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Escuela de Ciencias Matemáticas y Computacionales

On the main Vekua equation

Trabajo de integración curricular presentado como requisito para la obtención del título de Matemático

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Urcuquí, Abril de 2023

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Dedication

”To my dear niece Alessia, my brother Diego (†) who watches over me from heaven, and those who helped me become involved in the beautiful world of Mathematical research.”

Guido Samuel Tapia Riera

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Guido Samuel Tapia Riera

Resumen

Esta tesis consta de dos partes. En la primera parte, consideramos un sistema elíptico de ecuaciones en el plano denominado ecuación principal de Vekua

$$\partial_{\bar{z}} w = \frac{f_{\bar{z}}}{f} \overline{w}, \quad \text{en } \Omega, \tag{1}$$

donde $\partial_{\bar{z}} = \frac{1}{2}(\partial_x + i\partial_y)$, Ω es un conjunto abierto en $\mathbb{C}$, $f \in C^2(\Omega)$ es una función real sin ceros definida en Ω y $w \in C^1(\Omega)$ es la función desconocida. Estudiamos aspectos teóricos de la teoría de funciones pseudoanalíticas respecto a la ecuación principal de Vekua. En concreto, partiendo del concepto de par generador desarrollamos la teoría de diferenciación e integración para (1). Luego, presentamos un método para la construcción de un sistema infinito de soluciones de (1), denominado potencias formales, que en cierto sentido generalizan las potencias usuales del análisis complejo. Además, estudiamos la conexión entre la ecuación principal de Vekua y la ecuación de Schödinger estacionaria bidimensional. En la segunda parte extendemos algunos de los resultados anteriores para la ecuación principal de Vekua en su forma matricial

$$\partial_{\bar{z}} W = (\partial_{\bar{z}} F) F^{-1} \overline{W}, \quad \text{en } \Omega,$$

donde $F \in C^2_{n\times n}(\Omega)$ es una función matricial real invertible sin ceros definida Ω y $W \in C^1_{n\times n}(\Omega)$.

Palabras Clave: Ecuación principal de Vekua, teoría de funciónes pseudoanalíticas, potencias formales, ecuación estacionaria de Schödinguer.

Abstract

This thesis consists of two parts. In the first part, we consider an elliptic system of equations in the plane called main Vekua equation

$$\partial_{\bar{z}} w = \frac{f_{\bar{z}}}{f} \overline{w}, \quad \text{in } \Omega, \tag{2}$$

where $\partial_{\bar{z}} = \frac{1}{2}(\partial_x + i\partial_y)$, Ω is an open set in $\mathbb{C}$, $f \in C^2(\Omega)$ is a given nonvanishing real-valued function in Ω and $w \in C^1(\Omega)$ is the unknown function. We study theoretical aspects of pseudoanalytic function theory for the main Vekua equation. Specifically, starting with the concept of generating pair we develop the theory of differentiation and integration for (2). Then, we present a method for the construction of an infinite system of solutions of (2), called formal powers, that in some sense generalizes the usual powers of complex analysis. Also, we study the connection between the main Vekua equation and the two-dimensional stationary Schödinguer equation. In the second part we extend some of the previous results for the main Vekua equation in its matrix form

$$\partial_{\bar{z}} W = (\partial_{\bar{z}} F) F^{-1} \overline{W}, \quad \text{in } \Omega,$$

where $F \in C^2_{n\times n}(\Omega)$ is a given nonvanishing invertible real matrix function in Ω and $W \in C^1_{n\times n}(\Omega)$.

Keywords: Main Vekua equation, pseudoanalytic function theory, formal powers, stationary Schödinguer equation.

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Chapter 1

Introduction

In this work, we focused on studying an elliptic system of equations in the plane called main Vekua equation

$$\partial_{\bar{z}} w = \frac{f_{\bar{z}}}{f}\overline{w}, \quad \text{in } \Omega, \tag{1.1}$$

where $\partial_{\bar{z}} = \frac{1}{2}(\partial_x + i\partial_y)$, Ω is an open set in $\mathbb{C}$, $f \in C^2(\Omega)$ is a given nonvanishing real-valued function in Ω , $w \in C^1(\Omega)$ is the unkown function and $\overline{w}$ is the conjugate of w . The interest in studying (1.1) is due to the fact that under the general conditions the study of several equations of mathematical physics such as the Dirac equation, the stationary Schrödinger equation or the Beltrami fields can be reduced to the study of the main Vekua equation (see [7]). In this work we are mainly interested in study and apply several concepts from the L. Bers theory for the main Vekua equation (1.1), such as: generating pair , differentiation and integration with respect to a generating pair, generalization of Cauchy and Morera's integral theorems, generating sequence, and construction of formal powers. An important part of this work is that we have extended some of this results for the main Vekua equation in its matrix form (Section 4)

$$\partial_{\bar{z}} W = (\partial_{\bar{z}} F)F^{-1}\overline{W}, \quad \text{in } \Omega, \tag{1.2}$$

where $F \in C^2_{n\times n}(\Omega)$ is a given nonvanishing invertible real matrix function in Ω and $W \in C^1_{n\times n}(\Omega)$.

This work is organized as follows. Chapter 2 is devoted to provide several results and concepts of mathematical analysis which are needed throughout the subsequent chapters. In Chapter 3 we present the pseudoanalytic function theory for the main Vekua equation. The first important tool in this theory is the so called generator pair (Section 3.2) that play a role in pseudoanalytic function theory similar to the role of 1 and i in the theory of analytic functions. An important feature when dealing with equation (1.1) is that a generating pair can be constructed in closed form, namely $(f, i/f)$ (Proposition 6). This allows one to develop the results regarding differentiation (Section 3.3) and integration (Section 3.4) for solutions of (1.1) with respect to this generating pair. Among other results, we prove the analog of the Cauchy's integral theorem (Corollary 8) and Morera's theorem (Proposition 12) for solutions of (1.1). One of the main difficulties in the study of (1.1) is that the derivative of a solution of (1.1) is a solution of (in general) another Vekua equation (see Proposition 9) called successor Vekua equation. This lead us to introduce the concept of generating sequence, a special sequence of generating pairs needed to deal in a suitable manner with higher order derivatives of solutions of (1.1) (see Definition 15). In Section 3.5 we deal with the construction of an infinite system of solutions of (1.1), called formal powers, that in some sense generalize the usual powers of complex analysis. When f has a separable form, $f(x,y) = \sigma(x)\tau(y)$ we provide a method that allows one to construct explicitly the formal powers in terms of a simple algorithm based on recursive integration (Section 3.5.2). Finally, in Section 3.6 we give a complex factorization of the two-dimensional stationary Schödinger equation in terms of Vekua type operators. This factorization is important to us because it help us to construct solutions of the Schödinger equation by means of the solutions of the main Vekua equation (see Proposition 18).

In Chapter 4 we study the main Vekua equation in its matrix form. One of the main difficulties in the study of (1.2) is the noncommutativity of the matrix product. By introducing the concept of compatible pair (Definition 18) we extend some of the results present in Chapter 3 to the solutions of (1.2).

Chapter 2

Preliminaries

In this Chapter we present some results from mathematical analysis needed throughout the manuscript. For further details see [1], [11], [10], and [4].

2.1 Line integrals, path independence, and conservative vector fields

Along this manuscript we denote $\mathbb{N} = \{1, 2, 3, \dots\}$ and $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. Before introducing the definition of line integral, let us enunciate the curve and path definitions.

Definition 1. *We call path in $\mathbb{R}^2$ to a continuous function $\gamma : [a, b] \subseteq \mathbb{R} \to \mathbb{R}^2, t \longmapsto \gamma(t)$. While a curve Γ in $\mathbb{R}^2$ is the one that is defined by a path γ such that $\Gamma = \gamma([a, b])$. In this case, we say that Γ is parameterized by γ (or γ is a parameterization of Γ).*

Let a curve $\Gamma \subseteq \mathbb{R}^2$ be parameterized by $\gamma : [a, b] \to \mathbb{R}^2$. In order to give the line integral definition we first need to enunciate some concepts:

- Let $n \in \mathbb{N}_0$ and take a partition of the interval $[a, b]$ as follows

$$a = t_0 < t_1 < t_2 < \cdots < t_{n-1} < t_n = b.$$

  This generates a partition over the curve Γ

$$A = P_0, P_1, P_2, \dots, P_n = B,$$

  where $A = \gamma(a)$ is the initial point and $B = \gamma(b)$ the terminal point of Γ.

- Let us denote $P_0, \ldots, P_n$ as partition points. These divide Γ into n sub-curves of length Δs_i where $i = 1, \ldots, n$. Also, note that Δx_i and Δy_i are the projection of each sub-curve onto the x and y axes, respectively.

- $\|P\|$ denotes the length of the longest sub-curve.

- $\gamma(t_i^*) = (x(t_i^*), y(t_i^*))$ denotes some point over the i-th sub-curve where $t_i^* \in [t_{i-1}, t_i]$ and $i = 1, \ldots, n$. For simplicity we denote $(x(t_i^*), y(t_i^*))$ as (x_i^*, y_i^*).

Without loss generality, we assume that the domains where the functions used in this section are defined are open.

Definition 2. *Let $f : \Omega \subseteq \mathbb{R}^2 \to \mathbb{R}$ be a continuous function and $\Gamma \subseteq \Omega$ be a curve parameterized by $\gamma : [a, b] \subseteq \mathbb{R} \to \mathbb{R}^2$. Then*

(a) The line integral of f with respect to x along Γ from $A = \gamma(a)$ to $B = \gamma(b)$ is given as

$$\int_\Gamma f(x,y)dx = \lim_{\|P\| \to 0} \sum_{i=1}^n f\left(x_i^*, y_i^*\right) \Delta x_i.$$

(b) The line integral of f with respect to y along Γ from $A = \gamma(a)$ to $B = \gamma(b)$ is given by

$$\int_\Gamma f(x,y)dy = \lim_{\|P\| \to 0} \sum_{i=1}^n f\left(x_i^*, y_i^*\right) \Delta y_i.$$

Remark 1. *It is important to mention that a line integral is independent of the parameterization of the curve Γ as long as Γ is equipped with the same orientation through all parametric equations defining the curve Γ.*

Now, we extend the line integral concept to vector fields. Formally, we get

Definition 3. *Let $F = (P, Q) : \Omega \subseteq \mathbb{R}^2 \to \mathbb{R}^2$ be a continuous vector field and $\Gamma \subseteq \Omega$. Then, the line integral of F along Γ is defined as follows*

$$\int_\Gamma F \cdot d\gamma := \int_\Gamma P(x,y)dx + \int_\Gamma Q(x,y)dy.$$

Proposition 1. *Let $F = (P, Q) : \Omega \subseteq \mathbb{R}^2 \to \mathbb{R}^2$ be a continuous vector field and a smooth curve (or piece-wise smooth) $\Gamma \subseteq \Omega$ be parameterized by $\gamma = (x(t), y(t)) : [a, b] \subseteq \mathbb{R} \to \mathbb{R}^2$. Then*

$$\int_\Gamma P(x,y)dx + \int_\Gamma Q(x,y)dy = \int_a^b P(\gamma(t))x'(t)dt + \int_a^b Q(\gamma(t))y'(t)dt.$$

Definition 4. *We say that a vector field* $F : \Omega \subseteq \mathbb{R}^2 \to \mathbb{R}^2$ *is conservative if there exists* $\varphi : \Omega \subseteq \mathbb{R}^2 \to \mathbb{R} \in C^1(\Omega)$ *such that* $F = \nabla\varphi$ *in* Ω*. Moreover, in this case we say that* φ *is the potential of* F*.*

Theorem 1 (Fundamental Theorem for Line Integrals)**.** *Let* $F = (P, Q) : \Omega \subseteq \mathbb{R}^2$ *be a vector field and a smooth curve (or piece-wise smooth)* $\Gamma \subseteq \Omega$ *be parameterized by* $\gamma : [a, b] \subseteq \mathbb{R} \to \mathbb{R}^2$*. If* F *is a conservative vector field and* $\varphi : \Omega \to \mathbb{R}$ *is the corresponding potential function, then the integral of* F *along* Γ *depends only of the initial and terminal point and*

$$\int_\Gamma F \cdot d\gamma = \varphi(B) - \varphi(A),$$

where $A = \gamma(a)$ *and* $B = \gamma(b)$*.*

See Ruiz [10, Th. 7.4.1, pp. 703] for a proof.

Corollary 1. *Under the conditions of Theorem 1 for a closed curve* Γ *we have that*

$$\oint_\Gamma P(x, y)dx + Q(x, y)dy = 0.$$

Note that if Corollary 1 is fulfilled for every closed curve in the domain of definition of F, then we have that F is a conservative vector field.

Theorem 2. *Let a vector field* $F : \Omega \subseteq \mathbb{R}^2 \to \mathbb{R}^2$ *of class* C^1 *where* Ω *is a simply connected domain. We have that* F *is conservative if and only if*

$$P_y = Q_x, \quad \text{in } \Omega. \tag{2.1}$$

See Ruiz [10, Th. 7.4.4, pp. 714] for a proof.

Theorem 2 give us a tool to construct potential functions of conservative fields. To end this section we introduce the real version of the Green's formula which will be important for section 3.4

Theorem 3. *Let a regular domain* Ω[1] $\subseteq \mathbb{R}^2$ *and a continuously differentiable vector field* $F = (P, Q) : \overline{\Omega} \to \mathbb{R}^2$*. Then the following formula is valid:*

$$\int_\Omega \left(P_x(x, y) - Q_y(x, y)\right) dxdy = \int_{\partial\Omega} (P(x, y)dy + Q(x, y)dx).$$

See Tutschke [11, Lemma. 6, pp. 106] for a proof.

[1]We say that Ω is a regular domain if Ω is bounded and $\partial\Omega$ is formed by a finite number of piecewise continuously differentiable simple closed Jordan curves.

2.2 Cauchy-Riemann operators and related topics

2.2.1 The operators $\partial_{\bar{z}}$ and ∂_z

Let Ω be an open set in $\mathbb{C}$.

Definition 5. *The linear operators $\partial_{\bar{z}}, \partial_z : C^1(\Omega) \to C(\Omega)$ given by*

$$\partial_{\bar{z}} := \frac{1}{2}(\partial_x + i\partial_y) \quad \text{and} \quad \partial_z := \frac{1}{2}(\partial_x - i\partial_y)^{2},$$

are known as the Cauchy-Riemann operators.

Remark 2. *The operator $\partial_{\bar{z}}$ acts on a complex function $\Phi = \Phi_1 + i\Phi_2$ as follows*

$$\partial_{\bar{z}}\Phi := \frac{1}{2}(\partial_x\Phi_1 - \partial_y\Phi_2 + i(\partial_y\Phi_1 + \partial_x\Phi_2)).$$

Proposition 2. *Let $\Phi, \Psi \in C^1(\Omega)$ be complex valued functions, then:*

(a) $\partial_{\bar{z}}\partial_z\Phi = \partial_z\partial_{\bar{z}}\Phi = \frac{1}{4}\Delta\Phi$, *where $\Delta = \partial_x^2 + \partial_y^2$ is the Laplacian operator (here $\Phi \in C^2(\Omega)$)*

(b) $\partial_{\bar{z}}C\Phi = C\partial_z\Phi$, *where $C\Phi = \overline{\Phi}$ is the conjugation operator*

(c) $\partial_{\bar{z}}(\Phi\Psi) = (\partial_{\bar{z}}\Phi)\Psi + \Phi\partial_{\bar{z}}(\Psi)$

Proof. Let us prove item *(a)*. We have

$$\begin{aligned}\partial_{\bar{z}}\partial_z\Phi =& \frac{1}{2}\partial_{\bar{z}}(\partial_x\Phi - i\partial_y\Phi)\\ =& \frac{1}{4}(\partial_x^2\Phi - i\partial_x\partial_y\Phi + i\partial_y\partial_x\Phi + \partial_y^2\Phi)\end{aligned}$$

note that $\partial_x\partial_y\Phi = u_{yx} + iv_{yx}$ and $\partial_y\partial_x\Phi = u_{xy} + iv_{xy}$. By Clairaut's Theorem we get $\partial_x\partial_y\Phi = \partial_y\partial_x\Phi$. Thus

$$\partial_{\bar{z}}\partial_z\Phi = \frac{1}{4}(\partial_x^2\Phi + \partial_y^2\Phi) = \frac{1}{4}\Delta\Phi.$$

Analogous for $\partial_z\partial_{\bar{z}}\Phi$. Now we prove item *(b)*. We have

$$\partial_{\bar{z}}C\Phi = \partial_{\bar{z}}(u - iv) = \frac{1}{2}(\partial_x u + \partial_y v + i(\partial_y v - \partial_x v)) \tag{2.2}$$

$$\begin{aligned}C\partial_z\Phi = C\partial_z(u + iv) =& C\frac{1}{2}(\partial_x u + \partial_y v + i(\partial_x v - \partial_y u))\\ =& \frac{1}{2}(\partial_x u + \partial_y v + i(\partial_y v - \partial_x v))\end{aligned} \tag{2.3}$$

[2] ∂_x and ∂_y denote the partial derivative respect to x and y, respectively.

