UNIVERSIDAD DE INVESTIGACIÓN DE TECNOLOGÍA EXPERIMENTAL YACHAY

Escuela de Ciencias Matemáticas y Computacionales

GLOBAL IN TIME REGULARITY FOR THE 2D BOUSSINESQ SYSTEM

Trabajo de integración curricular presentado como requisito para la obtención
del título de Matemático

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Abstract

In this work we study partial cases of the 2D Boussinesq system and the relation between the accumulation of gradients of temperature and the global in time regularity. Initially we make the deduction of the Navier-Stokes equations as well as the equations involved in thermal convection. We developed the mathematical framework presented by Dongho Chae related to the formulation of the Boussinesq system in terms of vorticity and its relation to a blow-up criterion for smooth solutions of the zero-viscosity-thermal Boussinesq system. We establish the relation between the accumulation of gradients of temperature and the behavior of solutions at finite time. Moreover, using the blow-up criterion developed for the zero-viscosity-thermal Boussinesq system we go over the zero-thermal case of the Boussinesq system and we show that provided of initial data belonging to a particular Sobolev space, then we can prove that the solutions remains bounded for any time. That is, we prove the global in time regularity to the zero-thermal 2D Boussinesq case. Finally, we study the limit case when the thermal diffusivity constant of the complete Boussinesq system tends to zero and its convergence to the solutions of the partial zero-thermal Boussinesq system.

Keywords: Partial Differential Equations, Boussinesq system, blow-up for solutions, global regularity.

Resumen

En este trabajo estudiamos casos parciales del sistema 2D de Boussinesq y su relación entre la acumulacion de gradientes de temperatura y la regularidad global en tiempo. Inicialmente se realiza la deducción de las ecuaciones de Navier-Stokes así como las ecuaciones que involucran el fenómeno físico de convección natural. Desarrollamos el marco de referencia matemático desarrollado por Dongho Chae en cuanto a la formulación en términos de vorticidad del sistema de Boussinesq y su relación con un criterio de explosión para soluciones suaves del sistema cero-viscosidad-termal de Boussinesq. Se establece la relación entre la acumulacion de gradientes de temperatura y el comportamiento de las soluciones en tiempo finito. Además, usando el criterio de explosión desarrollado para el sistema cero-viscosidad-termal de Boussinesq abordamos el caso parcial zero-termal de Boussinesq en el cual mostramos que provisto de datos iniciales en un espacio de Sobolev apropiado, entonces podemos probar que las soluciones permanecen acotadas para cualquier tiempo. Es decir, probamos la regularidad global en tiempo para el caso parcial zero-termal 2D de Boussinesq. Finalmente, se estudia el caso límite en el cual la constante de difusividad termal del sistema completo de Boussinesq tiende a cero y su convergencia hacia las soluciones del caso parcial zero-termal de Boussinesq.

Palabras claves: Ecuaciones Diferenciales Parciales, Sistema de Boussinesq, explosión de soluciones, regularidad global.

Contents

1 General introduction

Fluid Mechanics studies the behaviour and the motion of fluid flows present in nature. When we refer to a fluid we think about either liquid, gas or plasma. For more than one hundred years mathematicians have developed a general framework to study the equations arising in Fluid Mechanics. Precisely, these equations are some of the most relevant in the field of Partial Differential Equations (PDEs) and present links to other branches of mathematics. PDEs usually arise to mathematically study nature phenomena. To study mathematically a PDE we mean to show crucial aspects like existence, uniqueness and regularity of the solutions. In most of the cases it is not possible to analytically find an explicit solution to a PDE and we turn to construct appropriate numerical methods.

Among the most important PDEs arising in Fluid Mechanics we have the so-called Navier-Stokes equations, [1]. These equations describe the motion of incompressible fluid flows when we consider velocity, pressure and density. The names of brilliant mathematicians like Leray, Ladyzhenskaya, Fefferman and others are linked to the study of Navier-Stokes equations. However, until now there are still open problems related to existence and regularity for the 3D case, [2].

Another very useful partial differential equation is the so-called convection-diffusion equation. This equation models the evolution of some quantity when it is transported at some velocity with the presence of a diffusion effect, [3]. In fact, a simplification of the convection-diffusion equation leads to consider the diffusion and heat equations. The heat equation describes the distribution of temperature over a determined region when time is evolving. The first mathematician who studied the heat equation was Joseph Fourier , who analytically found an explicit solution to this equation in one dimension.

Taking into account the equations for the motion of incompressible fluids and convection-diffusion we can obtain a system of PDEs which models the interaction of cold air masses at higher altitudes with hotter air masses at lower altitudes, [4]. In this direction, Joseph Valentine Boussinesq[1] at the nineteenth century, realized that the system governing natural thermal convection consists into a coupling between Navier-Stokes and convection-diffusion equations. Precisely, this coupling was called Boussinesq approximation, [4]. Several applications of practical importance of Boussinesq equations are: moving of fluids to transport or remove heat, thermal process in meteorology, atmospheric sciences, air and water pollution and chemical engineering, [5]. Studies for the Boussinesq system started with a paper by Canon and Dibedenetto of 1980, [6], and until now there are still open problems for partial cases, [7].

The 2D Boussinesq system for an incompressible fluid flow in $\mathbb{R}^2$ is

$$\text{2D Boussinesq system}\quad \begin{cases} \dfrac{\partial \mathbf{u}}{\partial t} + (\mathbf{u}\cdot\nabla)\mathbf{u} + \nabla p - \nu\Delta\mathbf{u} = \theta e_2, \\ \dfrac{\partial \theta}{\partial t} + \mathbf{u}\cdot\nabla\theta = k\Delta\theta, \\ \operatorname{div}\mathbf{u} = 0, \\ \mathbf{u}(x,0) = u_0(x) \quad \theta(x,0) = \theta_0(x), \end{cases} \tag{1.1}$$

where $u = (u_1, u_2)$ is the velocity vector field of the fluid, and $u_i = u_i(x,t), i = 1,2,\ (x,t) \in \mathbb{R}^2 \times (0,\infty)$, $p(x,t)$ is the scalar pressure, $\theta(x,t)$ the real valued temperature, $\nu \geq 0$ the viscosity coefficient and $k \geq 0$ the constant of thermal diffusivity. We have introduced system (1.1) in a compact way, however it consists of four scalar equations for the four unknown variables. As it was said before, the first equation of (1.1) describes the motion of an incompressible fluid and second equation of (1.1) governs the transport and distribution of fluid's temperature.

The case where $\nu > 0$ and $k > 0$ was studied in the seminal work, [6], where the authors dealt with the Cauchy problem (1.1) with initial data in L^p, finding a unique global weak solution.

[1] 1842 –1929 Joseph Valentine Boussinesq was a French mathematician and physicist who made significant contributions to the theory of hydrodynamics, vibration, light, and heat.

We can consider partial situations of the Boussinesq equations (1.1) by setting to zero both or at least one of the viscosity or thermal diffusivity constants. Studies in this direction was considered out by Cordoba et al. in [8], where the authors work with the case $k = 0, \nu > 0$ and showing that a type of *squirt singularities* are absent. Dongho Chae[2] inspired in [8] and using [9] proves in [10] the global-in-time regularity for partial cases $k = 0, \nu > 0$ and $k > 0,\ \nu = 0$. Independently, Thomas Y. Hou and Congming Li, [11], show the global well-posedness for the partial case $k = 0,\ \nu > 0$. Finally, studies for the global regularity for $k = \nu = 0$ constitute an open problem for the Boussinesq system, [7].

In this work we study partial cases $k = \nu = 0$ and $k = 0,\ \nu > 0$ of the Boussinesq system. We develop the mathematical framework presented by Dongho Chae in [9] and [10] in the following way. First, we deduce the Navier-Stokes equations and in particular we present the linearized forms of these equations based on [1] , [3], [4] and [2]. We give an introduction to the well known heat and transport equations which allow us to introduce the Boussinesq approximation. Then, based on [12] and [13] we introduce an appropriate functional approach, that is, using Lebesgue, Distributions and Sobolev spaces. By using these functional spaces we give a schematic way to construct solutions for Navier-Stokes and we establish the relation between accumulation of vorticity and blow-up for smooth solutions. Finally, we use all the functional analysis approach developed previously to show a blow up criterion for the zero-viscosity-thermal Boussinesq system which later allows us to prove the global in time regularity for the zero-thermal Boussinesq system.

2 Preliminaries

In this section we introduce the basic notation that will be useful for the rest of this work. Through this work we are dealing with the whole space $\mathbb{R}^n$ and in particular the two dimensional space $\mathbb{R}^2$. However for practical examples we sometimes consider $\Omega \subset \mathbb{R}^n$ to be a bounded domain and denote by $\partial\Omega$ its boundary.

2.1 Basic notations

We use notation " $:=$ " for the expression "by definition", and we consider the spatial variable $\mathbf{x} = (x_1, ..., x_n)$ together with the time variable $t \in [0, \infty)$ as the independent variables for the functions introduced later. To avoid overload notation and to give a realistic physical intuition we denote the spacial components in three dimensions by x, y and z respectively.

For $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$, we denote:

$$\mathbf{x} \cdot \mathbf{y} = \sum_{i=1}^{n} x_i y_i,$$
$$|\mathbf{x}| = \left(\sum_{i=1}^{n} x_i^2 \right)^{\frac{1}{2}},$$
$$e_n = (0\ 0\ \cdots\ 1)^t,$$

where w^t denotes the transpose of the corresponding vector/matrix w.

In particular for the two dimensional Euclidean space $\mathbb{R}^2$ we have:

$$|\mathbf{x}| = \sqrt{x^2 + y^2}$$
$$e_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, e_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

We denote the $i - th$ partial derivative by $D_i = \frac{\partial}{\partial_{x_i}} = \partial_{x_i}$ and use indistinctly D or ∇ to express the derivative or gradient of a function.

[2]Dongho Chae is a Korean mathematician developing theoretical studies of Partial Differential Equations arising in Mathematical Physics (Fluid mechanics, Gauge theories, Relativity).

We define the vector $\mathbf{u}(x,t) = (u_1(x,t), u_2(x,t))$ as the velocity of a fluid at $x \in \mathbb{R}^n$ and $t \geq 0$. For $\mathbf{u}$ we can assume that $\mathbf{u}(t,x) \to 0$ as $|x| \to \infty$ which later will be expressed simply by belonging to a certain functional space. By simplicity, in the notation, sometimes we do not write explicitly the dependence on x and t.

For the vector function $\mathbf{u}$ we denote $\mathbf{u}(x,0) = \mathbf{u}_0(x)$ and the following standard operations in $\mathbb{R}^2$ are considered:

$$\begin{aligned}
\nabla &= \begin{pmatrix} \partial_x \\ \partial_y \end{pmatrix}, \\
\nabla^\perp &= \begin{pmatrix} -\partial_y \\ \partial_x \end{pmatrix}, \\
\operatorname{div} \mathbf{u} = \nabla \cdot \mathbf{u} &= \partial_x u_1 + \partial_y u_2, \\
\Delta\, \mathbf{u} &= (\partial_x^2 + \partial_y^2)\, \mathbf{u}, \\
(\mathbf{u} \cdot \nabla)\, \mathbf{u} &= (u_1 \partial_x + u_2 \partial_y)\, \mathbf{u}, \\
\nabla\, \mathbf{u} &= \begin{pmatrix} \partial_x u_1 & \partial_y u_1 \\ \partial_x u_2 & \partial_y u_2 \end{pmatrix}.
\end{aligned}$$

2.2 The Navier-Stokes equations

In the study of Fluid Mechanics we describe the motion of a fluid by means of Euler or Lagrange coordinates, [4]. On one side, the so-called Eulerian framework consists on looking the fluid's motion at a referenced time and considering the evolution of an infinitesimal portion of the fluid. For example, it can be interpreted like being on the bank of a river and see the water passing, [4]. On the other side, Lagrangian coordinates consists in describing the motion of a fluid by looking at a fluid particle and following it through its evolution.

The Navier-Stokes equations is a system of non-linear PDEs describing the motion of an incompressible fluid, [4]. These equations owe its name to Claude-Louis Marie Henri Navier[3] and Sir George Gabriel Stokes[4]. Physically the Navier-Stokes equations consist of a formulation of Newton's law for the motion of a continuous distribution of the matter in fluid state. Based on the work of Charles Doering, [4], and Juan Luis Vasquez, [2], we shall use the Eulerian coordinates to show a deduction of the Navier-Stokes equations.

Initially we consider a fluid which is not affected by the presence of viscosity, i.e. an ideal inviscid fluid. With the aim of having a better physical intuition we consider the three dimensional case, that is $\mathbf{x} = (x,y,z) \in \mathbb{R}^3$ and $t \geq 0$ representing the space and time variables. In the Eulerian description the dependent variables are:

$$\begin{aligned}
\rho = \rho(x,t) &:= \text{density of the fluid}, \\
\mathbf{u} = \mathbf{u}(x,t) &:= \text{velocity vector field of the fluid}, \\
p = p(x,t) &:= \text{scalar pressure}.
\end{aligned}$$

Let us think about an infinitesimal element of the fluid having volume δV and located at position $\mathbf{x} = (x,y,z)$ at time t with mass $\delta m = \rho(x,t)\delta V$ (see Figure 1). This infinitesimal element of fluid is moving with velocity $\mathbf{u}(x,t)$ and momentum $\delta m \mathbf{u}(x,t)$. The normal force directed into δV, of area $n\delta a$ centered at $\mathbf{x}$, is $-np(x,t)\delta a$, where n is the outward directed unit vector normal to the upper face of δV. Moreover, the pressure $p(x,t)$ is the unit of force per unit of area imposed on elements of the fluid from neighboring elements, [4].

Remark. *The use of $\delta V, \delta m, \delta n, \delta a$ comes from the Engineering and Physics point of view to express small changes or small sizes. In the subject of Variational Calculus it gives us a convenient way to express admissible increments. For further references see for example [14].*

[3]1785-1836 Claude-Louis Marie Henri Navier was a French physicist, mathematician and engineer.
[4]1819-1903 Sir George Gabriel Stokes was a Anglo-Irish physicist and mathematician.

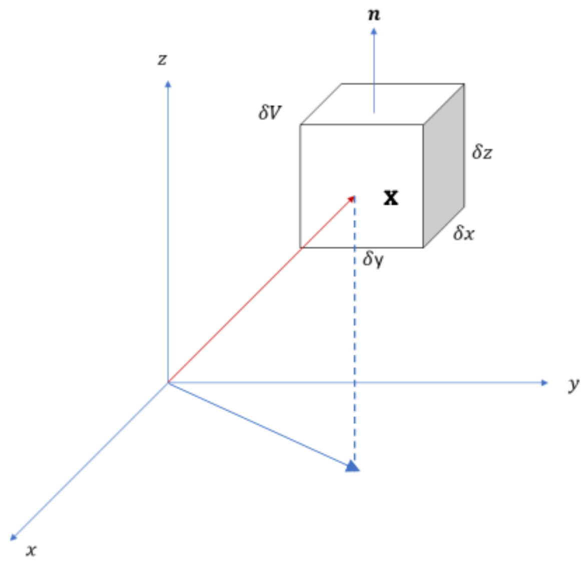

Figure 1: Infinitesimal element of fluid δV.

The rate of change of a quantity $f(x,t)$ with respect to time at fixed position is given by $\frac{\partial f(x,t)}{\partial t}$. Additionally, we introduce the rate of change of a quantity seeing it with respect to an observer with velocity $\mathbf{u}(x,t)$ by

$$\frac{df(x,t)}{dt} := \frac{\partial f(x,t)}{\partial t} + \mathbf{u} \cdot \nabla f(x,t), \tag{2.1}$$

where (2.1) is referred as the convective or material derivative of $\mathbf{u}$, [4].

The equations for the motion of an incompressible fluid comes from the Newton's second law and we take into account physical parameters like density, velocity and pressure. The law of conservation of mass states that δm can not be created or destroyed, even if it can be expanded or contracted, so that we must have:

$$\frac{d\delta m}{dt} = 0. \tag{2.2}$$

The rate of change of the infinitesimal volume $\delta V = \delta x \delta y \delta z$ occupied by δm is physically obtained by

$$\frac{d\delta V}{dt} = \frac{d\delta x}{dt}\delta y \delta z + \frac{d\delta y}{dt}\delta x \delta z + \frac{d\delta z}{dt}\delta x \delta y. \tag{2.3}$$

Since velocity is equal to distance over time, it is natural to think that the rate of change of $\delta x, \delta y, \delta z$ can be computed as follows:

$$\begin{aligned}
\frac{d\delta x}{dt} &= u_1(x + \tfrac{\delta x}{2}, y, z, t) - u_1(x - \tfrac{\delta x}{2}, y, z, t) &&= \frac{\partial u_1}{\partial x}\delta x,\\
\frac{d\delta y}{dt} &= u_2(x, y + \tfrac{\delta y}{2}, z, t) - u_2(x, y - \tfrac{\delta y}{2}, z, t) &&= \frac{\partial u_2}{\partial y}\delta y,\\
\frac{d\delta z}{dt} &= u_3(x, y, z + \tfrac{\delta z}{2}, t) - u_3(x, y, z - \tfrac{\delta z}{2}, t) &&= \frac{\partial u_3}{\partial z}\delta z.
\end{aligned} \tag{2.4}$$

Combining (2.3) and (2.4) we get:

$$\begin{aligned}
\frac{d\delta V}{dt} &= \frac{\partial u_1}{\partial x}\delta x \delta y \delta z + \frac{\partial u_2}{\partial y}\delta x \delta y \delta z + \frac{\partial u_3}{\partial z}\delta x \delta y \delta z,\\
\frac{d\delta V}{dt} &= \nabla \cdot \mathbf{u} \delta V.
\end{aligned} \tag{2.5}$$

Thus, the divergence of the velocity vector field is the local rate of change of the volume of elements having mass δm, [4]. Moreover, the rate of change of the density ρ in terms of δV and δm is given by

$$\begin{aligned}\frac{d\rho(x,t)}{dt} &= \frac{d}{dt}\frac{\delta m}{\delta V} \\ &= -\frac{\delta m}{\delta V^2}\frac{d\delta V}{dt} \\ &= -\frac{\rho\delta V}{\delta V^2}\nabla\cdot\mathbf{u}\delta V \\ &= -\rho\nabla\cdot\mathbf{u}.\end{aligned} \tag{2.6}$$

On the other side, using the definition of convective derivative (2.1) for the density ρ, we get

$$\frac{d\rho(x,t)}{dt} = \frac{\partial\rho(x,t)}{\partial t} + \mathbf{u}\cdot\nabla\rho(x,t). \tag{2.7}$$

Using (2.6) and (2.7), it follows that

$$\begin{aligned}\frac{\partial\rho}{\partial t} + \mathbf{u}\cdot\nabla\rho &= -\rho\nabla\cdot\mathbf{u}, \\ \frac{\partial\rho}{\partial t} + \mathbf{u}\cdot\nabla\rho + \rho\nabla\cdot\mathbf{u} &= 0, \\ \frac{\partial\rho}{\partial t} + \nabla(\mathbf{u}\rho) &= 0.\end{aligned} \tag{2.8}$$

We refer to (2.8) as the continuity equation.

Newton's second law states that the rate of change of the momentum equals to the net applied force. And, in the absence of external forces, the net force δF is provided by the pressure field on each element of the mass.

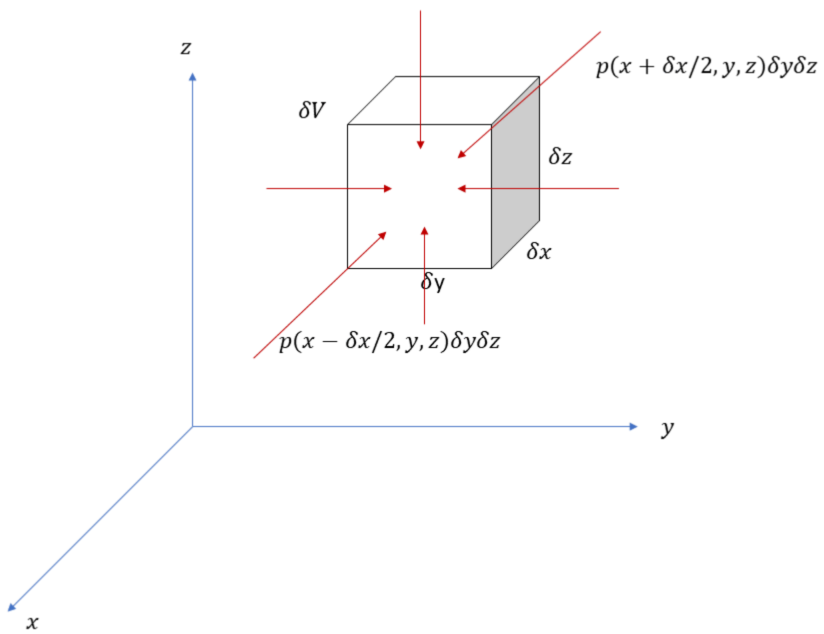

Figure 2: Net force provided by pressure.

The components of δF: $\delta F_1, \delta F_2, \delta F_3$ in the x, y, z- directions can be seen in Figure 2 and are represented by the following expression

$$\begin{aligned}\delta F_1 &= p(x-\frac{\delta x}{2},y,z,t)\delta y\delta z - p(x+\frac{\delta x}{2},y,z,t)\delta y\delta z,\\ &= -\frac{\partial p}{\partial x}\delta V. \end{aligned} \tag{2.9}$$

In the same way, doing the force decomposition in (2.9) for δF_2 and δF_3 we get

$$\begin{aligned}\delta F_2 &= -\frac{\partial p}{\partial y}\delta V,\\ \delta F_3 &= -\frac{\partial p}{\partial z}\delta V.\end{aligned}$$

Thus, the Newton second law for an element of fluid mass δm takes the form

$$\begin{aligned}\frac{d}{dt}(\delta m\,\mathbf{u}) &= \delta F,\\ \frac{d}{dt}\delta m\,\mathbf{u}+\delta m\frac{d}{dt}\mathbf{u} &= \delta F,\\ \delta m\frac{d}{dt}\mathbf{u} &= \delta F = -\delta V(\nabla p).\end{aligned} \tag{2.10}$$

Taking in to account the definition of convective derivative (2.1) for the velocity field $\mathbf{u}$

$$\frac{d\mathbf{u}(x,t)}{dt} = \frac{\partial \mathbf{u}(x,t)}{\partial t} + \mathbf{u}\cdot\nabla\mathbf{u}(x,t),$$

which, combined with (2.10) gives the so-called Euler's equations:

$$\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u}\cdot\nabla\mathbf{u} = -\frac{1}{\rho}(\nabla p). \tag{2.11}$$

Summarizing equations (2.8) and (2.11) we have

$$\begin{aligned}\frac{\partial \rho}{\partial t} + \nabla(\mathbf{u}\rho) &= 0 \quad \text{(Continuity equation)},\\ \frac{\partial \mathbf{u}}{\partial t} + \mathbf{u}\cdot\nabla\mathbf{u} &= -\frac{1}{\rho}(\nabla p) \quad \text{(Euler's equations)}.\end{aligned}$$

Note that when $n = 3$ we have 4 equations for the 5 unknowns $(\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \rho, p)$. This motivates to look for a relationship between pressure p and density ρ. From the continuity equation (2.8) we can assume that density of the fluid to be constant. This assumption gives what is known as the incompressibility condition:

$$\nabla\cdot\mathbf{u} = 0, \tag{2.12}$$

which physically means that all the relevant velocities of the fluid, are much smaller than the sound speed, [4].

Therefore, Euler's equations for an incompressible homogeneous fluid are

$$\begin{aligned}\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u}\cdot\nabla\mathbf{u} &= -\frac{1}{\rho}(\nabla p),\\ \nabla\cdot\mathbf{u} &= 0,\end{aligned}$$

where ρ is a parameter and in three dimensions we have 4 unknowns and 4 equations. Note that boundary conditions are necessary for both $\mathbf{u}$ and p. Those conditions are imposed by the particular physical conditions of the problem.

During the deduction of Euler's equations we assumed only that the pressure was acting over the fluid, i.e. the absence of external forces. Considering the study done for the pressure p, we can extend the deduction to the case when external forces f_{ex} are acting over the fluid, [2]. In this case we have that Euler's equations in the presence of an external force are given by

$$\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} = -\frac{1}{\rho}(\nabla p) + \frac{1}{\rho} f_{ex},$$
$$\nabla \cdot \mathbf{u} = 0.$$

In order to derive the full Navier-Stokes equations it is necessary to consider the viscosity present in the fluid. We define viscosity as the property of fluids to resist shear stress and it is analogous to the friction force of solid mechanics, [4]. Consequently, we are going to consider another force imposed over the fluid, namely, viscosity. The force due to viscosity f_{vis}, a shear stress, is defined by

$$f_{vis} = \nabla \tau \delta V = \nabla \tau \delta x \delta y \delta z, \tag{2.13}$$

where τ is a shear stress.

