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TÍTULO: Study of the DKP equation for the hyperbolic tangent potential

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Dedication

I dedicate my dissertation work to all my family. A special feeling of gratitude to my loving parents, Oscar and Mónica, with their words of encouragement and support always have motivated me to keep moving forward. To my siblings, Dany and Nico, for their unconditional friendship and always being there for me. Finally, my lovely partner, Gaby, for her help and support in accomplishing my goals.

This thesis is for you all...

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Resumen

En este trabajo de tesis presentamos la solución analítica de la ecuación DKP en presencia de un potencial tangente hiperbólico. La ecuación DKP se utiliza para describir partículas de espín uno o espín cero. En este trabajo hemos mostrado cómo, bajo la presencia de un potencial unidimensional, ambos sectores de la teoría DKP son equivalentes. Por esa razón, se eligió la teoría del espín uno para un análisis más detallado. Para resolver la ecuación DKP para partículas de espín uno, podemos dividir el espinor de diez componentes para simplificar el cálculo o resolverlo con las matrices completas de 10x10. Como resultado, se utilizan funciones hipergeométricas para deducir las soluciones de dispersión. Además, se explora el comportamiento asintótico de la solución. Esto nos permite calcular la corriente de probabilidad, paso necesario para obtener los coeficientes R y T. Esos coeficientes se calcularon en términos de funciones Gamma. Por último, dividimos el potencial tangente hiperbólico en cinco regiones diferentes. En la Región III del potencial, el coeficiente T es menor que cero. En consecuencia, para preservar la condición unitaria, el coeficiente R está obligado a ser mayor que uno. Conocido como la paradoja de Klein, este fenómeno ocurre cuando se reflejan más partículas de un potencial que el número de partículas incidentes en él, produciendo un efecto conocido como superradiancia.

Palabras Clave: Superradiancia, Coeficiente de reflexión, Coeficiente de transmisión, Ecuación de DKP, Paradoja de Klein.

Abstract

In this dissertation work, we present the analytical solution of the Duffin–Kemmer–Petiau (DKP) equation in the presence of a hyperbolic tangent potential. The DKP equation is used to describe particles either spin-one or spin-zero. In this work, we have shown how, under the presence of a one-dimension potential, both sectors of the DKP theory are equivalent. For that reason, the spin-one theory was chosen for further analysis. To solve the DKP equation for spin-one particles, we can partition the ten-component spinor to simplify the calculus or solve it with the full 10×10 matrices. As a result, hypergeometric functions are used to deduce the scattering solutions. Furthermore, the solution's asymptotic behavior is explored. This allows us to calculate the probability current, a necessary step to obtain the reflection (R) and transmission (T) coefficients. Those coefficients were calculated in terms of Gamma functions. At last, we divide how dividing the hyperbolic tangent potential into five different regions. In Region III of the potential, the T coefficient is less than zero. Consequently, to preserve the unitary condition, the R coefficient is forced to be greater than one. Known as the Klein Paradox, this phenomenon occurs when more particles reflect off a potential than the number of incident particles on it, producing an effect known as superradiance.

Keywords: Superradiance, Reflection coefficient, Transmission coefficient, DKP equation, Klein Paradox.

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1. Valladares, S.; Rojas, C. The superradiance phenomenon in spin–one particles. *Int. J. Mod. Phys. A*. **38**, 2350020, (**2023**).

Chapter 1

Introduction

When we want to study a particle that moves close to the speed of light ($\boldsymbol{c}$), the theory of Quantum Mechanics is no longer valid. In this case, we must recur to a more complex theory called Relativistic Quantum Mechanics. As a result of studying relativistic wave equations, we are able to understand a number of physical phenomena, including bound states, transmission resonances, and superradiance. In Relativistic Quantum Mechanics, the most common equation is the Klein–Gordon (KG) equation, which is a second-order differential equation that describes particles of spin-zero such as the Higgs boson, and the Dirac equation, which is a first-order differential equation that describes spin$-1/2$ particles. Nevertheless, there is one more first-order relativistic equation in the literature besides Dirac's. It is the Duffin–Kemmer–Petiau (DKP) equation[1]. The structure of the DKP equation is given in Eq (1.2). From now on, we will use natural units ($\hbar = 1$, $c = 1$).

$$\textit{Dirac equation}: \quad \left[i\gamma^{\mu}\partial_{\mu} - m\right]\psi(x) = 0. \tag{1.1}$$

$$\textit{DKP equation}: \quad \left[i\beta^{\mu}\partial_{\mu} - m\right]\psi(x) = 0. \tag{1.2}$$

There is a close resemblance of these two equations, as can be seen in equations (1.1) and (1.2). In this case, the gamma matrices are replaced by beta matrices which follow a more complex algebra; known as the DKP algebra.[2,3]. In contrast to the Dirac equation, which only applies to particles with spin-1/2, the DKP equation explains particles with spin–zero (scalar) and spin–one (vector)[4]. Interestingly, in the DKP equation, when the potential is one dimensional, the formalism for the spin-one is identical to the formalism for the spin-zero[5].

Due to its potential applications in a number of fields, such as nuclear and particle physics, cosmology, meson spectroscopy, and even nuclear-hadron interactions, the analytical solution of the DKP equation for a wide range of potential wells and barriers has attracted significant attention in recent years.[3,4,6–17]. In particular, a more detailed background of spin-one particle interactions can be provided by studying the DKP equation[18].

Several research studies have also been carried out on the superradiance phenomenon[6,8,19–22] in Relativistic Quantum Mechanics, which is a well-known phenomenon that occurs when step barriers are involved[8]. During superradiance, the reflection (R) coefficient becomes greater than one ($R > 1$), so the transmission (T) coefficient is less than zero ($T < 0$). In spite of this, the sum of both coefficients maintains the unitary relation $T + R = 1$. This phenomenon has been studied in the KG equation[20], the Dirac equation[19], and the DKP equation[6,8,21]. More in detail, the superradiance phenomenon describes the situation when the reflected current is greater than the incident current[22]. This phenomenon is closely related to the pair creation since more particles are reflected from the potential barrier than incident to it.

Scattering events attract attention in physics, especially to describe the atomic nucleus[23]. Among the several applications of the superradiance phenomenon in physics, the most important one, according to Mauricio Richartz et al., is the scattering events in rotating black holes and cylinders made of electrical conductive material. Nevertheless, in areas such as optics, quantum mechanics, astrophysics, and condensed matter physics, the superradiance phenomenon can also have different applications[22,24]. Therefore, a better understanding of this phenomenon is of high interest, mainly because of the other areas of physics it can be involved in.

There exist many kinds of smooth potentials. In particular, the hyperbolic tangent potential is a smooth barrier[20] in which its limiting case takes the form of a step potential. This potential is defined by,

$$V(x) = a\ \tanh(b\,x), \tag{1.3}$$

where a represents the height of the potential, and b gives the smoothness of the curve[3]. The shape of this potential is represented in Fig. (1.1) for different values of b. We can note that as b increases, the form of the potential changes into the well-known step potential.

1.1 Problem Statement

The behavior of spin–one and spin-zero particles can be obtained by studying the DKP equation. Even more, the superradiance effect can be observed under the influence of a smooth potential. This work aims to calculate an analytical solution for this equation, exploring the behavior of the reflection and transmission

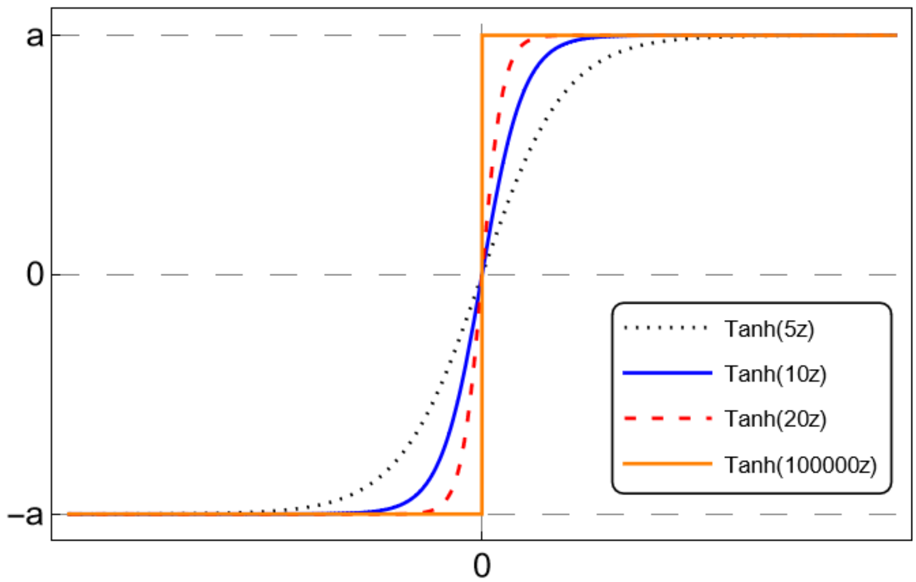

Figure 1.1: Hyperbolic tangent potential for $a = 1$ with $b = 1$ (black dotted line), $b = 5$ (blue solid line), $b = 20$ (red dashed line) and $b = 100000$ (orange solid line)

coefficients. It is also explained that for both sectors, the spin–zero and spin–one, there exists a relation between the DKP equation and the KG equation.

