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TÍTULO: Properties of cones of nuclear operators working on some separable Hilbert spaces

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Dedication

To my parents and sisters, family and friends who helped and supported me through this path and inspired me to never give up.

Antonio Xavier Villagómez Chiluisa

Acknowledgment

First of all, I want to thank god which is our main support and provided me of many blessings the most important my parents who never gave up to the problems and always supported me despite of them. Your advices were essential in my development and without you I would have ended giving me up time ago. I really do not know what is for me from now on, but I promise that I will do my best for you I really thank you.

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Antonio Xavier Villagómez Chiluisa

Abstract

Analogues of the Sobolev space H^1 are defined at the level of nuclear operators. These sets of operators are no longer normed linear spaces but cones equipped with a concept of total energy that replaces the role of the square of a norm. Using Operator Theory, we obtain properties similar to those obtained by Mayorga et al. when the pivot space $\mathrm{L}^2(\mathbb{R}^N)$ is replaced by another separable Hilbert space, such as $\mathrm{H}^1(\mathbb{R}^N)$, with $N > 4$.

The work is related to the stability of quantum systems (represented by nuclear operators), and therefore, properties of free energy functionals defined on the operator cone are also studied. In this context, Gagliardo-Nirenberg type inequalities for operators are proven.

Keywords:

Sobolev-like cone, free energy functional, nuclear operator.

Resumen

Se definen análogos del espacio de Sobolev H^1 a nivel de operadores nucleares. Estos conjuntos de operadores ya no son espacios lineales normados, sino conos equipados con un concepto de energía total que reemplaza el papel del cuadrado de una norma. Utilizando la Teoría de Operadores, obtenemos propiedades similares a las obtenidas por Mayorga et al. cuando el espacio pivote $\mathrm{L}^2(\mathbb{R}^N)$ se reemplaza por otro espacio de Hilbert separable, como $\mathrm{H}^1(\mathbb{R}^N)$, con $N > 4$.

El trabajo está relacionado con la estabilidad de sistemas cuánticos (representada por los operadores nucleares) y por ello, también se estudian propiedades de funcionales de energía libre definidos sobre el cono de operadores. En este contexto, se prueban desigualdades tipo Gagliardo-Nirenberg para operadores.

Palabras Clave:

Cono tipo Sobolev, funcional de energía libre, operadores nucleares

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Chapter 1

Introduction

For the development of Quantum Mechanics it was necessary Functional Analysis. In particular, as they help to describe positive self-adjoint trace class operators systems in Quantum Mechanics; for example, in the description of a system of gravitating quantum particles [2]. The set of self-adjoint trace class operators is denoted by $\mathscr{I}_1$.

The importance of these operators in Quantum Mechanics relies on the Riesz-Schauder and Hilbert-Schmidt theorems, which for $T \in \mathscr{I}_1$ justifies the existence of a sequence of eigenvalues $(\nu_{i,T})_{i\in\mathbb{N}} \subseteq \mathbb{R}_+$ and a sequence of eigenfunctions $(\psi_{i,T})_{i\in\mathbb{N}} \subseteq \mathrm{H}^1(\mathbb{R}^N) \subseteq \mathrm{L}^2(\mathbb{R}^N)$ of T such that

$$B = \{\psi_{i,T} : \ i \in \mathbb{N}\}$$

is a Hilbert basis of $\mathrm{H}^1(\mathbb{R}^N)$. In the context of Quantum Mechanics provided by [5] and [14], these $\psi_{i,T}$ are referred to as wave functions and $\nu_{i,T}$ as ocupation numbers. Furthermore,every pair $(\nu_{i,T}, \psi_{i,T})$ is said to be a mixed state. Some stability results and interpolation inequalities of mixed states were proved in [7] and they were brought to an operator setting in [8] and [18]. Moreover, in [8] and [18] it's proved a compactness theorem (generalized in [17]) for operators and it served as the main tool for the minization of free energy functionals.

In this work we answer some questions presented in [8] and [17], for the whole domain with $N > 4$. We consider a potential $V : \mathbb{R}^N \to \mathbb{R}$ with the following properties

- $V \in \mathrm{C}(\mathbb{R}^N)$,
- $\lim\limits_{|x|\to\infty} V(x) = \infty$,

- $V_0 = \inf_{x\in\mathbb{R}^N} V(x) > 0$.

Our operator setting consists of self-adjoint trace class operators $T \in \mathscr{I}_1$ such that $(\psi_{i,T})_{i\in\mathbb{N}} \subseteq \mathrm{H}^2_V(\mathbb{R}^N)$
where

$$\mathrm{H}^2_V(\mathbb{R}^N) = \{u \in \mathrm{H}^2(\mathbb{R}^N) : \int_{\mathbb{R}^N} V(x)|u(x)|^2 dx < \infty\},$$

and

$$\langle\langle T\rangle\rangle_{V,2} = \sum_{i=1}^{\infty} |\nu_{i,T}| \left(\int_{\mathbb{R}^N} |\triangle\psi_{i,T}(x)|^2 + V(x)|\psi_{i,T}(x)|^2 dx\right) < \infty. \tag{1.1}$$

In this case, we say that T belongs to the *Sobolev-like cone* $\mathscr{H}^2_V$ and define (1.1) as its total energy. Furthermore, we denote $\mathscr{H}^2_{V,+} = \{T \in \mathscr{H}^2_V / T \geq 0\}$.

We observe that the energy of an operator $T \in \mathscr{H}^2_{V,+}$ can be written as the sum of the kinetic and potential energy

$$\mathscr{K}(T) = \sum_{i=1}^{\infty} \nu_{i,T} \int_{\mathbb{R}^N} |\triangle\psi_{i,T}(x)|^2 dx, \quad \mathscr{P}_V(T) = \sum_{i=1}^{\infty} \nu_{i,T} \int_{\mathbb{R}^N} V(x)|\psi_{i,T}(x)|^2 dx,$$

respectively. Moreover, we define the density function associated to $T \in \mathscr{H}^2_{V,+}$ as the function $\rho_T : \mathbb{R}^N \to \mathbb{R}$ given by

$$\rho_T(x) = \sum_{i=1} \nu_{i,T} |\psi_{i,T}(x)|^2,$$

and prove that it belongs to a range of Lebesgue and Sobolev spaces; in particular, to the space $\mathrm{W}^{2,1}(\mathbb{R}^N) \cap \mathrm{L}^s(\mathbb{R}^N)$ with $s \in [1, N/(N-4)]$. To achieve this, we first prove that given $T \in \mathscr{H}^2_{V,+}$ the operator $L = T^{r^2}$ belongs to $\mathrm{W}^{2,r}(\mathbb{R}^N) \cap \mathrm{L}^s(\mathbb{R}^N)$ for $r \in [1, N(N-1)]$ and $s \in [1, N/(N-4)]$ then take $r = 1$.

We also study two type of functionals acting on $\mathscr{H}^2_{V,+}$ such as the entropy functionals:

$$\mathcal{S}_\beta = \mathrm{Tr}(\beta(T)) = \sum_{i=1}^{\infty} \beta(\nu_{i,t}), \quad T \in \mathscr{H}^2_{V,+}$$

where $\beta : \mathbb{R} \to \mathbb{R} \cup \{+\infty\}$ is a convex function such that $\beta(0) = 0$ and the second kind of functionals are the $\beta-$free energy functionals,

$$\mathscr{F}_{V,\beta}(T) = \mathcal{S}_\beta(T) + \langle\langle T\rangle\rangle_{V,2}, \quad T \in \mathscr{H}^2_{V,+}.$$

We also show that $\mathscr{F}_{V,\beta}$ is bounded from below and that the families $(\mathscr{K}(T_\theta))_{\theta\in\Lambda}$, $(\mathcal{S}_\beta(T_\theta))_{\theta\in\Lambda}$, $(||T_\theta||_1)_{\theta\in\Lambda}$, $(\langle\langle T_\theta\rangle\rangle_{V,2})_{\theta\in\Lambda}$ and $(\mathscr{P}_V(T_\theta)_{\theta\in\Lambda}$ are also bounded in $\mathbb{R}$. These facts will be useful if we want to minimize a free energy functional in $\mathscr{H}^2_{V,+}$.

Finally, we prove some Gagliardo-Nirenberg type inequalities for operators explicitly given β an entropy seed generated by $F \in \mathscr{C}_V$. We assume that the functions τ, G are such that $\tau(s) = -(-G)^*(s),\ s \in \mathbb{R}$, and

$$\operatorname{Tr}\Big(F(\triangle^2 + V\Big) \leq \int_{\mathbb{R}^N} G(V(x))dx. \tag{1.2}$$

Then, for every $T \in \mathscr{H}^2_{V,+}$,

$$\mathcal{S}_\beta(T) + \mathscr{K}(T) \geq \int_{\mathbb{R}^N} \tau(\rho_T(x))dx.$$

We give a brief description of this work

Summary of Chapter 2

In this chapter, we present useful definitions and theorems of Functional Analysis which will help us to develop our work. We begin with some basics as metric, Banach and Hilbert spaces. In Section 2.3, we introduce Sobolev spaces where we can find some useful inequalities such as the Poincare's inequality essential to prove the regularity results of Section 4.1.

In Section 2.4, we give some details of spectral theory a powerful tool for our work. We also talk about trace-class operators and compact operators in Section 2.5 this kind of operators will be the most important for our definitions. Section 2.6 is devoted to give some generalizations of the spectral theory but using unbounded operators this kind of operators is very useful for Quantum Mechanics such as the position and momentum operators. We end the chapter with some spectral theory using self-adjoint operators this will be important for us since we work with the operator $\triangle^2 + V$ which is self-adjoint.

Summary of Chapter 3

In Section 3.1 we give a brief introduction to Quatum Mechanics. Section 3.2 is devoted to introduce some important operators and the Heisenberg uncertainty principle. We end the chapter with a short presentation of the Schrödinger equation.

Summary Chapter 4

In Chapter 4 we present the main results of this work. Section 4.1 introduces some definitions and preliminary results. We define the Sobolev-like cone $\mathscr{H}_V^2$ and the energy of an operator. We prove some basic properties of $\mathscr{H}_V^2$. We end this section proving a regularity result of the density function associated to an operator $T \in \mathscr{H}_{V,+}^2$. This results states that for every $T \in \mathscr{H}_{V,+}^2$ its density function

$$\rho_T(x) = \sum_{i=1} \nu_{i,T} |\psi_{i,T}(x)|^2,$$

belongs to $\mathrm{W}^{2,r}(\mathbb{R}^N) \cap \mathrm{L}^s(\mathbb{R}^N)$ for certain values of r and s.

In Section 4.2, we define the Cassimir class functions that let us define trace-class operators of the form

$$F(\triangle^2 + V).$$

We also define the β-free energy functionals acting on $\mathscr{H}_{V,+}^2$ as the sum of the total energy and entropy of an operator $T \in \mathscr{H}_{V,+}^2$. We end the chapter proving that there exists a lower bound for the free energy and some Gagliardo-Nirenberg type inequalities.

Summary Chapter 5

We present our conclusions and recommendations.

Chapter 2

Mathematical framework

In this chapter we introduce some fundamental concepts of Functional Analysis which will be used through this work. We start introducing some basic concepts of mathematical analysis. Then we present some properties of Lebesgue and Sobolev spaces. Finally, we will study linear operators and their spectral properties.

Most of time we will consider and work with linear spaces over the field $\mathbb{R}$, except in Subsection 2.4, where $\mathbb{C}$ is prefered. The main references are [5], [13], [16],[3], [20], [14] and [6]

2.1 Some concepts of Functional Analysis

We will review some definitions and theorems of Functional Analysis and Operators Theory, as they will help us later to our work.

2.1.1 Metric and normed spaces

Let's introduce metric spaces wich have some good properties to work with.

Definition 2.1. A metric space is a pair (X, d), where X is a set and $d : X \times X \to \mathbb{R}$ is a real-valued function on $X \times X$ such that for all $x, y, z \in X$ it follows that:

- $d(x, y) \geq 0$,
- $d(x, y) = 0$ iff x=y,
- $d(x, y) = d(y, x)$,

- $d(x, z) \leq d(x, y) + d(y, z)$.

In particular, d is refered to be a metric on X. Note that the notion of metric deals in essence with the distance between the elements of the set X, for more details we refer to [6].

Remark 2.1. Given a set X we can employ different metrics d to become X into a metric space, when there is no confussion of the metric used we simply denoted the metric space (X,d) as X.

Thanks to our notion of metric we can now introduce some other definitions such that convergence

Definition 2.2. A sequence of elements $(x_n)_{n\in\mathbb{N}}$ of a metric space (X,d) is said to converge to an element $x \in X$ if $d(x, x_n) \to 0$ as $n \to \infty$ or

$$\lim_{n\to\infty} x_n = x.$$

Definition 2.3. A sequence of elements $(x_n)_{n\in\mathbb{N}}$ of a metric space (X,d) is called a Cauchy sequence if

$$\forall \epsilon > 0,\ \exists N \in \mathbb{N} :\ n, m \geq N \Rightarrow d(x_n, x_m) < \epsilon$$

for every $n, m \in \mathbb{N}$.

Remark 2.2. Any convergent sequence is a Cauchy sequence.

Proof. In fact given $\epsilon > 0$ and $(x_n)_{n\in\mathbb{N}}$ such that $x_n \to x$ as $n \to \infty$. We can find $N \in \mathbb{N}$ such that $n \geq N$ implies $d(x_n, x) < \epsilon/2$. Then using triangle inequality we have that $n, m \geq N$ implies

$$d(x_n, x_m) \leq d(x_n, x) + d(x_m, x) < \epsilon/2 + \epsilon/2 = \epsilon.$$

□

However, the inverse of this remark is not necessarily true, i.e. it could happen that there exists a Cauchy sequence which is not convergent. The importance of Cauchy sequences relies on the following definition

Definition 2.4. A metric space in which all Cauchy sequences converge is called complete.

In a metric spaces we can talk easily of the continuity which is defined as

Definition 2.5. A mapping T from a metric space (X,d) to a metric space (Y,ρ) is called continuous at x_0 if and only if $x_n \to x_0$ implies $T(x_n) \to T(x_0)$.

We say that a function is a bijection if it is injective and onto at the same time. Moreover, a bijection T from a metric space (X,d) to a metric space (Y, ρ) is called an *isometry* if it preserves the metric, i.e.

$$\rho(T(x), T(y)) = d(x, y). \tag{2.1}$$

Remark 2.3. An isometry is continuous. Moreover, if there exists an isometry between the metric spaces (X,d) and (Y,ρ) then they are said to be isometric

Under certain conditions we can always complete an incomplete space, as the following theorem retrieved from [20] shows

Theorem 2.1. *If (X,d) is an incomplete metric space, it is possible to find a complete metric space $\bar{X}$ so that X is isometric to a dense subset of $\bar{X}$.*

To end up with this part, we give a brief description and some properties of open and closed sets in metric spaces.

Definition 2.6. Let (X,d) be a metric space we have the following:

1. The set $\mathrm{B}(x_0, r) = \{x \in X : \ d(x, x_0) < r\}$ is called the open ball with center x_0 and radius $r > 0$.
2. A set $\mathcal{O} \subseteq X$ is called open if $\forall x_0 \in O, \exists r > 0 : \ \mathrm{B}(x_0, r) \subseteq O$.
3. A set $\mathcal{N} \subseteq X$ is called a neighborhood of $x_0 \in N$ if $\mathrm{B}(x_0, r) \subseteq N$ for some $r > 0$.
4. Let $E \subseteq X$. A point x is called a limit point of E, if $\forall r > 0 : \ \mathrm{B}(x, r) \cap E \backslash \{x\} \neq \emptyset$.
5. A set $F \subseteq X$ is called closed if F contains all its limit points.
6. If $G \subseteq X$, $x \in G$ is called an interior point of G, if G is a neighborhood of x.

Theorem 2.2. *A function T from a metric space (X,d) to another space (Y,ρ) is continuous if and only if for all open sets $O \subseteq Y$, $T^{-1}(O)$ (the inverse image of O) is open.*

Similarly to metric spaces we can now introduce *normed spaces*

Definition 2.7. (Norm and normed space) Let X be a linear space. The mapping $|| \cdot || : X \to \mathbb{R}$ is called a *norm* if the following properties hold:

1. $\forall x \in X: \quad ||x|| \geq 0,$

2. $\forall x \in X: \quad ||x|| = 0 \Leftrightarrow x = 0,$

3. $\forall \alpha \in \mathbb{R}, \quad \forall x \in X: \quad ||\alpha x|| = |\alpha| ||x||,$

4. $\forall x, y \in X: \quad ||x + y|| \leq ||x|| + ||y||.$

A linear space X on which a norm is defined is called a **normed space** and is denoted by (X, $|| \cdot ||$) or simply by X.

Remark 2.4. The norm of a normed space X, defines a metric d on X, that is the concept of metric space is more general than normed space. In fact, for each x and y in X we define

$$d(x, y) = ||x - y||.$$

This is called the *metric induced by the norm.* So that, any normed space is a metric space.

Remark 2.5. The norm is a continuous mapping from X into $\mathbb{R}$.

In a normed linear space, we can find more than one norm defined on it so that

Definition 2.8. Let $|| \cdot ||_0$ and $|| \cdot ||_1$ be two norms defined on X. We say that $|| \cdot ||_0$ and $|| \cdot ||_1$ are equivalent if there are positive numbers a and b such that

$$\forall x \in X: \ a||x||_0 \leq ||x|| \leq b||x||_1.$$

If two norms are equivalent on X then they define the same topology on X, i.e, the open subsets of X are the same.

Remark 2.6. In a normed space, we say that a sequence $(x_n)_{n \in \mathbb{N}} \subseteq X$ is convergent to $x \in X$ iff

$$\lim_{n \to \infty} ||x_n - x|| = 0.$$

An important property of normed spaces is compacteness which for finite dimensional normed spaces means that the space is bounded and closed and conversly. Nevertheless, for the infinite dimensional case we require of more general conditions some of them can be found in [16, Sec. 2.8] or [10]. Compact sets are important since they behave similarly to finite sets, so that one of their fundamental properties is the following:

Theorem 2.3. *Let X and Y metric spaces and $T: X \to Y$ a continuous mapping. If $M \subseteq X$ is compact then $T(M)$ is compact.*

The proof of this result is given in [13, Th 2.5-6].

As a consequence of the previous theorem we have the following result well-known from Calculus:

Corollary 2.1. *Let X be a metric space and $M \subseteq X$ compact. If $T : M \subseteq X \to \mathbb{R}$ is a continuous mapping then T assumes a maximum and a minimum at some points of M.*

Proof. By Theorem 2.3 we have that $T(M) \subseteq \mathbb{R}$ is compact then closed and bounded since $\mathbb{R}$ is a metric space. So that $\inf T(M) \in T(M)$ and $\sup T(M) \in T(M)$, and the inverse images of these two points are points x and y of M such that one is a point of minimum and maximum, respectively. □

2.1.2 Banach and Hilbert spaces

Through this work Banach and Hilbert spaces play an important role, so let's introduce some defintions.

Definition 2.9. Let X be a *complete* normed linear space then we say that X is a Banach space.

Remark 2.7. In a Banach space every Cauchy sequence has a limit.

Let's give some examples of Banach spaces

Example 2.1. (Space C[a,b])

The space of continuous functions from $[a, b]$ to $\mathbb{R}$ is a Banach space with the norm given by

$$||x|| = \max_{t \in [a,b]} |x(t)|.$$

Note that a sequence of elements $(x_n)_{n \in \mathbb{N}}$ in a normed linear space X is called summable if the series

$$\sum_{n=1}^{N} x_n$$

converges as $N \to \infty$ to an $x \in X$ and it is absolutely summable if

$$\sum_{n=1}^{\infty} ||x_n|| < \infty.$$

Thanks to this we have the following criterion to determine if a normed linear space X is complete and can be found in [20. Th. III.3].

Theorem 2.4. *A normed linear space is complete if and only if every absolutely summable sequence is summable.*

Now, we introduce a special kind of metric spaces known as *inner product spaces* wich have some nice geometric properties

Definition 2.10. Let X be a linear space. An **inner product** is a mapping of $X \times X$ into the scalar field K of X, i.e., $(\cdot,\cdot) : X \times X \to K$. Let x and y $\in X$ we denoted the inner product of x and y by

$$(x, y) \tag{2.2}$$

and it is such that for all $\alpha \in K$ and x, y and $z \in X$ we have

1. $(x + y, z) = (x, z) + (y, z)$,
2. $(\alpha x, y) = \alpha(x, y)$,
3. $(x, y) = \overline{(y, x)}$,
4. $(x, x) \geq 0$,
5. $(x, x) = 0 \quad \Leftrightarrow \quad x = 0$.

A linear space X with an inner product defined on it is said to be an *inner product space* or pre-Hilbert space.

Remark 2.8. In condition 3 of the previous definition the bar denotes the complex conjugate.

An inner product on a linear space X defines a norm and a metric on X given by

$$||x|| = \sqrt{(x, x)} \quad \text{and} \quad d(x, y) = ||x - y|| = \sqrt{(x - y, x - y)} \tag{2.3}$$

respectively.

Definition 2.11. (Hilbert space) Let X be an inner product space. If X is complete in the metric defined by (2.3), then X is called a Hilbert space.

If we have that a norm is induced by an inner product then it satifies the **parallelogram equality**

$$||x + y||^2 + ||x - y||^2 = 2(||x||^2 + ||y||^2). \tag{2.4}$$

Two elements x and y of an inner product space are said to be orthogonal if

$$(x, y) = 0.$$

Moreover, a collection of elements $(x_i)_{i\in\mathbb{N}}$ of H (a Hilbert space) is called an orthonormal set if

$$\begin{cases} (x_i, x_j) = 1, & \text{if } i = j, \\ (x_i, x_j) = 0, & \text{if } i \neq j. \end{cases}$$

In an inner product space we have two important inequalities

Lemma 2.1. *(Schwartz inequality, triangle inequaility) Any inner product space with their corresponding norm satisfy the Schwartz inequality*

$$|(x, y)| \leq ||x||||y||$$

and the triangle inequality

$$||x + y|| \leq ||x|| + ||y||.$$

The proof of this inequalities are given in [13, Lem. 3.2-1].

Remark 2.9. As a result of the previous lemma we get that the inner product is a continuous mapping.

As with Banach spaces we have that there exists a Hilbert space H and an isomorphism A from X onto a dense subspace $W \subseteq$ H for any inner product space X. In this case H is the completion of X and is unique except for isomorphisms. For the proof and more details about this fact we can refer to [13, Th. 3.2-3].

Definition 2.12. (Direct sum) A linear space is said to be a the *direct sum* of two subspaces Y and Z of X, and it is denoted by

$$X = Y \oplus Z,$$

if every $x \in X$ can be represented in a unique way as $x = y + z, \quad y \in Y, z \in Z$. Here Z is called an algebraic complement of Y in X and vice versa. Moreover, Y and Z are the complementary pair of subspaces in X.

In particular, for a general Hilbert space H we define its orthogonal complement as

$$Y^{\perp} = \{z \in H |\ z \perp Y\}$$

the set of all elements of H which are orthogonal to Y. Furthermore, if Y is closed so is $Y^\perp$.

Then for any Hilbert space H the following holds

Theorem 2.5. *(Direct sum) Let H be a Hilbert space and Y a closed subspace of H. Then*

$$H = Y \oplus Z$$

where $Z = Y^\perp$. Furthermore, each element $x \in H$ can be uniquely written as $x = y + z$, $y \in Y$ and $z \in Z = Y^\perp$.

A proof of this fact is given in [13, Th. 3.3-4].

For finite dimensional inner product spaces, the idea of orthonormal sets is interesting since allows us to approximate or represent every element in the space by the use of them, in this case it is enough to use an othonormal set set of n elements. But for the infinite dimensional case we need to understand the idea of total orthonormal set or Hilbert basis.

Definition 2.13. (Total orthonormal set) Let X be a normeed space and $M \subseteq X$. If the span of M is dense in X, then M is called a total set in X. Accordingly, if M is an orthonormal set in an inner product sapce X, and it is total in X, i.e.,

$$\langle \bar{M} \rangle = X$$

then M is called a total orthonormal set in X.

Remark 2.10. Every non trivial Hilbert space H has a total orthonormal set.

