UNIVERSIDAD DE INVESTIGACIÓN DE TECNOLOGÍA EXPERIMENTAL YACHAY

Escuela de Ciencias Matemáticas y Computacionales

Existence of Solutions and Controllability of Time Varying Semilinear Systems with Impulses, Delays, and Nonlocal Conditions

Trabajo de integración curricular presentado como requisito para la obtención
del título de Matemático

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Dedication

"To my father,
mother, brother and sisters,
who have given me their confidence
and unconditional support"

Dalia Nathaly Cabada Pesantez

Acknowledgments

“Thank the flame for its light, but do not forget the lamp holder standing in the shade with constancy of patience” (Rabindranath Tagore).

Rabindranath quote reflects entirely my point of view. My gratitude will not only be addressed to those who helped directly in my academic development, but also to all those who have held that lamp for that light shines.

In this way, I thank God for the wisdom to learn from each person with whom I have been surrounded during these past five years at Yachay Tech and also for the necessary strength to complete all what I have started.

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Dalia Nathaly Cabada Pesantez

Resumen

Este trabajo está basado en probar la siguiente conjetura: los impulsos, los retardos, y las condiciones no locales, bajo algunas suposiciones, no destruyen ciertas propiedades de un sistema puesto que ellas mismas son fenómenos intrínsicos al sistema como tal. Es por ello que en este trabajo estudiamos, en primer lugar, la existencia y unicidad de soluciones de ecuaciones diferenciales con impulsos, retardos, y condiciones no locales vistas como perturbaciones del sistema. En segundo lugar, se verifica que la propiedad de controlabilidad es robusta bajo este tipo de perturbaciones. Es decir, la propiedad de controlabilidad se preserva al añadirle, como perturbaciones, un término no lineal sujeto a retardo en el estado, impulsos, retardos y condiciones no locales. Así también se puede ver que si debilitamos las condiciones sobre las perturbaciones podemos conseguir una controlabilidad aproximada.

Palabras clave Existencia de soluciones, Controlabilidad, Ecuaciones semilineales no-autónomas con impulsos, Retardos, Condiciones no locales, Teorema de punto fijo de Rothe, Teorema de punto fijo de Karakostas.

Abstract

This work proves the following conjecture: impulses, delays, and nonlocal conditions, under some assumptions, do not destroy some posed system qualitative properties since they are themselves intrinsic to it.
In this work we study, first of all, the existence and uniqueness of solutions for differential equations with impulses, delays, and nonlocal conditions as perturbations of the system. Secondly, we verified that the property of controllability is robust under this type of disturbances. That is, if a linear system is controllable, and we add a nonlinear term subject to delays in the state, impulses, and nonlocal conditions like disturbances, then the controllability of the linear system is preserved under certain conditions. Moreover, weakening conditions on the perturbations, then the approximate controllability is achieved.

Keywords Existence of solutions, Controllability, Impulsive semilinear time varying equation, Delays, Nonlocal conditions, Rothe's Fixed Point Theorem, Karakostas Fixed Point Theorem.