1.2 General and Specific Objectives

This thesis project aims at analyzing the behavior of the reflection (R) and transmission (T) coefficients for the hyperbolic tangent potential by using the Duffin–Kemmer–Petiau (DKP) equation, with the objective of affirming the presence of the superradiance phenomenon under the introduction of smooth barriers. Moreover, the procedure to achieve this objective is split into the following specific tasks:

- To derive the DKP equation for a generic potential $V(x)$ to obtain the reduced representation of the equation in terms of one independent Klein–Gordon type equation and two dependent equations.
- To solve the DKP equation for one sector, the spin–one case, by adding the hyperbolic tangent potential.
- To study the asymptotic behavior of the solution and obtain both the R and T coefficients.

- To explore the solution by splitting the potential into different regions according to their energy.
- To evaluate the presence of superradiance phenomenon and in which region of the potential it appears.

The following work is divided into four chapters. Chapter 2 shows the mathematical framework of the DKP equation for spin-one particles, as well as for spin-zero particles, explicitly demonstrating the correlation between the DKP equation and the Klien–Gordon equation in both cases. In Chapter 3, we introduce the hyperbolic tangent potential into the DKP equation and derive its solution in terms of hyperbolic functions. Additionally, the solution's asymptotic behavior and the R and T coefficients are derived. In Chapter 4, an analysis of the results is given, explaining the conditions for superradiance and showing the region where it occurs. Finally, Chapter 5, constitutes the conclusion chapter and possible further research related to this topic.

Chapter 2

Methodology

2.1 The Duffin–Kemmer–Petiau equation

With the introduction of an electromagnetic field interaction[6], the Duffin–Kemmer–Petiau (DKP) equation is calculated as follows:

$$\left[i\beta^{\mu}\left(\partial_{\mu}+ieA_{\mu}\right)-m\right]\Psi(x,t)=0, \tag{2.1}$$

where ∂_{μ} is the partial derivative, A_{μ} is the external potential, e is the elementary charge, m is the particle mass, and the matrices β^{μ} are a special type of matrices that can have one or more dimensions depending on the field of study. Moreover, β^{μ} accomplish the following relation[5,6,8,21]

$$\beta^{\mu}\beta^{\nu}\beta^{\lambda}+\beta^{\lambda}\beta^{\nu}\beta^{\mu}=g^{\mu\nu}\beta^{\lambda}+g^{\lambda\nu}\beta^{\mu}, \tag{2.2}$$

and $g^{\mu\nu}$ is the well known metric tensor with diagonal components $(1,-1,-1,-1)$. As said previously, the matrices β^{μ} generate a more complex algebra known as the DKP algebra which has three irreducible representations: a trivial, a five-dimensional, and a ten-dimensional representation. Nevertheless, the two last are the only ones that have a physical meaning since they describe the spin–zero and spin–one particles, respectively.

Even more, notice that a particle under this potential only moves along the x-axis. Hence we get

$$\begin{cases}A_0(x)=V(x)=a\tanh(bx),\\ A_1(x)=0,\end{cases}\quad,\quad\begin{cases}\partial_0=\partial t,\\ \partial_1=\partial x.\end{cases} \tag{2.3}$$

This research project aims to solve equation (2.1) for both types of particles for the hyperbolic tangent potential (Eq. 1.3). Remarkably, this kind of potential is time-independent. Therefore, the stationary solution for the DKP equation has the form $\Psi(x) = e^{iEt}\phi(x)$[6,21].

Therefore, we arrive to

$$\left\{i\beta^0\left[\partial_0 + ieA_0(x)\right] + i\beta^1\left[\partial_1 + ieA_1(x)\right] - m\right\}\phi(x) = 0. \tag{2.4}$$

$$\left[i\beta^0\partial_0 - e\beta^0 A_0(x) + i\beta^1\partial_1 - e\beta^1 A_1(x) - m\right]\phi(x) = 0. \tag{2.5}$$

$$\left\{\beta^0\left[E - V(x)\right] + i\beta^1\frac{d}{dx} - m\right\}\phi(x) = 0. \tag{2.6}$$

Using Eq. (2.3), we arrive to Eq. (2.6) which is the form of the DKP equation we are going to work on in the following sections*. Finally, it is worth mentioning that our wave function $\phi(x)$, similar to the Dirac equation, is a spinor, which will have 5 or 10 components depending on the particle type of work. There is, however, no independent relationship between the components of the DKP spinor, as opposed to the Dirac spinor[25].

2.2 Spin–zero particles

Independent on the sector of the DKP theory, the only component that changes is the one with the matrices β^μ. In the case of spin-zero, the β matrices are 5×5 dimension and are given by

$$\beta^0 = \begin{pmatrix} \theta & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix}. \tag{2.7}$$

$$\beta^i = \begin{pmatrix} \mathbf{0} & \rho^i \\ -\rho^i_T & \mathbf{0} \end{pmatrix}, \quad i = 1, 2, 3. \tag{2.8}$$

Moreover, each component in (2.7) and (2.8) correspond to a matrix. Explicitly, these elements are

$$\theta = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \tag{2.9}$$

$$\rho^1 = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \rho^2 = \begin{pmatrix} 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \rho^3 = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \end{pmatrix}. \tag{2.10}$$

*Remember that: $\hat{E} \rightarrow i\hbar\frac{\partial}{\partial_t}$; $\partial_\mu = (\partial_t, \partial_x)$

The components ρ_T and $\mathbf{0}$ are the transposed matrices of ρ and the zero matrix, respectively. By introducing these matrices into Eq. 2.6, we arrived to

$$\left\{\begin{pmatrix}0&1&0&0&0\\1&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0\end{pmatrix}[E-V(x)] + i\frac{d}{dx}\begin{pmatrix}0&0&-1&0&0\\0&0&0&0&0\\1&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0\end{pmatrix} - m\,\mathbf{I}\right\}\begin{pmatrix}\varphi_1\\\varphi_2\\\varphi_3\\\varphi_4\\\varphi_5\end{pmatrix} = 0, \tag{2.11}$$

where $\mathbf{I}$ is the identity matrix of 5×5 dimension. Then, by solving this algebraic equation, we arrive at the following system of equations.

$$\begin{cases}[E-V(x)]\,\varphi_2 - i\dfrac{d}{dx}\varphi_3 - m\,\varphi_1 = 0,\\ [E-V(x)]\,\varphi_1 - m\,\varphi_2 = 0,\\ i\dfrac{d}{dx}\varphi_1 - m\,\varphi_3 = 0,\\ -m\,\varphi_4 = 0,\\ -m\,\varphi_5 = 0.\end{cases} \tag{2.12}$$

There is no problem at this stage in determining that the time-independent DKP equation can be decomposed into

$$\begin{cases}\left\{\dfrac{d^2}{dx^2} + [E-V(x)]^2 - m^2\right\}\varphi_1 = 0,\\ \varphi_2 = \dfrac{1}{m}\,[E-V(x)]\,\varphi_1,\\ \varphi_3 = \dfrac{i}{m}\dfrac{d}{dx}\varphi_1.\end{cases} \tag{2.13}$$

The dependence on the first component of the spinor is clear in Eq. (2.13)[†]. Even more, by analyzing the obtained solution, we observe that the first equation in Eq. (2.13) corresponds to the Klein–Gordon (KG) equation. In other words, if φ_1 is a solution of the KG equation, then it is also a solution for the DKP equation. Considering that both equations are equivalent under minimal coupling, this correlation is not surprising[8,26].

[†]Notice that $\varphi_4 = \varphi_5 = 0$.

2.3 Spin–one particles

2.3.1 Direct calculation

Following a similar procedure as before, let us calculate the DKP equation for the spin–one sector. In this case, the β^μ matrices are 10×10 dimensions and are given by the following representation

$$\beta^0 = \begin{pmatrix} 0 & \bar{0} & \bar{0} & \bar{0} \\ \bar{0}^T & \mathbf{0} & \mathbf{1} & \mathbf{0} \\ \bar{0}^T & \mathbf{1} & \mathbf{0} & \mathbf{0} \\ \bar{0}^T & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{pmatrix}, \quad \beta^i = \begin{pmatrix} 0 & \bar{0} & e_i & \bar{0} \\ \bar{0}^T & \mathbf{0} & \mathbf{0} & -is_i \\ -e_i^T & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \bar{0}^T & -is_i & \mathbf{0} & \mathbf{0} \end{pmatrix}, \quad i = 1, 2, 3. \tag{2.14}$$

where the component s_i are 3×3 matrices which take the form $(s_i)_{jk} = -i\epsilon_{ijk}$ depending on an even permutation, an odd permutation, and repeated indices, respectively[8,21,26]. Moreover, $\mathbf{1}$ and $\mathbf{0}$ are the 3×3 unity and zero matrices, $\bar{0} = (0, 0, 0)$, and e_i are

$$e_1 = (1, 0, 0), \quad e_2 = (0, 1, 0), \quad e_3 = (0, 0, 1).$$

Now, let us recall Eq. (2.6) and introduce the respective β^0 and β^1 matrices into it. In that sense, the DKP equation reduces to

$$\left\{ \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} [E - V(x)] + i\frac{d}{dx} \begin{pmatrix} 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} - m\,\mathbf{I} \right\} \begin{pmatrix} \varphi_1 \\ \varphi_2 \\ \varphi_3 \\ \varphi_4 \\ \varphi_5 \\ \varphi_6 \\ \varphi_7 \\ \varphi_8 \\ \varphi_9 \\ \varphi_{10} \end{pmatrix} = 0, \tag{2.15}$$

on this sector of the theory $\mathbf{I}$ is the 10×10 identity matrix. As a result of solving Eq. (2.15), we obtain the following explicit form.