In Hilbert spaces we have another criterion for totality, namely

Theorem 2.6. *(Totality) Let H be a Hilbert space and $M \subset H$ be an orthonormal set in H. Then M is total in H iff for all $x \in H$ the Parseval relation holds:*

$$||x||^2 = \sum_{n\in\mathbb{N}} |(x, e_n)|^2, \quad e_n \in M \textit{ for each } n \textit{ in } \mathbb{N}. \tag{2.5}$$

Even more every $x \in H$ can be represented as

$$x = \sum_{n\in\mathbb{N}} (ex, e_n) e_n$$

where the coefficients (x, e_n) are called Fourier coefficcients of x with respect to the total set M.

2.2 Bounded linear operators

In this subsection, we explore properties of linear operators such that boundedness and continuity which are useful in Functional Analysis and will have a great impact on our further work. Recall that a linear operator T is a mapping which goes from a linear space X into a linear space Y such that the following holds

$$T(\alpha x + \beta y) = \alpha T x + \beta T y$$

for any $x, y \in \mathcal{D}(T) \subseteq X$ and scalars α, β (Along this section we will use the notation of [13] to write the image of an element in X through T).

Now, we recall the concepts of norm and normed spaces given before to introduce some new properties.

2.2.1 Definition and properties

A linear operator defined on a normed space X into a normed space Y is said to be bounded if and only if the norm of the image of an element $x \in X$ through T is controlled by the norm of that element. Formally, we have the following definition

Definition 2.14. Let X and Y be normed spaces and $T : \ \mathcal{D}(T) \subseteq X \to Y$ a linear operator. We say that T is bounded iff

$$\exists c \in \mathbb{R}, \forall x \in \mathcal{D}(T) : \ ||Tx||_Y \leq c||x||_X. \tag{2.6}$$

Remark 2.11. From (2.6) we note that a bounded linear operator maps bounded sets onto bounded sets.

From Definition 2.14 we state the *norm of an operator* as

$$||T|| = \sup_{x \in \mathcal{D}(T)} \frac{||Tx||}{||x||} \tag{2.7}$$

with $x \neq 0$.

In particular, if $||x|| = 1$ then (2.7) is equivalent to

$$||T|| = \sup_{x \in \mathcal{D}(T)} ||Tx||$$

Lemma 2.2. *If T is a bounded linear operator as defined in (2.14), then the norm of T defined by (2.7) satisfies the properties of a norm.*

The proof of this lemma can be found in [13, Lem. 2.7-2].

As we mentioned before, operators are mappings so that we can define continuity on them. Generally, we say that an operator $T : \mathcal{D}(T) \subseteq X \to Y$, where X and Y are normed spaces, is continuous at $x_0 \in \mathcal{D}(T)$ if

$$\forall \epsilon > 0,\ \exists \delta > 0 :\ ||x - x_0|| < \delta \Rightarrow ||Tx - Tx_0|| < \epsilon.$$

T is continuous if T is continuous at every x on the Domain. This is a usual definition of continuity, but note that it is for a general operator; in contrast, when we work with linear operators continuity and boundedness become the same

Theorem 2.7. *(Continuity and boundedness) Let X and Y be normed spaces and $T : \mathcal{D}(T) \subseteq X \to Y$ a linear operator. Then T is bounded iff T continuous. Moreover, if T is continuous at one point then it is continuous.*

A very detail proof can be found in [13, Th. 2.7-9].

Let X, Y and Z normed spaces. Then for any bounded linear operator $T : X \to X$, we have the following useful inequality

$$||T^n|| \leq ||T||^n, \quad n \in \mathbb{N}.$$

In addition, for $T_1 : X \to Y$ and $T_2 : Y \to Z$

$$||T_1 T_2|| \leq ||T_1|| ||T_2||.$$

Let any two normed spaces X and Y, we define

$$\mathscr{L}(X, Y)$$

as the set of all bounded linear operators from X into Y when $X = Y$ we simply put $\mathscr{L}(X)$. This set becomes a normed space endowed with the norm defined in (2.7). Moreover, we have the following result

Theorem 2.8. *If Y is a Banach space, then $\mathscr{L}(X, Y)$ is also Banach.*

Proof. Let's consider a Cauchy sequence $(T_n)_{n\in\mathbb{N}}$ in $\mathrm{B}(X,Y)$, generic. Then

$$\forall\epsilon > 0, \exists N \in \mathbb{N} :\ n, m > N \Rightarrow ||T_n - T_m|| < \epsilon. \tag{2.8}$$

Let $x \in X$, we have that

$$||T_n x - T_m x|| = ||(T_n - T_m)x|| \leq ||T_n - T_m||||x||, \tag{2.9}$$

i.e., $(T_n x)_{n\in\mathbb{N}}$ is Cauchy in Y by (2.8). But Y is a Banach space so that there exists $Tx \in Y$ such that $T_n x \to Tx$ as $n \to \infty$. This defines an linear operator $T : X \to Y$. So that, letting $m \to \infty$ in (2.9) and for $n > N$ and $x \in X$

$$||(T_n - T)x|| \leq \epsilon||x||. \tag{2.10}$$

By the arbitrariness of x, the last implies that $T \in \mathrm{B}(X,Y)$. Finally, we also have that $\epsilon \in \mathcal{O}(T_n - T)$ so that

$$n > N \Rightarrow ||T_n - T|| \leq \epsilon. \tag{2.11}$$

Hence, $(T_n)_{n\in\mathbb{N}}$ is convergent to an element of $\mathrm{B}(X,Y)$ and since it was chosen arbitrarily we are done.

□

Remark 2.12. To this point, we have mentioned properties of operators in general defined from a normed space X into a normed space Y. In particular, when $Y = \mathbb{R}$ the operator is called a functional.

The Dual space of a normed space X is the set of all bounded linear functionals on X and it is denoted X', it is a normed space whenever it is endowed with the norm given by

$$\forall f \in X' :\ ||f|| = \sup_{x\in X} \frac{|f(x)|}{||x||}, \quad x \neq 0.$$

Theorem 2.9. *Let X be a normed space. Then its dual X' is a Banach space.*

To end this subsection we state a characterization theorem for bounded linear functionals namely it is known as the Riesz's representation theorem

Theorem 2.10. *(Riesz's theorem) Let H be a Hilbert space and $f \in H'$ a bounded linear functional. Then f can be represented in terms of the inner product as*

$$f(x) = (x, z)$$

for every $x \in H$ and where z depends on f. In addition $||f|| = ||z_f||$.

The proof of this theorem is given in [13, Th. 3.8-1].

2.2.2 Closed graph theorem

In practice not all operators are bounded, in work some of them are just closed linear operators. Some examples in Quatum mechanics are unbounded operators. In this subsection, we define closed linear operators on normed spaces and state some of their properties, in particular thanks to the closed graph theorem (state below) we can give sufficient conditions under which a closed linear operator on a Banach space is bounded.

Definition 2.15. (Closed linear operator) Let X and Y be normed spaces and $T:\ \mathcal{D}(T) \to Y$ a linear operator with domain $\mathcal{D}(T) \subset X$. Then T is said to be a closed linear operator if its graph

$$\mathcal{G}(T) = \{(x, y)|\ x \in \mathcal{D}(T),\ y = Tx\}$$

is closed in the normed spcae $X \times Y$ which norm is defined by

$$||(x, y)|| = ||x|| + ||y||. \tag{2.12}$$

The closedness of a linear operator is an important property which can also be expressed using the following criterion

Theorem 2.11. *(Closed linear operator) Let X and Y be normed spaces. A linear operator $T:\ \mathcal{D}(T) \subseteq X \to Y$ is said to be closed if and only if a sequence $x_n \in \mathcal{D}(T)$ is such that $x_n \to x$ and $Tx_n \to y$ then $x \in \mathcal{D}(T)$ and $Tx = y$.*

Theorem 2.12. *(Closed graph theorem) Let X and Y be Banach spaces and $T:\ \mathcal{D}(T) \to Y$ a closed linear operator, where $\mathcal{D}(T) \subseteq X$. Then if the domain of T is closed in X, the operator is bounded.*

Remark 2.13. If a linear operator is closed it does not imply that it is bounded, and conversely if a linear operator is bounded it does not imply that it is closed.

Despite the last remark, we can play with the domain of the operator and obtain a useful result

Lemma 2.3. *(Closed operator) Let X and Y be normed spaces. Let $T:\ \mathcal{D}(T) \subseteq X \to Y$ be a bounded linear operator, then:*

1. *If $\mathcal{D}(T)$ is a closed subset of X, then T is closed.*

2. *If T is closed and Y is complete, then $\mathcal{D}(T)$ is a closed subset of X.*

Proof. (1) If (x_n) is in $\mathcal{D}(T)$ and converges, say, $x_n \to x$ and is such that (Tx_n) also converges, then $x \in \overline{\mathcal{D}(T)} = \mathcal{D}(T)$ since $\mathcal{D}(T)$ is closed, and $Tx_n \to Tx$ since T is continuous. Hence T is closed by Theorem 2.11.

(2) Let's take $x \in \overline{\mathcal{D}(T)}$ there is a sequence (x_n) in $\mathcal{D}(T)$ such that $x_n \to x$. Since T is bounded we have that

$$||Tx_n - Tx_m|| = ||T(x_m - x_n)|| \leq ||T||||x_n - x_m||.$$

This shows that $(Tx_n)_{n\in\mathbb{N}}$ is Cauchy. Since Y is complete we have that this sequence converges, say, $Tx_n \to y \in Y$. As T is closed the last assertions implies that $x \in \mathcal{D}(T)$ by Theorem 2.11. Since x was arbitrary we are done. □

2.3 Lebesgue and Sobolev spaces

In Analysis, some of the most known spaces are Lebesgue and Sobolev spaces. In this subsection, we introduce them and give some of their properties.

2.3.1 Lebesgue spaces

Let's start with some definitions

Definition 2.16. Let $f: \Omega \to \mathbb{R}$ be a continuos functional, the set given by

$$\text{supp}(f) = \overline{\{x \in \Omega / \ f(x) \neq 0\}}.$$

is called the support of f, and is the smallest closed subset of Ω where the function does not vanish.

We denote by $\mathrm{C}_0^\infty(\Omega)$ the space of functions $f \in \mathrm{C}^\infty$ such that they have compact support. This space is a normed space whenever is equipped with the norm given by

$$\|f\|_{L^\infty(\Omega)} = \sup_{x\in\Omega}|f(x)|$$

For $1 \leq p \leq \infty$, we have that the functional $|| \cdot || : \ \mathrm{C}^\infty(\mathbb{R}) \to \mathbb{R}$ given by

$$\|f\|_{L^p(\Omega)} = \left(\int_\Omega |f(t)|^p\right)^{1/p}$$

is a norm.

Let's consider the following space

$$\tilde{L^p}(\mathbb{R}) = (C_0(\Omega), \|f\|_{L^p(\Omega)}) \tag{2.13}$$

it is not complete. However, by the Completion Theorem [16, Th 3.11] there exists a Banach space which we will denoted by $L^p(\Omega)$ and it is the completion of $\tilde{L^p}(\Omega)$, i.e.,

$$L^p(\Omega) = \overline{C_0(\Omega)}$$

and it is called the *Lebesgue space* $L^p(\Omega)$. Note that a function of $L^p(\Omega)$ can be approximated as much as we want by continuous functions with compact support.

Remark 2.14. The space $L^p(\Omega)$ is the set of equivalence classes determined by the following equivalence relation

$$f \sim g \Leftrightarrow \|f - g\|_{L^p(\Omega)} = 0. \tag{2.14}$$

It's is important to recall the following integration results since they are useful and must be known for a mathematician.

Theorem 2.13. *(Monotone convergence theorem-Beppo Levi) Let* $(f_n)_{n\in\mathbb{N}} \subseteq L^1(\Omega)$ *such that* $f_n(x) \le f_{n+1}(x)$ *a.e. for every* $n \in \mathbb{N}$. *Assume that*

$$\sup_{n\in\mathbb{N}} \int_\Omega f_n(x)dx < \infty.$$

Then $f_n(x)$ *converges a.e. on* Ω *to a* finite limit *in* $L^1(\Omega)$, *say, f(x) and*

$$\|f_n - f\|_{L^1(\Omega)} \to 0.$$

Theorem 2.14. *(Dominated convergence theorem-Lebesgue) Let* $(f_n)_{n\in\mathbb{N}} \subset L^1(\Omega)$ *such that it converges almost everywhere to a function f. If there exists* $g \in L^1(\Omega)$ *such that*

$$|f_n(x)| \le g(x) \ \textit{a.e for each}\ n \in \mathbb{N},$$

then $f \in L^1(\Omega)$ *and*

$$\|f_n - f\|_{L^1(\Omega)} \to 0.$$

Lemma 2.4. *(Fatou's Lemma) Let* $(f_n)_{n\in\mathbb{N}}$ *be a sequence in* $\mathrm{L}^1(\Omega)$ *such that for every* $n \in \mathbb{N}$*,* $f_n \geq 0$ *a.e. and* $\sup\limits_{n\in\mathbb{N}} \int f_n(x)\, dx < \infty$*. Then*

$$\int_\Omega \liminf_{n\to\infty} f_n(x)\, dx \leq \liminf_{n\to\infty} \int_\Omega f_n(x)\, dx. \tag{2.15}$$

These three are the principal theorems on convergence of integrable functions their detail demostrations and other results such that Fubbini and Tonelli theorems can be found in [5, Sec. 4.1] and [6, Sec. 2.8]. Now, we state some important inequalities.

Remark 2.15. Let $1 \leq p \leq \infty$, we say that p' is the *conjugate exponent* of p if

$$\frac{1}{p} + \frac{1}{p'} = 1.$$

Theorem 2.15. *(Hölder's inequality) Let* $1 \leq p \leq \infty$ *and* p' *its conjugate exponent. Assume that* $f \in \mathrm{L}^p(\Omega)$ *and* $g \in \mathrm{L}^{p'}(\Omega)$*, then* $fg \in \mathrm{L}^1(\Omega)$ *and*

$$\int_\Omega |f(x)g(x)|dx \leq \|f\|_{L^p(\Omega)}\|g\|_{L^{p'}(\Omega)} \tag{2.16}$$

For a proof of Theorem 2.15 see [5, Th. 4.6].

As a result of Hölder's inequality, we have the following interpolation inequality.

Corollary 2.2. *Let* $1 \leq p \leq q \leq \infty$ *and* $f \in \mathrm{L}^p(\Omega) \cap \mathrm{L}^q(\Omega)$*, then* $f \in \mathrm{L}^r(\Omega)$*, for any r such that* $p \leq r \leq q$*, and*

$$\|f\|_{L^r(\Omega)} \leq \|f\|^{\alpha}_{L^p(\Omega)}\|f\|^{1-\alpha}_{L^q(\Omega)},$$

where $\frac{1}{r} = \frac{\alpha}{p} + \frac{1-\alpha}{q}$, $0 \leq \alpha \leq 1$.

Proof. Let $f \in \mathrm{L}^p(\Omega) \cap \mathrm{L}^q(\Omega)$ and let r, $p \leq r \leq q$, such that

$$\frac{1}{r} = \frac{\alpha}{p} + \frac{1-\alpha}{q} \tag{2.17}$$

with $0 \leq \alpha \leq 1$. From (2.17) we have that

$$1 = \frac{1}{p/r\alpha} + \frac{1}{q/(r(1-\alpha))},$$

so that using Theorem 2.15 we get that

$$
\begin{aligned}
\|f\|^r_{L^r(\Omega)} &= \int_\Omega |f(x)|^r dx \\
&= \int_\Omega |f(x)|^r dx \\
&= \int_\Omega |f(x)|^{r\alpha} |f(x)|^{r-r\alpha} dx \\
&\leq \left(\int_\Omega ||f(x)^{r\alpha}|^{\frac{p}{r\alpha}}\right)^{\frac{r\alpha}{p}} \left(\int_\Omega ||f(x)^{r(1-\alpha)}|^{\frac{q}{r(1-\alpha)}}\right)^{\frac{r(1-\alpha)}{q}} \\
&= \left(\int_\Omega |f(x)|^p\right)^{\frac{r\alpha}{p}} \left(\int_\Omega |f(x)|^q\right)^{\frac{r(1-\alpha)}{q}} \\
&= \|f\|^{r\alpha}_{L^p(\Omega)} \|f\|^{r(1-\alpha)}_{L^q(\Omega)}.
\end{aligned}
$$

Since f was chosen arbitrarily, we have proved the interpolation inequality. □

The L^p spaces enjoy some good properties when $1 < p < \infty$, such that separability, reflexivity even more the dual of $\mathrm{L}^p(\Omega)$ can be identified as $\mathrm{L}^{p'}(\Omega)$.This last is thanks to the *Riesz representation theorem* which say that any continuous functional on $\mathrm{L}^p(\Omega)$ can be represented in a unique way as an integral, i.e.,

Theorem 2.16. *(Riesz representation theorem) Let $1 < p < \infty$ and $\varphi \in (\mathrm{L}^p(\Omega))^{\circledast}$. Then there exists one and only one function $u \in \mathrm{L}^{p'}(\Omega)$ such that*

$$\langle \varphi, f \rangle = \int_\Omega u(x) f(x) dx \quad \textit{for all } f \in \mathrm{L}^p(\Omega). \tag{2.18}$$

Moreover,

$$\|u\|_{L^{p'}(\Omega)} = \|\varphi\|_{(L^p(\Omega))^{\circledast}}.$$

A detail proof of Theorem 2.16 can be found in [5, Th. 4.11].

Remark 2.16. Thanks to Riesz theorem we can identify $(\mathrm{L}^p(\Omega))^{\circledast}$ with $\mathrm{L}^{p'}$ since the mapping $\varphi \to u$ is a surjective isometry, that is we have that

$$(\mathrm{L}^p(\Omega))^{\circledast} \cong \mathrm{L}^{p'}(\Omega).$$

As we mentioned before, the conditions of separability and reflexivity are both present when $1 < p < \infty$. Is that so that in the case p=1 we have only separability of the space; however, we still can made the identification

$$(\mathrm{L}^1(\Omega))^{\circledast} \cong L^\infty(\Omega).$$

this is a result of the Theorem 2.16 which say

Corollary 2.3. *Let* $\varphi \in (\mathrm{L}^1(\Omega))^{\circledast}$. *Then there exists one and only one function* $u \in \mathrm{L}^\infty(\Omega)$ *such that*

$$\langle \varphi, f\rangle = \int_\Omega u(x)f(x)dx \quad \textit{for all } f \in \mathrm{L}^1(\Omega). \tag{2.19}$$

Moreover,

$$\|u\|_{L^\infty(\Omega)} = \|\varphi\|_{(L^1(\Omega))^{\circledast}}.$$

Remark 2.17. For $p = \infty$, the space is not separable nor reflexive, even more the dual space $(\mathrm{L}^\infty(\Omega))^{\circledast}$ contains $\mathrm{L}^1(\Omega)$ so that the identification with $\mathrm{L}^1(\Omega)$ is not true.

Now, we shall present the space of locally integrable functions wich will be useful in the introduction of the next section about Sobolev spaces

Definition 2.17. Let A be a set, we define the characteristic function of A as

$$\chi_A(x) = \begin{cases} 1, & \text{if} \quad x \in A, \\ 0, & \text{if} \quad x \notin A. \end{cases}$$

We say that a functional $f : \Omega \to \mathbb{R}$ belongs to the space of locally integrable functions $\mathrm{L}^p_{loc}(\Omega)$ if $f\chi_A \in \mathrm{L}^p(\Omega)$ for any compact set A contained in Ω.

To end this subsection we make a little comment on Measure Theory. Let's consider

$$\begin{aligned} M : \mathbb{R}^N &\to \mathbb{R} \\ x &\mapsto M(x) \end{aligned}$$

a measurable function such that

$$\forall x \in \mathbb{R}^N : \; M(x) \geq 0$$

and let $\mathscr{A} \subseteq \mathscr{P}(\mathbb{R}^N)$ the σ-algebra of Lebesgue's measurable sets. We define a measure

$$\begin{aligned} \mu : \mathscr{A} &\to \mathbb{R} \cup \{+\infty\} \\ A \mapsto \mu(A) &= \int_A M(x)dx = \int_A dM. \end{aligned}$$

If we consider the set

$$\hat{\mathrm{L}}^2(\mathbb{R}^N; dM) \;\; = \left\{ f : \mathbb{R}^N \to \mathbb{R} / f \text{ is measurable and } \int_{\mathbb{R}^N} |f(x)|^2 dM < +\infty \right\}$$

then by the completion theorem we have that

$$\mathrm{L}^2(\mathbb{R}^N; dM) = \overline{\hat{\mathrm{L}}^2(\mathbb{R}^N; dM)}$$

is a Hilbert space with an inner product $(\cdot,\cdot) : \mathrm{L}^2(\mathbb{R}^N; dM) \times \mathrm{L}^2(\mathbb{R}^N; dM) \to \mathbb{R}$ given by

$$(u, v) = \int_{\mathbb{R}^N} u(x)v(x)dM = \int_{\mathbb{R}^N} u(x)v(x)M(x)dx$$

and its corresponding norm

$$||u||_{\mathrm{L}^2(\mathbb{R}^N;dM)} = \sqrt{(u, u)}. \tag{2.20}$$

If the reader is not concerned with some of the definitions just presented or needs to remember some of them we recommend the references [6] and [20].

2.3.2 Sobolev spaces and Sobolev embeddings

In Functional Analysis, there exists certain Banach and Hilbert spaces such that $\mathrm{W}^{1,p}$ and H^1 as important as L^p spaces and these are called Sobolev spaces. These spaces have some applications in physics for example in the study of partial differential equations (PDE) since the problem to find a strong or classical solution is not a trivial work. Thus we require from Sobolev spaces to define the notion of weak solution of a PDE, this is important since in the case of this solution to be C^2, then we have that is is actually a classical solution of the PDE.

Let's start with the definitions. Let $I = (a, b)$ be an open interval, bounded or not, and let $p \in \mathbb{R}$ with $1 \leq p \leq \infty$.

Definition 2.18. (The Sobolev space $\mathrm{W}^{1,p}(I)$) We define the set

$$\mathrm{W}^{1,p}(I) = \left\{ u \in L^p(I) : \ \exists g \in \mathrm{L}^p(I) \text{ such that } \int_I u\varphi' = -\int_I g\varphi, \ \forall \phi \in \mathrm{C}_0^1(\Omega) \right\}$$

and call it the Sobolev space $\mathrm{W}^{1,p}(I)$. For an element $u \in \mathrm{W}^{1,p}$ we denote $g = u'$. Moreover, if p=2 then we set

$$\mathrm{H}^1(I) = \mathrm{W}^{1,2}(I)$$

We equipped the space $\mathrm{W}^{1,p}$ with the norm given by

$$||u||_{\mathrm{W}^{1,p}(I)} = \|u\|_{L^p(I)} + \|u'\|_{L^p(I)},$$

and the space H^1 with the scalar product

$$(u,v)_{\mathrm{H}^1(I)} = (u,v)_{\mathrm{L}^2(I)} + (u',v')_{\mathrm{L}^2(I)} = \int_a^b (uv + u'v')$$

with the associated norm

$$||u||_{\mathrm{H}^1(I)} = \left(||u||^2_{\mathrm{L}^2(I)} + ||u'||^2_{\mathrm{L}^2(I)}\right)^{1/2}. \tag{2.21}$$

Proposition 2.1. *For $1 \leq p \leq \infty$, the space $\mathrm{W}^{1,p}(I)$ is a Banach space and $\mathrm{H}^1(I)$ is a separable Hilbert space. In addition, if $1 < p < \infty$ then $\mathrm{W}^{1,p}(I)$ is reflexive and separable if $1 \leq p < \infty$.*

A proof of this proposition can be found in [5].

Remark 2.18. Formally the elements of $\mathrm{W}^{1,p}$ are the primitives of the L^p functions.

As in Lebesgue space we can define the Sobolev spaces with a property of density, the following is one of several theorems of density in Sobolev spaces.

Theorem 2.17. *Let $1 \leq p < \infty$ and $u \in \mathrm{W}^{1,p}(I)$. Then*

$$\exists (u_n)_{n\in\mathbb{N}} \in \mathrm{C}_0^\infty(\mathbb{R}) : u_n \mid_I \to u \ \textit{in} \ W^{1,p}(I), \ \textit{as} \ m \to \infty. \tag{2.22}$$

Remark 2.19. In general, there not exists a sequence $(u_n)_{n\in\mathbb{N}} \in \mathrm{C}_0^\infty(I)$ such that $u_n \to u$ in $\mathrm{W}^{1,p}(I)$.

