UNIVERSIDAD DE INVESTIGACIÓN DE TECNOLOGÍA EXPERIMENTAL YACHAY

Escuela de Ciencias Matemáticas y Computacionales

TÍTULO: Multiplicity of solutions for the nonlinear biharmonic Schrödinger equation with critical frequency

Trabajo de integración curricular presentado como requisito para la obtención del título de Matemático

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Dedication

To the reader.

Acknowledgments

After almost five years of a really hard work in Yachay and at home (phandemic) I would like to thank my advisor Juan Mayorga-Zambrano. I shall not forget the importance of all my professors and classmates at the university that I began to consider as my second family, which I will always remember. In general, Yachay Tech University was an experience that changed my life, beginning with the fact that I had to live without my family, until the point of inspiring me to keep studying mathematics after graduating.

Resumen

Estudiamos la multiplicidad de soluciones del siguiente problema:

$$\left\{ \begin{array}{rll} -\varepsilon^4\Delta^2 u(x) + V(x)u(x) - |u(x)|^{p-1}u(x) & = 0, & x \in \mathbb{R}^N, \\ u(x) & \longrightarrow 0, & \text{cuando } |x| \longrightarrow +\infty, \end{array} \right. \tag{1}$$

donde $\Delta^2 = \Delta \circ \Delta$, y $1 < p+1 < 2^*$ con

$$2^* = \left\{ \begin{array}{ll} \dfrac{2N}{N-2}, & \text{if } N \geq 3; \\ +\infty, & \text{if } N = 1, 2, \end{array} \right.$$

bajo los siguientes supuestos:
(V1) $V \in C\left(\mathbb{R}^N\right)$ es una función no negativa,
(V2) $V(x) \longrightarrow +\infty$, cuando $|x| \longrightarrow +\infty$,
(V3) $\{V = \inf(V) = 0\} \neq \emptyset$.
Via un escalamiento adecuado, con $v(x) = u\left(\varepsilon^\alpha x\right)$ y $V_\varepsilon(x) = V\left(\varepsilon^{-\alpha} x\right)$, para $\varepsilon > 0$ y $\alpha = -1$, tratamos con la siguiente versión equivalente de (2):

$$(P'_\varepsilon) \left\{ \begin{array}{ll} -\Delta^2 u(x) + V_\varepsilon(x)u(x) - |u(x)|^{p-1}u(x) = 0, & x \in \mathbb{R}^N, \\ u(x) \longrightarrow 0, & \text{cuando } |x| \longrightarrow +\infty. \end{array} \right.$$

Consideramos el funcional asociado $I_\varepsilon : \mathcal{M}_\varepsilon \subseteq \mathrm{H}^2_\varepsilon \longrightarrow \mathbb{R}$, dado por

$$I_\varepsilon(u) = \frac{1}{2} \int_{\mathbb{R}^N} \left[|\Delta u(x)|^2 + V_\varepsilon(x)|u(x)|^2 \right] dx,$$

definido sobre la variedad

$$\mathcal{M}_\varepsilon = \left\{ u \in \mathrm{H}^2_\varepsilon \Big/ \int_{\mathbb{R}^N} |u(x)|^{p+1} dx = 1 \right\},$$

donde

$$\mathrm{H}^2_\varepsilon = \left\{ u \in \mathrm{H}^2\left(\mathbb{R}^N\right) \Big/ \|u\|_\varepsilon = \left(\int_{\mathbb{R}^N} \left[|\Delta u(x)|^2 + V_\varepsilon(x)|u(x)|^2 \right] dx \right)^{1/2} < +\infty \right\}.$$

Aplicamos un esquema de Ljusternik-Schnirelman para mostrar que el conjunto de puntos críticos tiene infinitos elementos, esto es realizado a través de escalamiento para obtener soluciones débiles de (P'_ε).

Palabras clave: Ecuación no lineal de Schrödinger, operador biarmónico de Laplace, multiplicidad de soluciones, esquema Ljusternik-Schnirelmann, Ecuaciones Diferenciales Parciales.

Abstract

This project deals with the multiplicity of solutions of the problem:

$$\begin{cases} -\varepsilon^4\Delta^2 u(x) + V(x)u(x) - |u(x)|^{p-1}u(x) & = 0, \qquad x \in \mathbb{R}^N, \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad u(x) & \longrightarrow 0, \quad \text{as } |x| \longrightarrow +\infty, \end{cases} \tag{2}$$

where $\Delta^2 = \Delta \circ \Delta$, and $1 < p+1 < 2^*$ with

$$2^* = \begin{cases} \dfrac{2N}{N-2}, & \text{if } N \geq 3; \\ +\infty, & \text{if } N = 1, 2, \end{cases}$$

under the following assumptions:
(V1) $V \in C\left(\mathbb{R}^N\right)$ is a nonnegative function,
(V2) $V(x) \longrightarrow +\infty$, when $|x| \longrightarrow +\infty$,
(V3) $\{V = \inf(V) = 0\} \neq \emptyset$.
By a suitable scaling, with $v(x) = u\left(\varepsilon^\alpha x\right)$ and $V_\varepsilon(x) = V\left(\varepsilon^{-\alpha}x\right)$, for $\varepsilon > 0$ and $\alpha = -1$, we deal with the following equivalent version of (1):

$$(P'_\varepsilon)\begin{cases} -\Delta^2 u(x) + V_\varepsilon(x)u(x) - |u(x)|^{p-1}u(x) = 0, & x \in \mathbb{R}^N, \\ \qquad\qquad\qquad\qquad\qquad\qquad u(x) \longrightarrow 0, & \text{as } |x| \longrightarrow +\infty. \end{cases}$$

We consider the associated functional $I_\varepsilon : \mathcal{M}_\varepsilon \subseteq \mathrm{H}^2_\varepsilon \longrightarrow \mathbb{R}$, given by

$$I_\varepsilon(u) = \frac{1}{2}\int_{\mathbb{R}^N} \left[|\Delta u(x)|^2 + V_\varepsilon(x)|u(x)|^2\right] dx,$$

defined on the manifold

$$\mathcal{M}_\varepsilon = \left\{u \in \mathrm{H}^2_\varepsilon \Big/ \int_{\mathbb{R}^N} |u(x)|^{p+1} dx = 1\right\},$$

where

$$\mathrm{H}^2_\varepsilon = \left\{u \in \mathrm{H}^2\left(\mathbb{R}^N\right) \Big/ \|u\|_\varepsilon = \left(\int_{\mathbb{R}^N} \left[|\Delta u(x)|^2 + V_\varepsilon(x)|u(x)|^2\right] dx\right)^{1/2} < +\infty\right\}.$$

We apply a Ljusternik-Schnirelman scheme to show that the set of critical points has infinite elements, this is done through scaling for get weak solutions of (P'_ε).

Keywords: Nonlinear Schrödinger equation, biharmonic Laplace operator, multiplicity of solutions, Ljusternik-Schnirelmann scheme, Partial Differential Equations.

Contents

Chapter 1

Introduction

In 1924 Louis de Broglie introduced the idea that particles, such as electrons, could be described not only as particles but also as waves. This was motivated by the way streams of electrons are reflected against crystals and spread through thin metal foils, [17]. The physicist Erwin Schrödinger (in 1925) adjusts de Broglie's theory and assigns a wave function to every quantum object. The evolution of a wave function, in the simplest situation, is described by a partial differential equation (PDE) called the linear Schrödinger equation,

$$i\hbar\frac{\partial}{\partial t}\psi(x,t)+\frac{\hbar^2}{2m}\Delta\psi(x,t)-V(x)\psi(x,t)=0,$$

where $i=\sqrt{-1}$, ψ is a state function, $x\in\mathbb{R}^N, t\geqslant 0; \hbar\approx 6.6255\times 10^{-27}$ Hz is the reduced Plank's constant, m is the particle's mass, $\Delta=\sum_{j=1}^{N}\frac{\partial^2}{\partial x_j^2}$ is the Laplace operator, and the real valued function V is the potential.

A typical form of the nonlinear Schrodinger equation is

$$i\hbar\frac{\partial\psi}{\partial t}(x,t)+\frac{\hbar^2}{2}\Delta\psi(x,t)-V_0(x)\psi(x,t)+|\psi(x,t)|^{p-1}\psi(x,t)=0, \tag{1.1}$$

where $p>1$. This is applied in several areas of physics, [26], like nonlinear optics to study light propagation in cubically nonlinear media, optical effects like four-wave mixing and self-(de)focusing, the presence of solitons, etc.

Additionally, the dynamics of a Bose-Einstein condensate at absolute zero is described by the Gross-Pitaevskii equation ($p=3$), [26]. In this context it is relevant to study the existence and properties of standing wave solutions of (1.1); in other words, solutions whose form is

$$\psi(x,t)=e^{-iEt/\hbar}v(x), \tag{1.2}$$

where

$$E=\inf_{x\in\mathbb{R}^{\mathbb{N}}}V_0(x)$$

is referred to as the energy.

Let's put $V=V_0-E$ and consider the critical frecuency situation, i.e., $\mathcal{Z}=\{V=0\}\neq\emptyset$. In [5] it is proved the existence of v_ε, a positive standing wave least energy solution for

$$\begin{cases} -\varepsilon^2\Delta v(x) + V(x)v(x) - |v(x)|^{p-1}v(x) &= 0, \qquad x \in \mathbb{R}^N, \\ v(x) &\longrightarrow 0, \quad \text{as } |x| \longrightarrow +\infty, \end{cases} \tag{1.3}$$

where the following behavior is verified

1. $$\lim_{\varepsilon\to 0} \|v_\varepsilon\|_{\mathrm{L}^\infty(\mathbb{R}^N)} = 0,$$

$$\lim_{\varepsilon\to 0}\inf \frac{\|v_\varepsilon\|_{\mathrm{L}^\infty(\mathbb{R}^N)}}{\varepsilon^{2/(p-1)}} > 0;$$

2. v_ε concentrates around an isolated component of $\mathcal{Z}$;
3. v_ε exponentially decays outside $\mathcal{Z}$;
4. There is not a unique limit problem and, therefore, neither is a unique profile. They depend on the behavior of V nearby $\mathcal{Z}$. Three cases were considered:

   Flat: $\mathrm{int}\mathcal{Z} = \mathcal{Z}$ is bounded;
   Finite: $\mathcal{Z}$ is finite and V vanishes polinomially around it;
   Infinite: $\mathcal{Z}$ is finite and V vanishes exponentially around it.

For these cases it was also proved that:

5. a scaling of the positive least-energy solution v_ε converges strongly to u, a positive least-energy solution of a corresponding limit problem;
6. the energy of v_ε converges to the energy of a solution of the corresponding limit problem.

In this work, we continue the study of the critical frequency situation, $\inf(V) = 0$, produced in [5], [1] and [13].

Let's consider the functional $I_\varepsilon : \mathcal{M}_\varepsilon \subseteq \mathrm{H}_\varepsilon \longrightarrow \mathbb{R}$, given by

$$I_\varepsilon(u) = \frac{1}{2}\int_{\mathbb{R}^N} \left[|\nabla u(x)|^2 + V_\varepsilon(x)|u(x)|^2\right] dx, \tag{1.4}$$

defined on the manifold

$$\mathcal{M}_\varepsilon = \left\{ u \in \mathrm{H}_\varepsilon \Big/ \int_{\mathbb{R}^N} |u(x)|^{p+1} dx = 1 \right\},$$

where

$$\mathrm{H}_\varepsilon = \left\{ u \in \mathrm{H}^1\left(\mathbb{R}^N\right) \Big/ \|u\|_\varepsilon = \left(\int_{\mathbb{R}^N} \left[|\nabla u(x)|^2 + V_\varepsilon(x)|u(x)|^2\right] dx \right)^{1/2} < +\infty \right\}.$$

In [13] and [1], for the Flat Case and the Infinite Case, respectively, the authors applied a Ljusternik-Schnirelman scheme to the even energy functional I_ε associated to (1.3) through rescaled solutions $\left(v_{k,\varepsilon}(x)\right)_{k\in\mathbb{N}} = u\left(\varepsilon^\alpha x\right)$. They proved that:

i) there exists a sequence of solutions $\left(v_{k,\varepsilon}\right)_{k\in\mathbb{N}}$ for (1.3);

ii) for a fixed k and as $\varepsilon \to 0$ the solution, not necessarily positive, $v_{k,\varepsilon}$ behaves like those found in [5]. That is, conditions 1), 2), 3), 5) and 6) hold.

This work considers a non-linear biharmonic Schrödinger equation, i.e., the Laplace operator Δ is replaced by $\Delta^2 = \Delta \circ \Delta$:

$$i\hbar\frac{\partial\psi}{\partial t}(x,t) + \frac{\hbar^4}{4}\Delta^2\psi(x,t) - V_0(x)\psi(x,t) + |\psi(x,t)|^{p-1}\psi(x,t) = 0. \tag{1.5}$$

Then, by substituting (1.2) in (1.5), we have that:

$$\begin{aligned}
&i\hbar\frac{\partial}{\partial t}e^{-iEt/\hbar}v(x) + \frac{\hbar^4}{4}\Delta^2 e^{-iEt/\hbar}v(x) - V_0(x)e^{-iEt/\hbar}v(x) + \left|e^{-iEt/\hbar}v(x)\right|^{p-1}e^{-iEt/\hbar}v(x) = 0\\
&\frac{\hbar^4}{4}e^{-iEt/\hbar}\Delta^2 v(x) - V_0(x)e^{-iEt/\hbar}v(x) + i\hbar v(x)\frac{\partial}{\partial t}e^{-iEt/\hbar} + \left|e^{-iEt/\hbar}\right|^{p-1}|v(x)|^{p-1}e^{-iEt/\hbar}v(x) = 0\\
&\frac{\hbar^4}{4}e^{-iEt/\hbar}\Delta^2 v(x) - V_0(x)e^{-iEt/\hbar}v(x) - i^2 Ev(x)\frac{\hbar}{\hbar}e^{-iEt/\hbar} + e^{-iqEt/\hbar}|v(x)|^{p-1}v(x) = 0\\
&\frac{\hbar^4}{4}\Delta^2 v(x) - V_0(x)v(x) + Ev(x) + |v(x)|^{p-1}v(x) = 0.
\end{aligned}$$

Then, by setting $V(x) = V_0(x) - E$, we see that v should satisfy the semilinear elliptic PDE,

$$\frac{1}{4}\hbar^4\Delta^2 v(x) - V(x)v(x) + |v(x)|^{p-1}v(x) = 0. \tag{1.6}$$

There exists considerable literature about the topic. For instance, under the non-critical condition

$$\forall x \in \mathbb{R}^N : E < V(x),$$

the authors in [14] used the Lyapunov-Schmidt reduction method to prove that, for sufficiently small $\hbar > 0$, there exists a solution $v_\hbar$ of (1.6) with

$$\liminf_{\hbar\to 0}\max_{x\in\mathbb{R}^N}|v_\hbar(x)| > 0,$$

which is concentrated around a non-degenerate critical point of V when $N = 1$, $p = 3$ and V is a bounded function. Further research was done e.g. in [2] and [30].

In this document we are interested in the problem:

$$\begin{cases} -\varepsilon^4\Delta^2 v(x) + V(x)v(x) - |v(x)|^{p-1}v(x) = 0, & x \in \mathbb{R}^N,\\ v(x) \longrightarrow 0, & \text{as } |x| \longrightarrow +\infty, \end{cases} \tag{1.7}$$

where $\varepsilon^4 = \dfrac{\hbar^4}{4}$. We assume that $1 < p+1 < 2^*$, where

$$2^* = \begin{cases} \dfrac{2N}{N-2}, & \text{if } N \geq 3;\\ +\infty, & \text{if } N = 1,2. \end{cases}$$

Note that the previous definition of 2^* has to see with the embedding theorems on the Sobolev space, for this see for example in [4].

We use the scaling $v(x) = u(\varepsilon^\alpha x)$ and $V_\varepsilon(x) = V\left(\varepsilon^{-\alpha} x\right)$, for $\varepsilon > 0$, $\alpha = -1$, and, in this way, we can focus on the following equivalent version of (1.7):

$$\begin{cases} -\Delta^2 u(x) + V_\varepsilon(x)u(x) - |u(x)|^{p-1}u(x) = 0, & x \in \mathbb{R}^N, \\ u(x) \longrightarrow 0, & \text{as } |x| \longrightarrow +\infty. \end{cases} \tag{P'_ε}$$

Moreover, we shall assume that
(V1) $V \in \mathrm{C}\left(\mathbb{R}^N\right)$ is a non-negative function,
(V2) $V(x) \longrightarrow +\infty$, as $|x| \longrightarrow +\infty$,
(V3) $\mathcal{Z} = \{V = \inf(V) = 0\} \neq \emptyset$.

As we already mentioned, associated to (P'_ε) is the functional I_ε which was given in (1.4). We will prove that the set of critical points of I_ε contains infinitely many elements which, up to a scaling, are weak solutions of (P'_ε).

The way we attack the problem is using a minimax scheme in the context of the Ljusternik-Schnirelmann theory, [23]. These kind of methods characterize a critical value c of a functional I as a minimax over a suitable class of sets S:

$$c = \inf_{A \in S} \max_{u \in A} I(u),$$

however there is no recipe for choosing S.

This document starts with a theoretical framework in Chapter 2. We introduce the concepts of linear, normed and Banach spaces. Then we introduce definitions of linearity and bounded of operators. Next, we present Lebesgue spaces. In the second section, we deal with some topics of PDEs, Sobolev spaces and embeddings. At the end, we deal with important tools of Calculus of Variations, that is, Gateaux and Fréchet differentiability and the Palais-Smale condition together with Krasnoselskii's genus.

In Chapter 3, we prove the main results of the research. We start by re-stating the problem and the Theorem 3.1 we shall apply. Next, we list and prove several propositions that prove that the hypotheses of the multiplicity theorem holds.

At the end, in Chapter 4, we state the conclusions of the current research and write some recommendations.

Chapter 2

Mathematical framework

In this chapter we introduce topics of Functional Analysis, PDE's, Nonlinear Analysis and Calculus of Variations which are relevant to our study.

2.1 Some topics of Functional Analysis

In this section we introduce several definitions and tools like normed spaces, continuity of operators, Lebesgue spaces and some important inequalities such as Cauchy-Bunyakovsky-Schwarz's, Hölder's and Minkowski's.

2.1.1 Normed linear spaces

For this section the main references are [20], [8], [12] and [22], were can be found all these contents fully explained. We assume that the reader has some knowledge of concepts related to topological spaces.

We call internal operation on a non-void set X to a function

$$\begin{aligned} + : & X \times X \longrightarrow X \\ & (u, v) \longmapsto u + v. \end{aligned}$$

On the other hand, given a non-void set A, an external operation on X, with help of A, is a function

$$\begin{aligned} \cdot : \quad & A \times X \longrightarrow X \\ & (\lambda, u) \longmapsto \lambda \cdot u. \end{aligned}$$

Let X be a non-void set and $+$ be an internal operation on X. We say that $(X, +)$ is a group iff it satisfies the following conditions:

1) **Additive asociativity.** $\forall x, y, z \in X : \quad x + (y + z) = (x + y) + z;$

2) **Additive neutral element.** $\exists 0 \in X, \forall x \in X : \quad x + 0 = 0 + x = x;$

3) **Additive inverses.** $\forall x \in X, \exists y \in X : \quad x + y = y + x = 0;$

$(X, +)$ it is said to be an Abelian group if, in addition, it verifies

4) **Additive commutativity.** $\forall x, y \in X : \quad x + y = y + x.$

Next, we state the concept of linear space. Let $(X, +)$ be an Abelian group and $\cdot$: $\mathbb{R} \times X \to X$ an external operation. We say that $(X, +, \cdot)$ is a linear space (over $\mathbb{R}$) iff it verifies the following properties:

1) **Mixed associativity.** $\forall x \in X, \forall \alpha, \beta \in \mathbb{R}: \quad \alpha \cdot (\beta \cdot x) = (\alpha \cdot \beta) \cdot x;$

2) **Scalar distributivity.** $\forall x, y \in X, \forall \alpha \in \mathbb{R}: \quad \alpha \cdot (x + y) = \alpha \cdot x + \alpha \cdot y;$

3) **Vector distributivity.** $\forall x \in X, \forall \alpha, \beta \in \mathbb{R}: \quad (\alpha + \beta) \cdot x = \alpha \cdot x + \beta \cdot x;$

4) **Harmlessness of 1.** $\forall x \in X: \quad 1 \cdot x = x.$

Before introducing the concept of normed space, it is important to take a look at metric spaces. Let X be a non-void set and $d : X \times X \to \mathbb{R}$. Then, (X, d) is a metric space iff

1) $\forall x, y \in X: \quad d(x, y) = 0 \Leftrightarrow x = y;$

2) $\forall x, y \in X: \quad d(x, y) = d(y, x);$

3) $\forall x, y, z \in X: \quad d(x, y) \leq d(x, z) + d(y, z).$

The real number $d(x, y)$ is referred to as the distance between x and y. Also notice that, these three properties imply that:

4) $\forall x, y \in X: \quad d(x, y) \geq 0.$

In a metric space (X, d), we call (open) ball with center $x_0 \in X$ and radius $r > 0$ to the set given by

$$B(x_0, r) = \{x \in X / d(x_0, x) < r\}.$$

Similarly, a closed ball is a set of the form

$$\overline{B}(x_0, r) = \{x \in X / d(x_0, x) \leq r\},$$

and a sphere is a set of the form

$$S(x_0, r) = \{x \in X / d(x_0, x) = r\}.$$

Let $\| \cdot \| : X \to \mathbb{R}$ be a function and X a linear space. Then, $(X, \| \cdot \|)$ is a normed linear space if it satisfies

N1) $\forall x \in X: \quad \|x\| \geq 0;$

N2) $\forall x \in X: \quad \|x\| = 0$ iff $x = 0;$

N3) $\forall x \in X, \forall \alpha \in \mathbb{R}: \quad \|\alpha x\| = |\alpha| \|x\|;$

N4) $\forall x, y \in X: \quad \|x + y\| \leq \|x\| + \|y\|$, (triangle inequality).