$$
\begin{cases}
i\dfrac{d}{dx}\varphi_5 - m\,\varphi_1 = 0,\\
[E - V(x)]\,\varphi_5 - m\,\varphi_2 = 0,\\
[E - V(x)]\,\varphi_6 - i\dfrac{d}{dx}\varphi_{10} - m\,\varphi_3 = 0,\\
[E - V(x)]\,\varphi_7 + i\dfrac{d}{dx}\varphi_9 - m\,\varphi_4 = 0,\\
[E - V(x)]\,\varphi_2 - i\dfrac{d}{dx}\varphi_1 - m\,\varphi_5 = 0,\\
[E - V(x)]\,\varphi_3 - m\,\varphi_6 = 0,\\
[E - V(x)]\,\varphi_4 - m\,\varphi_7 = 0,\\
-m\,\varphi_8 = 0,\\
-i\dfrac{d}{dx}\varphi_4 - m\,\varphi_9 = 0,\\
i\dfrac{d}{dx}\varphi_3 - m\,\varphi_{10} = 0.
\end{cases}
\tag{2.16}
$$

Let us use the clear dependence of φ_3, φ_4, and φ_5 with the objective of reducing Eq (2.16) into a more suitable equation. By doing that we obtain the following system

$$
\begin{cases}
[E - V(x)]\,\varphi_6 - i\dfrac{d}{dx}\varphi_{10} - m\,\varphi_3 = 0,\\
[E - V(x)]\,\varphi_7 + i\dfrac{d}{dx}\varphi_9 - m\,\varphi_4 = 0,\\
[E - V(x)]\,\varphi_2 - i\dfrac{d}{dx}\varphi_1 - m\,\varphi_5 = 0,\\
\varphi_1 = \dfrac{i}{m}\dfrac{d}{dx}\varphi_5,\\
\varphi_2 = \dfrac{1}{m}\,[E - V(x)]\,\varphi_5,\\
\varphi_6 = \dfrac{1}{m}\,[E - V(x)]\,\varphi_3,\\
\varphi_7 = \dfrac{1}{m}\,[E - V(x)]\,\varphi_4,\\
\varphi_9 = -\dfrac{i}{m}\dfrac{d}{dx}\varphi_4,\\
\varphi_{10} = \dfrac{i}{m}\dfrac{d}{dx}\varphi_3.
\end{cases}
\tag{2.17}
$$

Additionally, by replacing φ_1, φ_2, φ_6, φ_7, φ_9 and φ_{10} into the first three components of equation (2.17), we arrive to the final system

$$\begin{cases} \left[\dfrac{d^2}{dx^2} + [E - V(x)]^2 - m^2\right]\varphi_5 = 0, \\ \left[\dfrac{d^2}{dx^2} + [E - V(x)]^2 - m^2\right]\varphi_4 = 0, \\ \left[\dfrac{d^2}{dx^2} + [E - V(x)]^2 - m^2\right]\varphi_3 = 0, \\ \varphi_1 = \dfrac{i}{m}\dfrac{d}{dx}\varphi_5, \\ \varphi_2 = \dfrac{1}{m}[E - V(x)]\,\varphi_5, \\ \varphi_6 = \dfrac{1}{m}[E - V(x)]\,\varphi_3, \\ \varphi_7 = \dfrac{1}{m}[E - V(x)]\,\varphi_4, \\ \varphi_9 = -\dfrac{i}{m}\dfrac{d}{dx}\varphi_4, \\ \varphi_{10} = \dfrac{i}{m}\dfrac{d}{dx}\varphi_3. \end{cases} \tag{2.18}$$

Notice that once more, we have simplified the DKP equation into a KG equation, here the component $\varphi_8 = 0$. Even though this equivalence between equations has been shown by particles moving along the x-axis, which implies using the β^1 matrix, the same result can be obtained if the particle moves along the other axes, that is using the β^2 or β^3 matrices[26].

2.3.2 Partitioning the Duffin–Kemmer–Petiau spinor

It exists another way to address the case for the spin–one particles, which is more direct but requires us to make a partition of the DKP spin–one spinor into three subcomponents. First, let us start with the full form of the DKP spinor for spin-one particles.

$$\phi(x) = \begin{pmatrix} \varphi_1 \\ \varphi_2 \\ \varphi_3 \\ \varphi_4 \\ \varphi_5 \\ \varphi_6 \\ \varphi_7 \\ \varphi_8 \\ \varphi_9 \\ \varphi_{10} \end{pmatrix}. \tag{2.19}$$

To simplify the discussion, let us rewrite the spinor of Eq. (2.19) in a more compact form as follows in order to make the analysis as simple as possible[27]

$$\phi(x) = \begin{pmatrix} \omega \\ \mathbf{A} \\ \mathbf{B} \\ \mathbf{C} \end{pmatrix}, \tag{2.20}$$

where **A**, **B**, **C** are vector components of dimension 3×1, where each vector is made of subcomponents A_i, B_i and C_i with $i = 1, 2, 3$, respectively.[6,7,21,27]. Nevertheless, it is easy to verify that the components of each vector are given by

$$\omega = \varphi_1, \tag{2.21}$$

$$\mathbf{A} = \begin{pmatrix} \varphi_2 \\ \varphi_3 \\ \varphi_4 \end{pmatrix}, \quad \mathbf{B} = \begin{pmatrix} \varphi_5 \\ \varphi_6 \\ \varphi_7 \end{pmatrix}, \quad \mathbf{C} = \begin{pmatrix} \varphi_8 \\ \varphi_9 \\ \varphi_{10} \end{pmatrix}. \tag{2.22}$$

Now, let us use Eqs. (2.21) and (2.22) to obtain the following decomposition of the spin–one particles spinor‡.

$$\Psi(x) = \begin{pmatrix} A_2 \\ A_3 \\ B_1 \end{pmatrix}, \quad \Phi(x) = \begin{pmatrix} B_2 \\ B_3 \\ A_1 \end{pmatrix}, \quad \Theta(x) = \begin{pmatrix} C_3 \\ -C_2 \\ \omega \end{pmatrix}. \tag{2.23}$$

‡Notice that the subcomponent C_1 is automatically equal to zero according to Eq. 2.23[21,27]

Hence, we can rewrite the DKP spin–one spinor as the following.

$$\phi(x) = \begin{pmatrix} \Psi \\ \Phi \\ \Theta \end{pmatrix}. \tag{2.24}$$

Notice that Eq. (2.23) is the partition of the spin–one spinor that is needed for the calculations. However, it can be expressed in terms of the original spinor to make it simpler, giving the subsequent result[8]

$$\begin{aligned} \Psi^+(x) &= \begin{pmatrix} \varphi_3 \\ \varphi_4 \end{pmatrix}, \quad \Psi^-(x) = \varphi_5, \\ \Phi^+(x) &= \begin{pmatrix} \varphi_6 \\ \varphi_7 \end{pmatrix}, \quad \Phi^-(x) = \varphi_2, \\ \Theta^+(x) &= \begin{pmatrix} \varphi_{10} \\ -\varphi_9 \end{pmatrix}, \quad \Theta^-(x) = \varphi_1, \end{aligned} \tag{2.25}$$

and as found before $C_3 = \varphi_8 = 0$.

To understand better the correlation between this partition and the previous calculations, let us use the previous result

$$\begin{cases} \left\{ \dfrac{d^2}{dx^2} + [E - V(x)]^2 - m^2 \right\} \varphi_5 = 0, \\ \left\{ \dfrac{d^2}{dx^2} + [E - V(x)]^2 - m^2 \right\} \varphi_4 = 0, \\ \left\{ \dfrac{d^2}{dx^2} + [E - V(x)]^2 - m^2 \right\} \varphi_3 = 0, \\ \varphi_1 = \dfrac{i}{m} \dfrac{d}{dx} \varphi_5, \\ \varphi_2 = \dfrac{1}{m} [E - V(x)] \varphi_5, \\ \varphi_6 = \dfrac{1}{m} [E - V(x)] \varphi_3, \\ \varphi_7 = \dfrac{1}{m} [E - V(x)] \varphi_4, \\ \varphi_9 = -\dfrac{i}{m} \dfrac{d}{dx} \varphi_4, \\ \varphi_{10} = \dfrac{i}{m} \dfrac{d}{dx} \varphi_3, \\ -m \, \varphi_8 = 0. \end{cases} \tag{2.26}$$

By direct inspection of Eq. (2.26), it is clear that the first three equations correspond to the $\Psi(x)$ component of $\phi(x)$. Additionally, the following six equations in Eq. (2.26) correspond to Φ and Θ. Even more, this equation not only shows why the second subcomponent of Θ is negative but also confirms the $\varphi_8 = 0$ as it was expected.

Then, it is possible to reduce the time-independent one-dimensional DKP equation down to the following system of equations.

$$\begin{cases} \left\{ \dfrac{d^2}{dx^2} + [E - V(x)]^2 - m^2 \right\} \Psi^{\pm} = 0, \\ \Phi^{\pm} = \dfrac{1}{m} [E - V(x)] \Psi^{\pm}, \\ \Theta^{\pm} = \dfrac{i}{m} \dfrac{d}{dx} \Psi^{\pm}. \end{cases} \tag{2.27}$$

There is an explicit dependence on both Θ and Φ with Ψ. Furthermore, as each of the new components of $\phi(x)$ are made of three subcomponents, we recover the nine equations of Eq. (2.18). It shows the clear equivalence between both methods by direct calculations or reducing the problem using the partition of the DKP spin–one spinor.