The following theorem presents the Sobolev's inequality and a embedding of $\mathrm{W}^{1,p}(I)$ into the the Lebesgue space $\mathrm{L}^\infty(I)$

Theorem 2.18. *Let $1 \leq p \leq \infty$. Then there exists a constant C which depend on $|I|\infty$ such that*

$$\forall u \in \mathrm{W}^{1,p}(I), \ \|u\|_{L^\infty(I)} \leq C||u||_{\mathrm{W}^{1,p}(I)}, \tag{2.23}$$

i.e.,

$$\mathrm{W}^{1,p}(I) \subseteq \mathrm{L}^\infty(I)$$

with continuous injection.

A proof of this result can be found in [5, Th. 8.8].

To this point we have reviewed some definitions and properties of the Sobolev space $\mathrm{W}^{m,p}(I)$, m=1. However, this space can be generalized for any integer $m \geq 2$ and a real number $1 \leq p \leq \infty$ so that we get the space $\mathrm{W}^{m,p}(I)$, but even though this space is a one-dimensional one. So now we will state some results for a Sobolev space $\mathrm{W}^{m,p}(\Omega)$ where Ω is an open subset of $\mathbb{R}^N$ with p natural and following that $1 \leq p \leq \infty$. To do this let's start with some preliminaries

Definition 2.19. (Multiindex) A vector of the form $\alpha = (\alpha_1, \cdots, \alpha_n)$ where each component is such that $\alpha_i \geq 0$ is called a *multiindex* of order

$$|\alpha| = \sum_{k=1}^{n} \alpha_i.$$

Let α a multiindex of order k and $u \in \mathrm{C}^k(\Omega)$ then we define

$$\mathcal{D}^\alpha u(x) = \frac{\partial^{|\alpha|} u(x)}{\partial x_1^{\alpha_1} \cdots \partial x_n^{\alpha_n}}.$$

Finally, we define the Sobolev space $\mathrm{W}^{m,p}(\Omega)$ as

$$\mathrm{W}^{m,p}(\Omega) = \left\{ u \in \mathrm{L}^p(\Omega) |\ \forall |\alpha| \leq m, \exists g_\alpha \in \mathrm{L}^p(\Omega) : \ \int_\Omega u \mathcal{D}^\alpha \phi = (-1)^{|\alpha|} \int_\Omega g_\alpha \phi,\ \forall \phi \in \mathrm{C}_0(\Omega) \right\}. \tag{2.24}$$

If p=2, we write

$$\mathrm{H}^k(\Omega) = \mathrm{W}^{m,2}(\Omega)$$

where m is a non-negative integer. We set $\mathcal{D}^\alpha u = g_\alpha$. The space $\mathrm{W}^{m,p}(\Omega)$ is equipped with the norm

$$||u||_{\mathrm{W}^{m,p}(\Omega)} = \sum_{0 \leq |\alpha| \leq m} \|D^\alpha u\|_{L^p(\Omega)} \tag{2.25}$$

Theorem 2.19. *For $n \in \mathbb{N}$ and $p \in \mathbb{R}$ such that $1 \leq p \leq \infty$ the Sobolev space $\mathrm{W}^{m,p}(\Omega)$ is a Banach space. And the space $\mathrm{H}^m(\Omega)$ is a Hilbert space with the inner product*

$$\langle u, v \rangle_m = \sum_{0 \leq |\alpha| \leq m} \langle \mathcal{D}^\alpha u, \mathcal{D}^\alpha v \rangle_2.$$

Similarly to (2.25), we can define the norm of an element $u \in \mathrm{W}^{m,\infty}(\Omega)$ as

$$||u||_{\mathrm{W}^{m,\infty}(\Omega)} = \sum_{0\leq|\alpha|\leq k} \|\mathcal{D}^{\alpha}u\|_{L^{\infty}(\Omega)}.$$

The space of infinitely many differentiable functions with compact support $\mathrm{C}_0^{\infty}(\Omega)$ is not complete endowed with the norm above. So that, the completion of this space is denoted as $\mathrm{W}_0^{m,p}(\Omega)$, i.e.,

$$\exists (u_n)_{n\in\mathbb{N}} \subseteq \mathrm{C}_0^{\infty}(\Omega): \ u_n \to u \in \mathrm{W}_0^{m,p}(\Omega), \text{ as } n \to 0.$$

Again, for the case p=2 we write

$$\mathrm{H}_0^m(\Omega) = \mathrm{W}_0^{m,2}(\Omega).$$

While we are studying Sobolev spaces, we state the definition of the Sobolev critical exponent

Definition 2.20. Let $N \geq 2$ be the dimension of Ω and $1 \leq p \leq N$, we say that $p*$ is the *Sobolev critical exponent* of p and it is given by

$$p* = \frac{1}{p} - \frac{1}{N}.$$

The last definition allows us to state some inequalities and more embedding results on Sobolev spaces, for this part we will assume that $\Omega = \mathbb{R}^N$ and depending on the value of p we have the following:

Theorem 2.20. *(Sobolev, Gagliardo, Nirenberg) Let $1 \leq p < N$. Then there exists a constant C which depens on p and d such that*

$$\|u\|_{L^{p*}(\mathbb{R}^N)} \leq C\|\nabla u\|_{L^{p}(\mathbb{R}^N)} \tag{2.26}$$

and

$$\mathrm{W}^{1,p}(\mathbb{R}^N) \subseteq \mathrm{L}^{p*}(\mathbb{R}^N).$$

A proof of this inequality is given in [5, Th.9.9]. As a consequence of Theorem 2.20 we have the following Corolllary

Corollary 2.4. *Let* $1 \leq p < N$*. Then*

$$\mathrm{W}^{1,p}(\mathbb{R}^N) \subseteq \mathrm{L}^q(\mathbb{R}^N)$$

for every $q \in [p, p*]$ *with continuous injection.*

Proof. Let $q \in [p, p*]$, such that

$$\frac{1}{q} = \frac{\alpha}{p} + \frac{1-\alpha}{p*} \tag{2.27}$$

for some $\alpha \in [0,1]$. From (2.2) and Young's inequality we have that

$$\|u\|_{L^q(\mathbb{R}^N)} \leq \|u\|^{\alpha}_{L^p(\mathbb{R}^N)} \|u\|^{1-\alpha}_{L^{p*}(\mathbb{R}^N)} \leq \|u\|_{L^p(\mathbb{R}^N)} + \|u\|_{L^{p*}(\mathbb{R}^N)}.$$

Finally, using Theorem 2.20 we conclude that for every u in $\mathrm{W}_{1,p}(\mathbb{R}^N)$

$$\|u\|_{L^q(\mathbb{R}^N)} \leq C||u||_{\mathrm{W}^{1,p}(\mathbb{R}^N)}.$$

□

We have a similar result for the case $p = N$

Corollary 2.5. *Let* $p = N$ *and* $q \in [N, +\infty)$*. Then we have that*

$$\mathrm{W}^{1,p}(\mathbb{R}^N) \subset \mathrm{L}^q(\mathbb{R}^N).$$

The proof of this result can be found in [5, Cor. 9.11]. And finally we have the case where $p > d$, which is called the Morrey's theorem

Theorem 2.21. *(Morrey) Let* $p > N$*. Then*

$$\mathrm{W}^{1,p}(\mathbb{R}^N) \subseteq \mathrm{L}^{\infty}(\mathbb{R}^N)$$

with continuous injection.

A very detail proof of this fact can be found in [5, Th. 9.12]. To this point we have assumed that $\Omega = \mathbb{R}^N$; however, we can state some embeddings also for $\Omega \subseteq \mathbb{R}^N$. In order, to do this, let's consider the following extension operator P like in [5, Th. 9.7], this operator works from $\mathrm{W}^{1,p}(\Omega)$ to $\mathrm{W}^{1,p}(\mathbb{R}^N)$, where

$$\hat{u}(x) = Pu(x) = \begin{cases} u(x), & \text{if } x \in \Omega, \\ 0, & \text{if } x \in \mathbb{R}^N \backslash \Omega. \end{cases}$$

Now, let's assume that Ω is an open bounded set of class C^1. Then we have the following embeddings

Corollary 2.6. *Let $1 \leq p \leq \infty$ and $p*$ be the Sobolev critical exponent. Then*

- *if $p < N$, $\mathrm{W}^{1,p}(\Omega) \subset \mathrm{L}^{p*}(\Omega)$,*
- *if p=N, then for every $q \in [p, \infty)$ we have $\mathrm{W}^{1,p}(\Omega) \subset \mathrm{L}^{q}(\Omega)$,*
- *if $p > N$, $\mathrm{W}^{1,p}(\Omega) \subset \mathrm{L}^{\infty}(\Omega)$,*

with continuous injection.

For the case $p > N$ in the previous Corollary, we can see that

$$\forall u \in \mathrm{W}^{1,p}(\Omega) : |u(x) - u(y)| \leq C||u||_{\mathrm{W}^{1,p}(\Omega)}|x - y|^{\alpha}$$

for almost everywhere x,y$\in \Omega$, where $\alpha = 1 - (N/p)$ and C(Ω, p, N).

Remark 2.20. $\mathrm{W}^{1,p}(\Omega) \subseteq \mathrm{C}(\bar{\Omega})$.

The next result gives compact embeddings similarly to Corollary 2.6, the results are similar; however, we state only the remarkable changes

Theorem 2.22. *(Rellich-Kondrachov) Let Ω be bounded and of class C^1. Then*

- *if $p < N$, then for all $q \in [1, p*)$ we have that $\mathrm{W}^{1,p}(\Omega) \subset \mathrm{L}^{q}(\Omega)$,*
- *if $p > N$, $\mathrm{W}^{1,p}(\Omega) \subset \mathrm{C}(\bar{\Omega})$,*

with compact injection. Moreover, for all p and d, $\mathrm{W}^{1,p}(\Omega) \subset \mathrm{L}^{p}(\Omega)$ with compact injection.

This proof is given in [5, Th. 9.16].

Corollary 2.7. *(Poincaré's inequality) Let Ω be a bounded open set and $1 \leq p < \infty$. Then there is a constant C which depends on Ω and p such that*

$$\|u\|_{L^p(\Omega)} \leq C\|\nabla u\|_{L^p(\Omega)} \tag{2.28}$$

for each u in $\mathrm{W}_0^{1,p}(\Omega)$.

The proof of this fact appears on [5, Cor. 9.19]. A last remark about the previous inequality is that $\|\nabla u\|_{L^p(\Omega)}$ is a norm on $\mathrm{W}_0^{1,p}(\Omega)$ equivalent to the norm $||u||_{\mathrm{W}^{1,p}(\Omega)}$.

2.3.3 L^2-type Sobolev spaces in $\mathbb{R}^N$

In this subsection we introduce some basic concepts about the space $\mathrm{H}^s(\Omega)$, $\Omega \subset \mathbb{R}^N$ and $s \in \mathbb{R}$. We just have presented the multi-index notation in the previous section then we have

Definition 2.21. (Schwartz space) The Schwartz space $\mathscr{S}(\mathbb{R}^N)$ is the set of all $f \in \mathrm{C}^\infty(\mathbb{R}^N)$ such that

$$||f||_{\alpha\beta} = \sup_{x\in\mathbb{R}^N} |x^\alpha \partial^\beta f(x)| < \infty \tag{2.29}$$

for each $\alpha,\ \beta \in \mathbb{N}^d$. This space is also known as the space of rapidly decreasing C^∞ functions.

Remark 2.21. Since $\mathrm{C}_0^\infty(\mathbb{R}^N) \subset \mathscr{S}(\mathbb{R}^N)$, we have that $\mathscr{S}(\mathbb{R}^N)$ is dense in $\mathrm{L}^2(\mathbb{R}^N)$ for $1 \leq p < \infty$.

The dual space of the Schwartz space $\mathscr{S}(\mathbb{R}^N)$ is called the set of all temperate distributions and is denoted by $\mathscr{S}'(\mathbb{R}^N)$.

Definition 2.22. Let $s \in \mathbb{R}$. The (L^2-type) Sobolev space $\mathrm{H}^s(\mathbb{R}^N)$ is given by

$$\mathrm{H}^s(\mathbb{R}^N) = \left\{ f \in \mathscr{S}'(\mathbb{R}^N) :\ (1+|\xi|^2)^{s/2}\hat{f} \in \mathrm{L}^2(\mathbb{R}^N,\ d\xi) \right\},$$

where $\hat{f}$ is the usual Fourier transform of f.

By Definition 2.22 we have that $f \in \mathscr{S}'(\mathbb{R}^N)$ belongs to $\mathrm{H}^s(\mathbb{R}^N)$ if and only if $\hat{f} \in \mathrm{L}^2(\mathbb{R}^N)$ and

$$||f||_s^2 = \int_{\mathbb{R}^N} \left(1+|\xi|^2\right)^s |\hat{f}(\xi)|^2 d\xi < \infty. \tag{2.30}$$

Let $f \in \mathscr{S}'(\mathbb{R}^N)$. We define

$$(1-\Delta)^{s/2} f = \mathscr{F}^{-1}\left[(1+|\xi|^2)^{s/2}\hat{f}\right]$$

and then we can rewrite (2.30) as

$$||f||_s^2 = ||(1-\Delta)^{s/2} f||_0^2.$$

Note that

$$\mathscr{F}\left[\mathrm{H}^s(\mathbb{R}^N)\right] = \mathrm{L}^2\left(\mathbb{R}^N, (1+|\xi|^2)^s d\xi\right) = \mathrm{L}_s^2(\mathbb{R}^N),$$

and we will call this space as *weighted* L^2 *space*. As in any Sobolev space this have some important embeddings as mentioned in the next theorem:

Theorem 2.23. *Let $s \in \mathbb{R}$. Then $H^s(\mathbb{R}^N)$ is a Hilbert space with respect to the inner product*

$$(f,g)_s = \int_{\mathbb{R}^N} (1+|\xi|^2)^s \hat{f}(\xi)\overline{\hat{g}(\xi)}d\xi. \tag{2.31}$$

Moreover, the following properties hold.

1. *Let $s, r \in \mathbb{R}$ such that $s \geq r$ then $H^s(\mathbb{R}^N) \subseteq H^r(\mathbb{R}^N)$ with a continuous and dense injection.*

2. *The dual space $(H^s(\mathbb{R}^N))'$ is isometrically isomorphic to $H^{-s}(\mathbb{R}^N)$ for every $s \in \mathbb{R}$.*

3. *Let $m \in \mathbb{N}$, then $f \in H^m(\mathbb{R}^N)$ if and only if $\partial^\alpha f \in L^2(\mathbb{R}^N)$ for each α such that $|\alpha| \leq m$ and then (2.30) is equivalent to*

$$|||f|||_s = \left(\sum_{j=0}^{m} ||\partial^\alpha f||^2_{L^2(\mathbb{R}^N)} \right)^{1/2}.$$

4. *For all $s > \frac{d}{2}$, it holds that $H^s(\mathbb{R}^N) \subseteq C_\infty(\mathbb{R}^N)$, where $C_\infty(\mathbb{R}^N)$ is the set of all continuous functions that tend to zero at infinity.*

A proof of this result can be found in [12, Th. 7.75].

Remark 2.22. If $s \geq 0$ then $H^s(\mathbb{R}^N) \hookrightarrow L^2(\mathbb{R}^N)$, i.e., with a continuous and dense embedding.

2.4 Basics on Spectral Theory

This section introduces some basics about the spectrum of operators, in order to ease the study of their properties.

Let X be any non-trivial complex space and $T : \mathcal{D}(T) \subseteq X \to X$ a linear operator. Given $\lambda \in \mathbb{C}$, we associate the operator

$$T_\lambda = T - \lambda I$$

with T. In case that T_λ has an inverse, we write

$$T_\lambda^{-1} = (T - \lambda I)^{-1} = R_\lambda(T) \tag{2.32}$$

for the *resolvent* of T. The study of the resolvent operator of T is useful for a better understanding of the operator T itself.

Remark 2.23. If it exists $R_\lambda(T)$ then it is a linear operator.

Spectral theory is concerned with the properties of the resolvent operator of T such that existence, boundedness and density which depends on the value of λ.

Definition 2.23. Let $X \neq \{0\}$ be a complex normed space and $T : \mathcal{D}(T) \subseteq X \to X$. We say that $\lambda \in \mathbb{C}$ is a regular value of T iff $R_\lambda(T)$ exists, is bounded and is defined on a set which is dense in X.

The set of all regular values λ of T is called the resolvent set of T denoted by $\rho(T)$, and its complement $\sigma(T) = \mathbb{C} - \rho(T)$ is called the *spectrum* of T so that if $\lambda \in \sigma(T)$ then λ is called a spectral value of T. The spectrum $\sigma(T)$ is partitioned into three disjoint sets the point spectrum, the residual spectrum and the continuous spectrum. This is important since plenty of information about an operator can be studied from the analysis of its point spectrum.

Definition 2.24. Let X be a normed space and $T \in \mathscr{L}(X)$, we have that:

- The point spectrum or discrete spectrum $\sigma_p(T)$ is the set of all $\lambda \in \sigma(T)$ for which $R_\lambda(T)$ does not exist. A $\lambda \in \sigma_p(T)$ is called an eigenvalue of T.
- The continuous spectrum $\sigma_c(T)$ is the set of all $\lambda \in \sigma(T)$ for which $R_\lambda(T)$ exists and is defined on a dense set in X, but it is not bounded.
- The residual spectrum $\sigma_r(T)$ is the set of all $\lambda \in \sigma(T)$ for which $R_\lambda(T)$ exists and may be bounded or not, but the domain of $R_\lambda(T)$ is not dense in X.

Remark 2.24. $\sigma(T) = \sigma_p(T) \cup \sigma_c(T) \cup \sigma_r(T)$

From the open mapping theorem [13, Th. 4.12-2] we have that if X is a Banach space and $T : X \to X$ is a bounded linear operator, and if for some λ the resolvent $R_\lambda(T)$ exists and is defined on the whole space X, then for that λ the resolvent is bounded. Moreover, we have the following lemma

Lemma 2.5. *Domain of R_λ Let X be a complex Banach space, $T : X \to X$ a linear operator, and $\lambda \in \rho(T)$. If T is closed or bounded then the domain of $R_\lambda(T)$ is $\mathcal{D}(R_\lambda(T)) = X$ and bounded.*

A proof of this lemma can be found in [13, Lem. 7.2-3].

Remark 2.25. The resolvent set of a bounded linear operator on a complex Banach space X is open which implies that the spectrum is closed.

The following invertibility result gives us a way to prove some other properties of the resolvent and the spectrum

Theorem 2.24. *Let X be a Banach space and $T \in \mathscr{L}(X, X)$ such that $||T|| < 1$. Then*

$$(I - T)^{-1} = \sum_{n=0}^{\infty} T^n. \tag{2.33}$$

Proof. By 2.2.1 we have that the geometric series $\sum_{n=0}^{\infty} T^n$ converges for $||T|| < 1$, even more, it converges absolutely. Since $\mathscr{L}(X)$ is a Banach space (see 2.8) absolute converge implies convergence. That is the series is convergent in $\mathscr{L}(X)$. Let's denote the partial sum up to n of the series by S_n then we have

$$(I - T)(S_n) = (S_n)(I - T) = I - T^{n+1}.$$

Taking $n \to \infty$ shows us that $T^{n+1} \to 0$ since $||T|| < 1$. Then the previous part implies that

$$(I - T)S = S(I - T) = I,$$

i.e, $S = (I - T)^{-1}$ and we are done. □

As an application of this theorem we have that a useful representation of the resolvent. For a Banach space X and $T \in \mathscr{L}(X, X)$ and every $\lambda_0 \in \rho(T)$ we have that

$$R_\lambda = \sum_{n=0}^{\infty} (\lambda - \lambda_0)^n R_{\lambda_0}^{n+1}$$

this series is absolutely convergent for every λ such that

$$|\lambda - \lambda_0| < \tfrac{1}{||R_{\lambda_0}||}.$$

Furthermore, we have that the spectrum of T on a complex Banach space X is compact and $|\lambda| \leq ||T||$. This fact leads us to the following definition.

Definition 2.25. Let X be a complex Banach space. The spectral radius $r_\sigma(T)$ of and operator $T \in \mathrm{B}(X)$ is the smallest closed disk center at the origin ofthe complex λ-plane containing $\sigma(T)$ written

$$r_\sigma(T) = \sup_{\lambda \in \sigma(T)} |\lambda|.$$

In practice the spectral radius can be computed as

$$r_\sigma(T) = \lim_{n\to\infty} ||T^n||^{1/n}. \tag{2.34}$$

We end this section introducing the spectral mapping theorem. For this let's consider an eigenvalue λ of a square matrix A then $\mathrm{A}x = \lambda x$ for some $x \neq 0$.Repeated applications of A gives

$$\mathrm{A}^m x = \lambda^m x$$

that is λ^m is an eigenvalue of A^m if λ is an eigenvalue of A. This idea can be generalized to a polinomial where we have that

$$p(\lambda) = \alpha_n\lambda^n + \alpha_{n-1}\lambda^{n-1} + \cdots + \alpha_0$$

is an eigenvalue of the matrix

$$p(\mathrm{A}) = \alpha_n\mathrm{A}^n + \alpha_{n-1}\mathrm{A}^{n-1} + \cdots + \alpha_0\mathrm{I}.$$

As we can see this fact works for finite dimensional spaces, but it can be generalized for spaces of any dimension. Form [13] we have that

Theorem 2.25. *Let X be a complex Banach space, T a bounded linear operator in X and*

$$p(\lambda) = \alpha_n\lambda^n + \alpha_{n-1}\lambda^{n-1} + \cdots + \alpha_0, \quad (\alpha_n \neq 0).$$

Then

$$\sigma(p(T)) = p(\sigma(T)); \tag{2.35}$$

that is, the spectrum $\sigma(p(T))$ of the operator

$$p(\mathrm{T}) = \alpha_n\mathrm{T}^n + \alpha_{n-1}\mathrm{T}^{n-1} + \cdots + \alpha_0\mathrm{I}$$

consists precisely of all those values which the polynomial p assumes on the spectrum $\sigma(T)$ of T.

Remark 2.26. In the previous theorem we have that $p(\sigma(T))$ is the set of all complex numbers ρ such that $\rho = p(\lambda)$ for some $\lambda \in \sigma(T)$, i.e.

$$p(\sigma(T)) = \{\rho \in \mathbb{C}/\rho = p(\lambda),\ \lambda \in \sigma(T)\}.$$

The proof of Theorem 2.25 can be found in [13, Th. 7.4-2].

2.5 Adjoint, compact and trace-class operators

This section introduces some definitions and theorems about a useful kind of operators such that adjoint operators, compact operators which allows us to retrieve similar characteristics of operators on finite dimension and trace-class operators.

2.5.1 Adjoint operators

Using bounded linear operators we can define operators which are called adjoint or Hilbert-adjoint operators, these kind of operators was suggested by problems with matrices, linear differential or integral equations.

Definition 2.26. (Hilbert-adjoint operator T^*)

Let H_1, H_2 be Hilbert spaces and $T:\ H_1 \to H_2$ be a bounded linear operator. Then the Hilbert-adjoint operator T^* of T is the operator

$$T^*:\ H_2 \to H_1$$

such that for all $x \in H_1$ and $y \in H_2$,

$$\langle Tx, y\rangle = \langle x, T^* y\rangle. \tag{2.36}$$

The definition above makes sense thanks to the following theorem

Theorem 2.26. *(Existence) The Hilbert-adjoint operator T^* of T in (2.26) exists, is unique and is a bounded linear operator with norm given by*

$$||T^*|| = ||T||. \tag{2.37}$$

A proof of (2.26) is given in [13, th.3.9-2.]

We have to keep in mind the following lemma which will be a great tool to demostrate some important properties of Hilbert-adjoint operators

Lemma 2.6. *(Zero operator) Let X and Y be inner product spaces and $C : X \to Y$ a bounded linear operator. We say that $R = 0$ iff $(Rx, y) = 0$ for every $x \in X$ and $y \in Y$. Even more, if $X = Y = \mathbb{C}$ and $(Rx, x) = 0$ for all x in X then R=0.*

The following are some general properties of Hilbert-adjoint operators

Theorem 2.27. *Let H_1, H_2 be Hilbert spaces, $S : H_1 \to H_2$ and $T : H_1 \to H_2$ bounded linear operators and α any scalar. Then we have*

1. $(T^*y, x) = (y, Tx)$,

2. $(S + T)^* = S^* + T^*$,

3. $(\alpha T)^* = \overline{\alpha} T^*$,

4. $(T^*)^* = T$,

5. $||T^*T|| = ||TT^*|| = ||T||^2$,

6. $T^*T = 0 \iff T = 0$,

7. $(ST)^* = T^*S^*$.