Contents

GLOSSARY OF NOTATION

Symbol	Meaning
$\max\{l, m\}$	The maximum between the numbers l and m.
$\min\{r, s\}$	The minimum between the numbers r and s.
$\dim(\mathcal{A})$	Dimension of the matrix $\mathcal{A}$.
$\text{rank}(\mathcal{A})$	Number of linearly independent column vectors in the matrix $\mathcal{A}$.
$span\{v_1, v_2, \ldots, v_n\}$	Subspace generated by $v_1, v_2, \ldots, v_n$.
$\langle \cdot, \cdot \rangle_Z$	The Z-inner product.
$\text{Dom}(\mathcal{G})$	Domain of $\mathcal{G}$.
$\text{Ran}(\mathcal{G})$	Range or Image of $\mathcal{G}$.
$\ker(\mathcal{G})$	Set of elements from the domain of $\mathcal{G}$ whose image is zero.
$[\ker(\mathcal{G})]^{\perp}$	Orthogonal set of $\ker(\mathcal{G})$.
$\{t_1, t_2, \ldots, t_p\}$	Set of points where the impulses (jumps) occur.
z_t	The t-translation of the function z
$z(t_k^+)$	Right-hand limit of $z(t)$ at t_k.
$z(t_k^-)$	Left-hand limit of $z(t)$ at t_k.
$z(t_k) = z(t_k^+)$	The function z is right-continuous at t_k.
$PC_{t_1..t_p}(-r, \tau; \mathbb{R}^n)$	Set of functions from $[-r, \tau]$ to $\mathbb{R}^n$, continuous on $[-r, \tau] \backslash \{t_1, t_2, \ldots, t_p\}$, and right-continuous at $t_1, t_2, \ldots, t_p$.
$PC_{t_1..t_p}(-r, \infty; \mathbb{R}^n)$	Set of functions from $[-r, \infty)$ to $\mathbb{R}^n$, continuous on $[-r, \infty) \backslash \{t_1, t_2, \ldots, t_p\}$, and right-continuous at $t_1, t_2, \ldots, t_p$.
$L(W, Z)$	Space of bounded and linear operators from W to Z.
$L(\mathbb{R}^n)$	Space of bounded and linear operators from $\mathbb{R}^n$ to $\mathbb{R}^n$.
$C([a, b])$	Set of continuous functions from $[a, b]$ to $\mathbb{R}$.
$C(-r, 0; \mathbb{R}^n)$	Set of continuous functions from $[-r, 0]$ to $\mathbb{R}^n$.
$C(0, \tau; \mathbb{R}^m)$	Set of continuous functions from $[0, \tau]$ to $\mathbb{R}^n$.
$(\mathbb{R}^n)^q)$	The Cartesian product of q-spaces $\mathbb{R}^n$.
$C(-r, 0; (\mathbb{R}^n)^q)$	Set of continuous functions from $[-r, 0]$ to $(\mathbb{R}^n)^q$).
$L^2(0, \tau; \mathbb{R}^m)$	L^2-space from $[0, \tau]$ to $\mathbb{R}^m$.
$\mathcal{U}'(t, s)$	Partial derivative with respect t of the operator $\mathcal{U}(t, s)$.
$e^{\mathcal{A}t}$	The exponential matrix of $\mathcal{A}$ $(\sum_{k=0}^{\infty} \frac{1}{k!} t^k \mathcal{A}^k)$.
∂B	Boundary of the set B.
$\overline{B}$	Closure of the set B.
$\mathbb{R}_+$	Positive real numbers.
$\mathcal{G}^*$	Adjoint operator of $\mathcal{G}$.

1 Introduction

When we want to analyze mathematically a real life problem presented to us, the first step is to look for a model that represents problem. These models often are given by differential equations, equations that involve unknown functions and their derivatives. Usually, the differential equations describing the process by itself are not enough because there are intrinsic phenomena to the real problem that these equations do not consider. These phenomena could be of various nature, they may involve impulses, delays, nonlocal conditions, or random variables, such as noise that the model should include in order to represent the real problem.

Once the model for the real life problem is obtained, the existence of solutions for the differential equations is studied, a task that in some cases is very complicated to solve. Additionally, many properties of the model can be studied based on the type of problem that is being considered. For example, some people are interested in studying the stability, the invertibility, the causality, the controllability, or other properties of the model.

There are several studies concerning semilinear systems for diverse real life problems. However, only few of them about time varying (non-autonomous) semilinear systems. Some of these studies are focused on the existence of solutions or the controllability of the system considering some disturbances of it. We find researches on semilinear systems considering nonlocal conditions [1, 2, 3] or impulses [4, 5, 6, 7], impulses and nonlocal conditions [8, 9, 10], others considering delays and nonlocal conditions [11], others impulses and delays [12, 13]. However, to our knowledge, there are no studies considering these three disturbances together at the same time.

In this regard, this work is based on the study of the time varying semilinear differential equations under the influence of impulses, delays, and nonlocal conditions occurring simultaneously. This allows us to have a better approximation between the model and the real life problem. Here we focus on the study of the existence of solutions of the system and its controllability, specifically.

The non-autonomous semilinear system with impulses, delays, and nonlocal conditions that we will study is given by:

$$\begin{cases} z'(t) = \mathcal{A}(t)z + \mathcal{F}(t, z_t), & t \in (0, \tau], \quad t \neq t_k, \\ z(s) + \mathcal{H}(z_{\tau_1}, z_{\tau_2}, \dots, z_{\tau_q})(s) = \phi(s), & s \in [-r, 0], \\ z(t_k^+) = z(t_k^-) + \mathcal{I}_k(z(t_k)), & k = 1, 2, 3, \dots, p, \end{cases} \tag{1}$$

where $0 < t_1 < t_2 < t_3 < \cdots < t_p < \tau$, $0 < \tau_1 < \tau_2 < \cdots < \tau_q < \tau$, $\mathcal{A}(t)$ is a $n \times n$ continuous matrix; z_t, $\mathcal{F}$, $\mathcal{H}$, ϕ, and $\mathcal{I}_k$ are continuous and suitable functions, related with the delays, the non-linear part of the system, the nonlocal conditions, and the impulses, respectively.