X becomes a metric space with $d : X \times X \longrightarrow \mathbb{R}$ defined by

$$d(x, y) = \|x - y\|.$$

Given two norms $\|\cdot\|$ and $\|\cdot\|_1$ on a vector space X, we call them equivalent iff

$$\exists c_1, c_2 > 0, \forall x \in \Omega : \quad c_1\|x\|_1 \leq \|x\| \leq c_2\|x\|_1. \tag{2.1}$$

Next, we introduce the concept of convergence in norm. Let $(X, \|\cdot\|)$ be a normed space. A sequence $(x_n)_{n\in\mathbb{N}} \subseteq X$, is called convergent if there exists $a \in X$ such that

$$\forall \varepsilon > 0, \exists N_0 \in \mathbb{N} : \quad n > N_0 \Rightarrow \|x_n - a\| < \varepsilon,$$

which is denoted by

$$\lim_{n\to+\infty} x_n = a,$$

as well as

$$x_n \longrightarrow a, \quad \text{as } n \longrightarrow +\infty.$$

Remark 2.1. The convergence in norm is called too by strong convergence.

A sequence $(x_n)_{n\in\mathbb{N}} \subseteq X$ is called a Cauchy sequence if

$$\forall \varepsilon > 0, \exists N_0 \in \mathbb{N} : \quad n, m > N_0 \Rightarrow \|x_n - x_m\| < \varepsilon.$$

Theorem 2.1. *Let X be a normed space. If the sequence $(x_n)_{n\in\mathbb{N}}$ converges to both $L_1 \in X$ and $L_2 \in X$, then $L_1 = L_2$, that is, the limit is unique.*

Proof. Aiming for a contradiction, we suppose that:

$$\lim_{n\to+\infty} x_n = L_1$$

and

$$\lim_{n\to+\infty} x_n = L_2$$

such that

$$L_1 \neq L_2.$$

Let $\varepsilon = \dfrac{\|L_1 - L_2\|}{3}$. From the norm axioms it follows that $\varepsilon > 0$, and by definition:

$$\exists N_1 \in \mathbb{N} : \forall n > N_1 : \|x_n - L_1\| < \varepsilon,$$
$$\exists N_2 \in \mathbb{N} : \forall n > N_2 : \|x_n - L_2\| < \varepsilon.$$

Choose $n > N_1 + N_2$. By the triangle inequality, it follows that

$$\begin{aligned} \|L_1 - L_2\| &= \|L_1 - x_n + x_n - L_2\| \\ &\leq \|L_1 - x_n\| + \|x_n - L_2\| \\ &< \varepsilon + \varepsilon \\ &= \frac{2}{3}\|L_1 - L_2\| \end{aligned}$$

which implies that $1 < \frac{2}{3}$, a contradiction. Hence $L_1 = L_2$.

□

A subset $B \subseteq X$ is bounded in X if there exists $M > 0$ such that $\|x\| \leq M$, for every $x \in B$. In a similar way, a sequence $(x_n)_{n\in\mathbb{N}} \subseteq X$ is called bounded if

$$\sup_{n\in\mathbb{N}} \|x_n\| < +\infty.$$

Definition 2.2. [Complete normed spaces] We say that a normed space $(X, \|\cdot\|)$ is a Banach space if it's complete, i.e., if every Cauchy sequence is convergent.

Given two normed spaces $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ where $X \subseteq Y$, X is said to be continuously embedded in Y iff

$$\exists C > 0, \forall x \in X : \quad \|u\|_Y \leq C\|u\|_X \tag{2.2}$$

and it is denoted by $X \hookrightarrow Y$. Also, X is said to be compactly embedded in Y, denoted by $X \subset\subset Y$, if $X \hookrightarrow Y$ and all bounded sequence $(u_m)_{m\in\mathbb{N}} \subseteq X$ has a convergent subsequence in Y.

2.1.2 Bounded linear operators

Let E and F be two linear spaces. An operator S is a linear aplication such that $S : E \to F$, for every $x \in E$ and with $S(x) \in F$. We say that the operator $S : E \to F$ is linear iff

$$\forall x, y \in E, \forall \lambda \in \mathbb{R} : \quad S(\lambda x + y) = \lambda S(x) + S(y).$$

Then, we define a linear space

$$L(E, F) = \{S : E \to F / S \text{ is linear}\},$$

where, for every $U, V \in L(E, F)$ and $\lambda \in \mathbb{R}$, we use the natural definitions:

$$(U + V)(x) = U(x) + V(x), \quad x \in E,$$
$$(\lambda U)(x) = \lambda U(x), \quad x \in E.$$

Let E and F be normed spaces. An operator $S \in L(E, F)$ is bounded if there exists $c > 0$ such that

$$\forall x \in E : \quad \|S(x)\|_F \leq c\|x\|_E.$$

An operator $T \in L(E, F)$ is continuous at given $x, y \in E$, if

$$\forall \varepsilon > 0, \exists \delta > 0 : \|x - y\|_E < \delta \Rightarrow \|T(x) - T(y)\|_F < \varepsilon.$$

Then, we define

$$\mathcal{L}(E, F) = \{S \in L(E, F)/S \text{ is bounded }\},$$

which becomes a normed space with the norm $\|\cdot\| : \mathcal{L}(E, F) \to \mathbb{R}$, given by

$$\|S\| = \sup_{x\in E} \frac{\|Sx\|}{\|x\|} = \sup_{\|x\|=1} \|Sx\|_F. \tag{2.3}$$

The norm (2.3) can also be computed in the following way

$$\|T\| = \inf\{\mathcal{O}_T\} \tag{2.4}$$

where

$$\mathcal{O}_T = \{c > 0/ \quad \forall x \in E : \quad \|T(x)\|_F \leq c\|x\|_E\}.$$

Theorem 2.3. *Let E and F be normed spaces and $T \in \mathcal{L}(E, F)$. Then,*

$$\forall u \in E : \|T(u)\| \leq \|T\|\|u\|. \tag{2.5}$$

Proof. Let $u \in E$, generic. Then, for each $c \in \mathcal{O}_T$, it holds

$$\forall u \in E, \quad \|T(u)\| \leq c\|u\|. \tag{2.6}$$

Then, by taking infimum in (2.6), it follows that

$$\forall u \in E, \quad \|T(u)\| \leq \inf_{c \in \mathcal{O}_T} c \cdot \|u\|,$$

that is,

$$\forall u \in E, \quad \|T(u)\| \leq \|T\| \cdot \|u\|.$$

So, we have proved (2.5). □

Proposition 2.1.1. *The operator $\|\cdot\| : \mathcal{L}(E, F) \to \mathbb{R}$ where*

$$\forall S \in \mathcal{L}(E, F) : \quad \|S\| = \sup_{x \in E} \frac{\|Sx\|}{\|x\|} = \sup_{\|x\|=1} \|Sx\|_F,$$

satisfies the properties of a norm.

Proof. 1. Let's prove that

$$\forall S \in \mathcal{L}(E, F) : \quad \|S\| = 0 \Longleftrightarrow S = 0. \tag{2.7}$$

Let $S \in \mathcal{L}(E, F)$ be generic.

a) Let's assume that $\|S\| = 0$, then

$$\|S\|_F = 0,$$

then $S(u) = 0$, for every $u \in E$, i.e., $S = 0$.

b) Let's assume that $S = 0$. Therefore, $\|S\|_F = 0$.

Since S was chosen arbitrarily, we have proved (2.7).

c) Also, let's see that $\|S\| \geq 0$, in fact by the definition of $\|S\|$, (2.3), we have for every $u \in E$ and for every $\|u\| = 1$,

$$\|S(u)\|_F \geq 0,$$

then

$$\sup_{\|u\|=1} \|S(u)\|_F \geq 0.$$

2. Let's prove that

$$\forall \lambda \in \mathbb{R}, \forall S \in \mathcal{L}(E, F) : \quad \|\lambda \cdot S\| = |\lambda| \cdot \|S\|. \tag{2.8}$$

Let $\lambda \in \mathbb{R}$ and $S \in \mathcal{L}(E, F)$, be generic. If $\lambda = 0$, then (2.8) is trivial. Assume that $\lambda \neq 0$. Then,

$$\|\lambda \cdot S\| = \sup_{\|u\|=1} \|\lambda S(u)\| = |\lambda| \sup_{\|u\|=1} \|S(u)\| = |\lambda| \cdot \|S\|.$$

Since S was chosen arbitrarily, then we have proved (2.8).

3. Let's prove that

$$\forall T, S \in \mathcal{L}(E,F): \quad \|T+S\| \leq \|T\| + \|S\|. \tag{2.9}$$

Let $T, S \in \mathcal{L}(E,F)$ be generic. Let $u \in E$, be generic. Then, by Theorem 2.3, we get

$$\begin{aligned}\|(T+S)(u)\| &= \|T(u)+S(u)\| \leq \|T(u)\| + \|S(u)\| \\ &\leq \|T\|\|u\| + \|S\|\|u\| = (\|T\|+\|S\|)\,\|u\|.\end{aligned}$$

By using (2.4) and since T and S were chosen arbitrarily, we have proved (2.9).

□

Theorem 2.4. *Let $X \neq \{0\}$ be a normed space and Y be a normed space. Then $\mathcal{L}(X,Y)$ is a Banach space, if and only if, Y is a Banach space.*

Next, we prove only the direction if of the previous result, for more details see [20].

Proof. We have to prove that every Cauchy sequence of $\mathcal{L}(X,Y)$ is convergent.

1. Let $(T_n)_{n\in\mathbb{N}} \subseteq \mathcal{L}(X,Y)$ be a Cauchy sequence. Then for every $\varepsilon > 0$, there is $N \in \mathbb{N}$ such that $m, n > N$ implies that $\|T_n - T_m\| < \varepsilon$. Let $u \in X$ be generic, then

$$\|T_n(u) - T_m(u)\| = \|(T_n - T_m)(u)\| \leq \|T_n - T_m\|\,\|u\| < \varepsilon\|u\|, \tag{2.10}$$

then the sequence $(T_n(u))_{n\in\mathbb{N}} \subseteq Y$ is a Cauchy sequence. Since Y is complete, for every $u \in X$, there is a unique element $T(u) \in Y$ such that

$$\lim_{n\to+\infty} T_n(u) = T(u). \tag{2.11}$$

2. By (2.11), it follows that the operator $T : X \longrightarrow Y$ is linear. In fact, let $u, v \in X$ and $\lambda \in \mathbb{R}$ be generic. We have, by the linearity of T_n, that

$$\begin{aligned}T(\lambda u + v) &= \lim_{n\to+\infty} T_n(\lambda u + v) = \lim_{n\to+\infty} [\lambda T_n(u) + T_n(v)] \\ &= \lambda \lim_{n\to+\infty} T_n(u) + \lim_{n\to+\infty} T_n(v) = \lambda T(u) + T(v).\end{aligned} \tag{2.12}$$

Since u, v and λ were chosen arbitrarily, we have proved that T is linear.

3. Let's retake point 1 . By (2.12), we can let $m \longrightarrow +\infty$ in (2.10) to obtain, for $n > N$ and all $u \in X$,

$$\|(T_n - T)(u)\| \leq \varepsilon\|u\|, \tag{2.13}$$

which implies that $T_n - T \in \mathcal{L}(X,Y)$, for $n > N$. Since $\mathcal{L}$ is a linear space and $T_n \in \mathcal{L}(X,Y)$, it follows that $T \in \mathcal{L}(X,Y)$.
Since all Cauchy sequence is bounded, there exists a constant $M > 0$ such that

$$\|T(u)\| \leq \|T_n(u)\| + \varepsilon\|u\| \leq (M+\varepsilon)\|u\|,$$

then, T is bounded.

4. Let $\varepsilon > 0$ be generic. By (2.13), we have, for $n > N$, that

$$\|T_n - T\| < \varepsilon.$$

Since ε was chosen arbitrarily, this proves that $T_n \longrightarrow T$, as $n \longrightarrow +\infty$.

□

Remark 2.2. As a consequence of the previous theorem, the dual space of X,

$$X' := \mathcal{L}(X, \mathbb{R}),$$

is a Banach space; its norm, called the dual norm, can also be computed as

$$\|T\|_{X'} = \sup_{\|x\| \neq 0} \frac{|T(x)|}{\|x\|_X} = \sup_{\|x\|=1} |T(x)|.$$

Let X be a normed space. For $x \in X$ and $\nu \in X'$, we denote

$$\langle \nu, x \rangle = \nu(x),$$

and call $\langle \cdot, \cdot \rangle$ the dual product.

Theorem 2.5. *Let X and Y be normed spaces and $T \in \mathcal{L}(X, Y)$. Then T is bounded iff T is continuous.*

Proof. The case of $T = 0$ is trivial. Thus, let's assume that $T \neq 0$.
$\Rightarrow$) Suppose that T is a bounded linear operator. Then there exists $M > 0$ such that for all $x \in X$ we have

$$\|T(x)\| \leq M\|x\|.$$

Let's prove that T is continuous at given $x, y \in X$, that is,

$$\forall \varepsilon > 0, \exists \delta > 0 : \|x - y\| < \delta \Rightarrow \|T(x) - T(y)\| < \varepsilon.$$

Consider $\varepsilon > 0$ and define $\delta = \dfrac{\varepsilon}{M} > 0$. Then, if $\|x - y\| < \delta$, we have that:

$$\|T(x) - T(y)\| = \|T(x - y)\| \leq \|T\|\|x - y\| \leq M\|x - y\| < M\delta = M \cdot \frac{\varepsilon}{M} = \varepsilon.$$

Therefore T is continuous.
$\Leftarrow$) Suppose that $T : X \to Y$ is continuous on X. We have to prove that:

$$\exists c > 0, \forall x \in X : \|Tx\| \leq c\|x\|. \tag{2.14}$$

Since T is continuous at $0 \in X$, for $\varepsilon_0 = 1 > 0$ there exists $\delta > 0$ such that, if $\|x\| = \|x - 0\| < \delta$, then

$$\|T(x)\| = \|T(x) - T(0)\| < 1.$$

If $x \neq 0$ and $\delta > 0$ then

$$\left\| \frac{\delta x}{2\|x\|} \right\| = \frac{\delta}{2} \frac{\|x\|}{\|x\|} < \delta.$$

Also, if

$$\overline{x} = \frac{\delta}{2}\frac{x}{\|x\|}$$

then,

$$\|\overline{x}\| < \delta.$$

Thus

$$\left\| T\left(\frac{\delta x}{2\|x\|}\right) \right\| \leq 1.$$

Hence, for all $x \in X$ with $x \neq 0$, we get

$$\frac{\delta}{2}\frac{\|T(x)\|}{\|x\|} \leq 1 \quad \text{iff} \quad \|T(x)\| \leq \frac{2}{\delta}\|x\|.$$

Then, (2.21) holds for

$$c = \frac{2}{\delta} > 0.$$

□

2.1.3 Weak and weak * convergence

In this section we make a review of weak and weak * convergence and some related topics having as references [20], [8] and [12].

Proposition 2.1.2. *Let $(Y, \mathcal{T})$ be a topological space and W a set equipped with the initial topology Γ, generated by the functions*

$$f_\lambda : W \to Y, \quad \lambda \in \Lambda.$$

Then, a sequence $(x_n)_{n\in\mathbb{N}} \subseteq W$ converges to $x \in W$ in Γ iff

$$\forall \lambda \in \Lambda : \lim_{n\to+\infty} f_\lambda(x_n) = f_\lambda(x).$$

Let X be a normed space. Using the dual we can create two useful topologies, the weak topology on X and the weak * topology on X'. As mentioned in [12], let's denote by $\sigma(X, X')$ the initial topology associated to the family $(\ell)_{\ell\in X'}$, i.e. the weakest topology where each map $\ell \in X'$ is still continuous. We call $\sigma(X, X')$ the weak topology on X.

Let $(x_n)_{n\in\mathbb{N}}$ be a sequence in X. It converges in the topology $\sigma(X, X')$ to some $x \in X$ if and only if

$$\forall \ell \in X' : \quad \lim_{n\to+\infty} \langle \ell, x_n \rangle = \langle \ell, x \rangle. \tag{2.15}$$

The previous result is a consequence of the Proposition 2.1.2 which in its turn is derived from Corollary 6.4 in [12].

Let's consider the canonical mapping:

$$\begin{aligned} J : & X \longrightarrow X'' \\ & u \longmapsto J(u) = \phi_u, \end{aligned}$$

where X'' is the double dual of X and where $\phi_u : X' \to \mathbb{R}$ is defined by

$$\forall \ell \in X' : \langle \phi_u, \ell \rangle_{X'',X'} = \langle \ell, u \rangle_{X',X}.$$

Proposition 2.1.3. *Let $l \in X'$ and let $u \in X$. Then, J is an isometry.*

Proof. We have to prove that $\|J(u)\| = \|l(u)\| = \|u\|$ for each $u \in X$.
In fact, we have that

$$\|u\|_{X''} = \sup_{l \in X'} \frac{|l(u)|}{\|l\|} \leq \sup_{l \in X'} \frac{\|l\| \cdot \|u\|}{\|l\|} = \|u\|,$$

thus $\|u\|_{X''} \leq \|u\|$. But now, by the Hahn-Banach theorem, (for more details of this see [4]), there exists $g \in X'$ such that $g(u) = 1$ and $\|g\| = \dfrac{1}{\|u\|}$, so that

$$\|u\|_{X''} = \sup_{l \in X'} \frac{|l(u)|}{\|l\|} \geq \frac{1}{\|g\|} = \|u\|,$$

and we are done.

□

The weak * topology, $\sigma(X', X)$, is the initial topology associated to the family $(\phi_u)_{u \in X}$.

A sequence ℓ_n converges to ℓ in $\sigma(X', X)$ if and only if for all $u \in X$

$$\lim_{n \to +\infty} \langle \ell_n, u \rangle = \langle \ell, u \rangle$$

this is called weak * convergence. The previous result is a consequence of the Proposition 2.1.2 which in its turn is derived from Corollary 6.4 in [12].

Definition 2.6. (Weak convergence). Let X be a normed space. There exist a sequence $(u_k)_{k \in \mathbb{N}} \subseteq X$ and some $u \in X$ such that $u_k \rightharpoonup u$, as $k \longrightarrow +\infty$, i.e.,

$$\forall \nu \in X' : \langle \nu, u_k - u \rangle \longrightarrow 0 \quad as \quad k \longrightarrow +\infty. \tag{2.16}$$

Remark 2.3. If J is surjective, then the space X is called reflexive. In this case, $\sigma(X', X) = \sigma(X', X'')$.

Remark 2.4. Note that $\sigma(X', X)$ has fewer open sets than the topology $\sigma(X', X'')$, which in its turn has fewer open sets than the strong topology of X'.

Now, we have that strong convergence implies weak convergence.

Theorem 2.7. *(Strong and weak convergence). Let E be a normed space, $(x_n)_{n \in \mathbb{N}} \subseteq E$ be a sequence that converges strongly. Then, $(x_n)_{n \in \mathbb{N}}$ converges weakly with the same limit. That is, strong convergence implies weak convergence.*

The proof of the previous result can be found e.g. in [4].

Now, we present two theorems that involve reflexivity in Banach spaces.

