eta \in T_g^*G$ and

$$\begin{array}{rccc} \eta \colon & T_gG & \longrightarrow & \mathbb{R} \\ & v_g & \longmapsto & \eta(v_g). \end{array}$$

Hence, $(D_gL_{g^{-1}})^*(\omega)(v_g) := \omega\left(D_gL_{g^{-1}}(v_g)\right) = \omega(g^{-1}v_g)$.
Thus, $(D_gL_{g^{-1}})^*(\omega) = \omega g^{-1}$. It follows that

$$\begin{array}{ccc} G \times \mathfrak{g}^* & \cong & T^*G \\ (g, \omega) & \mapsto & (g, \omega g^{-1}) \\ (g, \mu g) & \leftarrow\!\shortmid & (g, \mu g^{-1}). \end{array}$$

□

Theorem 3.2.7. *$T(G \times \mathfrak{g}^*)$ and $TG \times T\mathfrak{g}^*$ are isomorphic. In fact,*

$$\begin{array}{ccc} T(G \times \mathfrak{g}^*) & \cong & TG \times T\mathfrak{g}^* \\ (g, \omega, \dot{g}, \dot{\omega}) & \longleftrightarrow & (g, \dot{g}, \omega, \dot{\omega}) \end{array}$$

Theorem 3.2.8. *TT^*G and $T(G \times \mathfrak{g}^*)$ are isomorphic and*

$$\begin{array}{ccc} T(G \times \mathfrak{g}^*) & \cong & TT^*G \\ (g, \eta, v, \xi) & \mapsto & (g, g^{*^{-1}}\xi, \dot{g}, g^{*^{-1}}\xi - g^{*^{-1}}\dot{g}^* g^{*^{-1}}\eta) \\ (g, g^*\mu, \dot{g}, \dot{g}^*\mu + g^*\dot{\mu}) & \leftarrow\!\shortmid & (g, \mu, \dot{g}, \dot{\mu}). \end{array}$$

Theorem 3.2.9. *We have that*

$$\begin{array}{rcl} TG \times T\mathfrak{g}^* & \cong & G \times \mathfrak{g} \times \mathfrak{g}^* \times \mathfrak{g}^* \\ (g, \dot{g}, \omega, \dot{\omega}) & \mapsto & (g, g^{-1}\dot{g}, \omega, \dot{\omega}). \end{array}$$

and

$$\begin{array}{rcl} G \times \mathfrak{g} \times \mathfrak{g}^* \times \mathfrak{g}^* & \cong & G \times \mathfrak{g}^* \times \mathfrak{g} \times \mathfrak{g}^* \\ (g, \xi, \mu, \eta) & \mapsto & (g, \mu, \xi, \eta). \end{array}$$

Corollary 3.2.2. *The spaces TT^*G and $G \times \mathfrak{g}^* \times \mathfrak{g} \times \mathfrak{g}^*$ are isomorphic and the isomorphism is defined by*

$$\begin{array}{rcl} TT^*G & \cong & G \times \mathfrak{g}^* \times \mathfrak{g} \times \mathfrak{g}^* \\ (g, p, \dot{g}, \dot{p}) & \mapsto & (g, g^*p, g^{-1}\dot{g}, \dot{g}^*p + g^*\dot{p}). \end{array}$$

3.2.2 Discrete Euler-Poincaré Equations

Consider the discrete Lie group G and a discrete curve $\mathcal{C}_d(G) = \{q_d : \{g_k\}_{k=0}^n \to G\}$. Using the left action on G, we can see $f_{k,k+1} := g_k^{-1} g_{k+1}$ as an arrow from g_k to g_{k+1}. We define the variation of g_k by

$$\left.\frac{d}{dt} g_k(\varepsilon)\right|_{\varepsilon=0} = \left.\frac{d}{dt} g_k^{\varepsilon}\right|_{\varepsilon=0} = g_k \delta g_k$$

such that $\delta g_0 = \delta g_n = 0$. Moreover, we define the variation of $f_{k,k+1}$ as

$$\delta f_{k,k+1} = \delta g_{k+1} - Ad(f_{k+1,k})(\delta g_k). \tag{3.35}$$

We define a discrete Lagrangian $\ell : G \to \mathbb{R}$ such that $\mathbb{L} = \ell \circ \pi$, where $\mathcal{L}$ is the Lagrangian used to find the discrete Euler-Lagrange equations DEL. Consider the sum action

$$\mathcal{A}_d(\{g_k\}_{k=0}^n) = \sum_{k=0}^{n} \Big(\ell(g_k) + \ell(f_{k,k+1}) \Big). \tag{3.36}$$