Consequently, by (2.2) and (2.3) we get that $\partial_{\bar{z}} C\Phi = C\partial_z \Phi$. Finally, let us prove *(c)*. Applying $\partial_{\bar{z}}$ to $\Phi\Psi$ we get

$$\begin{aligned}
\partial_{\bar{z}}(\Phi\Psi) &= \frac{1}{2}\left(\partial_x + i\partial_y\right)(\Phi\Psi) \\
&= \frac{1}{2}\left(\partial_x(\Phi\Psi) + i\partial_y(\Phi\Psi)\right) \\
&= \frac{1}{2}\left((\partial_x\Phi)\Psi + \Phi(\partial_x\Psi) + i(\partial_y\Phi)\Psi + i\Phi(\partial_y\Psi)\right) \\
&= \frac{1}{2}\left((\partial_x\Phi)\Psi + i(\partial_y\Phi)\Psi\right) + \frac{1}{2}\left(\Phi(\partial_x\Psi) + i\Phi(\partial_y\Psi)\right) \\
&= (\partial_{\bar{z}}\Phi)\Psi + \Phi(\partial_{\bar{z}}\Psi).
\end{aligned}$$

This completes the proof of Proposition 2. □

2.2.2 The operators A and $\overline{A}$

Let us consider the equation

$$\partial_z \varphi = \Phi, \quad \text{in } \Omega, \tag{2.4}$$

(which will be needed in Section 3.4). Here $\Phi \in C^1(\Omega)$ is a given complex-valued function $\Phi = \Phi_1 + i\Phi_2$, the unknown function φ is real valued and Ω is a simply connected domain. Taking into consideration the real and the imaginary parts of (2.4) we realize that it is equivalent to the following system

$$\partial_x \varphi = 2\Phi_1, \ \ \partial_y \varphi = -2\Phi_2$$

which tell us that φ is a potential of the vectorial field $F := (2\Phi_1, -2\Phi_2)$. According to Theorem 2 this is possible if the compatibility condition

$$\partial_y \Phi_1 + \partial_x \Phi_2 = 0 \tag{2.5}$$

holds on Ω. Moreover, φ can be recovered up to a constant by the formula

$$\varphi(x,y) = 2\left(\int_\Gamma \Phi_1(x,y)dx - \Phi_2(x,y)dy\right) + c \tag{2.6}$$

where $\Gamma \subseteq \Omega$ is some curve joining an arbitrary fixed point (x_0, y_0) to (x, y).

Definition 6. *For a complex function $\Phi = \Phi_1 + i\Phi_2$ fulfilling (2.5) let us define*

$$A[\Phi](x,y) = 2\left(\int_\Gamma \Phi_1(x,y)dx - \Phi_2(x,y)dy\right). \tag{2.7}$$

Note that the previous operator can be also written as follows:

$$A[\Phi](x,y) = 2\,\mathrm{Re}\int_\Gamma (\Phi_1(x,y) + i\Phi_2(x,y))\,(dx + idy) = 2\,\mathrm{Re}\int_\Gamma \Phi(z)dz.$$

By construction we have proved the following proposition.

Proposition 3. *If $\Phi = \Phi_1 + i\Phi_2$ fulfills (2.5), then $\partial_z A[\Phi] = \Phi$.*

Remark 3. *The previous proposition tell us that in some sense the operator A is the right inverse of ∂_z.*

Remark 4. *By analogy it is easy to see that a kind of right inverse for $\partial_{\bar{z}}$ can be constructed by the formula*

$$\bar{A}[\Phi](x,y) = 2\left(\int_\Gamma \Phi_1(x,y)dx + \Phi_2(x,y)dy\right)$$

assuming that the real and imaginary parts of Φ enjoy the compatibility condition $\partial_y\Phi_1 - \partial_x\Phi_2 = 0$.

2.2.3 Green-Gauss Integral Theorem

The following theorem is the complex version of the Green's formula.

Theorem 4. *Let $\Omega \subseteq \mathbb{C}$ be a regular domain and a continuously differentiable complex function Φ defined in $\overline{\Omega}$. Then the following formulas are valid:*

$$\int_\Omega \partial_{\bar{z}}\Phi(z)dxdy = \frac{1}{2i}\int_{\partial\Omega}\Phi(z)dz, \tag{2.8}$$

and

$$\int_\Omega \partial_z\Phi(z)dxdy = -\frac{1}{2i}\int_{\partial\Omega}\Phi(z)d\bar{z}. \tag{2.9}$$

Proof. Let us prove (2.8). Setting $\Phi = \Phi_1 + i\Phi_2$, from the definition of $\partial_{\bar{z}}$ we get

$$2\int_\Omega \partial_{\bar{z}}\Phi(z)dxdy = \int_\Omega \left(\partial_x\Phi_1(x,y) - \partial_y\Phi_2(x,y)\right)dxdy \\ + i\int_\Omega \left(\partial_x\Phi_2(x,y) + \partial_y\Phi_1(x,y)\right)dxdy$$

From the Green's formula (Theorem 3) we get

$$\begin{aligned}2\int_{\Omega}\partial_{\bar{z}}\Phi(z)dxdy =& -\int_{\partial\Omega}(\Phi_1(x,y)dy+\Phi_2(x,y)dx) - i\int_{\partial\Omega}(\Phi_2(x,y)dy-\Phi_1(x,y)dx)\\ =& i\int_{\partial\Omega}(i\Phi_1(x,y)dy+i\Phi_2(x,y)dx) + \int_{\partial\Omega}(\Phi_1(x,y)dx-\Phi_2(x,y)dy)\\ =& i\int_{\Omega}(\Phi_1(x,y)dx+i\Phi_1(x,y)dy+i\Phi_2(x,y)dx-\Phi_2(x,y)dy)\\ =& i\int_{\partial\Omega}\Phi(z)dz\end{aligned}$$

Thus

$$\int_{\Omega}\partial_{\bar{z}}\Phi(z)dxdy = \frac{1}{2i}\int_{\partial\Omega}\Phi(z)dz.$$

To prove formula (2.9) note that $\overline{\partial_{\bar{z}}\Phi} = \partial_z\overline{\Phi}$. Then, replacing $\overline{\Phi}$ by Φ we easily obtain (2.9). □

Corollary 2. *Under the conditions of Theorem 4, for a domain $\Omega \subseteq \mathbb{C}$ bounded by a simple closed smooth curve Γ we have the Cauchy theorem for analytic functions*

$$\oint_{\Gamma}\Phi(z)dz = 0.$$

2.3 Analytic functions

Let Ω be an open set in $\mathbb{C}$ and Φ a complex-valued function defined in Ω.

Definition 7. *We say that Φ has complex derivative at $z_0 \in \Omega$ if and only if the following limit exists*

$$\Phi'(z_0) = \lim_{z\to z_0}\frac{\Phi(z)-\Phi(z_0)}{z-z_0}.$$

If Φ has a complex derivative at z_0 and at every point of some neighborhood of z_0, then we say that Φ is analytic at z_0. Also, we say that Φ is analytic on Ω if Φ is analytic at each point of Ω. By analogy to the theory of functions of real variable for analytic functions we have:

Proposition 4. *If Φ has complex derivative at $z_o \in \Omega$, then Φ is continuous at that point.*

See Asmar [1, Th. 2.3.4, pp. 115] for a proof.

Proposition 5. *Assume that Φ and Ψ are analytic functions in Ω and let c_1 and c_2 complex constants, then* *(a)* $c_1\Phi + c_2\Psi$ *is analytic on* Ω *and*

$$(c_1\Phi + c_2\Psi)'(z) = c_1\Phi'(z) + c_2\Psi'(z)$$

(b) $\Phi\Psi$ *is analytic on* Ω *and*

$$(\Phi\Psi)'(z) = \Phi'(z)\Psi(z) + \Phi(z)\Psi'(z).$$

(c) Φ/Ψ *is analytic on* $\tilde{\Omega}$*, where* $\tilde{\Omega} = \Omega \setminus \{z \in \Omega : \ \Psi(z) = 0\}$ *and*

$$\left(\frac{\Phi}{\Psi}\right)'(z) = \frac{\Phi'(z)\Psi(z) - \Phi(z)\Psi'(z)}{\Psi^2(z)}.$$

See Asmar [1, Th. 2.3.5, pp. 116] for a proof.

Theorem 5 (Cauchy-Riemann Equations)**.** *Let* $\Phi(z) = u(x,y) + iv(x,y)$ *be a complex function defined in* Ω *and* $z_0 = x_0 + iy_0 \in \Omega$*. If* Φ *has complex derivative at* z_0*, then at* z_0 *the first order partial derivatives of* u *and* v *exist and satisfy the following equations*

$$u_x(x_0,y_0) = v_y(x_0,y_0) \quad \textit{and} \quad u_y(x_0,y_0) = -v_x(x_0,y_0). \tag{2.10}$$

Proof. Let $z_0 \in \Omega$ and assume that $\Phi = u + iv$ has complex derivative at z_0. Then

$$\begin{aligned} \Phi'(z_0) &= \lim_{z\to z_0} \frac{\Phi(z) - \Phi(z_0)}{z - z_0}, \\ &= \lim_{(x,y)\to(x_0,y_0)} \frac{u(x,y) - u(x_0,y_0) + i(v(x,y) - v(x_0,y_0))}{x - x_0 + i(y - y_0)}. \end{aligned} \tag{2.11}$$

Since $\Phi'(z_0)$ exists, the value of (2.11) is the same for any direction. Particularly, by setting $y = y_0$ and $x \to x_0$, we get

$$\begin{aligned} \Phi'(z_0) &= \lim_{y=y_0, x\to x_0} \left(\frac{u(x,y_0) - u(x_0,y_0)}{x - x_0} + i\frac{v(x,y_0) - v(x,y_0)}{x - x_0} \right) \\ &= u_x(x_0,y_0) + iu_x(x_0,y_0). \end{aligned} \tag{2.12}$$

Analogously, setting $x = x_0$ and $y \to y_0$ we get

$$\Phi'(z_0) = v_y(x,y) - iv_y(x_0,y_0). \tag{2.13}$$

Then, by (2.12) and (2.13) we obtain (2.10). □

Equations on (2.10) are known as the Cauchy-Riemann equations. These equations also provide a helpful rule to compute the complex derivative, let us see it in the following corollary.

Corollary 3. *If $\Phi'(z_0)$ exists, then*

$$\Phi'(z_0) = u_x(x_0, y_0) + iv_x(x_0, y_0) = v_y(x_0, y_0) - iu_y(x_0, y_0).$$

By themselves, the Cauchy-Riemann equations are not a sufficient condition to guarantee the existence of the complex derivative. Let us see this in the next example.

Example 1. *Let us consider $\Phi(z) = u(x, y) + iv(x, y)$ where*

$$u(x, y) = \frac{x^3 - y^3}{x^2 + y^2} \quad \text{and} \quad v(x, y) = \frac{x^3 + y^3}{x^2 + y^2}$$

for $(x, y) \neq (0, 0)$ and $u(0, 0) = v(0, 0) = (0, 0)$. Let us compute the partial derivatives of u and v at $(0, 0)$. We have

$$\begin{aligned} u_x(0, 0) &= \lim_{h \to 0} \frac{u(0 + h, 0) - u(0, 0)}{h} \\ &= \lim_{h \to 0} \frac{\frac{h^3}{h^2} - 0}{h} = 1. \end{aligned}$$

By analogy we also get $u_y(0, 0) = -1, v_x(0, 0) = 1$, and $v_y(0, 0) = 1$. Thus, the Cauchy-Riemann equations are satisfied at $(0, 0)$. However, Φ has no complex derivative at that point. To see it, note that

$$\Phi'(0) = \frac{\Phi(z) - \Phi(0)}{z - 0} = \lim_{z \to 0} \frac{x^3 - y^3 + i(x^3 + y^3)}{(x^2 + y^2)(x + iy)},$$

and taking the directional limit along the lines $y = 0$ and $y = x$ we get

$$\lim_{x \to 0, y = 0} \frac{x^3(1 + i)}{x^3} = 1 + i, \qquad \lim_{x \to 0, y = x} \frac{i2x^3}{2x^3(1 + i)} = \frac{i}{1 + i}.$$

Therefore, Φ at $z = 0$ does not derivative in that point.

As we shall see later, if u and v are sufficiently smooth, then via Cauchy-Riemann equations, we can guarantee the existence of complex derivative for Φ. First let us state some auxiliary definitions and facts.

Definition 8. *Let $\Omega \subseteq \mathbb{R}^2$ be an open set and $(x_0, y_0) \in \Omega$. A scalar field $u : \Omega \to \mathbb{R}$ is $\mathbb{R}^2$-differentiable at (x_0, y_0) if the partial derivatives exist and*

$$\lim_{(x,y)\to(x_0,y_0)} \frac{u(x,y) - u(x_0,y_0) - u_x(x_0,y_0)(x-x_0) - u_y(x_0,y_0)(y-y_0)}{\sqrt{(x-x_0)^2 + (y-y_0)^2}} = 0.$$

Lemma 1. *Let $z_0 \in \Omega$ and Φ, Ψ be complex functions defined in a neighborhood of z_0. Then, $\lim_{z\to z_0} \frac{\Phi(z)}{\Psi(z)} = 0$ if and only if $\lim_{z\to z_0} \frac{\Phi(z)}{|\Psi(z)|} = 0$.*

Proof. The proof of this Lemma is straightforward using $\varepsilon - \delta$ definition of limit. □

Theorem 6. *A function $\Phi = u + iv$ defined in Ω has a complex derivative at $z_0 = x_0 + iy_0 \in \Omega$ if and only if both conditions hold:*

(a) u and v are $\mathbb{R}^2-$differentiable at (x_0, y_0);

(b) u and v satisfy the Cauchy-Riemann equations (2.10) at (x_0, y_0).

Proof. Assume that Φ has a complex derivative at z_0. Then from Theorem 5 we have that *(b)* holds. Also, from definition of complex derivative we get

$$\lim_{z\to z_0} \frac{\Phi(z) - \Phi(z_0) - \Phi'(z_0)(z-z_0)}{z - z_0} = 0. \tag{2.14}$$

By (2.14) and Lemma 1 we deduce

$$\lim_{z\to z_0} \frac{\Phi(z) - \Phi(z_0) - \Phi'(z_0)(z-z_0)}{|z - z_0|} = 0,$$

or equivalently

$$\lim_{z\to z_0} \frac{\Phi(z) - \Phi(z_0) - \Phi'(z_0)(z-z_0)}{\sqrt{(x-x_0)^2 + (y-y_0)^2}} = 0. \tag{2.15}$$

Taking into account that $\Phi'(z_0) = u_x(x_0,y_0) + iv_x(x_0,y_0)$ that u, v satisfy the Cauchy-Riemann equations the numerator of (2.15) can be written as follows

$$\begin{aligned} &\Phi(z) - \Phi(z_0) - \Phi'(z_0)(z-z_0) = \\ =&u(x,y) - u(x_0,y_0) - u_x(x_0,y_0)(x-x_0) - u_y(x_0,y_0)(y-y_0) \\ &+ i[v(x,y) - v(x_0,y_0) - v_x(x_0,y_0)(x-x_0) - v_y(x_0,y_0)(y-y_0)]. \end{aligned} \tag{2.16}$$

Joining (2.16) with (2.15) we easily deduce

$$\lim_{(x,y)\to(x_0,y_0)} \frac{u(x,y) - u(x_0,y_0) - u_x(x_0,y_0)(x-x_0) - u_y(x_0,y_0)(y-y_0)}{\sqrt{(x-x_0)^2 + (y-y_0)^2}} = 0, \tag{2.17}$$

$$\lim_{(x,y)\to(x_0,y_0)} \frac{v(x,y) - v(x_0,y_0) - v_x(x_0,y_0)(x-x_0) - v_y(x_0,y_0)(y-y_0)}{\sqrt{(x-x_0)^2 + (y-y_0)^2}} = 0. \tag{2.18}$$

From (2.17) and (2.18), we have proved *(a)*. Conversely, from the Cauchy-Riemann equations we see that (2.16) holds if we set $\Phi'(z_0) = u_x(x_0,y_0) + iv_x(x_0,y_0)$. Moreover, joining (2.17), (2.18), and (2.16) we get (2.15) which is equivalent to (2.14). Finally from (2.14) we conclude that Φ has a complex derivative at z_0.

□

Chapter 3

Pseudoanalytic function theory for the main Vekua equation

3.1 Main Vekua equation

Let Ω be an open set in $\mathbb{C}$. An equation of the form

$$\partial_{\bar{z}} w = aw + b\overline{w}, \qquad \text{in } \Omega, \tag{3.1}$$

is called Vekua equation. Here $\partial_{\bar{z}} = \frac{1}{2}(\partial_x + i\partial_y)$ is the Cauchy-Riemann operator, $\overline{w}$ is the conjugate of w, a and $b \in C(\Omega)$ are given functions (called coefficients) and $w \in C^1(\Omega)$ is the unknown function. All the involved functions are complex valued.