Theorem 2.8. *Let X be a Banach space. Then X is reflexive if and only if the set*

$$\{x \in X : \|x\| \leqslant 1\}$$

is compact in the weak topology $\sigma(X, X')$.

The proof of the previous result can be found e.g. in [20].

Corollary 2.1.1. *Assume that X is a reflexive Banach space and let $(x_n)_{n\in\mathbb{N}}$ be a bounded sequence in X. Then there exists a subsequence $\left(x_{n_k}\right)_{k\in\mathbb{N}}$ that converges in the weak topology $\sigma(X, X')$.*

The proof of the previous result follows immediately from Theorem 2.8.

2.2 Lebesgue spaces

In this section we introduce Lebesgue spaces as well as some inequalities and other results that are used in this document. We assume that the reader is familiar with concepts from measure theory such as $\sigma-$ algebra, measure space, measurable set, measurable function, Lebesgue measure, and integrable function. All these concepts are taken from [6], [16], [20] and [21], where they are fully explained.

Let $\Omega \subseteq \mathbb{R}^N$ be open. We denote by $\tilde{\mathrm{L}}^1(\Omega)$ the space of integrable functions on Ω:

$$\tilde{\mathrm{L}}^1(\Omega) := \left\{ f : \Omega \to \mathbb{R} \text{ measurable } / \quad \|f\|_{\mathrm{L}^1(\Omega)} := \int_\Omega |f(x)| dx < +\infty \right\}.$$

If $1 \leq p < +\infty$, the space of p integrable functions is

$$\tilde{\mathrm{L}}^p(\Omega) := \left\{ f : \Omega \to \mathbb{R} / f \text{ is measurable and } |f|^p \in \tilde{\mathrm{L}}^1(\Omega) \right\}.$$

Finally, if we take $p = \infty$

$$\tilde{\mathrm{L}}^\infty(\Omega) := \left\{ f : \Omega \to \mathbb{R} \;\middle|\; \begin{array}{l} f \text{ is measurable and} \\ \exists C > 0 : \quad |f(x)| \leq C \text{ a.e. on } \Omega \end{array} \right\}$$

and

$$\|f\|_{\tilde{\mathrm{L}}^\infty(\Omega)} := \inf\{C > 0 : \quad |f(x)| \leq C \text{ a.e. on } \Omega\}.$$

Let's consider the following equivalence relation on $\tilde{\mathrm{L}}^p(\Omega), 1 \leq p < \infty$,

$$f \sim g \quad \text{iff} \quad \|g - f\|_{\tilde{\mathrm{L}}^p(\Omega)} = 0. \tag{2.17}$$

In definition to (2.17), we denote by $[f]$ the equivalence class associated to $f \in \tilde{\mathrm{L}}^p(\Omega)$. The Lebesgue space is

$$\mathrm{L}^p(\Omega) = \{[f] / f \in \tilde{\mathrm{L}}^p(\Omega)\}.$$

From now on, we take a representative element of each equivalence class $[f] \in \mathrm{L}^p(\Omega)$ and consider $[f] = f$ as an abuse of notation. The norm corresponding to the Lebesgue space for $1 \leq p < +\infty$, are defined as $\|\cdot\|_{\mathrm{L}^p(\Omega)} : \mathrm{L}^p(\Omega) \longrightarrow \mathbb{R}$,

$$\|f\|_{\mathrm{L}^p} = \left(\int_\Omega |f(x)|^p dx \right)^{\frac{1}{p}}.$$

Finally, if we take $p = +\infty$, then

$$\mathrm{L}^\infty(\Omega) := \left\{ f : \Omega \to \mathbb{R} \;\middle|\; \begin{array}{l} f \text{ is measurable and} \\ \exists t > 0 : \quad |f(x)| \le t \text{ a.e. on } \Omega \end{array} \right\},$$

and

$$\|f\|_{\mathrm{L}^\infty} = \operatorname*{ess\,sup}_{x\in\Omega} |f(x)| = \inf\{t > 0 : \mu(\{x : f(x) > t\}) = 0\},$$

where μ denotes the Lebesgue's measure.

Remark 2.5. The function $\|\cdot\|_{\mathrm{L}^p(\Omega)}$ defines a seminorm on $\tilde{\mathrm{L}}^p(\Omega)$, but as we will see in Theorem 2.10, using the equivalence relation (2.17) it defines a norm on $\mathrm{L}^p(\Omega)$.

Definition 2.9. (Conjugate exponent). Let $1 < p < +\infty$. We say that p' is the conjugate of p, if it satisfies

$$\frac{1}{p} + \frac{1}{p'} = 1.$$

Additionally, in the case that $p = 1$ its conjugate is $p' = +\infty$ and viceversa.

Remark 2.6. (Young's inequality). Let $1 < p < +\infty$. Then, the following inequality holds

$$\forall a, b \ge 0 : \quad ab \le \frac{a^p}{p} + \frac{b^{p'}}{p'}. \tag{2.18}$$

Inequality (2.17) is a straightforward consequence of the concavity of the log function on $(0, \infty)$:

$$\log\left(\frac{1}{p}a^p + \frac{1}{p'}b^{p'}\right) \ge \frac{1}{p}\log a^p + \frac{1}{p'}\log b^{p'} = \log ab. \tag{2.19}$$

Theorem 2.10. *(Hölder's inequality). Assume that* $1 \le p \le +\infty$. *Let* $f \in \mathrm{L}^p(\Omega)$, *and* $g \in \mathrm{L}^{p'}(\Omega)$. *Then* $fg \in \mathrm{L}^1(\Omega)$ *and*

$$\|fg\|_{\mathrm{L}^1(\Omega)} \le \|f\|_{\mathrm{L}^p(\Omega)} \|g\|_{\mathrm{L}^{p'}(\Omega)}.$$

Proof. The cases $p = 1$ and $p = +\infty$ are immediate. Also, it's assumed that when $p = 1$ we have that $p' = +\infty$ and viceversa; therefore we assume that $1 < p < \infty$. We have, by Young's inequality, that

$$|f(x)g(x)| \le \frac{1}{p}|f(x)|^p + \frac{1}{p'}|g(x)|^{p'} \text{ a.e. } x \in \Omega. \tag{2.20}$$

It follows that $fg \in \mathrm{L}^1(\Omega)$ and

$$\int_\Omega |f(x)g(x)|dx \le \frac{1}{p}\|f\|_p^p + \frac{1}{p'}\|g\|_{p'}^{p'}. \tag{2.21}$$

Replacing f by $\lambda f, (\lambda > 0)$, in (2.21), yields

$$\int_\Omega |f(x)g(x)|dx \le \frac{\lambda^{p-1}}{p}\|f\|_p^p + \frac{1}{\lambda p'}\|g\|_{p'}^{p'}. \tag{2.22}$$

Choosing $\lambda = \|f\|_p^{-1}\|g\|_p^{p'/p}$, we get the result.

□

Remark 2.7. Let $a, b \in \mathbb{R}$ and $1 \leq p < +\infty$, then

$$|a+b|^p \leq 2^p(|a|^p + |b|^p).$$

A proof can be found e.g. in [4].

Theorem 2.11. *Let* $1 \leq p \leq +\infty$. *Then* $\mathrm{L}^p(\Omega)$ *is a normed linear space where* $\|\cdot\|_p$ *is a norm. In particular, for* $f, g \in \mathrm{L}^p(\Omega)$,

$$\|f+g\|_{\mathrm{L}^p(\Omega)} \leq \|f\|_{\mathrm{L}^p(\Omega)} + \|g\|_{\mathrm{L}^p(\Omega)}.$$

Proof. 1) For $p = +\infty$; let $f \in \mathrm{L}^\infty(\Omega)$ be generic and let $x \in \Omega$ be generic.
By the properties of a norm, we have that:
for (N1), it's immediate that

$$\sup_{x\in\Omega} |f(x)| \geq 0.$$

For (N2), let $\lambda \in \mathbb{R}$. Then,

$$\|\lambda \cdot f\|_{\mathrm{L}^\infty(\Omega)} = \sup_{x\in\Omega} |\lambda f(x)| = |\lambda| \sup_{x\in\Omega} |f(x)| = |\lambda| \cdot \|f\|_{\mathrm{L}^\infty(\Omega)}.$$

For (N3);
$\Rightarrow$) Let's assume that $\|f\| = 0$. Then,

$$\|f\|_{\mathrm{L}^\infty(\Omega)} = 0,$$

then, $f(x) = 0$, for every $x \in \Omega$, i.e., $f(x) = 0$.
$\Leftarrow$) If $f = 0$ then $\|f\|_{\mathrm{L}^\infty(\Omega)} = 0$.
For (N4), we have

$$|f(x) + g(x)| \leq \max(|f(x)|, |g(x)|) \leq (|f(x)| + |g(x)|).$$

By taking the supremum on the previous inequality, we get that

$$\|f+g\|_{\mathrm{L}^\infty(\Omega)} \leq \|f\|_{\mathrm{L}^\infty(\Omega)} + \|g\|_{\mathrm{L}^\infty(\Omega)}.$$

2) Let $1 \leq p < +\infty$ and $f, g \in \mathrm{L}^p(\Omega)$, generic. It is immediate that

$$\int_\Omega |f(x)|^p dx \geq 0.$$

$\Rightarrow$) Let's assume that $\|f\| = 0$. Then,

$$\|f\|_{\mathrm{L}^p(\Omega)} = 0,$$

then, $f(x) = 0$, for every $x \in \Omega$, i.e., $f(x) = 0$.
$\Leftarrow$) If $f = 0$ then $\|f\|_{\mathrm{L}^p(\Omega)} = 0$.
Then $f \in \mathrm{L}^p(\Omega)$ satisfies $(N1)$ and $(N2)$.
By taking $\alpha \in \mathbb{R}$, generic, we have that

$$\int_\Omega |\alpha f(x)|^p dx = |\alpha|^p \int_\Omega |f(x)|^p dx = |\alpha|^p \|f\|_{\mathrm{L}^p(\Omega)},$$

satisfying $(N3)$.
In the case $p = 1$; for $x \in \Omega$ and by the triangle inequality, we have that

$$\|f + g\|_{\mathrm{L}^1(\Omega)} \leq \|f\|_{\mathrm{L}^1(\Omega)} + \|g\|_{\mathrm{L}^1(\Omega)}$$

.
For the triangle inequality (N4) and by Remark 2.7 , for $1 < p < +\infty$, it is needed to show that $f + g$ belongs to $\mathrm{L}^p(\Omega)$. For $x \in \Omega$ we have

$$\begin{aligned} |f(x) + g(x)|^p &\leq (2\max(|f(x)|, |g(x)|))^p \\ &\leq 2^p \max(|f(x)|^p, |g(x)|^p) \\ &\leq 2^p(|f(x)|^p + |g(x)|^p). \end{aligned}$$

The previous inequality implies that $f + g \in \mathrm{L}^p(\Omega)$. Since $p' = \dfrac{p}{p-1}$ and $(p-1)p' = p$

$$|f(x)+g(x)|^p = |f(x)+g(x)||f(x)+g(x)|^{p-1} \leq |f(x)||f(x)+g(x)|^{p-1}+|g(x)||f(x)+g(x)|^{p-1}.$$

Notice that $|f(x) + g(x)|^{p-1} \in \mathrm{L}^{p'}(\Omega)$, since $(p-1)p' = p$, and $\left\| |f(x) + g(x)|^{p-1} \right\|_{\mathrm{L}^{p'}(\Omega)} = \|f + g\|^{p/p'}_{\mathrm{L}^p(\Omega)}$. By using Hölder's inequality, we have

$$\begin{aligned} \|f + g\|^p_{\mathrm{L}^p(\Omega)} &\leq \|f\|_{\mathrm{L}^p(\Omega)} \|f + g\|^{\frac{p}{p'}}_{\mathrm{L}^p(\Omega)} + \|g\|_{\mathrm{L}^p(\Omega)} \|f + g\|^{\frac{p}{p'}}_{\mathrm{L}^p(\Omega)} \\ &= \left(\|f\|_{\mathrm{L}^p(\Omega)} + \|g\|_{\mathrm{L}^p(\Omega)}\right) \|f + g\|^{\frac{p}{p'}}_{\mathrm{L}^p(\Omega)}. \end{aligned}$$

Dividing the previous inequality by $\|f + g\|^{p/p'}_{\mathrm{L}^p(\Omega)}$ we get

$$\|f + g\|^{p-\frac{p}{p'}}_{\mathrm{L}^p(\Omega)} \leq \|f\|_{\mathrm{L}^p(\Omega)} + \|g\|_{\mathrm{L}^p(\Omega)}.$$

Since $p - \dfrac{p}{p'} = p\left(1 - \dfrac{1}{p'}\right) = 1$, given the arbitrariness of f, g and α the proof is done. □

Next, we make a review of the main theorems relevant to Lebesgue spaces.

Theorem 2.12. *(Monotone convergence theorem). Let $(f_n)_{n\in\mathbb{N}} \subseteq \mathrm{L}^1(\Omega)$ be a sequence that satisfies*

a) (f_n) is an increasing sequence,

b) $\sup\limits_{n\in\mathbb{N}} \displaystyle\int_\Omega f_n(x)dx < +\infty$.

Then, there exists $f \in \mathrm{L}^1(\Omega)$, such that,

$$\lim_{n\to+\infty} f_n(x) = f(x), \qquad a.e\ x \in \Omega,$$

and

$$\|f_n - f\|_{\mathrm{L}^1(\Omega)} \longrightarrow 0, \qquad as\ n \longrightarrow +\infty.$$

Theorem 2.13. *(Dominated convergence theorem). Let* $(f_n)_{n\in\mathbb{N}} \subseteq \mathrm{L}^1(\Omega)$ *such that* $f_n(x) \to f(x)$*, for a.e.* $x \in \Omega$*, as* $n \longrightarrow +\infty$*. Assume that there exists* $\varphi \in \mathrm{L}^1(\Omega)$ *that*

$$\forall n \in \mathbb{N}: \quad |f_n(x)| \leq \varphi(x), \qquad a.e.\ \Omega,$$

Then, $f \in \mathrm{L}^1(\Omega)$ *and*

$$\|f_n - f\|_{\mathrm{L}^1(\Omega)} \longrightarrow 0, \ \text{as } n \longrightarrow +\infty.$$

Lemma 2.14. *(Fatou's lemma). Let* $(f_n)_{n\in\mathbb{N}} \subseteq \mathrm{L}^1(\Omega)$ *be a sequence of functions that satisfy*

i) $\forall n \in \mathbb{N} : f_n > 0$*;*

ii) $\sup_{n\in\mathbb{N}} \int_\Omega f_n(x)dx < +\infty$*.*

Then, the function given by $f(x) = \liminf_{n\to+\infty} f_n(x)$ *is integrable and*

$$\liminf_{n\to+\infty} \int_\Omega f_n(x)dx \geq \int_\Omega f(x)dx.$$

Theorem 2.15. *(Fubini). Let* Ω_1 *and* Ω_2 *open subsets of* $\mathbb{R}^N$ *and* $f \in \mathrm{L}^1(\Omega_1 \times \Omega_2)$ *nonnegative. Then, it holds that*

$$\int_{\Omega_1}\int_{\Omega_2} f(x,y)dydx = \int_{\Omega_2}\int_{\Omega_1} f(x,y)dxdy = \iint_{\Omega_1\times\Omega_2} f(x,y)d(x,y).$$

A proof of the last results can be found in [4].

Once Lebesgue spaces were introduced, it is useful to state some inequalities that hold over these spaces and that are used along this thesis.

Remark 2.8. An extension of Hölder inequality is as follows. Assume $f_i \in \mathrm{L}^{p_i}(\Omega)$, $1 \leq i \leq k$ with

$$\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2} + \ldots + \frac{1}{p_k} \leq 1.$$

Then, the product $f = f_1 f_2 \ldots f_k$ belongs to $\mathrm{L}^p(\Omega)$ and

$$\|f\|_{\mathrm{L}^p(\Omega)} \leq \|f_1\|_{\mathrm{L}^{p_1}(\Omega)} \|f_2\|_{\mathrm{L}^{p_2}(\Omega)} \cdots \|f_k\|_{\mathrm{L}^{p_k}(\Omega)}$$

On the other hand, we have an inverse of the Hölder's inequality when $0 < p < 1$:

Theorem 2.16. *Let* $0 < p < 1$ *and* $f \in \mathrm{L}^p(\Omega)$ *and* $g \in \mathrm{L}^{p'}(\Omega)$ *both* $f, g > 0$*, then we have*

$$\int_\Omega f(x)g(x)dx \geq \left(\int_\Omega f(x)^p dx\right)^{1/p} \left(\int_\Omega g(x)^{p'} dx\right)^{1/p'}.$$

In the previous result, notice that $p' < 0$.

Next, we present a theorem that involves almost everywhere convergence.

Theorem 2.17. *Let $1 \leq p \leq +\infty$. Let $(f_n)_{n\in\mathbb{N}} \subset \mathrm{L}^p(\Omega)$, $f \in \mathrm{L}^p(\Omega)$ such that*

$$\|f_n - f\|_{\mathrm{L}^p(\Omega)} \longrightarrow 0.$$

Then, there exists a subsequence $(f_{n_k})_{k\in\mathbb{N}} \subseteq \mathrm{L}^p(\Omega)$ and a function $h \in \mathrm{L}^p(\Omega)$ such that

i) $f_{n_k}(x) \longrightarrow f(x)$ almost everywhere on Ω;

ii) $\forall k \in \mathbb{N} : |f_{n_k}| \leq h(x)$ almost everywhere on Ω.

A proof of this theorem can be found in [4].

Remark 2.9. If Ω is bounded, by Hölder inequality, we have

$$\mathrm{L}^\infty(\Omega) \subseteq \mathrm{L}^q(\Omega) \subseteq \mathrm{L}^p(\Omega) \subseteq \mathrm{L}^1(\Omega)$$

for $1 \leq p < q < +\infty$.

Definition 2.18. Let $\Omega \subseteq \mathbb{R}^N$ be open and $f : \Omega \to \mathbb{R}$ measurable. We say that f is locally integrable in Ω and write $f \in \mathrm{L}^p_{\mathrm{loc}}(\Omega)$, for $1 \leq p \leq +\infty$ iff

$$\int_K |f(x)|^p dx < +\infty$$

for all compact sets $K \subseteq \Omega$.

Definition 2.19. Let $\Omega \subseteq \mathbb{R}^N$ be a open set.
$\mathrm{C}(\Omega)$ is the space of continuous functions on Ω.
$\mathrm{C}^k(\Omega)$ is the space of functions k times continuously differentiable on Ω, ($k \geq 1$ is an integer).
$\mathrm{C}^\infty(\Omega) = \cap_k \mathrm{C}^k(\Omega)$.
$\mathrm{C}_0(\Omega)$ is the space of continuous functions on Ω with compact support in Ω, i.e., which vanish outside some compact set $K \subseteq \Omega$.
Then,

$$\mathrm{C}_0^\infty(\Omega) = \mathrm{C}^\infty(\Omega) \cap \mathrm{C}_0(\Omega).$$

Theorem 2.20. *Let $f \in \mathrm{L}^1_{loc}(\Omega)$. Then if*

$$\forall \varphi \in \mathrm{C}_0^\infty(\Omega) : \quad \int_\Omega f(x)\varphi(x)dx = 0,$$

we have that $f = 0$ a.e. Ω.

A proof of this Theorem can be found in [20].

2.3 Hilbert spaces

In this section we introduce the Hilbert spaces. A complete presentation can be found in [4] and [20].

Definition 2.21. Let V be a (real) linear space. We say that $(V, (\cdot, \cdot))$ is an inner-product space iff the functional $(\cdot, \cdot) : V \times V \to \mathbb{R}$ verifies:

H1) $\forall u, v, w \in V : (u + v, w) = (u, w) + (v, w)$;

H2) $\forall \lambda \in \mathbb{R}, \forall u, v \in V : (\lambda u, v) = \lambda(u, v)$;

H3) $\forall u, v \in V : (u, v) = (v, u)$;

H4) $\forall u \in V : (u, u) = 0$ iff $u = 0$.

The functional $(\cdot, \cdot)$ is known as an inner-product.