Proof. We just proof items (4) and (5).

4) Let $x \in H_1$ and $y \in H_2$, generic. Using (1) and (2.26) we have

$$\begin{aligned} ((T^*)^*x, y) = (x, T^*y) = & \ (Tx, y) \\ ((T^*)^*x, y) - (Tx, y) = & \ 0 \\ ([(T^*)^* - T]x, y) = & \ 0 \end{aligned}$$

by the arbitrariness of x and y taking $R = (T^*)^* - T$ in Lemma 2.6 we conclude that

$$(T^*)^* = T.$$

5) Let $x \neq 0 \in H_1$. From Schwarz inequality we have that

$$\begin{aligned} ||Tx||^2 = (Tx, Tx) = (T^*Tx, x) \leq & \ ||T^*T|| ||x||^2 \\ \tfrac{||Tx||^2}{||x||^2} \leq & \ ||T^*T|| \end{aligned}$$

taking the supremum on the right hand side of the last inequality gives us $||T||^2 \leq ||T^*T||$. By (2.2.1) the last implies that

$$||T||^2 \leq ||T^*T|| \leq ||T^*|| ||T|| = ||T||^2,$$

i.e., $||T^*T|| = ||T||^2$. Using $T = T^*$ and repeating the proccess gives us the othe inequality and then we have

$$||T^*T|| = ||TT^*|| = ||T||^2.$$

□

Thanks to Hilbert-adjoint operators we can define some other operators which will has practical importance for us.

Definition 2.27. Let H be a Hilbert space and $T : H \to H$ a bounded linear operator then if $T^* = T$ we say that t is self-adjoint or Hermitian, if T is bijective and $T^* = T^{-1}$ we say that T is unitary and normal if $TT^* = T^*T$.

A useful criterion which gives sufficient conditions to determine if a bounded linear operator $T : H \to H$ on a Hilbert space H is self-adjoint is given considering the inner product as follows

Theorem 2.28. *If T is self-adjoint then (Tx, x) is real for each x in H. Moreover, if H is complex and (Tx, x) is real for every x in H then T is self-adjoint.*

The proof of this fact is easy and is provided in [13, Th. 3.10-3].

To end this section we prove that any sequence of bounded self-adjoint linear operators converge to a bounded self-adjoint linear operator, i.e.

Theorem 2.29. *Let H be a Hilbert space and $(T_n)_{n\in\mathbb{N}}$ a sequence of bounded self-adjoint linear operators on H. If $(T_n)_{n\in\mathbb{N}}$ converges to and operator T, then T is a bounded self-adjoint linear operator on H.*

Proof. Let $(T_n)_{n\in\mathbb{N}} \subseteq \mathscr{L}(H, H)$, generic. Assume that

$$T_n \to T, \quad n \to \infty, \tag{2.38}$$

we have to prove that $T^* = T$. Note that by (2.26) and (2) we have that

$$\begin{aligned} ||T - T^*|| \leq & \; ||T - T_n|| + ||T_n - T_n^*|| + ||T_n^* - T^*|| \\ = & \; ||T - T_n|| + ||(T_n - T)^*|| \\ = & \; 2||T_n - T|| \end{aligned}$$

taking $n \to \infty$ in the last inequality gives us that $||T - T^*|| = 0$ as in consequence $T^* = T$. By the arbitrainess of $(T_n)_{n\in\mathbb{N}}$ we are done.

□

2.5.2 Compact linear operators

Another important property for linear operators such as boundedness is compactness. Compact linear operators are essential in applications since they behave such as operators working on a finite dimensional space.

Definition 2.28. (Compact linear operator) Let X and Y be normed spaces. A linear operator $T: X \to Y$ is said to be compact if for every bounded set $M \subseteq X$, the image of M through T is relatively compact, i.e., the closure of T(M) is compact.

Remark 2.27. An operator $T \in \mathscr{L}(X, Y)$ whose range has finite dimension is known as an operator of finite rank.

Compact linear operators was also known by *completely continuous* linear operators this term was motivated for the following lemma which gives a characterization of the continuity on these operators.

Lemma 2.7. *Let X and Y be normed spaces. Then every compact linear operator T is bounded this implies that T is continuos. Furthermore, if dim(X)=∞, then the identity operator is not compact.*

A proof of this result is given in [13, Lem. 8.1-2].

Remark 2.28. The last part of the lemma gives an example of a continuous linear operator which is not compact, so that continuity is not a sufficient condition.

From [13, Sec. 2.5] we retrive the definition of a compact set that states that a set A is compact if every sequence of elements of A has convergent subsequence to an element on A from this definition, we get easily a criterion for operators such as:

Theorem 2.30. *(Compactness criterion) Let X and Y be normed spaces and $T : X \to Y$ a compact linear operator. Then T is compact iff for every $(x_n)_{n\in\mathbb{N}} \subseteq X$ we have that $(Tx_n)_{n\in\mathbb{N}}$ has a convergent subsequence.*

One interesting fact about compact linear operators is the fact about the convergence of a sequence of them which say that the limit is a compact linear operator if the sequence converges uniformly, i.e.,

Theorem 2.31. *Let X be a normed space and Y be a Banach space. If $(T_n)_{n\in\mathbb{N}}$, a sequence of compact linear operators from X to Y, is uniformly convergent to a limit say it T, then T is also a compact linear operator.*

The proof of this result can be found in [13, Th. 8.1-5] and [20, Th. VI.12].

A compact linear operator $T : X \to Y$ defined on a normed space X and Y a Banach space has an extension $\tilde{T}$ defined on a $\hat{X}$ we is the completion of X and this operator is also compact and linear.

The relevance of compact linear operators in the solvability of operator equations is of great interest, and it has an important connection with adjoint-operators for example we have that the adjoint of a compact linear operator T is itself adjoint. If X and Y are normed spaces and $T : X \to Y$ is a linear opeartor then its adjoint is defined on the dual of X and Y that is $T^* : Y' \to X'$. In addition to the spectral theory, compact linear operators make a simple generalization of this theory for finite dimensional spaces.

As a first property of the spectral study of compact linear operators, we have the following result about the eigenvalues

Theorem 2.32. *Let $T : X \to Y$ be a compact linear operator then $\sigma_p(T)$ is countable.*

In addition, if $T : X \to X$ is a compact linear operator on a normed space X then for every $\lambda \neq 0 \in \mathbb{C}$ the Null space $\mathrm{Ker}(T_\lambda)$ of $T_\lambda = (T - \lambda I)$ has finite dimension. This is the reason why the spectral theory of compact linear operators is considered similar to eigenvalue theory of finite matrices. This affirmation is characterized by the following theorem.

Theorem 2.33. *Let $T : X \to X$ be a compact linear operator defined on a Banach space X. Then every $\lambda \neq 0 \in \sigma(T)$ is an eigenvalue of T.*

Proof. If $\mathrm{Ker}(T_\lambda) \neq \{0\}$, then $T_\lambda x = 0$ for $x \neq 0$, i.e.,

$$(T - \lambda I)x = 0, \quad x \neq 0$$

which implies that $\lambda \in \sigma_p(T)$. Now, suppose that $\mathrm{Ker}(T_\lambda) = \{0\}$, where $\lambda \neq 0$ then $T_\lambda x = 0$ implies that $x = 0$. Since T_λ is injective T_λ^{-1} exists and $\{0\} = \mathrm{Ker}(I) = \mathrm{Ker}(T_\lambda^0) = \mathrm{Ker}(T_\lambda)$, we have that the minimum r of the range is $r = 0$, so that $X = T_\lambda^0(X) = T_\lambda(X)$. By the open mapping theorem and since X is complete we have that T_λ^{-1} is bounded; hence, $\lambda \in \rho(T)$. □

2.5.3 Trace-class and Hilbert-Schmidt operators

In the previous subsection we have introduced compact operators and stablish some of their properties. Here, we will deal with the trace which is a generalization of the usual sum of the diagonal elements of a matrix, this will be an important tool for stablish a new compactness criterion and a special kind of operators. Let's give some definitions

Definition 2.29. Let H be a Hilbert space. An operator $T \in \mathscr{L}(H)$ is called positive if $(Tx, x) \geq 0$ for all $x \in H$. We write $T \geq 0$ if T is positive and $T \leq R$ if $R - T \geq 0$.

Thanks to *square root lemma* [20, Th. VI.9] we have that

Definition 2.30. Let $T \in \mathscr{L}(H)$. Then $|T| = \sqrt{T^*T}$. T^* is the adjoint operator of T.

Now, let's first introduce the trace for positive operators

Definition 2.31. Let H be a separable Hilbert space and $(\varphi_n)_{n=1}^{\infty}$ a Hilbert basis of H. Then for any positive operator $A \in \mathscr{L}(H)$ we define the trace of A as

$$\mathrm{Tr}(A) = \sum_{n=1}^{\infty} (\varphi_n, A\varphi_n),$$

and it is independent of the Hilbert basis chosen.

Theorem 2.34. *The trace has the following properties:*

1. $\mathrm{Tr}(A + B) = \mathrm{Tr}(A) + \mathrm{Tr}(B)$,
2. $\mathrm{Tr}(\lambda A) = \lambda\ \mathrm{Tr}(A)$, *for all* $\lambda \geq 0$,
3. $\mathrm{Tr}(UAU^{-1}) = \mathrm{Tr}(A)$ *for any unitary operator* U,
4. *If* $0 \leq A \leq B$, *then* $\mathrm{Tr}(A) \leq \mathrm{Tr}(B)$.

Proof. Let H be a Hilbert separable space and $(\varphi_n)_{n=1}^{\infty}$ a total orthonormal set of H. Let's prove point (1). Let A and B be two positive bounded linear operators on H, generic. Then

$$\begin{aligned} \mathrm{tr}(A + B) &= \sum_{n=1}^{\infty} (\varphi_n, (A + B)\varphi_n) \\ &= \sum_{n=1}^{\infty} [(\varphi_n, A\varphi_n) + (\varphi_n, B\varphi_n)] \\ &= \sum_{n=1}^{\infty} (\varphi_n, A\varphi_n) + \sum_{n=1}^{\infty} (\varphi_n, B\varphi_n) \\ & \mathrm{tr}(A) + \mathrm{tr}(B). \end{aligned}$$

We conclude by the arbitrariness of A and B.

- Let's prove point (2). Let A be positive bounded linear operator on H and $\lambda \geq 0$, generic. Then

$$\begin{aligned} \mathrm{Tr}(\lambda A) &= \sum_{n=1}^{\infty} (\varphi_n, \lambda A\varphi_n) \\ &= \lambda \sum_{n=1}^{\infty} (\varphi_n, \lambda A\varphi_n) \\ &= \lambda\ \mathrm{Tr}(A). \end{aligned}$$

We conclude by the arbitrariness of A and λ.

- Let's prove point (3). Let A be positive bounded linear operator on H and U a unitary operator, generic. Then by definition of adjoint operator and noticing that $(U\varphi_n)$ is also a total orthonormal set of H we have that

$$\begin{aligned}
\mathrm{Tr}(\lambda UAU^{-1}) &= \sum_{n=1}^{\infty}(\varphi_n, UAU^{-1}\varphi_n) \\
&= \sum_{n=1}^{\infty}(U\varphi_n, UA\varphi_n) \\
&= \sum_{n=1}^{\infty}(U^*U\varphi_n, A\varphi_n) \\
&= \sum_{n=1}^{\infty}(\varphi_n, A\varphi_n) \\
&= \mathrm{Tr}(A)
\end{aligned}$$

We conclude by the arbitrariness of A and U.

- Let's prove point (4). Let A and B be two positive bounded linear operators such that $0 \leq A \leq B$. Then

$$\begin{aligned}
\mathrm{Tr}(A) &= \sum_{n=1}^{\infty}(\varphi_n, A\varphi_n) \\
&\leq \sum_{n=1}^{\infty}(\varphi_n, B\varphi_n) \\
&= \mathrm{Tr}(B).
\end{aligned}$$

□

Definition 2.32. We say that an operator $A \in \mathscr{L}(H)$ is *trace class* iff $\mathrm{Tr}(|A|) < \infty$. We denote by $\mathscr{I}_1$ the family of all trace class operators.

We define a norm on $\mathscr{I}_1$ by

$$||A||_{t1} = tr(|A|). \tag{2.39}$$

Theorem 2.35. *$\mathscr{I}_1$ endowed with the norm (2.39) is a Banach space and $||A|| \leq ||A||_{t1}$*

In connection with the canonical form of compact operators [20, Th. VI.17] we have the following characterization

Theorem 2.36. *Let $A \in \mathscr{I}_1$ then A is compact. Moreover, a compact operator A is in $\mathscr{I}_1$ iff $\sum_{n=1}^{\infty} \lambda_n < \infty$ where $(\lambda_n)_{n\in\mathbb{N}}$ are the singular values of A.*

Proof. We just prove the first part. Let $\{\varphi_n\}_{n=1}^{\infty}$ be a total orthonormal set, generic. Since A is a trace class, $|A|^2 \in \mathscr{I}_1$. Then

$$\operatorname{Tr}(|A|^2) = \sum_{n=1}^{\infty} ||A\varphi_n||^2 < \infty.$$

Suppose that $\eta \in [\varphi_1, \ldots, \varphi_N]^{\perp}$ for some fixed N and $||\eta|| = 1$, then

$$||A\eta||^2 \leq tr(|A|^2) - \sum_{n=1}^{N} ||A\varphi_n||^2,$$

hence $[\varphi_1, \ldots, \varphi_N]^{\perp} \ni ||A\eta||\eta \to 0$ as $N \to \infty$. This last is particularly true for the supremum, so that

$$\sum_{n=1}^{N} (\varphi_n, \cdot)A\varphi_n \tag{2.40}$$

converges in norm to A. Therefore A is compact. □

Note that

Theorem 2.37. *If $A \in \mathscr{I}_1$ and $(\varphi_n)_{n\in\mathbb{N}}$ is any Hilbert space, then $\sum_{n=1}^{\infty}(\varphi_n, A\varphi_n)$ converges absolutely and the limit is independent of the choice basis.*

The proof of this theorem can be found in [20, Th VI.24].

Now, we are ready to define the trace on $\mathscr{I}_1$

Definition 2.33. The map tr : $\mathscr{I}_1 \to \mathbb{C}$ given by $\operatorname{tr}(A) = \sum_{n=1}^{\infty}(\varphi_n, A\varphi_n)$ where $(\varphi_n)_{n\in\mathbb{N}}$ is any Hilbert basis is called the trace.

Given $A \in \mathscr{I}_1$ it holds that $\operatorname{Tr}(A^*) = \overline{\operatorname{Tr}(A)}$ and i $B \in \mathscr{L}(H)$ then

$$tr(AB) = tr(BA),$$

i.e., it is symmetric.

The proof of this properties are given in [20, TH. VI.25].

We end this subsection with other class of operators known as Hilbert-Schmidt operators.

Definition 2.34. let H be a separable Hilbert space. We say that $A \in \mathscr{L}(H)$ is a Hilbert-Schmidt operator iff $\operatorname{Tr}(A^*A) < \infty$. The family of all Hilbert-Schmidt operators is denoted by $\mathscr{I}_2$.

Remark 2.29. The space $\mathscr{I}_2$ is a Hilbert space endowed with the inner product of L^2.

Whenever $H = L^2(M, d\mu)$ the Hilbert-Schmidt operators have a concrete realization [20, Th. VI.23]

Theorem 2.38. *Let (M, μ) be a measure space and $H = L^2(M, d\mu)$. Then $A \in \mathrm{B}(H)$ is Hilbert-Schmidt iff there is a function $K(x, y) \in L^2(M \times M)$ with*

$$(Au)(x) = \int_M K(x, y)u(y)d\mu(y).$$

A proof of this result can be found in [20, Th. VI.23].

2.6 Unbounded linear operators

As we mentioned before not all linear operators are bounded, such operators are called unbounded operators and appear on problems of differential equations and Quatum mechanics. The theory of them is a little more complicated than that bounded operators so that along this section we will consider only Hilbert spaces.

Let's consider a linear operator $T : \mathcal{D}(T) \to H$ where $\mathcal{D}(T) \subseteq H$ and H is a complex Hilbert space. This operator may or not be bounded.

Theorem 2.39. *(Boundedness) Let $T \in \mathscr{L}(H)$ defined on all of the Hilbert space H which is self-adjoint then T is bounded.*

The proof of this theorem can be found in [13, Th.10.1-1].

The conclusion of this theorem implies that $\mathcal{D}(T) = H$ cannot hold for unbounded operators which are self-adjoint. Therefore, we have to find suitable domains and this will also lead us to extension problems. We shall denote

$$S \subseteq T$$

to mean that T is an extension of the operator S; then

$$\mathcal{D}(S) \subseteq \mathcal{D}(T) \quad \text{and} \quad S = T|_{\mathcal{D}(S)}. \tag{2.41}$$

If $\mathcal{D}(S)$ is a proper subset of $\mathcal{D}(T)$ then T is a proper extension of S, i.e., $\mathcal{D}(T) - \mathcal{D}(S) \neq \emptyset$. As with bounded operators, the Hilbert-adjoint operators plays an important role for the spectral theory. In this case to define them we need operators densely defined that is if $\mathcal{D}(T)$ is dense in H

Definition 2.35. Let $T : \mathcal{D}(T) \to H$ a densely defined linear operator in a complex Hilbert space H. We define the Hilbert-adjoint operator $T^* : \mathcal{D}(T^*) \to H$ of T as follows

$$\mathcal{D}(T^*) = \{y \in H |\ \exists y^* \in H : (Tx, y) = (x, y^*)\ \forall x \in \mathcal{D}(T)\}$$

the Hilbert-adjoint operator T^* is defined in terms of y^* by

$$y^* = T^* y.$$

To end this section we state some extension properties and the definition of self-adjoint operator

Proposition 2.2. *Let two linear operators T and S densely defined in H, we have that if $S \subseteq T$ then $T^* \subseteq S^*$; moreover, if $\mathcal{D}(T^*)$ is dense in H then $T \subseteq T^{**}$.*

The proof of this proposition is found in [13, Th. 10.2-1].

Theorem 2.40. *We say that a linear operator T densely defined in H is symmetric if*

$$\forall x, y \in \mathcal{D}(T) :\ \ (Tx, y) = (x, Ty).$$

this is true in particular if $T \subseteq T^*$.

Proof. Let T be a densely defined operator in H, by Def. 2.35 we have that

$$(Tx, y) = (x, T^* y),$$

for all $x \in \mathcal{D}(T)$ and $y \in \mathcal{D}(T^*)$. Assume that T^* is an extension of T then the last is true also for $y \in \mathcal{D}(T)$ that is $T^* y = Ty$ so that

$$(Tx, y) = (x, ty).$$

Therefore, T is symmetric. □

Remark 2.30. Conversely, if T^* extends T then it is symmetric.

Definition 2.36. A linear operator T densely defined in a complex Hilbert space H is called a self-adjoint linear operator if $T^* = T$.

From (2.40) we have that every self-adjoint linear operator is symmetric. Moreover, a densely defined linear operator T in a complex Hilbert space H is symmetric iff $(Tx, x) \in \mathbb{R}$ for all $x \in \mathcal{D}(T)$.

2.7 Spectral properties of self-adjoint operators

We have already talk about of bounded self-adjoint linear operators on Hilbert spaces in Section 2.5. In this section, we will study their spectral theory such as the spectral representation and the spectrum.

Let's recalll that a self adjoint operator T is such that

$$T = T^*.$$

If in addition, it is bounded and linear then its spectrum has several properties which are useful in practice. Let's remark that a bounded sel-fadjoint operator $T : H \to H$ on a Hilbert space H may not have eigenvalues that is $\sigma_p(T) = \emptyset$.

Theorem 2.41. *If the eigenvalues of T exists then we have that they are real and its corresponding eigenvectors for different eigenvalues are orthogonal.*

Proof. Let's first prove that given any eigenvalue of T it is real. Let $\lambda \in \sigma_p(T)$ and x its corresponding eigenvector, i.e., $x \neq 0$ and $Tx = \lambda x$. Then by Definition 2.26 and since T is self-adjoint we have that

$$\begin{aligned} \lambda(x, x) = (\lambda x, x) = & \ (Tx, x) \\ = & \ (x, Tx) = (x, \lambda x) = \bar{\lambda}(x, x). \end{aligned} \tag{2.42}$$

Since $x \neq 0$, the inner product of x in the last equality is not zero, hence $\lambda = \bar{\lambda}$ that is λ is real.

For the second part, we consider two different eigenvalues and again apply the definition of self-adjoint. Let λ and ν two different eigenvalues of T. By part a) we have that $\nu \in \mathbb{R}$, so

$$\begin{aligned} \lambda(x, y) = (\lambda x, y) = & \ (Tx, y) \\ = & \ (x, Ty) = (x, \nu x) = \nu(x, y). \end{aligned}$$

The last implies that $(\lambda - \nu)(x, y) = 0$ since the eigenvalues are different we have that $(x, y) = 0$, i.e.the eigenvectors associated to λ and ν respectively, are orthogonal. □

This result can be extend to the whole spectrum thanks to the following proposition

Proposition 2.3. *Let H be a Hilbert space and $T \in \mathscr{L}(H)$ self-adjoint. We say that $\lambda \in \mathbb{C}$ belongs to $\rho(T)$ iff there exists a constant $c > 0$ such that*

$$\forall x \in H: \ ||T_\lambda x|| \geq c||x||.$$

A proof of this proposition is given in [13, Th.9.1-2].

Theorem 2.42. *Let H be a complex Hilbert space and $T \in \mathscr{L}(H)$ a self-adjoint operator. Then $\sigma(T)$ is real.*

Proof. Let $x \neq 0 \in H$ and $\lambda \in \mathbb{C}$, generic. Then we have

$$(T_\lambda x, x) = ((T - \lambda I)x, x) = (Tx, x) - \lambda(x, x) \tag{2.43}$$

Since T is self adjoint (Tx, x) is real and so

$$\overline{(T_\lambda x, x)} = ((T - \lambda I)x, x) = (Tx, x) - \bar{\lambda}(x, x) \tag{2.44}$$

where $\bar{\lambda} = a - ib$ $(a, b \in \mathbb{R})$. Substracting (2.43) with (2.44) give us

$$\begin{aligned} -2i \cdot \mathrm{Im}(T_\lambda x, x) = & \ \overline{(T_\lambda x, x)} - (T_\lambda x, x) \\ = & \ (\lambda - \bar{\lambda})(x, x) \\ = & \ 2ib||x||^2 \end{aligned}$$

Taking absolute value both sides and diving by 2 we obtain

$$\begin{aligned} |b|||x||^2 = & \ |Im(T_\lambda x, x)| \\ \leq & \ |(T_\lambda x, x)| \\ \leq & \ ||T_\lambda x||||x||. \end{aligned}$$

Since $x \neq 0$ the last implies

$$|b|||x|| \leq ||T_\lambda x|| \tag{2.45}$$

then if $b \neq 0$ by the Proposition 2.3 we have that $\lambda \in \rho(T)$. Hence, if $b = 0$ then $\lambda \in \sigma(T)$ and $\lambda \in \mathbb{R}$. □

As we have mentioned in Section 2.4, the spectrum $\sigma(T)$ of a bounded linear operator T is compact; however, thanks to the last result it implies more things such as the spectrum of T lies in a closed interval $[m, M] \subseteq \mathbb{R}$ where m and M are given by

$$m = \inf_{||x||=1} (Tx, x), \quad M = \sup_{||x||=1} (Tx, x)$$

and are spectral values of T. Moreover, we have that

$$||T|| = \max(|m|, |M|) = \sup_{||x||=1} |(Tx, x)|.$$

Remark 2.31. Remember that $\sigma(T)$ is the union of the point, the continuous and the residual spectrum but this last is empty for bounded self-adjoint linear operators.

We say that a bounded self-adjoint linear operator T is positive if $T \geq 0$, i.e.

$$\forall x \in H : \ (Tx, x) \geq 0.$$

In particular, if T is self-adjoint then we have that T^2 is positive because $(T^2x, x) = (Tx, Tx) \geq 0$. Let T be a positive bounded self-adjoint operator on a complex Hilbert space H we call A the square root of T if $A^2 = T$, furthermore, if A is positive then we call it the positive square root of T, which exists and is unique and is denoted by $A = T^{1/2}$.