2.4 Spin–one and spin–zero equivalence

In sections 2.2 and 2.3, we have derived the DKP equation with the presence of a general one-dimensional potential $V(x)$ in which the particle is restricted to move along the x-axis. In light of the analysis, let us explore both solutions and describe their properties.

First of all, let us explore the result obtained in equation (2.13), which corresponds to the spin–zero sector. As mentioned above, the result for the spin–zero was a system of three equations in which only three components of the spinor, $\phi(x)^T = (\varphi_1, ..., \varphi_5)$, survive, being those φ_1, φ_2 and φ_3. From it, we have mentioned the explicit dependence of φ_2 and φ_3 on φ_1. The first equation in (2.13) corresponds to the KG equation, a well-known relativistic equation. Remarkably in equation (2.13), the spinor components are scalar values. Because of that, when studying spin–zero particles under the DKP equation, they are also called scalar particles[21].

Similarly, we can analyze the result obtained for the spin–one sector. Even though the result can be derived from two possible paths, we arrived at to a unique and equal final solution. One is expressed in its explicit form in terms of a system of equations with nine components, and the other one is made of a system of three equations, as can be seen in equation (2.27). By using the reduced solution, we once more notice that the first component of Eq. (2.27) is the KG equation and that the two other spinor components depend on the first one. In contrast to the spin–zero, these spinor components are vectorial values. Therefore, to refer to the case of spin–one particles under the DKP equation, we also call them vectorial particles[21].

Both results are shown below.

$$\begin{cases} \left\{ \dfrac{d^2}{dx^2} + [E - V(x)]^2 - m^2 \right\} \varphi_1 = 0, \\ \varphi_2 = \dfrac{1}{m} [E - V(x)] \varphi_1, \\ \varphi_3 = \dfrac{i}{m} \dfrac{d}{dx} \varphi_1. \end{cases} \qquad \begin{cases} \left\{ \dfrac{d^2}{dx^2} + [E - V(x)]^2 - m^2 \right\} \Psi^\pm = 0, \\ \Phi^\pm = \dfrac{1}{m} [E - V(x)] \Psi^\pm, \\ \Theta^\pm = \dfrac{i}{m} \dfrac{d}{dx} \Psi^\pm. \end{cases} \tag{2.28}$$

More than this, by comparing both results, we arrive at the conclusion that the scalar spinor components φ_1, φ_2 and φ_3 behave as the spinors $\Psi^\pm$, $\Phi^\pm$, and $\Theta^\pm$, respectively. This result is true for a one-dimensional potential, and it is telling us that the spin-one sector is equivalent to the spin-zero sector in the DKP theory[5,8].

It is clear that as both sectors of the theory arrive at the same set of equations, the solution will be equal in both cases. Because of it, we will restrict our calculations to the vectorial, spin–one, sector of the theory in all the forthcoming sections of this research work.

Chapter 3

Results & Discussion

3.1 The DKP equation for the hyperbolic tangent potential

Let us recall the result of the Duffin–Kemmer–Petiau (DKP) equation for the spin–one particles under an arbitrary potential $V(x)$.

$$\begin{cases} \left\{ \dfrac{d^2}{dx^2} + [E - V(x)]^2 - m^2 \right\} \Psi^{\pm} = 0, \\ \Phi^{\pm} = \dfrac{1}{m}[E - V(x)]\Psi^{\pm}, \\ \Theta^{\pm} = \dfrac{i}{m}\dfrac{d}{dx}\Psi^{\pm}. \end{cases} \tag{3.1}$$

3.1.1 Solving the first component of equation (3.1)

As discussed in the previous section, Eq. (3.1) gives us the set of equations we need to solve, in which there is a clear dependence of the second and third component on the first one[6]. The first step we need to execute is to solve this first equation for $\Psi(x)$, which is a Klein–Gordon (KG) type equation. Thereby, replacing $V(x) = a\tanh(bx)$, we get the equation we have to work on first.

$$\left\{ \frac{d^2}{dx^2} + [E - a\tanh(bx)]^2 - m^2 \right\} \Psi = 0. \tag{3.2}$$

Moreover, we know that the $\tanh(bx)$ can be expressed as

$$\tanh(bx) = \frac{e^{bx} - e^{-bx}}{e^{bx} + e^{-bx}}. \tag{3.3}$$

Notice that, by now, our problem has been reduced to solve a KG equation. To assess this second-order differential equation, let us use the following change of variables[20] $y = -e^{2bx}$. By using the change of variable, we obtain that

$$\frac{d\Psi}{dx} = \frac{d\Psi}{dy}\frac{dy}{dx} = 2be^{y}\frac{d\Psi}{dy}, \tag{3.4}$$

$$\frac{d^2\Psi}{dx^2} = \frac{d}{dy}\left(2be^{y}\frac{d\Psi}{dy}\right)\frac{dy}{dx} = 4b^2 y\frac{d}{dy}\left(y\frac{d\Psi}{dy}\right), \tag{3.5}$$

and the hyperbolic tangent expression given in Eq. (3.3) is transformed into

$$\begin{aligned} y^{1/2} = ie^{b}x &\implies e^{b}x = -iy^{1/2}, \\ y^{-1/2} = -ie^{-bx} &\implies e^{-bx} = -iy^{1/2}, \\ &\implies \tanh(bx) = -\frac{1+y}{1-y}. \end{aligned} \tag{3.6}$$

Hence, we can rewrite Eq. (3.2) to a more suitable representation to work

$$4b^2 y\frac{d}{dy}\left(y\frac{d\Psi}{dy}\right) + \left\{\left[E + a\left(\frac{1+y}{1-y}\right)\right]^2 - m^2\right\}\Psi = 0. \tag{3.7}$$

Taking common factor $(1-y)$, we get

$$4b^2 y(1-y)^2\frac{d}{dy}\left(y\frac{d\Psi}{dy}\right) + \left\{\left[E(1-y) + a\,(1+y)\right]^2 - m^2(1-y)^2\right\}\Psi = 0. \tag{3.8}$$

To solve Eq. (3.8) let us propose the following substitution $\Psi = y^{\alpha}(1-y)^{\beta}f(y)$. It is not difficult to verify that we arrive to

$$\begin{gathered} y^{\alpha+1}(1-y)^{\beta+1}\left\{y(1-y)f''(y) + (\alpha - \alpha y - \beta y + \alpha + 1 - \alpha y - y - \beta y)f'(y)\right\} + y^{\alpha}(1-y)^{\beta} \\ \underbrace{\left\{\alpha^2(1-y)^2 - \alpha(1-y) - \beta(\alpha+1)y(1-y) + \beta(\beta-1)y^2 + \frac{1}{4b^2}\left[(E(1-y) + a(1+y)^2)^2 - m^2(1-y)^2\right]\right\}}_{Ay(1-y)} f(y) = 0. \end{gathered} \tag{3.9}$$

In this way, we will be able to obtain the following expression

$$y(1-y)f''(y) + [(2\alpha+1) - (2\alpha+2\beta+1)y]f'(y) + (\alpha+\beta-\gamma)(\alpha+\beta+\gamma)f(y) = 0. \tag{3.10}$$

Where the parameters α, β and γ are defined as

$$\alpha = i\nu, \quad \text{and} \quad \nu = \frac{\sqrt{(E+a)^2 - m^2}}{2b},$$
$$\beta = \lambda, \quad \text{and} \quad \lambda = \frac{b + \sqrt{b^2 - 4a^2}}{2b},$$
$$\gamma = i\mu, \quad \text{and} \quad \mu = \frac{\sqrt{(E-a)^2 - m^2}}{2b}.$$

Notice that Eq. (3.10) has the form of a hypergeometric differential equation, that has a known solution. Therefore, the solution for the first component of Eq.(3.1) is given by

$$\begin{aligned}\Psi(y) = [&c_1\, y^{\alpha}(1-y)^{\beta}\, {}_2F_1(\alpha+\beta-\gamma, \alpha+\beta+\gamma, 1+2\alpha, y) \\ &+ c_2\, y^{-\alpha}(1-y)^{\beta}\, {}_2F_1(-\alpha+\beta+\gamma, -\alpha+\beta-\gamma, 1-2\alpha, y)]\mathbf{V},\end{aligned} \tag{3.11}$$

where $\mathbf{V}$ is a 3×1 vector needed to recover the three spin-one directions[21], and ${}_2F_1$ corresponds to the Gaussian or ordinary hypergeometric function. Equation (3.11) is the solution for the KG type equation. As we saw, the other components depend on this result.

3.1.2 Solving the second and third component of equation (3.1)

The next step is to plug the result obtained in Eq. (3.11) into the two other components of Eq. (3.1). Notice that for the second component in Eq. (3.1), the substitution can be done straightforwardly with no special treatment. However, the same statement is not true for the third component.

The last component of Eq. (3.1) is given by

$$\Theta^{\pm} = \frac{i}{m}\frac{d}{dx}\Psi^{\pm}. \tag{3.12}$$

This is the derivative of Eq. (3.11), which requires careful analysis. To solve Eq.(3.12) let us use the following hypergeometric function property

$$\frac{d}{dy}\, {}_2F_1(\alpha,\beta,\gamma,y) = \frac{\alpha\beta}{\gamma}\, {}_2F_1(\alpha+1,\beta+1,\gamma+1,y).$$

Therefore, we can arrive at the solution of the DKP equation for the hyperbolic tangent potential, which has the following form

$$\begin{pmatrix}\Psi\\ \Phi\\ \Theta\end{pmatrix} = c1\; y^{\alpha}(1-y)^{\beta}\left[\,{}_2F_1(\alpha_1,\beta_1,\gamma_1,y)M_1(y) + \,{}_2F_1(\alpha_1+1,\beta_1+1,\gamma_1+1,y)N_1(y)\right] + \\ c2\; y^{-\alpha}(1-y)^{\beta}\left[\,{}_2F_1(\alpha_2,\beta_2,\gamma_2,y)M_2(y) + \,{}_2F_1(\alpha_2+1,\beta_2+1,\gamma_2+1,y)N_2(y)\right]. \tag{3.13}$$

Table (3.1) provides an important notation to understand the solution given in Eq. (3.13), where the value of the parameters for the hypergeometric functions are explained*.