According to the literature, the natural space to study this type of problems is the Banach space $PC_{t_1..t_p}$ endowed with the supremum norm and given by

$$PC_{t_1..t_p}(-r, \tau; \mathbb{R}^n) = \{z : [-r, \tau] \to \mathbb{R}^n : z \in C(J'; \mathbb{R}^n) : \forall\, k, z(t_k^+), z(t_k^-) \text{ exist, and } z(t_k) = z(t_k^+)\},$$

with $J' = [-r, \tau] \backslash \{t_1, t_2, \dots, t_p\}$.

The control problem corresponding to the semilinear system (1) is given by:

$$\begin{cases} z'(t) = \mathcal{A}(t)z(t) + \mathcal{B}(t)u(t) + f(t, z_t, u(t)), & t \in (0, \tau], \quad t \neq t_k, \\ z(s) + \mathcal{H}(z_{\tau_1}, z_{\tau_2}, \dots, z_{\tau_q})(s) = \phi(s), & s \in [-r, 0], \\ z(t_k^+) = z(t_k^-) + \mathcal{I}_k(z(t_k), u(t_k)), & k = 1, 2, 3, \dots, p, \end{cases} \tag{2}$$

where $\mathcal{B}(t)$ is a continuous matrix of dimension $n \times m$ and u is the control function which is continuous. In this control problem, the non-linear part $\mathcal{F}$ presented in (1) is given by

$$\mathcal{F}(t, z_t) = \mathcal{B}(t)u(t) + f(t, z_t, u(t)).$$

1.1 Notions of Control Theory

In recent decades, control theory has gained more importance as a discipline for engineers, mathematicians, physicist, economist, biologist and other scientists. There are some examples in control theory ranging from the simplest, as the heat conduction through a bar, to more complex cases, as the landing of a vehicle on the moon, the control of the economy, the control of epidemics, among others. There are an extensive bibliography on control theory for continuous systems. To mention some authors we have the works carried out by Barnett

[14], Curtain & Pritchard [15], Curtain & Zwart [16], Lee & Markus [17], and Zuazua [18]. This theory is based mainly on the study of the controllability for systems, which are also called control systems.

In order to figure out the control problem and its goal, let us define what a dynamical system is. A dynamical system can be defined conceptually as an entity that receives external actions or input variables, and whose response to these external actions are the called output variables [19]. The external actions of the system are divided into two groups, control variables, that can be manipulated, and perturbations which cannot be manipulated. The behaviour of a system can be represented in a conceptual way in the figure 1.

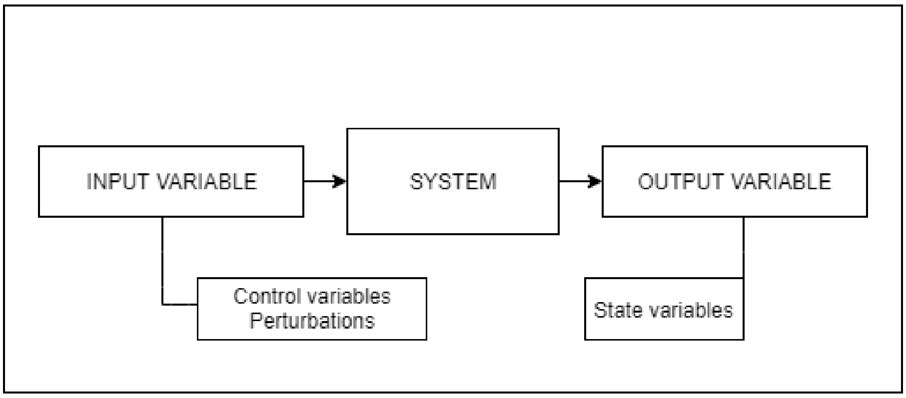

Figure 1: General scheme of a system.

A control system is a type of dynamical system which is characterized by the presence of elements that allow us to influence the behaviour of the system. The purpose of a control system is to achieve, by manipulating the control variables, a domain on the output variables, so they achieve desired values in a finite time.

An ideal control system must be able to achieve its objective with the following requirements [19]:

- Be robust when there are disturbances and errors in the models.
- Be efficient in the sense that the action of control over the input variables is reachable, avoiding abrupt and unreal behaviors.
- Be easily implemented and realizable in real time with the help of a computer.

Generally speaking, controllability is a qualitative property of dynamical systems and the basis of the study in control theory. In general, it refers that the system can be steered from an arbitrary initial state to an arbitrary final state using the set of admissible controls in finite time ([20], pp. 5).