Unlike pressure which has one component of force per direction, shear stress of viscosity is a tensor which has 3 forces in each direction (see Figures 3 and 4).

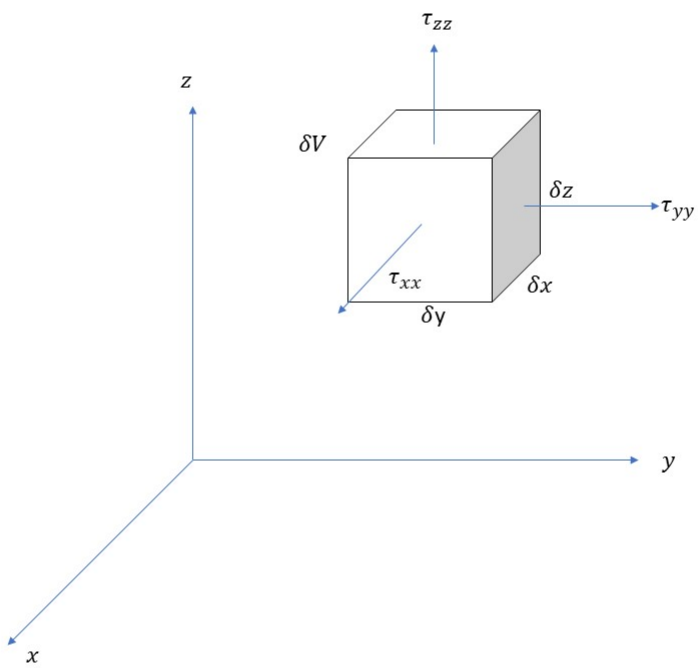

Figure 3: Orthogonal components of the forcing viscosity terms in a small region of volume δV.

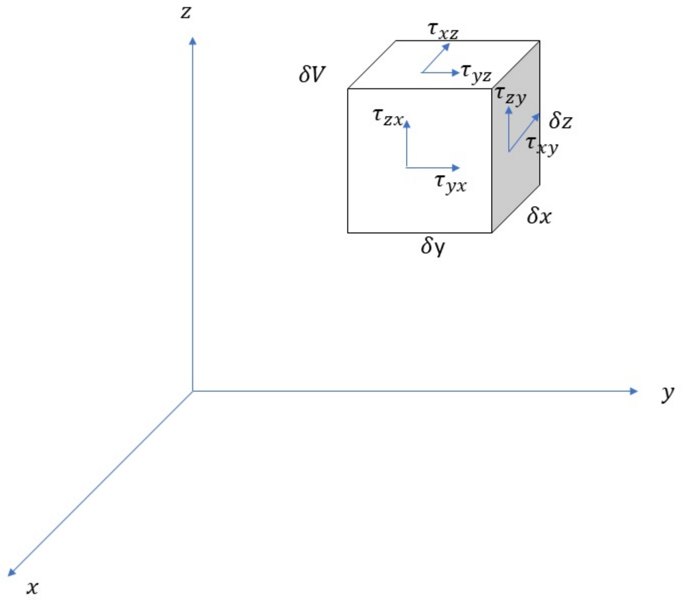

Figure 4: planar components of the forcing viscosity terms in a small region of volume δV.

For each direction the sum of forces due to viscosity with respect to x, y, z axis are given by

$$\frac{\partial \tau_{xx}}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z} = f_{vis_x}, \quad (x\text{-direction}), \tag{2.14}$$

$$\frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \tau_{yy}}{\partial y} + \frac{\partial \tau_{zy}}{\partial z} = f_{vis_y} \quad (y\text{-direction}), \tag{2.15}$$

$$\frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + \frac{\partial \tau_{zz}}{\partial z} = f_{vis_z} \quad (z\text{-direction}). \tag{2.16}$$

For a Newtonian fluid (constant viscosity), we denote by μ the viscosity. The shear stress is proportional to the shear strain rate over the time. For arbitrary t and distance y from the reference point x we have the strain rate given as follows

$$\begin{aligned} \dot{\epsilon}(y,t) &= \left(\frac{\partial}{\partial t}\frac{\partial x}{\partial y}\right)(y,t) \\ &= \left(\frac{\partial}{\partial y}\frac{\partial x}{\partial t}\right)(y,t) \\ &= \frac{\partial \mathbf{u}}{\partial y}(y,t). \end{aligned}$$

For each component of the shear stress we have

$$\begin{aligned} \tau_{xy} = \tau_{yx} &= (\dot{\epsilon}_{xy} + \dot{\epsilon}_{yx})\mu \\ &= \left(\frac{\partial}{\partial t}\frac{\partial y}{\partial x} + \frac{\partial}{\partial t}\frac{\partial x}{\partial y}\right)\mu \\ &= \left(\frac{\partial}{\partial x}u_2 + \frac{\partial}{\partial x}u_1\right)\mu, \end{aligned}$$

$$
\begin{aligned}
\tau_{yz} = \tau_{zy} &= (\dot{\epsilon}_{yz} + \dot{\epsilon}_{zy})\mu \\
&= \left(\frac{\partial}{\partial t}\frac{\partial z}{\partial y} + \frac{\partial}{\partial t}\frac{\partial y}{\partial z}\right)\mu \\
&= \left(\frac{\partial}{\partial y}u_3 + \frac{\partial}{\partial z}u_2\right)\mu,
\end{aligned}
$$

$$
\begin{aligned}
\tau_{xz} = \tau_{zx} &= (\dot{\epsilon}_{xz} + \dot{\epsilon}_{zx})\mu \\
&= \left(\frac{\partial}{\partial t}\frac{\partial z}{\partial x} + \frac{\partial}{\partial t}\frac{\partial x}{\partial z}\right)\mu \\
&= \left(\frac{\partial}{\partial x}u_3 + \frac{\partial}{\partial z}u_1\right)\mu,
\end{aligned}
$$

$$
\begin{aligned}
\tau_{xx} &= 2\mu\frac{\partial}{\partial x}u_1, \\
\tau_{yy} &= 2\mu\frac{\partial}{\partial y}u_2, \\
\tau_{zz} &= 2\mu\frac{\partial}{\partial z}u_3.
\end{aligned}
$$

Substituting the above stress-strain relations into (2.14), (2.15), (2.16) we get for the x-direction

$$
\begin{aligned}
f_{vis_x} &= \frac{\partial \tau_{xx}}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z} \\
&= \frac{\partial}{\partial x}\left(\mu\frac{\partial}{\partial x}u_1\right) + \frac{\partial}{\partial y}\left(\mu\left(\frac{\partial}{\partial x}u_2 + \frac{\partial}{\partial y}u_1\right)\right) + \frac{\partial}{\partial z}\left(\mu\left(\frac{\partial}{\partial x}u_3 + \frac{\partial}{\partial z}u_1\right)\right) \\
&= \mu\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)u_1 + \mu\frac{\partial}{\partial x}\left(\frac{\partial}{\partial x}u_1 + \frac{\partial}{\partial y}u_2 + \frac{\partial}{\partial z}u_3\right) \\
&= \mu\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)u_1 + \mu\frac{\partial}{\partial x}\nabla\cdot\mathbf{u} \\
&= \mu\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)u_1,
\end{aligned}
$$

where we have used the condition $\nabla\cdot\mathbf{u} = 0$. In the y and z directions respectively we get

$$
\begin{aligned}
f_{vis_y} &= \mu\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)u_2, \\
f_{vis_z} &= \mu\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)u_3.
\end{aligned}
$$

Putting all equations above together and considering (2.13) we get

$$f_{vis} = \mu\Delta\mathbf{u}\delta V.$$

Using Newton's second law in the presence of pressure p, viscosity μ and an external force f_{ex} we have

$$
\begin{aligned}
\frac{1}{\delta m}\sum f &= \frac{d}{dt}\mathbf{u}, \\
\frac{1}{\delta m}\left(-\nabla p\delta V + f_{ex}\delta V + \mu\Delta\mathbf{u}\delta V\right) &= \frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u}\cdot\nabla)\mathbf{u}, \\
\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u}\cdot\nabla)\mathbf{u} &= -\frac{1}{\rho}\nabla p + \frac{1}{\rho}f_{ex} + \frac{\mu}{\rho}\Delta\mathbf{u}.
\end{aligned}
$$

Defining $\nu = \dfrac{\mu}{\rho}$ we have the Navier-Stokes equations with constant density and viscosity, under the action of an external force f_{ex}, given by

$$\text{Navier-Stokes equations} \begin{cases} \dfrac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla)\mathbf{u} + \dfrac{1}{\rho}\nabla p - \nu \Delta \mathbf{u} = \dfrac{1}{\rho} f_{ex}, \\ \nabla \cdot \mathbf{u} = 0. \end{cases} \tag{2.17}$$

Notice that in a domain Ω with boundary condition and initial data we have what is known as a Navier-Stokes problem.

By simplicity, in most of the cases we assume $\rho = 1$ and we call ν the viscosity constant of a fluid. Moreover, we just consider the Navier-Stokes equations in two dimensions, namely, $\mathbf{u} = (u_1(x,t), u_2(x,t))$, $x \in \mathbb{R}^2$ and the time variable $t \in [0, \infty)$.

Albeit that we will work with equations (2.17) which are nonlinear, we present some useful simplifications of Navier-Stokes equations. We consider $\Omega \subset \mathbb{R}^{\mathrm{n}}$ be a bounded domain and its boundary given by $\partial\Omega$. Moreover, we assume the presence of an arbitrary external force $\mathbf{f}$. These simplifications are needed because the general problem has no global solutions and it is difficult to treat, [2].

The Stokes system

The Stokes equations are the linear form of the complete Navier-Stokes equations. In this case we consider very viscous fluids which produce the motion terms to vanish. Adding the assumption of constant density $\rho = 1$ and Dirichlet boundary condition we obtain

$$\text{Stokes system} \begin{cases} -\nu \Delta \mathbf{u} + \nabla p = \mathbf{f} & \text{in} \quad \Omega, \\ \nabla \cdot \mathbf{u} = 0 & \text{in} \quad \Omega, \\ \mathbf{u} = 0 & \text{on} \quad \partial\Omega. \end{cases} \tag{2.18}$$

Steady-State Navier-Stokes system

At this point we allow to assume certain kind of transport given by the nonlinear term $(\mathbf{u}\cdot\nabla)\mathbf{u}$. Adding Dirichlet boundary condition, the Steady-State Navier-Stokes system is given by

$$\text{Steady-State Navier-Stokes} \begin{cases} (\mathbf{u} \cdot \nabla)\mathbf{u} - \nu \Delta \mathbf{u} + \nabla p = \mathbf{f} & \text{in} \quad \Omega, \\ \nabla \cdot \mathbf{u} = 0 & \text{in} \quad \Omega, \\ \mathbf{u} = 0 & \text{on} \quad \partial\Omega. \end{cases} \tag{2.19}$$

In the previous equations we have considered Dirichlet boundary conditions. This condition on the boundary can be seen as the adherence of the fluid to the boundary of Ω. For studies related to the existence, uniqueness and regularity of solutions for the systems (2.18) and (2.19) we recommend the book by R. Temam, [1].

2.3 The heat equation

We are going to describe, briefly, one of the most important PDEs, hat is, the problem of describing the diffusion of heat over a region. The study of this problem has its origin with Joseph Fourier[5] who, in 1807 proposed what is known as Fourier's law of heat conduction, whence he obtained the heat equation. Fourier also proved that a solution of the heat equation can be represented in an infinite sum of sine-cosine functions which is known as a Fourier series.

Let $\Omega \subset \mathbb{R}^{\mathrm{n}}$, be an open bounded set, which is made of a homogeneous material having constant thermal diffusivity k. We denote the heat external source by $f(x,t)$ and by $\theta(x,t)$ the temperature at $x \in \Omega$ and time $t \geq 0$. We also consider the initial distribution of heat given by $\theta_0(x)$, $x \in \Omega$. Then from the conservation of

[5] Jean-Baptiste Joseph Fourier (1768 – 1830) was a French mathematician and physicist born in Auxerre and best known for initiating the investigation of Fourier series and their applications to problems of heat transfer and vibrations.

energy and the Fourier law, we present the equation modeling the evolution of the heat diffusion in Ω, and we complement it with a Dirichlet boundary condition and an initial heat distribution θ_0.

$$\text{The heat equation} \begin{cases} \dfrac{\partial\theta(x,t)}{\partial t} - k\Delta\theta(x,t) = f(x,t) & \text{in} \quad \Omega\times\mathbb{R}_*^+, \\ \theta(x,t) = 0 & \text{on} \quad \partial\Omega\times\mathbb{R}_*^+, \\ \theta(x,0) = \theta_0(x) & \text{in} \quad \Omega. \end{cases} \tag{2.20}$$

Note that in equation (2.20) we denote $t \in \mathbb{R}_*^+$ and we provide a condition over the boundary of Ω which gives a boundary value problem. The Dirichlet boundary condition physically means that the domain Ω is surrounded by constant temperature equal zero. Moreover, the assumption of initial conditions lead us to consider what is known as Cauchy problem. The problem (2.20) can be roughly explained as the evolution of heat with respect to time when a effect of diffusion is presented and a source provides heat externally.

Some important remarks about the heat equation are described in the following:

- The quantities involved in equation (2.20) have physical units. Since our objective is to mathematically study equations like (2.20), we neglect this dependence of physical units.
- The case of Dirichlet boundary condition in (2.20) can be extended depending of the physical context of the problem, [3]. For example when heat flux from Ω to the exterior is not allowed, then we introduce another boundary condition, namely, $\nabla\theta\cdot\mathbf{n} = 0$, where $\mathbf{n}$ is the outward unit vector to Ω.
- The model introduced in equation (2.20) is not only describing the heat propagation. In fact, the heat equation can be easily generalized to the case of modeling the dissipation of any quantity over a region.
- Several applications of equation (2.20) can be mentioned. For example it is linked to probability, random walks, Brownian motion, financial applications. Moreover, several modifications of the heat equation can be found in the literature (see for example [15, Chapter 7]).
- The heat propagation in a fixed medium can be extended to the case when we have a phenomenon of convection or transport through a non fixed medium. For example, we want to determine the distribution of heat trough the motion or transport of fluid. The introduction of a transport phenomenon requires the addition of the velocity $\mathbf{u}$ of the medium in which the heat propagates, [3]. Under the conditions described above we present the convection-diffusion equation,

$$\text{The convection-diffusion equation} \begin{cases} \dfrac{\partial\theta(x,t)}{\partial t} + \mathbf{u}\cdot\nabla\theta(x,t) - k\Delta\theta(x,t) = f(x,t) & \text{in} \quad \Omega\times\mathbb{R}_*^+, \\ \theta(x,t) = 0 & \text{on} \quad \partial\Omega\times\mathbb{R}_*^+, \\ \theta(x,0) = \theta_0(x) & \text{in} \quad \Omega. \end{cases} \tag{2.21}$$

From equation (2.21) we can consider a simplification by setting the thermal-diffusivity constant k to zero. For the case $k = 0$, we have the convection or transport equation that will be introduced in the next section.

2.4 The transport equation

The transport or convection equation is a first order linear PDE. The term convection is typically used in Fluid Mechanics. This equation can be obtained from the convection-diffusion equation (2.21) by setting the thermal diffusivity k to be zero. To introduce the transport equation we consider a scalar field θ, which by simplicity can be though as the unknown distribution of a physical quantity that we want to determine. In this sense, under the presence of an external force f we have the transport or convection equation given by

$$\text{The convection or transport equation} \begin{cases} \dfrac{\partial\theta(x,t)}{\partial t} + \mathbf{u}\cdot\nabla\theta(x,t) = f(x,t) & \text{in} \quad \Omega\times\mathbb{R}_*^+, \\ \theta(x,t) = 0 & \text{on} \quad \partial\Omega\times\mathbb{R}_*^+, \\ \theta(x,0) = \theta_0(x) & \text{in} \quad \Omega. \end{cases} \tag{2.22}$$

Physically we can describe equation (2.22) as the evolution in time of some quantity θ when it is transported proportional to the velocity $\mathbf{u}$ and the transport phenomenon is enhanced by an external force f.

So far, we have introduced the equations for the motion for incompressible fluids, the convection-diffusion equation and the heat and transport equations separately. From a mathematical and physical point of view it is important to consider cases of coupling between equations described before. Precisely, the Navier-Stokes equations and the convection-diffusion equation gives the core of this memory, namely, the Boussinesq equations.

2.5 The Boussinesq approximation

At the beginning of this section we have considered equations governing motion for incompressible fluids, namely, Navier-Stokes equation for the velocity field $\mathbf{u}$, and we neglected temperature and density variations of the fluid. Here we deal with the problem of thermal convection by an incompressible fluid. That is, we consider the case when motion of fluid is caused by an external heat source. Thermal convection or heat transfer was analyzed by Joseph Valentine Boussinesq who realized that variations in density are produced by the presence of gravitational force. According to [4, Sec. 1.5] thermal conduction between neighboring fluid elements is carried out by including a diffusive term, introducing the thermal diffusion coefficient and taking into account a buoyancy force into the velocity evolution equation. The inclusion of a buoyancy force comes from the fact that temperature variations lead to density variations.

Based on [4] and apart from the previous equations, we consider the whole space $\mathbb{R}^{\mathrm{n}}$ and in that case we present a sketchy idea for the deduction of the equations in the Boussinesq approximation. From (2.8) we have the continuity equation given by:

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho\, \mathbf{u}) = 0.$$

If density variations are neglected we have:

$$\operatorname{div} \mathbf{u} = 0. \tag{2.23}$$

From Navier-Stokes equations (2.17) with an external force $\mathbf{F}$ we have

$$\frac{\partial\, \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla)\, \mathbf{u} + \frac{1}{\rho} \nabla p = \nu \Delta\, \mathbf{u} + \frac{1}{\rho} \mathbf{F}. \tag{2.24}$$

Assuming that density variations are product of temperature variations, that is,

$$\rho - \rho_0 = -\alpha (T - T_0),$$

and $\mathbf{F}$ being a gravitational force i.e, $\mathbf{F} = \rho \mathbf{g}$, where $\mathbf{g} = -e_2 g$. Then $\mathbf{F}$ results into

$$\mathbf{F} = -(\rho_0 - \alpha(T - T_0)) e_2 g.$$

Replacing $\mathbf{F}$ in (2.24) gives us

$$\frac{\partial\, \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla)\, \mathbf{u} + \nabla p = \nu \Delta\, \mathbf{u} + e_2 g \alpha (T - T_0),$$

where ρ_0, T_0, are the density and temperature reference, α the thermal expansion coefficient, and we have included additive constants to p.

Moreover, the equation for the heat flow is given by

$$\frac{\partial T}{\partial t} + (\mathbf{u} \cdot \nabla) T = k \Delta T, \tag{2.25}$$

with k being the thermal diffusion coefficient.

In fact, the relative temperature $\theta = T - T_0$ can be put into the previous equations because T_0 can be considered as an arbitrary additive constant and derivatives in (2.25) depends only on T. In this way, equations

(2.23), (2.24) and (2.25) constitute the Boussinesq approximation. Indeed, setting $g\alpha = 1$ and providing initial conditions we write the Boussinesq equations as follows:

$$\text{The Boussinesq equations}\begin{cases} \dfrac{\partial \mathbf{u}}{\partial t} + (\mathbf{u}\cdot\nabla)\,\mathbf{u} + \nabla p = \nu\Delta\,\mathbf{u} + \theta e_2, \\ \dfrac{\partial \theta}{\partial t} + (\mathbf{u}\cdot\nabla)\theta = k\Delta\theta, \\ \operatorname{div}\mathbf{u} = 0, \\ \mathbf{u}(x,0) = \mathbf{u}_0(x), \quad \theta(x,0) = \theta_0(x). \end{cases} \tag{2.26}$$

Once we have introduced the Boussinesq equations we can go further and we can consider partial cases for the Boussinesq system. These partial cases are introduced by setting either viscosity or thermal diffusivity to zero. In the next section we present the most representative cases.

2.6 Partial Boussinesq approximations

The key feature in this section is the fact that we can consider cases when physical constants associated to (2.26) are allowed to be zero. Beyond the physical meaning of this new models, the simplifications introduced allow a theoretical treatment of this systems as we will see in the outcoming sections.

We introduce the case when $\nu, k = 0$. In this case we have a coupling between the equations for the motion of an inviscid fluid (Euler) and a transport equation. We present the zero-viscosity-thermal Boussinesq system.

$$\text{The zero-viscosity-thermal Boussinesq equations}\begin{cases} \dfrac{\partial \mathbf{u}}{\partial t} + (\mathbf{u}\cdot\nabla)\,\mathbf{u} + \nabla p = \theta e_2, \\ \dfrac{\partial \theta}{\partial t} + (\mathbf{u}\cdot\nabla)\theta = 0, \\ \operatorname{div}\mathbf{u} = 0, \\ \mathbf{u}(x,0) = \mathbf{u}_0(x), \quad \theta(x,0) = \theta_0(x). \end{cases} \tag{2.27}$$

The partial case $\nu = 0$ and $k > 0$ is referred to the zero-viscosity Boussinesq system. In this case we have a coupling between the equations for the motion of an inviscid fluid and the convection diffusion equation.

$$\text{The zero-viscosity Boussinesq equations}\begin{cases} \dfrac{\partial \mathbf{u}}{\partial t} + (\mathbf{u}\cdot\nabla)\,\mathbf{u} + \nabla p = \theta e_2, \\ \dfrac{\partial \theta}{\partial t} + (\mathbf{u}\cdot\nabla)\theta = k\Delta\theta, \\ \operatorname{div}\mathbf{u} = 0, \\ \mathbf{u}(x,0) = \mathbf{u}_0(x), \quad \theta(x,0) = \theta_0(x). \end{cases} \tag{2.28}$$

Finally, setting $\nu > 0$ and $k = 0$ we obtain a coupled system between the Navier-Stokes equations and the transport equation. We denote this case as zero-thermal Boussinesq system.

$$\text{The zero-thermal Boussinesq equations}\begin{cases} \dfrac{\partial \mathbf{u}}{\partial t} + (\mathbf{u}\cdot\nabla)\,\mathbf{u} + \nabla p = \theta e_2 + \nu\Delta\,\mathbf{u}, \\ \dfrac{\partial \theta}{\partial t} + (\mathbf{u}\cdot\nabla)\theta = 0, \\ \operatorname{div}\mathbf{u} = 0, \\ \mathbf{u}(x,0) = \mathbf{u}_0(x), \quad \theta(x,0) = \theta_0(x). \end{cases} \tag{2.29}$$

3 Functional Analysis approach

In order to mathematically study the Boussinesq system (2.26) we need to introduce some modern mathematical facts arising in the study of fluid mechanics PDE's. First, we present some explicit solutions for the equations mentioned before. Then we review the functional spaces on which the theory of PDE's is developed and its applications to the Boussinesq system. Finally we give a schematic way to construct smooth solutions for the Navier-Stokes equations and the relation between the accumulation of vorticity and the existence of global smooth solutions for Navier-Stokes.

3.1 Fundamental solutions

For the study of PDEs is important to look for explicit solutions given by the nature of the problem. For example after looking for properties like invariance under rotation and translation of the equations, we can suggest a simple solution that can be later improved, [15]. In this sense, we introduce the concept of fundamental solution for PDEs. The concept of fundamental solution is closely related to the space of distributions that shall be introduced later. For the moment we just concentrate on explicit solutions for partial differential equations.

Lemma 1 (Explicit solution for the convection-diffusion equation in $\mathbb{R}$)**.** *Let us consider the convection-diffusion equation* (2.21) *with continuous and bounded initial data. If we assume $\Omega = \mathbb{R}$, u be a constant velocity and $f = 0$. Then the solution of*

$$\begin{cases} \dfrac{\partial\theta(x,t)}{\partial t} + u\dfrac{\partial}{\partial x}\theta(x,t) - k\dfrac{\partial^2}{\partial x^2}\theta(x,t) = 0, \\ \theta(x,0) = \theta_0(x), \end{cases} \tag{3.1}$$

is given by the expression

$$\theta(x,t) = \frac{1}{\sqrt{4\pi\nu t}}\int_{\mathbb{R}} \theta_0(y)\exp\left(-\frac{|x-ut-y|^2}{4\nu t}\right)dy. \tag{3.2}$$

For a complete proof see [3, Exercise 1.2.1].

Now we summarize classic results for the heat and Laplace equations. For further studies of these equations we recommend the book of L. Evans, [15].

Lemma 2 (Explicit solution for the homogeneous heat equation in $\mathbb{R}^2$)**.** *Let us consider the heat equation* (2.20) *provided of continuous and bounded initial data. If $\Omega = \mathbb{R}^2$ and $f = 0$. Then the solution of the problem*

$$\begin{cases} \dfrac{\partial\theta(x,t)}{\partial t} - k\Delta\theta(x,t) = 0, \\ \theta(x,0) = \theta_0(x), \end{cases}$$

is given by the expression

$$\theta(x,t) = \frac{1}{4\pi t}\int_{\mathbb{R}^2} \exp\left(-\frac{|x-y|^2}{4t}\right)\theta_0(y)dy, \qquad (x\in\mathbb{R}^2, t>0).$$

The proof is presented in [15, Section 2.3].