Historically, the mathematical theory for (3.1) was mainly developed by Lipman Bers (Theory of pseudoanalytic functions, see [2]) and by Ilya Vekua (Generalized Analytic Functions, see [12]).

This chapter is devoted to the study of an important special Vekua equation introduced and studied in [7], called the main Vekua equation

$$\partial_{\bar{z}} w = \frac{f_{\bar{z}}}{f}\overline{w}. \tag{3.2}$$

Here $f \in C^2(\Omega)$ is a given real-valued function such that $f(z) \neq 0$ in Ω. Note that (3.2) is a particular case of (3.1) with $a = 0$ and $b = \frac{f_{\bar{z}}}{f}$ and that setting $w = u + iv$ (3.2), is equivalent to the following first order elliptic system of equations

$$\begin{aligned} u_x - v_y - \frac{f_x}{f}u - \frac{f_y}{f}v &= 0, \\ u_y + v_x - \frac{f_y}{f}u + \frac{f_x}{f}v &= 0. \end{aligned} \tag{3.3}$$

Observe that if we choose $f = 1$ then the main Vekua equation (3.2) is $\partial_{\bar{z}} w = 0$ and (3.3) is the Cauchy Riemann system defining the classical analytic functions. We tress that, regarding possible applications, the main Vekua equation is far-reaching than the Cauchy-Riemann system. In fact, as shown in [7], under the general conditions the study of several equations of mathematical physics such as the Dirac equation (see [6]), the stationary Schrödinger equation (see Section 3.6) or the Beltrami fields (see [9]) can be reduced to the study of the main Vekua equation.

In our study of the main Vekua equation we will consider mainly the constructive techniques developed by L. Bers, known as Pseudoanalytic function theory, since they are more close generalizations from complex analysis. In the next section we start by introducing the first important tool of this theory, the so-called generator pair. In our exposition we have followed mainly [2] and [7]

3.2 Generating pair

Definition 9. *Let F and G be complex functions*[1] *defined in Ω. We say that (F, G) is a generating pair corresponding to* (3.1) *in Ω if*

(a) $F, G \in C^1(\Omega)$ and they are solutions of (3.1).

(b) For $z_0 \in \Omega$ we have that

$$\forall m \in \mathbb{C}, \exists \lambda, \beta \in \mathbb{R} : \quad m = \lambda F(z_0) + \beta G(z_0).$$

Up to this point, we note that the generating pair play a role in the pseudoanalytic function theory similar to the role of 1 and i in the classical theory of complex analysis.

[1] In order to unify notation, we use a slight abuse of notation for writing complex functions. Specifically, for a complex function $\Phi = u + iv$ we write $u(x, y) = u(x + iy) = u(z)$ and $v(x, y) = v(x + iy) = v(z)$.

An important fact is that in the case of the main Vekua equation, a generating pair can be constructed in explicit form as we explain next.

Proposition 6. *Let $f \in C^2(\Omega)$ be a nonvanishing function in Ω. Then, $\left(f, \frac{i}{f}\right)$ is a generating pair corresponding to the main Vekua equation (3.2).*

Proof. Clearly f and i/f satisfy (3.2). Let $z_0 = x_0 + iy_0 \in \Omega$. Let us take any $m = m_1 + im_2 \in \mathbb{C}$. We choose $\lambda = m_1/f(z_0)$ and $\beta = m_2 f(z_0)$. Then, we get

$$m = \lambda f(z_0) + \beta \frac{i}{f(z_0)}.$$

Therefore, by arbitrariness of m we have proved that $\left(f, \frac{i}{f}\right)$ is a generating pair for (3.2). □

3.3 Differentiation with respect to generating pair $\left(f, \frac{i}{f}\right)$

Consider the generating pair $(f, i/f)$ corresponding to the main Vekua equation (3.2) given as in Proposition 6. Let $w : \Omega \to \mathbb{C}$ an arbitrary function. With the help of the generating pair $(f, i/f)$, w can written as

$$w = \varphi f + \psi \frac{i}{f}, \quad \text{in } \Omega,$$

where $\varphi = \text{Re}(w)/f$ and $\psi = \text{Im}(w) f$ are real-valued. Now, we introduce the concept of $(f, i/f)$-derivative. Formally, we get

Definition 10. *Let $w : \Omega \to \mathbb{C}$ and $z_0 = x_0 + iy_0 \in \Omega$. We say that w has $\left(f, \frac{i}{f}\right)$-derivative at z_0 if the following limit*

$$\dot{w}(z_0) = \lim_{z \to z_0} \frac{w(z) - \left(\varphi(z_0) f(z) + \psi(z_0) \frac{i}{f(z)}\right)}{z - z_0} \tag{3.4}$$

exists and is finite.

If $\dot{w}$ exists everywhere in Ω, we say that w is $(f, i/f)$-pseudoanalytic of first kind on Ω (or simply pseudoanalytic if there is no confusion). The following auxiliary function

(for a fixed z_0 in Ω)

$$\widetilde{w} := w - \left(\varphi(z_0) f + \psi(z_0) \frac{i}{f} \right), \quad \text{in } \Omega, \tag{3.5}$$

will be useful for what follows next.

Proposition 7. *A complex function w defined in Ω has $(f, i/f)$-derivative at $z_0 = x_0 + iy_0 \in \Omega$ if and only if a complex function $\widetilde{w}$ defined in Ω given by (3.5) has complex derivative at $z_0 \in \Omega$. Moreover, $\dot{w}(z_0) = \widetilde{w}'(z_0)$.*

Proof. Assume that w has $\left(f, \dfrac{i}{f}\right)$-derivative at z_0, that is,

$$\dot{w}(z_0) = \lim_{z \to z_0} \frac{w(z) - \left(\varphi(z_0) f(z) + \psi(z_0) \dfrac{i}{f(z)} \right)}{z - z_0}.$$

From (3.5) we get $\widetilde{w}(z_0) = 0$, it follows that

$$\dot{w}(z_0) = \lim_{z \to z_0} \frac{\widetilde{w}(z)}{z - z_0} = \lim_{z \to z_0} \frac{\widetilde{w}(z) - \widetilde{w}(z_0)}{z - z_0}.$$

Then, $\dot{w}(z_0) = \widetilde{w}'(z_0)$. Conversely, the other direction is clear. □

Corollary 4. *Let $w : \Omega \to \mathbb{C}$ and $z_0 = x_0 + iy_0 \in \Omega$. If w has $(f, i/f)$-derivative, then for $\widetilde{w} : \Omega \to \mathbb{C}$ given by (3.5) we have:*

(a) By previous Proposition $\widetilde{w}$ has complex derivative at z_0. Thus $\widetilde{w}$ satisfies the Cauchy-Riemann equations at z_0, which is equivalent to $\widetilde{w}_{\bar{z}}(z_0) = 0$.

(b) The complex derivative of $\widetilde{w}$ at z_0 is computed by $\widetilde{w}_z(z_0)$.

Theorem 7. *Let a complex function $w = \varphi f + i\psi/f \in C^1(\Omega)$ and $z_0 = x_0 + iy_0 \in \Omega$. Then, w has $(f, i/f)$-derivative at z_0 if and only if*

$$\varphi_{\bar{z}}(z_0) f(z_0) + \psi_{\bar{z}}(z_0) \frac{i}{f(z_0)} = 0. \tag{3.6}$$

Moreover, we have that

$$\dot{w}(z_0) = \varphi_z(z_0) f(z_0) + \psi_z(z_0) \frac{i}{f(z_0)}.$$

Proof. Let us consider the function $\widetilde{w} : \Omega \to \mathbb{C}$ given by

$$\widetilde{w}(z) = w(z) - \left(\varphi\left(z_0\right) f(z) + \psi(z_0) \frac{i}{f(z)} \right).$$

Note that

$$\widetilde{w}(z) = (\varphi(z) - \varphi(z_0)) f(z) + (\psi(z) - \psi(z_0)) \frac{i}{f(z)}.$$

Then, applying $\partial_{\bar{z}}$ we get

$$\begin{aligned}\widetilde{w}_{\bar{z}}(z) =& \partial_{\bar{z}} \left(\left(\varphi(z) - \varphi(z_0)\right) f(z) + \left(\psi(z) - \psi(z_0)\right) \frac{i}{f(z)} \right) \\ =& \varphi_{\bar{z}}(x, y) f(z) + \left(\varphi(z) - \varphi(z_0)\right) f_{\bar{z}}(z) + \psi_{\bar{z}}(z) \frac{i}{f(z)} \\ & - \left(\psi(z) - \psi(z_0)\right) \frac{i f_{\bar{z}}(z)}{f^2(z)}\end{aligned}$$

evaluating at z_0 we get

$$\widetilde{w}_{\bar{z}}(z_0) = \varphi_{\bar{z}}(z_0) f(z_0) + \psi_{\bar{z}}(z_0) \frac{i}{f(z_0)}. \tag{3.7}$$

In a similar way we get

$$\widetilde{w}_{z}(z_0) = \varphi_{z}(z_0) f(z_0) + \psi_{z}(z_0) \frac{i}{f(z_0)}. \tag{3.8}$$

Assume that w has $(f, i/f)$-derivative at z_0. Then, from Corollary 4 follows that $\widetilde{w}_{\bar{z}}(z_0) = 0$, joining this with (3.7) we get

$$\varphi_{\bar{z}}(z_0) f(z_0) + \psi_{\bar{z}}(z_0) \frac{i}{f(z_0)} = 0.$$

Moreover, again by item (b) of Corollary 4 and Proposition 7 we get

$$\dot{w}(z_0) = \varphi_{z}(z_0) f(z_0) + \psi_{z}(z_0) \frac{i}{f(z_0)}.$$

Conversely, if (3.6) holds then by (3.7) we have that $\widetilde{w}_{\bar{z}}(z_0) = 0$ and from (3.8) we get $\widetilde{w}'(z_0) = \dot{w}(z_0)$. This completes the proof. □

Proposition 8. *A complex function $w \in C^1(\Omega)$ has $(f, i/f)$-derivative at $z_0 = x_0 + iy_0 \in \Omega$ if and only if it satisfies the main Vekua Equation (3.2) at z_0.*

Proof. Assume that w has $(f, i/f)$-derivative at z_0. From Theorem 7 we get

$$\varphi_{\bar{z}}(z_0)f(z_0) + \psi_{\bar{z}}(z_0)\frac{i}{f(z_0)} = 0. \tag{3.9}$$

Note that

$$\begin{aligned}
0 =&\varphi_{\bar{z}}(z_0)f(z_0) + \psi_{\bar{z}}(z_0)\frac{i}{f(z_0)} + \varphi(z_0)f_{\bar{z}}(z_0) - \varphi(z_0)f_{\bar{z}}(z_0)\\
&- \psi(z_0)\frac{if_{\bar{z}}(z_0)}{f^2(z_0)} + \psi(z_0)\frac{if_{\bar{z}}(z_0)}{f^2(z_0)}\\
=&\varphi_{\bar{z}}(z_0)f(z_0) + \varphi(z_0)f_{\bar{z}}(z_0) + \psi_{\bar{z}}(z_0)\frac{i}{f(z_0)} - \psi(z_0)\frac{if_{\bar{z}}(z_0)}{f^2(z_0)}\\
&- \varphi(z_0)f_{\bar{z}}(z_0) + \psi(z_0)\frac{if_{\bar{z}}(z_0)}{f^2(z_0)}\\
=&w_{\bar{z}}(z_0) - \frac{f_{\bar{z}}(z_0)}{f(z_0)}\left(\varphi(z_0)f(z_0) - \psi(z_0)\frac{i}{f(z_0)}\right)\\
=&w_{\bar{z}}(z_0) - \frac{f_{\bar{z}}(z_0)}{f(z_0)}\overline{w}(z_0).
\end{aligned}$$

Thus,

$$w_{\bar{z}}(z_0) = \frac{f_{\bar{z}}(z_0)}{f(z_0)}\overline{w}(z_0).$$

Conversely, we assume that w satisfies the main Vekua equation (3.2) at z_0. By the previous direction we easily obtain that

$$\varphi_{\bar{z}}(z_0)f(z_0) + \psi_{\bar{z}}(z_0)\frac{i}{f(z_0)} = 0,$$

and the result follows from Theorem 7. □

Corollary 5. *Under the conditions of Proposition 8, $w \in C^1(\Omega)$ is a solution of the main Vekua equation in Ω if and only if*

$$\varphi_{\bar{z}}(z)f(z) + \psi_{\bar{z}}(z)\frac{i}{f(z)} = 0, \quad \forall z \in \Omega.$$

Proposition 9. *If a function $w : \Omega \to \mathbb{C}$ is a solution of the main Vekua equation in Ω, then the $(f, i/f)$-derivative of w is a solution of the following Vekua equation*

$$(\dot{w})_{\bar{z}} = -\frac{f_z}{f}\overline{\dot{w}}, \quad \text{in } \Omega. \tag{3.10}$$

Proof. Let $z = x + iy \in \Omega$. Since w is solution of the main Vekua equation at z, we get

$$\dot{w}(z) = \varphi_z(z) f(z) + \psi_z \frac{i}{f(z)}. \tag{3.11}$$

By from Corollary 5 we get

$$\varphi_{\bar{z}}(z) f(z) + \psi_{\bar{z}}(z) \frac{i}{f(z)} = 0 \tag{3.12}$$

taking the conjugate to (3.12), we have that

$$\varphi_z(z) f(z) - \psi_z(z) \frac{i}{f(z)} = 0. \tag{3.13}$$

Solving (3.11) and (3.13), we get

$$\varphi_z(z) = \frac{\dot{w}(z)}{2f(z)} \quad \text{and} \quad \psi_z(z) = \frac{f(z)\dot{w}(z)}{2i}. \tag{3.14}$$

Applying ∂_z to (3.12) we get

$$\varphi_{\bar{z}z}(z) f(z) + \varphi_{\bar{z}}(z) f_z(z) + \psi_{\bar{z}z} \frac{i}{f(z)} - \frac{i\psi_{\bar{z}}(z) f_z(z)}{f^2(z)} = 0. \tag{3.15}$$

Applying $\partial_{\bar{z}}$ to (3.11) we get

$$\dot{w}_{\bar{z}}(z) = \varphi_{\bar{z}z}(z) f(z) + \varphi_z(z) f_{\bar{z}}(z) + \psi_{\bar{z}z} \frac{i}{f(z)} - \frac{i\psi_z(z) f_{\bar{z}}(z)}{f^2(z)} \tag{3.16}$$

Replacing (3.15) in (3.16) we get

$$\begin{aligned} \dot{w}_{\bar{z}}(z) =& \varphi_z(z) f_{\bar{z}}(z) - \frac{i\psi_z(z) f_{\bar{z}}(z)}{f^2(z)} - \varphi_{\bar{z}}(z) f_z(z) + \frac{i\psi_{\bar{z}}(z) f_z(z)}{f^2(z)} \\ =& \varphi_z(z) f_{\bar{z}}(z) - \frac{i\psi_z(z) f_{\bar{z}}(z)}{f^2(z)} - \overline{\left(\varphi_z(z) f_{\bar{z}}(z) - \frac{i\psi_z(z) f_{\bar{z}}(z)}{f^2(z)} \right)} \end{aligned} \tag{3.17}$$

Replacing (3.14) in (3.17) we have that

$$\begin{aligned} \dot{w}_{\bar{z}}(z) =& \frac{\dot{w}(z)}{2f(z)} f_{\bar{z}}(z) - \frac{f(z)\dot{w}(z)}{2i} \frac{i f_{\bar{z}}(z)}{f^2(z)} - \overline{\left(\frac{\dot{w}(z)}{2f(z)} f_{\bar{z}}(z) + \frac{f(z)\dot{w}(z)}{2i} \left(\frac{i f_{\bar{z}}(z)}{f^2(z)} \right) \right)} \\ =& \frac{\dot{w}(z)}{2f(z)} f_{\bar{z}}(z) - \frac{\dot{w}(z)}{2f(z)} f_{\bar{z}}(z) - \frac{\bar{\dot{w}}(z) f_z(z)}{2f(z)} - \frac{\bar{\dot{w}}(z) f_z(z)}{2f(z)} \\ =& -\frac{f_z(z)\bar{\dot{w}}(z)}{f(z)} \end{aligned}$$

Therefore, the proof is done. □

Definition 11. *Note that (3.10) is a Vekua equation. More precisely, let $v = u + iv : \Omega \to \mathbb{C}$ and $f \in C^2(\Omega)$ be nonvanishing function in Ω . The successor Vekua equation of (3.2) is given by*

$$v_{\overline{z}} = -\frac{f_z}{f}\overline{v}, \quad \text{in } \Omega. \tag{3.18}$$

3.4 Integration with respect to the generating pair $\left(f, \frac{i}{f}\right)$

By equation (3.14) we get

$$\varphi_z = \frac{\dot{w}}{2f} \quad \text{and} \quad \psi_z = -\frac{if\dot{w}}{2}, \quad \text{in } \Omega. \tag{3.19}$$

To recover φ and ψ we use the operator A (defined in section 2.2.2). Then, applying operator A to each equation in (3.19) we recover φ and ψ as follows

$$\varphi = A\left[\frac{\dot{w}}{2f}\right] \quad \text{and} \quad \psi = -A\left[\frac{if\dot{w}}{2}\right].$$

Hence, we write any complex-valued function w defined in Ω in terms of operator A in the following way:

$$w = fA\left[\frac{\dot{w}}{2f}\right] - \frac{i}{f}A\left[\frac{if\dot{w}}{2}\right] \tag{3.20}$$

Consequently, in (3.20) appears the additive term $c_1 f + ic_2/f$ for $c_1, c_2 \in \mathbb{R}$. Fixing $w(z_0)$ where $z_0 \in \Omega$, we have that $c_1 = \varphi(z_0)$ and $c_2 = \psi(z_0)$.