Lemma 2.22. *(Cauchy-Bunyakovsky-Schwarz's inequality). Let $(V, (\cdot))$ be an inner-product space. Then,*

$$\forall x, y \in V : |(x, y)| \leq \sqrt{(x, x)}\sqrt{(y, y)}. \tag{2.23}$$

Proof. Let $x, y \in V$ be generic and denote

$$\|x\| = (x, x)^{1/2}.$$

Then,

$$0 \leq \|x - y\|^2 = \|x\|^2 - 2(x, y) + \|y\|^2 \Longrightarrow (x, y) \leq \frac{1}{2}\left(\|x\|^2 + \|y\|^2\right)$$
$$0 \leq \|x + y\|^2 = \|x\|^2 + 2(x, y) + \|y\|^2 \Longrightarrow (x, y) \geq -\frac{1}{2}\left(\|x\|^2 + \|y\|^2\right).$$

Then,

$$|(x, y)| \leq \frac{1}{2}\left(\|x\|^2 + \|y\|^2\right).$$

Therefore,

$$\begin{aligned} |(x, y)| &= \left|\left(\lambda x, \lambda^{-1} y\right)\right| \\ &\leq \frac{1}{2}\lambda^2\|x\|^2 + \frac{1}{2\lambda^2}\|y\|^2, \end{aligned}$$

being $\lambda \in \mathbb{R}^+$. By taking, in particular, $\lambda = \|y\|\|x\|^{-1}$ with $x \neq 0$, we get our result. We conclude by the arbitrariness of x and y. □

Proposition 2.3.1. $\|x\| = (x, x)^{1/2}$ *is a norm in* H.

Definition 2.23. A Hilbert space is a vector space H equipped with a scalar product such that H is complete endowed with the norm $\|\cdot\|$ induced by the inner−product, that is

$$\|\cdot\| = \sqrt{(\cdot, \cdot)}.$$

Theorem 2.24. *(Riesz-Fréchet representation theorem). Let* H *be a Hilbert space and* $\psi \in \mathrm{H}'$. *Then*

$$\exists! v \in \mathrm{H}, \forall u \in \mathrm{H} : \langle \psi, u \rangle = (u, v),$$

$$\|\psi\|_{\mathrm{H}'} = \|v\|_{\mathrm{H}}.$$

A proof of this Theorem can be found in [4].

Remark 2.10. Let $N \in \mathbb{N}$. Let $\Omega \subseteq \mathbb{R}^N$ be open and let $x \in \Omega$. $\mathrm{L}^2(\Omega)$ equipped with the scalar product

$$(u, v) = \int_\Omega u(x)v(x)dx,$$

for $u, v \in \mathrm{L}^2(\Omega)$, is a Hilbert space.

2.4 Sobolev spaces and Sobolev embeddings

In this section, we introduce Sobolev spaces and Sobolev embeddings. A complete presentation can be found in [4], [11] and [18]. Let $\Omega \subseteq \mathbb{R}^N$ be open and $1 \le p \le \infty$.

Definition 2.25. The Sobolev space $\mathrm{W}^{1,p}(\Omega)$ is defined by

$$\mathrm{W}^{1,p}(\Omega) = \left\{ u \in \mathrm{L}^p(\Omega) \;\middle|\; \begin{array}{l} \exists g_1, g_2, \ldots, g_N \in \mathrm{L}^p(\Omega) \quad \text{such that} \\ \displaystyle\int_\Omega u \frac{\partial \varphi}{\partial x_i} = -\int_\Omega g_i \varphi \quad \forall \varphi \in C_0^\infty(\Omega), \quad \forall i = 1, 2, \ldots, N \end{array} \right\}.$$

For $u \in \mathrm{W}^{1,p}(\Omega)$, we denote $\dfrac{\partial u}{\partial x_i} = g_i$, and

$$\nabla u = \left(\frac{\partial u}{\partial x_1}, \cdots, \frac{\partial u}{\partial x_N} \right).$$

The norm on $\mathrm{W}^{1,p}(\Omega)$ is defined by

$$\|u\|_{\mathrm{W}^{1,p}(\Omega)} = \|u\|_{\mathrm{L}^p(\Omega)} + \sum_{i=1}^N \left\| \frac{\partial u}{\partial x_i} \right\|_{\mathrm{L}^p(\Omega)},$$

for $1 \le p < \infty$.

Definition 2.26. Let $1 \le p < +\infty$. Let $\Omega \subseteq \mathbb{R}^N$ be open and let $x \in \Omega$. Let $u \in \mathrm{L}^2(\Omega)$. Let's consider that $\mathrm{H}_0^1(\Omega) = \{u \in \mathrm{L}^2(\Omega), \partial_{x_i} u \in \mathrm{L}^2(\Omega)/u = 0 \text{ on } \partial\Omega\}$, and $\mathrm{W}_0^{1,p}(\Omega)$ is the closure of $\mathrm{C}_0^1(\Omega)$ in $\mathrm{W}^{1,p}(\Omega)$, such that

$$\mathrm{W}_0^{1,p}(\Omega) = \{u \in \mathrm{W}^{1,p}(\Omega)/u = 0 \text{ on } \partial\Omega\},$$

and notice that

$$\mathrm{H}_0^1(\Omega) = \mathrm{W}_0^{1,2}(\Omega).$$

Proposition 2.4.1. *Given* $\Omega \subseteq \mathbb{R}^N$ *open, the Sobolev space* $\mathrm{W}^{1,p}(\Omega)$

i) is a Banach space for every $1 \leq p \leq \infty$*;*

ii) is reflexive for every $1 < p < \infty$*;*

iii) is separable for every $1 \leq p < \infty$*.*

Let's consider the set $\mathbb{N}_* = \mathbb{N} \cup \{0\}$, and let $N \in \mathbb{N}$.

Remark 2.11. Next, we have the multi-index notation to the Sobolev space $\mathrm{W}^{m,p}(\Omega)$: Let $\alpha = (\alpha_1, \ldots, \alpha_N) \in \mathbb{N}_*^N$ with $\alpha_i \geq 0$ an integer,

$$|\alpha| = \sum_{i=1}^{N} \alpha_i, \quad D^\alpha \varphi = \frac{\partial^{|\alpha|} \varphi}{\partial x_1^{\alpha_1} \partial x_2^{\alpha_2} \cdots \partial x_N^{\alpha_N}}.$$

The Sobolev space with the m^{th} partial derivatives is denoted by $\mathrm{W}^{m,p}(\Omega)$ for $m \geq 2$. We inductively define

$$\mathrm{W}^{m,p}(\Omega) = \left\{ u \in \mathrm{W}^{m-1,p}(\Omega), \frac{\partial u}{\partial x_i} \in \mathrm{W}^{m-1,p}(\Omega) \quad \forall i = 1, 2, \ldots, N \right\}.$$

Then, an alternative to this definition of the Sobolev space $W^{m,p}$ is given by using the standard multi-index notation

$$\mathrm{W}^{m,p}(\Omega) = \left\{ u \in \mathrm{L}^p(\Omega) \;\middle|\; \begin{array}{l} \forall \alpha \text{ with } |\alpha| \leq m, \exists g_\alpha \in L^p(\Omega) \text{ such that} \\ \displaystyle\int_\Omega u D^\alpha \varphi = (-1)^{|\alpha|} \int_\Omega g_\alpha \varphi \quad \forall \varphi \in C_c^\infty(\Omega) \end{array} \right\},$$

where we denote by $D^\alpha u = g_\alpha$.

Remark 2.12. (Weak partial derivative). Let $u, v \in \mathrm{L}^1_{loc}(\Omega)$ and α a multi-index. Then, v is said to be the α^{th}-weak partial derivative of u provided

$$\forall \phi \in C_c^\infty(\Omega): \quad \int_\Omega u(x) D^\alpha \phi(x) dx = (-1)^{|\alpha|} \int_\Omega v(x) \phi(x) dx.$$

In the case that $\Omega \subseteq \mathbb{R}$ and $|\alpha| = 1$ in equation (2.12), it becomes the formula of integration by parts.

Definition 2.27. (Space $\mathrm{H}^k(\Omega)$, with $\Omega \subseteq \mathbb{R}^N$). For every integer $k \geq 1$, the following space is called the classical Sobolev space of order k over Ω.

$$\mathrm{H}^k(\Omega) = \mathrm{W}^{k,2}(\Omega).$$

It is defined on $\mathrm{H}^k(\Omega)$ the scalar product

$$(u, v)_{\mathrm{H}^k(\Omega)} = \sum_{|\alpha| \leq k} (\partial^\alpha u, \partial^\alpha v)_{\mathrm{L}^2(\Omega)}$$

and for $v \in \mathrm{H}^k(\Omega)$ is denoted the norm $\| \cdot \|_{\mathrm{H}^k(\Omega)} : \mathrm{H}^k(\Omega) \longrightarrow \mathbb{R}$ by

$$\|v\|_{\mathrm{H}^k(\Omega)} = (v, v)_{\mathrm{H}^k(\Omega)}^{\frac{1}{2}}. \tag{2.24}$$

Proposition 2.4.2. *The functional* $|||\cdot|||_{\mathrm{H}^2(\mathbb{R}^N)} : \mathrm{H}^2\left(\mathbb{R}^N\right) \longrightarrow \mathbb{R}$ *given by*

$$|||u|||_{\mathrm{H}^2(\mathbb{R}^N)} = \left(\int_{\mathbb{R}^N} \left(|\Delta u(x)|^2 + |u(x)|^2\right) dx\right)^{1/2}. \tag{2.25}$$

is a norm.

Proof. Let $u, v \in \mathrm{H}^2(\mathbb{R}^{\mathbb{N}})$ be generic.

i) The idea for the proofs of $(N1)$ and $(N3)$ for the Proposition 2.4.2, is similar to the items 1.c) and 2) of the proof of the Proposition 2.1.1, which satisfies $(N1)$ and $(N3)$.

ii) $\Leftarrow$) When $u = 0$, we have $|||u|||_{\mathrm{H}^2(\mathbb{R}^N)} = 0$.
$\Rightarrow$) If $|||u|||_{\mathrm{H}^2(\mathbb{R}^N)} = 0$, then we have that

$$\int_{\mathbb{R}^N} |\Delta u(x)|^2 dx + \int_{\mathbb{R}^N} |u(x)|^2 dx = 0.$$

Since $|||u|||_{\mathrm{H}^2(\mathbb{R}^N)} = 0$, it follows that $\|u\|_{\mathrm{L}^2(\mathbb{R}^N)} = 0$. Then $u = 0$.

iii) We prove the triangular inequality by using Minkowski's inequality

$$\begin{aligned}
|||u+v|||_{\mathrm{H}^2(\mathbb{R}^N)} &= \left(\int_{\mathbb{R}^N} \left(|\Delta u(x) + \Delta v(x)|^2 + (u(x)+v(x))^2\right) dx\right)^{1/2} \\
&\leq \left(\int_{\mathbb{R}^N} |\Delta u + \Delta v|^2 dx + \int_{\mathbb{R}^N} (u(x)+v(x))^2 dx\right)^{1/2} \\
&\leq \left(\int_{\mathbb{R}^N} |\Delta u|^2 dx + \int_{\mathbb{R}^N} |\Delta v|^2 dx + \int_{\mathbb{R}^N} u^2(x) dx + \int_{\mathbb{R}^N} v^2(x) dx\right)^{1/2} \\
&\leq \left(\int_{\mathbb{R}^N} \left(|\Delta u|^2 + u^2(x)\right) dx\right)^{1/2} + \left(\int_{\mathbb{R}^N} \left(|\Delta v|^2 + v^2(x)\right) dx\right)^{1/2} \\
&= |||u|||_{\mathrm{H}^2(\mathbb{R}^N)} + |||v|||_{\mathrm{H}^2(\mathbb{R}^N)}.
\end{aligned}$$

Since u and v were chosen arbitarily the proof is done.

□

Remark 2.13. Let $\Omega \subseteq \mathbb{R}^N$ be open and bounded. Then, for $\mathrm{H}^2(\Omega) \cap \mathrm{H}^1_0(\Omega)$

$$\mathcal{N}_2(u) = \left(\int_{\Omega} |\Delta u|^2 dx\right)^{\frac{1}{2}} \tag{2.26}$$

is a norm.

Now, we introduce Poincaré's inequality.

Theorem 2.28. *(Poincaré's inequality). Let $1 \leq p < +\infty$. Let $\Omega \subseteq \mathbb{R}^N$ be open, connected and bounded. Then there exists a constant $C = C(N, \Omega)$ such that*

$$\|u\|_{\mathrm{L}^p(\Omega)} \leq C\|\nabla u\|_{\mathrm{L}^p(\Omega)} \tag{2.27}$$

for each $u \in \mathrm{W}_0^{1,p}(\Omega)$.

Proposition 2.4.3. *Let $\Omega \subseteq \mathbb{R}^N$ be open, connected and bounded. There is an equivalence of norms between (2.25) and (2.26). That is,*

$$\forall v \in \mathrm{H}^2(\Omega): \quad |||v|||_{\mathrm{H}^2(\Omega)} \leq c_0 \cdot \mathcal{N}_2(v) \leq c_1 \cdot |||v|||_{\mathrm{H}^2(\Omega)} \tag{2.28}$$

for some constants $c_0, c_1 > 0$.

Next, we prove the Proposition 2.4.3, for more details see [24] which justifies the following proof.

Proof. Let $\Omega \subseteq \mathbb{R}^N$ be open and bounded. We have to prove that the norm (2.26) is equivalent to (2.25). To do this, we will look for lower and upper bounds of the terms on (2.26).
By using integration by parts and the CSB's inequality, for all u in $\mathrm{H}^2(\Omega) \cap \mathrm{H}_0^1(\Omega)$, we get the following upper bound:

$$\begin{aligned}
\sum_{|\alpha| \leq 2} \|\partial^\alpha u\|_{\mathrm{L}^2(\Omega)} &= \sum_{i,j=1}^n \left(\int_\Omega u_{x_i x_j}^2 dx\right)^{\frac{1}{2}} = \sum_{i,j=1}^n \left(\int_\Omega u_{x_i x_j x_i} u_{x_j} dx\right)^{\frac{1}{2}} \\
&= \sum_{i,j=1}^n \left(\int_\Omega u_{x_i x_i} u_{x_j x_j} dx\right)^{\frac{1}{2}} \leq \sum_{i,j=1}^n \left(\left(\int_\Omega u_{x_i x_i}^2 dx\right)^{\frac{1}{2}} \left(\int_\Omega u_{x_j x_j}^2 dx\right)^{\frac{1}{2}}\right)^{\frac{1}{2}} \\
&\leq \sum_{i,j=1}^n 1 \cdot \left(\left(\int_\Omega |\Delta u|^2 dx\right)^{\frac{1}{2}} \left(\int_\Omega |\Delta u|^2 dx\right)^{\frac{1}{2}}\right)^{\frac{1}{2}} = n^2 \mathcal{N}_2(u).
\end{aligned}$$

and by using the facts that:

$$\left(\sum_{i=1}^n x_i\right)^2 \leq c \sum_{i=1}^n x_i^2, \quad c \in \mathbb{R}^+,$$

$$\left(\sum_{i=1}^n x_i\right)^{\frac{1}{2}} \leq \sum_{i=1}^n x_i^{\frac{1}{2}}, \quad x_i \in \mathbb{R}^+,$$

we have a lower bound:

$$\begin{aligned}
\mathcal{N}_2(u) = \left(\int_\Omega |\Delta u|^2 dx\right)^{\frac{1}{2}} &= \left(\int_\Omega \left|\sum_{i=1}^n u_{x_i x_i}\right|^2 dx\right)^{\frac{1}{2}} \leq c \left(\sum_{i=1}^n \int_\Omega u_{x_i x_i}^2 dx\right)^{\frac{1}{2}} \\
&\leq c \sum_{i=1}^n \left(\int_\Omega u_{x_i x_i}^2 dx\right)^{\frac{1}{2}} \leq c \sum_{i,j=1}^n \left(\int_\Omega u_{x_i x_j}^2 dx\right)^{\frac{1}{2}} = c \sum_{|\alpha| \leq 2} \|\partial^\alpha u\|_{\mathrm{L}^2(\Omega)}.
\end{aligned}$$

Therefore, we have proved that a norm equivalent to that of $H^2(\Omega) \cap H_0^1(\Omega)$ is the following:

$$|||u|||_{H^2(\Omega)} = \|u\|_{L^2(\Omega)} + \|\Delta u\|_{L^2(\Omega)}. \tag{2.29}$$

Then,

$$|||u|||_{H^2(\Omega)} \leq c_0 \cdot \mathcal{N}_2(u) \leq c_1 \cdot |||u|||_{H^2(\Omega)},$$

for some constants c_0, $c_1 > 0$.

□

Remark 2.14. Notice that if $\Omega = \mathbb{R}^N$ and $u \in H^2(\mathbb{R}^N) \cap H_0^1(\mathbb{R}^N)$ it is true that

$$\begin{aligned}\|\nabla u\|^2_{L^2(\mathbb{R}^N)} &= \int_{\mathbb{R}^N} \nabla u \cdot \nabla u dx = \sum_{i=1}^n \int_{\mathbb{R}^N} u_{x_i}^2 dx = \sum_{i=1}^n \int_{\mathbb{R}^N} u_{x_i} u_{x_i} dx = -\sum_{i=1}^n \int_{\mathbb{R}^N} u u_{x_i x_i} dx \\ &\leq \sum_{i=1}^n \int_{\mathbb{R}^N} \left| u u_{x_i x_i} \right| dx \leq \sum_{i=1}^n \left(\int_{\mathbb{R}^N} u^2 dx \right)^{\frac{1}{2}} \left(\int_{\mathbb{R}^N} u_{x_i x_i}^2 dx \right)^{\frac{1}{2}} \\ &\leq \sum_{i=1}^n 1 \cdot \left(\int_{\mathbb{R}^N} u^2 dx \right)^{\frac{1}{2}} \left(\int_{\mathbb{R}^N} |\Delta u|^2 dx \right)^{\frac{1}{2}} = n \|u\|_{L^2(\mathbb{R}^N)} \mathcal{N}_2(u).\end{aligned}$$

Therefore, using Poincaré's inequality (2.27), we get

$$\|\nabla u\|_{L^2(\mathbb{R}^N)} \leq c\mathcal{N}_2(u). \tag{2.30}$$

Combining (2.29) and (2.30), we can write the following two inequalities:

$$\begin{aligned}|||u|||_{H^2(\mathbb{R}^N)} &= \|u\|_{L^2(\mathbb{R}^N)} + \|\Delta u\|_{L^2(\mathbb{R}^N)} \\ &\leq c\|\Delta u\|_{L^2(\mathbb{R}^N)} + \|\Delta u\|_{L^2(\mathbb{R}^N)} \leq c\mathcal{N}_2(u), \\ \mathcal{N}_2(u) &\leq \|u\|_{L^2(\mathbb{R}^N)} + \|\Delta u\|_{L^2(\mathbb{R}^N)} = |||u|||_{H^2(\mathbb{R}^N)}.\end{aligned}$$

Remark 2.15. Let $\Omega = \mathbb{R}^N$ be open. For $k = 2$ in (2.24), let's define the norm

$$\|\cdot\|_{H^2(\mathbb{R}^N)} : H^2\left(\mathbb{R}^N\right) \longrightarrow \mathbb{R}$$

given by

$$\|u\|_{H^2(\mathbb{R}^N)} = \left(\int_{\mathbb{R}^N} \left(|\Delta u(x)|^2 + |\nabla u(x)|^2 + |u(x)|^2 \right) dx \right)^{1/2}.$$

Proposition 2.4.4. *The functional* $|||\cdot|||_{H^2(\mathbb{R}^N)} : H^2\left(\mathbb{R}^N\right) \longrightarrow \mathbb{R}$ *given by*

$$|||u|||_{H^2(\mathbb{R}^N)} = \left(\int_{\mathbb{R}^N} \left(|\Delta u(x)|^2 + |u(x)|^2 \right) dx \right)^{1/2}. \tag{2.31}$$

is a norm equivalent to $\|\cdot\|_{H^2(\mathbb{R}^N)}$.

Next, we prove the Proposition 2.4.4, for more details see [24].