An important particularization of the heat equation is given by its stationary version. This stationary version of the heat equation does not consider the evolution in time of θ and in the homogeneous case it is typically known as the Laplace[6] equation. Under the presence of an external force f the stationary case of heat equation is known as the Poisson[7] equation.

[6] 1749-1827 Pierre-Simon Laplace was a French scholar whose work was important to the development of engineering, mathematics, statistics, physics, astronomy, and philosophy.

[7] 1781-1840 Siméon Denis Poisson was a French mathematician, engineer, and physicist, who made several scientific advances.

Lemma 3 (Solution of the Poisson equation)**.** *Let f be a smooth function in $\mathbb{R}^2$, vanishing sufficiently rapidly as $|x| \to \infty$. Then the solution to the Poisson equation*

$$-\Delta\theta = f,$$

is given by

$$\theta(x) = \int_{\mathbb{R}^2} N(x-y)f(y)dy,$$

where the fundamental solution N, also called kernel is

$$N(x) = -\frac{1}{2\pi}\ln|x|.$$

The proof is presented in [15, Section 2.2].

Leaving aside our aim of looking explicit solutions of partial differential equations in the whole space, we present the Grönwall inequality. This is also called Grönwall's lemma and allows us to bound a function assumed to satisfy certain differential or integral inequality.

Lemma 4 (Grönwall's inequality, differential form)**.** *Let u, β be real valued functions defined on $[0,T]$ such that u, β are continuous at $[0,T]$ and differentiable at $(0,T)$, satisfying the inequality*

$$u'(t) \leq \beta(t)u(t) \quad t \in (0,T),$$

then

$$u(t) \leq u(0)\exp\left(\int_0^t \beta(s)ds\right). \tag{3.3}$$

For the proof see [15, Appendix B].

Lemma 5 (Grönwall's inequality, integral form)**.** *Let α, β and u be real-valued functions defined on $[0,T]$. Assume that β and u are continuous such that β is non negative and if u satisfies the integral inequality*

$$u(t) \leq \alpha(t) + \int_a^t \beta(s)u(s)ds, \quad \forall t \in [0,T],$$

then

$$u(t) \leq \alpha(t) + \int_a^t \alpha(s)\beta(s)\exp\left(\int_s^t \beta(r)dr\right)ds, \quad t \in [0,T].$$

Moreover, if α is non-decreasing, then

$$u(t) \leq \alpha(t)\exp\left(\int_a^t \beta(s)ds\right), \quad t \in [0,T]. \tag{3.4}$$

For the proof see [15, Appendix B].

In the next sections we give an introduction to the functional spaces used to study PDEs. These functional spaces make use of the integration theory developed by Lebesgue[8] with the construction of Lebesgue measurable functions and L^p spaces. We give an elementary review to the space of generalized functions or distributions developed by Schwartz[9]. Then, the L^p and Distributions spaces reached the pinnacle of its development with the theory presented by Sobolev[10] with the so-called Sobolev spaces.

[8] 1875-1941 Henri León Lebesgue was a French mathematician with contributions to integral and measure theory.
[9] 1915-2002 Laurent-Moïse Schwartz was a French mathematician pioneering in the theory of Distributions or generalized functions.
[10] 1908-189 Serguéi Sóbolev was a Russian mathematician who worked in mathematical analysis and PDE's.

3.2 L^p spaces

In this section we present some elementary facts about L^p-spaces. There is an extensive classical literature for this topic, and we refer for example [16] and [17] for a rigorous treatment about Functional Analysis and Measure Theory. In this matter, by simplicity we consider the whole space $\mathbb{R}^n$ and we show results that will be useful for further considerations. The next results will be based on [12].

Let $1 \leq p < \infty$. We denote by $L^p(\mathbb{R}^n)$ the Banach space of all (equivalence classes of) Lebesgue measurable real functions u defined on $\mathbb{R}^n$ which have finite norm

$$\|u\|_{L^p(\mathbb{R}^n)} = \|u\|_{L^p} := \left(\int_{\mathbb{R}^n} |u(x)|dx\right)^{\frac{1}{p}},$$

where two functions u and v are in the same equivalence class if and only if $u = v$ almost everywhere. For sake of simplicity we shall identify an equivalence class with one of its representatives. We also write "a.e" for "almost everywhere" concerning the Lebesgue measure.

Let $1 \leq p < \infty$. We define $L^p_{loc}(\mathbb{R}^n)$, the set of Lebesgue measurable functions such that

$$\|u\|_{L^p(\mathbb{R}^n)} = \|u\|_{L^p} := \left(\int_{K} |u(x)|dx\right)^{\frac{1}{p}} < +\infty, \ \forall K \subset X \ compact.$$

$L^2(\mathbb{R}^n)$ is a Hilbert space whenever is endowed with the scalar product given by

$$(u, v)_{L^2(\mathbb{R}^n)} = (u, v)_{L^2} := \int_{\mathbb{R}^n} u(x)v(x)dx,$$

for $u, v \in L^2(\mathbb{R}^n)$.

If $p = \infty$, we define $L^\infty(\mathbb{R}^n)$ be the Banach space of all measurable functions u with finite essential supremum

$$\|u\|_{L^\infty(\mathbb{R}^n)} = \|u\|_{L^\infty} := ess \sup_{x \in \mathbb{R}^n} |u(x)|.$$

Let $p \in [1, \infty]$. We denote by q the conjugate exponent,

$$\frac{1}{p} + \frac{1}{q} = 1.$$

Theorem 1 (A density result)**.** *The space of test functions C_c^∞ (infinitely differentiable functions with compact support) is dense in $L^p(1 \leq p < \infty)$.*

A proof is presented in [18, Proposition 8.17].

We collect some classical results on L^p-spaces for real valued functions that later can be extended to vector functions. The proof of the next results can be found in any Functional Analysis book, in particular we recommend [16].

Theorem 2 (Hölder's inequality)**.** *If $u \in L^p(\mathbb{R}^n)$, $v \in L^q(\mathbb{R}^n)$, then $uv \in L^1(\mathbb{R}^n)$, and*

$$\|uv\|_{L^1} \leq \|u\|_{L^p}\|v\|_{L^q}.$$

By using Hölder's inequality we get the following interpolation inequality.

Corollary 1 (Interpolation inequality)**.** *Let $u \in L^p \cap L^r$ with $1 \leq p \leq \gamma \leq r \leq \infty$, $0 \leq \alpha \leq 1$ such that*

$$\frac{1}{\gamma} = \frac{\alpha}{p} + \frac{1-\alpha}{r}.$$

Then $u \in L^\gamma$ and

$$\|u\|_{L^\gamma} \leq \|u\|^{\alpha}_{L^p}\|u\|^{1-\alpha}_{r} \leq \|u\|_{L^p} + \|u\|_{L^r}. \tag{3.5}$$

Theorem 3 (Young's inequality)**.** *Let a and b two nonnegative real numbers such that $1 < p, q < \infty$ and*

$$\frac{1}{p} + \frac{1}{q} = 1,$$

then

$$ab \leq \frac{a^p}{p} + \frac{b^q}{q}. \tag{3.6}$$

Theorem 4 (A generalized Minkowski inequality)**.** *Let F be a measurable function on the measurable product space $X \times Y$ with dx and dy the measures in X and Y respectively. Then*

$$\left(\int_Y \left(\int_X |F(x,y)| dx \right)^p dy \right)^{1/p} \leq \int_X \left(\int_Y |F(x,y)|^p dy \right)^{1/p} dx, \quad 1 \leq p < \infty. \tag{3.7}$$

Corollary 2 (Minkowski inequality)**.** *Let f and g in L^p. Then $f + g \in L^p$, and we have the triangle inequality*

$$\|f + g\|_{L^p} \leq \|f\|_{L^p} + \|g\|_{L^p}, \quad 1 \leq p \leq \infty. \tag{3.8}$$

We can go into several directions when studying L^p spaces. For example: reflexivity, separability and duality. For treatments in these directions, we refer again to the classical book of Brezis, [16].

Let $(u_j) = (u_j)_{j=1}^{\infty}$ be a sequence in $L^p(\mathbb{R}^{\mathrm{n}})$. Then we write the convergence of u_j in this space as

$$u = \lim_{j \to \infty} u_j \quad \text{in } L^p(\mathbb{R}^{\mathrm{n}})$$

meaning that $u \in L^p(\mathbb{R}^{\mathrm{n}})$ and

$$\lim_{j \to \infty} \|u - u_j\|_{L^p(\mathbb{R}^{\mathrm{n}})} = 0.$$

Taking into account the Boussinesq equations in two dimensions we have to deal with L^p spaces for vector fields in two dimensions $\mathbf{u} = (u_1, u_2)$. We define $(L^p(\mathbb{R}^{\mathrm{n}}))^2$ as the Banach space of vector functions such that each component belongs to $L^p(\mathbb{R}^{\mathrm{n}})$, namely,

$$(L^p(\mathbb{R}^{\mathrm{n}}))^2 := \left\{ (u_1, u_2); u_j \in L^p(\mathbb{R}^{\mathrm{n}}), j = 1, 2 \right\}.$$

Moreover, $(L^p(\mathbb{R}^{\mathrm{n}}))^2$ is endowed with the norm

$$\| \mathbf{u} \|_{L^p(\mathbb{R}^{\mathrm{n}})^2} = \| \mathbf{u} \|_{L^p(\mathbb{R}^{\mathrm{n}})} = \left(\|u_1\|_{L^p}^p + \|u_2\|_{L^p}^p \right)^{\frac{1}{p}}.$$

In particular, the space $(L^2(\mathbb{R}^{\mathrm{n}}))^2$ is a Hilbert space with scalar product

$$(\mathbf{u}, \mathbf{v})_{L^2(\mathbb{R}^{\mathrm{n}})} = (\mathbf{u}, \mathbf{v})_{L^2} := \int_{\mathbb{R}^n} \mathbf{u} \cdot \mathbf{v} dx, \tag{3.9}$$

with $\mathbf{u} \cdot \mathbf{v} = u_1 v_1 + u_2 v_2$.

Finally observe thatn the previous results for real valued functions, Hölder, interpolation and Minkowski inequalities in L^p spaces naturally extend to the case $(L^p(\mathbb{R}^2))^2$.

3.3 Distributions

Let $\Omega \subset \mathbb{R}^n$ be an open set. We denote by

$$D(\Omega) = C_c^{\infty}(\Omega),$$

the class of continuous infinitely differentiable functions with compact support. We say that a function f has compact support if there is some compact set $K \subsetneq \Omega$ such that f is identically zero outside of K.

Let α be a multiindex in $\mathbb{R}^n$, $n \in \mathbb{N}$ that is α is a vector in $\mathbb{R}^n$ given by

$$\alpha = (\alpha_1, \alpha_2, \cdots, \alpha_n),$$

where $\alpha_i \in \mathbb{N}$. Then the derivative of order $|\alpha|$, where

$$|\alpha| = \alpha_1 + \alpha_2 + \cdots + \alpha_n,$$

is defined by

$$D^\alpha := \frac{\partial^{|\alpha|}}{\partial_{x_1}^{\alpha_1}\partial_{x_2}^{\alpha_2}\cdots\partial_{x_n}^{\alpha_n}}.$$

We consider a notion of convergence in the space $D(\Omega)$ as follows: let $\{\phi_k\}_{k\in\mathbb{N}} \subset D(\Omega)$. We say that $\{\phi_k\}_k$ converges to $\phi \in D(\Omega)$ if

1. $\exists\ K \subset \Omega : supp(\phi_k) \subset K \quad \forall k \in \mathbb{N}$.
2. $\forall \alpha \in \mathbb{N}^n : \quad D^\alpha \phi_k \to D^\alpha \phi$.

Definition (Space of distributions)**.** *We define the space of distributions as the topological dual space of $D(\Omega)$, that is*

$$D'(\Omega) := \{T : D(\Omega) \to \mathbb{R} / T \textit{ is linear and continuos}\}.$$

If $T \in D'(\Omega)$ and $\phi \in D(\Omega)$ we denote the duality product as

$$\langle T, \phi\rangle_{D'(\Omega)\times D(\Omega)} = T(\phi).$$

In $D'(\Omega)$ we say that $\{T_k\}_{k\in\mathbb{N}} \subset D'(\Omega)$ converges to $T \in D'(\Omega)$ if and only if

$$\forall \phi \in D(\Omega) : \langle T_k, \phi\rangle \to \langle T, \phi\rangle.$$

Note that the notion of convergence given above makes sense due to the fact that $\langle T, \phi\rangle$ is a real number.

Example

- Given a function $f \in L^1_{\text{loc}}(\Omega)$, f induces an operator $T_f \in D'(\Omega)$ given by

$$\langle T_f, \phi\rangle := \int_\Omega f\phi dx, \quad \forall \phi \in D(\Omega).$$

Indeed, if $supp(f) \subset K \subset \Omega$ then

$$\begin{aligned} \left|\langle T_f, \phi\rangle\right| &= \left|\int_\Omega f\phi dx\right| \\ &\leq \int_\Omega |f||\phi| dx \\ &\leq \|\phi\|_{L^\infty} \int_K |f| dx < \infty. \end{aligned}$$

Therefore by taking $c = \int_K |f| dx$ we have

$$\left|T_f(u)\right| \leq c\|\phi\|_{L^\infty}.$$

So, T_f is well defined and it is continuous in $D(\Omega)$. Obviously it is linear in ϕ. We make an abuse of notation and denote $f \in D'(\Omega)$.

Example

- Let us see that $L^2(\Omega)$ can be seen as a subspace of $D'(\Omega)$. Remember that $L^2(\Omega)$ is a Hilbert space and by Theorem 1 $D(\Omega)$ is dense in $L^2(\Omega)$. Given a function $f \in L^2(\Omega)$, we associate the distribution $T_f \in D'(\Omega)$ as
$$\langle T_f, \phi\rangle := \int_\Omega f\phi dx, \quad \forall \phi \in D(\Omega).$$
Note that $T_f = 0$ is equivalent to say,
$$\int_\Omega f\phi dx = 0, \quad \forall \phi \in D(\Omega).$$
Which implies $f = 0$. Namely, T_f is injective, or $L^2(\Omega)$ is a subspace of $D'(\Omega)$. Moreover the injection $L^2(\Omega) \hookrightarrow D'(\Omega)$ is continuous.

Remark. *Note that the condition of L^2 being a subspace of the space of distributions gives us a characterization for the condition of functions vanishing sufficiently rapidly as $|x| \to \infty$. That is, a function vanishing sufficiently rapidly can be though as being in L^2 in such a way that in the case of integration by parts the border terms neglect.*

Definition (Distributional derivative)**.** *Let $T \in D'(\Omega)$. We define $\dfrac{\partial}{\partial_{x_i}} = \partial_i T \in D'(\Omega)$ as*
$$\langle \partial_i T, \phi\rangle = -\langle T, \partial_i \phi\rangle \text{ for any } \phi \in D(\Omega).$$

Observe that the last definition makes sense for $f \in C^1(\Omega)$. If $f \in C^1(\Omega)$ then $\partial_i f \in C(\Omega) \subset D'(\Omega)$, and we have:

$$\begin{aligned}\langle \partial_i T_f, \phi\rangle; &= -\langle T_f, \partial_i \phi\rangle \\ &= -\int_\Omega f\partial_i\phi dx \\ &= \int_\Omega \partial_i f \phi dx \\ &= \langle T_{\partial_i f}, \phi\rangle, \quad \forall \phi \in D(\Omega).\end{aligned}$$

Then, the derivative in the sense of distributions and the usual derivative coincide for $f \in C^1(\Omega)$.

Proposition 1 (Properties of distributional derivatives)**.** *Let $\{T_k\}_{k\in\mathbb{N}} \subset D'(\Omega)$ and $T \in D'(\Omega)$. We have:*

i) $\partial_i(\partial_j T) = \partial_j(\partial_i T), \quad \forall i, j = 1, \cdots, n.$

ii) $\langle D^\alpha T, \phi\rangle = (-1)^{|\alpha|}\langle T, D^\alpha \phi\rangle, \quad \forall \alpha \in \mathbb{N}_0^n.$

iii) If $T_k \xrightarrow{D'(\Omega)} T$ *then* $D^\alpha T_k \xrightarrow{D'(\Omega)} D^\alpha T, \quad \forall \alpha \in \mathbb{N}_0^n.$

Proof.

i) Let $T \in D'(\Omega)$ and $\phi \in D'(\Omega)$, by using the integration by parts formula we have, for all $i, j = 1, \cdots, n:$

$$\begin{aligned}\partial_i(\partial_j T) &= \langle \partial_i(\partial_j T), \phi\rangle \\ &:= -\langle \partial_j T, \partial_i \phi\rangle \\ &:= \langle T, \partial_j(\partial_i \phi)\rangle \\ &= \langle T, \partial_i(\partial_j \phi)\rangle \\ &:= -\langle \partial_i T, \partial_j \phi\rangle \\ &:= \langle \partial_j(\partial_i T), \phi\rangle \\ &= \partial_j(\partial_i T).\end{aligned}$$

ii) Let $T \in D'(\Omega)$ and $\phi \in D(\Omega)$ with α a multiindex. Then, if $\alpha = (\alpha_1, \cdots, \alpha_n)$, $|\alpha| = \sum_{i=1}^n \alpha_i$,

$$\begin{aligned}\langle D^\alpha T, \phi\rangle &= \langle \partial_1^{\alpha_1} \cdots \partial_n^{\alpha_n} T, \phi\rangle \\ &\overset{(i)}{=} (-1)^{\alpha_1+\cdots+\alpha_n} \langle T, \partial_1^{\alpha_1} \cdots \partial_n^{\alpha_n} \phi\rangle \\ &= (-1)^{|\alpha|}\langle T, D^\alpha \phi\rangle.\end{aligned}$$

iii) Let $\{T_k\}_{k\in\mathbb{N}} \subset D'(\Omega)$, $T \in D'(\Omega)$ and for any $\phi \in D(\Omega)$, we assume that $T_k \xrightarrow{D'(\Omega)} T$, i.e.

$$\lim_{k\to\infty} \left|T_k(\phi) - T(\phi)\right| = 0.$$

By property (ii) we have

$$\begin{aligned}\left|D^\alpha T_k(\phi) - D^\alpha T(\phi)\right| &= \left|\langle D^\alpha T_k, \phi\rangle - \langle D^\alpha T, \phi\rangle\right| \\ &= \left|(-1)^{|\alpha|}\langle T_k, D^\alpha \phi\rangle - (-1)^{|\alpha|}\langle T, D^\alpha\phi\rangle\right| \\ &= \left|\langle T_k, D^\alpha \phi\rangle - \langle T, D^\alpha\phi\rangle\right| \\ &= \left|\langle T_k - T, D^\alpha\phi\rangle\right|.\end{aligned}$$

Thus,

$$\lim_{k\to\infty} \left|D^\alpha T_k(\phi) - D^\alpha T(\phi)\right| = \lim_{k\to\infty} \left|\langle T_k - T, D^\alpha\phi\rangle\right| = 0,$$

since $\phi \in D(\Omega)$ implies $D^\alpha\phi \in D(\Omega)$.

□

3.4 Sobolev Spaces

Sobolev spaces provide a convenient framework when working with PDEs. These spaces use the already mentioned distributional derivative and the L^p theory. Let us fix $p \in [1, \infty]$ and let m be a nonegative integer. We define function spaces whose members have distribution or weak derivatives of various orders and its derivative lies in L^p spaces. By simplicity we consider the whole space $\mathbb{R}^n$, however the next results can be stated over a domain $\Omega \subset \mathbb{R}^n$; these are based on [12], [15] and [16].

Definition. *The Sobolev space $W^{m,p}(\mathbb{R}^n)$ consist of functions u belonging to $L^p(\mathbb{R}^n)$ such that for each multi-index α with $|\alpha| \le m$, $D^\alpha u$ exists in the distribution sense and belongs to $L^p(\mathbb{R}^n)$.*

The norm in $W^{m,p}(\mathbb{R}^n) = W^{m,p}$ is defined by

$$\|u\|_{W^{m,p}} := \begin{cases} \left(\sum_{|\alpha|\le m} \int_{\mathbb{R}^n} |D^\alpha u|^p \, dx\right)^{\frac{1}{p}}, & (1 \le p < \infty), \\ \sum_{|\alpha|\le m} \operatorname{ess\,sup} |D^\alpha u|, & (p = \infty). \end{cases}$$

We denote $L^p = W^{0,p}$. The cases of $m = 1, 2$ for $1 \le p < \infty$ will be frequently used and we use notations

$$\nabla u = Du := (D_j u)_{j=1}^n, \quad \|\nabla u\|_{L^p} = \|Du\|_{L^p} := \left(\sum_{j=1}^n \|D_j u\|_{L^p}^p\right)^{\frac{1}{p}},$$

similarly

$$\nabla^2 u = D^2 u := (D_j D_l u)_{j,l=1}^n, \quad \|\nabla^2 u\|_{L^p} = \|D^2 u\|_{L^p} := \left(\sum_{j,l=1}^n \|D_j D_l u\|_{L^p}^p\right)^{\frac{1}{p}}.$$

And for $p = \infty$

$$\|\nabla u\|_{L^\infty} := \max_{j=1,\cdots,n} \|D_j u\|_{L^\infty}, \quad \|\nabla^2 u\|_{L^\infty} := \max_{j,l=1,\cdots,n} \|D_j D_l u\|_{L^\infty}.$$

Sometimes we use convenient equivalent norms. For instance,

$$\|u\|_{L^p} + \|\nabla u\|_{L^p} \equiv |||u|||_{W^{1,p}},$$

and

$$\|u\|_{L^p} + \|\nabla u\|_{L^p} + \|\nabla^2 u\|_{L^p} \equiv |||u|||_{W^{2,p}}.$$

In particular if $p = 2$, we denote,

$$W^{m,2}(\mathbb{R}^n) := H^m(\mathbb{R}^n) := \Big\{ u \in L^2(\mathbb{R}^n) : D^\alpha u \in L^2(\mathbb{R}^n), \quad \forall\, 0 \leq |\alpha| \leq m \Big\}.$$

The importance of this space is given itself by the well know properties of the space L^2, which is a Hilbert space. In, in $H^m(\mathbb{R}^n) = H^m$ we define the inner product,

$$(u, v)_{H^m} := \sum_{0 \leq |\alpha| \leq m} \int_{\mathbb{R}^n} D^\alpha u D^\alpha v dx, \tag{3.10}$$

and the induced norm

$$\|u\|_{H^m} = \left(\sum_{0 \leq |\alpha| \leq m} \|D^\alpha u\|_{L^2}^2 \right)^{\frac{1}{2}}. \tag{3.11}$$

We present now some basic properties of Sobolev spaces, which can be found in several books (see for example [15], [16].)

Theorem 5. *H^m is a Hilbert space with the inner product $(\cdot, \cdot)_{H^m}$ defined in (3.10).*

Proof. Note that H^m is prehilbert because it is endowed with the inner product $(\cdot, \cdot)_{H^m}$. Thus, we have to prove that the H^m is complete with respect to $\|\cdot\|_{H^m}$.

Let $\{v_n\}_n \subset H^m(\mathbb{R}^n)$ be a Cauchy sequence. Using the definition of H^m norm (3.11) we have

$$\|v_n - v_m\|_{L^2} \leq \|v_n - v_m\|_{H^m},$$
$$\|D^\alpha v_n - D^\alpha v_m\|_{L^2} \leq \|D^\alpha v_n - D^\alpha v_m\|_{H^m}, \quad \forall\, |\alpha| \leq m.$$

Namely, $\{v_n\}_n$ and $\{D^\alpha v_n\}_n$ are Cauchy sequences in $L^2(\mathbb{R}^n)$ which is complete. Then, there is $v, w \in L^2(\Omega)$ such that

$$v_n \to v, \quad \text{in } L^2(\mathbb{R}^n),$$
$$D^\alpha v_n \to w \quad \text{in } L^2(\mathbb{R}^n), \quad \forall\, |\alpha| \leq m.$$

Thus, if we prove $w = D^\alpha v$ in the distribution sense, we would have $D^\alpha v \in L^2(\mathbb{R}^n)$ which implies $v \in H^m(\mathbb{R}^n)$ and $v_n \to v$ in $H^m(\mathbb{R}^n)$. We know

$$\|v_n - v\|_{H^m}^2 = \|v_n - v\|_{L^2}^2 + \sum_{1 \leq |\alpha| \leq m} \|D^\alpha v_n - D^\alpha v\|_{L^2}^2. \tag{3.12}$$

We mentioned above that injection $L^2(\mathbb{R}^n) \hookrightarrow D'(\mathbb{R}^n)$ is continuous. Then,

$$v_n \to v, \quad \text{in } D'(\mathbb{R}^n),$$
$$D^\alpha v_n \to w \quad \text{in } D'(\mathbb{R}^n), \quad \forall\, |\alpha| \leq m.$$

Since the derivative in the distribution sense defines a continuous linear operator in $D'(\Omega)$, and $v_n \to v$ in $D'(\Omega)$, then $D^\alpha v_n \to D^\alpha v$ in $D'(\Omega)$ and in particular $w = D^\alpha v$.