For our purpose of getting a spectral representation of bounded self-adjoint operators we need of *projection operators*. We already know that any Hilbert space H can be represented as a direct sum

$$H = Y \oplus Y^{\perp}$$
$$x = y + z$$

$y \in Y$ and $z \in Y^{\perp}$ where Y is closed and $Y^{\perp}$ is its orthogonal complement. By the uniqueness of y given $x \in H$ we define the projection of H onto Y as

$$P : H \to H$$
$$x \mapsto Px = y$$

In the other hand, we can rewrite the representation as $x = y + z = Px + (I - P)x$, showing that the projection of H onto $Y^{\perp}$ is (I-P). Alternatively, a bounded linear operator $P : H \to H$ on a Hilbert space is a projection iff P is self-adjoint and $P^2 = P$. The study of the spectrum gives us tools to get a representation of T using projections as they are simpler operators. This representation of T is easy to get for finite Hilbert spaces since it reduces to a sum over the projections associated to an eigenvalue. For the infinite case we have to consider spectral families which are families of one-parameter family of projections defined on a Hilbert space H.

Roughly speaking, we have that for H a complex Hilbert space and $\lambda \in \mathbb{R}$, we defined the operator $E_\lambda : H \to H$ by

$$E_\lambda = \sum_{\lambda_j \leq \lambda} P_{\lambda_j}$$

and is called a one-parameter family of projection. This operator is the projection of H onto the subspace V_λ sppaned by all eigenvectors for which $\lambda_j \leq \lambda$.

Definition 2.37. A spectral family is a one-parameter family $\mathcal{E} = (E_\lambda)_{\lambda \in \mathbb{R}}$ of projections E_λ defined on H such that for every $x \in H$ it follows

- for $\lambda < \mu$ we have $E_\lambda \leq E_\mu$
- $\lim_{\lambda \to -\infty} E_\lambda x = 0$
- $\lim_{\lambda \to +\infty} E_\lambda x = x$
- $E_{\lambda+0} x = \lim_{\mu \to \lambda+0} E_\mu x = E_\lambda x.$

this definition suggests a mapping $\mathbb{R} \ni \lambda \mapsto E_\lambda \in \mathscr{L}(H, H)$.

To get our spectral representation of a bounded self-adjoint linear operator $T : H \to H$ on a complex Hilbert space H we have to associate a suitable spectral family $\mathcal{E}$ to T. In order to do this, we have to define the positive and negative part of T, respectively

$$T^+ = \tfrac{1}{2}(B + T) \text{ and } T^- = \tfrac{1}{2}(B - T)$$

where $B = |T| = (T^2)^{1/2}$.

Remark 2.32. Note that $T = T^+ - T^-$ and $B = T^+ + T^-$.

Here, we state a few properties of these operators but more about them is find in [13].

Lemma 2.8. *The operators jus introduced follows the following properties*

- B, T^+, T^- *are bounded and self-adjoint,*
- $T^+ T^- = 0$ *and* $T^+, T^- \geq 0$.

These are true using $T_\lambda = T - \lambda I$ *instead of* T.

A very detail proof of this lemma can be find in [13, Lem. 9.8-1].

Theorem 2.43. *Let H be a complex Hilbert space and a self-adjoint linear operator $T \in \mathscr{L}(H,H)$. And let $\lambda \in \mathbb{R}$ such that E_λ is the projection of H onto the null space* $\operatorname{Ker}(T_{\lambda^+}) = Y_\lambda$ *of the positive part of T_λ. Then $\mathcal{E} = (E_\lambda)_{\lambda\in\mathbb{R}}$ is a spectral family on the interval $[m, M] \subset \mathbb{R}$, where m and M are defined such in 2.7.*

The proof of this result can be found in [13, Th. 9.8-3].

Thanks to this result we can associate a spectral family $\mathcal{E} = (E_\lambda)_{\lambda\in\mathbb{R}}$ to any self-adjoint linear operator $T \in \mathscr{L}(H,H)$, so that we can get a spectral representation of T with the Riemman-Stieltjes integral

Theorem 2.44. *(Spectral theorem V1) Let H be a complex Hilbert space and a self-adjoint linear operator $T \in \mathscr{L}(H,H)$. Then T has a spectral representation*

$$T = \int_{m-0}^{M} \lambda dE_\lambda \tag{2.46}$$

where $\mathcal{E} = (E_\lambda)$ is the spectral family associated with T. Even more,

$$(Tx, y) = \int_{m-0}^{M} \lambda dw(\lambda) \tag{2.47}$$

where $w(\lambda) = (E_\lambda x, y)$.

In particular this theorem can be used to makes sense evaluation of operators into polynomials as follows

Lemma 2.9. *Let p be a polynomial with real coefficients, i.e,*

$$p(\lambda) = \alpha_n\lambda^n + \alpha_{n-1}\lambda^{n-1} + \cdots + \alpha_0,$$

then the operator p(T) defined by

$$p(T) = \alpha_n T^n + \alpha_{n-1}T^{n-1} + \cdots + \alpha_0 I,$$

has a sprectral representation

$$p(T) = \int_{m-0}^{M} p(\lambda) dE_\lambda \tag{2.48}$$

and

$$\forall x, y \in H: \ (p(T)x, y) = \int_{m-0}^{M} p(\lambda) dw(\lambda). \tag{2.49}$$

A proof of these results can be found in [13, Th. 9.9-2].

Remark 2.33. We have used m-0 previously to denote

$$\int_a^M \lambda dE_\lambda = \int_{m-0}^M \lambda dE_\lambda = mE_m + \int_m^M \lambda dE_\lambda$$

for $m \neq 0,\ E_m \neq 0$ and $a < m$.

The last lemma is of great practical importance since allows us to makes sense $f(T)$ where f is a continuous function and $T \in \mathscr{L}(H, H)$ a self-adjoint operator defined on a Hilbert space H. Recall that thanks to Weierstrass theorem the space of polynomials is dense in the space of continuous functions.

To end this section we introduce another formulation of spectral theorem so as the one introduced before. It is just one of its several formulations and they are equivalent in some sense.

Theorem 2.45. *(Spectral theorem V2) Let H be a Hilbert space and $A : \mathcal{D}(A) \subseteq H \to H$ a bounded self-adjoint linear operator. Then there exists a measure space $(\mathcal{M}, \beta, \mu)$, a unitary operator*

$$U : H \to L^2(\mathcal{M}, d\mu)$$

and a function $a : \mathcal{M} \to \mathbb{R}$ finite a.e., such that

$$\Phi \in \mathcal{D}(A) \text{ iff } a \cdot U\Phi \in L^2(\mathcal{M}, d\mu), \tag{2.50}$$

and

$$A\Phi = U^{-1} M_a U \Phi, \tag{2.51}$$

where M_a is the operator of multiplication by a, i.e.

$$\mathcal{D}(M_a) = \{f \in L^2(\mathcal{M}, d\mu) :\ af \in L^2(\mathcal{M}, d\mu)\},$$

$$(M_a f)(m) = a(m) f(m),\ \mu - a.e.$$

This formulation is something more concerning to measure theory but gives us a powerful representation of a self-adjoint linear operator T defined on a Hilbert space since it transforms T into an algebraic multiplication which of course simplify things such as the previous formulation. Moreover, this provides a functional calculus such as the result of

Lemma 2.9 but a little bit more general. In fact, let $F : \mathbb{R} \to \mathbb{C}$ measurable then

$$F(A) : \mathcal{D}(F(A)) \subseteq H \to H$$

is defined by

$$\begin{aligned} \mathcal{D}(F(A)) = &\ \{\phi \in H |\ F(a(\cdot))(U\phi)(\cdot) \in \mathrm{L}^2(\mathcal{M}, d\mu)\}, \\ F(A)\phi = &\ U^{-1}F(a(\cdot))U\phi, \\ UF(A)\phi = &\ F(a(\cdot)) \cdot U\phi. \end{aligned}$$

where $a(\cdot)$ is a function which depends on A.

Remark 2.34. If $F \in \mathrm{L}^\infty(\mathbb{R})$ then $F(A) \in \mathrm{B}(H)$ and

$$||F(A)\phi||_H \leq \|F\|_{L^\infty(\mathbb{R})}||\phi||_H.$$

Chapter 3

A brief introduction to Quantum Mechanics

The results we present in this work are linked to Quantum Mechanics. Therefore, in this chapter we provide a very short introduction to it. We will not go deep, but just concern ourselves to some main topics in connection with operators. Our point of start will be a brief explanation of how was conceived Quantum Mechanics, then we shall see how operators appear in the formulation of Quantum Mechanics. Finally we state the Heinsenberg uncertainty principle. The main references in this chapter are [3], [13] and [14].

3.1 Where does it come from?

The experiments described by [3] and [14] try to explain the nature of Quatum Mechanics and the first attempts to use mathematical objects such as probabilities intrepet the strange exprimental results. In [13] it is mentioned that the concept of quantum given by Max Planck was revolutionary and rised a new research area known as Quantum Mechanics. It is believed that this event caused the division between classical and modern physics, which was the most important tool for that time to explain the majority of physical phenomena, in this period many discoveries were conceived such as X rays and radioactivity concepts that rose contradictions when classical principles were applied.

Since Quantum Mechanics works in regions of small dimensions, the constant h found by Planck in his research about the properties of thermal radiation in 1900 helped Schrödinger and many others to the development of the field and its better understanding.

3.2 Operators in Quantum Mechanics

Hilbert space theory is important to give sense to many quantum concepts, so in this section we shall see how Hilbert spaces, self-adjoint operators and Quantum Mechanics work together. First, we explain some basic definitions of Quantum Mechanics using a single particle system in one dimesion. Then, we present two self-adjoint linear operators which are important in the study of Quantum Mechanics: the position and momentum operators. Additionally, we give two fundamental postulates for Quatum mechanics. Finally, we state the Heisenberg uncertainty principle and a brief introduction to the Schrödinger equation. We refer the reader to [5] and [13] for more details.

3.2.1 Basic concepts. The position and momentum operator

We consider a single particle system in $\mathbb{R}$ and fixed at some instant in time. In Classical Mechanics the state of our system can be described using the position and the velocity of the particle but in Quantum Mechanics it is not possible to give such a description because of the uncertainty principle.

Thus for Quantum Mechanics we describe the state of the system by a function $\Psi : \mathbb{R} \to \mathbb{C}$ which is not-time dependent because of the definition of the system. Let's assume that $\psi \in \mathrm{L}^2(\mathbb{R})$ then for $A \subseteq \mathbb{R}$ we have that

$$\int_A |\psi(q)|^2 \; dq \tag{3.1}$$

defines the probability to find the particle in A. If we want to extend this idea to the whole real line we have to impose the following normalizing condition

$$||\psi||^2 = \int_{\mathbb{R}} |\psi(q)|^2 \; dq = 1.$$

This allows us to replace the deterministic description of a state given in the classical sense by a probabilistic one for Quantum Mechanics. Thus we define a state by

Definition 3.1. We say that ψ is a **state** of the physical system at some instant if and only if $\psi \in \mathrm{L}^2(\mathbb{R})$ and $||\psi|| = 1$. From this we can define an equivalence relation by

$$\psi_1 \sim \psi_2 \quad \Leftrightarrow \quad \psi_1 = \alpha\psi_2,\ |\alpha| = 1$$

It's important to note that ψ in the last definition generates a subspace Y of $L^2(\mathbb{R})$ where

$$Y = \{\varphi :\ \varphi = \beta\psi,\ \beta \in \mathbb{C}\},$$

hence we can define a probability distribution for a particle in our system as in (3.1) using $\varphi \in Y$ such that φ has unit norm.

It's clear by (3.1) that $|\psi(\cdot)|^2$ represents a probability density function on $\mathbb{R}$. Then we can define the mean value and the variance of the distribution by

$$\mu_\psi = \int_{\mathbb{R}} q|\psi(q)|^2\ dq \quad \text{and} \quad \mathrm{var}_\psi = \int_{\mathbb{R}} (q - \mu_\psi)^2 |\psi(q)|^2\ dq \tag{3.2}$$

respectively. We can also get the standard deviation by $\mathrm{sd}_\psi = \sqrt{\mathrm{var}_\psi}$. As usual, the mean represents the central location and the variance the spread of the distribution. Thus given a state ψ the mean μ_ψ characterizes the *average position* of a particle for the state ψ. It is worth to see that the mean value (3.2) can be written as

$$\mu_\psi(Q) = \langle Q\psi, \psi\rangle = \int_{-\infty}^{\infty} Q\psi(q)\overline{\psi(q)}dq$$

where $Q : \mathcal{D}(Q) \to \mathrm{L}^2(\mathbb{R})$ is given by

$$Q[\psi](q) = q\psi(q) \tag{3.3}$$

and this is known as the *position operator*.

Remark 3.1. Note that $\mathcal{D}(Q)$ is the set of all $\psi \in \mathrm{L}^2(\mathbb{R})$ such that $Q\psi \in \mathrm{L}^2(\mathbb{R})$.

Using the position operator, the variance (3.2) now becomes

$$\mathrm{var}_\psi(Q) = \langle (Q - \mu I)^2\psi, \psi\rangle = \int_{\mathbb{R}} (Q - \mu I)^2 \psi(q)\overline{\psi(q)}\ dq$$

The *position operator* is a linear unbounded self-adjoint operator whose domain is dense in $\mathrm{L}^2(\mathbb{R})$.

In general, a state ψ provides theoretical information of our system, in the other hand, the information which is observed experimentally from ψ is called an observable as examples we have position, momentum and energy. Such as above we can use suitable self-adjoint operators to solve problems of Quantum mechanics then we define

Definition 3.2. An **observable** is a self-adjoint linear operator $T : \mathcal{D}(T) \to \mathrm{L}^2(\mathbb{R})$, where $\mathcal{D}(T)$ is dense in the space $\mathrm{L}^2(\mathbb{R})$.

The mean and the variance of T is defined by

$$\mu_\psi(T) = (T\psi, \psi) = \int_{\mathbb{R}} T\psi(q)\overline{\psi(q)}dq \tag{3.4}$$

and

$$\mathrm{var}_\psi(T) = \Big((T - \mu I)^2\psi, \psi\Big) = \int_{\mathbb{R}} (T - \mu I)^2\psi(q)\overline{\psi(q)}dq. \tag{3.5}$$

To end this section we give a brief definition of another useful operator. The *momentum operator* $D : \mathcal{D}(D) \to \mathrm{L}^2(\mathbb{R})$ is given by

$$D[\psi] = \frac{h}{2\pi i}\frac{d\psi}{dq}$$

where h is Planck's constant.

Remark 3.2. Note that $\mathcal{D}(D) \subseteq \mathrm{L}^2(\mathbb{R})$ is formed by all absolutely continuous functions $\psi \in \mathrm{L}^2(\mathbb{R})$ defined on a compact interval on $\mathbb{R}$ such that $D\psi \in \mathrm{L}^2(\mathbb{R})$.

3.2.2 Heisenberg uncertainty principle

As it is shown by [5, Intro.] and [14, Ch. 1] in Quantum Mechanics there is no such a thing as the path of a particle, that is, we cannot determine the position and momentum of a particle simultaneously as in Classical Mechanics where we can give a complete description of a state given its coordinates and velocity at any instant and thus determing its behaviour in a subsequent instant. Even if we had the state of an electron completely described in Quantum Mechanics we could not know the behaviour at a subsequence instant because it is uncertain, however, it is possible to do in a range of possible values. This is what is called the uncertainty principle of Heisenberg.

Definition 3.3. (Conmutator) Let S and T be self-adjoint linear operators defined on the same complex Hilbert space. We define

$$C = ST - TS$$

which is called the **conmutator operator** of S and T with $\mathcal{D}(C) = \mathcal{D}(ST) \cap \mathcal{D}(TS)$.

By differentiation, we can easily find the conmutator of the position and momentum operator. In fact,

$$\begin{aligned} DQ[\psi](q) = D(q\psi(q)) = \; & Dq \cdot \psi(q) + q \cdot D[\psi](q) \\ = \; & \tfrac{h}{2\pi i}\psi(q) + q\tfrac{h}{2\pi i}\tfrac{d\psi(q)}{dq} \\ = \; & \tfrac{h}{2\pi i}\psi(q) + QD[\psi](q) \end{aligned}$$

hence we have that

$$(DQ - QD)[\psi](q) = \tfrac{h}{2\pi i}\psi(q)$$

and it follows that $C_{pm} = DQ - QD = \frac{h}{2\pi i}\bar{I}$ where $\bar{I}$ is the identity operator on $\mathcal{D}(C_{pm})$.

In our way to get the Heisenberg uncertainty principle, we first prove the following:

Theorem 3.1. *(Conmutator) Let S and T be self-adjoint linear operators with domain and range in* $\mathrm{L}^2(\mathbb{R})$. *Then the conmutator operator of S and T satisfies*

$$|\mu_\psi(C)| \leq 2sd_\psi(S)sd_\psi(T) \tag{3.6}$$

for every $\psi \in \mathcal{D}(C)$.

Proof. Let $\psi \in \mathcal{D}(C)$, generic. Let's denote $\mu_1 = \mu_\psi(S)$ and $\mu_2 = \mu_\psi(T)$ and consider

$$A = S - \mu_1 I, \quad B = T - \mu_2 I$$

it is easy to see with a simple calculation that

$$C = AB - BA = ST - TS$$

By (3.4) we have that $\mu_1,\ \mu_2$ are real and since S and T are self-adjoint we have that A and B are self-adjoint as well. Then by definition

$$
\begin{aligned}
\mu_\psi(C) = & \ \langle (AB - BA)\psi, \psi \rangle \\
= & \ \langle AB\psi, \psi \rangle - \langle BA\psi, \psi \rangle \\
= & \ \int_{\mathbb{R}} AB\psi(q)\overline{\psi(q)}dq - \int_{\mathbb{R}} BA\psi(q)\overline{\psi(q)}dq \\
= & \ \int_{\mathbb{R}} B\psi(q)\overline{A\psi(q)}dq - \int_{\mathbb{R}} A\psi(q)\overline{B\psi(q)}dq \\
= & \ \langle B\psi, A\psi \rangle - \langle A\psi, B\psi \rangle .
\end{aligned}
$$

Since A and B are self-adjoint, they are symmetric and $|(B\psi, A\psi)| = |(A\psi, B\psi)|$. Then by the triangle and Cauchy-Schwartz inequality we have that

$$
|\mu_\psi(C)| \leq |\langle B\psi, A\psi \rangle| + |\langle A\psi, B\psi \rangle| = 2\,|\langle B\psi, A\psi \rangle| \leq 2||B\psi||||A\psi||
$$

which by (3.5) gives us

$$
||B\psi|| = \left\langle (T - \mu_2 I)^2 \psi, \psi \right\rangle^{1/2} = \sqrt{\mathrm{var}_\psi(T)} = \mathrm{sd}_\psi(T)
$$

and the same for $||A\psi||$. Since ψ was taken arbitrarily, we have proved (3.6). □

Note that $|\mu_\psi(C_{pm})| = \dfrac{h}{2\pi}$ since $||\psi||_{\mathrm{L}^2(\mathbb{R})} = 1$, so that the last theorem implies that

Corollary 3.1. *(Heisenberg uncertainty principle) For the position operator Q and the momentum operator D,*

$$
sd_\psi(D) sd_\psi(Q) \geq \frac{h}{4\pi} \tag{3.7}
$$

This last implies that we cannot measure position and momentum simultaneously, not only by the imperfection in the precision of the measurement methods but this precision is in principle limited.

3.3 The Schrödinger equation

In this section we give a brief description of the time-independent Schrödinger equation. In our way to deduce the equation we use the famous wave equation, which is used in some optical phenomenons, given by

$$
\Psi_{tt} = \gamma^2 \Delta \Psi \tag{3.8}
$$

where $\Psi_{tt} = \partial^2\Psi/\partial t^2$, $\gamma \in \mathbb{R}$ and $\Delta\Psi$ is the Laplacian operator applied to Ψ, i.e.,

$$
\Delta\Psi = \frac{\partial^2\Psi}{\partial x_1^2} + \frac{\partial^2\Psi}{\partial x_2^2} + \cdots + \frac{\partial^2\Psi}{\partial x_N^2} \tag{3.9}
$$

for every $x \in \mathbb{R}^N$.

Let's assume a simple and periodic time dependance of the form

$$\Psi(x_1, x_2, \cdots, x_N; t) = \psi(x_1, x_2, \cdots, x_N)e^{-i\omega t}.$$

Replacing it into (3.8), we get

$$\begin{aligned} \Psi_{tt} - \gamma^2 \Delta\Psi &= 0 \\ -\omega^2 e^{-i\omega t}\psi - \gamma^2 e^{-i\omega t}\Delta\psi &= 0 \\ -\omega^2\psi - \gamma^2\Delta\psi &= 0 \\ \Delta\psi + \left(\tfrac{\omega}{\gamma}\right)^2\psi &= 0 \\ \Delta\psi + k^2\psi &= 0. \end{aligned} \tag{3.10}$$

This last is known as the *Helmhotz equation*, where

$$k = \frac{\omega}{\gamma} = \frac{2\pi\nu}{\gamma} = \frac{2\pi}{\Lambda}$$

and ν is the frecuency.

Considering $\Lambda = \frac{h}{m\nu}$ the de Broglie wave length of matter waves (3.10) becomes

$$\Delta\psi + \tfrac{4\pi^2 m^2 \nu^2}{h^2}\psi = \Delta\psi + \tfrac{8\pi^2 m}{h^2} \cdot \tfrac{m\nu^2}{2}\psi = 0.$$

Since the total energy E of the system is the sum of $\frac{m\nu^2}{2}$ the kinetic energy and the potential energy V, we have that

$$\frac{m\nu^2}{2} = E - V$$

and then

$$\Delta\psi + \tfrac{8\pi^2 m}{h^2}(E - V)\psi = 0.$$

This last is called the time-independent *Schrödinger equation* which can be rewritten as

$$\left(-\frac{h^2}{8\pi^2 m}\Delta + V\right)\psi = E\psi, \tag{3.11}$$

so that the possible energy levels of the system are associated to the spectrum of the left-hand side operator of (3.11). This is why that equation is fundamental in quantum mechanics.

Chapter 4

Results

In this chapter we answer a question proposed in [8], [17] and [18] about defining and working with partially the kind of cones presented in those works, but using a Hilbert separable space other than of $L^2(\mathbb{R}^N)$, as our pivote space. In particular, we will use $H=H^1(\mathbb{R}^N)$ with $N > 4$.

4.1 Preliminaries

In this section, we set definitions. We denote by $\mathscr{L} = \mathscr{L}(H)$ the set of linear bounded operators acting on $H = H^1(\mathbb{R}^N)$. By $\mathscr{I}_\infty$ and $\mathscr{S}$ we denote, respectively, the spaces of compact operators and bounded self-adjoint operators on H. We also write $\mathscr{S}_\infty = \mathscr{I}_\infty \cap \mathscr{S}$. Thanks to the Riesz-Schauder and Hilbert-Schmidt theorems (see e.g. [20]), for a given $T \in \mathscr{S}_\infty$ there exists $(\nu_{i,T})_{i\in\mathbb{N}} \subseteq \mathbb{R}$, a sequence of eigenvalues of T, and $B = \{\psi_{i,T}/i \in \mathbb{N}\} \subseteq H\backslash\{0\}$, a Hilbert basis of H, such that for each $i \in \mathbb{N}$,

$$T\psi_{i,T} = \nu_{i,T}\psi_{i,T}, \tag{4.1}$$

and that is why we say that B is an *eigenbasis* of T.

Remark 4.1. Given $k \in \mathbb{N}$, let's define the k-trace of an operator $T \in \mathscr{L}$ as

$$\mathrm{Tr}_k(T) = \sum_{i=1}^{\infty}(\psi_{i,T}, T\psi_{i,T})_{H^k(\mathbb{R}^N)} \tag{4.2}$$

for each $i \in \mathbb{N}$, $\psi_{i,T} \in \mathrm{H}^k(\mathbb{R}^N)$. We denote

$$\mathrm{Tr}(T) = \mathrm{Tr}_0(T) = \sum_{i=1}^{\infty} (\psi_{i,T}, T\psi_{i,T})_{\mathrm{L}^2(\mathbb{R}^N)}$$

.

From Definition 2.32 and Remark 4.2 we have that the trace of an operator T is given by

$$\mathrm{Tr}_1(T) = \sum_{i=1}^{\infty} (\psi_{i,T}, T\psi_{i,T})_{\mathrm{H}^1(\mathbb{R}^N)}. \tag{4.3}$$

and it is basis-independent thanks to Theorem 2.37.

Moreover, we shall assume that the sequence $(\nu_{i,T})_{i\in\mathbb{N}} \subseteq \mathbb{R}$ is ordered, that is

$$|\nu_{i,T}| \geq |\nu_{j,T}|, \text{ for all } i, j \in \mathbb{N},\ i \leq j;$$

and if both $\nu_{i,T}$ and $-\nu_{i,T}$ are eigenvalues, $-|\nu_{i,T}|$ comes first.