Definition 12. *Let w a complex-value function defined in Ω and Γ a rectifiable curve leading from $z_0 = x_0 + iy_0$ to $z_1 = x_1 + iy_1$ in Ω. Then, we have that*

(a) The $\left(f, \frac{i}{f}\right)$--integral is defined as*

$$*\int_\Gamma w(z)d_{(f,i/f)}z = \operatorname{Re}\int_\Gamma \frac{1}{f(z)}w(z)dz - i\operatorname{Re}\int_\Gamma if(z)w(z)dz.$$

(b) The $\left(f, \frac{i}{f}\right)$-integral is defined by

$$\int_\Gamma w(z)d_{(f,i/f)}z = f(z_1)\operatorname{Re}\int_\Gamma \frac{1}{f(z)}w(z)dz - \frac{i}{f(z_1)}\operatorname{Re}\int_\Gamma if(z)w(z)dz. \tag{3.21}$$

Definition 13. *Let $w : \Omega \to \mathbb{C}$ be a continuous function. We say that w is $(f, i/f)$-integrable in Ω if for every closed curve Γ lying in a simply connected subdomain of Ω we have*

$$\oint_\Gamma w(z) d_{(f,i/f)} z = 0. \tag{3.22}$$

Proposition 10. *Let $w : \Omega \to \mathbb{C}$ be a solution of the main Vekua equation in Ω where Ω is simply connected domain. Then its $(f, i/f)$-derivative is $(f, i/f)$-integrable in Ω.*

Proof. Let us take a closed curve Γ such that it lies in Ω. It is enough to prove that

$$* \oint_\Gamma \dot{w}(z) d_{(f,i/f)} z = 0. \tag{3.23}$$

From Definition 12 item *(a)* we get

$$* \oint_\Gamma \dot{w}(z) d_{(f,i/f)} z = \text{Re} \oint_\Gamma \frac{1}{f(z)} \dot{w}(z) dz + \text{Re} \oint_\Gamma -if(z)\dot{w}(z) dz.$$

Since $\dot{w}(z) = \varphi_z(z) f(z) + \psi_z(z) i/f(z)$, it follows that

$$\begin{aligned} * \oint_\Gamma \dot{w}(z) d_{(f,i/f)} z =& \text{Re} \oint_\Gamma \frac{1}{f(z)} \left(\varphi_z(z) f(z) + \psi_z(z) \frac{i}{f(z)} \right) dz \\ &+ \text{Re} \oint_\Gamma -if(z) \left(\varphi_z(z) f(z) + \psi_z(z) \frac{i}{f(z)} \right) dz \\ =& \text{Re} \oint_\Gamma \left(\varphi_z(z) + \psi_z(z) \frac{i}{f^2(z)} \right) dz + \text{Re} \oint_\Gamma \left(-i\varphi_z(z) f^2(z) + \psi_z(z) \right) dz \end{aligned}$$

From (3.13) we have that $\varphi_z(z) f(z) = \psi_z(z) i/f(z)$. Then

$$\begin{aligned} * \oint_\Gamma \dot{w}(z) d_{(f,i/f)} z =& \text{Re} \oint_\Gamma 2\varphi_z(z) dz + \text{Re} \oint_\Gamma 2\psi_z(z) dz \\ =& \text{Re} \oint_\Gamma (\varphi_x(z) - i\varphi_y(z)) (dx + idy) + \text{Re} \oint_\Gamma (\psi_x(z) - i\psi_y(z)) (dx + idy) \\ =& \oint_\Gamma (\varphi_x(z) dx + \varphi_y(z) dy) + \oint_\Gamma (\psi_x(z) dx + \psi_y(z) dy) \end{aligned} \tag{3.24}$$

By Theorem 1 we have that integrals in (3.24) are path-independent. Consequently, we get

$$\oint_\Gamma \dot{w}(z) d_{(f,i/f)} z = 0.$$

This completes the proof. □

Corollary 6. *Under the hypothesis that Proposition 10, we have that for $\Gamma \subseteq \Omega$ a rectifiable* *curve from z_0 to z in Ω, the following equality is valid*

$$\int_\Gamma \dot{w}(z) d_{(f,i/f)} z = w(z) - \varphi(z_0) f(z) - \psi(z_0) \frac{i}{f(z)}. \tag{3.25}$$

Formula (3.25) is known as the $\left(f, \frac{i}{f}\right)$-antiderivative of $\dot{w}$.

Proposition 11. *Let $v : \Omega \to \mathbb{C}$ be a continuous function where Ω is a simply connected domain. If v is $(f, i/f)$-integrable in Ω, then there exists a solution w of the main Vekua equation in Ω such that*

$$v = \frac{d_{(f,i/f)} w}{dz}, \quad in\ \Omega.$$

Proof. Let $z_0 = x_0 + iy_0$ and $z = x + iy$ both in Ω. Assume that v is $\left(f, \frac{i}{f}\right)$-integrable. From Definition 12 item (a) we have that

$$* \int_\Gamma v(z) d_{(f,i/f)} z = \operatorname{Re} \int_\Gamma \frac{1}{f(z)} v(z) dz - i \operatorname{Re} \int_\Gamma i f(z) v(z) dz,$$

where Γ a rectifiable curve leading from z_0 to z. Let us denote

$$\varphi(z) = \operatorname{Re} \int_\Gamma \frac{1}{f(z)} v(z) dz, \tag{3.26}$$

$$\psi(z) = \operatorname{Re} \int_\Gamma -i f(z) v(z) dz. \tag{3.27}$$

Note that for (3.26) we get

$$\varphi(z) = \operatorname{Re} \int_\Gamma \frac{1}{f(z)} v(z) dz \tag{3.28}$$

$$= \frac{2}{2} \operatorname{Re} \int_\Gamma \frac{1}{f(z)} (v_1(z) + iv_2(z))(dx + idy)$$

$$= \int_\Gamma \frac{1}{2f(z)} (2v_1(z)dx - 2v_2(z)dy) \tag{3.29}$$

$$= \int_\Gamma \frac{1}{2f(z)} (2v_1(z)dx + iv_1(z)dy - iv_1(z)dy)$$

$$+ \int_\Gamma \frac{1}{2f(z)} (iv_2(z)dx - iv_2(z)dx - 2v_2(z)dy)$$

$$= \int_\Gamma \frac{1}{2f(z)} (v_1(z)dx + iv_1(z)dy + iv_2(z)dx - v_2(z)dy)$$

$$+ \int_\Gamma \frac{1}{2f(z)} (v_1(z)dx - iv_1(z)dy - iv_2(z)dx - v_2(z)dy)$$

$$= \int_\Gamma \frac{1}{2f(z)} (v(z)dz + \overline{v}(z)d\overline{z}) \tag{3.30}$$

Analogously, we have that (3.27) is equal to

$$\psi(z) = \int_{\Gamma} \frac{f(z)}{2i}(v(z)dz - \overline{v}(z)d\overline{z}) \tag{3.31}$$

Taking ∂_z to (3.30) and (3.31) we get

$$\varphi_z(z) = \frac{v(z)}{2f(z)}, \tag{3.32}$$

$$\psi_z(z) = \frac{f(z)v(z)}{2i}. \tag{3.33}$$

Hence adding (3.32) and (3.33)

$$\varphi_z(z)f(z) + \psi_z \frac{i}{f(z)} = \frac{v(z)}{2f(z)}f(z) + \frac{f(z)v(z)}{2i}\frac{i}{f(z)} = v(z). \tag{3.34}$$

Then taking the conjugate to (3.32) and (3.33) and adding them we get

$$\varphi_{\overline{z}}(z)f(z) + \psi_{\overline{z}}(z)\frac{i}{f(z)} = \frac{\overline{v}(z)}{2f(z)}f(z) + \frac{if(z)\overline{v}(z)}{2}\frac{i}{f(z)} = 0.$$

From Theorem 7 and (3.34) we have that $\dot{w} = v$. While, by Proposition 8 we have that $\dot{w}$ is solution of the main Vekua equation. Therefore, by arbitrariness of z_0 and z the proof is done. □

Corollary 7. *Let a complex-valued function v a solution of the successor Vekua equation defined in a simply connected domain Ω. Then for $z_0 = x_0 + iy_0$ and $z = x + iy$ in Ω we have that*

$$w(z) = \int_{z_0}^{z} v(\zeta)d_{(f,i/f)}\zeta,$$

is a solution of the main Vekua equation in Ω.

Proposition 12. *Let $v : \Omega \to \mathbb{C}$ where Ω is a simply connected domain. If v is a solution of the successor Vekua equation in Ω, then v is $(f, i/f)$-integrable in Ω.*

Proof. Let us take $\Omega_1 \subseteq \Omega$. From Proposition 11 note that is enough to prove that if Ω_1 is a regular domain and $\overline{\Omega}_1 \subseteq \Omega$, then

$$* \int_{\partial\Omega_1} v(z)_{d(f,i/f)}z = 0. \tag{3.35}$$

By Definition 12 item *(a)*, we have that

$$\operatorname{Re}\left(*\int_{\partial\Omega_1} v(z)d_{(f,i/f)}z\right) = \operatorname{Re}*\int_{\partial\Omega_1}\frac{1}{f(z)}v(z)dz,$$

and

$$\operatorname{Im}\left(*\int_{\partial\Omega_1} v(z)d_{(f,i/f)}z\right) = \operatorname{Re}*\int_{\partial\Omega_1} -if(z)v(z)dz.$$

From Theorem 4 we get

$$\begin{aligned}\int_{\partial\Omega_1} -if(z)v(z)dz =& -2i\int_{\Omega_1}(if(z)v(z))_{\overline{z}}dxy\\ =&2i\int_{\Omega_1}(if_{\overline{z}}(z)v(z)+if(z)v_{\overline{z}}(z))dxdy\end{aligned}$$

Since v is solution of the successor Vekua equation, we get

$$\begin{aligned}\int_{\partial\Omega_1} -if(z)v(z)dz =& -2i\int_{\Omega_1}(if_{\overline{z}}(z)v(z)-if_z(z)\overline{v}(z))dxdy\\ =& -2i\int_{\Omega_1}\left(if_{\overline{z}}(z)v(z)+\overline{if_{\overline{z}}(z)v(z)}\right)dxdy\\ =& -2i\int_{\Omega_1}2\operatorname{Re}(if_{\overline{z}}(z)\overline{v}(z))dxy\\ =& -4i\int_{\Omega_1}\operatorname{Re}(if_{\overline{z}}(z)\overline{v}(z))dxy.\end{aligned} \tag{3.36}$$

Note that (3.36) is pure imaginary hence

$$\operatorname{Re}\int_{\partial\Omega_1} -if(z)v(z) = 0. \tag{3.37}$$

Analogously, we obtain that

$$\operatorname{Re}\int_{\partial\Omega_1}\frac{1}{f(z)}v(z) = 0. \tag{3.38}$$

From (3.37) and (3.38), we have proved (3.35). Therefore, v is $(f, i/f)$-integrable. □

Corollary 7 established a relation about how to pass from a solution of the successor Vekua equation to one of the main Vekua equation through an integration process. Moreover, by Propositions 12, 11,and 10 we obtain the following result

Corollary 8. *Let $v:\Omega\to\mathbb{C}$ a solution of the successor Vekua equation in Ω where Ω is a* *simply connected domain. Then every closed curve* Γ *situated in a subdomain of* Ω *satisfies*

$$\operatorname{Re}\oint_{\Gamma}\frac{v(z)}{f(z)}dz + i\operatorname{Im}\oint_{\Gamma}f(z)v(z)dz = 0. \tag{3.39}$$

Note that, Corollary 8 is a generalization of the well-known Cauchy Integral Theorem.

3.5 Formal Powers for the main Vekua equation

In complex analysis the nonnegative usual powers

$$\alpha(z-z_0)^n, \quad n = 0,1,2\ldots$$

are a system of analytic functions of great importance. For example, they allow us to expand an analytic function as a Taylor series. In this section we deal with the construction of a system of solutions of the main Vekua equation

$$\partial_{\bar{z}}w = \frac{f_{\bar{z}}}{f}\overline{w}, \quad \text{in } \Omega, \tag{3.40}$$

that generalize in some sense the usual powers. We call it formal powers and denote it by

$$Z^{(n)}(a,z_0;z), \quad n = 0,1,2\ldots$$

Here $a \in \mathbb{C}$, $z_0 \in \Omega$, and $n \in \mathbb{N}_0$ are parameters called the coefficient, center and grade of the formal power, respectively. The function $Z^{(n)}(a,z_0;z)$ should be a solution of the main Vekua equation in the variable z and near the center it behaves like the usual powers, that is,

$$Z^{(n)}(a,z_0;z) \sim a(z-z_0)^n \qquad \text{as} \quad z \to z_0.$$

In this section we present an elegant method to construct explicitly the system of formal powers for (3.40) when f has a separable form, that is, when $f(x,y) = \sigma(x)\tau(y)$. This method was proposed by L. Bers (see [2], [3], [7], and [5]) and it can be applied when we have at our disposal a special sequence of generating pairs called generating sequence.

3.5.1 Generating sequence. The special case $f(x,y) = \sigma(x)\tau(y)$

Recall that by Proposition 9 we have that the $(f, i/f)$-derivative of a solution of the main Vekua equation is a solution of the successor Vekua equation. Motivated by this, we arrive at the following definition.

Definition 14. *Let (F, G) and (F_1, G_1) be two generating pairs defined in Ω. We say that (F_1, G_1) is the successor of (F, G) if given a solution of the Vekua equation corresponding to (F, G) its $(F, G)-$derivative is a solution of the Vekua equation corresponding to (F_1, G_1).*

This process of constructing new generating pairs associated with the previous one via Vekua equations can be continued and we arrive naturally at the next definition.

Definition 15. *A sequence of generating pairs $\{(F_m, G_m)\}$, $m \in \mathbb{Z}$, is called a generating sequence if (F_{m+1}, G_{m+1}) is a successor of (F_m, G_m). Moreover, if $(F_0, G_0) = (F, G)$, we say that (F, G) is embedded in $\{(F_m, G_m)\}$.*

Definition 16. *A generating sequence $\{(F_m, G_m)\}$ has period $\mu > 0$ if $(F_{m+\mu}, G_{m+\mu}) = (F_m, G_m)$.*

Let us consider the main Vekua equation (3.40) with $f(x,y) = \sigma(x)\tau(y)$ and the corresponding generating pair

$$\left(f, \frac{i}{f}\right) = \left(\sigma\tau, \frac{i}{\sigma\tau}\right). \tag{3.41}$$

Next we are going to construct a generating sequence embedding (3.41).

Proposition 13. *Let $(F, G) = (\sigma\tau, i/\sigma\tau)$. Then $(F_1, G_1) = (\tau/\sigma, i\sigma/\tau)$ is the successor of (F, G).*

Proof. Let us prove that (F_1, G_1) is a generating pair corresponding to the successor Vekua equation (3.18). Then

Step 1. Let us replace F_1 in (3.18). Then

(i) $\partial_{\bar{z}}\left(\frac{\tau}{\sigma}\right) = \frac{1}{2}\left(\frac{i\partial_y\tau\sigma - \partial_x\sigma\tau}{\sigma^2}\right)$

(ii) $-\frac{\partial_z f}{f}\left(\overline{\frac{\tau}{\sigma}}\right) = -\frac{1}{2}\left(\frac{\partial_x\sigma\tau - i\sigma\partial_y\tau}{\sigma\tau}\right)\left(\frac{\tau}{\sigma}\right) = \frac{1}{2}\left(\frac{i\partial_y\tau\sigma - \partial_x\sigma\tau}{\sigma^2}\right)$

From (i) and (ii) F_1 satisfies (3.18).

Step 2. Similar to the previous item we replace G_1 in (3.18). Then

(i) $\partial_{\bar{z}}\left(i\frac{\sigma}{\tau}\right) = \frac{1}{2}\left(\frac{\sigma\partial_y\tau + i\partial_x\sigma\tau}{\tau^2}\right)$

(ii) $-\frac{\partial_z f}{f}\left(\overline{i\frac{\sigma}{\tau}}\right) = \frac{1}{2}\left(\frac{\partial_x\sigma\tau - i\sigma\partial_y\tau}{\sigma\tau}\right)\left(i\frac{\sigma}{\tau}\right) = \frac{1}{2}\left(\frac{\sigma\partial_y\tau + i\partial_x\sigma\tau}{\tau^2}\right)$

From (i) and (ii) G_1 satisfies (3.18).