Finally, (3.12) implies that $\lim_{n\to\infty} \|v_n - v\|_{H^m} = 0$. □

One question which arises when studying Sobolev spaces is the approximation of functions in H^m by smooth functions, as is possible for L^p spaces. In this direction we present the following density result.

Theorem 6 (Density result)**.** *In the whole space* $\mathbb{R}^n$ *we have that the closure of smooth functions* $D(\mathbb{R}^n)$ *gives* $H^m(\mathbb{R}^n)$*, namely,*

$$\overline{C_c^\infty(\mathbb{R}^n)} = H^m(\mathbb{R}^n).$$

In other words, given $u \in H^m(\mathbb{R}^n)$ *there exist* $(\phi_n)_n \subset D(\mathbb{R}^n)$ *such that* $\phi_n \to \phi$ *in* $H^m(\mathbb{R}^n)$*.*

The proof uses the techniques of convolution(to get C^∞ functions) and cut off (to get compact support). The details of the proof can be found in [15].

Since we are working on the whole space $\mathbb{R}^n$, we make use of the Fourier transform and its properties to get an equivalent definition of the space $H^m(\mathbb{R}^n)$. As is pointed in [19] we define the Sobolev space $H^m(\mathbb{R}^n)$ by

$$H^m(\mathbb{R}^n) := \left\{ u \in L^2(\mathbb{R}^n) : (1+|\xi|^2)^{\frac{m}{2}} \hat{u} \in L^2(\mathbb{R}^n) \right\}, \tag{3.13}$$

and the norm

$$\|u\|_{H^m(\mathbb{R}^n)} = \|(1+|\xi|^2)^{m/2} \hat{u}\|_{L^2(\mathbb{R}^n)}, \tag{3.14}$$

where $\hat{u}$ denotes the Fourier transform of the function u. That is,

$$\hat{u}(\xi) = \int_{\mathbb{R}^n} u(x) e^{-2\pi i x\cdot\xi} dx, \quad \text{for } \xi \in \mathbb{R}^n.$$

We can also consider the case when $m = s \in \mathbb{R}$ and the previous definition remains valid.

Now we present a result which shows us that if we consider the space $H^m(\mathbb{R}^n)$ with m large enough, then the functions belong to spaces with classical derivatives.

Theorem 7. *If* $m > \frac{n}{2}$ *then* $H^m(\mathbb{R}^n)$ *is continuously embedded in* $C(\mathbb{R}^n)$*, i.e. there exist a constant* $c > 0$ *such that*

$$\|u\|_{L^\infty} \le c\|u\|_{H^m}.$$

Proof. Let $u \in H^m(\mathbb{R}^n)$, $m > \frac{n}{2}$. By means of the Fourier transform we have

$$\hat{u}(\xi) = (1+|\xi|^2)^{-\frac{m}{2}} (1+|\xi|^2)^{\frac{m}{2}} \hat{u}(\xi). \tag{3.15}$$

Since $u \in H^m(\mathbb{R}^n)$, then $(1+|\xi|^2)^{\frac{m}{2}} \hat{u}(\xi) \in L^2(\mathbb{R}^n)$. On the other hand, using polar coordinates we obtain

$$\begin{aligned} \int_{\mathbb{R}^n} \frac{d\xi}{(1+|\xi|^2)^m} &= \int_0^\infty \int_{S^{n-1}} \frac{1}{(1+|r|^2)^m} r^{n-1} d\sigma(x') dr \\ &= m(S^{n-1}) \int_0^\infty \frac{r^{n-1}}{(1+|r|^2)^m} dr, \end{aligned} \tag{3.16}$$

where $m(S^{n-1})$ is the measure of the $n-1$ unit sphere.

Note that expression (3.16) is finite if and only if

$$\begin{aligned} 2m - n + 1 &> 1 \\ m &> \frac{n}{2}. \end{aligned}$$

Namely, $(1+|\xi|^2)^{-\frac{m}{2}} \in L^2(\mathbb{R}^n)$ and using (3.15) together with the Cauchy-Schwartz inequality we have $\hat{u} \in L^1(\mathbb{R}^n)$. Thus $u = \check{\hat{u}} \in C(\mathbb{R}^n)$, since the Fourier transform of a L^1-function is continuous.

Moreover,

$$\begin{aligned}|u(x)| &= \left|\int_{\mathbb{R}^n} \hat{u}(\xi) e^{2\pi i x\cdot\xi} d\xi\right| \\ &\leq \|\hat{u}(\xi)\|_{L^1} \\ &\leq C(n,m)\|(1+|\xi|^2)^{\frac{m}{2}}\hat{u}(\xi)\|_{L^2} \\ &\leq C(n,m)\|u\|_{H^m},\end{aligned}$$

where we used the alternative definition for the norm H^m given in (3.14).

Finally, note that $\sup_{x\in\mathbb{R}^n}|u(x)|\leq c\|u\|_{H^m}$, which give us the result. □

Corollary 3. *If $k \in \mathbb{Z}^+$ and $m > \frac{n}{2} + k$, then*

$$H^m(\mathbb{R}^n) \hookrightarrow C^k(\mathbb{R}^n).$$

Proof. Since $m - k > \frac{n}{2}$, Theorem 7 gives the continuous injection,

$$H^{m-k}(\mathbb{R}^n) \hookrightarrow C(\mathbb{R}^n).$$

If $u \in H^m(\mathbb{R}^n)$, then $D^\alpha u \in H^{m-k}(\mathbb{R}^n)$, $\forall\ |\alpha|\leq k$. Moreover using the above continuous injection we have $D^\alpha u \in C(\mathbb{R}^n)$ which give us $u \in C^k(\mathbb{R}^n)$. □

Example

- In two dimensions ($n = 2$) and for $k = 1$ we have the continuous injection

  $$H^m\left(\mathbb{R}^2\right) \hookrightarrow C^1\left(\mathbb{R}^2\right), \quad m > 2,$$

  and the estimate

  $$\|\nabla u\|_{L^\infty} \leq \|u\|_{H^m}. \tag{3.17}$$

Sobolev inequalities

Once we have presented some elementary results for the space Hilbert space H^m, we deal with the general Banach space $W^{m,p}$. The space $W^{m,p}$ inherits the completeness, reflexivity and separability from L^p. The most important facts related to $W^{m,p}$ are called Sobolev embeddings. These embeddings are summarized in the following theorem. For a good treatment of Sobolev embeddings see for example [15].

Theorem 8 (Sobolev inequalities)**.** *In the whole space $\mathbb{R}^n$ we have*

$$\begin{aligned}&if\ p < n \quad W^{1,p}(\mathbb{R}^n) \subset L^q(\mathbb{R}^n),\ \forall q \in [1, p^*],\ \frac{1}{p^*} = \frac{1}{p} - \frac{1}{n}, \\ &if\ p = n \quad W^{1,p}(\mathbb{R}^n) \subset L^q(\mathbb{R}^n),\ \forall q \in [1, +\infty], \\ &if\ p > n \quad W^{1,p}(\mathbb{R}^n) \subset C(\mathbb{R}^n),\end{aligned}$$

with continuous injection, that is, $W^{1,p}(\mathbb{R}^n) \hookrightarrow E$, with E denoting the corresponding space. Moreover, there exists a constant $C > 0$ such that, for all $u \in W^{1,p}(\mathbb{R}^n)$,

$$\|u\|_E \leq C\|u\|_{W^{1,p}(\mathbb{R}^n)}.$$

The proof is presented in [15].

More general, if $k > l$ and $1 \leq p < q < \infty$ such that

$$\frac{1}{p} - \frac{k}{n} = \frac{1}{q} - \frac{l}{n}$$

then we have the continuous injection

$$W^{k,p}(\mathbb{R}^n) \hookrightarrow W^{l,q}(\mathbb{R}^n).$$

Example

- In two dimensions (n=2), for $m > 2$ we have the continuous injection

$$H^m(\mathbb{R}^2) \hookrightarrow W^{2,p}(\mathbb{R}^2),$$

provided p satisfies,

$$\frac{1-m}{2} = \frac{1-p}{p} < -\frac{1}{2},$$
$$1-p < -\frac{p}{2},$$
$$2 < p.$$

Thus, for $m > 2$ and $p > 2$ we have the estimate for $u \in H^m(\mathbb{R}^2)$,

$$\|u\|_{W^{2,p}(\mathbb{R}^2)} \leq C\|u\|_{H^m(\mathbb{R}^2)}. \tag{3.18}$$

Theorem 9 (Gagliardo-Nirenberg interpolation inequality)**.** *Let $1 \leq q, r \leq \infty$ and let $m \in \mathbb{N}$. Assume $\alpha, j \in \mathbb{N}$ are such that*

$$\frac{1}{p} = \frac{j}{n} + \left(\frac{1}{r} - \frac{m}{n}\right)\alpha + \frac{1-\alpha}{q}, \tag{3.19}$$

and

$$\frac{j}{m} \leq \alpha \leq 1.$$

Then, there exist a constant C depending only on m, n, j, q, r and α such that

$$\|D^j \mathbf{u}\|_{L^p(\mathbb{R}^n)} \leq C\|D^m \mathbf{u}\|^{\alpha}_{L^r(\mathbb{R}^n)} \|\mathbf{u}\|^{1-\alpha}_{L^q(\mathbb{R}^n)}. \tag{3.20}$$

The proof is presented in [15].

Examples

- In two dimensions ($n = 2$) and for j, p, q, r satisfying (3.19), we have

$$\|\mathbf{u}\|_{L^\infty} \leq C\|\mathbf{u}\|^{\frac{p-2}{2p-2}}_{L^2} \|\nabla \mathbf{u}\|^{\frac{p}{2p-2}}_{L^p}, \quad p > 2, \tag{3.21}$$

$$\|\nabla \mathbf{u}\|_{L^\infty} \leq C\|\nabla \mathbf{u}\|^{\frac{p-2}{2p-2}}_{L^2} \|D^2 \mathbf{u}\|^{\frac{p}{2p-2}}_{L^p}, \quad p > 2. \tag{3.22}$$

- In $\mathbb{R}^n$, for $0 < k < m$, it holds

$$\|D^k \mathbf{u}\|_{L^2} \leq C\|\mathbf{u}\|^{1-k/m}_{L^2} \|D^m \mathbf{u}\|^{k/m}_{L^2}. \tag{3.23}$$

- For $\omega = \operatorname{curl} \mathbf{u}$ in two dimensions, for $q = 2, r = 2, j = 0$ and $m > 2$, it remains true that for

$$\alpha = \frac{p-2}{(m-1)p},$$
$$1-\alpha = \frac{2}{p} + \frac{(m-2)(p-2)}{(m-1)p}.$$

Then,

$$\|\omega\|_{L^p} \leq C\|\omega\|^{\frac{2}{p}+\frac{(m-2)(p-2)}{(m-1)p}}_{L^2} \|D^{m-1}\omega\|^{\frac{p-2}{(m-1)p}}_{L^2}. \tag{3.24}$$

- For a function θ in two dimensions and $j = 1, r = 2, q = 2, m > 2$. It is true that

$$\|\nabla \theta\|_{L^p} \leq C\|D^m \theta\|^{2(1-\frac{1}{p})}_{L^2} \|\theta\|^{\frac{2}{p}-1}_{L^2}. \tag{3.25}$$

3.5 Mollifiers

Now we give an introduction to the regularization technique by mollifiers. This regularization involves convolution of a non regular function with a function having better properties. Mollifiers have the well known property of smoothing things. For further references in this way see for example [16].

We define a regularizing operator called *mollifier*, and then we show how to use this tool to answer important questions for Euler and Navier-Stokes equations. Given any radial function

$$\rho(|x|) \in C_c^\infty(\mathbb{R}^2), \quad \rho \geq 0, \quad \int_{\mathbb{R}^2} \rho dx = 1,$$

we define the mollification $\mathcal{J}_\epsilon u$ of a function $u \in L^p(\mathbb{R}^2)$, $1 \leq p \leq \infty$, by

$$(\mathcal{J}_\epsilon u)(x) = \epsilon^{-2} \int_{\mathbb{R}^2} \rho\left(\frac{x-y}{\epsilon}\right) u(y)dy, \quad \epsilon > 0. \tag{3.26}$$

We present now, some of the main properties of mollifiers.

Lemma 6 (Properties of mollifiers). *Let $u \in L^p(\mathbb{R}^2)$ and $\mathcal{J}_\epsilon$ be the mollifier defined in (3.26). Then $\mathcal{J}_\epsilon u$ is a C^∞ function and the following statements hold:*

(i) Assume u is continuous in $\mathbb{R}^2$. Then

$$\| \mathcal{J}_\epsilon u\|_{L^\infty} \leq \|u\|_{L^\infty} \quad \text{and} \quad \lim_{\epsilon \to 0} \mathcal{J}_\epsilon u = u$$

on any compact set Ω in $\mathbb{R}^2$.

(ii) Mollifiers commute with distribution derivatives,

$$D^\alpha \mathcal{J}_\epsilon u = \mathcal{J}_\epsilon D^\alpha u, \quad \forall |\alpha| \leq m, \quad u \in H^m.$$

(iii) Mollifiers commute with integrals. For every $u \in L^p$, $v \in L^q$ with $\frac{1}{p} + \frac{1}{q} = 1$, we have

$$\int_{\mathbb{R}^2} (\mathcal{J}_\epsilon u)v dx = \int_{\mathbb{R}^2} u(\mathcal{J}_\epsilon v)dx.$$

The proof is presented in [13]. (Appendix of chapter 3).

3.6 Linear transport equation

In this section we study the homogeneous linear transport problem (2.22) in the case $\Omega = \mathbb{R}^n$, $t > 0$, provided with an initial condition θ_0 and variable velocity $\mathbf{u} = \mathbf{u}(x,t)$:

$$\begin{cases} \dfrac{\partial \theta}{\partial t} + \mathbf{u} \cdot \nabla \theta = 0, & \mathbb{R}^n \times \mathbb{R}_*^+, \\ \theta(x, 0) = \theta_0(x), & x \in \mathbb{R}^n. \end{cases} \tag{3.27}$$

Here $\mathbf{u}$ represents the velocity vector field of a fluid. Moreover, instead of considering an infinitesimal volume of fluid at fixed time, we present now the concept of particle trajectories mapping which makes use of the Lagrangian coordinates (see [13] for details).

Definition (Particle trajectories mapping). *Given a fluid particle with velocity field $\mathbf{u}$, we define the particle trajectories mapping $\Psi(\cdot,t)$ as the function which takes a marker $\alpha \in \mathbb{R}^n$ and give us the location $\Psi(\alpha,t) = (x_1,\cdots,x_n)^t$ of a fluid particle at time t that was initially located at α. Moreover, this mapping $\Psi(\cdot,t)$ satisfies the following differential equation:*

$$\frac{d}{dt}\Psi(\cdot,t) = \mathbf{u}(\Psi(\cdot,t),t),$$
$$\Psi(\cdot,0) = \alpha,$$

where α is known as the Lagrangian particle marker.

Theorem 10. *Let $\theta_0 \in C^1(\mathbb{R}^n)$ and Ψ be the particle trajectories mapping associated to $\mathbf{u}$. Then the problem (3.27) has a unique solution given by*

$$\theta(t,x) = \theta_0(\Psi(-t,x))$$

Proof. Let us consider the single valued function $\psi(t) = \theta(t, \Psi(\cdot,t))$ depending on t. By using the chain rule we have:

$$\psi'(t) = \theta_t(t, \Psi(\cdot,t)) + \frac{d}{dt}\Psi(\cdot,t) \cdot \nabla\theta(t, \Psi(\cdot,t)).$$

Since $\Psi(\cdot,t)$ satisfies the differential equation of the particle trajectories mapping and $\Psi(\cdot,t)$ is a diffeomorphism such that $\Psi(\cdot,t)^{-1} = \Psi(\cdot,-t)$,we have

$$\psi(t) = \psi(0) = \theta(0,\alpha) = \theta_0(\alpha).$$

If we define $\alpha := \Psi(-t,x)$, then

$$\theta(t,x) = \theta_0(\Psi(-t,x)).$$

□

Theorem 11 (Basic estimate)**.** *If $p \in [1,\infty]$ and $\operatorname{div}\mathbf{u} \in L^1(0,T;L^\infty(\mathbb{R}^n))$, then the solution of (3.27) satisfies*

$$\|\theta(t)\|_{L^p} \leq \|\theta_0\|_{L^p}.$$

Proof. Note that the condition $\operatorname{div}\mathbf{u} \in L^1(0,T;L^\infty(\mathbb{R}^n))$ naturally implies that the flux velocity between a temporal average $[0,T]$ is finite and it is bounded spatially. So that, the assumption $\operatorname{div}\mathbf{u} \in L^1(0,T;L^\infty(\mathbb{R}^n))$ has clearly a reasonable physical support.

If $p = \infty$, using Theorem 10, we have

$$\theta(\Psi(x,t),t) = \theta_0(x).$$

Taking supremum over the trajectories we have

$$\|\theta(t)\|_{L^\infty} \leq \|\theta_0\|_{L^\infty}.$$

If $p < \infty$, multiplying by $\theta|\theta|^{p-2}$ we observe that

$$\left(\frac{\partial\theta}{\partial t} + \mathbf{u}\cdot\nabla\theta\right)\ \theta|\theta|^{p-2} = 0,$$
$$\frac{\partial}{\partial t}|\theta|^p + \mathbf{u}\cdot\nabla|\theta|^p = 0.$$

Integrating over $\mathbb{R}^n$ and using integration by parts we have

$$\frac{d}{dt}\int_{\mathbb{R}^n}|\theta|^p dx = \int_{\mathbb{R}^n}\operatorname{div}\mathbf{u}|\theta|^p dx.$$

Since $\operatorname{div}(\mathbf{u}) \in L^1(0,T;L^\infty(\mathbb{R}^n))$, it follows that

$$\begin{aligned}\frac{d}{dt}\|\theta(t)\|_{L^p} &\leq \|\operatorname{div}\mathbf{u}\|_{L^\infty}\|\theta(t)\|_{L^p}\\ &\leq c\|\theta(t)\|_{L^p}.\end{aligned}$$

Moreover, by using Grönwall's inequality we get

$$\|\theta(t)\|_{L^p} \leq c\|\theta_0\|_{L^p},$$

where c depends on $\mathbf{u}$.

□

3.7 Leray's formulation of incompressible flows

As it is pointed out on [13], the Navier-Stokes equation contains only three derivatives in time of the four unknown variables. This fact motivates the search of an equation $p = p(\mathbf{u})$ to eliminate the pressure from the Navier-Stokes equations. We recall the Navier-Stokes equations in two dimensions

$$\text{Navier-Stokes equations} \begin{cases} \dfrac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla)\mathbf{u} + \dfrac{1}{\rho}\nabla p - \nu\Delta\mathbf{u} = \dfrac{1}{\rho} f_{ex}, \\ \nabla \cdot \mathbf{u} = 0, \\ \mathbf{u}|_{t=0} = \mathbf{u_0}. \end{cases} \tag{3.28}$$

Proposition 2 (Leray's Formulation of the Navier-Stokes equations)**.** *Solving the Navier-Stokes equations* (3.28) *with smooth velocity* $\mathbf{u_0}$*,* $\operatorname{div}\mathbf{u_0} = 0$ *is equivalent to solving the equation*

$$\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla)\mathbf{u} = -\frac{1}{2\pi}\int_{\mathbb{R}^2} \frac{x-y}{|x-y|^2} \operatorname{tr}(\nabla\mathbf{u}(y,t))^2 dy + \nu\Delta\mathbf{u}, \tag{3.29}$$
$$\mathbf{u}|_{t=0} = \mathbf{u_0}.$$

The pressure $p(\cdot,\cdot)$ *can be recovered from velocity* $\mathbf{u}(\cdot,\cdot)$ *by the Poisson equation*

$$-\Delta p = \operatorname{tr}(\nabla\mathbf{u})^2, \tag{3.30}$$

where tr *denotes the trace of the corresponding matrix.*

Proof. To set notation let us assume that $\mathbf{u} \in L^2$. We introduce a subspace of L^2 which contains all vector field with null divergence in the sense of distributions. Notice that if $\mathbf{u}$ is a C^1 vector field with zero divergence, then for any test function ϕ we have

$$0 = \int_{\mathbb{R}^2} \operatorname{div}\mathbf{u}\phi dx = \int_{\mathbb{R}^2} \operatorname{div}(\mathbf{u}\phi) dx - \int_{\mathbb{R}^2} \mathbf{u} \cdot \nabla\phi dx.$$

Thus, we introduce the following subspace $E \subset L^2$, such that it contains all vector fields $\mathbf{u}$ whose divergence (in the distribution sense) vanishes.

$$E = \left\{ \mathbf{u} \in L^2 : \int_{\mathbb{R}^2} \mathbf{u} \cdot \nabla\phi dx = 0 \quad \text{for any } \phi \in D(\mathbb{R}^2) \right\}.$$

Note that if $\mathbf{u}(t) \in E \quad \forall t \geq 0$, then its partial derivative with respect to time also belongs to E for all times. Moreover, considering the condition $\operatorname{div}\mathbf{u} = 0$ into the first equation of Navier-Stokes, we observe that

$$\begin{aligned} \operatorname{div}\mathbf{u} = 0 &\implies \operatorname{div}\Delta\mathbf{u} = 0, \\ \operatorname{div}\mathbf{u} = 0 &\not\Longrightarrow \operatorname{div}(\mathbf{u} \cdot \nabla\mathbf{u}) = 0. \end{aligned}$$

And in particular,

$$\begin{aligned} \operatorname{div}(\mathbf{u} \cdot \nabla\mathbf{u}) &= \operatorname{tr}(\nabla\mathbf{u})^2 + (\mathbf{u} \cdot \nabla\mathbf{u}) \operatorname{div}\mathbf{u} \\ &= \operatorname{tr}(\nabla\mathbf{u})^2. \end{aligned}$$

We now search for a scalar function p such that the equation

$$\frac{\partial \mathbf{u}}{\partial t} = (\mathbf{u} \cdot \nabla)\mathbf{u} + \nu\Delta\mathbf{u} - \nabla p,$$

lies in the subspace E. Observing that the previous equation lies on E if the identity

$$\operatorname{div}[(\mathbf{u} \cdot \nabla)\mathbf{u} + \nabla p] = 0,$$

is fulfilled, the following Poisson equation

$$-\Delta p = \mathrm{tr}(\nabla \mathbf{u})^2, \tag{3.31}$$

has to be satisfied.

From Lemma 3 we can solve (3.31), and compute the pressure as follows,

$$p(t,x) = \int_{\mathbb{R}^2} \frac{1}{2\pi} \ln|x-y| \mathrm{tr}(\nabla \mathbf{u}(t,y))^2 dy.$$

And the gradient of p takes the form

$$\nabla p(x,t) = \frac{1}{2\pi} \nabla \ln|x| * \mathrm{tr}(\nabla \mathbf{u})^2(x).$$

Computing $\nabla \ln|x|$ we have

$$\nabla p(x,t) = \frac{1}{2\pi} \int_{\mathbb{R}^2} \frac{x-y}{|x-y|^2} \mathrm{tr}(\nabla \mathbf{u}(t,y))^2 dy,$$

Using this formula we can drop the pressure from the standard formulation of Navier-Stokes equation and we obtain a system of equations for $\mathbf{u}$ given by (3.29).

□

Remark. *Note that if $\mathbf{u}(t,\cdot) \in E$ for all $t \geq 0$, in particular we have that $\mathrm{div}\, \mathbf{u}_0 = 0$ is enough to have $\mathrm{div}\, \mathbf{u} = 0$ for all times, so the incompressiblity condition can be dropped when we are solving (3.30). That is, we have shown that we can recover the scalar pressure(up to additive constants) from a closed evolution equation for $\mathbf{u}$. In what follows we will assume that pressure can be recovered from (3.29) and we will not go into its explicit computation.*

In the solution of the Poisson equation for the pressure (3.30) we were dealing with a kernel which has a singularity at the origin i.e. this kernel is not locally integrable and we need to take into account the cancellation of the integral of N over any sphere centered at zero. Thus, we have to consider the integral over $|x| > \epsilon$, for a given $\epsilon > 0$ and then let $\epsilon \to 0$. Namely, we consider the expression

$$\lim_{\epsilon \to 0} \int_{|x|>\epsilon} N(y) f(x-y) dy, \tag{3.32}$$

which is referred as a singular integral.