We define the space of *nuclear operators* as $\mathscr{S}_1 = \mathscr{I}_1 \cap \mathscr{L}_S \subseteq \mathscr{S}_\infty$ which is a Banach space [20, Th. VI.20] whenever it is endowed with the trace norm

$$||T||_1 \equiv \mathrm{Tr}_1(|T|) = \sum_{i=1}^{\infty} |\nu_{i,T}| < \infty. \tag{4.4}$$

4.2 Sobolev-like cones

Now, we shall introduce our cone of operators. Let's consider a potential $V : \mathbb{R}^N \to \mathbb{R}$ verifying the following conditions:

(V1) $V \in \mathrm{C}(\mathbb{R}^N)$;

(V2) $\lim_{|x|\to\infty} V(x) = \infty$;

(V3) $V_0 = \inf_{x\in\mathbb{R}^N} V(x) > 0$.

Remark 4.2. From now on, we shall assume that V verifies (V1)-(V3).

Proposition 4.1. *Let's consider the functional* $(\cdot,\cdot)_{V,2} : \mathrm{C}_0^\infty(\mathbb{R}^N) \times \mathrm{C}_0^\infty(\mathbb{R}^N) \to \mathbb{R}$ *given by*

$$(u, w)_{V,2} = \int_{\mathbb{R}^N} [\triangle u(x) \triangle w(x) + V(x)u(x)w(x)]\, dx. \tag{4.5}$$

Then $(\cdot,\cdot)_{V,2}$ *is an inner product.*

Proof. We have to show that $(\cdot,\cdot)_{V,2}$ verifies the points of (2.2). Points (1)-(3) are easy by the linearity of $\triangle$. So, let's prove that

$$\forall u \in \mathrm{C}_0^\infty(\mathbb{R}^N): \; (u,u)_{V,2} \geq 0$$

and that

$$(u,u)_{V,2} = 0 \Leftrightarrow u = 0.$$

Let $u \in \mathrm{C}_0^\infty(\mathbb{R}^N)$, generic. By definition we have that

$$(u,u)_{V,2} = \int_{\mathbb{R}^N} |\triangle u(x)|^2 + V(x)|u(x)|^2 \, dx \geq 0.$$

and

- If $u = 0$, it immediately follows that $(u,u)_{V,2} = 0$.
- If $(u,u)_{V,2} = 0$, we have that

$$(u,u)_{V,2} = \int_{\mathbb{R}^N} |\triangle u(x)|^2 \, dx + \int_{\mathbb{R}^N} V(x)|u(x)|^2 \, dx = 0, \tag{4.6}$$

whence

$$||u||_{\mathrm{L}^2(\mathbb{R}^N;dV)} = 0. \tag{4.7}$$

By (2.20) the last implies that $u = 0$. Since u was chosen arbitrarily, we are done.

□

Thanks to Proposition 4.1 we can define a norm on $\mathrm{C}_0^\infty(\mathbb{R}^N)$ by

$$||u||_{V,2} = \left(\int_{\mathbb{R}^N} |\triangle u(x)|^2 + V(x)|u(x)|^2 \, dx \right)^{1/2}. \tag{4.8}$$

The completion of $\mathrm{C}_0^\infty(\mathbb{R}^N)$ in the norm $||u||_{V,2}$ is the Hilbert space

$$\mathrm{H}_V^2 = \left\{ u \in \mathrm{H}^2(\mathbb{R}^N) : \; \int_{\mathbb{R}^N} V(x)|u(x)|^2 \, dx < \infty \right\}. \tag{4.9}$$

Remark 4.3. The embedding

$$\mathrm{H}_V^2 \subseteq \mathrm{L}^q(\mathbb{R}^N) \tag{4.10}$$

is continuous and compact (see e.g. [4]) for all $q \in [2, 2^{**}[$ where

$$2^{**} = \frac{2N}{N-4}.$$

The embedding $\mathrm{H}^2_V \subseteq \mathrm{L}^{2^{**}}(\mathbb{R}^N)$ is continuous.

Now, we are ready to introduce our cone of operators

Definition 4.1. An operator $T \in \mathscr{I}_1$ is in the Sobolev-like cone $\mathscr{H}^2_V$ iff the sequence of eigenvectors of T, $(\psi_{i,T})_{i\in\mathbb{N}}$, belongs to $\mathrm{H}^2_V(\mathbb{R}^N)$ and

$$\langle\langle T\rangle\rangle_{V,2} \equiv \sum_{i=1}^{\infty} |\nu_{i,T}| \cdot ||\psi_{i,T}||^2_{V,2} < \infty. \tag{4.11}$$

We call $\langle\langle T\rangle\rangle_{V,2}$ the total energy of the operator T.

We give and prove some properties of this cone.

Proposition 4.2. *Let $T \in \mathscr{H}^2_V$ and $\alpha \in \mathbb{R}$. Then $\alpha T \in \mathscr{H}^2_V$,*

$$\langle\langle \alpha T\rangle\rangle_{V,2} = |\alpha|\langle\langle T\rangle\rangle_{V,2}, \tag{4.12}$$

and

$$\langle\langle \alpha T\rangle\rangle_{V,2} = 0 \Leftrightarrow (\alpha = 0 \vee T = 0). \tag{4.13}$$

Proof. Note that for any $i \in \mathbb{N}$ we have that

$$\alpha T\psi_{i,T} = \alpha\nu_{i,T}\psi_{i,T}$$

whence

$$\forall i \in \mathbb{N}: \ \nu_{i,\alpha T} = \alpha\nu_{i,T}, \ \psi_{i,\alpha T} = \psi_{i,T},$$

so that $\alpha T \in \mathscr{H}^2_V$. Moreover, we have that

$$\begin{aligned} \langle\langle \alpha T\rangle\rangle_{V,2} &= \textstyle\sum_{i=1}^{\infty} |\alpha\nu_{i,T}|||\psi_{i,T}||^2_{V,2} \\ &= |\alpha| \textstyle\sum_{i=1}^{\infty} |\nu_{i,T}|||\psi_{i,T}||^2_{V,2} \\ &= |\alpha|\langle\langle T\rangle\rangle_{V,2}. \end{aligned} \tag{4.14}$$

From (4.12) we see that

$$(\alpha = 0 \vee T = 0) \Rightarrow \langle\langle \alpha T\rangle\rangle_{V,2} = 0$$

and

$$\langle\langle \alpha T\rangle\rangle_{V,2} = 0 \Rightarrow (\alpha = 0 \vee \langle\langle T\rangle\rangle_{V,2} = 0).$$

We conclude by proving that

$$\langle\langle T\rangle\rangle_{V,2} = 0 \Rightarrow T = 0,$$

assuming $\langle\langle T\rangle\rangle_{V,2} = 0$ yields

$$\sum_{i=1}^{\infty} |\nu_{i,T}| ||\psi_{i,T}||_{V,2}^2 = 0$$

since this holds for any $i \in \mathbb{N}$, the above implies

$$|\nu_{i,T}| ||\psi_{i,T}||_{V,2}^2 = 0.$$

The $\psi_{i,T}$'s are eigenvalues, so that $\psi_{i,T} \neq 0$ and we get

$$\forall i \in \mathbb{N} : \ |\nu_{i,T}| = 0$$

whence

$$||T||_1 = \sum_{i=1}^{\infty} |\nu_{i,T}| = 0 \Rightarrow T = 0.$$

□

Proposition 4.3. *Let $M > 0$ and $\beta \in \mathrm{C}([-M, M])$ such that $\beta(0) = 0$. Let $T \in \mathscr{H}_V^2$ such that its spectrum is contained in $[-M, M]$. Assume that there exist $d_1 > 0$, $\alpha \geq 1$ and $t_0 \in]0, M]$ such that*

$$\forall t \in [-t_0, t_0] : \ |\beta(t)| \leq d_1 |t|^{\alpha}. \tag{4.15}$$

Then there exists $D^* > 0$ such that

$$\langle\langle \beta(T)\rangle\rangle_{V,2} \leq D^* \langle\langle T\rangle\rangle_{V,2},$$

whence $\beta(T) \in \mathscr{H}_V^2$.

Proof. Let $t_1 = \min\{1, t_0\}$. Since $\sigma(T) \in [-M, M]$, we have that $\#\{i \in \mathbb{N} : \ |\nu_{i,T}| > t_1\} < \infty$ and we can choose $\hat{d} > 0$ such that

$$|\beta(\nu_{j,T})| \leq \hat{d} |\nu_{j,T}|,$$

for each $j \in \{i \in \mathbb{N} : \ |\nu_{i,T}| > t_1\}$. By the Spectral Theorem, [20, Th. VII.2], and (4.15), we get

$$\begin{aligned} \langle\langle \beta(T)\rangle\rangle_{V,2} &= \sum_{i \in \mathbb{N}} |\beta(\nu_{i,T})| ||\psi_{i,T}||_{V,2}^2 \\ &= \sum_{|\nu_{i,T}| \leq t_1} |\beta(\nu_{i,T})| ||\psi_{i,T}||_{V,2}^2 + \sum_{|\nu_{i,T}| > t_1} |\beta(\nu_{i,T})| ||\psi_{i,T}||_{V,2}^2 \\ &\leq d_1 \sum_{|\nu_{i,T}| \leq t_1} |\nu_{i,T}|^{\alpha} ||\psi_{i,T}||_{V,2}^2 + \hat{d} \sum_{|\nu_{i,T}| > t_1} |\nu_{i,T}| ||\psi_{i,T}||_{V,2}^2 \end{aligned} \tag{4.16}$$

Since $\alpha \geq 1$, by (4.16) we have that

$$\begin{aligned}\langle\langle\beta(T)\rangle\rangle_{V,2} &\leq D^* \sum_{i\in\mathbb{N}} |\nu_{i,T}| ||\psi_{i,T}||^2_{V,2} \\ &= D^* \langle\langle T\rangle\rangle_{V,2}\end{aligned}$$

where $D^* = \max\{d_1, \hat{d}\}$. □

Remark 4.4. From Sobolev embeddings we have a useful estimative of the trace norm of an operator $T \in \mathscr{H}^2_V$ by its energy; this is a Poincaré-type inequality but at operators level: there exists a constant $C > 0$ such that

$$||T||_1 \leq C\langle\langle T\rangle\rangle_{V,2}, \text{ for all } T \in \mathscr{H}^2_V. \tag{4.17}$$

Proof. Let $T \in \mathscr{H}^2_V$, generic. By the Sobolev embbedding $\mathrm{H}^2(\mathbb{R}^N) \subseteq \mathrm{H}^1(\mathbb{R}^N)$, there exists $C > 0$ such that

$$|\nu_{i,T}| ||\psi_{i,T}||^2_{H^1(\mathbb{R}^N)} \leq C|\nu_{i,T}| ||\psi_{i,T}||^2_{V,2}, \text{ for all } i \in \mathbb{N},$$

whence

$$||T||_1 = \sum_{i=1}^{\infty} |\nu_{i,T}| \leq C\langle\langle T\rangle\rangle_{V,2}$$

,

since $(\psi_{i,T})_{i\in\mathbb{N}}$ is a Hilbert basis of $\mathrm{H}^1(\mathbb{R}^N)$. Since T was chosen arbitrarily, we have proved (4.17). □

Let's now introduce the concepts of kinetic and potential energy for operators that belong to the positive cone

$$\mathscr{H}^2_{V,+} = \{T \in \mathscr{H}^2_V : \ T \geq 0\}.$$

Note that if $T \in \mathscr{H}^2_{V,+}$, then every eigenvalue $\nu_{i,T}$ is non-negative and

$$||T||_1 = \mathrm{Tr}_1(T) = \sum_{i=1}^{\infty} \nu_{i,T}.$$

Definition 4.2. Let $T \in \mathscr{H}^2_{V,+}$. The kinetic energy and the potential energy of T are given by

$$\mathscr{K}(T) = \sum_{i=1}^{\infty} \nu_{i,T} \int_{\mathbb{R}^N} |\triangle\psi_{i,T}(x)|^2 \, dx \tag{4.18}$$

and

$$\mathscr{P}_V(T) = \sum_{i=1}^{\infty} \nu_{i,T} \int_{\mathbb{R}^N} V(x)|\psi_{i,T}(x)|^2 \, dx, \tag{4.19}$$

respectively.

It is not difficult to see that

$$\langle\langle T\rangle\rangle_{V,2} = \mathscr{K}(T) + \mathscr{P}_V(T), \quad \text{for every } T \in \mathscr{H}^2_{V,+}.$$

Moreover, by (4.2) and integration by parts, we formally have that

$$\begin{aligned}
\operatorname{Tr}(\triangle^2 T) &= \sum_{i=1}^{\infty} \left(\psi_{i,T}, \triangle^2 T\psi_{i,T}\right)_{\mathrm{L}^2(\mathbb{R}^N)} \\
&= \sum_{i=1}^{\infty} \nu_{i,T} \int_{\mathbb{R}^N} \psi_{i,T}(x)\triangle^2\psi_{i,T}(x)dx \\
&= \sum_{i=1}^{\infty} \nu_{i,T} \int_{\mathbb{R}^N} \nabla\psi_{i,T}(x)\nabla(-\triangle\psi_{i,T}(x))dx \\
&= \sum_{i=1}^{\infty} \nu_{i,T} \left(-\int_{\mathbb{R}^N} \triangle\psi_{i,t}(x)(-\triangle\psi_{i,T}(x))dx\right) \\
&= \sum_{i=1}^{\infty} \nu_{i,T} \int_{\mathbb{R}^N} |\triangle\psi_{i,T}(x)|^2 dx \\
&= \mathscr{K}(T)
\end{aligned}$$

Remark 4.5. Note that $\triangle^2 = (-\triangle) \circ (-\triangle)$ is self-adjoint, so that the previous equality makes sense. Therefore, the total energy is formally given by

$$\langle\langle T\rangle\rangle_{V,2} = \operatorname{Tr}\left(\left(\triangle^2 + V\right) T\right). \tag{4.20}$$

It is important to see that given $T \in \mathscr{S}_1$ such that $T \geq 0$ we can associate a function $\rho_T : \mathbb{R}^N \to \mathbb{R}$ given by

$$\rho_T(x) = \sum_{i=1}^{\infty} \nu_{i,T}|\psi_{i,T}(x)|^2, \tag{4.21}$$

and is called the *density function* associated to T. Observe that ρ_T does not depend of the

Hilbert basis chosen (see Definition 2.31). It is easy to see that $\rho_T \in \mathrm{L}^1(\mathbb{R}^N)$:

$$\begin{aligned} ||\rho_T||_{\mathrm{L}^1(\mathbb{R}^N)} &= \int_{\mathbb{R}^N} \rho_T(x)\ dx \\ &= \sum_{i=1}^{\infty} \nu_{i,T} \int_{\mathbb{R}^N} |\psi_{i,T}(x)|^2\ dx \\ &\leq \sum_{i=1}^{\infty} \nu_{i,T} (\psi_{i,T}, \psi_{i,T})_{\mathrm{H}^1(\mathbb{R}^N)} \\ &= ||T||_1. \end{aligned}$$

Now, let's state some regularity properties of ρ_T, for $T \in \mathscr{H}^2_{V,+}$; in particular that $\rho_T \in \mathrm{W}^{2,r}(\mathbb{R}^N) \cap \mathrm{L}^s(\mathbb{R}^N)$. First we prove the following

Theorem 4.1. *Let $N > 4$ and $T \in \mathscr{H}^2_{V,+}$. Then there exists $Z > 0$ such that*

$$\forall r \in [1, N/(N-1)] : ||\triangle \rho_L||_{\mathrm{L}^r(\mathbb{R}^N)} \leq Z \langle\langle T \rangle\rangle_{V,2}^{\frac{r^2(r-1)+1}{r}}, \tag{4.22}$$

where $L = T^{r^2}$.

Proof. We have that

$$\begin{aligned} \triangle \rho_T(x) &= \nabla \cdot \nabla \rho_T(x) \\ &= \nabla \left[\sum_{i=1}^{\infty} \nu_{i,T} \nabla (\psi_{i,T}(x) \cdot \psi_{i,T}(x)) \right] \\ &= \nabla \left[\sum_{i=1}^{\infty} \nu_{i,T} (\nabla \psi_{i,T}(x) \cdot \psi_{i,T}(x) + \psi_{i,T}(x) \cdot \nabla \psi_{i,T}(x)) \right] \\ &= \nabla \left[2 \sum_{i=1}^{\infty} \nu_{i,T} (\psi_{i,T}(x) \cdot \nabla \psi_{i,T}(x)) \right] \\ &= 2 \left[\sum_{i=1}^{\infty} \nu_{i,T} (\psi_{i,}(x) \triangle \psi_{i,T}(x) + \nabla \psi_{i,T}(x) \nabla \psi_{i,T}(x)) \right] \\ &= 2 \left[\sum_{i=1}^{\infty} \nu_{i,T} (|\nabla \psi_{i,T}(x)|^2 + \psi_{i,T}(x) \triangle \psi_{i,T}(x)) \right] \end{aligned}$$

Now, let's assume that $T \neq 0$ and $r \in [1, N/(N-1)]$. Since $r \geq 1$, it follows by Proposition 4.3 that $L = T^{r^2} \in \mathscr{H}^2_{V,+}$. By the Spectral Theorem, $\nu_{i,L} = \nu_{i,T}^{r^2}$, for each

$i \in \mathbb{N}$. Then

$$\begin{aligned}
||\triangle\rho_L||^r_{\mathrm{L}^r(\mathbb{R}^N)} &= \int_{\mathbb{R}^N} |\triangle\rho_L(x)|^r dx \\
&= 2^r \int_{\mathbb{R}^N} \left|\sum_{i=1}^{\infty} \nu_{i,L}(|\nabla\psi_{i,L}(x)|^2 + \psi_{i,L}(x)\triangle\psi_{i,L}(x))\right|^r dx \\
&\leq 2^r \int_{\mathbb{R}^N} \left[\sum_{i=1}^{\infty} \left|\nu_{i,L}(|\nabla\psi_{i,L}(x)|^2 + \psi_{i,L}(x)\triangle\psi_{i,L}(x))\right|\right]^r dx \\
&= 2^r \int_{\mathbb{R}^N} \left[\sum_{i=1}^{\infty} \nu_{i,L}\left||\nabla\psi_{i,L}(x)|^2 + \psi_{i,L}(x)\triangle\psi_{i,L}(x)\right|\right]^r dx \\
&\leq 2^r \int_{\mathbb{R}^N} \left[\sum_{j=1}^{\infty} \nu_{j,L} \sum_{i=1}^{\infty} \left(\frac{\nu_{i,L}}{\sum_{j=1}^{\infty}\nu_{j,L}}\right)\left||\nabla\psi_{i,L}(x)|^2 + \psi_{i,L}(x)\triangle\psi_{i,L}(x)\right|\right]^r dx \\
&\leq 2^r ||L||_1^r \int_{\mathbb{R}^N} \left[\sum_{i=1}^{\infty} \left(\frac{\nu_{i,L}}{\sum_{j=1}^{\infty}\nu_{j,L}}\right)\left||\nabla\psi_{i,L}(x)|^2 + \psi_{i,L}(x)\triangle\psi_{i,L}(x)\right|\right]^r dx
\end{aligned}$$

By the convexity of $t \mapsto t^r$ and Jensen's inequality, we have that

$$\begin{aligned}
\int_{\mathbb{R}^N} |\triangle\rho_L(x)|^r dx &\leq 2^r ||L||_1^r \int_{\mathbb{R}^N} \sum_{i=1}^{\infty} \left(\frac{\nu_{i,L}}{\sum_{j=1}^{\infty}\nu_{j,L}}\right)\left||\nabla\psi_{i,L}(x)|^2 + \psi_{i,L}(x)\triangle\psi_{i,L}(x)\right|^r dx \\
&= 2^r ||L||_1^{r-1} \sum_{i=1}^{\infty} \nu_{i,L} \int_{\mathbb{R}^N} \left||\nabla\psi_{i,L}(x)|^2 + \psi_{i,L}(x)\triangle\psi_{i,L}(x)\right|^r dx
\end{aligned}$$

Note that

$$\int_{\mathbb{R}^N} \left||\nabla\psi_{i,L}(x)|^2 + \psi_{i,L}(x)\triangle\psi_{i,L}(x)\right|^r dx = |||\nabla\psi_{i,L}|^2 + \psi_{i,L}\triangle\psi_{i,L}||^r_{\mathrm{L}^r(\mathbb{R}^N)}$$

whence, using Poincaré and triangle inequalities we have that

$$\begin{aligned}
\int_{\mathbb{R}^N} |\triangle\rho_L(x)|^r dx &\leq 2^r ||L||_1^{r-1} \sum_{i=1}^{\infty} \nu_{i,L} \left(\left(\int_{\mathbb{R}^N} |\nabla\psi_{i,L}(x)|^{2r} dx\right)^{1/r}\right. \\
&\quad \left. + \left(\int_{\mathbb{R}^N} |\psi_{i,L}(x)\triangle\psi_{i,L}(x)|^r dx\right)^{1/r}\right)^r \\
&\leq 2^r ||L||_1^{r-1} \sum_{i=1}^{\infty} \nu_{i,L} \left(C_0 \left(\int_{\mathbb{R}^N} |\nabla\cdot|\nabla\psi_{i,L}(x)|^2|^r dx\right)^{1/r}\right. \\
&\quad \left. + \left(\int_{\mathbb{R}^N} |\psi_{i,L}(x)\triangle\psi_{i,L}(x)|^r dx\right)^{1/r}\right)^r \\
&= 2^r ||L||_1^{r-1} \sum_{i=1}^{\infty} \nu_{i,L} \left(2C_0 \left(\int_{\mathbb{R}^N} |\nabla\psi_{i,L}(x)|^r |\triangle\psi_{i,L}(x)|^r dx\right)^{1/r}\right. \\
&\quad \left. + \left(\int_{\mathbb{R}^N} |\psi_{i,L}(x)\triangle\psi_{i,L}(x)|^r dx\right)^{1/r}\right)^r
\end{aligned}$$

Since $r \leq \frac{N}{N-1}$ it follows that

$$P = \tfrac{2}{r} \geq 2 - \tfrac{2}{N} \geq 1 \quad \text{and} \quad P' = \tfrac{2}{2-r}$$

then from Hölder and Poincaré inequalities we have that

$$\begin{aligned}
\int_{\mathbb{R}^N} |\triangle \rho_L(x)|^r dx &\leq 2^r ||L||_1^{r-1} \sum_{i=1}^{\infty} \nu_{i,L} \left(2C_0 \left(\int_{\mathbb{R}^N} |\triangle \psi_{i,L}(x)|^{rP} dx \right)^{1/P} \left(\int_{\mathbb{R}^N} |\nabla \psi_{i,L}(x)|^{rP'} dx \right)^{1/P'} \right. \\
&\quad \left. + \left(\int_{\mathbb{R}^N} |\triangle \psi_{i,L}(x)|^{rP} dx \right)^{1/P} \left(\int_{\mathbb{R}^N} |\psi_{i,L}(x)|^{rP'} dx \right)^{1/P'} \right)^r \\
&= 2^r ||L||_1^{r-1} \sum_{i=1}^{\infty} \nu_{i,L} \left(2C_0 \left(\int_{\mathbb{R}^N} |\triangle \psi_{i,L}(x)|^2 dx \right)^{r/2} \left(\int_{\mathbb{R}^N} |\nabla \psi_{i,L}(x)|^{\frac{2r}{2-r}} dx \right)^{\frac{2-r}{2}} \right. \\
&\quad \left. + \left(\int_{\mathbb{R}^N} |\triangle \psi_{i,L}(x)|^2 dx \right)^{r/2} \left(\int_{\mathbb{R}^N} |\psi_{i,L}(x)|^{\frac{2r}{2-r}} dx \right)^{\frac{2-r}{2}} \right)^r
\end{aligned} \tag{4.23}$$

Note that

$$\left(\int_{\mathbb{R}^N} |\psi_{i,L}(x)|^{2r/(2-r)} dx \right)^{(2-r)/2} \leq C_1^r \left(\int_{\mathbb{R}^N} |\nabla \psi_{i,L}(x)|^{2r/(2-r)} \right)^{(2-r)/2} dx. \tag{4.24}$$