Within the study of the controllability of a system, there are two concepts that can be distinguished, the exact controllability and approximate controllability. In general terms, the exact controllability is related to the existence of a control steering the system from one state to another state in finite time, while the approximate controllability allows the system to go from one state to just a neighborhood of the desired state.

The controllability of the linear system is very well known and there is a broad reference about it, including books such as [21, 22, 23]. Unlike for linear systems, the bibliography of semilinear systems is limited, in this regard we can mention the work done by Lukes in [24], J.C. Coron in [25](see Theorem 3.40 and Corollary 3.41) and Vidyasager in [26] where he studied the case when the function f does not depend on the parameter $u \in \mathbb{R}^m$, and proved the controllability by using Schauder Fixed Point Theorem.

Dauer in [27] obtained several sufficient conditions on the function f for the controllability of perturbed systems without impulses and nonlocal conditions (2). In some works the non-linear perturbation f is subject to the linear system, which is natural when a linear system is perturbed. In this sense, V. N. Do in [28] found a weaker condition on the nonlinear term f for the controllability of the system (2) containing Dauer's conditions; however this condition depends strongly on the linear system (14), particularly, on the fundamental matrix $\Phi(t)$ of the uncontrolled linear system, which is in general not available in closed form.

The following subsection builds the notions and concepts required to understand the mathematical model studied here.

1.2 Notions of Impulses, delays, and nonlocal conditions

1.2.1 Impulsive differential equations

Impulsive differential equations are those involving impulse or jump effect. These differential equations usually appear as a characterization of observed time varying phenomena where an abrupt change occurs. There are many physical phenomena exhibiting a sudden change in their states. For instance, such as mechanical systems with impact, biological systems such as heart beats, blood flows, population dynamics where appropriated mathematical models of such processes are impulsive differential equations [29, 30].

As stated in [31], an impulsive differential equation is described by three components: a continuous-time differential equation, which governs the state of the system between impulses; an impulse equation, which models an impulsive jump defined by a jump function at the instant where an impulse occurs; and a jump criterion, which defines a set of jump events in which the impulse equation is active. According to our mathematical model, the impulsive differential equation take the form

$$\begin{aligned} z'(t) &= \mathcal{A}(t)z + \mathcal{F}(t,z), \qquad t \in (0,\tau], \quad t \neq t_k, \\ z(t_k^+) &= z(t_k^-) + \mathcal{I}_k(z(t_k)), \quad k = 1,2,3,\dots,p, \end{aligned}$$

where $z'(t) = \mathcal{A}(t)z + \mathcal{F}(t,z)$ is the differential equation, $\mathcal{I}_k(z(t_k)) = z(t_k^+) - z(t_k^-)$ is the impulse equation and $k = 1,2,3,\dots,p$ is the jump criterion. For more information about impulsive differential equations the reader can see [5].

1.2.2 Delay differential equations

According to [32], delay differential equations are a type of differential equations in which the derivative of the unknown function at a certain time is given in terms of the values of the function at previous times. In order to illustrate a tangible application, let us present a simple real life model of this type of systems, the mixing problems.

Example 1.1. *Suppose that three connected tanks each containing 100 gls of solution of a certain chemical substance. Starting at a certain instant, a solution of the same chemical, with a concentration $u(t)$ lb/gal, flows into the first tank at the rate of R gl/mi. The mixture is drained at the same rate into the second tank; from this tank the chemical pass to the third one at the same rate; finally the solution is drained off from this tank at the same rate.*

This problem can be formulated as a cascade control system for the amount of chemical in these three tanks at time t. In fact, the three tanks can be denoted respectively by T_1, T_2 and T_3 and $u(t)$, the concentration of the chemical that flows into T_1, acts as a control, $z_i(t), i = 1,2,3$ the amount of chemical into T_i in time t and $\frac{z(t)_i}{100}$ the concentration of the chemical into T_i in time t if the mixture is instantaneous (see Figure 2).