The kernel N mentioned above and $\dfrac{1}{2\pi} \dfrac{x_i}{|x|}$ for $i = 1, 2.$, are typical examples of *Calderón-Zygmund kernels* which include any function $N(\cdot)+$ in $\mathbb{R}^n$ of the form

$$N(x) = \frac{P(x)}{|x|^{n+d}},$$

where $P(x)$ is a homogeneous polynomial of degree $d \geq 1$ with $\int_{|x|=1} P = 0$.

In this context we present a fundamental result by Calderón and Zygmund. This result lies outside the scope of this memory and is presented as a result that lies in the subject of singular integrals and differentiability properties of functions.

Theorem 12 (Calderón-Zygmund theory)**.** *Suppose $N(\cdot)$ is a Calderón-Zygmund kernel in $\mathbb{R}^n$. For any function $g \in L^p(\mathbb{R}^n)$, $1 < p < \infty$, the limit*

$$(Tg)(x) := \lim_{\epsilon \to 0} \int_{|x|>\epsilon} N(y) f(x-y) dy$$

converges a.e. in L^p and

$$\|Tg\|_{L^p} \leq C(n,p) \|g\|_{L^p}.$$

For a proof see for example [20, Section 3.2].

3.8 Vorticity stream formulation for Euler and Navier-Stokes equations

We turn out to the study of the vorticity associated to a fluid. Roughly speaking, vorticity is a particular form in the motion of fluid, which has its origin in the rotation of fluid elements. Physically we can say that vorticity is the quantity that characterizes the rotation of a fluid. Mathematically we define the vorticity as $\omega = \nabla \times \mathbf{u}$, where $\mathbf{u}$ is the velocity field of the fluid. In this manner, we can also define *vortex* as the region of the fluid which has high concentration of fluid rotation in comparison of its surroundings. Even when the motion of the fluid implies the origin of small vortex, this region have essential importance and it has been called by several authors as *"the voice of the fluid"*, [21].

In this section, based on [13], we present the vorticity stream formulation for the Euler and Navier-Stokes equations. We show that, in two dimensions, we can transform the Navier-Stokes equations into a self contained equation for the vorticity ω. In particular, we present the so-called vorticity-transport formula, which states that for inviscid fluid we can integrate exactly by means of the particle-trajectory mapping.

For two dimensional flows, we can transform the Navier-Stokes equations into a vorticity equation by means of the curl operator. Seen the velocity field as $\mathbf{u} = (u_1, u_2, 0)^t$, we can define the vorticity as $\omega = (0, 0, \partial_x u_2 - \partial_y u_1)^t$, which is orthogonal to $\mathbf{u}$. Thereby, taking curl operation to both sides of Navier-Stokes equations we have

$$\nabla \times \left(\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla)\, \mathbf{u} + \nabla p - \nu \Delta\, \mathbf{u} \right) = 0,$$

since the dimension is two, we denote $\omega = \text{curl } \mathbf{u} = \left(\partial_x u_2 - \partial_y u_1\right)$.

For each term

$$\begin{aligned} \nabla \times \left(\frac{\partial \mathbf{u}}{\partial t} \right) = \frac{\partial}{\partial t}\left(\partial_x u_2 - \partial_y u_1\right) &= \frac{\partial}{\partial t}\omega, \\ \nabla \times (\mathbf{u} \cdot \nabla)\, \mathbf{u} &= (\mathbf{u} \cdot \nabla)\, \omega, \\ \nabla \times \nabla p &= 0, \\ \nabla \times \nu \Delta\, \mathbf{u} &= \nu \Delta \omega, \\ \nabla \times \mathbf{u}_0 &= \omega_0. \end{aligned}$$

Putting all above computations together we get

$$\begin{cases} \dfrac{\partial \omega}{\partial t} + (\mathbf{u} \cdot \nabla)\omega = \nu \Delta \omega, \\ \qquad\qquad \omega|_{t=0} = \omega_0. \end{cases} \tag{3.33}$$

Let us note that for inviscid fluids we have $\nu = 0$. In this case we state the following result which is called the vorticity-transport in two dimensions.

Proposition 3. *Let $\Psi(\alpha, t)$ be the smooth trajectories corresponding to a divergence-free velocity field. Then for $\nu = 0$, the vorticity $\omega(\cdot, \cdot)$ in equation (3.33) satisfies*

$$\omega(\Psi(\alpha, t), t) = \omega_0(\alpha), \quad \alpha \in \mathbb{R}^2, \tag{3.34}$$

and the vorticity $\omega_0(\alpha)$ is conserved along particle trajectories for two-dimensional (2D) inviscid fluid flows.

Proof. Note that for $\nu = 0$ we have a linear transport equation in terms of ω. Then using Theorem 10 we have the result. □

In general for fluid flows in two dimensions we can render vorticity equation (3.33) into a self contained equation for ω.

Proposition 4. *For 2D flows vanishing sufficiently rapidly as $|x| \to \infty$, the Navier-Stokes equations (2.17) are equivalent to the vorticity-stream formulation,*

$$\frac{\partial \omega}{\partial t} + (\mathbf{u} \cdot \nabla)\omega = \nu \Delta \omega, \quad (x,t) \in \mathbb{R}^2 \times [0,\infty),$$
$$\omega|_{t=0} = \omega_0,$$

where the velocity $\mathbf{u}$ *is determined from the vorticity* ω *by*

$$\mathbf{u}(x,t) = \int_{\mathbb{R}^2} K_2(x-y)\omega(y,t)dy, \quad x \in \mathbb{R}^2,$$

involving the kernel

$$K_2(x) = \frac{1}{2\pi}\left(\frac{-x_2}{|x|^2}, \frac{x_1}{|x|^2}\right)^t.$$

Proof. From the condition of incompressibility we have div $\mathbf{u} = 0$. Since we are in two dimensions it implies the existence of a stream function ψ defined by

$$u_1 = -\frac{\partial \psi}{\partial y}, \quad u_2 = \frac{\partial \psi}{\partial x}.$$

Note that with the previous definition, ψ makes sense, and it is valid for all fluid flows. The equation that the stream function ψ has to satisfy comes from the vorticity equation as follows

$$\mathbf{u} = \nabla^{\perp}\psi, \qquad (3.35)$$
$$\omega = \Delta\psi,$$

where we have applied curl operation to both sides of the equation for the velocity field $\mathbf{u}$.

Thus, we have arrived to a Poisson equation $\omega = \Delta\psi$ in $\mathbb{R}^2$ in terms of the vorticity and the stream function. The fundamental solution of this Poisson problem is given by (see Lemma 3)

$$\psi(x,t) = \frac{1}{2\pi}\int_{\mathbb{R}^2} \ln|x-y|\omega(y,t)dy.$$

Since we can define ψ as the product of convolution, we can compute the gradient of ψ by differentiation under the integral. Thus, velocity can be recovered from ψ by applying orthogonal gradient to equation (3.35) as

$$\mathbf{u}(x,t) = \int_{\mathbb{R}^2} K_2(x-y)\omega(y,t)dy. \qquad (3.36)$$

Where $K_2(\cdot)$ is given by

$$K_2(x) = \frac{1}{2\pi}\left(\frac{-x_2}{|x|^2}, \frac{x_1}{|x|^2}\right)^t. \qquad (3.37)$$

□

Remark. *Note that* K_2 *(3.37) defines a Calderón-Zygmund operator and (3.36) a singular integral operator. Then, using Theorem 3 we obtain*

$$\|\nabla \mathbf{u}\|_{L^p} \leq C_p\|\omega\|_{L^p}, \quad 1 < p < \infty. \qquad (3.38)$$

Moreover if $\omega \in W^{1,p}(\mathbb{R}^2)$,

$$\|D^2\mathbf{u}\|_{L^p} \leq C_p\|\nabla\omega\|_{L^p}, \quad 1 < p < \infty. \qquad (3.39)$$

3.9 Other useful inequalities

Using the theory of Sobolev spaces and the vorticity formulation for incompressible fluids in two dimensions we present the following useful results.

Theorem 13 (Potential theory estimate). *Let* $\mathbf{u}$ *be a smooth velocity field in* $L^2(\mathbb{R}^2) \cap L^\infty(\mathbb{R}^2)$, $\operatorname{div} \mathbf{u} = 0$. *Then, for* $m > 2$

$$\|\nabla \mathbf{u}\|_{L^\infty} \leq C \left(1 + \log^+ \|\mathbf{u}\|_{H^m} + \log^+ \|\omega\|_{L^2}\right) \left(1 + \|\omega\|_{L^\infty}\right), \tag{3.40}$$

here $\log^+(x)$ *denotes* $\log(x)$ *for* $x > 1$ *and* 0 *otherwise.*

A proof is presented in [13].

Theorem 14 (Brezis-Wainger inequality). *For* $\mathbf{u} \in L^2(\mathbb{R}^2) \cap W^{1,p}(\mathbb{R}^2)$ *and* $p > 2$ *we have the inequality*

$$\| \mathbf{u} \|_{L^\infty} \leq C \left(1 + \|\nabla \mathbf{u}\|_{L^2}\right) \left[1 + \log^+ \left(\|\nabla \mathbf{u}\|_p\right)\right]^{\frac{1}{2}} + C\| \mathbf{u} \|_{L^2}. \tag{3.41}$$

Proof. For the proof see [22]. □

Proposition 5. *Let* $\mathbf{u}$ *be a solution of the Navier-Stokes equations* (2.17). *If the norms* $\|u(t)\|_{H^m}$ *and* $\|\nabla \mathbf{u}(t)\|_{L^\infty}$ *remain bounded. Then there exist a constant* c_m *such that*

$$\frac{d}{dt}\| \mathbf{u}(t)\|_{H^m}^2 \leq c_m \|\nabla \mathbf{u}(t)\|_{L^\infty} \| \mathbf{u}(t)\|_{H^m}^2. \tag{3.42}$$

Proof. The proof takes D^α of the Navier-Stokes equations and multiply it by $D^\alpha \mathbf{u}$. After we integrate over $\mathbb{R}^n$ and make use of Sobolev inequalities. A complete proof is presented in [13]. □

Proposition 6. *Let* $\theta, \mathbf{u}$ *be a solution to the Boussinesq system* (2.27) *such that the norms* $\|u(t,\cdot)\|_{H^m}$, $\|\nabla \mathbf{u}(t,\cdot)\|_{L^\infty}$ *and* $\|\theta(t,\cdot)\|_{H^m}$ *remains bounded. Then there exists* c_m *such that*

$$\frac{d}{dt}\|\theta(t)\|_{H^m} \leq c_m \|\nabla \mathbf{u}(t)\|_{L^\infty} (\| \mathbf{u}(t)\|_{H^m} + \|\theta(t)\|_{H^m}). \tag{3.43}$$

Proof. Similar to Proposition 5 we take D^α of the Boussinesq equations and multiply it by $D^\alpha \theta$. Then we integrate over $\mathbb{R}^n$ and make use of Sobolev inequalities. A complete proof is presented in [9]. □

3.10 Energy methods, stability and uniqueness for Euler and Navier-Stokes equations

In this section we give an introduction to a fundamental method for the study of PDEs, that is, we describe the concept of the energy method. Energy methods arise from questions like, [13]:

- Given an initial data. Is it possible to find a solution at any time?
- If there is a solution. Is it unique?
- Does the solution present a singularity or experiment a blow-up at finite time?
- Is there a criterion for the blow-up of smooth solutions at later times?

To answer this type of questions we construct finite estimates for physical quantities like kinetic energy

$$E := \frac{1}{2} \int_{\mathbb{R}^2} |\mathbf{u}|^2 dx$$

and we use approximation schemes associated with the equations, [13]. To be more intuitive, we consider the prototype case of Euler and Navier-Stokes equations and we show a sketchy process for the construction of its solutions. Moreover, we state the relation of the accumulation of vorticity and the existence of smooth solutions globally in time.

In a general framework, we construct energy estimates by multiplying a PDE by a function which is assumed to be smooth enough, and then integrating it over the corresponding domain. In most of the cases we use integration by parts and in the case of being working with the hole space we neglect the border terms by recalling the condition of rapidly decreasing as $|x| \to \infty$, say for example being in L^2, [13].

It is important to remark that multiplying by a smooth function and estimating its norm in a determined functional space will give the regularity of a determined function in a problem. Moreover, any estimate has to involve an appropriate initial condition, say for example, being in L^2 or having distribution derivatives in L^p.

We present an standard argument to show uniqueness of the Navier-Stokes equations. It makes use of the ideas described above.

3.10.1 Uniqueness of solutions

Let us consider the Navier-Stokes equation for an incompressible fluid in two dimensions

$$\begin{aligned} \frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla)\, \mathbf{u} + \nabla p &= \nu \Delta\, \mathbf{u}, \\ \operatorname{div} \mathbf{u} &= 0, \\ \mathbf{u}(x,0) &= \mathbf{u}_0(x). \end{aligned} \tag{3.44}$$

Theorem 15 (Uniqueness of solutions)**.** *Let $\mathbf{u}_1$ and $\mathbf{u}_2$ be solutions of the Navier-Stokes problem (3.44), having bounded energy with the same initial conditions. And if $\mathbf{u}_2$ satisfies*

$$\int_0^T \|\nabla \mathbf{u}_2\|_{L^\infty} dt < \infty, \tag{3.45}$$

then $\mathbf{u}_1(t,\cdot) = \mathbf{u}_2(t,\cdot)$ for all $t \in [0,T]$.

Proof. We assume that $\mathbf{u}_1, \mathbf{u}_2 \in L^2$ and p_1, p_2 are two solutions of (3.44) with the same initial conditions and satisfying the condition of rapidly decreasing as $|x| \to \infty$. If we make the difference of the equations satisfying the corresponding velocities , and we define $\tilde{\mathbf{u}} := \mathbf{u}_1 - \mathbf{u}_2$, $\tilde{p} = p_1 - p_2$. Then $\tilde{\mathbf{u}}$ and $\tilde{p}$ satisfy the following partial differential equation

$$\frac{\partial \tilde{\mathbf{u}}}{\partial t} + (\mathbf{u}_1 \cdot \nabla)\tilde{\mathbf{u}} + (\tilde{\mathbf{u}} \cdot \nabla)\, \mathbf{u}_2 + \nabla \tilde{p} = \nu \Delta \tilde{\mathbf{u}}. \tag{3.46}$$

Since we have $\mathbf{u}_1, \mathbf{u}_2$ having bounded energy, we take L^2 inner product of (3.46) with $\tilde{\mathbf{u}}$, that is

$$\begin{aligned} \frac{1d}{2dt}\int_{\mathbb{R}^2} \tilde{\mathbf{u}}^2 + \nu \int_{\mathbb{R}^2} |\nabla \tilde{\mathbf{u}}|^2 dx &= -\int_{\mathbb{R}^2} (\tilde{\mathbf{u}} \cdot \nabla)\, \mathbf{u}_2 \cdot \tilde{\mathbf{u}} dx, \\ \operatorname{div} \tilde{\mathbf{u}} &= 0, \end{aligned} \tag{3.47}$$

where we used integration by parts and the following facts

$$\begin{aligned} \int_{\mathbb{R}^2} (\mathbf{u}_1 \cdot \nabla)\tilde{\mathbf{u}} \cdot \tilde{\mathbf{u}} dx &= \sum_{i=1}^2 \int_{\mathbb{R}^2} \mathbf{u}_1 \cdot \nabla \left(\frac{|\tilde{\mathbf{u}}^i|}{2} \right) = \sum_{i=1}^2 \int_{\mathbb{R}^2} \operatorname{div} \left(\mathbf{u}_1 \frac{|\tilde{\mathbf{u}}^i|}{2} \right) = 0, \\ \int_{\mathbb{R}^2} \nabla \tilde{p} \cdot \tilde{\mathbf{u}} &= -\int_{\mathbb{R}^2} \tilde{p} \operatorname{div} \tilde{\mathbf{u}} = 0. \end{aligned}$$

From (3.47) we obtain

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^2} \tilde{\mathbf{u}}^2 + \nu \int_{\mathbb{R}^2} |\nabla \tilde{\mathbf{u}}|^2 dx &\le \int_{\mathbb{R}^2} |\nabla \mathbf{u}_2| |\tilde{\mathbf{u}}|^2 dx, \\ \frac{d}{dt}\|\tilde{\mathbf{u}}(t)\|_{L^2}^2 &\le 2\|\nabla \mathbf{u}_2(t)\|_{L^\infty} \|\tilde{\mathbf{u}}(t)\|_{L^2}^2, \end{aligned}$$

where we used Cauchy-Schwartz inequality.

Using Grönwall's inequality we have

$$\|\tilde{\mathbf{u}}(t)\|_{L^2}^2 \leq \|\tilde{\mathbf{u}}(0)\|_{L^2}^2 \exp\left(\int_0^t 2\|\nabla \mathbf{u}_2(s)\|_{L^\infty} ds\right).$$

Since we assume $\mathbf{u}_1$ and $\mathbf{u}_2$ having the same initial condition and using estimate (3.45), then we conclude that

$$\mathbf{u}_1(t,\cdot) = \mathbf{u}_2(t,\cdot) \quad \forall t \in [0,T].$$

□

3.10.2 Construction of solutions

Now we present an argument to show the existence of solutions to the Navier-Stokes equations, locally in time. The key step is related with the construction of finite estimates for the velocity in the $\|\cdot\|_{H^m}$ norm. The next energy estimates assume the existence of smooth solutions and then we construct a priori estimates about them. In particular based in the book by Andrew Majda, [13], we show that for smooth initial data belonging to some functional space, we can prove that solutions exists on a finite-time interval. We emphasize that the following results will be presented in a sketchy way and it does not constitute a rigorous proof for the existence of solutions to the Navier-Stokes equations.

The scheme for proving the existence of solutions is summarized as follows: first we prove existence for regularized Navier-Stokes equations, and then we make use of a priori H^m estimates to pass to the limit in the approximation scheme which give us a solution to the Navier-Stokes equations.

We present the arguments used for the construction of solutions to the Navier-Stokes equations in two steps.

Step one

Let us consider the Navier-Stokes equations

$$\begin{aligned}\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u}\cdot\nabla)\,\mathbf{u} + \nabla p &= \nu\Delta\,\mathbf{u},\\ \operatorname{div}\mathbf{u} &= 0,\\ \mathbf{u}(x,0) &= \mathbf{u}_0(x).\end{aligned}$$

By using the properties of mollifiers presented in Lemma 6 , we apply $\mathcal{J}_\epsilon$ to the above equations. After some computations we obtain

$$\begin{aligned}\mathbf{u}_t^\epsilon + \mathcal{J}_\epsilon\left[(\mathcal{J}_\epsilon\,\mathbf{u}^\epsilon)\cdot\nabla(\mathcal{J}_\epsilon\,\mathbf{u}^\epsilon)\right] &= -\nabla p^\epsilon + \nu\,\mathcal{J}_\epsilon(\mathcal{J}_\epsilon\,\Delta\,\mathbf{u}^\epsilon),\\ \operatorname{div}\mathbf{u}^\epsilon &= 0,\\ \mathbf{u}^\epsilon|_{t=0} &= \mathbf{u}_0\,.\end{aligned}$$

Using the Leray formulation for incompressible fluids, we can drop the dependence of p in the above equations. That is, we project into the space of free divergence functions. Then the regularized Navier-Stokes equations produces an ordinary differential equation in the Banach space $E \subset L^2$,

$$\begin{cases}\dfrac{d\,\mathbf{u}_t^\epsilon}{dt} = F_\epsilon(\mathbf{u}^\epsilon),\\ \mathbf{u}^\epsilon|_{t=0} = \mathbf{u}_0,\end{cases} \tag{3.48}$$

where $F_\epsilon(\mathbf{u}^\epsilon) = \nu\,\mathcal{J}_\epsilon{}^2\,\Delta\,\mathbf{u}^\epsilon - P\,\mathcal{J}_\epsilon[(\mathcal{J}_\epsilon\,\mathbf{u}^\epsilon)\cdot\nabla(\mathcal{J}_\epsilon\,\mathbf{u}^\epsilon)]$ and P denotes the Leray projection operator.

Since the right hand side of (3.48) is Lipschitz because of the presence of mollifiers, then we can use Picard's theorem for ODE's in Banach spaces (see [23]) to prove existence, uniqueness and global persistence of solutions in the space $C^1([0,T]; C^2 \cap V^m)$, with $V^m := \left\{\mathbf{u} \in H^m(\mathbb{R}^n) : \operatorname{div} \mathbf{u} = 0\right\}$.

Step two

From Proposition 5 we have for regularized solutions,

$$\frac{d}{dt}\| \mathbf{u}^\epsilon(t)\|_{H^m}^2 \leq c_m \|\nabla \mathbf{u}^\epsilon(t)\|_{L^\infty} \| \mathbf{u}^\epsilon(t)\|_{H^m}^2.$$

Using Corollary 3, provided that $m > \frac{n}{2} + 1$, we get

$$\frac{d}{dt}\| \mathbf{u}^\epsilon(t)\|_{H^m}^2 \leq c_m \| \mathbf{u}^\epsilon(t)\|_{H^m}^3.$$

Definig $z(t) := \| \mathbf{u}^\epsilon(t)\|_{H^m}^2$, the corresponding ODE takes the form:

$$z'(t) \leq c_m z(t)^{3/2}.$$

Then for an appropriate small enough interval $[0,T]$, the Grönwall inequality gives us that $\| \mathbf{u}^\epsilon(t)\|_{H^m}^2$ remains bounded.

Step three

We claim that the sequence $\mathbf{u}^\epsilon : [0,T] \to E \subset L^2$, is Cauchy when $\epsilon \to 0$. Thus for $\epsilon, \epsilon' > 0$ it is possible to show that (see [13, Section 3.2] for details)

$$\frac{d}{dt}\| \mathbf{u}^\epsilon - \mathbf{u}^{\epsilon'} \|_{L^2} \leq c(1 + \| \mathbf{u}^\epsilon \|_{H^m} + \| \mathbf{u}^{\epsilon'} \|_{H^m}) \left[(\epsilon + \epsilon') + \| \mathbf{u}^\epsilon - \mathbf{u}^{\epsilon'} \|_{L^2}\right].$$

By step two $\| \mathbf{u}^\epsilon \|_{H^m}$ is uniformly bounded for $t \in [0,T]$ and using Grönwall's inequality we obtain

$$\| \mathbf{u}^\epsilon(t) - \mathbf{u}^{\epsilon'}(t)\|_{L^2} \leq (\epsilon + \epsilon') \exp(kt), \quad t \in [0,T].$$

Passing to the limit when $\epsilon \to 0$ we conclude that $\mathbf{u}^\epsilon$ is a Cauchy sequence. Namely, there is $\mathbf{u} \in C([0,T]; L^2(\mathbb{R}^n))$ such that

$$\lim_{\epsilon \to 0} \| \mathbf{u}^\epsilon(t) - \mathbf{u}(t)\|_{L^2} = 0.$$

Step four

We claim that the derivatives $D^\alpha \mathbf{u}^\epsilon$ with $|\alpha| \leq m$ converge to the corresponding derivatives of $\mathbf{u}$. For $0 < k < m$ we make use of the Gagliardo- Nirenberg interpolation inequality (3.23) which give us

$$\lim_{\epsilon \to 0} \| \mathbf{u}^\epsilon - \mathbf{u} \|_{H^k} \leq C_k \lim_{\epsilon \to 0} \| \mathbf{u}^\epsilon - \mathbf{u} \|_{L^2}^{1-k/m} \lim_{\epsilon \to 0} \| \mathbf{u}^\epsilon - \mathbf{u} \|_{H^m}^{k/m} = 0,$$

where we used step three and the Banach[11]-Alaoglu[12] theorem, [18, Theorem 5.18], which in particular states that a bounded sequence in H^m has a convergent subsequence in H^m.

In particular, if $m > \frac{n}{2}$, we have the following uniform estimates for $t \in [0,T]$

$$\|(\mathbf{u}^\epsilon \cdot \nabla \mathbf{u}^\epsilon) - (\mathbf{u} \cdot \nabla) \mathbf{u} \|_{L^2} \to 0,$$
$$\|\Delta \mathbf{u}^\epsilon - \Delta \mathbf{u} \|_{L^2} \to 0.$$

[11] 1892-1945 Stefan Banach was a Polish mathematician who is generally considered one of the world's most important and influential 20th-century mathematicians. He was the founder of modern functional analysis.

[12] Leonidas Alaoglu was a Greek mathematician, known for his result, called Alaoglu's theorem on the weak-star compactness of the closed unit ball in the dual of a normed space.

Step five

From equation (3.48) we get

$$\mathbf{u}^{\epsilon}(t) = \mathbf{u}_0 + \int_0^t \nu\, \mathcal{J}_{\epsilon}{}^2\, \Delta\, \mathbf{u}^{\epsilon}(s) ds - \int_0^t P\, \mathcal{J}_{\epsilon}[(\mathcal{J}_{\epsilon}\, \mathbf{u}^{\epsilon}(s)) \cdot \nabla(\mathcal{J}_{\epsilon}\, \mathbf{u}^{\epsilon}(s))] ds.$$

Moreover, using step four and letting $\epsilon \to 0$, we have that there is $u \in E$ such that

$$\mathbf{u}(t) = \mathbf{u}_0 + \int_0^t \left\{ P[(\mathbf{u}(s) \cdot \nabla)\, \mathbf{u}(s)] + \nu \Delta\, \mathbf{u}(s) \right\} ds, \quad t \in [0, T]. \tag{3.49}$$

Finally, all the above steps can be summarized in the next theorem.