Step 3. Fix $z_0 = x_0 + iy_0 \in \Omega$. Let $m = m_1 + im_2 \in \mathbb{C}$ we take $\lambda = m_1\sigma(x_0)\tau(y_0)$ and $\beta = m2/\sigma(x_0)\tau(y_0)$. Then

$$m = \lambda\frac{1}{\sigma(x_0)\tau(y_0)} + i\beta\sigma(x_0)\tau(y_0).$$

By Proposition 9 we have that (F_1, G_1) is the successor of (F, G). This completes the proof. □

Next we are going to construct a successor for $(F_1, G_1) = (\tau/\sigma, i\sigma/\tau)$.

Proposition 14. *The generating pair* $(F_2, G_2) = (\sigma\tau, i/\sigma\tau)$ *is the successor of* $(F_1, G_1) = (\tau/\sigma, i\sigma/\tau)$.

Proof. To prove this is enough show that

$$\frac{\partial_{\bar{z}} f}{f} = -\frac{\partial_z \widetilde{f}}{\widetilde{f}},$$

where $\widetilde{f} = \tau/\sigma$.Then

$$\frac{\partial_{\bar{z}} f}{f} = \frac{1}{2}\left(\frac{\partial_x\sigma\tau + i\sigma\partial_y\tau}{\sigma\tau}\right)$$

and

$$\frac{\partial_z \widetilde{f}}{\widetilde{f}} = -\frac{1}{2}\left(\frac{-\partial_x\sigma\tau - i\sigma\partial_y\tau}{\sigma^2}\right)\frac{\sigma}{\tau} = \frac{1}{2}\left(\frac{\partial_x\sigma\tau + i\sigma\partial_y\tau}{\sigma\tau}\right)$$

Therefore, (F_2, G_2) is the successor of (F_1, G_1). □

Note that by Proposition 13 we have that $(F_3, G_3) = (\tau/\sigma, i\sigma/\tau)$ is the successor of $(F_2, G_2) = (\sigma\tau, i/\sigma\tau)$. Then, by construction we have proved the following proposition.

Proposition 15. *The generating pair (3.41) is embedded in the generating sequence* $\{(F_m, G_m)\}$ *of period two given by*

$$(F, G) = \left(\sigma\tau, \frac{i}{\sigma\tau}\right),\ (F_1, G_1) = \left(\frac{\tau}{\sigma}, i\frac{\sigma}{\tau}\right),\ (F_2, G_2) = (F, G),\ (F_3, G_3) = (F_1, G_1)\ \ldots$$

3.5.2 Definition and construction of formal powers

Let us consider the main Vekua equation (3.40) with $f(x,y) = \sigma(x)\tau(y)$ and let $\{(F_m, G_m)\}$ be the corresponding generating sequence given in Proposition 15. Let us define:

Definition 17. *The formal power with center at $z_0 \in \Omega$, coefficient $a \in \mathbb{C}$, and grade $n \in \mathbb{N}_0$ is given by*

$$Z_m^{(0)}(a, z_0; z) = \lambda F_m(z) + \mu G_m(z) \quad \text{such that} \quad \lambda F_m(z_0) + \mu G_m(z_0) = a \tag{3.42}$$

and

$$Z_m^{(n)}(a, z_0; z) = n \int_{z_0}^{z} Z_{m+1}^{(n-1)}(a, z_0; \zeta)\, d_{(F_m, G_m)}\zeta, \quad \forall n \in \mathbb{N}. \tag{3.43}$$

This definition implies the following properties:

(a) $Z_m^{(n)}(a, z_0; z)$ is a solution of the Vekua equation corresponding to the generating pair (F_m, G_m) as function of z.

(b) If a' and a'' are real constant, then

$$Z_m^{(n)}(a' + ia'', z_0; z) = a' Z_m^{(n)}(1, z_0; z) + a'' Z_m^{(n)}(i, z_0; z)$$

(c) The formal powers satisfy the following differential relation

$$\frac{d_{(F_m, G_m)} Z_m^{(n)}(a, z_0; z)}{dz} = n Z_{m+1}^{(n-1)}(a, z_0; z).$$

Remark 5. *Note that from property (b) the formal power $Z^{(n)}(a, z_0; z)$ can be written through $Z^{(n)}(1, z_0; z)$ and $Z^{(n)}(i, z_0; z)$. Therefore, it is enough calculate only these two formal powers for any grade n and center z_0.*

The case $f(x,y) = \tau(y)$

Consider the main Vekua equation (3.40) with $f(x,y) = \tau(y)$ and let $\{(F_m, G_m)\}$ be the corresponding generating sequence given in Proposition 15. For this case is easy to see that the generating sequence has period one, that is,

$$(F, G) = \left(\tau, \frac{i}{\tau}\right), \quad (F_1, G_1) = (F, G), \quad (F_2, G_2) = (F, G), \quad \ldots$$

and the main Vekua equation take the following form

$$\partial_{\bar{z}} w = \frac{i\tau_y}{2\tau}\overline{w}. \tag{3.44}$$

In this subsection we consider the formal powers with the center at the point $z_0 = x_0 + iy_0 \in \Omega$. We assume that $\tau(y_0) = 1$ and define the system of functions $\widetilde{Y}^{(n)}$ and $Y^{(n)}$ constructed in terms of τ through the following recursive relations

$$Y^{(0)} \equiv 1, \quad \widetilde{Y}^{(0)} \equiv 1, \tag{3.45}$$

$$Y^{(n)}(y) = \begin{cases} n\displaystyle\int_{y_0}^{y} Y^{(n-1)}(\eta)\tau^2(\eta)d\eta, & n \text{ odd}, \\ n\displaystyle\int_{y_0}^{y} Y^{(n-1)}(\eta)\frac{d\eta}{\tau^2(\eta)}, & n \text{ even}. \end{cases} \tag{3.46}$$

$$\widetilde{Y}^{(n)}(y) = \begin{cases} n\displaystyle\int_{y_0}^{y} \widetilde{Y}^{(n-1)}(\eta)\frac{d\eta}{\tau^2(\eta)}, & n \text{ odd}, \\ n\displaystyle\int_{y_0}^{y} \widetilde{Y}^{(n-1)}(\eta)\tau^2(\eta)d\eta, & n \text{ even}. \end{cases} \tag{3.47}$$

Remark 6. *If τ is chosen as $\tau \equiv 1$, then $Y^{(n)}(y) = \widetilde{Y}^{(n)}(y) = y^n$ are the usual powers.*

Let us calculate $Z^{(n)}(a, z_0; z)$ of[2] grades $n = 0, 1, 2$ for the Vekua equation (3.44) where $a = a_1 + ia_2 \in \mathbb{C}$. From Definition 17 property (b) we need calculate $Z^{(n)}(1, z_0; z)$ and $Z^{(n)}(i, z_0; z)$. By formula (3.42) and since $\tau(y_0) = 1$ it is easy to see that

$$Z^{(0)}(1, z_0; z) = \tau(y) \quad \text{and} \quad Z^{(0)}(i, z_0; z) = \frac{i}{\tau(y)}.$$

In order to construct $Z^{(1)}(\alpha, z_0; z)$ for $\alpha = 1, i$ recall that $(F_m, G_m) = (\tau, i/\tau)$ for any m in $\mathbb{Z}$. Then formula (3.43) give us

$$Z^{(1)}(\alpha, z_0; z) = \int_{z_0}^{z} Z^{(0)}(\alpha, z_0; \zeta) d_{(f, i/f)}\zeta, \quad \alpha = 1, i.$$

We calculate these two integrals using (3.21). Denoting $\zeta := \kappa + i\xi$ we get

$$Z^{(1)}(1, z_0; z) = \tau(y)\,\mathrm{Re}\int_{z_0}^{z} d\zeta - \frac{i}{\tau(y)}\,\mathrm{Re}\int_{z_0}^{z} i\tau^2(\kappa)d\zeta$$

[2] The absence of the subindice m means that all the formal powers correspond to the same generator pair.

$$=\tau(y)\int_{z_0}^{z} d\kappa + \frac{i}{\tau(y)}\int_{z_0}^{z}\tau^2(\xi)d\xi$$

We use the curve Γ from $z_0 = x_0 + iy_0$ to $z = x + iy$ which consists of $\Gamma_1 \cup \Gamma_2$ parameterized by

$$\gamma_1(t) = (t, y_0) \quad \text{for} \;\; x_0 \le t \le x \quad \text{and} \quad \gamma_2(t) = (x, t) \quad \text{for} \;\; y_0 \le t \le y, \tag{3.48}$$

respectively. Then, we get

$$\begin{aligned} Z^{(1)}(1, z_0; z) =& \tau(y)\int_{x_0}^{x} d\kappa + \frac{i}{\tau(y)}\int_{y_0}^{y}\tau^2(\xi)d\xi \\ =& \tau(y)(x - x_0) + \frac{i}{\tau(y)} Y^{(1)}(y) \\ =& \tau(y)\begin{pmatrix}1\\0\end{pmatrix}(x - x_0) + \frac{i}{\tau(y)}\begin{pmatrix}1\\1\end{pmatrix} Y^{(1)}(y) \end{aligned}$$

Similarly,

$$\begin{aligned} Z^{(1)}(i, z_0; z) =& \tau(y)\operatorname{Re}\int_{z_0}^{z}\frac{i}{\tau^2(\xi)}d\zeta + \frac{i}{\tau(y)}\operatorname{Re}\int_{z_0}^{z} d\zeta \\ =& -\tau(y)\int_{z_0}^{z}\frac{1}{\tau^2(\xi)}d\xi + \frac{i}{\tau(y)}\int_{z_0}^{z} d\kappa \end{aligned}$$

using (3.48) we obtain

$$\begin{aligned} Z^{(1)}(i, z_0; z) =& -\tau(y)\int_{y_0}^{y}\frac{1}{\tau^2(\xi)}d\xi + \frac{i}{\tau(y)}\int_{x_0}^{x} d\kappa \\ =& \tau(y)\widetilde{Y}^{(1)}(y) + \frac{i}{\tau(y)}(x - x_0) \end{aligned}$$

Analogously we construct $Z^{(2)}(\alpha, z_0; z)$ for $\alpha = 1, i$. By formula (3.43) we get

$$Z^{(2)}(\alpha, z_0; z) = 2\int_{z_0}^{z} Z^{(1)}(\alpha, z_0; \zeta) d_{(f, i/f)}\zeta, \quad \alpha = 1, i.$$

We calculate these two integrals using *(b)*. Then

$$\begin{aligned} Z^{(2)}(1, z_0; z) =& 2\tau(y)\operatorname{Re}\int_{z_0}^{z}\frac{1}{\tau(\xi)}\left[\tau(\xi)(\kappa - x_0) + \frac{i}{\tau(\xi)} Y^{(1)}(\xi)\right] d\zeta \\ &- \frac{2i}{\tau(y)}\operatorname{Re}\int_{z_0}^{z} i\tau(\xi)\left[\tau(\xi)(\kappa - x_0) + \frac{i}{\tau(\xi)} Y^{(1)}(\xi)\right] d\zeta \end{aligned}$$

$$
\begin{aligned}
&=2\tau(y)\operatorname{Re}\int_{z_0}^{z}(\kappa-x_0)d\zeta+2\tau(y)\operatorname{Re}\int_{z_0}^{z}\frac{i}{\tau^2(\xi)}Y^{(1)}(\xi)d\zeta\\
&\quad-\frac{2i}{\tau(y)}\operatorname{Re}\int_{z_0}^{z}i\tau^2(\xi)(\kappa-x_0)d\zeta+\frac{2i}{\tau(y)}\operatorname{Re}\int_{z_0}^{z}Y^{(1)}(\xi)d\zeta\\
&=2\tau(y)\int_{z_0}^{z}(\kappa-x_0)d\kappa-2\tau(y)\int_{z_0}^{z}\frac{1}{\tau^2(\xi)}Y^{(1)}(\xi)d\xi\\
&\quad+\frac{2i}{\tau(y)}\int_{z_0}^{z}\tau^2(\xi)(\kappa-x_0)d\xi+\frac{2i}{\tau(y)}\int_{z_0}^{z}Y^{(1)}(\xi)d\kappa
\end{aligned}
$$

Using (3.48) we obtain

$$
\begin{aligned}
Z^{(2)}(1,z_0;z)=&2\tau(y)\int_{x_0}^{x}(t-x_0)dt-2\tau(y)\int_{y_0}^{y}\frac{1}{\tau^2(t)}Y^{(1)}(t)dt+\frac{2i}{\tau(y)}(x-x_0)\int_{y_0}^{y}\tau^2(t)dt\\
&+\frac{2i}{\tau(y)}\int_{y_0}^{y}Y^{(1)}(y_0)dt\\
=&\tau(y)(x-x_0)^2-\tau(y)Y^{(2)}(y)+\frac{2i}{\tau(y)}(x-x_0)Y^{(1)}(y)\\
=&\tau(y)(x-x_0)^2+\frac{2i}{\tau(y)}(x-x_0)Y^{(1)}(y)-\tau(y)Y^{(2)}(y)\\
=&\tau(y)\binom{2}{0}(x-x_0)^2+\frac{i}{\tau(y)}\binom{2}{1}(x-x_0)Y^{(1)}(y)-\tau(y)\binom{2}{2}Y^{(2)}(y)
\end{aligned}
$$

Similarly, we calculate

$$
\begin{aligned}
Z^{(2)}(i,z_0;z)=&2\tau(y)\operatorname{Re}\int_{z_0}^{z}\frac{1}{\tau(\xi)}\left[-\tau(\xi)\widetilde{Y}^{(1)}(\xi)+\frac{i}{\tau(\xi)}(\kappa-x_0)\right]d\xi\\
&-\frac{2i}{\tau(y)}\operatorname{Re}\int_{z_0}^{z}i\tau(\xi)\left[-\tau_0\tau(\xi)\widetilde{Y}^{(1)}(\xi)+\frac{i\tau_0}{\tau(\xi)}(\kappa-x_0)\right]d\zeta\\
=&2\tau(y)\operatorname{Re}\int_{z_0}^{z}-\widetilde{Y}^{(1)}(\xi)d\zeta+2\tau(y)\operatorname{Re}\int_{z_0}^{z}\frac{i}{\tau^2(\xi)}(\kappa-x_0)d\zeta\\
&+\frac{2i}{\tau(y)}\operatorname{Re}\int_{z_0}^{z}i\tau^2(\xi)\widetilde{Y}^{(1)}(\xi)d\zeta+\frac{2i}{\tau(y)}\operatorname{Re}\int_{z_0}^{z}(\kappa-x_0)d\zeta\\
=&-2\tau(y)\int_{z_0}^{z}\widetilde{Y}^{(1)}(\xi)d\kappa-2\tau(y)\int_{z_0}^{z}\frac{1}{\tau^2(\xi)}(\kappa-x_0)d\xi\\
&-\frac{2i}{\tau(y)}\int_{z_0}^{z}\tau^2(\xi)\widetilde{Y}^{(1)}(\xi)d\xi+\frac{2i}{\tau(y)}\int_{z_0}^{z}(\kappa-x_0)d\kappa
\end{aligned}
$$

Using (3.48) we obtain

$$
\begin{aligned}
Z^{(2)}(i,z_0;z)=&-2\tau(y)\int_{y_0}^{y}Y^{(1)}(y_0)dt-2\tau(y)(x-x_0)\int_{y_0}^{y}\frac{1}{\tau^2(t)}dt\\
&-\frac{2i\tau_0}{\tau(y)}\int_{y_0}^{y}\tau^2(t)\widetilde{Y}^{(1)}(t)dt+\frac{2i\tau_0}{\tau(y)}\int_{x_0}^{x}(t-x_0)dt
\end{aligned}
$$

$$
\begin{aligned}
&= -2\tau(y)(x-x_0)\widetilde{Y}^{(1)}(y) - \frac{i}{\tau(y)}\widetilde{Y}^{(2)}(y) + \frac{i}{\tau(y)}(x-x_0)^2\\
&= \frac{i}{\tau(y)}(x-x_0)^2 - 2\tau(y)(x-x_0)\widetilde{Y}^{(1)}(y) - \frac{i}{\tau(y)}\widetilde{Y}^{(2)}(y)\\
&= \frac{i}{\tau(y)}\binom{2}{0}(x-x_0)^2 - \tau(y)\binom{2}{1}(x-x_0)\widetilde{Y}^{(1)}(y) - \frac{i}{\tau(y)}\binom{2}{2}\widetilde{Y}^{(2)}(y)
\end{aligned}
$$

By construction, the formal powers $Z^{(n)}$ computed previously are indeed solutions of the Vekua equation (3.44). Following the previous reasoning, to calculate formal powers, we arrive to the next proposition.