Table 3.1: Explanation of the notation used in Eq. (3.13).

$\alpha_1 = \alpha + \beta - \gamma$	$\beta_1 = \alpha + \beta + \gamma$	$\gamma_1 = 1 + 2\alpha$
$\alpha_2 = -\alpha + \beta + \gamma$	$\beta_2 = -\alpha + \beta - \gamma$	$\gamma_2 = 1 - 2\alpha$

Additionally, $M_1(y)$, $M_2(y)$, $N_1(y)$, and $N_2(y)$ are 9×1 the components of these vectors correspond to the solution of each component of Eq. (3.1), respectively. They are given by

$$\begin{aligned} M_1(y) &= \begin{pmatrix} 1 \\ \dfrac{E}{m} - \dfrac{a(1+y)}{m(1-y)} \\ \dfrac{2bi}{m}(1-y)^{-1}[\alpha - (\alpha+\beta)y] \end{pmatrix} \otimes \mathbf{V}; \quad & N_1(y) &= \begin{pmatrix} 0 \\ 0 \\ \dfrac{2bi}{m}\dfrac{\alpha_1\beta_1}{\gamma_1}y \end{pmatrix} \otimes \mathbf{V}, \\ M_2(y) &= \begin{pmatrix} 1 \\ \dfrac{E}{m} - \dfrac{a(1+y)}{m(1-y)} \\ \dfrac{-2bi}{m}(1-y)^{-1}[\alpha - (\alpha+\beta)y]) \end{pmatrix} \otimes \mathbf{V}; \quad & N_2(y) &= \begin{pmatrix} 0 \\ 0 \\ \dfrac{2bi}{m}\dfrac{\alpha_2\beta_2}{\gamma_2}y \end{pmatrix} \otimes \mathbf{V}. \end{aligned} \tag{3.14}$$

3.2 Asymptotic behavior of the solution

As said above, Eq. (3.13) is our solution for the DKP equation by using the hyperbolic tangent potential. Nevertheless, in order to study the reflection and transmission coefficients, this solution is incomplete. In fact, by studying Eq. (3.13), we notice that we arrive at two solutions where one is the incident wave, and

*Remember that $\alpha = i\nu$, $\beta = \lambda$, and $\gamma = i\mu$.

the other one is the reflected wave. To complete the solution, we need to construct the transmitted wave. To achieve it, let us use the following hypergeometric function property[20].

$$\begin{aligned}{}_2F_1(a,b,c,z) = &\frac{\Gamma(c)\Gamma(b-a)}{\Gamma(b)\Gamma(c-a)}(-z)^{-a}\ {}_2F_1\left(a, 1-c+a, 1-b+a, z^{-1}\right)\\ &+\frac{\Gamma(c)\Gamma(a-b)}{\Gamma(a)\Gamma(c-b)}(-z)^{-b}\ {}_2F_1\left(b, 1-c+b, 1-a+b, z^{-1}\right).\end{aligned} \tag{3.15}$$

By using Eq. (3.15) in our solution of the DKP equation for the hyperbolic tangent potential we get

$$\begin{aligned}\begin{pmatrix}\Psi\\ \Phi\\ \Theta\end{pmatrix} = &c_1\left(-e^{2bx}\right)^{i\nu}\left(1+e^{2bx}\right)^{\lambda}\left\{\left[\Gamma_1\left(e^{2bx}\right)^{-\alpha_1}\ {}_2F_1\left(\alpha_1, 1-\gamma_1+\alpha_1, 1-\beta_1+\alpha_1, -e^{-2bx}\right)\right]M_1(x)\right.\\ &\left.+\left[\Gamma_3\left(e^{2bx}\right)^{-\alpha_1-1}\ {}_2F_1\left(\alpha_1+1, 1-\gamma_1+\alpha_1, 1-\beta_1+\alpha_1, -e^{-2bx}\right)\right]N_1(x)\right\}\\ &+c_1\left(-e^{2}bx\right)^{i\nu}(1+e^{2bx})^{\lambda}\left\{\left[\Gamma_2\left(e^{2bx}\right)^{-\beta_1}\ {}_2F_1\left(\beta_1, 1-\gamma_1+\beta_1, 1-\alpha_1+\beta_1, -e^{-2bx}\right)\right]M_1(x)\right.\\ &\left.+\left[\Gamma_4(e^{2bx})^{-\beta_1-1}\ {}_2F_1\left(\beta_1+1, 1-\gamma_1+\beta_1, 1-\alpha_1+\beta_1, -e^{-2bx}\right)\right]N_1(x)\right\}\\ &+c_2\left(-e^{2bx}\right)^{-i\nu}(1+e^{2bx})^{\lambda}\left\{\left[\Gamma_5\left(e^{2bx}\right)^{-\alpha_2}\ {}_2F_1\left(\alpha_2, 1-\gamma_2+\alpha_2, 1-\beta_2+\alpha_2, -e^{-2bx}\right)\right]M_2(x)\right.\\ &\left.+\left[\Gamma_7\left(e^{2bx}\right)^{-\alpha_2-1}\ {}_2F_1\left(\alpha_2+1, 1-\gamma_2+\alpha_2, 1-\beta_2+\alpha_2, -e^{-2bx}\right)\right]N_2(x)\right\}\\ &+c_2\left(-e^{2bx}\right)^{-i\nu}(1+e^{2bx})^{\lambda}\left\{\left[\Gamma_6\left(e^{2bx}\right)^{-\beta_2}\ {}_2F_1\left(\beta_2, 1-\gamma_2+\beta_2, 1-\alpha_2+\beta_2, -e^{-2bx}\right)\right]M_2(x)\right.\\ &\left.+\left[\Gamma_8\left(e^{2bx}\right)^{-\beta_2-1}\ {}_2F_1\left(\beta_2+1, 1-\gamma_2+\beta_2, 1-\alpha_2+\beta_2, -e^{-2bx}\right)\right]N_2(x)\right\}.\end{aligned} \tag{3.16}$$

Hence, the transmitted wave becomes;

$$\begin{pmatrix}\Psi\\ \Phi\\ \Theta\end{pmatrix}_{\text{trans}} = c_1(-1)^{i\nu}\left(-e^{2bx}\right)^{-\lambda}\left(1+e^{2bz}\right)^{\lambda}\left(e^{2bx}\right)^{i\mu}\left\{\left[\Gamma_1\ {}_2F_1\left(\alpha 1, 1-\gamma 1+\alpha 1, 1-\beta 1+\alpha 1, -e^{-2bx}\right)\right]M_1(x)\right.$$
$$+\left[\Gamma_3\left(e^{-2bx}\right)\ {}_2F_1\left(\alpha 1+1, 1-\gamma 1+\alpha 1, 1-\beta 1+\alpha 1, -e^{-2bx}\right)\right]N_1(x)\Big\}$$
$$+c_2(-1)^{-i\nu}\left(-e^{2bx}\right)^{-\lambda}\left(1+e^{2bz}\right)^{\lambda}\left(e^{2bx}\right)^{i\mu}\left\{\left[\Gamma_6\ {}_2F_1\left(\beta_2, 1-\gamma_2+\beta_2, 1-\alpha_2+\beta_2, -e^{-2bx}\right)\right]M_2(x)\right.$$
$$\left.+\left[\Gamma_8\left(e^{-2bx}\right)\ {}_2F_1\left(\beta_2+1, 1-\gamma_2+\beta_2, 1-\alpha_2+\beta_2, -e^{-2bx}\right)\right]N_2(x)\right\}. \tag{3.17}$$

Even more, it is known that the incident wave is equal to the sum of the transmitted and reflected waves[20].

$$\begin{pmatrix}\Psi\\ \Phi\\ \Theta\end{pmatrix}_{\text{trans}} = A\begin{pmatrix}\Psi\\ \Phi\\ \Theta\end{pmatrix}_{\text{inc}} + B\begin{pmatrix}\Psi\\ \Phi\\ \Theta\end{pmatrix}_{\text{ref}}. \tag{3.18}$$