Theorem 16. *Let $u_0 \in H^m$ and $m > \frac{n}{2} + 2$. Then there exist a solution of the Navier-Stokes equations given by (3.49) for $t \in [0, T]$ with T sufficiently small.*

3.10.3 Accumulation of vorticity and existence of solutions

In the previous section we have constructed classical solutions for the Navier-Stokes equations. Namely, $\mathbf{u} \in C^1([0, T]; C^2 \cap V^m)$. A key feature for the proof of existence of solutions were related to the boundedness of $\|\, \mathbf{u}(\cdot, t)\|_{H^m}$. In fact, the solutions can be extended provided that $\|\, \mathbf{u}(\cdot, t)\|_{H^m}$ remains bounded. In this section we follow ideas presented in [13, Section 3.3] to show that the global existence is related with the accumulation of vorticity. We state that the maximum of H^m norm for $\mathbf{u}$ is controlled by the $L^1([0, T]; L^{\infty}(\mathbb{R}^n))$ of the vorticity.

Theorem 17 (L^{∞} Vorticity control and global existence)**.** *Let the initial velocity $\mathbf{u}_0 \in V^m$, $m > \frac{n}{2} + 2$, so that there exist a classical solution $\mathbf{u} \in C^1([0, T]; C^2 \cap V^m)$ to the 2D Navier-Stokes equation. Then:*

i) If for any $T > 0$ there exists a classical solution $\mathbf{u} \in C^1([0, T]; C^2 \cap V^m)$ such that the vorticity ω satisfies

$$\int_0^T \|\omega(\cdot, t)\|_{L^{\infty}} dt \leq M, \tag{3.50}$$

then the solution $\mathbf{u}$ exist globally in time, $\mathbf{u} \in C^1([0, \infty); C^2 \cap V^m)$.

ii) If the maximal time T of the existence of solutions $\mathbf{u} \in C^1([0, T]; C^2 \cap V^m)$ is finite, then necessarily the vorticity accumulates so rapidly that

$$\lim_{t \to T} \int_0^t \|\omega(\cdot, s)\|_{L^{\infty}} ds = \infty. \tag{3.51}$$

For a proof see [13, Theorem 3.6].

4 Blow-up criterion for the zero-viscosity-thermal Boussinesq system

In this section we mathematically study the zero viscosity-thermal Boussinesq system under the action of an external force $\mathbf{f}$ satisfying $curl(\mathbf{f}) = 0$. That is,

$$\begin{cases} \dfrac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} + \nabla p = \theta \mathbf{f}, \\ \dfrac{\partial \theta}{\partial t} + \mathbf{u} \cdot \nabla \theta = 0, \\ \nabla \cdot \mathbf{u} = 0, \\ \mathbf{u}(x,0) = u_0(x), \quad \theta(x,0) = \theta_0(x), \end{cases} \tag{4.1}$$

where θ is the scalar temperature depending on x and t, $\mathbf{u} = (u_1, u_2)$ and $(x,t) \in \mathbb{R}^2 \times (0, \infty)$.

We want to show mathematically that provided of suitable initial velocity and temperature $\mathbf{u}_0, \theta_0$, there exists a unique solution for the system (4.1) that is controlled by $\mathbf{u}_0$ and θ_0 . Moreover, we want to establish a blow-up or breakdown criterion for smooth solutions. Concretely, the solution for (4.1) remains bounded if we control the gradients of velocity and temperature. In this sense we present the following theorem

Theorem 18. *Suppose* $(u_0, \theta_0) \in H^m(\mathbb{R}^2)$, $\operatorname{div} \mathbf{u}_0 = 0$ *with* $m > 2$ *being an integer and suppose that* $\mathbf{f} \in L^\infty([0,T]; W^{m,\infty}((\mathbb{R}^2))$*. Then, there exist a unique local classical solution* $(u, \theta) \in C([0, T_1]; H^m(\mathbb{R}^2))$ *for some* $T_1 = T_1(\|u_0\|_{H^m}, \|\theta_0\|_{H^m})$*. Moreover, the solution remains in* $H^m(\mathbb{R}^2)$ *up to some time* $T > T_1$*, namely* $(u, \theta) \in C([0, T]; H^m(\mathbb{R}^2))$ *if and only if*

$$\int_0^T \|\nabla \theta(t)\|_{L^\infty} dt < \infty. \tag{4.2}$$

Proof. Existence and uniqueness of solutions follow ideas presented in Section 3.10.2. That is, we regularize the Boussinesq equations and we use properties of Sobolev spaces to show that there is a sequence converging to the solution of the Boussinesq system. A complete proof for existence and uniqueness of (4.1) is presented in [9]. The second part related to the breakdown of solutions is summarized in the next theorem. □

Theorem 19. *Let* $(\mathbf{u_0}, \theta_0) \in H^m(\mathbb{R}^2)$ *with* $\operatorname{div} \mathbf{u_0} = 0$ *for some* $m > 2$ *and suppose that* $\mathbf{f} \in L^\infty([0,T]; W^{m,\infty}(\mathbb{R}^2))$*. Then we have:*

$$\lim_{t \to T} \sup(\|\mathbf{u}(\cdot,t)\|_{H^m} + \|\theta(\cdot,t)\|_{H^m}) < \infty \quad \textit{if and only if} \quad \int_0^T \|\nabla \theta(\cdot, \tau)\|_{L^\infty} d\tau < \infty.$$

Proof. ($\Longrightarrow$) Let us assume that

$$\sup_{0 \le t \le T} (\|\mathbf{u}(\cdot,t)\|_{H^m} + \|\theta(\cdot,t)\|_{H^m}) \le C_T. \tag{4.3}$$

Assumption (4.3) implies that

$$\sup_{0 \le t \le T} \|\theta(\cdot,t)\|_{H^m} \le C_T.$$

In particular,

$$\|\theta(\cdot,t)\|_{H^m} \le C_T, \quad \forall t \in [0,T].$$

Since we are in two dimensions we can use Sobolev inequality (3.17) to get

$$\sup_{x \in \mathbb{R}^2} |\nabla \theta(\cdot,t)| \le \|\theta(\cdot,t)\|_{H^m(\mathbb{R}^2)},$$

moreover using the bound for $\|\theta(\cdot,t)\|_{H^m(\mathbb{R}^2)}$, we get:

$$\|\nabla \theta(\cdot,t)\|_{L^\infty(\mathbb{R}^2)} \le \|\theta(\cdot,t)\|_{H^m(\mathbb{R}^2)} \le C_T.$$

Integrating from 0 to T:

$$\int_0^T \|\nabla\theta(\cdot,t)\|_\infty dt \leq \int_0^T C_T dt,$$

which give us the result

$$\int_0^T \|\nabla\theta(\cdot,t)\|_\infty \leq M_T.$$

($\Longleftarrow$) Let us suppose that

$$\int_0^T \|\nabla\theta(\cdot,t)\|_\infty \leq M_T. \tag{4.4}$$

Applying the curl operator to the first equation of (4.1) and following the scheme presented in Section 3.8 we get a scalar equation in terms of the vorticity ω, namely,

$$\omega_t + \mathbf{u}\cdot\nabla\omega = (\nabla\theta \times \mathbf{f})_3\,, \tag{4.5}$$

with initial condition curl $\mathbf{u_0} = \nabla \times \mathbf{u_0} = \omega_0$ and $(\cdot)_3$ denoting the third component of a 3D vector.

Note that (4.5) can be seen as a first order equation in ω, more precisely, as a transport equation with velocity $\mathbf{u}$ and external force $(\nabla \times \mathbf{f})_3$. Thus, using Proposition 3 we obtain

$$\omega\left(\Psi_t(\alpha),t\right) = \omega_0(\alpha) + \int_0^t (\nabla\theta\times\mathbf{f})_3\left(\Psi_s(\alpha),s\right)ds, \tag{4.6}$$

where $\Psi_t(\alpha)$ is the particle trajectories mapping satisfying the following ordinary differential equation

$$\begin{aligned}&\frac{d}{dt}\Psi_t(\alpha) = \mathbf{u}\left(\Psi_t(\alpha),t\right),\\ &\Psi(\alpha)|_{t=0} = \alpha.\end{aligned}$$

Applying modulus to both sides and taking p power on (4.6) we get

$$\left|\omega\left(\Psi_t(\alpha),t\right)\right|^p = \left|\omega_0(\alpha) + \int_0^t (\nabla\theta\times\mathbf{f})_3\left(\Psi_s(\alpha),s\right)ds\right|^p \quad 1\leq p<\infty.$$

Integrating over $\mathbb{R}^2$ and taking p root,

$$\left(\int_{\mathbb{R}^2}\left|\omega\left(\Psi_t(\alpha),t\right)\right|^p dx\right)^{\frac{1}{p}} = \left(\int_{\mathbb{R}^2}\left|\omega_0(\alpha) + \int_0^t (\nabla\theta\times\mathbf{f})_3\left(\Psi_s(\alpha),s\right)ds\right|^p dx\right)^{\frac{1}{p}}. \tag{4.7}$$

Applying the classical Minkowski inequality (3.8) to the right hand side of (4.7),

$$\left(\int_{\mathbb{R}^2}\left|\omega_0(\alpha) + \int_0^t (\nabla\theta\times\mathbf{f})_3\left(\Psi_s(\alpha),s\right)ds\right|^p dx\right)^{\frac{1}{p}} \leq \left(\int_{\mathbb{R}^2}|\omega_0(\alpha)|^p\right)^{\frac{1}{p}} + \left(\int_{\mathbb{R}^2}\left|\int_0^t (\nabla\theta\times\mathbf{f})_3\left(\Psi_s(\alpha),s\right)ds\right|^p dx\right)^{\frac{1}{p}}.$$

Moreover, the Minkowski generalized inequality (3.7) gives us:

$$\left(\int_{\mathbb{R}^2}\left|\int_0^t (\nabla\theta\times\mathbf{f})_3\left(\Psi_s(\alpha),s\right)ds\right|^p dx\right)^{\frac{1}{p}} \leq \int_0^t\left(\int_{\mathbb{R}^2}\left|(\nabla\theta\times\mathbf{f})_3\left(\Psi_s(\alpha),s\right)\right|^p dx\right)^{\frac{1}{p}} ds.$$

Thus, equation (4.7) implies,

$$\left(\int_{\mathbb{R}^2}\left|\omega\left(\Psi_t(\alpha),t\right)\right|^p dx\right)^{\frac{1}{p}} \leq \left(\int_{\mathbb{R}^2}|\omega_0(\alpha)|^p\right)^{\frac{1}{p}} + \int_0^t\left(\int_{\mathbb{R}^2}\left|(\nabla\theta\times\mathbf{f})_3\left(\Psi_s(\alpha),s\right)\right|^p dx\right)^{\frac{1}{p}} ds.$$

In particular, inequality above can be written as

$$\left(\int_{\mathbb{R}^2} \left|\omega\left(\Psi_t(\alpha), t\right)\right|^p dx\right)^{\frac{1}{p}} \leq \left(\int_{\mathbb{R}^2} |\omega_0(\alpha)|^p\right)^{\frac{1}{p}} + \int_0^t \left(\int_{\mathbb{R}^2} \left|\left(\nabla^\perp\theta\cdot\mathbf{f}\right)\left(\Psi_s(\alpha), s\right)\right|^p dx\right)^{\frac{1}{p}} ds.$$

By using the Cauchy-Schwartz inequality and the assumption $\mathbf{f} \in L^\infty([0,T]; W^{m,\infty}(\mathbb{R}^2))$ we obtain

$$\begin{aligned}\left(\int_{\mathbb{R}^2} \left|\omega\left(\Psi_t(\alpha), t\right)\right|^p dx\right)^{\frac{1}{p}} &\leq \left(\int_{\mathbb{R}^2} |\omega_0(\alpha)|^p\right)^{\frac{1}{p}} + \int_0^t \left(\int_{\mathbb{R}^2} \left|\nabla^\perp\theta\left(\Psi_s(\alpha), s\right)\right|^p \left|\mathbf{f}\left(\Psi_s(\alpha), s\right)\right|^p dx\right)^{\frac{1}{p}} ds \\ &\leq \left(\int_{\mathbb{R}^2} |\omega_0(\alpha)|^p\right)^{\frac{1}{p}} + \int_0^t \left(\int_{\mathbb{R}^2} \left|\nabla^\perp\theta\left(\Psi_s(\alpha), s\right)\right|^p \sup_{x\in\mathbb{R}^2} \left|\mathbf{f}(x,s)\right|^p dx\right)^{\frac{1}{p}} ds \\ &\leq \left(\int_{\mathbb{R}^2} |\omega_0(\alpha)|^p\right)^{\frac{1}{p}} + \int_0^t \|\mathbf{f}(\cdot,s)\|_{L^\infty} \left(\int_{\mathbb{R}^2} \left|\nabla\theta\left(\Psi_s(\alpha), s\right)\right|^p dx\right)^{\frac{1}{p}} ds.\end{aligned}$$

Note that by definition of L^p norm we have

$$\|\omega(\cdot,t)\|_{L^p} \leq \|\omega_0\|_{L^p} + \int_0^T \|\nabla\theta(\cdot,s)\|_{L^p}\|\mathbf{f}(\cdot,s)\|_{L^\infty} ds. \tag{4.8}$$

On the other hand, since $\mathbf{f} \in L^\infty([0,T]; W^{m,\infty}(\mathbb{R}^2))$, it is possible to take L^∞ norm of (4.6) to get, as before,

$$\|\omega(\cdot,t)\|_{L^\infty} \leq \|\omega_0\|_{L^\infty} + \int_0^T \|\nabla\theta(\cdot,s)\|_{L^\infty}\|\mathbf{f}(\cdot,s)\|_{L^\infty} ds, \tag{4.9}$$

where $\|\omega_0(t)\|_{L^\infty}$ is finite because the Sobolev inequality (3.17) for $m > 2$ implies that

$$\|\omega_0\|_{L^\infty} \leq \|\mathbf{u_0}\|_{H^m} < \infty. \tag{4.10}$$

Moreover, by using (4.4) and (4.10) in (4.9) we obtain

$$\|\omega(\cdot,t)\|_{L^\infty} \leq C\left(\|\mathbf{u_0}\|_{H^m(\mathbb{R}^2)}, \sup_{0\leq t\leq T}\|\mathbf{f}(\cdot,s)\|_{L^\infty}, M_T\right). \tag{4.11}$$

Together with the above estimates, we can apply the orthogonal gradient $\nabla^\perp$ to second equation of (4.1) to obtain

$$\nabla^\perp\theta_t + (\mathbf{u}\cdot\nabla)\nabla^\perp\theta = \nabla\mathbf{u}\nabla^\perp\theta. \tag{4.12}$$

If we denote by $\Psi = \Psi(x,t)$ the particle trajectories associated to $\mathbf{u}$ and we evaluate (4.12) in Ψ we have

$$\frac{d}{dt}\nabla^\perp\theta\left(\Psi(x,t),t\right) = \nabla\mathbf{u}\left((\Psi(x,t),t\right)\nabla^\perp\theta\left((\Psi(x,t),t\right).$$

Integrating from 0 to t the above equation yields

$$\nabla^\perp\theta\left(\Psi(x,t),t\right) = \nabla^\perp\theta_0(x) + \int_0^t \nabla\mathbf{u}\left((\Psi(x,t),t\right)\nabla^\perp\theta\left((\Psi(x,t),t\right). \tag{4.13}$$

Taking into account the previous steps done for (4.6), we can estimate the L^p norm of $\nabla\theta(\cdot,t)$ in equation (4.13) as follows

$$\|\nabla\theta(\cdot,t)\|_{L^p} \leq \|\nabla\theta_0\|_{L^p} + \int_0^T \|\nabla\theta(\cdot,t)\|_{L^\infty}\|\nabla\mathbf{u}(\cdot,t)\|_{L^p} dt.$$

By using Calderón-Zygmund inequality (3.38) to the right hand side of the above inequality we get

$$\|\nabla\theta(\cdot,t)\|_{L^p} \leq \|\nabla\theta_0\|_{L^p} + C_p\int_0^T \|\nabla\theta(\cdot,t)\|_{L^\infty}\|\omega(\cdot,t)\|_{L^p} dt. \tag{4.14}$$

Adding (4.8) and (4.14), we obtain

$$\|\omega(\cdot,t)\|_{L^p}+\|\nabla\theta(\cdot,t)\|_{L^p} \leq \|\nabla\theta_0\|_{L^p}+\|\omega_0\|_{L^p}+C_p\int_0^T \left(\|\mathbf{f}(\cdot,s)\|_{L^\infty}+\|\nabla\theta(\cdot,s)\|_{L^\infty}\right)\left(\|\omega(\cdot,s)\|_{L^p}+\|\nabla\theta(\cdot,t)\|_{L^p}\right)ds. \tag{4.15}$$

Then using Grönwall's inequality to (4.15),

$$\begin{aligned}\|\omega(\cdot,t)\|_{L^p}+\|\nabla\theta(\cdot,t)\|_{L^p} &\leq \left(\|\omega_0\|_{L^p}+\|\nabla\theta_0\|_{L^p}\right)\exp\left[C_p\int_0^t \left(\|\mathbf{f}(\cdot,s)\|_{L^\infty}+\|\nabla\theta(\cdot,s)\|_{L^\infty}\right)ds\right]\\ &\leq C\left(\|\mathbf{u_0}\|_{H^m}+\|\theta_0\|_{H^m}\right)\exp\left[C_p\int_0^t \left(\|\mathbf{f}(\cdot,s)\|_{L^\infty}+\|\nabla\theta(\cdot,s)\|_{L^\infty}ds\right)\right]\\ &\leq C\left(\|\mathbf{u_0}\|_{H^m}+\|\theta_0\|_{H^m}\right)\exp\left[C_p\int_0^t \sup_{0\leq t\leq T}\|\mathbf{f}(\cdot,s)\|_{L^\infty}+\|\nabla\theta(\cdot,s)\|_{L^\infty}\right]\\ &\leq C\left(\|\mathbf{u_0}\|_{H^m},\|\theta_0\|_{H^m},\sup_{0\leq t\leq T}\|\mathbf{f}(\cdot,s)\|_{L^\infty},M_T,C_p\right),\end{aligned}$$

where we estimate $\|\omega_0\|_{L^p}$ and $\|\nabla\theta_0\|_{L^p}$ by means of the Gagliardo-Nirenberg interpolation inequalities (3.24) and (3.25) for $\mathbf{u}_0$ and θ_0 respectively.

From Theorem 13 we recall that

$$\|\nabla\mathbf{u}\|_{L^\infty} \leq C\left(1+\log^+\|\mathbf{u}\|_{H^m}+\log^+\|\omega\|_{L^2}\right)\left(1+\|\omega\|_{L^\infty}\right),$$

which together with estimates (4.8) and (4.11) gives

$$\|\nabla\mathbf{u}(\cdot,t)\|_{L^\infty} \leq C\left(\|\mathbf{u_0}\|_{H^m},\sup_{0\leq t\leq T}\|\mathbf{f}(\cdot,t)\|_{L^\infty},M_T,C_p\right)\left(1+\log^+\|\mathbf{u}\|_{H^m}\right). \tag{4.16}$$

Applying Proposition 5 to the first equation of (4.1) we have

$$\begin{aligned}\frac{d}{dt}\|\mathbf{u}(\cdot,t)\|_{H^m} &\leq C\left(\|\mathbf{f}(\cdot,t)\|_{W^{m,\infty}}+\|\nabla\mathbf{u}(\cdot,t)\|_{L^\infty}\right)\|\mathbf{u}(\cdot,t)\|_{H^m}\\ &\leq C\left(\|\mathbf{u_0}\|_{H^m},\|\theta_0\|_{H^m},\sup_{0\leq t\leq T}\|\mathbf{f}(\cdot,s)\|_{W^{m,\infty}},M_T,C_p\right)\left(1+\log^+\|\mathbf{u}(\cdot,t)\|_{H^m}\right)\|\mathbf{u}(\cdot,t)\|_{H^m},\end{aligned} \tag{4.17}$$

where we have used (4.16).

Note that it is possible to write (4.17) as follows

$$\frac{d}{dt}\left(1+\log^+\|\mathbf{u}\|_{H^m}\right) \leq C\left(1+\log^+\|\mathbf{u}\|_{H^m}\right),$$

and by means of Grönwall's inequality we have

$$\left(1+\log^+\|\mathbf{u}\|_{H^m}\right) \leq \left(1+\log^+\|\mathbf{u_0}\|_{H^m}\right)\exp\left(CT\right).$$

Therefore (4.17) can be written as

$$\frac{d}{dt}\|\mathbf{u}(\cdot,t)\|_{H^m} \leq C\left(1+\log^+\|\mathbf{u_0}\|_{H^m}\right)\exp\left(CT\right)\|\mathbf{u}(\cdot,t)\|_{H^m}.$$

Again, by means of Grönwall's we have

$$\|\mathbf{u}(\cdot,t)\|_{H^m} \leq \|\mathbf{u_0}\|_{H^m}\exp\left(C\left(1+\log^+\|\mathbf{u_0}\|_{H^m}\right)\exp\left(CT\right)\right) < \infty, \quad \forall\ 0<t<T$$

and thus

$$\sup_{0\leq t\leq T} \|\mathbf{u}\|_{H^m} \leq C\left(\|\mathbf{u_0}\|_{H^m}, \|\theta_0\|_{H^m}, \sup_{0\leq t\leq T} \|\mathbf{f}(\cdot,s)\|_{W^{m,\infty}}, M_T, C_p\right). \tag{4.18}$$

Taking in to account Proposition 6 together with the previous estimates (4.16) and (4.18) we can use Grönwall's inequality to get

$$\begin{aligned}
\frac{d}{dt}\|\theta(t)\|_{H^m} &\leq C\|\nabla\,\mathbf{u}(t)\|_{L^\infty}(\|\,\mathbf{u}(t)\|_{H^m} + \|\theta(t)\|_{H^m}),\\
\frac{d}{dt}\|\theta(\cdot,t)\|_{H^m} &\leq C\|\theta(\cdot,t)\|_{H^m},\\
\|\theta(\cdot,t)\|_{H^m} &\leq \|\theta_0\| \exp{(CT)} < \infty, \quad \forall\ 0<t<T,\\
\sup_{0\leq t\leq T}\|\theta(\cdot,t)\|_{H^m} &\leq C\left(\|\mathbf{u_0}\|_{H^m}, \|\theta_0\|_{H^m}, \sup_{0\leq t\leq T} \|\mathbf{f}(\cdot,s)\|_{W^{m,\infty}}, M_T, C_p\right).
\end{aligned} \tag{4.19}$$

Finally, using estimates (4.18) and (4.19) we obtain

$$\lim_{t\to T}\sup(\|\mathbf{u}(\cdot,t)\|_{H^m} + \|\theta(\cdot,t)\|_{H^m}) < \infty.$$

□

5 Global in time regularity for the zero-thermal Boussinesq system

We consider the 2D zero-thermal Boussinesq system (2.29). This system can be seen as a coupling between the Navier-Stokes equations and a linear transport equation:

$$\begin{cases} \dfrac{\partial \mathbf{u}}{\partial t} + (\mathbf{u}\cdot\nabla)\,\mathbf{u} + \nabla p = \theta e_2 + \nu\Delta\,\mathbf{u}, \\ \dfrac{\partial \theta}{\partial t} + (\mathbf{u}\cdot\nabla)\theta = 0, \\ \operatorname{div}\mathbf{u} = 0, \\ \mathbf{u}(x,0) = \mathbf{u}_0(x), \quad \theta(x,0) = \theta_0(x). \end{cases} \tag{5.1}$$

We want to prove mathematically that provided of suitable initial data, there exists a unique solution that can be extended to an arbitrary time $T > 0$. The fact that the solution of the zero-thermal Boussinesq system can be continued until any time give us what is know as the global in time regularity. The global in time regularity for the 2D zero-thermal Boussinesq system is summarized in the next theorem.