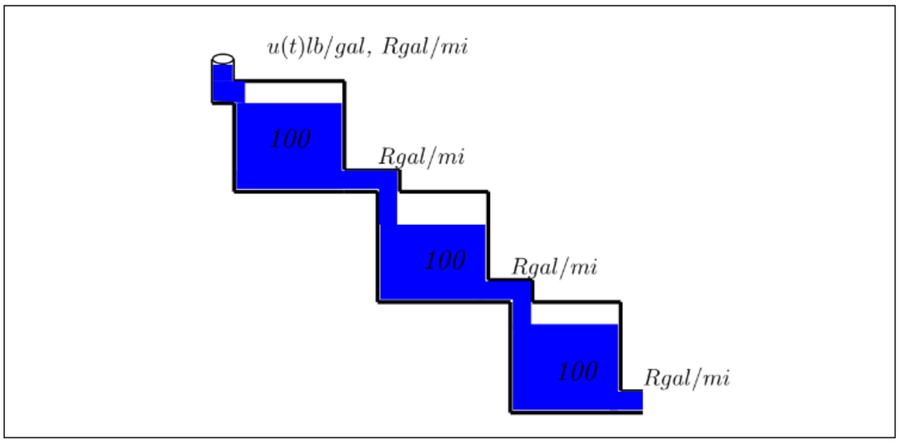

Figure 2: Scheme of the Example 1.1

If we assume that the mixture is instantaneous, the variation of the chemical in each tank is given by the following cascade system

$$\begin{cases} \dfrac{dz_1(t)}{dt} = Ru(t) \, - R\dfrac{z_1(t)}{100}, \\ \\ \dfrac{dz_2(t)}{dt} = R\dfrac{z_1(t)}{100} \, - \, R\dfrac{z_2(t)}{100}, \\ \\ \dfrac{dz_3(t)}{dt} = R\dfrac{z_2(t)}{100} \, - \, R\dfrac{z_3(t)}{100}. \end{cases} \tag{3}$$

Now, if we put $a = R$ and $b = R/100$, (3) can be written in the following way

$$\begin{cases} z_1'(t) &= & au(t) - bz_1(t) \\ z_2'(t) &= & bz_1(t) - bz_2(t) \\ z_3'(t) &= & bz_2(t) - bz_3(t) \end{cases} \tag{4}$$

and

$$z' = Az + Bu(t), \tag{5}$$

where

$$A = \begin{pmatrix} -b & 0 & 0 \\ b & -b & 0 \\ 0 & b & -b \end{pmatrix}, \quad B = \begin{pmatrix} a \\ 0 \\ 0 \end{pmatrix} \quad \text{and} \quad z = \begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix}.$$

Since it was assumed that the mixture is instantaneous, the above model is not realistic because in real world mixtures are not instantaneous. In fact, the concentration of the chemical that is calculated at time t is actually the one that was in the tank at a previous time $t - r$, where r is a positive number, we will call it delay. Thus the model with delay it would take the following form

$$\begin{cases} z_1'(t) &= & au(t) - bz_1(t-r) \\ z_2'(t) &= & bz_1 t - r) - bz_2(t-r) \\ z_3'(t) &= & bz_2(t-r) - bz_3(t-r) \end{cases}$$

and

$$z'(t) = Az(t-r) + Bu(t).$$

Hence, the Cauchy initial value problem needs to be formulated as follows:

$$\begin{cases} z'(t) = Az(t-r) + Bu(t), \\ z(s) = \phi(s), \qquad\qquad s \in [-r, 0], \end{cases} \tag{6}$$

where $\phi \in C([-r, 0]; \mathbb{R}^3)$ represents history of the concentration of the chemical into the tanks.
Additionally, defining the functions z_t from $[-r, 0]$ to $\mathbb{R}^3$ by $z_t(s) = z(t + s), -r \leq s \leq 0$ and $f : [0, \tau] \times C([-r, 0]; \mathbb{R}^3) \times \mathbb{R} \to \mathbb{R}^3$ by

$$f(t, \phi, u) = A\phi(-r) + Bu, \quad \forall (t, \phi, u) \in [0, \tau] \times C([-r, 0]; \mathbb{R}^3) \times \mathbb{R},$$

the Cauchy problem (6) takes the form:

$$\begin{cases} z'(t) = f(t, z_t, u(t)), \\ z(s) = \phi(s), \qquad\qquad s \in [-r, 0], \end{cases} \tag{7}$$

System (7) is called a state delay control system. In consonance with the mathematical model studied in this thesis, the delay differential equation is given by

$$z'(t) = \mathcal{A}(t)z + \mathcal{F}(t, z_t), \qquad t \in (0, \tau]$$

where $\mathcal{F}$ is defined from $[0, \tau] \times C(-r, 0; \mathbb{R}^n)$ to $\mathbb{R}^n$, z_t represents the trajectory of the solution in the past, z_t is defined as a function from $[-r, 0]$ to $\mathbb{R}^n$ given by $z_t(s) = z(t + s)$, $-r \leq s \leq 0$.