Therefore, let us use the property given in Eq. (3.15) once more we get

$$\begin{pmatrix}\Psi\\ \Phi\\ \Theta\end{pmatrix}_{\text{trans}} = \left(1+e^{2bx}\right)^{\lambda}\left(e^{2bx}\right)^{i\nu}\left[\Gamma_1'\ {}_2F_1\left(\alpha_1,\beta_1,\gamma_1,-e^{2bx}\right)M_1(x)+\Gamma_3'\ {}_2F_1\left(\alpha_1+1,\beta_1+1,\gamma_1+1,-e^{2bx}\right)N_1(x)\right]$$
$$+\left(1+e^{2bx}\right)^{\lambda}\left(e^{2bx}\right)^{-i\nu-1}\left[\left(e^{-2bx}\right)^{-1}\Gamma_2'\ {}_2F_1\left(1-\gamma_1+\alpha_1, 1-\alpha_1+\beta_1, 2-\gamma_1, -e^{2bx}\right)M_1(x)\right.$$
$$\left.+\Gamma_4'\ {}_2F_1\left(1-\gamma_1+\alpha_1, 1-\gamma_1+\beta_1, 1-\gamma_1, -e^{2bx}\right)N_1(x)\right]$$
$$\left(1+e^{2bx}\right)^{\lambda}\left(e^{2bx}\right)^{-i\nu}\left[\Gamma_1'\ {}_2F_1\left(\beta_2,\alpha_2,\gamma_2,-e^{2bx}\right)M_2(x)+\Gamma_7'\ {}_2F_1\left(\beta_2+1,\alpha_2+1,\gamma_2+1,-e^{2bx}\right)N_2(x)\right]$$
$$+\left(1+e^{2bx}\right)^{\lambda}\left(e^{2bx}\right)^{i\nu-1}\left[\left(e^{-2bx}\right)^{-1}\Gamma_6'\ {}_2F_1\left(1-\gamma_2+\beta_2, 1-\gamma_2+\alpha_2, 2-\gamma_2, -e^{2bx}\right)M_2(x)\right.$$
$$\left.+\Gamma_8'\ {}_2F_1\left(1-\alpha_2+\beta_2, 1-\gamma_2+\alpha_2, 1-\gamma_2, -e^{2bx}\right)N_2(x)\right]. \tag{3.19}$$

From Eq. (3.19), we can verify that the statement given in Eq. (3.18) is accomplished. The incident and reflection wave is recovered, then we may write them as

$$\begin{pmatrix}\Psi\\ \Phi\\ \Theta\end{pmatrix}_{\text{inc}} = (1+e^{2bx})^{\lambda}(e^{2bx})^{iv}\left[A\;{}_2F_1(\alpha_1,\beta_1,\gamma_1,-e^{2bx})M_1(x) + B\;{}_2F_1(\alpha_1+1,\beta_1+1,\gamma_1+1,-e^{2bx})N_1(x)\right], \tag{3.20}$$

$$\begin{pmatrix}\Psi\\ \Phi\\ \Theta\end{pmatrix}_{\text{ref}} = (1+e^{2bx})^{\lambda}(e^{2bx})^{-iv}\left[C\;{}_2F_1(\alpha_2,\beta_2,\gamma_2,-e^{2bx})M_2(x)D\;{}_2F_1(\alpha_2+1,\beta_2+1,\gamma_2+1,-e^{2bx})N_2(x)\right], \tag{3.21}$$

where,

$$\begin{aligned} A &= \frac{\Gamma(1-\beta_1+\alpha 1)\Gamma(1-\gamma_1)}{\Gamma(1-\gamma_1+\alpha_1)\Gamma(1-\beta_1)}, \quad & B &= \frac{\Gamma(1-\beta_1+\alpha_1)\Gamma(-\gamma_1)}{\Gamma(1-\gamma_1+\alpha_1)\Gamma(-\beta_1)},\\ C &= \frac{\Gamma(1-\alpha_2+\beta_2)\Gamma(1-\gamma_2)}{\Gamma(1-\gamma_2+\beta_2)\Gamma(1-\alpha_2)}, \quad & D &= \frac{\Gamma(1-\alpha_2+\beta_2)\Gamma(-\gamma_2)}{\Gamma(1-\gamma_2+\beta_2)\Gamma(-\alpha_2)}. \end{aligned} \tag{3.22}$$

Now, let us use these three equations to obtain the reflection (R) and transmission (T) coefficients. For that, we need to study the asymptotic behavior of the incident, reflected, and transmitted waves. For that, let us analyze the limiting case when $x \to \pm\infty$.[†]

Figure (3.1) gives us the shape of the change of variable introduced back in section (3.1.1). A quick analysis of this plot reveals to us that if $x \to \infty$ implies that $y \to -\infty$. On the other hand, when $x \to -\infty$ implies that $y \to 0$. Moreover, it is assumed that the wave travels from left to right. Then, the case when $x \to -\infty$ refers to the incident and reflected wave, whereas the transmitted wave is when $x \to \infty$.

First, let us analyze the incident wave ($x \to -\infty$). This solution needs the use of the following limits,

$$\lim_{y\to 0}(-y)^{iv} = e^{2ibvx}, \quad \lim_{y\to 0}(1-y)^{\lambda} = 1, \quad \lim_{y\to 0}{}_2F_1(a,b,c,y) = 1.$$

Therefore, the asymptotic behavior for the incident wave behaves as

$$\begin{pmatrix}\Psi\\ \Phi\\ \Theta\end{pmatrix}_{\text{inc}} = Ae^{2ibvx}\begin{pmatrix}1\\ \dfrac{E-a}{m}\\ \dfrac{2bi}{m}\alpha\end{pmatrix} \otimes \mathbf{V}. \tag{3.23}$$

[†]Remember that the change of variable used was $y = -e^{2bx}$.

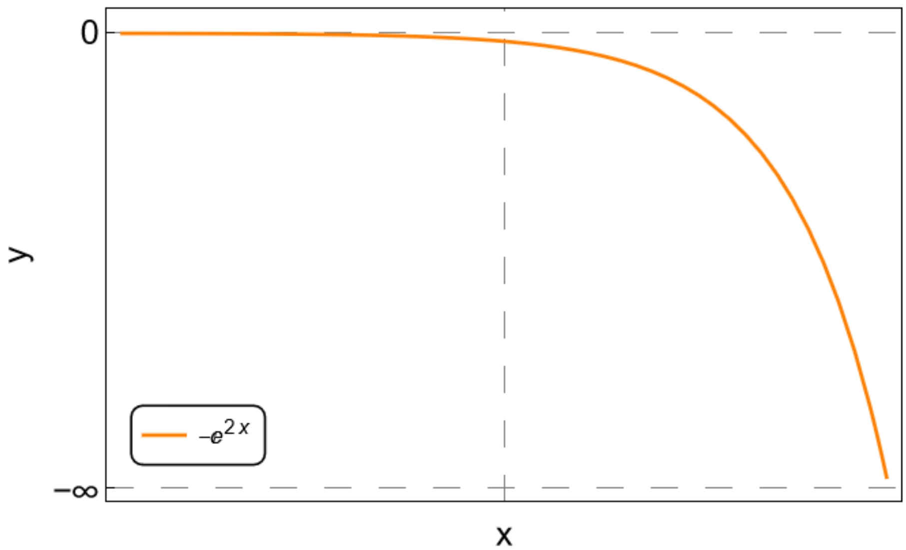

Figure 3.1: Plot of the change of variable† used at the beginning of the calculations, in the plot $b = 1$ was taken.

Next, for the reflected wave ($x \to -\infty$) the following limits were used

$$\lim_{y\to 0}(-y)^{-i\nu} = e^{-2ib\nu x}, \quad \lim_{y\to 0}(1-y)^{\lambda} = 1, \quad \lim_{y\to 0} {}_2F_1(a,b,c,y) = 1.$$

This gives us as a result an asymptotic behavior for the reflected wave of the following form

$$\begin{pmatrix} \Psi \\ \Phi \\ \Theta \end{pmatrix}_{\text{ref}} = Ce^{-2ib\nu x} \begin{pmatrix} 1 \\ \dfrac{E-a}{m} \\ \dfrac{-2bi}{m}\alpha \end{pmatrix} \otimes \mathbf{V}. \tag{3.24}$$

Finally, for the transmitted wave ($x \to \infty$) we used the limits

$$\lim_{y\to -\infty}(-y)^{\lambda} = e^{-2ib\lambda x}, \quad \lim_{y\to -\infty}(1-y)^{\lambda} = e^{2b\lambda x}, \quad \lim_{y\to -\infty}(-y)^{i\mu} = e^{2bi\mu x}, \quad \lim_{y\to -\infty} {}_2F_1(a,b,c,y^{-1}) = 1.$$

And so the asymptotic behavior for the transmitted wave is given by

$$\begin{pmatrix} \Psi \\ \Phi \\ \Theta \end{pmatrix}_{\text{trans}} = e^{2ib\mu x} \begin{pmatrix} 1 \\ \dfrac{E+a}{m} \\ \dfrac{2bi(\alpha+\beta)}{m} \end{pmatrix} \otimes \mathbf{V}. \tag{3.25}$$

3.3 Reflection and Transmission coefficients

As it is known, when studying the solutions of waves propagating through space, the reflection (R) and transmission (T) coefficients provide important information about the amplitude, intensity, and energy the wave will have once it propagates through the potential. The sum of both coefficients accomplishes what is known as the unitary relation, in which $R + T = 1$ [28,29]. To calculate these coefficients related to the hyperbolic tangential potential, let us use the conserved current for the Duffin–Kemmer–Petiau (DKP) equation.