Theorem 20. *Let $\nu > 0$ be fixed, and $\operatorname{div}\mathbf{u}_0 = 0$. Let $m > 2$ be an integer and $(\mathbf{u}_0, \theta_0) \in H^m(\mathbb{R}^2)$. Then, there exists a unique solution (u, θ) with $\theta \in C([0,\infty); H^m(\mathbb{R}^2))$ and $\mathbf{u} \in C([0,\infty); H^m(\mathbb{R}^2)) \cap L^2(0,T; H^{m+1}(\mathbb{R}^2))$ of the system (5.1). Moreover, for each $s < m$, the solutions $(\mathbf{u}, \theta)$ of (2.26) converges to the corresponding solutions of (5.1) in $C([0,T]; H^s(\mathbb{R}^2))$ as $k \to 0$.*

Taking into account the proof of Theorem 18 we can infer that a similar conclusion holds for system (5.1) when viscosity is included, [10]. Thus, for the proof of the global-in-time regularity, it suffices to prove that the L^∞ norm of the gradient of the temperature remains bounded for any $T \in (0,\infty)$ for the solutions $(\mathbf{u}, \theta)$ of (5.1). Namely, we have to prove that, for any $T > 0$,

$$\int_0^T \|\nabla\theta(\cdot,t)\|_{L^\infty}\,dt < \infty,$$

remains true.

This section is dedicated to the proof of Theorem 20

5.1 Preliminary estimates

Let $T > 0$ be a fixed time. From the second equation of (5.1) we have

$$\frac{\partial \theta}{\partial t} + \mathbf{u}\cdot\nabla\theta = 0. \tag{5.2}$$

We recall the following standard result for the linear transport equation (5.2) (see Theorem 11)

$$\|\theta(t)\|_{L^p} \le \|\theta_0\|_{L^p} \quad \forall t \in [0,T],\ p \in [1,\infty]. \tag{5.3}$$

Now, we consider the first equation of (5.1) and we take L^2 inner product with $\mathbf{u}$. For each term we have

$$\begin{aligned} \left(\frac{\partial \mathbf{u}}{\partial t}, \mathbf{u}\right)_{L^2} &= \int_{\mathbb{R}^2} \frac{\partial \mathbf{u}}{\partial t}\cdot \mathbf{u}\,dx, \\ &= \int_{\mathbb{R}^2} \left[\frac{1}{2}\frac{d}{dt}(u_1)^2 + \frac{1}{2}\frac{d}{dt}(u_2)^2\right] dx, \\ &= \frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^2} |\mathbf{u}|^2\,dx, \\ &= \frac{1}{2}\frac{d}{dt}\|\mathbf{u}(t)\|_{L^2}^2. \end{aligned} \tag{5.4}$$

For the nonlinear term we get, after integration by parts

$$\begin{aligned}
\big((\mathbf{u}\cdot\nabla)\,\mathbf{u},\mathbf{u}\big)_{L^2} &= \int_{\mathbb{R}^2} (\mathbf{u}\cdot\nabla)\,\mathbf{u}\cdot\mathbf{u}dx \\
&= \int_{\mathbb{R}^2} \big(u_1\partial_x u_1 u_1 + u_2\partial_y u_1 u_1 + u_1\partial_x u_2 u_2 + u_2\partial_y u_2 u_2\big)\,dx \\
&= \int_{\mathbb{R}^2} (u_1\frac{\partial_x}{2}|\mathbf{u}|^2{+}u_2\frac{\partial_y}{2}|\mathbf{u}|^2)dx \\
&= \frac{1}{2}\int_{\mathbb{R}^2} \mathbf{u}\cdot\nabla|\mathbf{u}|^2)dx \\
&= -\frac{1}{2}\int_{\mathbb{R}^2} \text{div }\mathbf{u}|\mathbf{u}|^2dx \\
&= 0.
\end{aligned} \tag{5.5}$$

For the gradient of the pressure, we find after integration by parts,

$$\begin{aligned}
(\nabla p,\mathbf{u})_{L^2} &= \int_{\mathbb{R}^2} \nabla p\cdot\mathbf{u}dx \\
&= -\int_{\mathbb{R}^2} \text{div}\mathbf{u}\;pdx \\
&= 0.
\end{aligned} \tag{5.6}$$

For the viscosity-diffusion term, using integration by parts

$$\begin{aligned}
(\nu\Delta\mathbf{u},\mathbf{u})_2 &= \int_{\mathbb{R}^2} \nu\Delta\mathbf{u}\cdot\mathbf{u} \\
&= \nu\int_{\mathbb{R}^2} \Big(u_1\partial_x^2 u_1 + u_1\partial_y^2 u_1 + u_2\partial_x^2 u_2 + u_2\partial_y^2 u_2\Big)\,dx \\
&= -\nu\int_{\mathbb{R}^2} \Big[(\partial_x u_1)^2 + \big(\partial_y u_1\big)^2 + (\partial_x u_2)^2 + \big(\partial_y u_2\big)^2\Big]\,dx \\
&= -\nu\int_{\mathbb{R}^2} (|\nabla u_1|^2{+}|\nabla u_2|^2)dx \\
&= -\nu\int_{\mathbb{R}^2} |\nabla\mathbf{u}|^2\,dx \\
&= -\nu\|\nabla\mathbf{u}(t)\|_{L^2}^2.
\end{aligned} \tag{5.7}$$

Using Hölder's inequality we obtain

$$\begin{aligned}
(\theta e_2,\mathbf{u})_{L^2} &= \int_{\mathbb{R}^2} \theta u_2 dx, \\
&\le \left(\int_{\mathbb{R}^2} \theta^2 dx\right)^{1/2}\left(\int_{\mathbb{R}^2} (u_2)^2\right)^{1/2} dx, \\
&\le \left(\int_{\mathbb{R}^2} |\theta|^2\,dx\right)^{1/2}\left(\int_{\mathbb{R}^2} |\mathbf{u}|^2 dx\right)^{1/2} \\
&\le \|\theta(t)\|_{L^2}\|\mathbf{u}(t)\|_{L^2}.
\end{aligned} \tag{5.8}$$

Joining (5.4),(5.5), (5.6) , (5.7) and (5.8) we have

$$\frac{1}{2}\frac{d}{dt}\|\mathbf{u}\|_{L^2}^2 + \nu\|\nabla\mathbf{u}\|_{L^2}^2 \le \|\theta\|_{L^2}\|\mathbf{u}\|_{L^2}.$$

Using estimate (5.3) for $p = 2$ in the above equation we get

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}\|\mathbf{u}\|_{L^2}^2 &\le \|\theta_0\|_{L^2}\|\mathbf{u}\|_{L^2}, \\
\frac{d}{dt}\|\mathbf{u}\|_{L^2} &\le \|\theta_0\|_{L^2}.
\end{aligned} \tag{5.9}$$

Integration from 0 to t equation (5.9) yields

$$\begin{aligned}\|\mathbf{u}(t)\|_{L^2} &\leq \|\mathbf{u}_0\|_{L^2} + \|\theta_0\|_{L^2} t \\ &\leq \|\mathbf{u}_0\|_{L^2} + \|\theta_0\|_{L^2} T, \quad \forall t \in [0,T].\end{aligned} \tag{5.10}$$

Now we transform the first equation of the Boussinesq system (5.1) into an equivalent vorticity formulation as was presented in Proposition 4:

$$\omega_t + (\mathbf{u}\cdot\nabla)\,\omega = \partial_x\theta + \nu\Delta\omega, \tag{5.11}$$

where $\omega = \partial_x u_2 - \partial_y u_1$.

We are going to construct a L^p estimate for the vorticity ω by multiplying equation (5.11) by $\omega\,|\omega|^{p-2}$ and integrating it over $\mathbb{R}^2$. After integration by parts for each term and $p \geq 2$ we get

$$\int_{\mathbb{R}^2} \omega_t \omega\,|\omega|^{p-2}\,dx = \frac{1}{p}\int_{\mathbb{R}^2} \frac{d}{dt}\,|\omega|^p\,dx. \tag{5.12}$$

$$\begin{aligned}\int_{\mathbb{R}^2} (\mathbf{u}\cdot\nabla)\,\omega\omega\,|\omega|^{p-2}\,dx &= \frac{1}{p}\int_{\mathbb{R}^2} \mathbf{u}\cdot\nabla\,|\omega|^p\,dx \\ &\overset{I.P.}{=} -\frac{1}{p}\int_{\mathbb{R}^2} \operatorname{div}\mathbf{u}\ \ |\omega|^p\,dx \\ &= 0.\end{aligned} \tag{5.13}$$

$$\begin{aligned}\int_{\mathbb{R}^2} \partial_x\theta\ \omega\,|\omega|^{p-2}\,dx &\overset{I.P.}{=} -\int_{\mathbb{R}^2} \theta\partial_x(\omega|\omega|^{p-2})dx \\ &= -(p-1)\int_{\mathbb{R}^2} \theta\ \partial_x\omega\,|\omega|^{p-2}\,dx.\end{aligned} \tag{5.14}$$

$$\begin{aligned}\int_{\mathbb{R}^2} \nu\Delta\omega\omega\,|\omega|^{p-2}\,dx &\overset{I.P.}{=} -\nu\int_{\mathbb{R}^2} \nabla\omega\cdot\nabla(\omega\,|\omega|^{p-2})dx \\ &= -\nu\int_{\mathbb{R}^2} \nabla\omega\cdot(\nabla\omega\,|\omega|^{p-2} + (p-2)\,|\omega|^{p-2}\,\nabla\omega) \\ &= -(p-1)\nu\int_{\mathbb{R}^2} |\nabla\omega|^2\,|\omega|^{p-2}\,dx.\end{aligned} \tag{5.15}$$

Combining (5.12), (5.13), (5.14) and (5.15) we can write (5.11) as follows

$$\begin{aligned}\frac{1}{p}\frac{d}{dt}\int_{\mathbb{R}^2} |\omega|^p\,dx + (p-1)\nu\int_{\mathbb{R}^2} |\nabla\omega|^2\,|\omega|^{p-2}\,dx &= -(p-1)\int_{\mathbb{R}^2} \theta\ \partial_x\omega\,|\omega|^{p-2}\,dx, \\ &\leq \frac{(p-1)}{2\nu}\int_{\mathbb{R}^2} \theta^2\,|\omega|^{p-2}\,dx + \frac{(p-1)\nu}{2}\int_{\mathbb{R}^2} (\partial_x\theta)^2\,|\omega|^{p-2}\,dx \\ &\leq \frac{(p-1)}{2\nu}\int_{\mathbb{R}^2} |\theta|^2|\omega|^{p-2}\,dx + \frac{(p-1)\nu}{2}\int_{\mathbb{R}^2} |\nabla\omega|^2\,|\omega|^{p-2}\,dx,\end{aligned}$$

where we have used Young's inequality

$$ab \leq \frac{1}{2\nu}a^2 + \frac{\nu}{2}b^2 \text{ for } \nu > 0.$$

Note also that using Hölder inequality for $\hat{p} = \frac{p}{2}, \hat{q} = \frac{p}{p-2}$ such that $\frac{1}{\hat{p}} + \frac{1}{\hat{q}} = 1$ we have

$$\int_{\mathbb{R}^2} \theta^2\,|\omega|^{p-2}\,dx \leq \|\theta\|_{L^p}^2\|\omega\|_{L^p}^{p-2}.$$

Replacing in the above equation we find that

$$\begin{aligned}
&\frac{1}{p}\frac{d}{dt}\int_{\mathbb{R}^2}|\omega|^p\,dx+(p-1)\nu\int_{\mathbb{R}^2}|\nabla\omega|^2\,|\omega|^{p-2}\,dx\leq\frac{(p-1)\nu}{2}\int_{\mathbb{R}^2}|\nabla\omega|^2\,|\omega|^{p-2}\,dx+\frac{(p-1)}{2\nu}\|\theta\|_{L^p}^2\|\omega\|_{L^p}^{p-2},\\
&\frac{1}{p}\frac{d}{dt}\|\omega\|_{L^p}^p dx+\frac{(p-1)\nu}{2}\int_{\mathbb{R}^2}|\nabla\omega|^2\,|\omega|^{p-2}\,dx\leq\frac{(p-1)}{2\nu}\|\theta\|_{L^p}^2\|\omega\|_{L^p}^{p-2}.
\end{aligned}\tag{5.16}$$

In particular for $p=2$ and using initial estimate (5.3) we obtain

$$\frac{d}{dt}\|\omega\|_{L^2}^2+\nu\|\nabla\omega\|_{L^2}^2\leq\frac{1}{\nu}\|\theta_0\|_{L^2}^2.$$

Integrating from 0 to t yields, for $t\in[0,T]$

$$\begin{aligned}
\|\omega(t)\|_{L^2}^2-\|\omega_0\|_{L^2}^2+\nu\int_0^t\|\nabla\omega(s)\|_{L^2}^2ds&\leq\frac{1}{\nu}\|\theta_0\|_{L^2}^2t,\\
\|\omega(t)\|_{L^2}^2-\|\omega_0\|_{L^2}^2+\nu\int_0^T\|\nabla\omega(t)\|_{L^2}^2dt&\leq\frac{1}{\nu}\|\theta_0\|_{L^2}^2T,\quad\forall t\in[0,T],\\
\|\omega(t)\|_{L^2}^2+\nu\int_0^T\|\nabla\omega(t)\|_{L^2}^2dt&\leq\|\omega_0\|_{L^2}^2+\frac{1}{\nu}\|\theta_0\|_{L^2}^2T\quad\forall t\in[0,T].
\end{aligned}$$

We find that, by Hölder inequality,

$$\int_0^T\|\nabla\omega(s)\|_{L^2}ds\leq C\sqrt{T}\left(\int_0^T\|\nabla\omega(s)\|_{L^2}^2ds\right)^{\frac{1}{2}},$$

and noting that

$$\left(\int_0^T\|\nabla\omega(s)\|_{L^2}^2ds\right)^{\frac{1}{2}}\leq C(\|\omega_0\|_{L^2}+\frac{1}{\sqrt{\nu}}\|\theta_0\|_{L^2}\sqrt{T})\quad\forall t\in[0,T],\tag{5.17}$$

we get

$$\int_0^T\|\nabla\omega(s)\|_{L^2}ds\leq C\|\omega_0\|_{L^2}\sqrt{T}+C\|\theta_0\|_{L^2}T\quad\forall t\in[0,T].\tag{5.18}$$

Note that expression (5.17) give us a priori that $\mathbf{u}$ belongs to $L^2(0,T;H^{m+1}(\mathbb{R}^2))$. This fact will be clarified in the outcoming steps.

On the other hand, from (5.16), we have for $p\in[2,\infty)$

$$\begin{aligned}
\frac{1}{p}\frac{d}{dt}\|\omega(t)\|_{L^p}^p&\leq\frac{(p-1)}{2\nu}\|\theta_0\|_{L^p}^2\|\omega(t)\|_{L^p}^{p-2},\\
\|\omega(t)\|_{L^p}^{p-2}\frac{d}{dt}\|\omega(t)\|_{L^p}^2&\leq\frac{(p-1)}{\nu}\|\theta_0\|_{L^p}^2\|\omega(t)\|_{L^p}^{p-2},\\
\frac{d}{dt}\|\omega(t)\|_{L^p}^2&\leq\frac{(p-1)}{\nu}\|\theta_0\|_{L^p}^2.
\end{aligned}$$

Integration from 0 to t gives

$$\begin{aligned}
\|\omega(t)\|_{L^p}^2&\leq\|\omega_0\|_{L^p}^2+\frac{(p-1)}{\nu}\|\theta_0\|_{L^p}^2t,\\
\|\omega(t)\|_{L^p}^2&\leq\|\omega_0\|_{L^p}^2+\frac{(p-1)}{\nu}\|\theta_0\|_{L^p}^2T\quad\forall t\in[0,T]\\
&\leq\left(\|\omega_0\|_{L^p}+\frac{\sqrt{p-1}}{\sqrt{\nu}}\|\theta_0\|_{L^p}\sqrt{T}\right)^2\quad\forall t\in[0,T],
\end{aligned}$$

so that,

$$\|\omega(t)\|_{L^p} \leq \|\omega_0\|_{L^p} + \frac{\sqrt{p-1}}{\sqrt{\nu}}\|\theta_0\|_{L^p}\sqrt{T} \quad \forall t \in [0,T], \quad p \in [2,\infty). \tag{5.19}$$

Now we recall the following Gagliardo-Nirenberg interpolation inequality (3.22) in $\mathbb{R}^2$:

$$\| \mathbf{u} \|_{L^\infty} \leq C \| \mathbf{u} \|_{L^2}^{\frac{p-2}{2p-2}} \| D \mathbf{u} \|_{L^p}^{\frac{p}{2p-2}}, \quad \mathbf{u} \in W^{1,p}(\mathbb{R}^2), \quad p > 2. \tag{5.20}$$

Thus using inequality (5.20) and the Calderón-Zygmund inequality (3.38) combined with estimates (5.10) and (5.19) for $p \in (2,\infty)$ we find

$$\begin{aligned}\|\mathbf{u}(t)\|_{L^\infty} &\leq \|\mathbf{u}(t)\|_{L^2}^{\frac{p-2}{2p-2}} \|\nabla \mathbf{u}(t)\|_{L^p}^{\frac{p}{2p-2}} \\ &\leq C\|\mathbf{u}(t)\|_{L^2}^{\frac{p-2}{2p-1}} \|\omega(t)\|_{L^p}^{\frac{p}{2p-2}} \\ &\leq C\left(\mathbf{u_0}, \theta_0, T, \nu, p\right) \quad \forall t \in [0,T].\end{aligned} \tag{5.21}$$

So far, L^2 estimate for $\mathbf{u}$ and L^p estimate for ω allow us to use Gagliardo-Nirenberg and Calderón-Zygmund inequalities such that $\mathbf{u}$ remains uniformly bounded. Namely,$\| \mathbf{u} \|_{L^\infty} \leq C$. In the next section we will see how the estimate for $\| \mathbf{u} \|_{L^\infty}$ allows us to bound $\|\nabla \mathbf{u} \|_{L^\infty}$ such that $\|\nabla\theta(t)\|_{L^p} < \infty$ remains finite.

5.2 $W^{2,p}$ estimate for u

We take derivative operation $D = \left(\partial_x, \partial_y\right)$ on the vorticity equation (5.11), and then we take L^2 inner product with $D\omega\,|D\omega|^{p-2}$ for $p > 2$. For each term, after integration by parts, we get

$$\begin{aligned}\left(D\omega_t, D\omega\,|D\omega|^{p-2}\right)_{L^2} &= \int_{\mathbb{R}^2} D\omega_t \cdot D\omega\,|D\omega|^{p-2}\,dx \\ &= \frac{1}{p}\frac{d}{dt}\int_{\mathbb{R}^2} |D\omega|^p\,dx \\ &= \frac{1}{p}\frac{d}{dt}\|D\omega(t)\|_{L^p}^p.\end{aligned}$$

$$\begin{aligned}\left(D\left(\mathbf{u}\cdot\nabla\right)\omega, D\omega\,|D\omega|^{p-2}\right)_{L^2} &= \int_{\mathbb{R}^2} \left[D\left(\mathbf{u}\cdot\nabla\right)\omega\right]\cdot D\omega\,|D\omega|^{p-2}\,dx \\ &= \int_{\mathbb{R}^2} [(\mathbf{u}\cdot\nabla)\,\omega]D\cdot(D\omega\,|D\omega|^{p-2})dx \\ &= -\left(p-1\right)\int_{\mathbb{R}^2} [(\mathbf{u}\cdot\nabla)\,\omega]|D\omega|^{p-2}D^2\omega dx.\end{aligned}$$

$$\begin{aligned}\left(D\theta_x, D\omega\,|D\omega|^{p-2}\right)_{L^2} &= \int_{\mathbb{R}^2} D\theta_x \cdot D\omega\,|D\omega|^{p-2}\,dx \\ &= \int_{\mathbb{R}^2} \theta_x D\cdot(D\omega\,|D\omega|^{p-2})dx \\ &= -\left(p-1\right)\int_{\mathbb{R}^2} \theta_x\,|D\omega|^{p-2}\,D^2\omega dx.\end{aligned}$$

$$\begin{aligned}\left(\nu D\Delta\omega, D\omega\,|D\omega|^{p-2}\right)_{L^2} &= \nu\int_{\mathbb{R}^2} D\Delta\omega\cdot D\omega\,|D\omega|^{p-2}\,dx \\ &= \nu\int_{\mathbb{R}^2} \Delta\omega D\cdot(D\omega\,|D\omega|^{p-2})dx \\ &= -\left(p-1\right)\nu\int_{\mathbb{R}^2} |D^2\omega|^2|D\omega|^{p-2}\,dx.\end{aligned}$$

Collecting the previous results, we get for $p > 2$

$$\frac{1}{p}\frac{d}{dt}\|D\omega\|_{L^p}^p + (p-1)\nu\int_{\mathbb{R}^2}|D^2\omega|^2|D\omega|^{p-2}\,dx = (p-1)\int_{\mathbb{R}^2}\left[(\mathbf{u}\cdot\nabla)\,\omega\right]|D\omega|^{p-2}\,D^2\omega dx - (p-1)\int_{\mathbb{R}^2}\theta_x\,|D\omega|^{p-2}\,D^2\omega dx.$$

Using the inequality $ab \leq \frac{\nu}{4}a^2 + \frac{1}{\nu}b^2$ in the right hand side of the preceding equation,

$$\begin{aligned}
&\frac{1}{p}\frac{d}{dt}\|D\omega\|_{L^p}^p + (p-1)\nu\int_{\mathbb{R}^2}|D^2\omega|^2|D\omega|^{p-2}\,dx\\
&\leq \frac{(p-1)\nu}{4}\int_{\mathbb{R}^2}|D^2\omega|^2|D\omega|^{p-2}dx + \frac{(p-1)}{\nu}\int_{\mathbb{R}^2}|\mathbf{u}(x)|^2|D\omega|^p dx\\
&+ \frac{(p-1)\nu}{4}\int_{\mathbb{R}^2}|D^2\omega|^2|D\omega|^{p-2}dx + \frac{(p-1)}{\nu}\int_{\mathbb{R}^2}|\nabla\theta|^2|D\omega|^{p-2}dx.
\end{aligned}$$

Grouping similar terms to the left hand side we obtain

$$\begin{aligned}
\frac{1}{p}\frac{d}{dt}\|D\omega\|_{L^p}^p + \frac{(p-1)\nu}{2}\int_{\mathbb{R}^2}|D^2\omega|^2|D\omega|^{p-2}\,dx &\leq \frac{(p-1)}{\nu}\int_{\mathbb{R}^2}|\mathbf{u}(x)|^2|D\omega|^p dx + \frac{(p-1)}{\nu}\int_{\mathbb{R}^2}|\nabla\theta|^2|D\omega|^{p-2}dx\\
&\leq \frac{(p-1)}{\nu}\int_{\mathbb{R}^2}\sup_{x\in\mathbb{R}^2}|\mathbf{u}(x)|^2|D\omega|^p dx + \frac{(p-1)}{\nu}\int_{\mathbb{R}^2}|\nabla\theta|^2|D\omega|^{p-2}dx\\
&\leq \frac{(p-1)}{\nu}\int_{\mathbb{R}^2}\|\mathbf{u}\|_{L^\infty}^2\|D\omega\|_{L^p}^p + \frac{2(p-1)}{p\nu}\|\nabla\theta\|_{L^p}^p + \frac{(p-1)(p-2)}{p\nu}\|D\omega\|_{L^p}^p,
\end{aligned}$$

where, in the last line, we used Young's inequality

$$a^2b^{p-2} \leq \frac{2}{p}a^p + \frac{p-2}{p}b^p \text{ for } p \geq 2.$$

Taking in to account estimate of $\|\mathbf{u}\|_{L^\infty}$ in (5.21), we get that

$$\begin{aligned}
\frac{1}{p}\frac{d}{dt}\|D\omega\|_{L^p}^p &\leq C\|D\omega\|_{L^p}^p + C\|\nabla\theta\|_{L^p}^p + C\|D\omega\|_{L^p}^p\\
\frac{d}{dt}\|D\omega\|_{L^p}^p &\leq C\|D\omega\|_{L^p}^p + C\|\nabla\theta\|_{L^p}^p \quad \forall t\in[0,T],
\end{aligned} \tag{5.22}$$

where $C(\|\,\mathbf{u}_0\,\|_{W^{1,p}}, \|\theta_0\|_{L^p}, T, \nu, p)$.