1.2.3 Differential equations with nonlocal conditions

This subsection is devoted to motivate the nonlocal condition in a Cauchy problem. For this purpose we use [33]. In the process for obtaining descriptions of the evolution of some chemical, physical, biological, economical phenomena that are as accurate as possible, there are at least two main types of problems that have been found and differ from other ones.
The first one is a initial value problem o Cauchy problem which consists of finding a function $z(t) = z(t, a, \xi)$ defined on an interval J, that satisfies a differential equation plus an initial condition

$$\begin{cases} z'(t) = f(t, z(t)), \quad t \in J, \\ z(a) = \xi, \end{cases} \tag{8}$$

where $a \in J$ is the so-called initial time and ξ the initial state, belonging to the state space, which could be a finite or an infinite-dimensional Banach space. This is a very simple deterministic model which amounts to saying that, once we know the initial state ξ of the system at the initial time a and the law which describes the dependence of the instantaneous rate of change $z'(t)$ of the state $z(t)$, we are able to predicts the complete evolution of the system on the entire interval on which it exists.
The second one is a so-called periodic problem which involves finding a function z satisfying:

$$\begin{cases} z'(t) = f(t, z(t)), \quad t \in \mathbb{R}_+, \\ z(t) = z(t+T), \qquad t \in \mathbb{R}_+. \end{cases} \tag{9}$$

In this case, in order to have a solution, it is almost necessary to assume at least that f is T-periodic with respect to its argument, i.e., $f(t, z) = f(t + T, z), \quad \forall t \in \mathbb{R}_+$ and each z in the state space. In the foregoing case, as far as the solution of the problem is concerned, although describing a deterministic model, we cannot speak about the initial time or the initial state notions which, in this case, are not a priory given. Nevertheless, due to some mathematical reasons, in many situations, we are led to fix an arbitrary initial time and to treat the problem (9) at least in the first stage as an initial value problem.

We emphasize, however, that in spite the two types of problems are completely different, from the mathematical point of view, they can be compared to each other, using the perspective of how accurate the description of the phenomenon is that each offers. We can say that an initial value problem better describes the evolution of a certain phenomenon than a periodic problem or conversely. Sometimes, starting from an initial value problem; for some initial data, the corresponding solutions are periodic or at least in the initial history, i.e., for $t \in [-r, 0]$, z satisfies the T-periodicity-like condition $z(t) = z(t+T)$, even though f fails to be periodic with respect to its first argument.

Furthermore, there are cases for which the prediction offered by an initial value problem is more accurate if instead of a single initial datum given at a single initial time $t = a$, more data, at certain different times, strictly greater than a, are collected and their weighted average is used.

For the sake of simplicity, let us assume that $J = \mathbb{R}_+$, $a = 0$ and we have the possibility to measure the values of Z -the exact solution- at some points $0 < t_1 < t_2 < \cdots < t_n$. It should be emphasized that the exact solution Z does not coincide with the solution z of the mathematical model (8), simply because in the construction of (8) it is impossible to take into consideration all the data involved in the evolution of the phenomenon considered.
Then, one may approximate the exact solution Z of the system by the solution v of the differential model below whose initial data, $v(0)$, is assumed to be the weighted average of the measured data $Z(t_1), Z(t_2), \cdots, Z(t_n)$, gathered at the specific moments $0 < t_1 < t_2 < \cdots < t_n$. We denote this weighted average by $g(Z)$, i.e.,

$$g(Z) = \sum_{k=1}^{n} \alpha_k Z(t_k),$$

where $\alpha_k \in (0, 1), \quad k = 1, 2, 3, \ldots, n$, are such that

$$\sum_{k=1}^{n} \alpha_k = 1.$$

Thus, instead of the initial value problem (8) with $J = \mathbb{R}_+$, $a = 0$ and $z(0) = \xi = z(0)$, it is more convenient to consider a variant, i.e.,

$$\begin{cases} v'(t) = f(t, v(t)), \quad t \in \mathbb{R}_+, \\ v(0) = g(Z(\cdot)). \end{cases} \tag{10}$$