Next, we minimize (3.36); i.e.,

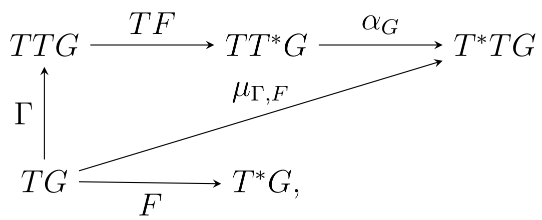
$$
\begin{aligned}
\frac{d}{d\varepsilon}\mathcal{A}_d(\{g_k^\varepsilon\}_{k=0}^n) &= \sum_{k=0}^n \left(\frac{\partial \ell}{\partial g}\frac{d}{d\varepsilon} g_k^\varepsilon \bigg|_{\varepsilon=0} + \frac{\partial \ell}{\partial f}\frac{d}{d\varepsilon} f_{k,k+1}^\varepsilon \bigg|_{\varepsilon=0} \right) \\
&= \sum_{k=0}^n \left(\frac{\partial \ell}{\partial g} g_k \delta g_k + \frac{\partial \ell}{\partial f} f_{k,k+1}\delta f_{k,k+1} \right) \\
&= \sum_{k=0}^n \left(\frac{\partial \ell}{\partial g} g_k \delta g_k + \frac{\partial \ell}{\partial f} f_{k,k+1}[\delta g_{k+1} - Ad(f_{k+1,k})\delta g_k] f_{k,k+1} \right) \\
&= \sum_{k=1}^n \frac{\partial \ell}{\partial g} g_k \delta g_k + \sum_{k=1}^n \frac{\partial \ell}{\partial f} f_{k-1,k}\delta g_k - \sum_{k=1}^n \frac{\partial \ell}{\partial f} f_{k,k+1} \left(Ad(f_{k+1,k})\delta g_k \right) \\
&= \sum_{k=1}^n \left(\frac{\partial \ell}{\partial g} g_k + \frac{\partial \ell}{\partial f} f_{k-1,k} - \frac{\partial \ell}{\partial f} f_{k,k+1} \left(Ad(f_{k+1,k}) \right) \right) \delta g_k \\
&= \sum_{k=1}^n \left(\frac{\partial \ell}{\partial g} g_k + \frac{\partial \ell}{\partial f} DL_{f_{k-1,k}} - \frac{\partial \ell}{\partial f} DL_{f_{k,k+1}} \circ Ad(f_{k+1,k}) \right) \delta g_k = 0.
\end{aligned}
$$

Thus, we get the discrete Euler-Poincaré equations

$$
\frac{\partial \ell}{\partial g} g_k + \frac{\partial \ell}{\partial f} DL_{f_{k-1,k}} = \frac{\partial \ell}{\partial f} DL_{f_{k,k+1}} \circ Ad(f_{k+1,k}) \tag{DEP}
$$

3.2.3 Inverse Problem for the Discrete Euler-Poincaré Equations

Before, we get into this problem, we need to translate the diagram



for Lie groups and algebras. In order to do so, we use the theorems introduced in this section.

Definition 3.2.1. *We define the second order differential equation*

$$
\begin{array}{rccc}
\widetilde{\Gamma}\colon & G \times \mathfrak{g} & \longrightarrow & G \times \mathfrak{g} \times \mathfrak{g} \times \mathfrak{g} \\
& (q,\xi) & \longmapsto & (g,\xi,\xi,g^{-1}\Gamma - \xi^2).
\end{array}
$$

Definition 3.2.2. *We define the local diffeomorphism*

$$
\begin{array}{rccc}
\widetilde{F}\colon & G \times \mathfrak{g} & \longrightarrow & G \times \mathfrak{g}^* \\
& (g,\xi) & \longmapsto & (g, g^* F(g, g\xi)).
\end{array}
$$

Note that we built this function using the previous theorems. In fact

$$\widetilde{F} : G \times \mathfrak{g} \to TG \to T^*G \to G \times \mathfrak{g}^*.$$