Now we take orthogonal gradient in the second equation of (5.1), similarly to (4.12) we obtain

$$\nabla^\perp\theta_t + (\mathbf{u}\cdot\nabla)\,\nabla^\perp\theta = \nabla\mathbf{u}\nabla^\perp\theta.$$

Taking L^2 inner product with $\nabla^\perp\theta|\nabla^\perp\theta|^{p-2}$, we deduce, for each term

$$\begin{aligned}
\left(\nabla^\perp\theta_t, \nabla^\perp\theta|\nabla^\perp\theta|^{p-2}\right)_{L^2} &= \int_{\mathbb{R}^2}\nabla^\perp\theta_t\cdot\nabla^\perp\theta|\nabla^\perp\theta|^{p-2}dx\\
&= \frac{1}{p}\frac{d}{dt}\int_{\mathbb{R}^2}|\nabla\theta|^p dx.
\end{aligned}$$

$$\begin{aligned}
\left((\mathbf{u}\cdot\nabla)\,\nabla^\perp\theta, \nabla^\perp\theta|\nabla^\perp\theta|^{p-2}\right)_{L^2} &= \int_{\mathbb{R}^2}(\mathbf{u}\cdot\nabla)\,\nabla^\perp\theta\cdot\nabla^\perp\theta|\nabla^\perp\theta|^{p-2}dx\\
&= \frac{1}{p}\int_{\mathbb{R}^2}\mathbf{u}\cdot\nabla|\nabla\theta|^p dx\\
&= -\frac{1}{p}\int_{\mathbb{R}^2}\text{div }\mathbf{u}|\nabla\theta|^p dx\\
&= 0.
\end{aligned}$$

$$\begin{aligned}\left(\nabla\mathbf{u}\nabla^{\perp}\theta,\nabla^{\perp}\theta|\nabla^{\perp}\theta|^{p-2}\right)_{L^2} &= \int_{\mathbb{R}^2}\nabla\mathbf{u}\nabla^{\perp}\theta\cdot\nabla^{\perp}\theta|\nabla^{\perp}\theta|^{p-2}dx\\ &\le\left|\int_{\mathbb{R}^2}\nabla\mathbf{u}\nabla^{\perp}\theta\cdot\nabla^{\perp}\theta|\nabla^{\perp}\theta|^{p-2}dx\right|\\ &\le\int_{\mathbb{R}^2}|\nabla\mathbf{u}||\nabla\theta|^p dx.\end{aligned}$$

Bringing together the above computations we find

$$\begin{aligned}\frac{1}{p}\frac{d}{dt}\int_{\mathbb{R}^2}|\nabla\theta|^p dx &\le \int_{\mathbb{R}^2}|\nabla\mathbf{u}(x)||\nabla\theta|^p dx,\\ \frac{d}{dt}\|\nabla\theta\|_{L^p}^p &\le p\int_{\mathbb{R}^2}\sup_{x\in\mathbb{R}^2}|\nabla\mathbf{u}(x)||\nabla\theta|^p dx\\ &\le p\|\nabla\mathbf{u}\|_{L^\infty}\|\nabla\theta\|_{L^p}^p.\end{aligned}\tag{5.23}$$

Note that for $\mathbf{u}\in L^2\left(\mathbb{R}^2\right)\cap W^{1,p}(\mathbb{R}^2)$ and $p>2$ we can use Brezis-Wainger inequality (3.41) presented in Theorem 14. Then we have

$$\begin{aligned}\|\nabla\mathbf{u}\|_{L^\infty} &\le C\left(1+\|D^2\mathbf{u}\|_{L^2}\right)\left[1+\log^+\left(\|D^2\mathbf{u}\|_p\right)\right]^{\frac{1}{2}}+C\|\nabla\mathbf{u}\|_{L^2}\\ &\le C(1+\|\nabla\mathbf{u}\|_{L^2}+\|D^2\mathbf{u}\|_{L^2})\left[1+\log^+\left(\|D^2\mathbf{u}\|_p\right)\right]^{\frac{1}{2}}\\ &\le C(1+\|\nabla\mathbf{u}\|_{L^2}+\|D^2\mathbf{u}\|_{L^2})\left[1+\log^+\left(\|D^2\mathbf{u}\|_p\right)\right],\end{aligned}\tag{5.24}$$

where we used the fact that $\forall a\ge 1:\sqrt{a}\le a$.

Moreover by using Calderón-Zygmund inequality (3.38) together with equations (5.23) and (5.24) for $p>2$ we get

$$\begin{aligned}\frac{d}{dt}\|\nabla\theta\|_{L^p}^p &\le p\|\nabla\mathbf{u}\|_{L^\infty}\|\nabla\theta\|_{L^p}^p\\ &\le C(1+\|\nabla\mathbf{u}\|_{L^2}+\|D^2\mathbf{u}\|_{L^2})\left[1+\log^+\left(\|D^2\mathbf{u}\|_{L^p}\right)\right]\|\nabla\theta\|_{L^p}^p\\ &\le C(1+\|\omega\|_{L^2}+\|D\omega\|_{L^2})\left[1+\log^+\left(\|D\omega\|_{L^p}^p+\|\nabla\theta\|_{L^p}^p\right)\right]\|\nabla\theta\|_{L^p}^p.\end{aligned}$$

The above inequality and estimate (5.19) gives

$$\frac{d}{dt}\|\nabla\theta\|_{L^p}^p\le C(1+\|D\omega\|_{L^2})\left[1+\log^+\left(\|D\omega\|_{L^p}^p+\|\nabla\theta\|_{L^p}^p\right)\right]\|\nabla\theta\|_{L^p}^p,\tag{5.25}$$

where $C(\|\,\mathbf{u}_0\,\|_{L^2},\|\theta_0\|_{L^2},T,\nu,p)$.

Adding (5.22) and (5.25) together, and defining

$$X(t)=\|\nabla\theta\|_{L^p}^p+\|D\omega\|_{L^p}^p,$$

we obtain

$$\begin{aligned}\frac{d}{dt}X &\le C(1+\|D\omega\|_{L^2})\left[1+\log^+\left(\|D\omega\|_{L^p}^p+\|\nabla\theta\|_{L^p}^p\right)\right]\|\nabla\theta\|_{L^p}^p\\ &\quad+C(1+\|D\omega\|_{L^2})\left[1+\log^+\left(\|D\omega\|_{L^p}^p+\|\nabla\theta\|_{L^p}^p\right)\right]\|D\omega\|_{L^p}^p\\ &\quad+C(1+\|D\omega\|_{L^2})\left[1+\log^+\left(\|D\omega\|_{L^p}^p+\|\nabla\theta\|_{L^p}^p\right)\right]X,\end{aligned}$$

which implies

$$\frac{d}{dt}X \leq C(1+\|D\omega\|_{L^2})(1+\log^+ X)X,$$

for all $t \in [0,T]$.

Using Gronwall's inequality we have

$$X(t) \leq X(0)\exp\left[\left(CT + C\int_0^T \|D\omega\|_{L^2}ds\right)\exp\left\{CT + C\int_0^T \|D\omega\|_{L^2}ds\right\}\right] \quad \forall t \in [0,T],$$

which combined with estimate (5.18) gives us for $p > 2$

$$\|D\omega(t)\|_{L^p} \leq C(\|\mathbf{u_0}\|_{W^{2,p}}, \|\theta_0\|_{W^{1,p}}, T, \nu, p) \quad \forall t \in [0,T]. \tag{5.26}$$

By the Gagliardo -Nirenberg (3.23) and Calderón-Zygmund inequality (3.39) we have

$$\begin{aligned}\|\nabla\mathbf{u}(t)\|_{L^\infty} &\leq C\|\nabla\mathbf{u}(t)\|_{L^2}^{\frac{p-2}{2p-2}}\|D^2\mathbf{u}(t)\|_{L^p}^{\frac{p}{2p-2}}\\ &\leq C\|\omega(t)\|_{L^2}^{\frac{p-2}{2p-2}}\|D\omega(t)\|_{L^p}^{\frac{p}{2p-2}}\\ &\leq C(\|\mathbf{u_0}\|_{W^{2,p}}, \|\theta_0\|_{W^{1,p}}, T, \nu, p), \quad \forall t \in [0,T], \quad p \in (2,\infty],\end{aligned} \tag{5.27}$$

where we used estimates (5.19) and (5.26).

Recall that from (5.23) we have that

$$\frac{d}{dt}\|\nabla\theta(t)\|_{L^p}^p \leq p\|\nabla\mathbf{u}\|_{L^\infty}\|\nabla\theta(t)\|_{L^p}^p;$$

in particular,

$$\frac{d}{dt}\|\nabla\theta\|_{L^p} \leq \|\nabla\mathbf{u}\|_{L^\infty}\|\nabla\theta\|_{L^p}.$$

Moreover, by Grönwall's inequality we get

$$\|\nabla\theta(t)\|_{L^p} \leq \|\nabla\theta_0\|_{L^p}\exp\left(\int_0^t \|\nabla\mathbf{u}(s)\|_{L^\infty}ds\right). \tag{5.28}$$

Note that estimate for $\|\nabla\,\mathbf{u}\,\|_{L^\infty}$ was possible because previous estimates for $\|\omega\|_{L^2}$ and $\|D\omega\|_{L^p}$. We also use the Gagliardo-Nirenberg and Calderón-Zygmund inequalities.

Finally, let us see that in equation (5.28) we can pass to the limit when $p \to \infty$. To do that we define the following region depending on t

$$A_{\epsilon,r}(t) = \left\{x \in \mathbb{R}^2 : |\nabla\theta(x,t)| > \|\nabla\theta(t)\|_{L^\infty} - \epsilon, |x| < R\right\}.$$

We apply interpolation inequality (3.5) presented in Corollary 1 to (5.28), in this way we have

$$\|\nabla\theta(t)\|_{L^p} \leq \|\nabla\theta_0\|_{L^2}^{\frac{2}{p}}\|\nabla\theta_0\|_{L^\infty}^{1-\frac{2}{p}}\exp\left(\int_0^t \|\nabla\mathbf{u}(s)\|_{L^\infty}ds\right).$$

If we restrict our attention to the set $A_{\epsilon,R}(t)$, we have

$$(\|\nabla\theta(t)\|_{L^\infty} - \epsilon)\left(\int_{A_{\epsilon,r}(t)} dx\right)^{\frac{1}{p}} < \|\nabla\theta(x,t)\|^p,$$

$$(\|\nabla\theta(t)\|_{L^\infty} - \epsilon)|A_{\epsilon,r}(t)|^{\frac{1}{p}} < \|\nabla\theta(x,t)\|^p,$$

where $|A_{\epsilon,R}(t)|$ is the Lebesgue measure of the set $A_{\epsilon,R}(t)$ which is finite because of $|x|< R$ and $t \in [0,T]$.

Combining computations above, together with (5.28) we obtain

$$(\|\nabla\theta(t)\|_{L^\infty} - \epsilon)|A_{\epsilon,r}(t)|^{\frac{1}{p}} \leq \|\nabla\theta_0\|_{L^2}^{\frac{2}{p}} \|\nabla\theta_0\|_{L^\infty}^{1-\frac{2}{p}} \exp\left(\int_0^t \|\nabla\mathbf{u}(s)\|_{L^\infty} ds\right). \tag{5.29}$$

Taking $p \to \infty$, and then $\epsilon \to 0$ in (5.29), we have

$$\begin{aligned}\|\nabla\theta(t)\|_{L^\infty} &\leq \|\nabla\theta_0\|_{L^\infty} \exp\left(\int_0^t \|\nabla\mathbf{u}(s)\|_{L^\infty} ds\right)\\ &\leq C \quad \forall t \in [0,T],\end{aligned}$$

where C depends on $\|\mathbf{u_0}\|_{W^{2,p}}, \|\theta_0\|_{W^{2,p}}, T, p, \nu$ and we used estimate (5.27).

Therefore, since we have the embedding, $H^m(\mathbb{R}^2) \hookrightarrow W^{2,p}(\mathbb{R}^2)$ (see inequality (3.18)) for all $m > 2$ and $p > 2$ we attained desired estimate for the gradient of the temperature for any given $T \in (0,\infty)$ and for all $\mathbf{u_0}, \theta_0 \in H^m(\mathbb{R}^2)$ with $m > 2$.

The last part of Theorem 20 is presented in the next section.

5.3 Vanishing diffusivity limit

Now we are in the position to deal with the problem of letting $k \to 0$ in the complete Boussinesq system. We make use of energy estimates, and we use similar techniques as before to get the convergence of solutions.

Let $(\mathbf{u}, p, \theta)$ and $(\tilde{\mathbf{u}}, \tilde{p}, \tilde{\theta})$ be solutions of the zero-thermal Boussinesq system (5.1) and the full Boussinesq system (2.26) respectively with the same initial conditions . We can see that all estimates done previously are still valid for the solutions $(\tilde{\mathbf{u}}, \tilde{p}, \tilde{\theta})$. We summarize the key estimates which are independent of thermal constant k.

From equation (5.27) we have

$$\|\nabla\tilde{\mathbf{u}}\|_{L^\infty} \leq C(\|\mathbf{u}_0\|_{W^{2,p}}, \|\theta_0\|_{W^{1,p}}, \nu, T, p).$$

Considering the second equation of (2.26), after applying orthogonal gradient and integrating over $\mathbb{R}^2$ we find that for $p > 2$,

$$\begin{aligned}\frac{d}{dt}\|\nabla\tilde{\theta}\|_{L^p}^p + (p-1)k\int_{\mathbb{R}^2}|D^2\tilde{\theta}|^2|\nabla^\perp\tilde{\theta}|^{p-2}dx &\leq \int_{\mathbb{R}^2}|\nabla\tilde{\mathbf{u}}||\nabla\tilde{\theta}|^p dx,\\ \frac{d}{dt}\|\nabla\tilde{\theta}\|_{L^p}^p &\leq \int_{\mathbb{R}^2}|\nabla\tilde{\mathbf{u}}||\nabla\tilde{\theta}|^p dx,\end{aligned}$$

which implies after computations

$$\|\nabla\tilde{\theta}\|_{L^\infty} \leq C(\|\mathbf{u}_0\|_{W^{2,p}}, \|\theta_0\|_{W^{2,p}}, \nu, T, p).$$

Note also that from (5.26) we have the following estimate

$$\|\tilde{\mathbf{u}}\|_{W^{2,p}} \leq C(\|\mathbf{u}_0\|_{W^{2,p}}, \|\theta_0\|_{W^{2,p}}, \nu, T, p),$$

which is independent of $\tilde{\theta}$.

And from (5.28) we get

$$\|\tilde{\theta}\|_{W^{2,p}} \leq C(\|\mathbf{u}_0\|_{W^{2,p}}, \|\theta_0\|_{W^{2,p}}, \nu, T, p).$$

Summarizing the previous estimates done for $\tilde{\theta}$ and $\tilde{\mathbf{u}}$ we have

$$\|\nabla\tilde{\mathbf{u}}\|_{L^\infty} + \|\nabla\tilde{\theta}\|_{L^\infty} + \|\tilde{\mathbf{u}}\|_{W^{2,p}} + \|\tilde{\theta}\|_{W^{2,p}} \leq C(\|\mathbf{u}_0\|_{W^{2,p}}, \|\theta_0\|_{W^{2,p}}, \nu, T, p). \tag{5.30}$$

For $\Theta = \theta - \tilde{\theta},\ P = p - \tilde{p},\ U = \mathbf{u} - \tilde{\mathbf{u}}$ we obtain

$$\Theta_t + (\mathbf{u} \cdot \nabla)\Theta + (U \cdot \nabla)\tilde{\theta} = k\Delta\Theta + k\Delta\tilde{\theta}, \tag{5.31}$$

and

$$U_t + (\mathbf{u} \cdot \nabla)U + (U \cdot \nabla)\tilde{\mathbf{u}} = -\nabla P + \Theta e_2 + \nu\Delta U, \tag{5.32}$$

with the incompressibility condition div $U = 0$.

Taking L^2 inner product (5.31) with Θ, after integration by parts we have

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}\|\Theta\|_{L^2}^2 + k\|\nabla\Theta\|_{L^2}^2 &= -\int_{\mathbb{R}^2} (U \cdot \nabla)\tilde{\theta}\Theta dx - k\int_{\mathbb{R}^2} \nabla\tilde{\theta} \cdot \nabla\Theta dx \\
&\le \left|\int_{\mathbb{R}^2} (U \cdot \nabla)\tilde{\theta}\Theta dx\right| + \left|k\int_{\mathbb{R}^2} \nabla\tilde{\theta} \cdot \nabla\Theta dx\right| \\
&\le \|\nabla\tilde{\theta}\|_{L^\infty}\int_{\mathbb{R}^2} |U\Theta| dx + k\int_{\mathbb{R}^2} |\nabla\tilde{\theta} \cdot \nabla\Theta| dx \\
&\le \|\nabla\tilde{\theta}\|_{L^\infty}\|U\|_{L^2}\|\Theta\|_{L^2} + k\|\nabla\tilde{\theta}\|_{L^2}\|\nabla\Theta\|_{L^2} \\
&\le C\|U\|_{L^2}^2 + C\|\Theta\|_{L^2}^2 + \frac{k}{2}\|\nabla\tilde{\theta}\|_{L^2}^2 + \frac{k}{2}\|\nabla\Theta\|_{L^2}^2,
\end{aligned}$$

where we used estimate (5.30) and $2ab \le a^2 + b^2$. Joining similar terms to the left hand side we obtain

$$\frac{d}{dt}\|\Theta\|_{L^2}^2 + k\|\nabla\Theta\|_{L^2}^2 \le C\|U\|_{L^2}^2 + C\|\Theta\|_{L^2}^2 + Ck\|\nabla\tilde{\theta}\|_{L^2}^2. \tag{5.33}$$

In the same way for equation (5.32), taking L^2 inner product with U, and integrating by parts, we have

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}\|U\|_{L^2}^2 + \nu\|\nabla U\|_{L^2}^2 &= -\int_{\mathbb{R}^2} (U \cdot \nabla)\tilde{\mathbf{u}} \cdot U dx + \int_{\mathbb{R}^2} \Theta e_2 U dx \\
&\le \|\nabla\tilde{\mathbf{u}}\|_{L^\infty}\|U\|_{L^2}^2 + \|\Theta\|_{L^2}\|U\|_{L^2} \\
&\le C(\|U\|_{L^2}^2 + \|\Theta\|_{L^2}^2),
\end{aligned} \tag{5.34}$$

where $C(\mathbf{u_0}, \theta_0, T, \nu)$ and we used estimate (5.30).

Combining (5.33) with (5.34), and denoting $X(t) = \|U(t)\|_{L^2}^2 + \|\Theta(t)\|_{L^2}^2$, we get

$$\frac{d}{dt}X(t) \le CX(t) + Ck\|\nabla\tilde{\theta}\|_{L^2}^2.$$

By using Gronwall's lemma we have

$$\begin{aligned}
X(t) &\le X(0)e^{Ct} + Ck\int_0^t \|\nabla\tilde{\theta}(s)\|_{L^2}^2 e^{C(t-s)} ds \\
&\le Ce^{CT}k\int_0^T \|\nabla\tilde{\theta}(t)\|_{L^2}^2 dt \\
&\le Ck,
\end{aligned}$$

where we used estimate (5.30) and the fact that both Boussinesq systems (2.26) and (5.1) have the same initial conditions .i.e., $X(0) = 0$. Thus we obtain the following estimate

$$\|U(t)\|_{L^2}^2 + \|\Theta(t)\|_{L^2}^2 \le Ck.$$

In particular

$$\begin{aligned}
&\|U(t)\|_{L^2}^2 \le \|U(t)\|_{L^2}^2 + \|\Theta(t)\|_{L^2}^2 \le Ck, \\
&\|U(t)\|_{L^2} \le C\sqrt{k}, \\
&\|\Theta(t)\|_{L^2} \le C\sqrt{k},
\end{aligned}$$

which implies that

$$\sup_{0\leq t\leq T}(\|U(t)\|_{L^2}+\|\Theta(t)\|_{L^2})\leq C\sqrt{k},$$
$$\sup_{0\leq t\leq T}(\|\mathbf{u}(t)-\tilde{\mathbf{u}}(t)\|_{L^2}+\|\theta(t)-\tilde{\theta}(t)\|_{L^2})\leq C\sqrt{k}, \tag{5.35}$$

where $C(\mathbf{u_0},\theta_0,T,\nu)$.

The next argument can be done exactly for Θ and U. Here we present the case for Θ. By (5.35) we have

$$\sup_{0\leq t\leq T}(\|\theta(t)-\tilde{\theta}(t)\|_{L^2})\leq C\sqrt{k}.$$

From the Gagliardo-Nirenberg interpolation inequality (3.21), estimate (5.30) and the embedding $H^m(\mathbb{R}^2)\hookrightarrow W^{2,p}(\mathbb{R}^2)$ for $m>2$ we infer that for $0\leq s<m$

$$\begin{aligned}\sup_{0\leq t\leq T}\|\theta(t)-\tilde{\theta}(t)\|_{H^s}&\leq \sup_{0\leq t\leq T}\|\theta(t)-\tilde{\theta}(t)\|_{L^2}^{\alpha}\|\theta(t)-\tilde{\theta}(t)\|_{H^m}^{1-\alpha}\\&\leq C(\|\theta_0\|_{H^m}+\|\tilde{\theta}_0\|_{H^m})^{1-\alpha}\sup_{0\leq t\leq T}\|\theta(t)-\tilde{\theta}(t)\|_{L^2}^{\alpha}\\&\leq Ck^{\frac{\alpha}{2}},\end{aligned}$$

where

$$\alpha=1-\frac{s}{m}\text{ and }C=C(\mathbf{u_0},\theta_0,T,\nu,s,m).$$

Similarly for U we have

$$\sup_{0\leq t\leq T}(\|\mathbf{u}(t)-\tilde{\mathbf{u}}(t)\|_{L^2})\leq Ck^{\frac{\alpha}{2}}.$$

Finally, making $k\to 0$ we obtain the convergence $(\tilde{\mathbf{u}},\tilde{\theta})\to(\mathbf{u},\theta)$ in $C([0,T];H^s(\mathbb{R}^2))$.

6 Conclusions

We have studied the global in time regularity for the two dimensional Boussinesq system in the following way. First, we have introduced the equations involved in the Boussinesq system, that is, the Navier-Stokes equations, together with the heat and transport equations. Second, we review the functional framework in which PDE's are developed, namely, L^p, distributions and Sobolev spaces, as well as, the Leray and vorticity formulation for two dimensional fluid flows. Third, we have presented a breakdown or blow-up criterion for the partial Boussinesq case when $k = \nu = 0$. Finally, we have proved the global in time regularity for the two dimensional Boussinesq case when $k = 0, \nu > 0$.

The Boussinesq equations constitute a coupling between the Navier-Stokes equations and the convection-diffusion equation, that is, we have dealt with a system of PDE's describing the case when temperature variations are allowed in the fluid. Beyond the physical meaning of the Boussinesq equations, constants like viscosity and thermal diffusivity of the fluid can be partially neglected to study it mathematically. In this sense we have considered the cases $k = \nu = 0$ and $k = 0, \nu > 0$.

Functional spaces like L^p, distributions and Sobolev spaces provide an appropriate framework to state certain inequalities like: Grönwall, Gagliardo-Nirenberg, Calderón-Zygmund and Brezis-Wainger that have been used frequently through this memory. The above inequalities have been useful to establish a priori estimates for velocity and temperature concerning the Boussinesq equations. Moreover, we have taken advantage of being working in two dimensions and we have transformed the Navier-Stokes equations into a self contained scalar equation in terms of vorticity. This vorticity formulation together with the Leray formulation have gave us a schematic way to construct solutions locally in time for the Euler and Navier-Stokes equations.

Based on [9], we have established a blow-up criterion for the partial Boussinesq case when $k = \nu = 0$. In this case, we have shown that the breakdown or blow-up of smooth solutions is closely related to the accumulation of gradients of temperature. That is, $\| \mathbf{u} \|_{H^m}$ and $\|\theta\|_{H^m}$ remains bounded if and only if $\nabla\theta \in L^1(0, T; L^\infty(\mathbb{R}^2))$, and T depending on $\| \mathbf{u}_0 \|_{H^m}, \|\theta_0\|_{H^m}$ for $m > 2$. In one hand, since we are in two dimensions and $m > 2$, we have the injection $H^m(\mathbb{R}^2) \hookrightarrow C^1(\mathbb{R}^2)$. On the other hand, energy estimate for $\|\nabla \mathbf{u}\|_{L^\infty}$ together with a priori estimates for $\mathbf{u}$ and θ gives $\| \mathbf{u} \|_{H^m} < \infty$ and $\|\theta\|_{H^m} < \infty$.

Based on [10], we have shown the global in time regularity for the two dimensional zero-thermal Boussinesq system. Namely, we can find solutions of the partial Boussinesq system $k = 0$ and $\nu > 0$ that can be extended for any time provided that $\mathbf{u}_0, \theta_0 \in H^m$ for $m > 2$. In this sense, we have realized that if we prove the blow-up criterion for any time then we are done with the global in time regularity for the case $k = 0$ and $\nu > 0$. For the proof of the global regularity we have combined a priori estimates for $\mathbf{u}$ and θ together with Gagliardo-Nirenberg, Calderón-Zygmund and Brezis-Wainger inequalities. That is, we estimate uniformly the velocity and its gradient, which together with estimates for vorticity gives a bound for the maximum norm of the temperature's gradient.

We conjecture that with the theory developed in this memory we can consider without any problem the remaining content presented in [10], which consider the case $k > 0$ and $\nu = 0$. Moreover, as we mentioned in the introduction, we believe that we can take advantage of the properties of the space H^m to go over the implementation of numerical methods for solving partial cases of the Boussinesq system. Aside of this, for further considerations we can show that in the partial case $k = \nu = 0$, the solutions can be extended for any time. That is, the global in time regularity for the zero-viscosity-thermal is still an open problem in the subject.

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