Proposition 16. *For $a = a' + ia'' \in \mathbb{C}$ and $z_0 = x_0 + iy_0 \in \Omega$ we have that $Z^{(n)}(a, z_0; z)$ is given by*

$$Z^{(n)}(a, z_0, z) = \tau(y)\,\mathrm{Re}\,{}_*Z^{(n)}(a, z_0, z) + \frac{i}{\tau(y)}\,\mathrm{Im}\,{}_*Z^{(n)}(a, z_0, z) \tag{3.49}$$

where

$${}_*Z^{(n)}(a, z_0; z) = a' \sum_{j=0}^{n} \binom{n}{j} (x-x_0)^{(n-j)} i^j Y^{(j)} + ia'' \sum_{j=0}^{n} \binom{n}{j} (x-x_0)^{(n-j)} i^j \widetilde{Y}^{(j)}. \tag{3.50}$$

The proof of this proposition is achieved by induction on n. Now, we give an example of constructed formal powers.

Example 2. *Consider the Yukawa equation*

$$\left(-\Delta + c^2\right) u = 0, \tag{3.51}$$

where is c a real constant. For (3.51) let us take a particular solution $f = e^{cy}$. The corresponding Vekua equation has the form

$$\partial_{\bar{z}} w = \frac{ic}{2}\overline{w}. \tag{3.52}$$

For this case the generating pair $(F, G) = (e^{cy}, ie^{-cy})$ is embedded into a sequence of period one, i.e., $(F_m, G_m) = (F, G)$ for any $m \in \mathbb{Z}$. Let us construct the first two formal powers with center at the origin. Then

- $Z^{(0)}(1, 0; z) = e^{cy}$
- $Z^{(0)}(i, 0; z) = ie^{-cy}$

- $$\begin{aligned} Z^{(1)}(1,0;z) =& e^{cy}\operatorname{Re}\left(x+iY^{(1)}(y)\right)+ie^{-cy}\operatorname{Im}\left(x+iY^{(1)}(y)\right)\\ =& xe^{cy}+ie^{-cy}Y^{(1)}(y)\\ =& xe^{cy}+ie^{-cy}\int_0^y e^{2c\eta}d\eta\\ =& xe^{cy}+ie^{-cy}\left(\frac{e^{2cy}-1}{2c}\right)\\ =& xe^{cy}+\frac{i\sinh(cy)}{c} \end{aligned}$$
- $$\begin{aligned} Z^{(1)}(i,0;z) =& e^{cy}\operatorname{Re}\left(ix-\widetilde{Y}^{(1)}(y)\right)+ie^{-cy}\operatorname{Im}\left(ix-\widetilde{Y}^{(1)}(y)\right)\\ =& ixe^{-cy}-e^{cy}\widetilde{Y}^{(1)}(y)\\ =& ixe^{-cy}-e^{cy}\int_0^y e^{-2c\eta}d\eta\\ =& ixe^{-cy}-e^{cy}\left(\frac{1-e^{-2cy}}{2c}\right)\\ =& ixe^{-cy}-\frac{\sinh(cy)}{c} \end{aligned}$$
- $$\begin{aligned} Z^{(2)}(1,0;z) =& e^{cy}\operatorname{Re}\left(x^2+2xiY^{(1)}(y)-Y^{(2)}(y)\right)\\ &+ie^{-cy}\operatorname{Im}\left(x^2+2xiY^{(1)}(y)-Y^{(2)}(y)\right)\\ =& x^2e^{cy}-e^{cy}Y^{(2)}(y)+2xie^{cy}Y^{(1)}(y)\\ =& x^2e^{cy}-e^{cy}\left(\left(\frac{e^{-2cy}-1}{2c^2}\right)+\frac{y}{c}\right)+2xie^{-cy}\left(\frac{e^{2cy}-1}{2c}\right)\\ =& \left(x^2-\frac{y}{c}\right)e^{cy}+\frac{2xi\sinh(cy)}{c}+\frac{\sinh(cy)}{c^2} \end{aligned}$$
- $$\begin{aligned} Z^{(2)}(i,0;z) =& e^{cy}\operatorname{Re}\left(ix-2x\widetilde{Y}^{(1)}(y)-i\widetilde{Y}^{(2)}(y)\right)\\ &+ie^{-cy}\operatorname{Im}\left(ix-2x\widetilde{Y}^{(1)}(y)-i\widetilde{Y}^{(2)}(y)\right)\\ =& 2x^2e^{cy}\widetilde{Y}^{(1)}(y)+ixe^{-cy}-ie^{-cy}\widetilde{Y}^{(2)}(y)\\ =& ix^2e^{-cy}-2xe^{cy}\left(\frac{1-e^{-2cy}}{2c}\right)-ie^{-cy}\left(\frac{e^{2cy}-1}{2c^2}-\frac{y}{c}\right)\\ =& i\left(\left(x^2+\frac{y}{c}\right)e^{-cy}-\frac{\sinh(cy)}{c^2}\right)-\frac{2x\sinh(cy)}{c} \end{aligned}$$

Note each of these function is a solution of (3.52). Taking real parts of formal powers we obtain a infinite system of solutions of the Yukawa equation

$$u_0(x,y)=e^{cy},\quad u_1(x,y)=xe^{cy},\quad u_2(x,y)=-\frac{\sinh(cy)}{c},$$

$$u_3(x,y) = \left(x^2 - \frac{y}{c}\right) e^{cy} + \frac{\sinh(cy)}{c^2}, \quad u_4(x,y) = -2x\frac{\sinh(cy)}{c} \dots$$

The case $f(x,y) = \sigma(x)\tau(y)$

Consider the main Vekua equation (3.40) with $f(x,y) = \sigma(x)\tau(y)$ and let $\{(F_m, G_m)\}$ be the corresponding generating sequence given in Proposition 15. For this case we know that the generating sequence has period two and the main Vekua equation take the following form

$$\partial_{\bar{z}} w = \left(\frac{\sigma_x \tau + i\sigma\tau_y}{2\sigma\tau}\right)\overline{w} \tag{3.53}$$

In this subsection we consider the formal powers with the centre at the point $z_0 = x_0 + iy_0 \in \Omega$. We assume that $\sigma(x_0) = \tau(y_0) = 1$ and define the system of functions $X^{(n)}$ and $\widetilde{X}^{(n)}$ constructed in terms of σ through the following recursive relations

$$X^{(0)} \equiv 1, \quad \widetilde{X}^{(0)} \equiv 1, \tag{3.54}$$

$$X^{(n)}(x) = \begin{cases} n\displaystyle\int_{x_0}^{x} X^{(n-1)}(\eta)\frac{d\eta}{\sigma^2(\eta)}, & n \text{ odd}, \\ n\displaystyle\int_{x_0}^{x} X^{(n-1)}(\eta)\sigma^2(\eta)d\eta, & n \text{ even}. \end{cases} \tag{3.55}$$

$$\widetilde{X}^{(n)}(x) = \begin{cases} n\displaystyle\int_{x_0}^{x} \widetilde{X}^{(n-1)}(\eta)\sigma^2(\eta)d\eta, & n \text{ odd}, \\ n\displaystyle\int_{y_0}^{y} \widetilde{X}^{(n-1)}(\eta)\frac{d\eta}{\sigma^2(\eta)}, & n \text{ even}. \end{cases} \tag{3.56}$$

Remark 7. *If σ is chosen as $\sigma \equiv 1$, then $X^{(n)}(x) = \widetilde{X}^{(n)}(x) = x^n$ are the usual powers.*

Now, we generalize Proposition 16 when $f(x,y) = \sigma(x)\tau(y)$.

Proposition 17. *For $a = a' + ia'' \in \mathbb{C}$ and $z_0 = x_0 + iy_0 \in \Omega$ we have that $Z^{(n)}(a, z_0; z)$ is given by*

$$Z^{(n)}(a,z_0,z) = \sigma(x)\tau(y)\,\mathrm{Re}\,{}_*Z^{(n)}(a,z_0,z) + \frac{i}{\sigma(x)\tau(y)}\,\mathrm{Im}\,{}_*Z^{(n)}(a,z_0,z) \tag{3.57}$$

where

$${}_*Z^{(n)}(a,z_0,z) = a'\sum_{j=0}^{n}\binom{n}{j}X^{(n-j)}i^jY^{(j)} + ia''\sum_{j=0}^{n}\binom{n}{j}\widetilde{X}^{(n-j)}i^j\widetilde{Y}^{(j)}, \quad n \text{ odd}$$

and

$$_*Z^{(n)}(a, z_0, z) = a' \sum_{j=0}^{n} \binom{n}{j} \widetilde{X}^{(n-j)} i^j Y^{(j)} + ia'' \sum_{j=0}^{n} \binom{n}{j} X^{(n-j)} i^j \widetilde{Y}^{(j)}, \quad n \text{ even}.$$

The proof of this proposition can be achieved by induction on n.

Remark 8. *If we choose $\sigma \equiv 1$ and $\tau \equiv 1$ formula (3.57) generalized the binomial representation of the analytic powers, i.e., $Z^{(n)}(a, z_0, z) = a(z - z_0)^n$.*

3.6 Connection between the main Vekua equation and the two-dimensional stationary Schödinger equation

Let us consider the stationary Schödinger equation

$$(-\Delta + \nu)\, u = 0, \quad \text{in } \Omega, \tag{3.58}$$

where $\Delta = \partial_x^2 + \partial_y^2$ and $\nu \in C(\Omega)$ is a real-valued function. In the following theorem we establish the complex factorization of the stationary Schödinger operator in terms of Vekua type operators.

Theorem 8. *Let f be a particular nonvanishing solution of (3.58) in Ω and denote by C the complex conjugation operator. Then*

$$\frac{1}{4}(\Delta - \nu)\varphi = \left(\partial_z + \frac{f_{\bar{z}}}{f} C\right)\left(\partial_{\bar{z}} - \frac{f_{\bar{z}}}{f} C\right)\varphi, \tag{3.59}$$

for all real-valued function $\varphi \in C^2(\Omega)$.

Proof. Let φ be a real valued twice continuously differentiable function. Then, we get

$$\begin{aligned}
\left(\partial_z + \frac{f_{\bar{z}}}{f} C\right)\left(\partial_{\bar{z}} - \frac{f_{\bar{z}}}{f} C\right)\varphi &= \left(\partial_z\partial_{\bar{z}} - \partial_z \frac{f_{\bar{z}}}{f} C + \frac{f_{\bar{z}}}{f} C \partial_{\bar{z}} - \frac{f_{\bar{z}} C f_{\bar{z}} C}{f^2}\right)\varphi \\
&= \frac{1}{4}\Delta\varphi - \partial_z \frac{f_{\bar{z}}}{f}\varphi + \frac{f_{\bar{z}}}{f}\partial_z\varphi - \frac{f_{\bar{z}} f_z}{f^2}\varphi \\
&= \frac{1}{4}\Delta\varphi - \left(\left(\partial_z \frac{f_{\bar{z}}}{f}\right)\varphi + \frac{f_{\bar{z}}}{f}\partial_z\varphi\right) + \frac{f_{\bar{z}}}{f}\partial_z\varphi - \frac{f_{\bar{z}} f_z}{f^2}\varphi \\
&= \frac{1}{4}\Delta\varphi - \left(\partial_z \frac{f_{\bar{z}}}{f}\right)\varphi - \frac{f_{\bar{z}} f_z}{f^2}\varphi
\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{4}\Delta\varphi - \left(\frac{(\frac{\Delta}{4} f) f - f_{\bar{z}} f_z}{f^2}\right)\varphi - \frac{f_{\bar{z}} f_z}{f^2}\varphi \\
&= \frac{1}{4}\Delta\varphi - \left(\frac{(\Delta f) f}{4 f^2} - \frac{f_{\bar{z}} f_z}{f^2}\right)\varphi - \frac{f_{\bar{z}} f_z}{f^2}\varphi \\
&= \frac{1}{4}\Delta\varphi - \frac{1}{4}\frac{(\Delta f) f}{f^2}\varphi \\
&= \frac{1}{4}\Delta\varphi - \frac{1}{4}\frac{\Delta f}{f}\varphi \\
&= \frac{1}{4}\left(\Delta - \nu\right)\varphi
\end{aligned}
$$

Since φ was taken arbitrarily, we have proved (3.59). □

Note that the kernel of the first operator appearing on the previous factorization, that is $ker\left(\partial_{\bar{z}} - \frac{f_{\bar{z}}}{f} C\right)$, is formed by the solutions of the main Vekua equation.

Proposition 18. *Let f be a nonvanishing solution of the Schrödinguer equation (3.58) in Ω. If w is a solution of the main Vekua equation*

$$
\partial_{\bar{z}} w = \frac{f_{\bar{z}}}{f}\overline{w},
$$

then we have:

(a) $u = Re(w)$ is solution of

$$
\left(-\Delta + q_1\right) u = 0, \quad in \ \ \Omega,
$$

where $q_1 = \Delta f / f$.

(b) $v = Im(w)$ is solution of

$$
\left(-\Delta + q_2\right) v = 0, \quad in \ \ \Omega,
$$

where $q_2 = 2(f_x^2 + f_y^2)/f^2 - q_1$.

See Kravchenko [8, Prop. 19, pp. 3955] for a proof.

Chapter 4

On the main matrix Vekua equation

In this Chapter we shall generalized some results of Chapter 3 for the main Vekua equation in its matrix form. We will mainly deal with the results regarding differentiation and integration.

4.1 Main matrix Vekua equation

Let $\mathcal{M}_{n\times n}(\mathbb{K})$ be the set of $n \times n$ matrices with entries in $\mathbb{K}$ ($\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$) and let Ω be an open set in $\mathbb{C}$. By a matrix function we mean an application $V : \Omega \to \mathcal{M}_{n\times n}(\mathbb{K})$ and by $C^k_{n\times n}(\Omega)$ we denote the class of matrix functions having $C^k(\Omega)$ entries.
Analogous to Chapter 3 we study an important special Vekua equation called the main matrix Vekua equation

$$\partial_{\bar{z}} W = (\partial_{\bar{z}} F) F^{-1} \overline{W}, \quad \text{in } \Omega, \tag{4.1}$$

where $F \in C^2_{n\times n}(\Omega)$ is a given invertible real matrix function in Ω and $W \in C^1_{n\times n}(\Omega)$ is the unknown matrix function.

Definition 18. *Let F and G be $n \times n$ invertible matrix functions defined in Ω. We say that G is compatible with F if*

$$G^T \partial_{\bar{z}} F - (\partial_{\bar{z}} G^T) F = 0. \quad \textit{in } \Omega. \tag{4.2}$$

Remark 9. *Note that equation (4.2) also works for ∂_z.*

4.2 Matrix generating pair

Definition 19. *Let F and G be $n \times n$ complex invertible matrix functions defined in Ω. We say that (F, G) is a generating pair corresponding to (4.1) if*

(a) $F, G \in C^1_{n\times n}(\Omega)$ and they are solutions of (4.1).

(b) For $z_0 \in \Omega$ we have that

$$\forall M \in \mathcal{M}_{n\times n}(\mathbb{C}), \exists \Lambda, B \in \mathcal{M}_{n\times n}(\mathbb{R}): \quad M = F(z_0)\Lambda + G(z_0)B.$$

Similarly to the complex case, we can construct a generating pair for the main matrix Vekua equation in explicit form, as we explain next.

Proposition 19. *Let F and G be an $n \times n$ invertible nonvanishing real-valued matrix functions defined in Ω such that G is compatible with F. Then, $\left(F, i(G^{-1})^T\right)$ is a generating pair corresponding to the main matrix Vekua equation (4.1).*

Proof. This proof is performed in the following steps:

Step 1. By replacing F in (4.1) we trivially obtain that F is solution of (4.1).

Step 2. By replacing $i(G^{-1})^T$ in (4.1) we get:

(a) In the left term

$$\partial_{\bar{z}}(i(G^{-1})^T) = -i(G^{-1})^T \partial_{\bar{z}} G^T (G^{-1})^T \tag{4.3}$$

(b) In the right term

$$(\partial_{\bar{z}}F)F^{-1}\overline{(i(G^{-1})^T)} = -i(\partial_{\bar{z}}F)F^{-1}(G^{-1})^T$$

Since G is compatible with F, by (4.2) we have that $(\partial_{\bar{z}}F)F^{-1} = (G^{-1})^T \partial_{\bar{z}} G^T$. Then

$$(\partial_{\bar{z}}F)F^{-1}\overline{(i(G^{-1})^T)} = -i(G^{-1})^T \partial_{\bar{z}} G^T (G^{-1})^T \tag{4.4}$$

Thus, from (4.3) and (4.4) we have proved that $i(G^{-1})^T$ is solution of (4.1).