Meanwhile, J^μ can be calculated by the following relation [8,30]

$$J^1 = \frac{1}{m}\Im\left[\Psi^{(+)\dagger}\frac{d\Psi^{(+)}}{dx} + \Psi^{(-)\dagger}\frac{d\Psi^{(-)}}{dx}\right]. \tag{3.26}$$

Let us use Eq. (3.26) to calculate each current.

$$J_{\text{inc}} = \frac{1}{m}\Im\left[A^* e^{-2ib\nu x}\begin{pmatrix}1 & 1\end{pmatrix}\cdot 2ib\nu A e^{2ib\nu x}\begin{pmatrix}1\\1\end{pmatrix} + A^* e^{-2ib\nu x}\left(-2ib\nu A e^{2ib\nu x}\right)\right], \tag{3.27}$$

$$J_{\text{ref}} = \frac{1}{m}\Im\left[C^* e^{2ib\nu x}\begin{pmatrix}1 & 1\end{pmatrix}\cdot -2ib\nu C e^{-2ib\nu x}\begin{pmatrix}1\\1\end{pmatrix} + C^* e^{2ib\nu x}\left(2ib\nu C e^{-2ib\nu x}\right)\right], \tag{3.28}$$

$$J_{\text{trans}} = \frac{1}{m}\Im\left[e^{-2ib\mu x}\begin{pmatrix}1 & 1\end{pmatrix}\cdot 2ib\mu e^{2ib\mu x}\begin{pmatrix}1\\1\end{pmatrix} + e^{-2ib\mu x}\left(2ib\mu e^{2ib\mu x}\right)\right]. \tag{3.29}$$

From Eqs. (3.27), (3.28), (3.29) give us as result the following currents.

$$J_{\text{inc}} = \frac{6AA^* b\nu}{m}, \tag{3.30}$$

$$J_{\text{ref}} = -\frac{6CC^* b\nu}{m}, \tag{3.31}$$

$$J_{\text{trans}} = \frac{6b\mu}{m}. \tag{3.32}$$

With the incident, reflected, and transmitted current, it is possible to calculate the R and T coefficients with no difficulty, giving us the following result.

$$R = \frac{|J_{\text{ref}}|}{|J_{\text{inc}}|} = \frac{\left|\frac{\Gamma(1-\alpha_2+\beta_2)\Gamma(1-\gamma_2)}{\Gamma(1-\gamma_2+\beta_2)\Gamma(1-\alpha 2)}\right|^2}{\left|\frac{\Gamma(1-\beta_1+\alpha 1)\Gamma(1-\gamma_1)}{\Gamma(1-\gamma_1+\alpha_1)\Gamma(1-\beta_1)}\right|^2}, \tag{3.33}$$

$$T = \frac{|J_{\text{trans}}|}{|J_{\text{inc}}|} = \frac{\mu}{\nu}\frac{1}{\left|\frac{\Gamma(1-\beta_1+\alpha 1)\Gamma(1-\gamma_1)}{\Gamma(1-\gamma_1+\alpha_1)\Gamma(1-\beta_1)}\right|^2}. \tag{3.34}$$

Eqs. (3.33) and (3.34) are the main result of this research work. Notice how the T coefficient has a term related to the dispersion relation, which will be used in the next section to describe a phenomenon that occurs in this potential. It is worth mentioning that the unitary relation $R + T = 1$ is satisfied, as can be seen in figure (3.2). Both coefficients are given in terms of Gamma functions, and their behavior has been explored by using the Wolfram Mathematica® Software.

At first glance of figure (3.2), we can notice something odd in the behavior of the R and T coefficients. As it is known, the value for these coefficients normally is greater than zero and less than one. However, we observe how the coefficients have values less than zero and greater than one. Therefore, this behavior requires further analysis.

3.4 Superradiance effect

Notice that as $x \to \pm\infty$, we are left with two dispersion relations, respectively,

$$x \to \infty : \mu = \sqrt{(E-a)^2 - m^2}, \tag{3.35}$$

$$x \to -\infty : \nu = \sqrt{(E+a)^2 - m^2}. \tag{3.36}$$

Therefore, according to the dispersion relation, the hyperbolic tangent potential can be split into five different regions. Figure (3.3) shows how the potential is divided.

As said previously, the incident particle travels left to right. Then, the dispersion relation (μ, ν) has to be positive. Due to this fact, it is the group velocity that describes the sign of the wave[31].

$$\frac{dE}{d\nu} = \frac{\nu}{E+a} \geq 0, \tag{3.37}$$

$$\frac{dE}{d\mu} = \frac{\mu}{E-a} \geq 0. \tag{3.38}$$

Therefore, by studying figure (3.3) it can be seen how in the region where $a + m > E > a - m$ and $-a + m > E > -a - m$, the dispersion relations (μ, ν) are imaginary. As the transmission (T) coefficient is given by Eq. (3.34), we can see how in either of those regions, as we have the solution of a planar wave Eq. (3.25), the transmitted wave is attenuated. It implies that the wave is completely reflected $(R = 1)$. On the other hand, in the regions $E > a + m$ and $E < -a - m$, the dispersion relation is real, and as μ/ν would be greater than zero, which means that for these regions, $T > 0$ and $R > 0$. This analysis can be better understood by studying the table 3.4.

In addition, notice that in the region where $a - m > E > -a + m$, the dispersion relation behaves $\mu < 0$ and $\nu > 0$. This implies that our T coefficient has to be less than zero in that region. Consequently, to still satisfy the unitary condition, the reflection (R) coefficient needs to be greater than one, producing in that region of the potential the phenomenon known as superradiance.

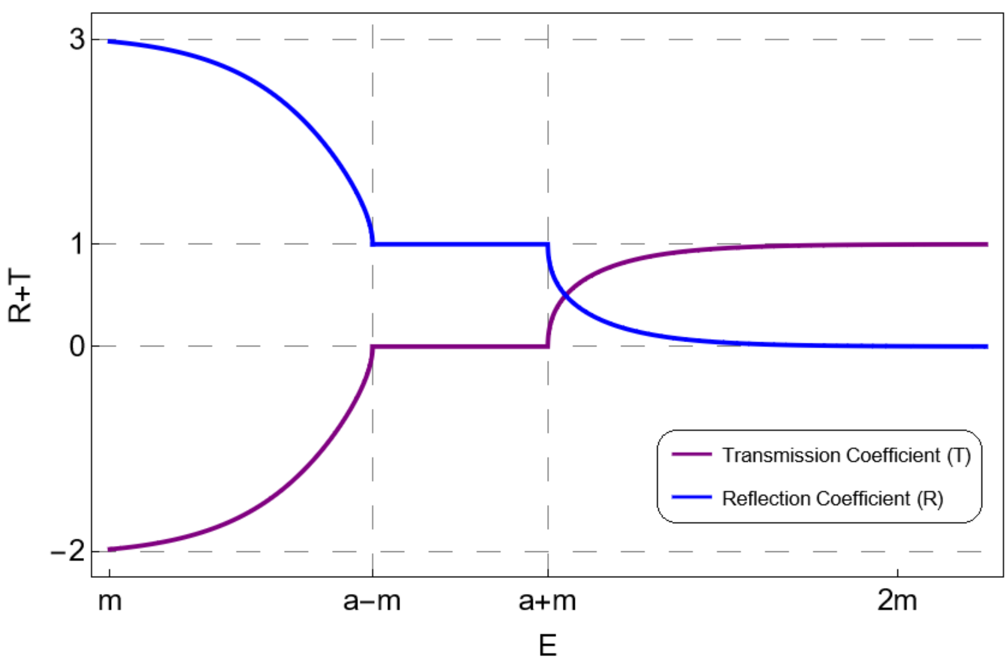

Figure 3.2: R (blue line) and T (purple line) coefficients varying energy E for the hyperbolic tangent potential for $a = 5$, $m = 1$, and $b = 5$.

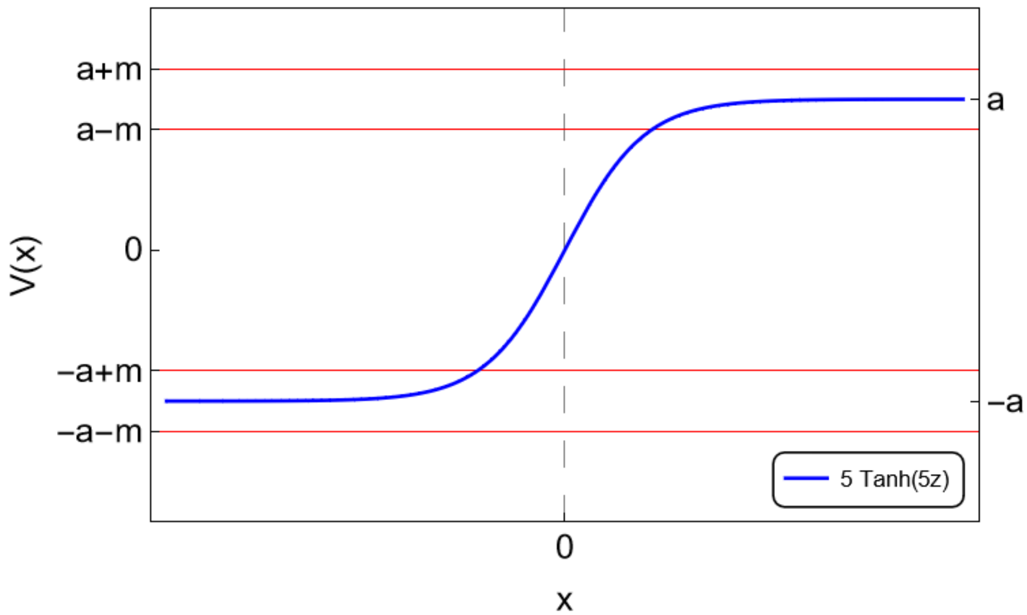

Figure 3.3: The hyperbolic tangent potential is divided into five regions. The parameters are $a = 5$, and $b = 5$.

Table 3.2: The five regions of the hyperbolic tangent potential.