Proof. We have to prove the existence of constants $c_0, c_1 > 0$ such that

$$\forall u \in \mathrm{H}^2(\mathbb{R}^N): \quad |||u|||_{\mathrm{H}^2(\mathbb{R}^N)} \leq c_0 \cdot \|u\|_{\mathrm{H}^2(\mathbb{R}^N)} \leq c_1 \cdot |||u|||_{\mathrm{H}^2(\mathbb{R}^N)}. \tag{2.32}$$

In fact, by applying (2.30), we obtain

$$\begin{aligned} |||u|||_{\mathrm{H}^2(\mathbb{R}^N)} &= \|u\|_{\mathrm{L}^2(\mathbb{R}^N)} + \|\Delta u\|_{\mathrm{L}^2(\mathbb{R}^N)} \\ &\leq \|u\|_{\mathrm{L}^2(\mathbb{R}^N)} + \|\nabla u\|_{\mathrm{L}^2(\mathbb{R}^N)} + \|\Delta u\|_{\mathrm{L}^2(\mathbb{R}^N)} = \|u\|_{\mathrm{H}^2(\mathbb{R}^N)} \leq c_0 \cdot \|u\|_{\mathrm{H}^2(\mathbb{R}^N)}, \\ \|u\|_{\mathrm{H}^2(\mathbb{R}^N)} &= \|u\|_{\mathrm{L}^2(\mathbb{R}^N)} + \|\nabla u\|_{\mathrm{L}^2(\mathbb{R}^N)} + \|\Delta u\|_{\mathrm{L}^2(\mathbb{R}^N)} \\ &\leq \|u\|_{\mathrm{L}^2(\mathbb{R}^N)} + c\|\Delta u\|_{\mathrm{L}^2(\mathbb{R}^N)} + \|\Delta u\|_{\mathrm{L}^2(\mathbb{R}^N)} \\ &\leq \|u\|_{\mathrm{L}^2(\mathbb{R}^N)} + c\|\Delta u\|_{\mathrm{L}^2(\mathbb{R}^N)} \leq c_1 \cdot |||u|||_{\mathrm{H}^2(\mathbb{R}^N)}. \end{aligned}$$

Then, by the previous inequalities, we have proved (2.32) and we conclude the proof. □

Once we have introduced Sobolev spaces, we make a review of some results about values of q for which the inclusion $W^{m,p}(\Omega) \subseteq \mathrm{L}^q(\Omega)$ holds; these are the Sobolev embeddings, for more details see [4]. Let $N \in \mathbb{N}$.

Theorem 2.29. *(Sobolev, Gagliardo, Nirenberg). Let* $1 \leq p < N$*. Then,*

$$\mathrm{W}^{1,p}(\mathbb{R}^N) \subseteq \mathrm{L}^{p^*}(\mathbb{R}^N), \text{ where } p^* \text{ is given by } \frac{1}{p^*} = \frac{1}{p} - \frac{1}{N},$$

and

$$\exists C = C(p, N) > 0, \forall u \in \mathrm{W}^{1,p}(\mathbb{R}^N): \quad \|u\|_{\mathrm{L}^{p^*}(\Omega)} \leq C\|\nabla u\|_{\mathrm{L}^p(\Omega)}.$$

A proof of this Theorem can be found in [4].

An extension of Theorem 2.29 is stated in the following corollary.

Corollary 2.4.1. *Let* $1 \leq p < N$*. Then,*

$$\forall q \in [p, p^*]: \quad \mathrm{W}^{1,p}(\mathbb{R}^N) \subseteq \mathrm{L}^q(\mathbb{R}^N)$$

with continuous embedding.

Proof. We take $q \in [p, p^\star]$, generic. Let $u \in \mathrm{W}^{1,p}(\mathbb{R}^N)$. Then, for some $\alpha \in [0, 1]$ we have that

$$\frac{1}{q} = \frac{\alpha}{p} + \frac{1-\alpha}{p^\star}.$$

By Young's inequality

$$\|u\|_{\mathrm{L}^q(\mathbb{R}^N)} \leq \|u\|^{\alpha}_{\mathrm{L}^p(\mathbb{R}^N)} \|u\|^{1-\alpha}_{\mathrm{L}^{p^*}(\mathbb{R}^N)} \leq \|u\|_{\mathrm{L}^p(\mathbb{R}^N)} + \|u\|_{\mathrm{L}^{p^*}(\mathbb{R}^N)}.$$

Using Theorem 2.29, we have that

$$\|u\|_{\mathrm{L}^q(\mathbb{R}^N)} \leq C\|u\|_{\mathrm{W}^{1,p}(\mathbb{R}^N)}.$$

Since q was chosen arbitrarily the proof is complete. □

Theorem 2.30. *Let $m \in \mathbb{N}$ and $1 \leq p < \infty$, then*

$$\begin{aligned} &\mathrm{W}^{m,p}\left(\mathbb{R}^N\right) \subseteq \mathrm{L}^q\left(\mathbb{R}^N\right) && \text{where } \frac{1}{q} = \frac{1}{p} - \frac{m}{N} \text{ if } \frac{1}{p} - \frac{m}{N} > 0, \\ &\mathrm{W}^{m,p}\left(\mathbb{R}^N\right) \subseteq \mathrm{L}^q\left(\mathbb{R}^N\right) && \forall q \in [p, +\infty) \text{ if } \frac{1}{p} - \frac{m}{N} = 0, \\ &\mathrm{W}^{m,p}\left(\mathbb{R}^N\right) \subseteq \mathrm{L}^\infty\left(\mathbb{R}^N\right) && \text{if } \frac{1}{p} - \frac{m}{N} < 0, \end{aligned}$$

and all these embeddings are continuos.

Theorem 2.31 (Rellich-Kondrachov). *Let $\Omega \subseteq \mathbb{R}^N$ be open and bounded. Then we have the following compact embeddings:*

$$\begin{aligned} &\mathrm{W}^{1,p}(\Omega) \subseteq L^q(\Omega), \quad \forall q \in [1, p^\star), \quad \text{where } \frac{1}{p^\star} = \frac{1}{p} - \frac{1}{N}, && \text{if } p < N, \\ &\mathrm{W}^{1,p}(\Omega) \subseteq L^q(\Omega), \quad \forall q \in [p, +\infty), && \text{if } p = N, \\ &\mathrm{W}^{1,p}(\Omega) \subseteq C(\bar{\Omega}), && \text{if } p > N. \end{aligned}$$

A proof of this Theorem can be found in [4]. Notice, that $\mathrm{W}^{1,p}(\Omega) \subseteq \mathrm{L}^p(\Omega)$ with compact embedding for all p and all N.

2.5 Topics of Calculus of Variations, Nonlinear Analysis and of non−linear PDEs

The main references for this section are [3] and [20].

2.5.1 Basic definitions

Let E and F be normed spaces and $f : \mathcal{O} \subseteq E \to F$, where $\mathcal{O}$ is open. Then we define the directional derivative of f at the point $u \in \mathcal{O}$ in the direction h as the following limit (whenever exists)

$$\partial_h f(u) = \lim_{t \to 0} \frac{1}{t}[f(u + th) - f(u)].$$

Proposition 2.5.1. *Let E and F be normed spaces and $f : \mathcal{O} \subseteq E \to F$, where $\mathcal{O}$ is open. Let $u \in \mathcal{O}$ and let $h \in E$. If $\partial_h f(u)$ is defined and $\lambda \in \mathbb{R}\backslash\{0\}$, then*

$$\partial_{\lambda h} f(u) = \lambda \partial_h f(u). \tag{2.33}$$

Proof. Let E and F be normed spaces and $f : \mathcal{O} \subseteq E \to F$, where $\mathcal{O}$ is open. Let $u \in \mathcal{O}$ and let $h \in E$. Notice that for $\lambda \in \mathbb{R}\backslash\{0\}$ and $\alpha := \lambda t$, we have that

$$\partial_{\lambda h} f(u) = \lim_{t \to 0} \frac{1}{t}[f(u + \lambda th) - f(u)] = \lambda \lim_{\alpha \to 0} \frac{1}{\alpha}[f(u + \alpha h) - f(u)] = \lambda \partial_h f(u).$$

□

Moreover, if $\partial_h f(u)$ exists for every nonzero $h \in E$ and there exists an operator $T \in \mathcal{L}(E, F)$ such that

$$\forall h \in E, \forall u \in \mathcal{O}: \quad \partial_h f(u) = Th,$$

then, we say that f is Gateaux differentiable (or weakly differentiable) and we denote by $f'_G(u) = T$, for every $u \in \mathcal{O}$, the Gateaux differential (or weak differential) of f at u.

Let $g : \mathcal{O} \subseteq E \to F$, where $0 \in \mathcal{O}$ and $g(0) = 0$. We say that g is a small o of $h \in E$, denoted by $g(h) = o(h)$, if there exists a mapping $\epsilon : \mathcal{O} \subseteq E \to F$ such that

$$\lim_{h \to 0} \epsilon(h) = 0$$

and

$$g(h) = \|h\|\epsilon(h).$$

A function $f : \mathcal{O} \subseteq X \to Y$ is Fréchet differentiable at $u \in \mathcal{O} \subseteq X$ if

$$\exists \phi \in \mathcal{L}(X, Y), \forall h \in X: \quad u + h \in \mathcal{O} \Rightarrow f(u + h) - f(u) = \phi(h) + g(h), \tag{2.34}$$

where $g(h) = o(h)$.

Remark 2.16. When not specified it will be understood by differentiable to Fréchet differentiable.

The uniqueness of the operator ϕ is stated in the following result:

Proposition 2.5.2. *The operator* $\phi \in \mathcal{L}(X, Y)$ *in* (2.34) *is unique.*

Proof. Let $\varphi \in \mathcal{L}(X, Y)$, such that,

$$\forall h \in E: \quad u + h \in \mathcal{O} \Rightarrow f(u + h) - f(u) = \varphi(h) + o(h). \tag{2.35}$$

Since $\mathcal{O} \subseteq X$ is open, we have that

$$\exists r > 0: \quad B(a, r) = a + B(0, r) \subseteq \mathcal{O}.$$

Then, by (2.35) and (2.34),

$$\forall h \in B(0, r): \quad \phi(h) + \|h\|e_1(h) = \varphi(h) + \|h\|e_2(h), \tag{2.36}$$

where

$$\lim_{h \to 0} e_1(h) = \lim_{h \to 0} e_2(h) = 0. \tag{2.37}$$

Let $u \in E$, generic. If $u = 0$ the proof is clearly because of the linearity of ϕ and φ. Instead, we consider $u \neq 0$ and choose $N \in \mathbb{N}$ such that

$$\forall n \in \mathbb{N}: \quad n > N \Longrightarrow h_n := \frac{1}{n} \cdot \frac{1}{\|u\|} u \in B(0, r).$$

By (2.36), we have that

$$\phi(h_n) - \varphi(h_n) = \|h_n\| (\epsilon_2(h_n) - \epsilon_1(h_n)).$$

Since $\phi, \varphi \in \mathcal{L}(E, F)$,

$$\left(\frac{1}{n} \cdot \frac{1}{\|u\|}\right)(\phi(u) - \varphi(u)) = \left(\frac{1}{n} \cdot \frac{1}{\|u\|}\right)\|u\|\left(\epsilon_2\left(h_n\right) - \epsilon_1\left(h_n\right)\right).$$

By letting $n \to 0$ and using (2.37), we get

$$\phi(u) = \varphi(u).$$

Since u was chosen arbitrarily the proof is complete.

□

Remark 2.17. By Proposition 2.5.2 we can write (2.35) as

$$f(u+h) - f(u) = f'(u)h + o(h),$$

for $u \in X, h \in X$, with $f'(u) \in \mathcal{L}(X, Y)$ referred to as the differential of f at u.

Remark 2.18. Notice that if a function is Fréchet differentiable, it is also Gateaux differentiable and there exist all its directional derivatives. Next we have an important result that relates the differentaibility and continuity.

Proposition 2.5.3. *Let X and Y be normed spaces and $f : \mathcal{O} \subseteq X \to Y$, where $\mathcal{O}$ is open. If f differentiable at $u \in \mathcal{O}$ then f is continuous at u.*

Proof. We take $u \in \mathcal{O}$, such that

$$\begin{aligned} \forall h \in X: \quad u + h \in \mathcal{O} &\Rightarrow f(u+h) - f(u) = f'(u)h + o(h) \\ &\Rightarrow f(u+h) = f(u) + f'(u)h + o(h). \end{aligned}$$

Given that $f' \in \mathcal{L}(X, Y)$ and $\lim_{h \to 0} o(h) = 0$, it is clear that

$$\lim_{h \to 0} f(u+h) = f(u). \tag{2.38}$$

Since u was chosen arbitrarily we have proved that f is continuous. □

Remark 2.19. We say that $f : \mathcal{O} \subseteq X \to Y$ belongs to the class $\mathrm{C}^1(\mathcal{O}, Y)$ iff

1. f is differentiable;
2. the function

$$\begin{aligned} f' : \mathcal{O} \subseteq X &\longrightarrow \mathcal{L}(X, Y) \\ x &\longmapsto f'(x), \end{aligned}$$

is continuous.

Remark 2.20. Let $\mathcal{O} \subseteq \mathbb{R}^N$ open and $f : \mathcal{O} \to \mathbb{R}$, if all its partial derivatives are defined and continuous on $\mathcal{O}$. Then $f \in \mathrm{C}^1(\mathcal{O}, Y)$.

Now, we introduce the concepts of critical and extremum points.

Definition 2.32. (**Extremum**) Let $f : X \to \mathbb{R}$ where X is a non-void set. Then,

1) A point $x_0 \in X$ is called a point of (global) minimum iff

$$\forall x \in X: \quad f(x_0) \leq f(x).$$

2) A point $x_0 \in X$ is called a point of (global) maximum iff

$$\forall x \in X: \quad f(x_0) \geq f(x).$$

If x_0 is a point of minimum or maximum, it is called a point of (global) extremum.

Theorem 2.33. *Let X be a normed space, $\mathcal{O} \subseteq X$ open and $f : \mathcal{O} \subseteq X \to \mathbb{R}$. Suppose that*

i) f has a local extremum at $x \in \mathcal{O}$,

ii) f is differentiable at x.

Then x is a critical point of f, i.e.

$$f'(x) = 0.$$

Proof. Let $x_0 \in \mathcal{O}$ be a local minimum and f differentiable at x_0, arbitrary, then:

$$\forall x \in X: \quad 0 \leq |f(x_0 + h) - f(x_0)| \leq |f(x + h) - f(x)|.$$

Dividing by h we get

$$0 \leq \frac{|f(x_0 + h) - f(x_0)|}{h} \leq \frac{|f(x + h) - f(x)|}{h},$$

Passing to the limit and given the continuity of every $x \in \Theta$

$$0 \leq \lim_{h \to 0} \frac{|f(x_0 + h) - f(x_0)|}{h} \leq \lim_{h \to 0} \frac{|f(x + h) - f(x)|}{h} \to 0$$

which shows that $f'(x_0) = 0$. The proof is similar when x_0 is a local maximum. □

Now, we introduce PDEs and tools to deal with them. For this section we used [3], [15] and [19].

Definition 2.34. Let $\Omega \subseteq \mathbb{R}^n$ open and $k \in \mathbb{N}$ fixed. The equation

$$Q(D^k(u(x)), D^{k-1}u(x), ..., Du(x), u(x), x) = 0$$

is referred to as a kth-order PDE, where the mapping

$$Q : R^{n^k} \times \mathbb{R}^{n^{k-1}} \times ... \times \mathbb{R}^n \times \mathbb{R} \times U \longrightarrow \mathbb{R}$$

is given and $u : \Omega \longrightarrow \mathbb{R}$ is the unknown.

We can classify the PDE′s by considering their linearity properties.

i. **Linear:** if the PDE is linear with respect of u and its derivatives:

$$\sum_{|\alpha|\leq k} a_\alpha(x)D^\alpha u(x) = f(x)$$

for given functions f, and a_α (with $|\alpha| \leq k$). In the case that $f = 0$ then the linear PDE is **homogeneous**.

ii. **Semilinear:** if the PDE is linear in the highest order derivatives of u, i.e.

$$\sum_{|\alpha|=k} a_\alpha(x)D^\alpha u + a_0\left(D^{k-1}u, \ldots, Du, u, x\right) = 0.$$

iii. **Quasi-linear** if the PDE is semilinear, and the function a_α depends on u and its derivatives upon order $k-1$ i.e. it has the following form

$$\sum_{|\alpha|=k} a_\alpha\left(D^{k-1}u(x), \ldots, Du(x), u(x), x\right) D^\alpha u(x) + a_0\left(D^{k-1}u(x), \ldots, Du(x), u(x), x\right) = 0.$$

iv. A PDE is said to be **fully nonlinear** if it is not linear and given whatever order of derivative is not linear too.

It doesn't exist an specific method for solving PDEs because it depends on the structure of each problem. Things get even more complicated when the PDE is nonlinear. One way to find solutions is by applying variational methods.

Let A an operator which could be nonlinear and u just unknown, we consider the following representation of a PDE

$$A(u) = 0, \tag{2.39}$$

this variational method consist on taking a convenient functional I, where the PDE in (2.39) must be its "derivative", i.e.,

$$I'(u) = A(u). \tag{2.40}$$

Then, the following is the new weak problem

$$I'(u) = 0. \tag{2.41}$$

The advantage of the previous formulation is that we now can recognize solutions as critical points of I. In certain circumstances, it is easier to find minimum (or maximum or other critical points) of the functional I, then the formulation of the weak problem (2.41) is valid and we have found weak solutions of the original PDE.

The previous can be seen as the following, in a formalized sense:
let $N \in \mathbb{N}$ and assume that $\Omega \subseteq \mathbb{R}^N$ is a bounded, open set with smooth boundary $\partial\Omega$. Then, we call the Lagrangian to a smooth function

$$\begin{aligned} L : \overline{\Omega} \times \mathbb{R} \times \mathbb{R}^N &\longrightarrow \mathbb{R} \\ (x, y, z) &\longmapsto L(x_1, ..., x_N, y, z_1, ..., z_N). \end{aligned} \tag{2.42}$$

We also denote:

1. $\nabla_x L = (L_{x_1}, ..., L_{x_N})$;
2. $L_y = \dfrac{\partial L}{\partial y}$;
3. $\nabla_z L = (L_{z_1}, ..., L_{z_N})$.

Let us assume for the moment that I has the explicit form:

$$I(w) = \int_\Omega L(x, w(x), \nabla w(x))dx, \tag{2.43}$$

for a smooth function $w : \overline{\Omega} \longrightarrow \mathbb{R}$, which satisfies the boundary condition $w = g$ on $\partial\Omega$, where g is given. Now suppose that u is a smooth function that satisfies the boundary condition. Therefore, u solves the quasilinear second-order PDE in divergence form:

$$L_y(x, u(x), \nabla u(x)) - \sum_{j=1}^{N} (L_{z_j}(x, u(x), \nabla u(x)))_{x_j} \quad \text{on} \quad \Omega,$$

which is better called as the Euler-Lagrange equation associated to the energy functional I defined in (2.43).

We work in this way in our thesis with a non-linear biharmonic Schrödinger equation. Consider the following examples:

Example 1. (Laplace's equation). Let $N \in \mathbb{N}$, $\Omega \subseteq \mathbb{R}^N$ open connected with smooth boundary. Let $u : \Omega \subseteq \mathbb{R}^N \to \mathbb{R}$, where u is a unknown. We look for solutions of the problem

$$\begin{cases} -\Delta u(x) &= 0, \quad x \in \Omega, \\ \quad u(x) &= 0, \quad x \in \partial\Omega, \end{cases} \tag{2.44}$$

where (2.44) is the Euler-Lagrange equation associated to the functional

$$I(u) = \frac{1}{2}\int_\Omega |\nabla u(x)|^2 dx.$$

Example 2. (Non-homogeneous PDE). Let $N \in \mathbb{N}$, $\Omega \subseteq \mathbb{R}^N$ open connected.
Let $u : \Omega \subseteq \mathbb{R}^N \to \mathbb{R}$ and $f : \Omega \subseteq \mathbb{R}^N \longrightarrow \mathbb{R}$, where u is a unknown and f is given. We have that

$$-\Delta u(x) = f(x), \quad x \in \Omega.$$

As we shall see, our equation is a semi-linear PDE, i.e., it verifies *ii*) in the previous PDE's classification.

Next, we have the notion of strong and weak solution, respectively, for a PDE. For more details of this see e.g. [4] or [7]. Also, for ilustrate the previous notions we will use a PDE in base from the Definition 2.34.

Let $N \in \mathbb{N}$. Let $\Omega \subseteq \mathbb{R}^N$ be open. Let $k \in \mathbb{N}_*$. Let's consider a mapping as in the Definition 2.34 with the respective conditions for u, togheter to a PDE as in *i*) on the PDE's classification for the following:
we have that

$$Lu = \sum_{|\alpha| \leq k} a_\alpha(x) D^\alpha u = f, \tag{2.45}$$

for given functions f, and a_α.

By the Definition 2.18, let's consider a locally integrable function u that has locally integrable derivatives of all orders less or equal than k, and which satisfies (2.45) almost-everywhere in Ω. Then, the notion of a strong solution can be seen as follows:

Definition 2.35. A function u is called a strong solution of (2.45) if there are sequences of smooth (for example, $C^\infty(\Omega)$) functions $(u_n)_{n\in\mathbb{N}}$, $(f_n)_{n\in\mathbb{N}}$ such that for each $n \in \mathbb{N}$,

$$u_n \longrightarrow u \quad \text{and} \quad f_n \longrightarrow f$$

as $n \longrightarrow +\infty$, this is, strong convergence and

$$Lu_n = f_n,$$

where the convergence is taken in $L^1_{loc}(K)$ for any compact set $K \subseteq \Omega$.