Step 3. Let $z_0 \in \Omega$. Let us take any $M \in \mathcal{M}_{n\times n}(\mathbb{C})$ and denote $\mathrm{Re}(M) = M_1$ and $\mathrm{Im}(M) = M_2$. We choose $\Lambda = F^{-1}(z_0)M_1$ and $B = G^T(z_0)M_2$. Then

$$M = F(z_0)\Lambda + i(G^{-1}(z_0))^T B.$$

By arbitrariness of M we have that it has a representation through the generating pair $\left(F, i(G^{-1})^T\right)$. Therefore from previous steps we have proved that $\left(F, i(G^{-1})^T\right)$ is a generating pair for (4.1). □

4.3 Differentiation with respect to generating pair $\left(F, i(G^{-1})^T\right)$

Throughout the rest of this chapter consider the generating pair $\left(F, i(G^{-1})^T\right)$ corresponding to the main matrix Vekua equation (4.1) given as in Proposition 19. Let W an arbitrary $n \times n$ matrix function defined in Ω. With the help of the generating pair $\left(F, i(G^{-1})^T\right)$, by analogy to the complex case, we have that W can be written as

$$W = F\Upsilon + i(G^{-1})^T\Psi, \quad \text{in } \Omega, \tag{4.5}$$

where $\Upsilon = F^{-1}\operatorname{Re}(W)$ and $\Psi = G^T \operatorname{Im}(W)$ are $n \times n$ real-valued matrix functions. Now, we introduce the concept of $\left(F, i(G^{-1})^T\right)$-derivative. Formally, we get

Definition 20. *Let W be an $n \times n$ complex matrix function defined in Ω and $z_0 \in \Omega$. We say that W has $\left(F, i(G^{-1})^T\right)$-derivative at z_0 if the following limit*

$$\dot{W}(z_0) = \lim_{z \to z_0} \frac{W(z) - \left(F(z)\Upsilon(z_0) + i(G^{-1}(z))^T\Psi(z_0)\right)}{z - z_0} \tag{4.6}$$

exists and is finite.

If $\dot{W}$ exists everywhere in Ω, we say that W is $\left(F, i(G^{-1})^T\right)$-pseudoanalytic of first kind on Ω (or simply pseudoanalytic if there is no confusion). The following auxiliary complex matrix function (for a fixed z_0 in Ω)

$$\tilde{W} := W - \left(F\Upsilon(z_0) + i(G^{-1})^T\Psi(z_0)\right), \quad \text{in } \Omega, \tag{4.7}$$

will be useful for what follows next.

Proposition 20. *An $n \times n$ complex matrix function W defined in Ω has $\left(F, i(G^{-1})^T\right)$-derivative at $z_0 \in \Omega$ if and only if an $n \times n$ complex matrix $\tilde{W}$ given by (4.7) has complex derivative at z_0. Moreover, $\dot{W}(z_0) = \tilde{W}'(z_0)$.*

Proof. Assume that W has $\left(F, i(G^{-1})^T\right)$-derivative at z_0, that is,

$$\dot{W}(z_0) = \lim_{z \to z_0} \frac{W(z) - \left(F(z)\Upsilon(z_0) + i(G^{-1}(z))^T\Psi(z_0)\right)}{z - z_0}.$$

From (4.7) we get $\tilde{W}(z_0) = 0$, it follows that

$$\dot{W}(z_0) = \lim_{z \to z_0} \frac{\tilde{W}(z)}{z - z_0} = \lim_{z \to z_0} \frac{\tilde{W}(z) - \tilde{W}(z_0)}{z - z_0}.$$

Then, $\dot{W}(z_0) = \tilde{W}'(z_0)$. Conversely, the other direction is clear. □

Corollary 9. *Let an $n \times n$ complex matrix function $W \in C^1_{n\times n}(\Omega)$ and $z_0 \in \Omega$. If W has $\big(F, i(G^{-1})^T\big)$-derivative, then for $\tilde{W}$ given by (4.7) we have:*

(a) By previous Proposition $\tilde{W}$ has complex derivative at z_0. Thus $\tilde{W}$ satisfies the Cauchy-Riemann equations at z_0, which is equivalent to $\partial_{\bar{z}}\tilde{W}(z_0) = 0$.

(b) The complex derivative of $\tilde{W}$ at z_0 is computed by $\partial_z \tilde{W}(z_0)$.

Proof. We denote $\mathrm{Re}(\tilde{W}) = U$ and $\mathrm{Im}(\tilde{W}) = V$. Let us prove *(a)*. Then

$$\begin{aligned}
\partial_{\bar{z}}\tilde{W}(z) =& \partial_{\bar{z}}\left(U(z) + iV(z)\right) \\
=& \partial_{\bar{z}}U(z) + i\partial_{\bar{z}}V(z) \\
=& \frac{1}{2}(\partial_x + i\partial_y)U(z) + \frac{i}{2}(\partial_x + i\partial_y)V(z) \\
=& \frac{1}{2}\partial_x U(z) + \frac{i}{2}\partial_y U(z) + \frac{i}{2}\partial_x V(z) - \frac{1}{2}\partial_y V(z)
\end{aligned}$$

assuming that $\partial_{\bar{z}}\tilde{W}(z_0) = 0$ we get

$$\frac{1}{2}\partial_x U(z_0) + \frac{i}{2}\partial_y U(z_0) + \frac{i}{2}\partial_x V(z_0) - \frac{1}{2}\partial_y V(z_0) = 0.$$

It follows that

$$\partial_x U(z_0) = \partial_y V(z_0) \quad \text{and} \quad \partial_y U(z_0) = -\partial_x V(z_0).$$

Conversely, the other direction is straightforward. Now, let us prove *(b)*. Proceeding similarly to the previous item we get

$$\partial_z \tilde{W}(z_0) = \frac{1}{2}\partial_x U(z_0) - \frac{i}{2}\partial_y U(z_0) + \frac{i}{2}\partial_x V(z_0) + \frac{1}{2}\partial_y V(z_0)$$

this means

$$\partial_z \tilde{w}_{ij}(z_0) = \frac{1}{2}\partial_x u_{ij}(z_0) - \frac{i}{2}\partial_y u_{ij}(z_0) + \frac{i}{2}\partial_x v_{ij}(z_0) + \frac{1}{2}\partial_y v_{ij}(z_0), \quad \forall i, j \in \mathbb{N}^{1}. \tag{4.8}$$

[1] $\tilde{w}_{ij}, u_{ij}$, and v_{ij} represent the i, j-th entry of $\tilde{W}, U$, and V, respectively.

Assume that $\tilde{W}$ has complex derivative at z_0. This means that each entry of $\tilde{W}$ satisfies Corollary 3 at z_0, that is,

$$\tilde{w}'_{ij}(z_0) = \partial_x u_{ij}(z_0) + i\partial_x v_{ij}(z_0) = \partial_y v_{ij}(z_0) - i\partial_y u_{ij}(z_0), \quad \forall i, j \in \mathbb{N}. \tag{4.9}$$

From (4.8) and (4.9) we get

$$\partial_z \tilde{w}_{ij}(z_0) = \tilde{w}'_{ij}(z_0), \quad \forall i, j \in \mathbb{N}.$$

Thus, we obtain

$$\partial_z \tilde{W}(z_0) = \tilde{W}'(z_0).$$

This completes the proof. □

Theorem 9. *Let an $n \times n$ complex matrix function $W = F\Upsilon + i(G^{-1})^T\Psi \in C^1_{n\times n}(\Omega)$ and $z_0 \in \Omega$. Then, W has $\left(F, i(G^{-1})^T\right)$-derivative at z_0 if and only if*

$$F(z_0)\partial_{\bar{z}}\Upsilon(z_0) + i(G^{-1}(z_0))^T\partial_{\bar{z}}\Psi(z_0) = 0. \tag{4.10}$$

Moreover, we have that

$$\dot{W}(z_0) = F(z_0)\partial_z\Upsilon(z_0) + i(G^{-1}(z_0))^T\partial_z\Psi(z_0).$$

Proof. Let us consider the $n \times n$ complex matrix function $\tilde{W}(z)$ given by (4.7), that is,

$$\tilde{W}(z) = W(z) - \left(F(z)\Upsilon(z_0) + i(G^{-1}(z))^T\Psi(z_0)\right)$$

Note that

$$\tilde{W}(z) = F(z)(\Upsilon(z) - \Upsilon(z_0)) + i(G^{-1}(z))^T(\Psi(z) - \Psi(z_0))$$

Then, applying $\partial_{\bar{z}}$ we get

$$\begin{aligned}\partial_{\bar{z}}\tilde{W}(z) =& \partial_{\bar{z}}\left((F(z)(\Upsilon(z) - \Upsilon(z_0)) + i(G^{-1}(z))^T(\Psi(z) - \Psi(z_0))\right)\\ =& \partial_{\bar{z}}F(z)(\Upsilon(z) - \Upsilon(z_0)) + F(z)\partial_{\bar{z}}\Upsilon(z) + i\partial_{\bar{z}}(G^{-1}(z))^T(\Psi(z) - \Psi(z_0))\\ &+ i(G^{-1}(z))^T\partial_{\bar{z}}\Psi(z)\end{aligned}$$

evaluating at z_0 we get

$$\partial_{\bar{z}}\tilde{W}(z_0) = F(z_0)\partial_{\bar{z}}\Upsilon(z_0) + i(G^{-1}(z_0))^T\partial_{\bar{z}}\Psi(z_0). \tag{4.11}$$

In a similar way we get

$$\partial_z \tilde{W}(z_0) = F(z_0)\partial_z \Upsilon(z_0) + i(G^{-1}(z_0))^T \partial_z \Psi(z_0). \tag{4.12}$$

Assume that $\tilde{W}$ has $\left(F, i(G^{-1})^T\right)$-derivative at z_0. Then, from Corollary 9 follows that $\partial_{\bar{z}} \tilde{W}(z_0) = 0$, joining this with (4.11) we get

$$F(z_0)\partial_{\bar{z}} \Upsilon(z_0) + i(G^{-1}(z_0))^T \partial_{\bar{z}} \Psi(z_0) = 0.$$

Moreover, again by item *(b)* of Corollary 9 and Proposition 20 we get

$$\dot{W}(z_0) = F(z_0)\partial_z \Upsilon(z_0) + i(G^{-1}(z_0))^T \partial_z \Psi(z_0).$$

Conversely, if (4.10) holds then by (4.11) we have that $\partial_{\bar{z}} \tilde{W}(z_0) = 0$ and from (4.12) we get $\tilde{W}'(z_0) = \dot{W}(z_0)$. This completes the proof. □

Corollary 10. *Under the conditions of Theorem 9 we have that*

$$\dot{W}(z_0) = \partial_z W(z_0) - (\partial_z F(z_0))F^{-1}(z_0)\overline{W}(z_0).$$

Proof. By equation (4.12) we get

$$\partial_z \widetilde{W}(z_0) = F(z_0)\partial_z \Upsilon(z_0) + i(G^{-1}(z_0))^T \partial_z \Psi(z_0)$$

Let us denote $\text{Re}(W) = W_1$ and $\text{Im}(W) = W_2$. From equation (4.5) we get

$$\begin{aligned}
\partial_z \widetilde{W}(z_0) =& F(z_0)\partial_z(F^{-1}(z_0)W_1(z_0)) + i(G^{-1}(z_0))^T \partial_z(G^T(z_0)W_2(z_0)) \\
=& F(z_0)(-F^{-1}(z_0)(\partial_z F(z_0))F^{-1}(z_0)W_1(z_0) + F^{-1}(z_0)\partial_z W_1(z_0)) \\
&+ i(G^{-1}(z_0))^T((\partial_z G^T(z_0))W_2(z_0) + G^T(z_0)\partial_z W_2(z_0)) \\
=& -(\partial_z F(z_0))F^{-1}(z_0)W_1(z_0) + \partial_z W_1(z_0) + i(G^{-1}(z_0))^T(\partial_z G^T(z_0))W_2(z_0) \\
&+ i\partial_z W_2(z_0)
\end{aligned}$$

By Remark 9 we have that $(G^{-1}(z_0))^T = (\partial_z F(z_0))F^{-1}(z_0)\partial_z(G^{-1}(z_0))^T$. Then

$$\begin{aligned}
\partial_z \widetilde{W}(z_0) =& \partial_z W_1(z_0) + i\partial_z W_2(z_0) - (\partial_z F(z_0))F^{-1}(z_0)W_1(z_0) \\
&+ i(\partial_z F(z_0))F^{-1}(z_0)\partial_z(G^{-1}(z_0))^T(\partial_z G^T(z_0))W_2(z_0) \\
=& \partial_z(W_1(z_0) + iW_2(z_0)) - (\partial_z F(z_0))F^{-1}(z_0)W_1(z_0) + i(\partial_z F(z_0))F^{-1}(z_0)W_2(z_0) \\
=& \partial_z(W_1(z_0) + iW_2(z_0)) - (\partial_z F(z_0))F^{-1}(z_0)(W_1(z_0) - iW_2(z_0))
\end{aligned}$$

$$=\partial_z W(z_0) - (\partial_z F(z_0))F^{-1}(z_0)\overline{W}(z_0)$$

From Corollary 9 item *(b)* and Proposition 20 we get

$$\dot{W}(z_0) = \partial_z W(z_0) - (\partial_z F(z_0))F^{-1}(z_0)\overline{W}(z_0).$$

□

Proposition 21. *An $n \times n$ complex matrix function $W \in C^1_{n\times n}(\Omega)$ has $\left(F, i(G^{-1})^T\right)$-derivative at $z_0 \in \Omega$ if and only if it satisfies the main matrix Vekua equation (4.1) at z_0.*

Proof. Assume that W has $\left(F, i(G^{-1})^T\right)$-derivative at z_0. From Theorem 9 we get

$$0 = F(z_0)\partial_{\bar{z}}\Upsilon(z_0) + i(G^{-1}(z_0))^T\partial_{\bar{z}}\Psi(z_0).$$

Note that

$$\begin{aligned}
0 =& F(z_0)\partial_{\bar{z}}\Upsilon(z_0) + i(G^{-1}(z_0))^T\partial_{\bar{z}}\Psi(z_0) + (\partial_{\bar{z}}F(z_0))\Upsilon(z_0) - (\partial_{\bar{z}}F(z_0))\Upsilon(z_0) \\
& - i(G^{-1}(z_0))^T(\partial_{\bar{z}}G^T(z_0))(G^{-1}(z_0))^T\Psi(z_0) + i(G^{-1}(z_0))^T(\partial_{\bar{z}}G^T(z_0))(G^{-1}(z_0))^T\Psi(z_0) \\
=& (\partial_{\bar{z}}F(z_0))\Upsilon(z_0) + F(z_0)\partial_{\bar{z}}\Upsilon(z_0) - i(G^{-1}(z_0))^T(\partial_{\bar{z}}G^T(z_0))(G^{-1}(z_0))^T\Psi(z_0) \\
& + i(G^{-1}(z_0))^T\partial_{\bar{z}}\Psi(z_0) + i(G^{-1}(z_0))^T(\partial_{\bar{z}}G^T(z_0))(G^{-1}(z_0))^T\Psi(z_0) - (\partial_{\bar{z}}F(z_0))\Upsilon(z_0) \\
=& (\partial_{\bar{z}}F(z_0))\Upsilon(z_0) + F(z_0)\partial_{\bar{z}}\Upsilon(z_0) + i\partial_{\bar{z}}(G^{-1}(z_0))^T\Psi(z_0) + i(G^{-1}(z_0))^T\partial_{\bar{z}}\Psi(z_0) \\
& + i(G^{-1}(z_0))^T(\partial_{\bar{z}}G^T(z_0))(G^{-1}(z_0))^T\Psi(z_0) - (\partial_{\bar{z}}F(z_0))\Upsilon(z_0) \\
=& \partial_{\bar{z}}W(z_0) + i(G^{-1}(z_0))^T(\partial_{\bar{z}}G^T(z_0))(G^{-1}(z_0))^T\Psi(z_0) - (\partial_{\bar{z}}F(z_0))\Upsilon(z_0)
\end{aligned}$$

from Definition 18 note that $(\partial_{\bar{z}}F(z))F^{-1}(z) = i(G^{-1}(z))^T\partial_{\bar{z}}G^T$. Then

$$\begin{aligned}
0 =& \partial_{\bar{z}}W(z_0) + i(G^{-1}(z_0))^T(\partial_{\bar{z}}G^T(z_0))(G^{-1}(z_0))^T\Psi(z_0) - (\partial_{\bar{z}}F(z_0))\Upsilon(z_0) \\
=& \partial_{\bar{z}}W(z_0) + i(\partial_{\bar{z}}F(z_0))F^{-1}(z_0)(G^{-1}(z_0))^T\Psi(z_0) - (\partial_{\bar{z}}F(z_0))\Upsilon(z_0) \\
=& \partial_{\bar{z}}W(z_0) + i(\partial_{\bar{z}}F(z_0))F^{-1}(z_0)(G^{-1}(z_0))^T\Psi(z_0) - (\partial_{\bar{z}}F(z_0))F^{-1}(z_0)F(z_0)\Upsilon(z_0) \\
=& \partial_{\bar{z}}W(z_0) - (\partial_{\bar{z}}F(z_0))F^{-1}(z_0)(F(z_0)\Upsilon(z_0) - i(G^{-1}(z_0))^T\Psi(z_0)) \\
=& \partial_{\bar{z}}W(z_0) - (\partial_{\bar{z}}F(z_0))F^{-1}(z_0)\overline{W}(z_0)
\end{aligned}$$

Thus,

$$\partial_{\bar{z}}W(z_0) = (\partial_{\bar{z}}F(z_0))F^{-1}(z_0)\overline{W}(z_0).$$