Hence, by (4.23), (4.24) and taking $C = \max\{2C_0, C_1^r\}$ we have that

$$\begin{aligned}
\int_{\mathbb{R}^N} |\triangle \rho_L(x)|^r dx &\leq (4C)^r ||L||_1^{r-1} \sum_{i=1}^{\infty} \nu_{i,L} \left(\left(\int_{\mathbb{R}^N} |\triangle \psi_{i,L}(x)|^2 dx \right)^{r/2} \left(\int_{\mathbb{R}^N} |\nabla \psi_{i,L}(x)|^{\frac{2r}{2-r}} dx \right)^{\frac{2-r}{2}} \right)^r \\
&= (4C)^r ||T||_1^{r^2(r-1)} \sum_{i=1}^{\infty} \nu_{i,L} \left(\left(\int_{\mathbb{R}^N} |\triangle \psi_{i,L}(x)|^2 dx \right)^{r/2} \left(\int_{\mathbb{R}^N} |\nabla \psi_{i,L}(x)|^{\frac{2r}{2-r}} dx \right)^{\frac{2-r}{2}} \right)^r
\end{aligned}$$

Since $2 \leq \frac{2r}{2-r} \leq 2^{**}$, using Poincaré's inequality and (4.17) we have that

$$\begin{aligned}
\int_{\mathbb{R}^N} |\triangle \rho_L(x)|^r dx &\leq (4C)^r ||T||_1^{r^2(r-1)} \sum_{i=1}^{\infty} \nu_{i,L} \left(\left(\int_{\mathbb{R}^N} |\triangle \psi_{i,L}(x)|^2 dx \right)^{r/2} C_2^r \left(\int_{\mathbb{R}^N} |\triangle \psi_{i,L}(x)|^2 dx \right)^{\frac{r}{2}} \right)^r \\
&= (4C)^r C_2^{r^2} ||T||_1^{r^2(r-1)} \sum_{i=1}^{\infty} \nu_{i,L} \left(\int_{\mathbb{R}^N} |\triangle \psi_{i,L}(x)|^2 dx \right)^{r^2} \\
&\leq Z_1 \langle\langle T \rangle\rangle_{V,2}^{r^2(r-1)} \sum_{i=1}^{\infty} \nu_{i,L} \left(\int_{\mathbb{R}^N} |\triangle \psi_{i,L}(x)|^2 dx \right)^{r^2} \\
&= Z_1 \langle\langle T \rangle\rangle_{V,2}^{r^2(r-1)} \sum_{i=1}^{\infty} \left(\nu_{i,T} \int_{\mathbb{R}^N} |\triangle \psi_{i,T}(x)|^2 dx \right)^{r^2}
\end{aligned} \tag{4.25}$$

where $Z_1 = (4C)^r C_2^{r^2}$. Since $T \in \mathscr{H}^2_{V,+}$ we have that

$$\sum_{i=1}^{\infty} \nu_{i,T} \int_{\mathbb{R}^N} |\triangle \psi_{i,T}(x)|^2 dx < \infty \Rightarrow \lim_{i \to \infty} \nu_{i,T} \int_{\mathbb{R}^N} |\triangle \psi_{i,T}(x)|^2 dx = 0.$$

Thus the sequence $(\nu_{i,T}||\triangle\psi_i||^2_{\mathrm{L}^2(\mathbb{R}^N)})_{i\in\mathbb{N}} \subseteq \mathbb{R}$ is bounded and we can take

$$J = \sup_{i\in\mathbb{N}} \nu_{i,T}||\triangle\psi_i||^2_{\mathrm{L}^2(\mathbb{R}^N)} < \infty$$

so that

$$\left(J^{-1}\nu_{i,T}||\triangle\psi_i||^2_{\mathrm{L}^2(\mathbb{R}^N)}\right)^{r^2} \leq J^{-1}\nu_{i,T}||\triangle\psi_i||^2_{\mathrm{L}^2(\mathbb{R}^N)}$$

as $r \geq 1$. Hence from (4.25) we get

$$\begin{aligned}
\int_{\mathbb{R}^N} |\triangle\rho_T(x)|^r dx &\leq Z_1\langle\langle T\rangle\rangle_{V,2}^{r^2(r-1)} \sum_{i=1}^{\infty} \left(\nu_{i,T}\int_{\mathbb{R}^N} |\triangle\psi_{i,T}(x)|^2 dx\right)^{r^2} \\
&= Z_1\langle\langle T\rangle\rangle_{V,2}^{r^2(r-1)} \sum_{i=1}^{\infty} \left(JJ^{-1}\nu_{i,T}\int_{\mathbb{R}^N} |\triangle\psi_{i,T}(x)|^2 dx\right)^{r^2} \\
&\leq Z_1 J^{r^2-1}\langle\langle T\rangle\rangle_{V,2}^{r^2(r-1)} \sum_{i=1}^{\infty} \nu_{i,T}\int_{\mathbb{R}^N} |\triangle\psi_{i,T}(x)|^2 dx \\
&= Z_1 J^{r^2-1}\langle\langle T\rangle\rangle_{V,2}^{r^2(r-1)} \mathscr{K}(T) \\
&\leq Z_1 J^{r^2-1}\langle\langle T\rangle\rangle_{V,2}^{r^2(r-1)+1}
\end{aligned}$$

whence

$$||\triangle\rho_{T^{r^2}}||_{\mathrm{L}^r(\mathbb{R}^N)} \leq Z_1^{1/r} J^{\frac{r^2-1}{r}} \langle\langle T\rangle\rangle_{V,2}^{\frac{r^2(r-1)+1}{r}}$$

taking $Z = Z_1^{1/r} J^{\frac{r^2-1}{r}}$ we get (4.22) and we conclude by the arbitrariness of r. □

By using the last theorem we can prove the following result.

Theorem 4.2. *Let $N > 4$ and $T \in \mathscr{H}^2_{V,+}$. Then*

$$\forall r \in \left[1, \frac{N}{N-1}\right],\ \forall s \in \left[1, \frac{N}{N-4}\right]: \quad \rho_L \in \mathrm{W}^{2,r}(\mathbb{R}^N) \cap \mathrm{L}^s(\mathbb{R}^N). \tag{4.26}$$

where $L = T^{r^2}$.

Proof. Let's prove that

$$\forall s \in [1, N/(N-4)] : \rho_L \in \mathrm{L}^s(\mathbb{R}^N). \tag{4.27}$$

By interpolation in Lebesgue spaces [5, pg. 93] we just have to prove that

$$\rho_L \in \mathrm{L}^1(\mathbb{R}^N) \cap \mathrm{L}^{N/(N-4)}(\mathbb{R}^N). \tag{4.28}$$

We already know that $\rho_L \in \mathrm{L}^1(\mathbb{R}^N)$. So let's prove that $\rho_L \in \mathrm{L}^{N/(N-4)}(\mathbb{R}^N)$. By choosing,

$$\hat{p} = \frac{N}{N-1}, \quad \hat{p}^{**} = \frac{N}{N-4},$$

from Poincaré's inequality and Theorem 4.22, we have that

$$\begin{aligned}||\rho_L||_{\mathrm{L}^{\hat{p}^{**}}(\mathbb{R}^N)} &\leq D_1||\nabla\rho_L||_{\mathrm{L}^{\hat{p}}(\mathbb{R}^N)}\\ &\leq D_1D_2||\triangle\rho_L||_{\mathrm{L}^{\hat{p}}(\mathbb{R}^N)}\\ &\leq D_1D_2Z\langle\langle T\rangle\rangle_{V,2}^{(r^2(r-1)+1)/r}\end{aligned}$$

where D_1, D_2 are the Poincaré's constants. Hence, we have proved (4.28). Finally, from Theorem 4.22 and (4.28) we conclude the proof of (4.27). □

We end this section with the following result:

Corollary 4.1. *Let $N > 4$ and $T \in \mathscr{H}^2_{V,+}$. Then there exists $\hat{Z} > 0$ such that*

$$||\triangle\rho_T||_{\mathrm{L}^1(\mathbb{R}^N)} \leq \hat{Z}\langle\langle T\rangle\rangle_{V,2},$$

hence $\rho_T \in \mathrm{W}^{2,1}(\mathbb{R}^N) \cap \mathrm{L}^s(\mathbb{R}^N)$, for every $s \in [1, N/(N-4)]$.

Proof. It follows from taking $r = 1$ in Theorems 4.1 and 4.2. □

4.3 Free energy functionals

In this section we introduce free energy functionals, but first we give some preliminaries.

Let's recall that the Legendre-Fenchel transform of $f : \mathbb{R} \to \mathbb{R} \cup \{\infty\}$, $f \not\equiv \infty$, is given by

$$f^*(x) = \sup_{\lambda\in\mathbb{R}}[x\lambda - f(\lambda)], \quad x \in \mathbb{R}. \tag{4.29}$$

Definition 4.3. Given $T \in \mathscr{H}^2_{V,+}$ and a convex function $\beta : \mathbb{R} \to \mathbb{R} \cup \{\infty\}$ such that $\beta(0) = 0$, we shall call the value

$$\mathcal{S}_\beta(T) = \mathrm{Tr}(\beta(T)) = \sum_{i=1}^{\infty}\beta(\nu_{i,T}) \tag{4.30}$$

the β-entropy of T provided $\mathcal{S}_\beta(T) \in]-\infty, +\infty]$. In this case we say that β is an entropy seed.

Remark 4.6. If for some function F, $\beta(x) = F^*(-x),\ x \in \mathbb{R}$, then we say that the entropy seed β is generated by the function F.

Now, given an entropy seed β we define the β-free energy functional by

$$\mathscr{F}_{V,\beta}(T) = \mathcal{S}_\beta(T) + \langle\langle T\rangle\rangle_{V,2}. \tag{4.31}$$

There is a class of functions which generate entropy seeds namely Cassimir class, so that we will introduce some results which will help us to define it.

Remark 4.7. (Homogeneous Dirichlet-like problem for the bi-Laplacian) Let $\Omega \subseteq \mathbb{R}^N$ be an open bounded set. We say that $u \in \mathrm{H}^2(\Omega)$ is a weak solution of the problem

$$\begin{cases} \triangle^2 u + Vu = f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{4.32}$$

if it verifies

$$\int_\Omega \left[\triangle u(x)\triangle w(x)dx + V(x)u(x)w(x)\right] dx = \int_\Omega f(x)w(x)dx, \quad \forall w \in \mathrm{H}^2(\Omega).$$

A classical solution of (4.32) is a function $u \in \mathrm{C}^4(\Omega)$ satisfying (4.32).

The existence and uniqueness of a weak solution for (4.32) follows from the next theorem.

Theorem 4.3. *Given any $f \in \mathrm{H}^1(\mathbb{R}^N)$, there exists a unique weak solution $u \in \mathrm{H}^2_V$ of (4.32).*

Proof. Let's prove that the bilinear form $a : \mathrm{H}^2_V \times \mathrm{H}^2_V \to \mathbb{R}$, given by

$$a(u, w) = \int_{\mathbb{R}^N} (\triangle u(x)\triangle w(x) + V(x)u(x)w(x))\, dx,$$

is continuous and coercive.

- First the continuity, let $u, w \in \mathrm{H}^2(\mathbb{R}^N)$, generic. Taking C=2 we have that

$$\begin{aligned} |a(u, w)| &= \left|\int_{\mathbb{R}^N} \triangle u(x)\triangle w(x) + V(x)u(x)w(x)dx\right| \\ &\leq \int_{\mathbb{R}^N} |\triangle u(x)||\triangle w(x)| + V(x)|u(x)||w(x)|dx \\ &\leq ||u||_{\mathrm{H}^2_V}||w||_{\mathrm{H}^2_V} + ||u||_{\mathrm{H}^2_V}||w||_{\mathrm{H}^2_V} \\ &\leq 2||u||_{\mathrm{H}^2_V}||w||_{\mathrm{H}^2_V} \end{aligned}$$

  we conclude by the arbitrariness of u and w that a is continuous.

- To prove the coercivity of a, let's consider

$$\Omega = \{x \in \mathbb{R}^N : \ V(x) \leq 1\} \text{ and } \Omega^c = \{x \in \mathbb{R}^N : \ V(x) > 1\}$$

and take $\alpha = C\min\{1, V_0\}$, so that

$$\begin{aligned} a(u,u) &= \int_{\mathbb{R}^N} \triangle u(x)|^2 + V(x)|u(x)|^2 dx \\ &\geq \int_{\mathbb{R}^N} |\triangle u(x)|^2 dx + \int_{\mathbb{R}^N} V_0(x)|u(x)|^2 dx \\ &\geq \int_{\mathbb{R}^N} |\triangle u(x)|^2 dx + \int_{\Omega} V_0(x)|u(x)|^2 dx + \int_{\Omega^c} |u(x)|^2 dx \\ &\geq \min\{1, V_0\} \int_{\mathbb{R}^N} \left(|\triangle u(x)|^2 + |u(x)|^2\right) dx. \end{aligned}$$

Since $\mathrm{H}^2(\mathbb{R}^N) \subseteq \mathrm{H}^1(\mathbb{R}^N)$, the last implies that

$$\begin{aligned} a(u,u) &\geq C\min\{1, V_0\}||u||_{\mathrm{H}^1(\mathbb{R}^N)} \\ &= \alpha||u||_{\mathrm{H}^1(\mathbb{R}^N)} \end{aligned}$$

(note that C is the constant appering in the embedding $\mathrm{H}^2(\mathbb{R}^N) \subseteq \mathrm{H}^1(\mathbb{R}^N)$) hence a is also coercive. By this two points we conclude that a is a continuous coercive bilinear form.

Finally, we get the conclusion of Theorem 4.3 applying Lax-Milgram theorem in the Hilbert space $H = \mathrm{H}^2_V$ with the bilinear form a(u,w) and the linear functional

$$\varphi : \ w \mapsto \int_\Omega f(x)w(x)dx$$

. □

The following theorem is a result concerning the eigenvalues and eigenvectors of the operator $\triangle^2 + V$.

Theorem 4.4. *There exists a Hilbert basis $\{e_n : n \in \mathbb{N}\}$ of $\mathrm{H}^1(\mathbb{R}^N)$ and a sequence $(\lambda_n)_{n\in\mathbb{N}} \subseteq]0, +\infty[$ such that $\lambda_n \to \infty$ and, for every $n \in \mathbb{N}$,*

$$\begin{aligned} e_n &\in \mathrm{H}^2_V \cap \mathrm{C}^\infty(\mathbb{R}^N), \\ \triangle^2 e_n + V(x)e_n &= \lambda_n e_n \quad \textit{in } \mathbb{R}^N. \end{aligned}$$

Proof. By Theorem 4.3 given $f \in \mathrm{H}^1(\mathbb{R}^N)$ we consider $u = Tf$ as the unique solution $u \in \mathrm{H}^2_V$ of the problem (4.32). We consider T as an operator from $\mathrm{H}^1(\mathbb{R}^N)$ into $\mathrm{H}^1(\mathbb{R}^N)$ and claim that T is a self-adjoint compact operator. In fact, let's first prove that it is compact:

- From the regularity results of [5, Th. 9.25] and [11, Th. 8.10] we have that $u \in \mathrm{H}^3(\mathbb{R}^N)$ and

$$||u||_{\mathrm{H}^2_V} \leq ||u||_{\mathrm{H}^3(\mathbb{R}^N)} \leq C||f||_{\mathrm{H}^1(\mathbb{R}^N)},$$

whence

$$||Tf||_{\mathrm{H}^2_V} \leq C||f||_{\mathrm{H}^1(\mathbb{R}^N)},$$

i.e.,

$$T(\mathrm{H}^1(\mathbb{R}^N)) \subseteq \mathrm{H}^2_V.$$

Since the injection of $\mathrm{H}^2(\mathbb{R}^N)$ into $\mathrm{H}^1(\mathbb{R}^N)$ is compact, the last implies that T is a compact operator from $\mathrm{H}^1(\mathbb{R}^N)$ into $\mathrm{H}^1(\mathbb{R}^N)$.

- Now we show that T is self-adjoint, i.e.,

$$\int_{\mathbb{R}^N} (Tf)g = \int_{\mathbb{R}^N} f(Tg), \ \forall f, g \in \mathrm{H}^1(\mathbb{R}^N).$$

Setting $u = Tf$ and $w = Tg$, we have

$$\triangle^2 u + Vu = f \tag{4.33}$$

$$\triangle^2 w + Vw = g. \tag{4.34}$$

Multiplying (4.33) by w and (4.34) by u and then integrating we obtain

$$\int_{\mathbb{R}^N} [\triangle u \triangle w + Vuw] \, dx = \int_{\mathbb{R}^N} fw \ dx = \int_{\mathbb{R}^N} gu \ dx$$

which is the desired conclusion.

Note that

$$\int_{\mathbb{R}^N} (Tf)f = \int_{\mathbb{R}^N} uf = \int_{\mathbb{R}^N} |\triangle u|^2 + V|u|^2 \geq 0, \quad \forall f \in \mathrm{H}^1(\mathbb{R}^N) \tag{4.35}$$

and also that $\mathrm{Ker}(T) = \{0\}$, since $Tf = 0$ implies $u = 0$ and so $f = 0$.

Finally, applying [5, Th. 6.11] we have that $\mathrm{H}^1(\mathbb{R}^N)$ admits a Hilbert basis $\{e_n : \ n \in \mathbb{N}\}$ consisting of eigenvectors of T with corresponding eigenvalues $(\mu_n)_{n\geq 1}$. By (4.35) and the last remark we have that $\mu_n > 0$ for every in $n \in \mathbb{N}$. Writing that $Te_n = \mu_n e_n$, we obtain

$$\int_{\mathbb{R}^N} \triangle e_n \triangle w + V e_n w = \lambda_n \int_{\mathbb{R}^N} e_n w, \ \forall w \in \mathrm{H}^1(\mathbb{R}^N),$$

with $\lambda_n = \dfrac{1}{\mu_n}$, so that we have that $\mu_n \to 0$. Therefore, $e_n \in \mathrm{H}^2_V$ and is a weak solution of $\triangle^2 e_n + V e_n = \lambda_n e_n$. By [5, Remark 25] we also have that $e_n \in \mathrm{C}^\infty(\omega)$ for all ω strictly contained in $\mathbb{R}^N$, i.e., $e_n \in \mathrm{C}^\infty(\mathbb{R}^N)$. □

Thanks to Theorem 4.4, for a positive number $\alpha > 0$, we consider the following condition.

(G_{V_α}): The operator $\alpha\triangle^2 + V$, with Dirichlet boundary conditions, has a sequence of eigenelements $((\lambda_{V,i}^{(\alpha)}, \phi_{V,i}^{(\alpha)}))_{i\in\mathbb{N}} \subseteq \mathbb{R} \times \mathrm{H}^2(\mathbb{R}^N)$ such that $(\phi_{V.i}^{(\alpha)})_{i\in\mathbb{N}}$ is a Hilbert basis of $\mathrm{H}^1(\mathbb{R}^N)$ and $\lambda_{V,i}^{(\alpha)} \to \infty$, as $i \to \infty$.

Definition 4.4. Assume (G_{V_α}). A function $F : \mathbb{R} \to \mathbb{R} \cup \{+\infty\}$ belongs to the Cassimir class $\mathscr{C}_V^\alpha$ if it is convex, non-increasing on $]0, +\infty[$ and

$$\mathrm{Tr}\left[F(\alpha\triangle^2 + V)\right] = \sum_{i\in\mathbb{N}} F(\lambda_i^{(\alpha)}) < \infty. \tag{4.36}$$

When $\alpha = 1$ we consider condition (G_{V_1}) and we shall simply write $\lambda_{V,i}$, $\phi_{V,i}$ and $\mathscr{C}_V$ instead of $\lambda_{V,i}^{(1)}$, $\phi_{V,i}^{(1)}$ and $\mathscr{C}_V^1$, respectively.

The previous background helps us to introduce a result about a lower bound for $\mathscr{F}_{V,\beta}$ which will be very useful to prove some Gagliardo-Nirenberg type inequalities in the context of the cone $\mathscr{H}_{V,+}^2$.

Theorem 4.5. *Let β be an entropy seed generated by $F \in \mathscr{C}_V$. Then*

$$\forall T \in \mathscr{H}_{V,+}^2 : \quad \mathscr{F}_{V,\beta}(T) \geq -\mathrm{Tr}\left(F(\triangle^2 + V)\right) \tag{4.37}$$

Proof. Let $T \in \mathscr{H}_{V,+}^2$ and $\{\phi_{V,j}) : j \in \mathbb{N}\} \subseteq \mathrm{H}^2(\mathbb{R}^N)$. Let $i \in \mathbb{N}$. We have that

$$\psi_{i,T} = \sum_{j\in\mathbb{N}} (\phi_{V,j}, \psi_{i,T})_{\mathrm{L}^2(\mathbb{R}^N)} \phi_{V,j}, \quad \sum_{j\in\mathbb{N}} |(\phi_{V,j}, \psi_{i,T})_{\mathrm{L}^2(\mathbb{R}^N)}|^2 = 1.$$

Then, as in [8, Lemma 3.1] we have that

$$\begin{aligned}
&\int_{\mathbb{R}^N} \left(|\triangle\psi_{i,T}(x)|^2 + V(x)|\psi_{i,T}(x)|^2\right) dx \\
&= \sum_{j\in\mathbb{N}} |(\phi_{V,j}, \psi_{i,T})_{\mathrm{L}^2(\mathbb{R}^N)}|^2 \left(\int_{\mathbb{R}^N} |\triangle\phi_{V,j}(x)|^2 + V(x)|\phi_{V,j}(x)|^2 dx\right) \\
&= \sum_{j\in\mathbb{N}} |(\phi_{V,j}, \psi_{i,T})_{\mathrm{L}^2(\mathbb{R}^N)}|^2 \lambda_{V,j} \int_{\mathbb{R}^N} |\phi_{V,j}(x)|^2 dx \\
&= \sum_{j\in\mathbb{N}} |(\phi_{V,j}, \psi_{i,T})_{\mathrm{L}^2(\mathbb{R}^N)}|^2 \lambda_{V,j} (\phi_{V,j}, \phi_{V,j})_{\mathrm{L}^2(\mathbb{R}^N)} \\
&= \sum_{j\in\mathbb{N}} |(\phi_{V,j}, \psi_{i,T})_{\mathrm{L}^2(\mathbb{R}^N)}|^2 \lambda_{V,j}
\end{aligned}$$

By the convexity of F and Jensen's inequality, we get

$$F\left(\int_{\mathbb{R}^N}\left(|\triangle\psi_{i,T}(x)|^2+V(x)|\psi_{i,T}(x)|^2\right)dx\right) \begin{aligned}&= F\left(\sum_{j\in\mathbb{N}}|(\phi_{V,j},\psi_{i,T})_{\mathrm{L}^2(\mathbb{R}^N)}|^2\lambda_{V,j}\right)\\ &\leq \sum_{j\in\mathbb{N}}|(\phi_{V,j},\psi_{i,T})_{\mathrm{L}^2(\mathbb{R}^N)}|^2F(\lambda_{V,j})\end{aligned} \tag{4.38}$$

By the spectral theorem we also have that

$$\forall j\in\mathbb{N}:\;\; F(\triangle^2+V)\phi_{V,j}=F(\lambda_{V,j})\phi_{V,j}.$$

Then,

$$\begin{aligned}&\sum_{j\in\mathbb{N}}|(\phi_{V,j},\psi_{i,T})_{\mathrm{L}^2(\mathbb{R}^N)}|^2F(\lambda_{V,j})\\ &=\sum_{j\in\mathbb{N}}|(\phi_{V,j},\psi_{i,T})_{\mathrm{L}^2(\mathbb{R}^N)}|^2F(\lambda_{V,j})(\phi_{V,j},\phi_{V,j})_{\mathrm{L}^2(\mathbb{R}^N)}\\ &=\left(\sum_{j\in\mathbb{N}}(\phi_{V,j},\psi_{i,T})_{\mathrm{L}^2(\mathbb{R}^N)}\phi_{V,j},\sum_{j\in\mathbb{N}}(\phi_{V,j},\psi_{i,T})_{\mathrm{L}^2(\mathbb{R}^N)}F(\lambda_{V,j})\phi_{V,j}\right)_{\mathrm{L}^2(\mathbb{R}^N)}\\ &=\left(\psi_{i,T},F(\triangle^2+V)\psi_{i,T}\right)_{\mathrm{L}^2(\mathbb{R}^N)}.\end{aligned} \tag{4.39}$$

Therefore, by (4.41) and (4.42), adding over $i\in\mathbb{N}$ yields

$$\sum_{i\in\mathbb{N}}F(||\psi_{i,T}||_{V,2}^2)\leq\mathrm{Tr}(F(\triangle^2+V)). \tag{4.40}$$

Since β is an entropy seed generated by F, we have that

$$\forall\nu,y\in\mathbb{R}:\quad\beta(\nu)+\nu\lambda\geq-F(\lambda).$$

Therefore, using (4.43) with $\nu=\nu_{i,T}$ and $y=||\psi_{i,T}||_{V,2}^2$, and adding over $i\in\mathbb{N}$, we get

$$\begin{aligned}\mathscr{F}_{V,\beta}(T)&=\mathcal{S}_\beta(T)+\langle\langle T\rangle\rangle_{V,2}\\ &=\sum_{i\in\mathbb{N}}\beta(\nu_{i,T})+\sum_{i\in\mathbb{N}}\nu_{i,T}||\psi_{i,T}||_{V,2}^2\\ &\geq-\sum_{i\in\mathbb{N}}F(||\psi_{i,T}||_{V,2}^2)\\ &\geq-\mathrm{Tr}(F(\triangle^2+V)).\end{aligned}$$

We conclude by the arbitrariness of T. □

In the other hand, if we consider condition (G_{V_α}) with $\alpha>0$ in the previous theorem we get the following proposition.