In the previous definition, $L^1_{loc}(K)$ can be replaced by $L^p_{loc}(K)$ functions whose values of p, for $1 < p \leq +\infty$, are locally integrable.

Definition 2.36. A weak solution to a PDE is a function u for which the derivatives may not all exist but which is nonetheless deemed to satisfy the equation in some precisely defined sense.

For more details of the previous definition, see [7].

Example 3. Also, for example, in (2.39) a solution u on $A(u)$ is called strong solution and a solution w for a functional I is a weak solution, where I satisfies (2.43).

2.5.2 Palais-Smale condition

In this section we introduce Palais-Smale condition and Krasnoselskii's genus. Our main reference is [19]. We consider that X is a Banach space for this whole section.

We present a compactness condition that our functional shall satisfy to prove multiplicity. Let $I : X \longrightarrow \mathbb{R}$.

Definition 2.37. A functional $I \in C^1(X)$ satisfies the Palais-Smale condition if each sequence $(u_n)_{n\in\mathbb{N}} \subseteq X$ such that

1. $(I(u_n))_{n\in\mathbb{N}}$ is bounded;
2. $I'(u_n) \longrightarrow 0$ as $n \longrightarrow +\infty$.

Then, I has a convergent subsequence in X.

2.5.3 Krasnoselskii's genus

Let E be a Banach space. We define the class of all closed, symmetric subsets $A \subseteq E$ that do not contain 0 as

$$\Sigma_E = \{A \in E : A = \overline{A}, A = -A, 0 \notin A\}. \tag{2.46}$$

Next, we have the concept of Krasnoselskii's genus which is a generalization of the concept of dimension of a vector space, see Example 3 below.

The genus of A, denoted $\gamma(A)$, is the smallest integer k such that there exists and odd mapping $h \in C(A, \mathbb{R}^k \backslash \{0\})$. That is, for a given $A \in \Sigma_E$ we set

$$K_A = \{k \in \mathbb{N} \big/ \exists f \in C(A, \mathbb{R}^k \backslash \{0\}) \quad \text{is an odd function}\},$$

and the Krasnoselskii genus as:

$$\gamma(A) := \inf(K) \quad \text{if} \quad K_A \neq \emptyset.$$

If $K_A = \emptyset$ then $\gamma(A) = \infty$ and $\gamma(\emptyset) = 0$.

Example 4. $\gamma(\mathbb{S}^{N-1}) = N$, and $\gamma(\mathbb{S}^\infty) = +\infty$, where $\mathbb{S}^{N-1}$ is the unit-sphere in $\mathbb{R}^N$ and $\mathbb{S}^\infty$ is the unit-sphere in an infinite-dimensional Banach space Y.

Example 5. Suppose $B \subseteq E$ is closed and $B \cap (-B) = \emptyset$. Let $A := B \cup (-B)$, then $\gamma(A) = 1$ since the function $f(x) := (\chi_B - \chi_{-B})(x)$ is odd and $f \in \mathrm{C}(A, \mathbb{R} \backslash \{0\})$.

To see the normal spaces, see [12].

Theorem 2.38. *(Tietze's Extension Theorem). Let X be a normal space, let $A \subseteq X$ be a closed subspace, and let $f : A \to [a; b]$ be a continuous function for some $[a; b] \subseteq \mathbb{R}$. There exits a continuous function $\overline{f} : X \to [a; b]$ such that $\overline{f}|_A = f$.*

The following lemma states some properties of the genus. Also, the previous Theorem 2.37 is useful to prove the next lemma.

Lemma 2.39. *Let $A, B \in \sum_E$. Then, we have that:*

1. *If $f \in C(A, E)$ is odd, then $\gamma(A) \leq \gamma(f(A))$.*
2. *If $A \subseteq B$ then $\gamma(A) \leq \gamma(B)$,*
3. *$\gamma(A \cup B) \leq \gamma(A) + \gamma(B)$,*
4. *If A is compact, then $\gamma(A) < +\infty$ and there exists $\delta > 0$ such that*

$$B_\delta(A) \in \Sigma_E \quad \textit{and} \quad \gamma\left(B_\delta(A)\right) = \gamma(A),$$

where $B_\delta(A)$ represents a uniform $\delta-$neighborhood of A, that is,

$$B_\delta(A) = \{x \in E / \|x - A\| \leq \delta\}.$$

Next, we prove the previous lemma, for more details see [19].

Proof. Let $A, B \in \Sigma_E$ and $\gamma(A), \gamma(B) < +\infty$.

1) Suppose $\gamma(B) = n$, then there exists $h \in \mathrm{C}(B, \mathbb{R}^n \backslash \{0\})$ so $h \circ f \in \mathrm{C}(A, \mathbb{R}^n \backslash \{0\})$ is odd. Therefore,
$$\gamma(A) \leq n = \gamma(B).$$

2) Let $f \in \mathrm{C}(A, \mathbb{R}^n \backslash \{0\})$. Taking $f = \mathrm{id}$, being id the identity function, in 1 we conclude 2.

3) Suppose $\gamma(A) = m$ and $\gamma(B) = n$, therefore there exist $g \in \mathrm{C}(A, \mathbb{R}^m \backslash \{0\})$ and $h \in \mathrm{C}(B, \mathbb{R}^n \backslash \{0\})$ odd. By Tietze's Extension Theorem, there exist $\hat{g} \in \mathrm{C}(E, \mathbb{R}^m)$ y $\hat{h} \in \mathrm{C}(E, \mathbb{R}^n)$ such that $\hat{g}|_A = g$ and $\hat{h}\big|_B = h$. From the above, we can assume that $\hat{g}$ and $\hat{h}$ are odd in E.
Let $f = (\hat{g}, \hat{h})$, then $f \in \mathrm{C}\left(A \cup B, \mathbb{R}^{m+n} \backslash \{0\}\right)$ and is odd, therefore
$$\gamma(A \cup B) \leq m + n = \gamma(A) + \gamma(B).$$

4) For $x \in A$, we define $r(x) := \frac{1}{2}\|x\| = r(-x)$ and $T_x := B_{r(x)}(x) \cup B_{r(x)}(-x)$. So $\gamma\left(\overline{T_x}\right) = 1$ by Example 5. Then
$$A \subseteq \bigcup_{x \in A} T_x$$
and, by the compactness of A, we conclude that
$$A \subseteq \bigcup_{i=1}^{k} T_{x_i}$$
for some finite set of points $x_1, ..., x_k \in A$. Therefore from 3 we get that
$$\gamma(A) < \infty.$$
If $\gamma(A) = n$, then there exists $f \in \mathrm{C}(A, \mathbb{R}^n \backslash \{0\})$ odd. We extend f to an odd function $\hat{f}$ as in 3. Since A is compact, there exists $\delta > 0$ such that $\hat{f} \neq 0$ over $B_\delta(A)$, then
$$\gamma(B_\delta(A)) \leq n = \gamma(A)$$
but by 2 we get the inequality
$$\gamma(A) \leq \gamma(B_\delta(A)).$$

□

2.5.4 Differentiable manifolds

We present some results on differentiable manifolds. The main reference for this section is [3].

Remark 2.21. Let E be a normed space and $T \in \mathrm{C}^1(E)$. We say that $a \in \mathrm{Im}(T)$ is a critical value of T if there is some $u \in T^{-1}(a)$ such that $T'(u) = 0$, we refer to such u as critical point of T. We call $a \in \mathrm{Im}(T)$ a regular value of T if it is not a critical value of T.

Definition 2.40. (C^n Manifold). Let H be a Hilbert space and $U \subseteq \mathrm{H}$ open. We say that a closed subset $\mathcal{M} \subseteq \mathrm{H}$ is a C^n manifold if there exists $L \in \mathrm{C}^n(U), a \in \mathbb{R}$, such that a is a regular value of L and

$$\mathcal{M} = L^{-1}(\{a\}).$$

For $u \in \mathcal{M}$, we say that

$$T_u\mathcal{M} := \mathrm{Ker}(L'(u))$$

is the tangent space of $\mathcal{M}$ at u.

Definition 2.41. (Constrained critical point). Let $J \in \mathrm{C}^1(\mathrm{H})$, where H is a Hilbert space and $\mathcal{M} \subseteq \mathrm{H}$ a C^1 manifold. We say that $u \in \mathcal{M}$ is a constrained critical point of J on $\mathcal{M}$ if $D_{\mathcal{M}}J(u) = 0$, i.e.,

$$\forall \psi \in T_u\mathcal{M} : \langle J'(u), \psi \rangle = 0.$$

Whenever $\mathrm{H} = \mathcal{M}$, we have $J'(u) = 0$.

Theorem 2.42. *(Lagrange multiplier). Let* H *be a Hilbert space and* $\mathcal{M} \subseteq \mathrm{H}$ *a* C^1 *manifold such that*

$$\mathcal{M} = L^{-1}(\{a\})$$

where a *is a regular value of* $L \in \mathrm{C}^1(\mathrm{H})$*. Let* $J \in \mathrm{C}^1(\mathrm{H})$ *and consider* u*, a constrained critical point of* J *on* $\mathcal{M}$*. Then*

$$\exists \lambda \in \mathbb{R} : \quad J'(u) = \lambda L'(u).$$

Proof. Let $J, L \in \mathrm{C}^1(\mathrm{H})$. Let u be a constrained critical point of J on $\mathcal{M}$, then

$$\forall \psi \in T_u\mathcal{M} : \langle J'(u), \psi \rangle = 0. \tag{2.47}$$

There are $J', L' \in \mathrm{H}$ and by the Riesz-Fréchet representation theorem, it follows that

$$\forall v \in \mathrm{H} : \langle J'(u), v \rangle = (J'(u), v), \langle L'(u), v \rangle = (L'(u), v).$$

Then, (2.47) can be rewritten as

$$\forall \psi \in \mathrm{H} : \langle L'(u), \psi \rangle = 0 \Rightarrow \langle J'(u), \psi \rangle = 0.$$

Therefore there is $\lambda \in \mathbb{R}$ such that

$$J'(u) = \lambda L'(u).$$

□

Chapter 3

Results

3.1 Problem statement

As mentioned in Chapter 1, the objective of this thesis is to prove multiplicity of solutions for the following problem:

$$\begin{cases} -\varepsilon^4\Delta^2 v(x) + V(x)v(x) - |v(x)|^{p-1}v(x) &= 0, \qquad\qquad x \in \mathbb{R}^N, \\ \qquad\qquad\qquad\qquad\qquad\qquad v(x) &\longrightarrow 0, \quad \text{as } |x| \longrightarrow +\infty, \end{cases} \tag{P_ε}$$

where $\varepsilon^4 = \dfrac{\hbar^4}{4}$ and the biharmonic-Laplace operator is given by

$$\Delta^2 = \Delta \circ \Delta.$$

We assume that $1 < p+1 < 2^*$, where

$$2^* = \begin{cases} \dfrac{2N}{N-2}, & \text{if } N \geq 3; \\ +\infty, & \text{if } N = 1, 2. \end{cases}$$

Let $\varepsilon > 0$. Let $u \in C^2(\mathbb{R}^N)$. Let's assume that $v \in C^2(\mathbb{R}^N)$ verifies (P_ε). By the scaling

$$v(x) = u(\varepsilon^\alpha x), \quad x \in \mathbb{R}^N, \alpha \in \mathbb{R},$$

it follows that

$$\begin{aligned} \nabla v(x) &= \varepsilon^\alpha \nabla u(\varepsilon^\alpha x), \\ \Delta v(x) &= \varepsilon^{2\alpha}\Delta u(\varepsilon^\alpha x), \\ \Delta^2 v(x) &= \varepsilon^{4\alpha}\Delta^2 u(\varepsilon^\alpha x). \end{aligned}$$

So that

$$-\varepsilon^{4+4\alpha}\Delta^2 u(\varepsilon^\alpha x) + V(x)u(\varepsilon^\alpha x) - |u(\varepsilon^\alpha x)|^{p-1}u(\varepsilon^\alpha x) = 0, \quad x \in \mathbb{R}^N. \tag{3.1}$$

Then, by choosing,

$$\begin{aligned} \alpha &= -1, \\ y &= \varepsilon^\alpha x = \varepsilon^{-1}x, \end{aligned}$$

we get

$$-\Delta^2 u(y) + V_\varepsilon(x)u(y) - |u(y)|^{p-1}u(y) = 0, \quad y \in \mathbb{R}^N,$$

with

$$V_\varepsilon(x) = V(\varepsilon^{-1}x).$$

Therefore, problem (P_ε) is equivalent to

$$\begin{cases} -\Delta^2 u(x) + V_\varepsilon(x)u(x) - |u(x)|^{p-1}u(x) & = 0, \qquad\qquad x \in \mathbb{R}^N, \\ \qquad\qquad\qquad\qquad u(x) & \longrightarrow 0, \quad \text{as } |x| \longrightarrow +\infty. \end{cases} \tag{P'_ε}$$

We shall assume that:
V1) $V \in \mathrm{C}\left(\mathbb{R}^N\right)$ is a non-negative function;
V2) $V(x) \longrightarrow +\infty$, as $|x| \longrightarrow +\infty$;
V3) $\{V = \inf\{V\} = 0\} \neq \emptyset$.

Let's consider the manifold

$$\mathcal{M}_\varepsilon = \left\{ u \in \mathrm{H}^2_\varepsilon / \int_{\mathbb{R}^N} |u(x)|^{p+1} dx = 1 \right\},$$

where

$$\mathrm{H}^2_\varepsilon := \{u \in \mathrm{H}^2(\mathbb{R}^N) \Big/ \int_{\mathbb{R}^N} [|\Delta u(x)|^2 + V_\varepsilon(x)u^2(x)]dx < +\infty\}$$

is equipped with the norm given by

$$\|u\|_\varepsilon = \left(\int_{\mathbb{R}^N} [|\Delta u(x)|^2 + V_\varepsilon(x)u^2(x)]dx \right)^{\frac{1}{2}}.$$

Let's define the functional $I_\varepsilon : \mathrm{H}^2_\varepsilon(\mathbb{R}^N) \longrightarrow \mathbb{R}$, given by

$$I_\varepsilon(u) = \frac{1}{2} \int_{\mathbb{R}^N} [|\Delta u(x)|^2 + V_\varepsilon(x)u^2(x)]dx. \tag{3.2}$$

The functional associated to (P'_ε) is $J_\varepsilon : \mathcal{M}_\varepsilon \subseteq \mathrm{H}^2_\varepsilon \longrightarrow \mathbb{R}$, given by

$$J_\varepsilon(u) = \frac{1}{2}\|u\|^2_\varepsilon = \frac{1}{2} \int_{\mathbb{R}^N} [|\Delta u(x)|^2 + V_\varepsilon(x)|u(x)|^2]dx.$$

Remark 3.1. Observe that $J_\varepsilon = I_\varepsilon|_{\mathcal{M}_\varepsilon}$.

Let's show that the critical points of J_ε provide weak solutions for (P'_ε) and with $v(x) = u(\varepsilon^{-1}x)$ and $V_\varepsilon(x) = V(\varepsilon x)$ on I_ε, also for (P_ε).

By (2.42), we have that the Lagrangian functional $L_\lambda : \mathrm{H}^2_\varepsilon(\mathbb{R}^N) \longrightarrow \mathbb{R}$ is given by

$$L_\lambda(v) = J_\varepsilon(v) + \lambda \left(\int_{\mathbb{R}^N} |v(x)|^{p+1} dx - 1 \right),$$

with $\lambda \in \mathbb{R}$ being the Lagrange's multiplier, in our context. In reference to [1], let's recall that

$$J'_\varepsilon(v) = 0 \Longleftrightarrow \begin{cases} L'_\varepsilon(v) = 0, \\ \displaystyle\int_{\mathbb{R}^N} |v(x)|^{p+1} dx = 1. \end{cases} \tag{3.3}$$

By using the Theorem 3.7 and by (3.3), let's observe that for $u, w \in \mathcal{M}_\varepsilon$,

$$L'_\lambda(v)w = \int_{\mathbb{R}^N} [\Delta v(x)\Delta w(x) + V_\varepsilon(x)v(x)w(x)]dx - \lambda(p+1)\int_{\mathbb{R}^N} |v(x)|^{p-1}v(x)w(x)dx. \quad (3.4)$$

In the following section we shall prove that J_ε is of class C^1 and we shall get its Fréchet differential.

Now if $L'_\lambda(v) = 0$, then we get, by choosing $w = v$ in (3.4),

$$\int_{\mathbb{R}^N} [\Delta v(x)\Delta v(x) + V_\varepsilon(x)v(x)v(x)]dx = \lambda(p+1)\int_{\mathbb{R}^N} |v(x)|^{p-1}v(x)v(x)dx,$$

then,

$$\int_{\mathbb{R}^N} [|\Delta v(x)|^2 + V_\varepsilon(x)v^2(x)]dx = \lambda(p+1)\int_{\mathbb{R}^N} |v(x)|^{p+1}dx. \quad (3.5)$$

Then, by the definition of J_ε and by (3.3) in (3.5), we get

$$\lambda(p+1) = 2c,$$

where

$$c = J_\varepsilon(v)$$

is the corresponding critical value. From this it follows that v is a weak solution of

$$-\Delta^2 v(x) + V_\varepsilon(x)v(x) - 2c|v(x)|^{p-1}v(x) = 0, \quad x \in \mathbb{R}^N. \quad (3.6)$$

Therefore, the function u defined by

$$v = \gamma u, \quad (3.7)$$

with

$$\gamma = (2c)^{-1/p},$$

is a weak solution of (P'_ε). In fact, by (3.7) on (3.6), we get that

$$\begin{aligned} -\Delta^2(\gamma u(x)) + V_\varepsilon(x)\gamma u(x) - 2c|\gamma u(x)|^{p-1}\gamma u(x) &= 0, \quad x \in \mathbb{R}^N, \\ -\Delta^2(\gamma u(x)) + V_\varepsilon(x)\gamma u(x) - 2c\gamma^{p-1}|u(x)|^{p-1}\gamma u(x) &= 0, \quad x \in \mathbb{R}^N, \\ -\gamma\Delta^2 u(x) + \gamma V_\varepsilon(x)u(x) - 2c(2c)^{-p/p}|u(x)|^{p-1}u(x) &= 0, \quad x \in \mathbb{R}^N, \\ -\gamma\Delta^2 u(x) + \gamma V_\varepsilon(x)u(x) - |u(x)|^{p-1}u(x) &= 0, \quad x \in \mathbb{R}^N. \end{aligned}$$

Proposition 3.1.1. *The functional* $\|\cdot\|_\varepsilon : \mathrm{H}^2_\varepsilon\left(\mathbb{R}^N\right) \longrightarrow \mathbb{R}$ *given by*

$$\|u\|_\varepsilon = \left(\int_{\mathbb{R}^N} \left(|\Delta u(x)|^2 + V_\varepsilon(x)|u(x)|^2\right)dx\right)^{1/2}. \quad (3.8)$$

is a norm.

Proof. The idea for the proofs of $(N1)$ and $(N3)$ for the Proposition 3.1.1, is similar to the item 1.c) and 2) of the proof of the Proposition 2.1.1, which satisfies $(N1)$ and $(N3)$. For $(N2)$ and $(N4)$ we divide the proof in two parts:

i) We prove that (3.8), satisfies ($N2$).