Region I	$E > a + m$	$\nu > 0$	$\nu \in \mathbb{R}$	$\mu > 0$	$\mu \in \mathbb{R}$
Region II	$a + m > E > a - m$	$\nu > 0$	$\nu \in \mathbb{R}$	–	$\mu \in \mathbb{I}$
Region III	$a - m > E > -a + m$	$\nu > 0$	$\nu \in \mathbb{R}$	$\mu < 0$	$\mu \in \mathbb{R}$
Region IV	$-a + m > E > -a - m$	–	$\nu \in \mathbb{I}$	$\mu < 0$	$\mu \in \mathbb{R}$
Region V	$E < -a - m$	$\nu < 0$	$\nu \in \mathbb{R}$	$\mu < 0$	$\mu \in \mathbb{R}$

Figures (3.4) and (3.5) show the R and T coefficients, respectively, for the hyperbolic tangent potential. We can observe that in the first region of the potential, $a - m > E > m$ is where the superradiance phenomenon is produced.

The famous Klein paradox covers two main phenomena: the superradiance phenomenon and pair creation. Bosons and fermions are said to exhibit these phenomena when their rest masses are greater than V. Meanwhile, superradiance occurs when the reflected current is greater than its incoming current[19].

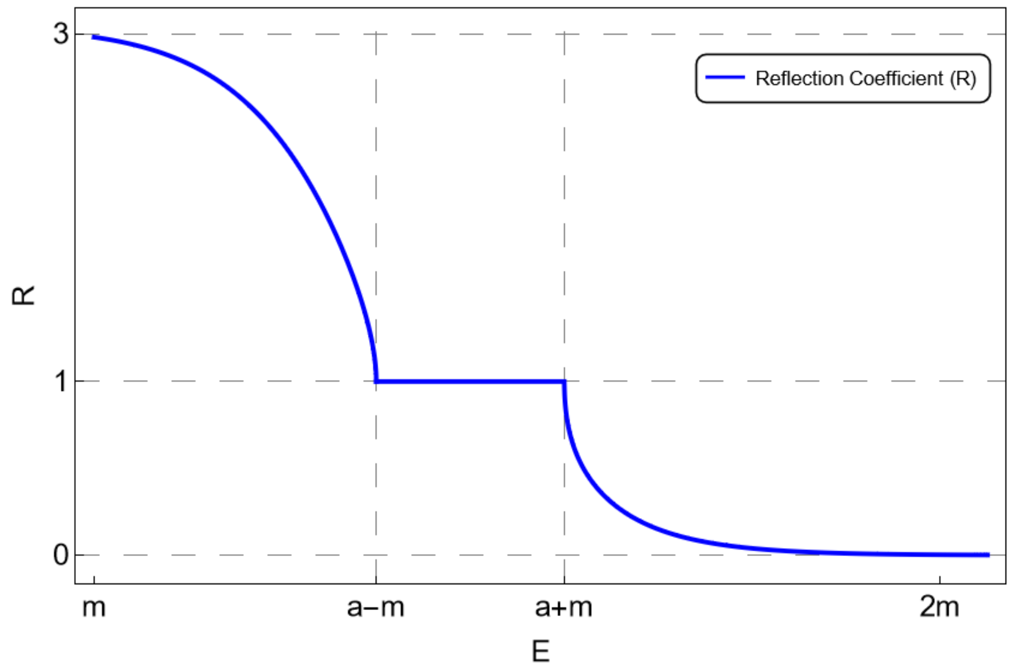

Figure 3.4: The reflection coefficient R varying energy E for the hyperbolic tangent potential for $a = 5$, $m = 1$, and $b = 5$.

Nevertheless, the asymptotic behavior of the solutions demonstrates that a scattering operator exists whose norm is strictly always greater than one. This phenomenon of superradiance is considered a radiation-enhancement phenomena[24,32]. It has been studied in black holes regarding static, stability, rotating, and energy extraction from black holes[33]. Moreover, it has been established that a dissipation mechanism and rotation are sufficient conditions to produce superradiance[23,33]. In addition, the existence of superradiance has been supported experimentally. The first experiment was done by using an apparatus that mimics a rotating black hole. It consists of a spinning cylinder, which creates a similar condition to an ergoregion[‡] in curved spacetime[33].

The results given in this research work allow us to understand how in the presence of a strong enough potential, the pair creation can occur and, with it, the superradiance. In fact, the superradiance phenomenon occurs due to spontaneous pair creation[33].

[‡]An ergoregion is a region where physical observers are incapable of remaining at rest[22].

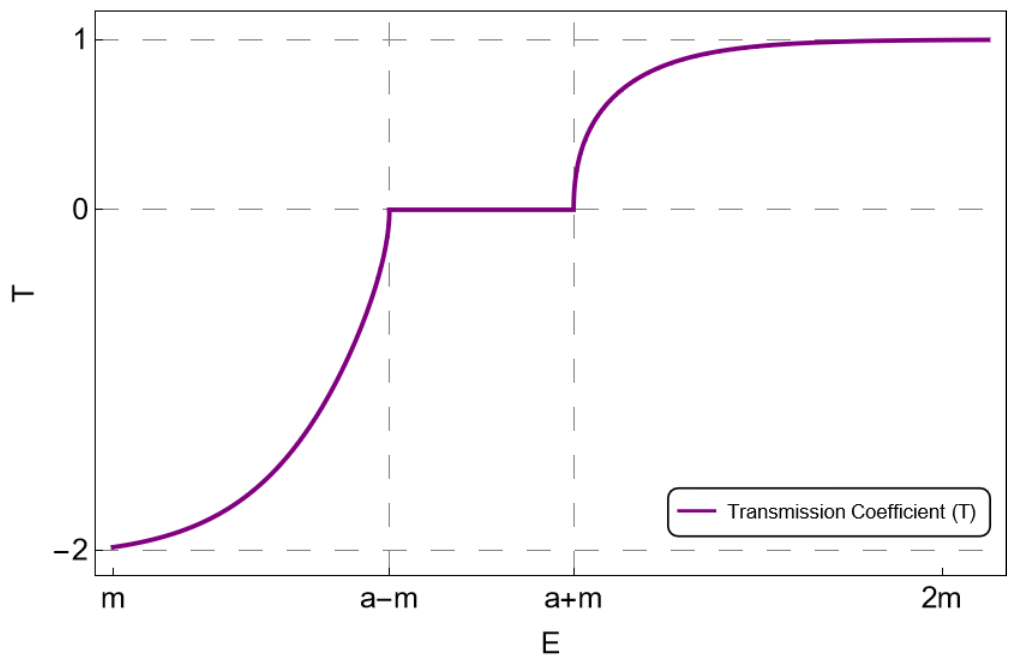

Figure 3.5: The transmission coefficient T varying energy E for the hyperbolic tangent potential for $a = 5$, $m = 1$, and $b = 5$.

Chapter 4

Conclusions & Outlook

In this research work, we have presented the analytical solution for the Duffin–Kemmer–Petiau (DKP) equation under the presence of a hyperbolic tangent potential force. The problem presented in this manuscript was analyzed from the most general case, solving the DKP equation for both types of particles, to our specific case of study, the spin–one particles. Following that analysis, the first calculation was to solve the DKP equation for an arbitrary potential $V(x)$. For this particular case, we have shown how the general solution for the DKP theory is the same for spin-one and spin-zero particles, where the result is a set of three coupled equations in both cases of the theory. The first solution is the famous Klein–Gordon (KG) relativistic equation, and the second and the third equations were simple equations that depend on the first solution, as can be seen in Eq. (2.28). It is necessary to emphasize that this solution is valid if and only if we are working with a one-dimension potential. Moreover, dependently of this, if the particle is restricted to move along the x, y, or z direction, the solution remains the same.

From that result, we could go further into the analysis and restrict the DKP theory for the spin-one particles. We used a partition of the DKP spin-one spinor to study the behavior of the solutions in a more compact form. The analytical solution of the DKP equation under the hyperbolic tangent potential is Eq. (3.13), which was obtained by solving Eq. (2.28). The results were expressed in terms of hypergeometric functions, from which we obtained the solution for the incident and reflected waves. To get the transmitted wave, we had to construct the solution by using a hypergeometric function property.

The reflection (R) and transmission (T) coefficients were also derived. Initially, the asymptotic behavior of the three solutions is examined (the incident, reflected, and transmitted wave). After that, we obtain the value for the fourth current in each case of the solution. Remarkably, as the particle was restricted to move only along the x-axis, we only needed to calculate the J^1 component. The R and T coefficients were given in terms of Gamma functions, and they satisfy the unitary condition $R + T = 1$. Nevertheless, as demonstrated in figure (3.2), the coefficients exhibit an odd behavior. From the dispersion relation, we show how the hyperbolic tangent potential can be divided into five different regions. The corresponding analysis demonstrated that for Region III, the T coefficient takes values below zero. Therefore, to maintain the unitary condition, the R coefficient has to be greater than one. This phenomenon is known as the superradiance effect. The results given in this research project aim to contribute to a better understanding of scattering events and the occurrence of the superradiance phenomenon. It can be especially helpful in studying superradiance in black holes or in another area of physics where this phenomenon appears.

Finally, future research could be considered to explore the limiting case when the parameter b in the hyperbolic tangent potential tends to infinity, transforming the smooth potential into a step potential. Remember that the parameter b describes the smoothness of the potential; as it grows, the potential becomes more sharp. A proper analysis of the behavior of its solutions in this particular case can be performed. The solution obtained for the DKP equation can be taken as a starting point. Moreover, it could be studied if the phenomenon of superradiance is still present when $b \to \infty$.

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Abbreviations

R reflection iii, xi, 2–4, 21, 23–26, 29, 30
T transmission iii, xi, 2–4, 21, 23–26, 29, 30

DKP Duffin–Kemmer–Petiau iii, 1–8, 10, 12–15, 18, 19, 23, 29, 30

KG Klein–Gordon 1–3, 7, 10, 14–17, 29