Of course, (10) is simply (8) with $v(0) = g(Z(\cdot))$. In some practical circumstances, the problem is that it is very hard, or even impossible, to make accurate measurements. Therefore, we have to choose a different approach. A possible strategy would be to replace $Z(t_k)$ by $v(t_k)$ for $k = 1, 2, \ldots, n$. Empirical studies have shown that the "model" thus obtained, i.e.,

$$\begin{cases} v'(t) = f(t, v(t)), & t \in \mathbb{R}_+, \\ v(0) = \sum_{k=1}^{n} \alpha_k v(t_k) = g(v(\cdot)), & \end{cases} \tag{11}$$

even less exact than the preceding one, is still reliable enough to be taken into consideration as an acceptable alternative. So, we are led to consider the nonlocal initial value problem above as a substitute for (10).
Clearly, the problem (11), involving a nonlocal initial condition, is completely different from its classical initial value counterpart (8) and is sufficiently complicated. Indeed, the difficulties come from the fact that we must find v satisfying not only the differential equation, but an implicit constraint as well, i.e., $v(0) = g(v(\cdot))$, which in its simplest case reduces to the T-periodicity condition, $v(0) = v(T)$.
Therefore, the mathematical machinery which is appropriate in the study of (8) is not longer useful in the case of (11). In addition, the problem of finding suitable methods for analyzing (11) is not as simple and it could be, in many cases, rather challenging.

Differential equations subject to nonlocal boundary conditions could be problems with feedback controls, such as the steady-states of a thermostat, where a controller at one of its ends adds or removes heat, depending upon the temperature registered in another point, or phenomena with functional dependence in the equation, or in the boundary conditions, with delays or advances, maximum or minimum arguments [34]. In line with system (1), the nonlocal conditions is given by

$$z(s) + \mathcal{H}(z_{\tau_1}, z_{\tau_2}, \ldots, z_{\tau_q})(s) = \phi(s), \quad s \in [-r, 0],$$

where $\phi(s) - \mathcal{H}(z_{\tau_1}, z_{\tau_2}, \ldots, z_{\tau_q})(s)$ collects the historical information of the solution on $[-r, 0]$. $\mathcal{H}$ goes from $C(-r, 0; (\mathbb{R}^n)^q)$ to $C(-r, 0; \mathbb{R}^n)$ and $\phi \in C(-r, 0; \mathbb{R}^n)$.

This thesis has been divided into the following below sections; aiming a better understanding.

Section 2 (Preliminaries): In this section, we present some results of dense range linear operators, as a particular case we consider surjective operators, which to be used in Section 3. Additionally, it provides some definitions and theorems that will be necessary in the next sections.

Section 3 (Controllability of linear system): In this section, the exact controllability for the linear system corresponding to system (2) is proved. Moreover, an algebraic characterization for the controllability of autonomous systems, due to Kalman, is shown.

Section 4 (Research Results): In this section, we present and prove the main results of this thesis: the existence of solutions, the exact controllability, and the approximate controllability for time varying systems with impulses delays and nonlocal conditions.

Section 5 (Conclusions and Open problems): This section is devoted to the conclusions and some open problems derived from this work.

2 Preliminaries

2.1 Definitions, Lemmas, and Theorems

Definitions, lemmas, and theorems used in this thesis are introduced. Other results are introduced in the relevant sections as necessary. These results can be found in [5, 35, 36, 37].

Definition 2.1. ***(Equicontractivity)*** *Let $\{T_\alpha\}_{\alpha \in I}$ be a family of mapping, $T_\alpha : Z \to Z$. T_α is said to be equicontractive if there exist $0 < L < 1$ such that:*

$$\|T_\alpha z_1 - T_\alpha z_2\| \leq L \|z_1 - z_2\| \quad \forall \alpha \in I, \quad \forall z_1, z_2 \in Z.$$

Theorem 2.2. ***(Arzelà-Ascoli)*** *If $\{f_n\} \subset C([a,b])$ is uniformly bounded and equicontinuous, then there is a subsequence $f_n^{'}$ converging to $f \in C([a,b])$.*

Lemma 2.3. ***(Gronwall)*** *Let $f : [a,b] \to \mathbb{R}$ and $g : [a,b] \to \mathbb{R}^+$ be continuous functions. Let $y : [a,b] \to \mathbb{R}$ be a continuous function such that for all $t \in [a,b]$*

$$y(t) \leq f(t) + \int_a^t g(s)y(s)ds.$$

Then, for all $t \in [a,b]$ we have

$$y(t)t \leq f(t) + \int_a^t f(s)g(s)exp\left(\int_s^t g(u)du\right)ds.$$

In particular, if $f(t) \equiv k$ is constant, then

$$y(t) \leq k\ exp\left(\int_a^t g(s)ds\right).$$

Lemma 2.4. ***(Generalized Gronwall-Bellman)*** *Let a non negative function $z \in PC_{t_1..t_p}(-r,\infty;\mathbb{R}^n)$ satisfy, for $t \geq t_0$, the inequality*