Definition 3.2.3. *Let us define the map*

$$\begin{array}{cccc} \widetilde{TF}\colon & G \times \mathfrak{g} \times \mathfrak{g} \times \mathfrak{g} & \longrightarrow & G \times \mathfrak{g}^* \times \mathfrak{g} \times \mathfrak{g}^* \\ & (g, \xi, \eta, \xi) & \longmapsto & \left(g, g^* F(g, g\xi), \eta, g^* \left(\frac{\partial F}{\partial g}\eta + \frac{\partial F}{\partial \xi}(\dot{\xi} + \eta\xi)\right)\right). \end{array}$$

We still need to trivialize α_G.

Definition 3.2.4. *Consider the maps*

$$A\colon G \times \mathfrak{g}^* \times \mathfrak{g} \times \mathfrak{g}^* \to TT^*G, \tag{3.37}$$

$$\alpha_G\colon T^*TG \to T^*TG. \tag{3.38}$$

$$B\colon T^*TG \to G \times \mathfrak{g} \times \mathfrak{g}^* \times \mathfrak{g}^*. \tag{3.39}$$

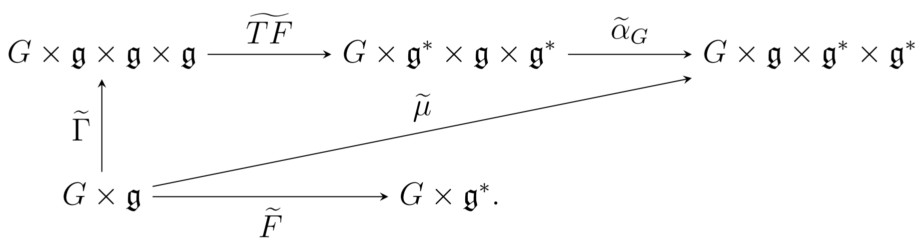
We denote the trivialized Tulczyjew isomorphism $\widetilde{\alpha}_G := B \circ \alpha_G \circ A$, which is defined by

$$\begin{array}{cccc} \widetilde{\alpha}_G\colon & G \times \mathfrak{g}^* \times \mathfrak{g} \times \mathfrak{g}^* & \longrightarrow & G \times \mathfrak{g} \times \mathfrak{g}^* \times \mathfrak{g}^* \\ & (g, \eta, \xi, \mu) & \longmapsto & (g, \xi, \mu - \mathsf{ad}^*_\xi(\eta), \eta). \end{array}$$

Let us denote $f(g, \xi) = g^* F(g, g\xi)$ and $\gamma(g, \xi) = g^{-1}\Gamma(g, g\xi) - \xi^2$. Finally, we have can define the symplectic structure $\widetilde{\mu}$.

$$\begin{array}{cccc} \widetilde{\mu}\colon & G \times \mathfrak{g} & \longrightarrow & G \times \mathfrak{g} \times \mathfrak{g}^* \times \mathfrak{g}^* \\ & (g, \xi) & \longmapsto & (g, \xi, Df(g, \xi)(\gamma(g, \xi)) - \mathsf{ad}^*_\xi(f(g, \xi)), f(g, \xi)). \end{array}$$

Using these definitions we can draw the diagram



Definition 3.2.5. *The trivialized version of ω_{TG} is given by $\widetilde{\omega}_{TG} : G \times \mathfrak{g} \times \mathfrak{g}^* \times \mathfrak{g}^* \times \mathfrak{g} \times \mathfrak{g} \times \mathfrak{g}^* \times \mathfrak{g}^* \to \mathbb{R}$. For any $\phi = (g, \xi, \mu, \eta, X, V, A, B), \psi = (g, \xi, \mu, \eta, Y, W, C, D)$ we have*

$$\widetilde{\omega}_{TG}(\phi, \psi) = \langle A, Y\rangle - \langle C, X\rangle - \langle \mu, [X, Y]_{\mathfrak{g}}\rangle + \langle B, W\rangle - \langle D, V\rangle. \tag{3.40}$$

For more details on this definition, we refer the reader to [19].