Let $u, v \in \mathrm{H}^2_\varepsilon(\mathbb{R}^\mathbb{N})$ be generic.
The case $u = 0$ is trivial. When $u = 0$ we have $\|u\|_\varepsilon = 0$.
Also, we have that

$$\exists c > 0 : \quad \int_{\mathbb{R}^N} |u(x)|^2 dx \leq c \int_{\mathbb{R}^N} V_\varepsilon(x)|u(x)|^2 dx.$$

Let $R > 0$. If we define

$$Q(0, R) := \{x \in \mathbb{R}^N : V_\varepsilon(x) \leq 1\},$$

then

$$\mathbb{R}^N \backslash Q(0, R) := \{x \in \mathbb{R}^\mathbb{N} : V_\varepsilon(x) > 1\}.$$

Let's assume that $\|u\|^2_\varepsilon = 0$ and $V_\varepsilon(x) > 1$, $\forall |x| \geq R$. Let's consider that

$$\forall x \in \mathbb{R}^N \backslash Q(0, R) : V_\varepsilon(x) \geq \inf\{V_\varepsilon / x \in \mathbb{R}^N \backslash Q(0, R)\} = (V_\varepsilon)_R,$$

and

$$\forall x \in \mathbb{R}^N \backslash Q(0, R) : \frac{V_\varepsilon}{(V_\varepsilon)_R} \geq 1,$$

then, by (3.8), we get

$$\begin{aligned}\int_{\mathbb{R}^N} V_\varepsilon(x)|u(x)|^2 dx &= \int_{Q(0,R)} V_\varepsilon(x)|u(x)|^2 dx + \int_{\mathbb{R}^N \backslash Q(0,R)} V_\varepsilon(x)|u(x)|^2 dx, \\ &\leq \|V_\varepsilon\|_{\mathrm{L}^\infty(Q(0,R))} \int_{Q(0,R)} |u(x)|^2 dx + \int_{\mathbb{R}^N \backslash Q(0,R)} \frac{1}{(V_\varepsilon)_R} |u(x)|^2 dx, \\ &\leq \max\left\{\frac{1}{(V_\varepsilon)_R}, \|V_\varepsilon\|_{\mathrm{L}^\infty(Q(0,R))}\right\} \int_{\mathbb{R}^N} |u(x)|^2 dx.\end{aligned}$$

Then,

$$\|u\|^2_{\mathrm{H}^2} \leq \max\left\{1, \frac{1}{(V_\varepsilon)_R}, \|V_\varepsilon\|_{\mathrm{L}^\infty(Q(0,R))}\right\} \|u\|^2_\varepsilon.$$

So, by the previous analysis, to see that in fact $u = 0$ if $\|u\|_\varepsilon = 0$, we have that

$$\int_{\mathbb{R}^N} |\Delta u(x)|^2 dx + \int_{\mathbb{R}^N} V_\varepsilon(x)|u(x)|^2 dx = 0,$$

that is, $\|u\|_{\mathrm{H}^2_\varepsilon(\mathbb{R}^N)} = 0$ and $\|u\|_{\mathrm{L}^2(\mathbb{R}^N; V_\varepsilon(x)dx)} = 0$. And by (2.1) and by the equivalence of norms of (3.8) and (2.31), we have that

$$0 = \|u\|^2_\varepsilon \geq c\|u\|^2_{\mathrm{H}^2(\mathbb{R}^N)} = 0,$$

for some constant $c > 0$. Then, $u = 0$.

ii) We prove the triangular inequality by using Minkowski's inequality

$$\begin{aligned}
\|u+v\|_\varepsilon &= \left(\int_{\mathbb{R}^N} \left(|\Delta u(x)+\Delta v(x)|^2 + V_\varepsilon(x)(u(x)+v(x))^2\right) dx\right)^{1/2} \\
&\leq \left(\int_{\mathbb{R}^N} |\Delta u+\Delta v|^2 dx + \int_{\mathbb{R}^N} (V_\varepsilon^{1/2}(x)(u(x)+v(x)))^2 dx\right)^{1/2} \\
&\leq \left(\int_{\mathbb{R}^N} |\Delta u|^2 dx + \int_{\mathbb{R}^N} |\Delta v|^2 dx + \int_{\mathbb{R}^N} V_\varepsilon(x) u^2(x) dx + \int_{\mathbb{R}^N} V_\varepsilon(x) v^2(x) dx\right)^{1/2} \\
&\leq \left(\int_{\mathbb{R}^N} \left(|\Delta u|^2 + V_\varepsilon(x)u^2(x)\right) dx\right)^{1/2} + \left(\int_{\mathbb{R}^N} \left(|\Delta v|^2 + V_\varepsilon(x)v^2(x)\right) dx\right)^{1/2} \\
&= \|u\|_\varepsilon + \|v\|_\varepsilon.
\end{aligned}$$

Since u and v were chosen arbitarily the proof is done.

□

Remark 3.2. Let's observe that, by the completion theorem (see e.g. [24]),

$$\overline{C_0^\infty(\mathbb{R}^N)} = \mathrm{H}^2_\varepsilon(\mathbb{R}^N), \text{ in the norm } \|\cdot\|_\varepsilon.$$

Remark 3.3. As a consequence for e.g. of Lemma 4.1 in [13], we have the following compact injection:

$$\forall q \in [p, p^*), \quad \mathrm{H}^2_\varepsilon(\mathbb{R}^N) \subseteq \mathrm{L}^q\left(\mathbb{R}^N\right),$$

with $1 < p < \infty$. In particular, there is $C_q > 0$ such that

$$\forall w \in \mathrm{H}^2_\varepsilon(\mathbb{R}^N) : \|w\|_{\mathrm{L}^q(\mathbb{R}^N)} \leqslant C_q \|w\|_\varepsilon.$$

3.2 Properties of the associated functional I_ε

We shall use the following theorem to prove multiplicity of critical points of the functional I_ε.

Let E be a Banach space and Σ_E be as in (2.46).

Theorem 3.1. *Let $M \in \Sigma_E$ be a C^1 sub-manifold of E and let $f \in \mathrm{C}^1(E)$ be even. Suppose that (M, f) satisfy the Palais-Smale condition and let*

$$C_k(f) = \inf_{A \in \mathcal{A}_k(M)} \max_{u \in A} f(u),$$

where

$$\mathcal{A}_k(M) = \{A \in \Sigma_E \cap M : \gamma(A) \geqslant k\}.$$

If $C_k(f) \in \mathbb{R}$, then $C_k(f)$ is a critical value for f. Moreover, if exists $m \in \mathbb{N}$ such that $c \equiv C_k(f) = \cdots = C_{k+m}(f)$, then $\gamma(K_c) \geqslant m+1$. In particular, if $m > 1$, then K_c, contains infinitely many elements.

Remark 3.4. In the previous theorem, (M, f) satisfies the Palais-Smale condition, where M is a manifold, and (M, f) means that a functional f is defined over a manifold M, for this see [19]. Also, the next is an abuse of notation, the intersection of (2.46) and of a manifold M is equal to $\Sigma_E \cap M$, with the respective conditions over M and f mentioned in the same Theorem 3.1.

The set

$$K_c = \left\{u \in \mathrm{H}^2_\varepsilon : I_\varepsilon(u) = c \wedge I'_\varepsilon(u) = 0\right\},$$

is the set of critical points corresponding to the value c and K_c is compact for any $c \in \mathbb{R}$ whenever I_ε satisfies the Palais-Smale condition, for this see [19].

First of all, we adapt the theorem notation to our problem. For every $k \in \mathbb{N}$ and $\varepsilon > 0$, we write:

a) $\Sigma_\varepsilon = \Sigma_{\mathrm{H}^2_\varepsilon}$,
b) $\mathcal{A}_{k,\varepsilon} = \mathcal{A}_k(\mathcal{M}_\varepsilon)$,
c) $c_{k,\varepsilon} = C_k(J_\varepsilon) = J_\varepsilon\left(u_{k,\varepsilon}\right)$, where $\left(u_{k,\varepsilon}\right)_{k\in\mathbb{N}} \subseteq \mathcal{M}_\varepsilon$.

We will first verify that $\mathcal{M}_\varepsilon$ is a sub-manifold of class C^1 in H^2_ε. Then, we prove that $I_\varepsilon \in \mathrm{C}^1\left(\mathrm{H}^2_\varepsilon\right)$. Finally we prove that I_ε satisfies the Palais-Smale (PS) sequence.

Proposition 3.2.1. *Let $\varepsilon > 0$. Let Σ_E be as in (2.46). Then, $\mathcal{M}_\varepsilon \in \Sigma_E$.*

1. *$\mathcal{M}_\varepsilon$ is closed in $\mathrm{H}^2_\varepsilon(\mathbb{R}^N)$,*
2. *$\mathcal{M}_\varepsilon = -\mathcal{M}_\varepsilon$,*
3. *$0 \notin \mathcal{M}_\varepsilon$.*

Proof. Let us show that $\mathcal{M}_\varepsilon \in \Sigma_\varepsilon$.

1. Let us prove that $\mathcal{M}_\varepsilon$ is closed in $\mathrm{H}^2_\varepsilon(\mathbb{R}^N)$. Let $u \in \overline{\mathcal{M}_\varepsilon}$, then there exists a sequence $(u_n)_{n\in\mathbb{N}} \in \mathcal{M}_\varepsilon$ such that, for all $n \in \mathbb{N}$:

$$\lim_{n\to+\infty} \|u_n - u\|_\varepsilon = 0.$$

By Remark 3.2 and by point (2.2), we know that the norm $\|\cdot\|_\varepsilon$ dominates $\|\cdot\|_{\mathrm{L}^{p+1}(\Omega)}$. Then

$$\lim_{n\to+\infty} \left|\|u_n\|_{\mathrm{L}^{p+1}(\Omega)} - \|u\|_{\mathrm{L}^{p+1}(\Omega)}\right| \leqslant \lim_{n\to+\infty} \|u_n - u\|_{\mathrm{L}^{p+1}(\Omega)} = 0,$$

thus

$$\|u\|_{\mathrm{L}^{p+1}(\Omega)} = 1.$$

Therefore, it is clear that $u \in \mathcal{M}_\varepsilon$ by arbitrariness of u. Therefore, we have proved that $\mathcal{M}_\varepsilon = \overline{\mathcal{M}_\varepsilon}$.

2. From the definition of $\mathcal{M}_\varepsilon$, we directly get that

$$\begin{aligned}\mathcal{M}_\varepsilon &= \left\{u \in \mathrm{H}^2_\varepsilon : \int_{\mathbb{R}^N} |u|^{p+1} dx = 1\right\}\\ &= \left\{-u \in \mathrm{H}^2_\varepsilon : \int_{\mathbb{R}^N} |-u|^{p+1} dx = 1\right\}\\ &= \left\{-u \in \mathrm{H}^2_\varepsilon : \int_{\mathbb{R}^N} |u|^{p+1} dx = 1\right\}\\ &= -\left\{u \in \mathrm{H}^2_\varepsilon : \int_{\mathbb{R}^N} |u|^{p+1} dx = 1\right\}\\ &= -\mathcal{M}_\varepsilon.\end{aligned}$$

3. From the norm definition we have that $\|u\|_{L^{p+1}} = 0$ if and only if $u = 0$ and then since $\mathcal{M}_\varepsilon = \left\{u \in \mathrm{H}^2_\varepsilon(\mathbb{R}^N) : \int_{\mathbb{R}^N} |u(x)|^{p+1} dx = 1\right\}$, it is clear that $0 \notin \mathcal{M}_\varepsilon$.

□

Proposition 3.2.2. *$\mathcal{M}_\varepsilon$ is a C^1 manifold.*

To prove Proposition 3.2.2, some previous results are required. We begin by considering the functional

$$\begin{aligned} L : \mathrm{H}^2_\varepsilon(\mathbb{R}^N) &\longrightarrow \mathbb{R}\\ w &\longmapsto L(w) = \frac{\|w\|^{p+1}_{\mathrm{L}^{p+1}} - 1}{p+1},\end{aligned}$$

for $1 < p < \infty$.
Since norms are continuous and the composition of continuous functions is continuous, L is continuous.

Lemma 3.2. *L is of class C^1. Furthermore, its Fréchet differential,* $L' : \mathrm{H}^2_\varepsilon(\mathbb{R}^N) \to \left(\mathrm{H}^2_\varepsilon(\mathbb{R}^N)\right)'$ *is given by*

$$\langle L'(u), h\rangle = \int_{\mathbb{R}^N} u(x)|u(x)|^{p-1} h(x) dx. \tag{3.9}$$

Proof. 1. Let us first compute the directional derivative of L at $u \in \mathrm{H}^2_\varepsilon(\mathbb{R}^N)$, in the direction $h \in \mathrm{H}^2_\varepsilon(\mathbb{R}^N)$,

$$\begin{aligned}\left.\frac{\partial}{\partial t} L(u+th)\right|_{t=0} &= \left.\frac{\partial}{\partial t}\frac{\|u+th\|_{\mathrm{L}^{p+1}} - 1}{p+1}\right|_{t=0}\\ &= \left.\int_{\mathbb{R}^N} |u(x)+th(x)|^p \frac{(u(x)+th(x))h(x)}{|u(x)+th(x)|} dx\right|_{t=0}\\ &= \int_{\mathbb{R}^N} u(x)|u(x)|^{p-1} h(x) dx.\end{aligned}$$

Then,

$$\frac{d}{dt}L(u+th)|_{t=0} = \int_{\mathbb{R}^N} u(x)|u(x)|^{p-1}h(x)dx,$$

and

$$\partial_h L(u) = \int_{\mathbb{R}^N} u(x)|u(x)|^{p-1}h(x)dx.$$

Then, by the Proposition 2.5.1, we get

$$\frac{d}{dt}L(u+th)|_{t=0} = \partial_h L(u).$$

We define the functional $\Phi : \mathrm{H}^2_\varepsilon(\mathbb{R}^N) \to \mathbb{R}$ given by

$$\Phi(h) = \int_{\mathbb{R}^N} u(x)|u(x)|^{p-1}h(x)dx.$$

Clearly Φ is linear, we shall prove that it's bounded, that is,

$$\exists c > 0, \forall h \in \mathrm{H}^2_\varepsilon(\mathbb{R}^N) : |\Phi(h)| \leq c\|h\|_\varepsilon.$$

Let's consider $C_q = C$ as in Remark 3.3. We choose $c > C\|u\|^p_\varepsilon$.
Let $h \in \mathrm{H}^2_\varepsilon(\mathbb{R}^N)$. By the Hölder's inequality, (2.10), CBS's inequality and Remark 3.3 we have that

$$\begin{aligned}
|\Phi(h)| &= \left|\int_{\mathbb{R}^N} u(x)|u(x)|^{p-1}h(x)dx\right| \\
&\leq \int_{\mathbb{R}^N} |u(x)|u(x)|^{p-1}h(x)|dx \\
&\leq \int_{\mathbb{R}^N} |u(x)|^p|h(x)|dx \\
&\leq \Big(\int_{\mathbb{R}^N} |u(x)|^{\frac{p(p+1)}{p}}dx\Big)^{\frac{p}{p+1}}\Big(\int_{\mathbb{R}^N} |h(x)|^{p+1}dx\Big)^{\frac{1}{p+1}} \\
&= \|u\|^p_{\mathrm{L}^{p+1}(\mathbb{R}^N)}\|h\|_{\mathrm{L}^{p+1}(\mathbb{R}^N)} \\
&\leq C\|u\|^p_\varepsilon\|h\|_\varepsilon.
\end{aligned}$$

Hence the functional Φ is bounded and therefore $\Phi \in \mathrm{H}^2_\varepsilon(\mathbb{R}^N)$ is Gateaux differentiable, namely

$$L'_G(u)h = \Phi(h).$$

The arbitrariness of u, h implies that the Gateaux differential of L exists at every $u \in \mathrm{H}^2_\varepsilon(\mathbb{R}^N)$.

2. For every $u, w \in \mathrm{H}^2_\varepsilon(\mathbb{R}^N)$, let us find an upper bound for $\|L'_G(u) - L'_G(w)\|_{(\mathrm{H}^2_\varepsilon(\mathbb{R}^N))'}$. Let $u, w, v \in \mathrm{H}^2_\varepsilon(\mathbb{R}^N)$, then, by Hölder inequality and Remark 3.3,

$$\begin{aligned}
|\langle L'_G(u) - L'_G(w), v\rangle| &= |\langle L'_G(u), v\rangle - \langle L'_G(w), v\rangle| \\
&\leq \int_{\mathbb{R}^N} \Big|u(x)|u(x)|^{p-1} - w(x)|w(x)|^{p-1}\Big||v(x)|dx \\
&\leq \Big\|u|u|^{p-1} - w|w|^{p-1}\Big\|_{\mathrm{L}^{(p+1)/p}(\mathbb{R}^N)}\|v\|_{\mathrm{L}^{p+1}(\mathbb{R}^N)} \\
&\leq C\Big\|u|u|^{p-1} - w|w|^{p-1}\Big\|_{\mathrm{L}^{(p+1)/p}(\mathbb{R}^N)}\|v\|_{\mathrm{H}^2_\varepsilon(\mathbb{R}^N)}.
\end{aligned}$$

By arbitrariness of u, w and v, we conclude that

$$\forall u, w \in \mathrm{H}^2_\varepsilon(\mathbb{R}^N) : \|L'_G(u) - L'_G(w)\|_{\left(\mathrm{H}^2_\varepsilon(\mathbb{R}^N)\right)'} \leq C \left\| u|u|^{p-1} - w|w|^{p-1} \right\|_{\mathrm{L}^{(p+1)/p}(\mathbb{R}^N)}. \tag{3.10}$$

3. Let us prove that L'_G is continuous, i.e., for all $u \in \mathrm{H}^2_\varepsilon(\mathbb{R}^N)$, and for all $(u_k)_{k\in\mathbb{N}} \subseteq \mathrm{H}^2_\varepsilon(\mathbb{R}^N)$

$$\lim_{k\to+\infty} \|u_k - u\|_{\mathrm{H}^2_\varepsilon(\mathbb{R}^N)} = 0 \Rightarrow \lim_{k\to+\infty} \left\| L'_G(u_k) - L'_G(u) \right\|_{\left(\mathrm{H}^2_\varepsilon(\mathbb{R}^N)\right)'} = 0. \tag{3.11}$$

Let $u \in \mathrm{H}^2_\varepsilon(\mathbb{R}^N)$ and $(u_k)_{k\in\mathbb{N}} \subseteq \mathrm{H}^2_\varepsilon(\mathbb{R}^N)$, such that

$$\lim_{k\to+\infty} \|u_k - u\|_{\mathrm{H}^2_\varepsilon(\mathbb{R}^N)} = 0. \tag{3.12}$$

Then, by, (3.10), triangle inequality, Hölder's inequality, and (3.11),

$$\begin{aligned}
&\left\| L'_G(u_k) - L'_G(u) \right\| \leq C \left\| u|u|^{p-1} - u_k|u_k|^{p-1} \right\|_{\mathrm{L}^{(p+1)/p}(\mathbb{R}^N)} \\
&\leq C \left\| u_k|u_k|^{p-1} - u|u_k|^{p-1} \right\|_{\mathrm{L}^{(p+1)/p}(\mathbb{R}^N)} \\
&+ C \left\| u|u_k|^{p-1} - u|u|^{p-1} \right\|_{\mathrm{L}^{(p+1)/p}(\mathbb{R}^N)} \\
&\leq C \left\| |u_k|^{p-1}(u - u_k) \right\|_{\mathrm{L}^{(p+1)/p}(\mathbb{R}^N)} \\
&+ C \left\| u|u_k|^{p-1} - u|u|^{p-1} \right\|_{\mathrm{L}^{(p+1)/p}(\mathbb{R}^N)} \\
&\leq C \left(\int_{\mathbb{R}^N} |u_k|^{\frac{p^2-1}{p}} |u_k - u|^{\frac{p+1}{p}}\, dx \right)^{\frac{p}{p+1}} \\
&+ C \left(\int_{\mathbb{R}^N} |u|^{\frac{p+1}{p}} \left| |u_k|^{p-1} - |u|^{p-1} \right|^{\frac{p+1}{p}}\, dx \right)^{\frac{p}{p+1}}
\end{aligned}$$

$$\begin{aligned}
&\leq C \left(\left\| u_k^{\frac{p^2-1}{p}} \right\|_{\mathrm{L}^{\frac{p}{p-1}}(\mathbb{R}^N)} \left\| (u_k - u)^{\frac{p+1}{p}} \right\|_{\mathrm{L}^p(\mathbb{R}^N)} \right)^{\frac{p}{p+1}} \\
&+ C_2 \|u\|^{(p-1)/(p+1)}_{\mathrm{L}^{(p+1)/(p-1)}(\mathbb{R}^N)} \left\| |u_k|^{p-1} - |u|^{p-1} \right\|^{p/(p+1)}_{\mathrm{L}^{(p+1)/p}(\mathbb{R}^N)} \\
&\leq C \left(\int_{\mathbb{R}^N} |u_k|^{p+1}\, dx \right)^{\frac{p-1}{p+1}} \left(\int_{\mathbb{R}^N} |u_k - u|^{p+1}\, dx \right)^{\frac{1}{p+1}} + C_3 \|u\|^{(p-1)/(p+1)}_{\mathrm{L}^{(p+1)/(p-1)}(\mathbb{R}^N)} \|u_k - u\|^{p/(p+1)}_{\mathrm{H}^2_\varepsilon(\mathbb{R}^N)} \\
&\leq C \|u_k\|^{p-1}_{\mathrm{L}^{p+1}(\mathbb{R}^N)} \|u_k - u\|_{\mathrm{L}^{p+1}(\mathbb{R}^N)} + C_3 \|u\|^{(p-1)/(p+1)}_{\mathrm{L}^{(p+1)/(p-1)}(\mathbb{R}^N)} \|u_k - u\|^{p/(p+1)}_{\mathrm{H}^2_\varepsilon(\mathbb{R}^N)} \\
&\leq C_1 \left(\|u_k\|^{p-1}_{\mathrm{H}^2_\varepsilon(\mathbb{R}^N)} \|u_k - u\|_{\mathrm{H}^2_\varepsilon(\mathbb{R}^N)} + \|u\|^{(p-1)/(p+1)}_{\mathrm{L}^{(p+1)/(p-1)}(\mathbb{R}^N)} \|u_k - u\|^{p/(p+1)}_{\mathrm{H}^2_\varepsilon(\mathbb{R}^N)} \right).
\end{aligned}$$

From (3.12), the previous inequality, and arbitrariness of $u, (u_k)_{k\in\mathbb{N}}$, we conclude that L'_G is continuous.