Conversely, we assume that W satisfies the main matrix Vekua equation at z_0. By the previous direction we easily obtain that

$$F(z_0)\partial_{\bar{z}}\Upsilon(z_0) + i(G^{-1}(z_0))^T\partial_{\bar{z}}\Psi(z_0) = 0,$$

and the result follows from Theorem 9. □

Corollary 11. *Under the conditions of Proposition 21, $W \in C^1_{n\times n}(\Omega)$ is a solution of the main matrix Vekua equation (4.1) in Ω if and only if*

$$F\partial_{\bar{z}}\Upsilon + i(G^{-1})^T\partial_{\bar{z}}\Psi = 0, \quad in \;\; \Omega. \tag{4.13}$$

Proposition 22. *If an $n \times n$ matrix function W defined in Ω is a solution of the main matrix Vekua equation (4.1) in Ω, then the $\left(F, i(G^{-1})^T\right)$-derivative of W is a solution of the following Vekua equation*

$$\partial_{\bar{z}}\dot{W} = -(\partial_z F)F^{-1}\overline{\dot{W}}, \quad in \;\; \Omega. \tag{4.14}$$

Proof. Let $z \in \Omega$. Since W is solution of the main matrix Vekua equation at z, we have that

$$\dot{W}(z) = F(z)\partial_z\Upsilon(z) + i(G^{-1}(z))^T\partial_z\Psi(z). \tag{4.15}$$

By Corollary 11 we get

$$F(z)\partial_{\bar{z}}\Upsilon(z) + i(G^{-1}(z_0))^T\partial_{\bar{z}}\Psi(z) = 0. \tag{4.16}$$

Taking the conjugate to (4.16), we have that

$$F(z)\partial_z\Upsilon(z) - i(G^{-1}(z))^T\partial_z\Psi(z) = 0. \tag{4.17}$$

Solving (4.15) and (4.17) we get

$$\partial_z\Upsilon(z) = \frac{1}{2}F^{-1}(z)\dot{W}(z) \quad \text{and} \quad \partial_z\Psi(z) = \frac{1}{2i}G^T(z)\dot{W}. \tag{4.18}$$

Applying ∂_z to (4.16) we get

$$\begin{aligned}0 =&\partial_z F(z)\partial_{\bar{z}}\Upsilon(z) + F(z)\partial_z\partial_{\bar{z}}\Upsilon(z) - i(G^{-1}(z))^T(\partial_z G^T(z))(G^{-1}(z))^T\partial_{\bar{z}}\Psi(z)\\ &+ i(G^{-1}(z))^T\partial_z\partial_{\bar{z}}\Psi(z).\end{aligned} \tag{4.19}$$

Applying $\partial_{\bar{z}}$ to (4.15) we get

$$\begin{aligned}\partial_{\bar{z}}\dot{W}(z) =&\partial_{\bar{z}}F(z)\partial_z\Upsilon(z) + F(z)\partial_{\bar{z}}\partial_z\Upsilon(z)\\ &- i(G^{-1}(z))^T(\partial_{\bar{z}}G^T(z))(G^{-1}(z))^T\partial_{\bar{z}}\Psi(z) + i(G^{-1}(z))^T\partial_z\partial_{\bar{z}}\Psi(z)\end{aligned} \tag{4.20}$$

Replacing (4.19) in (4.20) we get

$$\begin{aligned}\partial_{\bar{z}}\dot{W}(z) =&\partial_{\bar{z}}F(z)\partial_{z}\Upsilon(z) - i(G^{-1}(z))^{T}(\partial_{\bar{z}}G^{T}(z))(G^{-1}(z))^{T}\partial_{z}\Psi(z)\\ &- \partial_{z}F(z)\partial_{\bar{z}}\Psi(z) + i(G^{-1}(z))^{T}(\partial_{z}G^{T}(z))(G^{-1}(z))^{T}\partial_{\bar{z}}\Psi(z).\end{aligned} \tag{4.21}$$

From Definition 18 note that $(G^{-1}(z))^{T}\partial_{\bar{z}}G^{T}(z) = (\partial_{\bar{z}}F(z))F^{-1}(z)$. Then in (4.21) we get

$$\begin{aligned}\partial_{\bar{z}}\dot{W}(z) =&\partial_{\bar{z}}F(z)\partial_{z}\Upsilon(z) - i(\partial_{\bar{z}}F(z))F^{-1}(z)(G^{-1}(z))^{T}\partial_{z}\Psi(z) - (\partial_{z}F(z))\partial_{\bar{z}}\Upsilon(z)\\ &+ i(\partial_{\bar{z}}F(z))F^{-1}(z)(G^{-1}(z))^{T}\partial_{\bar{z}}\Psi(z)\\ =&\partial_{\bar{z}}F(z)\partial_{z}\Upsilon(z) - i(\partial_{\bar{z}}F(z))F^{-1}(z)(G^{-1}(z))^{T}\partial_{z}\Psi(z)\\ &- \overline{(\partial_{z}F(z)\partial_{\bar{z}}\Upsilon(z) + i(\partial_{\bar{z}}F(z))F^{-1}(z)(G^{-1}(z))^{T}\partial_{\bar{z}}\Psi(z))}\end{aligned} \tag{4.22}$$

by replacing (4.18) in (4.22) we obtain

$$\begin{aligned}\partial_{\bar{z}}\dot{W}(z) =&\frac{1}{2}(\partial_{\bar{z}}F(z))F^{-1}(z)\dot{W}(z) - \frac{1}{2}(\partial_{\bar{z}}F(z))F^{-1}(G^{-1}(z))^{T}G^{T}(z)\dot{W}(z)\\ &- \overline{\left(\frac{1}{2}(\partial_{\bar{z}}F(z))F^{-1}(z)\dot{W}(z) + \frac{1}{2}(\partial_{\bar{z}}F(z))F^{-1}(z)(G^{-1}(z))^{T}G^{T}(z)\dot{W}(z)\right)}\\ =&\frac{1}{2}(\partial_{\bar{z}}F(z))F^{-1}(z)\dot{W}(z) - \frac{1}{2}(\partial_{\bar{z}}F(z))F^{-1}\dot{W}(z)\\ &- \overline{\left(\frac{1}{2}(\partial_{\bar{z}}F(z))F^{-1}(z)\dot{W}(z) + \frac{1}{2}(\partial_{\bar{z}}F(z))F^{-1}(z)\dot{W}(z)\right)}\\ =&- \overline{\left(\frac{1}{2}(\partial_{\bar{z}}F(z))F^{-1}(z)\dot{W}(z) + \frac{1}{2}(\partial_{\bar{z}}F(z))F^{-1}(z)\dot{W}(z)\right)}\\ =&- \overline{((\partial_{\bar{z}}F(z))F^{-1}(z)\dot{W}(z))}\\ =&- (\partial_{z}F(z))F^{-1}(z)\overline{\dot{W}}(z)\end{aligned}$$

Therefore, we have that

$$\partial_{\bar{z}}\dot{W} = -(\partial_{z}F)F^{-1}\overline{\dot{W}}, \quad \text{in } \Omega.$$

This completes the proof. □

Definition 21. *Note that (4.14) is an $n \times n$ matrix Vekua equation. More precisely, let an $n \times n$ complex matrix function V defined in Ω. In a complete analogy with the complex case we say that the successor matrix Vekua equation of (4.1) is given by*

$$\partial_{\bar{z}}V = (-\partial_{z}F)F^{-1}\overline{V}, \quad \textit{in } \Omega. \tag{4.23}$$

4.4 Integration with respect to the generating pair $(F, i(G^{-1})^T)$

By equation (4.18) we get

$$\partial_z \Upsilon = \frac{1}{2}F^{-1}\dot{W} \quad \text{and} \quad \partial_z \Psi = -\frac{i}{2}G^T\dot{W}. \tag{4.24}$$

To recover Υ and Ψ we use the operator A (defined in section 2.2.2). Then, applying the operator A to each matrix equation in (4.24) we recover Υ and Ψ as follows

$$\Upsilon = A\left[\frac{1}{2}F^{-1}\dot{W}\right] \quad \text{and} \quad \Psi = -A\left[\frac{i}{2}G^T\dot{W}\right].$$

Hence, we write any $n \times n$ matrix complex-valued function W defined in Ω in terms of operator A in the following way:

$$W = FA\left[\frac{1}{2}F^{-1}\dot{W}\right] - i(G^{-1})^T A\left[\frac{i}{2}G^T\dot{W}\right]. \tag{4.25}$$

Consequently, in (4.25) appears the additive term $FC_1 + i(G^{-1})^T C_2$ for $C_1, C_2 \in \mathcal{M}_{n\times n}(\mathbb{R})$. Fixing $W(z_0)$ where $z_0 \in \Omega$, we have $C_1 = \Upsilon(z_0)$ and $C_2 = \Psi(z_0)$.

Definition 22. *Let W be an $n \times n$ complex matrix function defined in Ω and Γ a rectifiable curve leading from z_0 to z_1 in Ω. Then, we have that*

(a) The $(F, i(G^{-1})^T)$--integral is defined as*

$$* \int_\Gamma W(z) d_{(F,i(G^{-1})^T)} z = \operatorname{Re} \int_\Gamma F^{-1}(z)W(z)dz - i \operatorname{Re} \int_\Gamma iG^T(z)W(z)dz.$$

(b) The $(F, i(G^{-1})^T)$-integral is defined by

$$\int_\Gamma W(z) d_{(F,i(G^{-1})^T)} z = F(z_1) \operatorname{Re} \int_\Gamma F^{-1}(z)W(z)dz - i(G^{-1}(z_1))^T \operatorname{Re} \int_\Gamma iG^T(z)W(z)dz.$$

Definition 23. *Let a continuous $n \times n$ matrix function W defined in Ω. We say that W is $(F, i(G^{-1})^T)$-integrable in Ω if for every closed curve Γ lying in a simply connected subdomain of Ω we have*

$$\oint_\Gamma W(z) d_{(F,i(G^{-1})^T)} z = 0. \tag{4.26}$$

Proposition 23. *Let W be an $n \times n$ matrix function solution of the main matrix Vekua equation (4.1) in Ω where Ω is simply connected domain. Then its $(F, i(G^{-1})^T)$-derivative is $(F, i(G^{-1})^T)$-integrable in Ω.*

Proof. Let us take a closed curve Γ such that it lies in Ω. It is enough to prove that

$$* \oint_\Gamma \dot{W}(z) d_{(F,i(G^{-1})^T)} z = 0. \tag{4.27}$$

From Definition 22 item *(a)* we get

$$* \oint_\Gamma \dot{W}(z) d_{(F,i(G^{-1})^T)} z = \operatorname{Re} \oint_\Gamma F^{-1}(z) \dot{W}(z) dz - \operatorname{Re} \oint_\Gamma -iG^T(z) \dot{W}(z) dz.$$

Since $\dot{W}(z) = F(z)\partial_z \Upsilon(z) + i(G^{-1}(z))^T \partial_z \Psi(z)$, it follows that

$$\begin{aligned} * \oint_\Gamma \dot{W}(z) d_{(F,i(G^{-1})^T)} z =& \operatorname{Re} \oint_\Gamma F^{-1}(z) \left(F(z)\partial_z \Upsilon(z) + i(G^{-1}(z))^T \partial_z \Psi(z) \right) dz \\ & - \operatorname{Re} \oint_\Gamma iG^T(z) \left(F(z)\partial_z \Upsilon(z) + i(G^{-1}(z))^T \partial_z \Psi(z) \right) dz \\ =& \operatorname{Re} \oint_\Gamma \left(\partial_z \Upsilon(z) + iF^{-1}(z)(G^{-1}(z))^T \partial_z \Psi(z) \right) \\ & - \operatorname{Re} \oint_\Gamma \left(iG^T(z)F(z)\partial_z \Upsilon - \partial_z \Psi(z) \right) dz \end{aligned}$$

From (4.13) we have that $F(z)\partial_z \Upsilon(z) = i(G^{-1}(z))^T \partial_z \Psi(z)$. Then

$$\begin{aligned} * \oint_\Gamma \dot{W}(z) d_{(F,i(G^{-1})^T)} z =& \operatorname{Re} \oint_\Gamma 2\partial_z \Upsilon(z) dx + \operatorname{Re} \oint_\Gamma 2\partial_z \Psi(z) dz \\ =& \operatorname{Re} \oint_\Gamma (\partial_x \Upsilon(z) - i\partial_y \Upsilon(z)) (dx + idy) \\ & + \operatorname{Re} \oint_\Gamma (\partial_x \Psi(z) - i\partial_y \Psi(z)) (dx + idy) \\ =& \oint_\Gamma (\partial_x \Upsilon(z) dx + \partial_y \Upsilon(z) dy) + \oint_\Gamma (\partial_x \Psi(z) dx + \partial_y \Psi(z) dy) \end{aligned} \tag{4.28}$$

By Theorem 1 we have that the integrals in (4.28) are path-independent. Consequently, we get

$$\oint_\Gamma \dot{W}(z) d_{(F,i(G^{-1})^T)} z = 0.$$

This completes the proof. □

Proposition 24. *Let V be an $n \times n$ continuous complex-valued matrix function defined in Ω where Ω is a simply connected domain. If V is $(F, i(G^{-1})^T)$-integrable in Ω, then there exists* *a solution W of the main matrix Vekua equation* (4.1) *such that*

$$V = \frac{d_{(F,i(G^{-1})^T)} W}{dz}, \quad in \ \Omega.$$

Proof. Let z_0 and z in Ω. Assume that V is $(F, i(G^{-1})^T)$-integrable in Ω. From Definition 22 item *(a)* we have that

$$* \int_\Gamma V(z) d_{(F,i(G^{-1})^T)} z = \operatorname{Re} \int_\Gamma F^{-1}(z) V(z) dz - i \operatorname{Re} \int_\Gamma i G^T(z) V(z) dz,$$

where Γ a rectifiable curve leading from z_0 to z. Let us denote

$$\Upsilon(z) = \operatorname{Re} \int_\Gamma F^{-1}(z) V(z) dz, \tag{4.29}$$

$$\Psi(z) = \operatorname{Re} \int_\Gamma -i G^T(z) V(z) dz. \tag{4.30}$$

Note that for (4.29) we get

$$\begin{aligned}
\Upsilon(z) =& \operatorname{Re} \int_\Gamma F^{-1}(z) V(z) dz \\
=& \frac{2}{2} \operatorname{Re} \int_\Gamma F^{-1}(z) \left(V_1(z) + i V_2(z) \right) (dx + i dy) \\
=& \operatorname{Re} \int_\Gamma \frac{1}{2} F^{-1}(z) \left(2V_1(z) + 2i V_2(z) \right) (dx + i dy) \\
=& \int_\Gamma \frac{1}{2} F^{-1}(z) \left(2V_1(z) dx - 2V_2(z) dy \right) \\
=& \int_\Gamma \frac{1}{2} F^{-1}(z) \left(2V_1(z) dx + i V_1(z) dy - i V_1(z) dy \right) \\
& + \int_\Gamma \frac{1}{2} F^{-1}(z) \left(i V_2(z) dx - i V_2(z) dx - 2V_2(z) dy \right) \\
=& \int_\Gamma \frac{1}{2} F^{-1}(z) \left(V_1(z) dx + i V_1(z) dy + i V_2(z) dx - V_2(z) dy \right) \\
& + \int_\Gamma \frac{1}{2} F^{-1}(z) \left(V_1(z) dx - i V_1(z) dy - i V_2(z) dx - V_2(z) dy \right) \\
=& \int_\Gamma \frac{1}{2} F^{-1}(z) \left(V(z) dz + \overline{V}(z) d\overline{z} \right)
\end{aligned} \tag{4.31}$$

Analogously, we have that (4.30) is equal to

$$\Psi(z) = \int_\Gamma \frac{1}{2i} G^T(z) \left(V(z) dz - \overline{V}(z) d\overline{z} \right) \tag{4.32}$$

Taking ∂_z to (4.31) and (4.32) we get

$$\partial_z \Upsilon(z) = \frac{1}{2} F^{-1}(z) V(z), \tag{4.33}$$

$$\partial_z \Psi(z) = \frac{1}{2i} G^T(z) V(z). \tag{4.34}$$

Hence adding (4.33) and (4.34)

$$\begin{aligned} F(z)\partial_z \Upsilon(z) + i(G^{-1}(z))^T \partial_z \Psi &= \frac{1}{2} F(z) F^{-1}(z) V(z) + \frac{1}{2} (G^{-1}(z))^T G^T(z) V(z) \\ &= V(z) \end{aligned} \tag{4.35}$$

Then taking the conjugate to (4.33) and (4.34) and adding them we get

$$\begin{aligned} F(z)\partial_{\overline{z}} \Upsilon(z) + i(G^{-1}(z))^T \partial_{\overline{z}} \Psi(z) &= \frac{1}{2} F(z) F^{-1}(z) \overline{V}(z) + \frac{1}{2} (G^{-1}(z))^T G^T(z) \overline{V}(z) \\ &= 0 \end{aligned} \tag{4.36}$$

From Theorem 9 and (4.35) we have that $\dot{W} = V$. While, by Proposition 21 we have that $\dot{W}$ is solution of the main matrix Vekua equation in Ω. Therefore, by arbitrariness of z_0 and z the proof is done. □

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