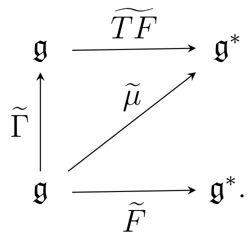
Theorem 3.2.10. *A second order differential equation $\widetilde{\Gamma}$ is variational if and only if there is a local diffeomorphism $\widetilde{F}$ such that $Im(\widetilde{\mu})$ is a Lagrangian submanifold of $(G \times \mathfrak{g} \times \mathfrak{g}^* \times \mathfrak{g}^*, \widetilde{\omega}_{TG})$.*

Proof (sketch). Suppose that $Im(\widetilde{\mu})$ is a Lagrangian submanifold, then we can find conditions that resemble the Helmholtz conditions for which the system is variational. In order to do that, we let $\widetilde{\omega}_{TG}(\phi, \psi) = 0$. □

If we consider that Γ is left invariant, we get that

$$\gamma(\xi) := \gamma(e, \xi) = \Gamma(e, \xi) - \xi^2.$$

So, γ is also invariant. If F is invariant, then $F(g, g\xi) = ((g^{-1})^*)F(e, \xi)$. Hence, we would have the reduced diagram



Chapter 4

Conclusions

We have derive four important equations. First, of all the Euler-Lagrange equations

$$\frac{\partial L}{\partial q^i} - \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial v^i}\right) = 0. \tag{4.1}$$

Then, we derived the discrete Euler-Lagrange equations

$$D_1\mathbb{L}_q(q_k, q_{k+1}) + D_2\mathbb{L}_q(q_{k-1}, q_k) = 0. \tag{4.2}$$

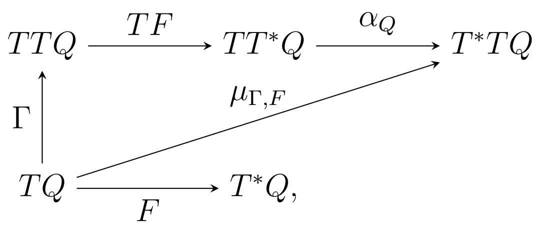
After that we found the Euler-Poincaré equations

$$L_g^*(\partial_g l) + \frac{d}{dt}\frac{\partial l}{\partial \xi} + ad^*[\partial_\xi l] = 0. \tag{4.3}$$

Finally, we derived the discrete Euler-Poincaré equations.

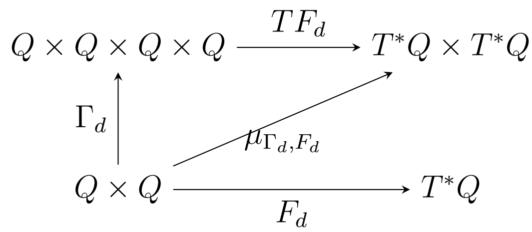
$$\mathcal{A}_d(\{g_k\}_{k=0}^n) = \sum_{k=0}^{n}\Big(\ell(g_k) + \ell(f_{k,k+1})\Big)\,. \tag{4.4}$$

Each of these equation has an associated inverse problem. Even though the inverse problem for Euler-Lagrange equations has many ways to be solved. The approach we emphasized on is the new geometric formulation. This theorem states that given a system of SODEs, this system is variational if and only if we can find a local diffeomorphism F such that $Im(\mu_{\Gamma,F})$ is a Lagrangian submanifold.
Furthermore we were able to find equivalent theorems for the other inverse problem. The best way to do so, was to translate the the diagram

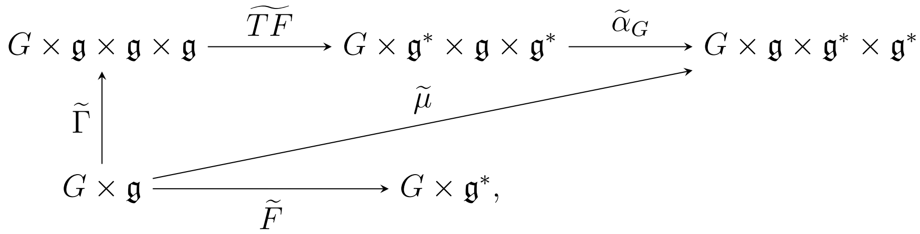


depending on the space we were working on.

In the inverse problem for discrete Euler-Lagrange equations, this diagram became



In the case of the discrete Euler-Poincaré equations we had the diagram



which was possible because of the trivialization introduced in the last chapter.

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