4. From (3.11), we get that $L \in C^1\left(\mathrm{H}^2_\varepsilon, \mathbb{R}\right)$. Since the Fréchet differential of L exists, it coincides with the Gateaux derivative, i.e., it is given by (3.9).

□

Let's prove Proposition 3.2.2.

Proof. From Lemma 3.2, and since $\mathcal{M} = L^{-1}(0)$, it suffices to prove that 0 is a regular value of L. Let us consider a critical value of L, i.e., $u_0 \in \mathcal{M}$ such that $L'(u_0) = 0$. Then $\forall v \in \mathrm{H}^2_\varepsilon$:

$$\left\langle L'(u_0), v\right\rangle = \int_{\mathbb{R}^N} u_0 \left|u_0\right|^{p-1} v dx = 0,$$

Then, by Riesz-Fréchet Representation Theorem, $u_0 = 0$.

□

Now, let us analyze the regularity of the functional through the following results.

Lemma 3.3. *The functional I_ε is of class C^1 and, for every $u, h \in \mathrm{H}^2_\varepsilon$*

$$\langle I_\varepsilon'(u), h\rangle = \int_{\mathbb{R}^N} [\Delta u(x)\Delta h(x) + V_\varepsilon(x)u(x)h(x)]dx = (u, h)_\varepsilon. \tag{3.13}$$

Proof. 1. First let's prove that I_ε is Gateaux differentiable. Let $u, h \in \mathrm{H}^2_\varepsilon$ and $\lambda \in \mathbb{R}$. By (3.2) we have that

$$\begin{aligned} I_\varepsilon(u+\lambda h) &= \frac{1}{2}\int_{\mathbb{R}^N} [|\Delta(u+\lambda h)(x)|^2 + V_\varepsilon(x)|(u+\lambda h)|^2]dx \\ &= \frac{1}{2}\int_{\mathbb{R}^N} |\Delta u(x)|^2 + 2\lambda\Delta u(x)\Delta h(x) + \lambda^2|h(x)|^2 + V_\varepsilon(x)|u(x)|^2 \\ &\quad + 2\lambda V_\varepsilon(x)u(x)h(x) + \lambda^2 V_\varepsilon(x)|h(x)|^2]dx, \end{aligned} \tag{3.14}$$

so that, by Fubini-Tonelli's theorem, we get

$$\frac{d}{d\lambda} I_\varepsilon(u+\lambda h) = \frac{1}{2}\int_{\mathbb{R}^N} [2\Delta u(x)\Delta h(x) + 2\lambda|h(x)|^2 + 2V_\varepsilon u(x)h(x) + 2\lambda V_\varepsilon|h(x)|^2]dx,$$

and

$$\frac{d}{d\lambda} I_\varepsilon(u+\lambda h)|_{\lambda=0} = \frac{1}{2}\int_{\mathbb{R}^N} [2\Delta u(x)\Delta h(x) + 2V_\varepsilon u(x)h(x)]dx.$$

Then,

$$\partial_h I_\varepsilon(u) = \int_{\mathbb{R}^{\mathbb{N}}} [\Delta u(x)\Delta h(x) + V_\varepsilon u(x)h(x)]dx.$$

Then, by the Proposition 2.5.1, we get

$$\frac{d}{d\lambda} I_\varepsilon(u+\lambda h)|_{\lambda=0} = \partial_h I_\varepsilon(u).$$

We define the functional $\Phi : \mathrm{H}^2_\varepsilon(\mathbb{R}^N) \to \mathbb{R}$ given by

$$\Phi(h) = \int_{\mathbb{R}} [\Delta u(x)\Delta h(x) + V_\varepsilon u(x)h(x)]dx.$$

Clearly Φ is linear.
We shall prove that it's bounded, that is,

$$\exists c > 0, \forall h \in \mathrm{H}^2_\varepsilon(\mathbb{R}^N) : |\Phi(h)| \leq c\|h\|_\varepsilon. \tag{3.15}$$

We choose $c > 2\|u\|_\varepsilon$.
Let $h \in \mathrm{H}^2_\varepsilon(\mathbb{R}^N)$. By the Hölder's inequality, (2.10), and CBS's inequality and we have that

$$\begin{aligned}
|\Phi(h)| &= \left|\int_{\mathbb{R}^N} [\Delta u(x)\Delta h(x) + V_\varepsilon u(x)h(x)]dx\right| \\
&\leq \int_{\mathbb{R}^N} [|\Delta u(x)\Delta h(x)| + \int_{\mathbb{R}^N} |V_\varepsilon u(x)h(x)|dx \\
&\leq \int_{\mathbb{R}^N} |\Delta u(x)||\Delta h(x)|dx + \int_{\mathbb{R}^N} |[V_\varepsilon]^{1/2}u(x)[V_\varepsilon]^{1/2}h(x)|]dx. \\
&\leq \left(\int_{\mathbb{R}^N} |\Delta u(x)|^2dx\right)^{1/2} \left(\int_{\mathbb{R}^N} |\Delta h(x)|^2dx\right)^{1/2} \\
&\quad + \left(\int_{\mathbb{R}^N} V_\varepsilon|u(x)|^2dx\right)^{1/2} \left(\int_{\mathbb{R}^N} V_\varepsilon|h(x)|^2dx\right)^{1/2} \\
&= \|\Delta u\|_{\mathrm{L}^2(\mathbb{R}^N)}\|\Delta h\|_{\mathrm{L}^2(\mathbb{R}^N)} + \|V_\varepsilon^{1/2}u\|_{\mathrm{L}^2(\mathbb{R}^N)}\|V_\varepsilon^{1/2}h\|_{\mathrm{L}^2(\mathbb{R}^N)} \\
&\leq 2\|u\|_\varepsilon\|h\|_\varepsilon.
\end{aligned}$$

Hence the functional Φ is bounded and therefore $\Phi \in \mathrm{H}^2_\varepsilon(\mathbb{R}^N)$ is Gateaux differentiable, namely

$$I'_{\varepsilon G}(u)h = \Phi(h).$$

2. Let's prove that I_ε is Fréchet differentiable. By (3.14) we have that

$$I_\varepsilon(u+h) = \frac{1}{2}\int_{\mathbb{R}^N} [|\Delta u(x) + \Delta h(x)|^2 + V_\varepsilon|u(x) + h(x)|^2]dx,$$

so that

$$\begin{aligned}
I_\varepsilon(u+h) - I_\varepsilon(u) &= \frac{1}{2}\int_{\mathbb{R}^N} [|\Delta u(x) + \Delta h(x)|^2 + V_\varepsilon|u(x) + h(x)|^2]dx - \\
&\quad - \int_{\mathbb{R}^N} |\Delta u(x)|^2 + V_\varepsilon|u(x)|^2]dx) \\
&= \int_{\mathbb{R}^N} [\Delta u(x)\Delta h(x) + V_\varepsilon(x)u(x)h(x)]dx \\
&\quad + \frac{1}{2}\int_{\mathbb{R}^N} [|\Delta h(x)|^2 + V_\varepsilon|h(x)|^2]dx \\
&= (u,h)_\varepsilon + \frac{1}{2}\|h\|^2_\varepsilon.
\end{aligned}$$

Now, since $(u,\cdot)_\varepsilon \in (\mathrm{H}^2_\varepsilon)'$, we denote by

$$g(h) = \frac{1}{2}\|h\|^2_\varepsilon.$$

Then,

$$\lim_{h\to 0}\frac{|g(h)|}{\|h\|_{\mathrm{H}_\varepsilon}} = \frac{1}{2}\lim_{h\to 0}\|h\|_{\mathrm{H}_\varepsilon} = 0.$$

Then,

$$g(h) = o(h).$$

Then we have that I_ε is differentiable and its Fréchet differential is given by (3.13). We conclude that I_ε is Fréchet differentiable by the arbitrariness of u, h, y and λ. Finally, by the CBS inequality, Lemma 2.22, we have that for any $u, v \in \mathrm{H}^2_\varepsilon$

$$\langle I'_\varepsilon(u), v\rangle = (u,v)_\varepsilon \leq \|u\|_\varepsilon \|v\|_\varepsilon,$$

which implies that

$$\forall u \in \mathrm{H}^2_\varepsilon : \|I'_\varepsilon(u)\|_{(\mathrm{H}^2_\varepsilon)'} \leq \|u\|_\varepsilon. \tag{3.16}$$

so that I'_ε is continuous and linear. Therefore I_ε is of class C^1.

□

Lemma 3.4. *I_ε satisfies the Palais-Smale condition.*

Proof. Let $(u_n)_{n\in\mathbb{N}} \subseteq \mathrm{H}^2_\varepsilon$, such that
i) $(I_\varepsilon(u_n))_{n\in\mathbb{N}}$ is bounded, i.e.,

$$\exists c > 0, \forall n \in \mathbb{N} : 0 \leq I_\varepsilon(u_n) \leq c;$$

ii) $I'_\varepsilon(u_n) \longrightarrow 0$, as $n \longrightarrow +\infty$ in $(\mathrm{H}^2_\varepsilon)'$.
We have to prove that there exists a convergent subsequence of $(u_n)_{n\in\mathbb{N}}$, say $(u_{n_k})_{k\in\mathbb{N}}$.
By (3.2), with $I_\varepsilon(u_n) = \|u_n\|_\varepsilon = (2c)^{\frac{1}{2}}$, where $c > 0$ and from assumption i), we have that $\|u_n\|_\varepsilon \leq (2c)^{\frac{1}{2}}$, for every $n \in \mathbb{N}$.
By (3.13) togheter with (3.15) and from assumption ii), we have that

$$\lim_{n\to+\infty} ||I'_\varepsilon(u_n)||_{(\mathrm{H}^2_\varepsilon(\mathbb{R}^N))'} = 0,$$

so that I'_ε converges to 0 strongly in the norm $\|\cdot\|_{(\mathrm{H}^2_\varepsilon(\mathbb{R}^N))'}$. Then, by Theorem 2.7, we have that I'_ε converges weakly to 0 in $\sigma(\mathrm{H}^2_\varepsilon(\mathbb{R}^N), (\mathrm{H}^2_\varepsilon(\mathbb{R}^N))')$.
Since $(\|u_n\|_\varepsilon)_{n\in\mathbb{N}}$ is bounded, there exist a subsequence $(u_{n_k})_{k\in\mathbb{N}} \subseteq \mathrm{H}^2_\varepsilon$ and some $u \in \mathrm{H}^2_\varepsilon$ such that $u_{n_k} \rightharpoonup u$, as $k \longrightarrow +\infty$, i.e.,

$$\forall \nu \in (\mathrm{H}^2_\varepsilon(\mathbb{R}^N))' : \langle \nu, u_{n_k} - u\rangle \longrightarrow 0 \quad as \quad k \longrightarrow +\infty. \tag{3.17}$$

From assumption ii) and by (3.17), it follows that

$$\langle I'_\varepsilon(u_{n_k}) - I'_\varepsilon(u), u_{n_k} - u\rangle \longrightarrow 0 \quad as \quad k \longrightarrow +\infty. \tag{3.18}$$

By using the definition of $I_{\varepsilon}^{'}(u)h$ in Lemma 3.3, we get

$$\begin{aligned}
\langle I_{\varepsilon}^{'}(u_{n_k}) - I_{\varepsilon}^{'}(u), u_{n_k} - u\rangle &= I_{\varepsilon}^{'}(u_{n_k})(u_{n_k} - u) - I_{\varepsilon}^{'}(u)(u_{n_k} - u) \\
&= \int_{\mathbb{R}^{\mathbb{N}}} (\Delta u_{n_k}\Delta u_{n_k} - \Delta u\Delta u_{n_k} + V_{\varepsilon}u_{n_k}u_{n_k} - V_{\varepsilon}u_{n_k}u)dx \\
&- \int_{\mathbb{R}^{\mathbb{N}}} (\Delta u_{n_k}\Delta u - \Delta u\Delta u + V_{\varepsilon}u_{n_k}u - V_{\varepsilon}uu)dx \\
&= \int_{\mathbb{R}^{\mathbb{N}}} [(\Delta u_{n_k})^2 - 2\Delta u_{n_k}\Delta u + (\Delta u)^2 + V_{\varepsilon}(u_{n_k}^2 - 2u_{n_k}u + u^2)]dx \\
&= \int_{\mathbb{R}^{\mathbb{N}}} (\Delta u_{n_k} - \Delta u)^2 dx + \int_{\mathbb{R}^{\mathbb{N}}} V_{\varepsilon}(u_{n_k} - u)^2 dx \\
&= \|u_{n_k} - u\|_{\varepsilon}^2.
\end{aligned}$$

Then, by point (3.18) and from the previous equality, we have that $(u_{n_k})_{k\in\mathbb{N}}$ converges to u in $\mathrm{H}_{\varepsilon}^2$ as $k \longrightarrow +\infty$.
Whence, the functional I_{ε} satisfies the PS sequence.

□

3.3 Multiplicity

In this section we summarize the results obtained and apply Theorem 3.1.

Remark 3.5. $I_{\varepsilon}^{'}$ is bounded from below, since its definition involves a norm explicitly. All the results of the previous section hold for the space $\mathrm{H}_{\varepsilon}^2$ and immediately are inherited to the manifold $\mathcal{M}_{\varepsilon}$ as referenced in [5].

Now we present the result that establishes the multiplicity of solutions for the problem of solving a non-linear biharmonic Laplace Schrödinger equation.

Proposition 3.3.1. *For $\varepsilon > 0$, I_{ε} has a sequence of infinitely many critical points.*

Proof. We obtain this result by verifying the hypothesis of Theorem 3.1.

1. In the Proposition 3.2.1, we show that $\mathcal{M}_{\varepsilon} \in \Sigma_{\varepsilon}$ is a C^1 sub-manifold of $\mathrm{H}_{\varepsilon}^2$.
2. $I_{\varepsilon} \in C^1\left(\mathrm{H}_{\varepsilon}^2\right)$ is verified first by finding the Gateaux differential as in Lemma 3.3 then proving that it is continuous, that is, Fréchet differentiability (3.13).
3. In the Lemma 3.4, we saw that $\left(\mathrm{H}_{\varepsilon}^2(\mathbb{R}^N), I_{\varepsilon}\right)$ satisfies the PS sequence, then by Remark 3.2, we have that $(\mathcal{M}_{\varepsilon}, J_{\varepsilon})$ also does it.
4. At this point, using the Theorem 3.1 we define

$$c_{k,\varepsilon} = \inf_{A\in\mathcal{A}_{k,\varepsilon}} \max_{u\in A} I_{\varepsilon}(u),$$

where

$$\mathcal{A}_{k,\varepsilon} = \{A \in \Sigma_{\varepsilon} \cap \mathcal{M}_{\varepsilon} : \gamma(A) \geqslant k\}.$$

It remains to prove that these critical values are real numbers.

5. Let us consider the definition of $c_{k,\varepsilon}$ in the Theorem 3.1 and used on the previous item (4). Since by Remark 3.1, I_ε (is a functional that) involves the norm $\|\cdot\|_\varepsilon$, for $1 < p < \infty$ we have that $\max_{u \in A} I_\varepsilon(u)$ is a non-negative real number. And, by taking the infimum over the set $A \in \mathcal{A}_{k,\varepsilon}$ then $c_{k,\varepsilon} \in \mathbb{R}^+$.

6. Thus, thanks to items (1)-(5) and applying Theorem 3.1, we fix a topological level $k \in \mathbb{N}$ and for $\varepsilon > 0$ we obtain $u_{k,\varepsilon}$ which converges, up to a scaling, to a corresponding solution of (P'_ε), this was showed by (3.7) in (3.6). And in that way, the set (K_c) contains infinitely elements, i.e., I_ε, has a sequence of infinitely many critical points.

$\square$

Chapter 4

Conclusions and recommendations

4.1 Conclusions

In this project we applied a Ljusternik-Schnirelman scheme to show the multiplicity of solutions of the quasi-linear problem:

$$\begin{cases} -\varepsilon^4\Delta^2 v(x) + V(x)v(x) - |v(x)|^{p-1}v(x) &= 0, \qquad x \in \mathbb{R}^N, \\ v(x) &\longrightarrow 0, \quad \text{as } |x| \longrightarrow +\infty, \end{cases} \tag{4.1}$$

where $\varepsilon^4 = \dfrac{\hbar^4}{4}$, and the biharmonic Laplace operator is given by

$$\Delta^2 = \Delta \circ \Delta.$$

We assumed that $1 < p+1 < 2^*$, where

$$2^* = \begin{cases} \dfrac{2N}{N-2}, & \text{if } N \geq 3; \\ +\infty, & \text{if } N = 1, 2, \end{cases}$$

We dealt with an equivalent version of (4.1):

$$(P'_\varepsilon) \begin{cases} -\Delta^2 u(x) + V_\varepsilon(x)u(x) - |u(x)|^{p-1}u(x) = 0, & x \in \mathbb{R}^N, \\ u(x) \longrightarrow 0, & \text{as } |x| \longrightarrow +\infty, \end{cases}$$

using the scaling $v(x) = u\left(\varepsilon^\alpha x\right)$ and $V_\varepsilon(x) = V\left(\varepsilon^{-\alpha}x\right)$, for $\varepsilon > 0$, $\alpha = -1$, and under the following assumptions for the potencial:
(V1) $V \in C\left(\mathbb{R}^N\right)$ is a non-negative function;
(V2) $V(x) \longrightarrow +\infty$, as $|x| \longrightarrow +\infty$;
(V3) $\{V = \inf\{V\} = 0\} \neq \emptyset$.
Associated to (P'_ε) is the functional $I_\varepsilon : \mathcal{M}_\varepsilon \subseteq \mathrm{H}^2_\varepsilon \longrightarrow \mathbb{R}$, given by

$$I_\varepsilon(u) = \frac{1}{2}\int_{\mathbb{R}^N} \left[|\Delta u(x)|^2 + V_\varepsilon(x)|u(x)|^2\right] dx,$$

defined on the manifold

$$\mathcal{M}_\varepsilon = \left\{u \in \mathrm{H}^2_\varepsilon \middle/ \int_{\mathbb{R}^N} |u(x)|^{p+1} dx = 1\right\},$$

with

$$\mathrm{H}_\varepsilon^2 = \left\{ u \in \mathrm{H}^2\left(\mathbb{R}^N\right) \big/ \|u\|_\varepsilon = \left(\int_{\mathbb{R}^N} \left[|\Delta u(x)|^2 + V_\varepsilon(x)|u(x)|^2 \right] dx \right)^{1/2} < +\infty \right\}.$$

Thanks to Theorem 3.1, and by the results obtained in the previous Chapter, we conclude that the set of critical points of the functional I_ε, has infinitely many elements which through scaling, are solutions of (P'_ε).

4.2 Recommendations

1) We suggest to substitute the operator Δ^2 in (1.5) by the operator Δ^m, with m even having its definition as in [30] and [29], and work with this operator in another thesis.

2) It would be interesting to analyze the concentration problem related to (P'_ε), under additional assumptions for potential V.
   For instance, by defining $\Omega = \operatorname{int}\left\{x \in \mathbb{R}^N : V(x) = 0\right\} \neq \varnothing$ connected and smooth, we suggest to study asymptotic estimates on the boundary of Ω, e.g., that $u_{k,\varepsilon}$ verifies

$$\forall k \in \mathbb{N} : \lim_{\varepsilon \to 0} \max_{x \in \partial\Omega} \left| u_{k,\varepsilon}(x) \right| = 0$$

3) Yachay Tech University has been facing difficult moments, for almost 9 years, just because of political disagreements and bad administrators that has been focused on discrediting its activities and achiviments. All this without taking into consideration the effort of all its students whom, despite the lack of support, have obtained places into prestigious master's and doctoral programs.

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