Proposition 4.4. *Let $\alpha > 0$ and assume (G_{V_α}). Consider an entropy seed generated by $F \in \mathscr{C}_V^\alpha$. Then, for any $T \in \mathscr{H}_{V,+}^2$, we have that*

$$\mathcal{S}_\beta(T) + (\alpha+1)\langle\langle T\rangle\rangle_{V,2} \geq -\mathrm{Tr}(F(\alpha\triangle^2 + V)).$$

Proof. Let $T \in \mathscr{H}_{V,+}^2$ and $\{\phi_{V,j} :\ j \in \mathbb{N}\} \subseteq \mathrm{H}^2(\mathbb{R}^N)$. Let $i \in \mathbb{N}$. By Theorem 4.4 we know that $\{\psi_{i,T}/\ i \in \mathbb{N}\}$ is a Hilbert basis of $\mathrm{H}^1(\mathbb{R}^N)$ we consider

$$\psi_{i,T} = \sum_{j\in\mathbb{N}} (\phi_{V,j}^{(\alpha)}, \psi_{i,T})_{\mathrm{L}^2(\mathbb{R}^N)}\phi_{V,j}^{(\alpha)},\ \ \sum_{j\in\mathbb{N}} |(\phi_{V,j}^{(\alpha)}, \psi_{i,T})_{\mathrm{L}^2(\mathbb{R}^N)}|^2 = 1$$

As in the proof of the previous theorem, we have

$$\begin{aligned}
&\int_{\mathbb{R}^N} \left(\alpha|\triangle\psi_{i,T}(x)|^2 + V(x)|\psi_{i,T}(x)|^2\right) dx \\
&= \sum_{j\in\mathbb{N}} |(\phi_{V,j}^{(\alpha)}, \psi_{i,T})_{\mathrm{L}^2(\mathbb{R}^N)}|^2 \left(\int_{\mathbb{R}^N} \alpha|\triangle\phi_{V,j}^{(\alpha)}(x)|^2 + V(x)|\phi_{V,j}^{(\alpha)}(x)|^2 dx\right) \\
&= \sum_{j\in\mathbb{N}} |(\phi_{V,j}^{(\alpha)}, \psi_{i,T})_{\mathrm{L}^2(\mathbb{R}^N)}|^2 \lambda_{V,j}^{(\alpha)} \int_{\mathbb{R}^N} |\phi_{V,j}^{(\alpha)}(x)|^2 dx \\
&= \sum_{j\in\mathbb{N}} |(\phi_{V,j}^{(\alpha)}, \psi_{i,T})_{\mathrm{L}^2(\mathbb{R}^N)}|^2 \lambda_{V,j}^{(\alpha)} (\phi_{V,j}^{(\alpha)}, \phi_{V,j}^{(\alpha)})_{\mathrm{L}^2(\mathbb{R}^N)} \\
&= \sum_{j\in\mathbb{N}} |(\phi_{V,j}^{(\alpha)}, \psi_{i,T})_{\mathrm{L}^2(\mathbb{R}^N)}|^2 \lambda_{V,j}^{(\alpha)}
\end{aligned}$$

By the convexity of F and Jensen's inequality, we get

$$\begin{aligned}
F\left(\int_{\mathbb{R}^N} \left(\alpha|\triangle\psi_{i,T}(x)|^2 + V(x)|\psi_{i,T}(x)|^2\right) dx\right) &= F\left(\sum_{j\in\mathbb{N}} |(\phi_{V,j}^{(\alpha)}, \psi_{i,T})_{\mathrm{L}^2(\mathbb{R}^N)}|^2 \lambda_{V,j}^{(\alpha)}\right) \\
&\leq \sum_{j\in\mathbb{N}} |(\phi_{V,j}^{(\alpha)}, \psi_{i,T})_{\mathrm{L}^2(\mathbb{R}^N)}|^2 F(\lambda_{V,j}^{(\alpha)})
\end{aligned} \tag{4.41}$$

From the spectral theorem we also have

$$\forall j \in \mathbb{N} :\ \ F(\alpha\triangle^2 + V)\phi_{V,j}^{(\alpha)} = F(\lambda_{V,j}^{\alpha)})\phi_{V,j}^{(\alpha)}.$$

Then,

$$\begin{aligned}
&\sum_{j\in\mathbb{N}} |(\phi^{(\alpha)}_{V,j}, \psi_{i,T})_{\mathrm{L}^2(\mathbb{R}^N)}|^2 F(\lambda^{(\alpha)}_{V,j}) \\
&= \sum_{j\in\mathbb{N}} |(\phi^{(\alpha)}_{V,j}, \psi_{i,T})_{\mathrm{L}^2(\mathbb{R}^N)}|^2 F(\lambda_{V,j})(\phi^{(\alpha)}_{V,j}, \phi^{(\alpha)}_{V,j})_{\mathrm{L}^2(\mathbb{R}^N)} \\
&= \left(\sum_{j\in\mathbb{N}} (\phi^{(\alpha)}_{V,j}, \psi_{i,T})_{\mathrm{L}^2(\mathbb{R}^N)} \phi^{(\alpha)}_{V,j}, \sum_{j\in\mathbb{N}} (\phi^{(\alpha)}_{V,j}, \psi_{i,T})_{\mathrm{L}^2(\mathbb{R}^N)} F(\lambda^{(\alpha)}_{V,j}) \phi^{(\alpha)}_{V,j}\right)_{\mathrm{L}^2(\mathbb{R}^N)} \\
&= \left(\psi^{(\alpha)}_{i,T}, F(\alpha\triangle^2 + V)\psi^{(\alpha)}_{i,T}\right)_{\mathrm{L}^2(\mathbb{R}^N)}.
\end{aligned} \tag{4.42}$$

Therefore, by (4.41) and (4.42), adding over $i \in \mathbb{N}$ yields

$$\sum_{j\in\mathbb{N}} F\left(\int_{\mathbb{R}^N} \alpha|\triangle\psi_{i,T}(x)|^2 + V(x)|\psi_{i,T}(x)|^2 dx\right) \leq \mathrm{Tr}(F(\alpha\triangle^2 + V)). \tag{4.43}$$

Since β is an entropy seed generated by F, we have that

$$\forall \nu, y \in \mathbb{R}: \quad \beta(\nu) + \nu y \geq -F(y).$$

Hence, using (4.43) with $\nu = \nu_{i,T}$ and $y = \int_{\mathbb{R}^N} \alpha|\triangle\psi_{i,T}(x)|^2 + V(x)|\psi_{i,T}(x)|^2 dx$, and adding over $i \in \mathbb{N}$, we get

$$\begin{aligned}
&\sum_{i\in\mathbb{N}} \beta(\nu_{i,T}) + \sum_{i\in\mathbb{N}} \nu_{i,T} \int_{\mathbb{R}^N} \alpha|\triangle\psi_{i,T}(x)|^2 + V(x)|\psi_{i,T}(x)|^2 dx \\
&\geq -\sum_{i\in\mathbb{N}} F(\int_{\mathbb{R}^N} \alpha|\triangle\psi_{i,T}(x)|^2 + V(x)|\psi_{i,T}(x)|^2 dx) \\
&\geq -\mathrm{Tr}(F(\alpha\triangle^2 + V)).
\end{aligned} \tag{4.44}$$

Finally, since $\alpha > 0$, we have that

$$(\alpha+1)\int_{\mathbb{R}^N} |\triangle\psi_{i,T}(x)|^2 + V(x)|\psi_{i,T}(x)|^2 dx \geq \int_{\mathbb{R}^N} \alpha|\triangle\psi_{i,T}(x)|^2 + V(x)|\psi_{i,T}(x)|^2 dx$$

so that

$$\begin{aligned}
\mathcal{S}_\beta(T) + (\alpha+1)\langle\langle T\rangle\rangle_{V,2} &= \sum_{i\in\mathbb{N}} \beta(\nu_{i,T}) + (\alpha+1)\sum_{i\in\mathbb{N}} \nu_{i,T}||\psi_{i,T}||^2_{V,2} \\
&\geq \sum_{i\in\mathbb{N}} \beta(\nu_{i,T}) + \sum_{i\in\mathbb{N}} \nu_{i,T} \int_{\mathbb{R}^N} \alpha|\triangle\psi_{i,T}(x)|^2 + V(x)|\psi_{i,T}(x)|^2 dx.
\end{aligned}$$

We conclude from this, (4.44) and the arbitrariness of T. □

Taking $V = 0$ in the previous theorem we have that

Remark 4.8. Let β be an entropy seed generated by $F \in \mathscr{C}_0^\alpha$. Then,

$$\mathcal{S}_\beta(T) + \alpha\mathscr{K}(T) \geq -\mathrm{Tr}\left(F(\alpha\triangle^2)\right).$$

We end this section with some Gagliardo-Nirenberg type inequalities and give some previous definitions

Definition 4.5. We say that a potential V is bounded away from zero if there exists $\gamma_V > 0$ such that

$$V(x) \geq \gamma_V > 0, \quad \forall x \in \mathbb{R}^N.$$

Definition 4.6. Assuming V bounded away from zero and $\lambda \leq \gamma_V$, we define a generalized free energy functional $\mathscr{F}^\lambda_{\beta,V} : \mathscr{H}^2_{V,+} \to \mathbb{R} \cup \{+\infty\}$ by

$$\mathscr{F}^\lambda_{\beta,V}(T) = \mathscr{F}_{\beta,V}(T) - \lambda||T||_1 = \mathcal{S}_\beta(T) + \langle\langle T\rangle\rangle_{V,2} - \lambda||T||_1.$$

The following result states that the generalized free energy functional is bounded from below

Theorem 4.6. *Assume that V is bounded away from zero and that $\lambda \leq \gamma_V$. Let β be an entropy seed generated by $F \in \mathscr{C}_0^{\epsilon/2}$ for some $\epsilon \in]0,1]$. Then, for every $T \in \mathscr{H}^2_{V,+}$,*

$$\mathscr{F}^\lambda_{\beta,V}(T) \geq -\mathrm{Tr}\left(F\left(\frac{\epsilon}{2}\triangle^2\right)\right) + \frac{\epsilon}{2}\mathscr{K}(T). \tag{4.45}$$

Proof. Let $T \in \mathscr{H}^2_{V,+}$, generic. We have

$$\begin{aligned}\mathscr{F}^\lambda_{\beta,V} = \;& \mathscr{F}_{\beta,V}(T) - \lambda||T||_1 \\ & = \mathcal{S}_\beta + \mathscr{K}(T) + \mathscr{P}_V(T) - \lambda||T||_1 \\ & = \mathcal{S}_\beta + \tfrac{\epsilon}{2}\mathscr{K}(T) + \tfrac{\epsilon}{2}\mathscr{K}(T) \\ & +\mathscr{K}(T) - \epsilon\mathscr{K}(T) + \mathscr{P}_V(T) - \lambda||T||_1.\end{aligned}$$

Since $F \in \mathscr{C}_0^{\epsilon/2}$, by Proposition 4.4 we have that

$$\mathcal{S}_\beta(T) + \frac{\epsilon}{2}\mathscr{K}(T) \geq -\mathrm{Tr}\left(F\left(\frac{\epsilon}{2}\triangle^2\right)\right),$$

whence,

$$\mathscr{F}^\lambda_{\beta,V}(T) \geq -\mathrm{Tr}\left(F\left(\frac{\epsilon}{2}\triangle^2\right)\right) + \frac{\epsilon}{2}\mathscr{K}(T) + (1-\epsilon)\mathscr{K}(T) + \mathscr{P}_V(T) - \lambda||T||_1. \tag{4.46}$$

We claim that

$$(1-\epsilon)\mathscr{K}(T)+\mathscr{P}_V(T)-\lambda||T||_1\geq 0 \tag{4.47}$$

which together with (4.46), imply (4.45).

If $\lambda \leq 0$, (4.47) is immediate. So let's assume that $\lambda > 0$. Since V is bounded away from zero and using $||\psi_{i,T}||_{\mathrm{L}^2(\mathbb{R}^N)}=1$, we have that

$$\begin{aligned}
\mathscr{P}_V(T)-\lambda||T||_1 &= \sum_{i=1}^{\infty}\nu_{i,T}\left(\int_{\mathbb{R}^N}V(x)|\psi_{i,T}(x)|^2dx\right)-\lambda\sum_{i=1}^{\infty}\nu_{i,T}\\
&= \sum_{i=1}^{\infty}\nu_{i,T}\left(\int_{\mathbb{R}^N}V(x)|\psi_{i,T}(x)|^2dx-\frac{\lambda}{||\psi_{i,T}||_{\mathrm{L}^2(\mathbb{R}^N)}}\int_{\mathbb{R}^N}|\psi_{i,T}(x)|^2dx\right)\\
&= \sum_{i=1}^{\infty}\nu_{i,T}\left(\int_{\mathbb{R}^N}(V(x)-\lambda)|\psi_{i,T}(x)|^2dx\right)\\
&\geq 0,
\end{aligned}$$

which implies (4.47). Since T was chosen arbitrarily, we are done.

□

As a consequence of Theorem 4.6 we have the following corollary which will be useful if we want to minimize free energy functionals.

Corollary 4.2. *Assume that V is bounded away from zero and that* $\lambda \leq \gamma_V$. *Let* β *be an entropy seed generated by* $F \in \mathscr{C}_0^{\epsilon/2}$ *for some* $\epsilon \in]0,1]$. *Let's assume that* $(T_\theta)_{\theta\in\Lambda} \subseteq \mathscr{H}^2_{V,+}$ *is such that* $(\mathscr{F}^\lambda_{\beta,0}(T_\theta))_{\theta\in\Lambda} \subseteq \mathbb{R}$ *is bounded. Then the families* $(\mathscr{K}(T_\theta))_{\theta\in\Lambda}$, $(\mathcal{S}_\beta(T_\theta))_{\theta\in\Lambda}$, $(||T_\theta||_1)_{\theta\in\Lambda}$, $(\langle\langle T_\theta\rangle\rangle_{V,2})_{\theta\in\Lambda}$ *and* $(\mathscr{P}_V(T_\theta)_{\theta\in\Lambda}$ *are also bounded in* $\mathbb{R}$.

Proof. Since $(\mathscr{F}^\lambda_{\beta,0}(T_\theta))_{\theta\in\Lambda}$ is bounded there exists $C_1 > 0$ such that

$$\mathscr{F}^\lambda_{\beta,0}(T_\theta)=\mathcal{S}_\beta(T_\theta)+\mathscr{K}(T_\theta)<C_1$$

for each $\theta \in \Lambda$.

By Remark 4.8, we have that

$$\mathcal{S}_\beta(T)+\frac{1}{2}\mathscr{K}(T)\geq -\mathrm{Tr}\left(F(\frac{1}{2}\triangle^2)\right). \tag{4.48}$$

As $F \in \mathscr{C}_0^{1/2}$, there exists $C_2 > 0$ such that

$$\mathrm{Tr}\left(F(\frac{1}{2}\triangle^2)\right)<C_2, \tag{4.49}$$

then by 4.48 and 4.49 we get

$$\begin{aligned} -C_2 + \tfrac{1}{2}\mathscr{K}(T_\theta) &< -\mathrm{Tr}\left(F(\tfrac{1}{2}\triangle^2)\right) + \tfrac{1}{2}\mathscr{K}(T_\theta) \\ &\leq \mathcal{S}_\beta(T_\theta) + \tfrac{1}{2}\mathscr{K}(T_\theta) + \tfrac{1}{2}\mathscr{K}(T_\theta) \\ &< C_1 \end{aligned}$$

which implies

$$\mathscr{K}(T_\theta) \leq 2(C_1 + C_2).$$

Then, we have proved that the sequence $(\mathscr{K}(T_\theta))_{\theta\in\Lambda}$ is bounded. The boundedness of $(\mathcal{S}_\beta(T_\theta))_{\theta\in\Lambda}$ immediately follows. Moreover, let $\theta \in \Lambda$, generic. Then from Poincaré's inequality we have that

$$\begin{aligned} ||T_\theta||_1 &= \sum_{i=1}^{\infty} \nu_{i,T_\theta} \\ &= \sum_{i=1}^{\infty} \nu_{i,T_\theta} \int_{\mathbb{R}^N} |\psi_{i,T_\theta}(x)|^2 dx \\ &\leq C_3 \sum_{i=1}^{\infty} \nu_{i,T_\theta} \int_{\mathbb{R}^N} |\nabla\psi_{i,T_\theta}(x)|^2 dx \\ &\leq C_4 C_3 \sum_{i=1}^{\infty} \nu_{i,T_\theta} \int_{\mathbb{R}^N} |\triangle\psi_{i,T_\theta}(x)|^2 dx \\ &= C_4 C_3 \mathscr{K}(T_\theta) \end{aligned}$$

by the arbitrariness of $\theta \in \lambda$ we conclude that $(||T_\theta||_1)_{\theta\in\Lambda}$ is also bounded. This easily gives the boundedness of $(\langle\langle T_\theta\rangle\rangle_{V,2})_{\theta\in\Lambda}$ and $(\mathscr{P}_V(T_\theta)_{\theta\in\Lambda}$. □

After we have proved the boundedness from below of our free energy functionals, we shall obtain some Gagliardo-Nirenberg type inequalities for operators.

Theorem 4.7. *Let β be an entropy seed generated by $F \in \mathscr{C}_V$. Let's assume that the functions τ, G are such that $\tau(s) = -(-G)^*(s),\ s \in \mathbb{R}$, and*

$$\mathrm{Tr}\left(F(\triangle^2 + V\right) \leq \int_{\mathbb{R}^N} G(V(x))dx. \tag{4.50}$$

Then, for every $T \in \mathscr{H}^2_{V,+}$,

$$\mathcal{S}_\beta(T) + \mathscr{K}(T) \geq \int_{\mathbb{R}^N} \tau(\rho_T(x))dx$$

Proof. Let $T \in \mathscr{H}^2_{V,+}$, generic. By Definition 4.29, for $x, \lambda \in \mathbb{R}$,

$$-\lambda s - G(\lambda) \geq -(-G)^*(s) = \tau(s).$$

If we choose $\lambda = V(x)$ and $s = \rho_T(x)$, we get, by (4.50) and Theorem 4.5 that

$$\begin{aligned}\mathcal{S}_\beta + \mathscr{K}(T) &\geq -\mathscr{P}_V(T) - \mathrm{Tr}(F(\triangle^2 + V)) \\ &\geq -\mathscr{P}_V(T) - \int_{\mathbb{R}^N} G(V(x))dx \\ &= \int_{\mathbb{R}^N} \left[-V(x)\rho_T(x) - G(V(x))\right] dx \\ &\geq \int_{\mathbb{R}^N} \tau(\rho_T(x))dx.\end{aligned}$$

Since T was chosen arbitrarily, we are done. □

Chapter 5

Conclusions & recommendations

5.1 Conclusions

Analogues of the Sobolev space H^1 were defined at the level of nuclear operators. Those sets of operators were no longer normed linear spaces but cones equipped with a concept of total energy that replaced the role of the square of a norm. Using Operator Theory, we obtained properties similar to those obtained by Mayorga et al. when the pivot space $\mathrm{L}^2(\mathbb{R}^N)$ was replaced by another separable Hilbert space, such as $\mathrm{H}^1(\mathbb{R}^N)$, with $N > 4$.

This work is related to the stability of quantum systems (represented by nuclear operators), and therefore, properties of free energy functionals defined on the operator cone were also studied. In this context, Gagliardo-Nirenberg type inequalities for operators were proven.

5.2 Recommendations

1. Similar to [17] and [18] we have some compactness results which are useful to minimize the total energy of an operator T, so we recommend to study those results, in principle, using the cone $\mathscr{H}^2_{V,+}$ and then trying to generalize them to a cone $\mathscr{H}^m_{V,+}$, $m \in \mathbb{N}$.

2. The lower bound for $\mathscr{F}_{V,\beta}$ can be extended to the context of the cone $\mathscr{H}^m_{V,+}$ with $m \in \mathbb{N}$ or even for $m \in \mathbb{R}$, so that it can be used for future works to prove some Gagliardo-Nirenberg type inequalities.

3. I also recommend to the math students at Yachay interested in this area to take a course of mathematical physics because this kind of courses will set basics which will help them to understand some definitions related to Quantum Mechanics.

Bibliography

[1] Adams, R. Sobolev Spaces, vol .64, Academic Press, 1 ed., 1975.

[2] G. L. Aki, J. Dolbeault, and C. Sparber, *Thermal effects in gravitational hartree systems*, Annales Henri Poincaré, 12 (2011), p. 1055-1079.

[3] L. Ballentine, Quantum Mechanics,World Scientific Publising, (2000).

[4] Bartsch, T., Pankov, A. & Wang, Z., *Non-linear Schrödinger equations with steep potential well*,*Communications in Contemporary Mathematics*, Vol. 3, No. 4, 549-569, (2001).

[5] H. Brezis, *Functional Analysis, Sobolev Spaces and Partial Differential Equations*, Universitext, Springer, 2011.
https://doi.org/10.1007/978-0-387-70914-7

[6] V. Bogachev, Measure Theory, Springer,Vol. 1,(2007).

[7] Dolbeault, J., Felmer, P. Loss, M., and Paturel, E., *Lieb-thirring type inequalities and gagliardo-nirenberg inequalities for systems*,Journal of Functional Analysis, 238, (2006), pp. 193-2220.

[8] Dolbeault, J., Felmer, P. & Mayorga-Zambrano, J., *Compactness properties for trace-class operators and applications to quatum mechanics*,Monatshefte fur Mathematik, (2008).

[9] Evans, L., , *Partial Differential Equations*, vol 19 of Graduate Studies in Mathematics, American Mathematical Society, Providence, RI, 1998.

[10] Fairchild, W. W., Tulcea, C. I., *Topology*, W. B. Saunders Company, 1971.

[11] Gilbarg, D. & Trudinger, N., Elliptic partial differential equations of second order. (springer,2015)

[12] Iorio, R. & de Magalhães, V. Fourier analysis and partial differential equations. Cambridge University Press, United Kingdom, 2001.

[13] Kreyszig, E., , *Introductory Functional Analysis with Applications*, vol. 1, Wiley & Sons, New York, 1978

[14] Landau, L. D. & Lifshitz, E. M., *Quantum Mechanics non-relativistic thoery*, vol. 3, BPCC Wheatons, Great Britain, 1977

[15] Lax, P., *Functional analysis*, John Wiley & Sons, vol. 55, 2002.

[16] Mayorga-Zambrano, J. , *A course of Functional Analysis with Calculus of Variations*, AMARUN, 2021.

[17] Mayorga-Zambrano, J., Castillo-Jaramillo, J. & Burbano-Gallegos, J., Compact embeddings of p-Sobolev-like cones of nuclear operators, Banach Journal of Mathematical Analysis, (2022).

[18] Mayorga-Zambrano, J. & Salina-Pillajo, Z., Sobolev-like cones of trace-class operators on unbounded domains: Interpolation inequalities and compactness properties, Nonlinear Analysis, (2013).

[19] Pedersen, K., *Analysis now*, Springer Science and Business Media, vol. 118, 2012.

[20] Reed, M. & Simon, B., Methods of Modern Mathematical Analysis Vols. 1-4, Academic Press, (1978).

[21] Salsa, S., *Partial differential equations in action: from modelling to theory*, Springer-Verlag Italia, Milan, 2008.

[22] Simon B., *Trace ideals and their applications*, Cambridge University Press, 1979.