$$z(t) \leq C + \int_{t_0}^t v(s)z(s)ds + \sum_{t_0<t_k<t} \beta_k u(t_k),$$

where $C \geq 0$, $\beta_k \geq 0$, $v(s) > 0$, and t_k's are the discontinuity points of first type for the function z. Then we have,

$$z(t) \leq C \prod_{t_0<t_k<t} (1+\beta_k)e^{\int_{t_0}^t v(s)ds}.$$

Definition 2.5. ***(Exact Controllability)*** *The system (2) is said to be controllable on $[0,\tau]$ if for every $\phi \in C(-r,0;\mathbb{R}^n)$, $z_1 \in \mathbb{R}^n$, there exists $u \in C(0,\tau;\mathbb{R}^m)$ such that the solution $z(t)$ of (2) corresponding to u verifies:* $z(0) + \mathcal{H}(z_{\tau_1}, z_{\tau_2}, \ldots, z_{\tau_q})(0) = \phi(0)$ *and* $z(\tau) = z_1$ (Figure 3).

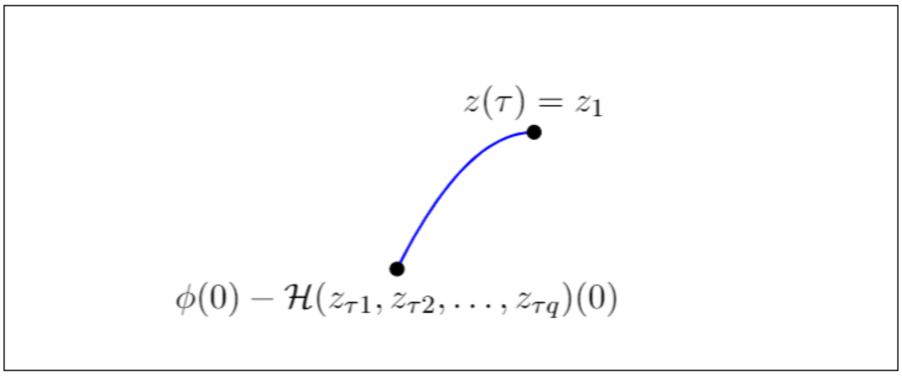

Figure 3: Scheme of exact controllability.

Definition 2.6. ***(Approximate Controllability)*** *The system (2) is said to be approximately controllable on $[0,\tau]$ if for every $\phi \in C(-r,0;\mathbb{R}^n)$, $z_1 \in \mathbb{R}^n$ and $\epsilon > 0$, there exists $u \in L^2(0,\tau;\mathbb{R}^m)$ such that the solution $z(t)$ of (2) corresponding to u verifies:*

$$z(0) + \mathcal{H}(z_{\tau_1}, \ldots, z_{\tau_q})(0) = \phi(0), \quad \textit{and} \quad \left\|z(\tau) - z_1\right\|_{\mathbb{R}^n} < \epsilon \quad \text{(Figure 4).}$$

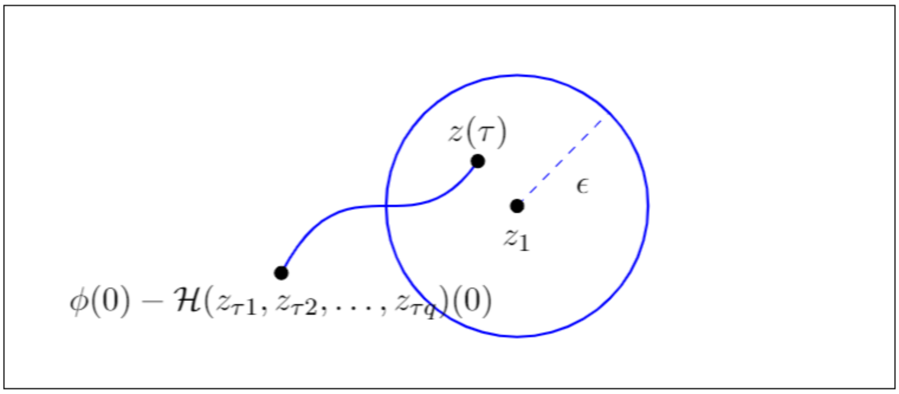

Figure 4: Scheme of approximate controllability.

2.1.1 Fixed Point Theorems

The following theorems are the main tool to prove the existence of solutions for the semilinear system (1) and its controllability, respectively.

Theorem 2.7. ***(G. L. Karakostas Fixed Point Theorem)*** *Let Z and Y be Banach spaces and D be a closed convex subset of Z, and let $\mathcal{C} : D \to Y$ be a continuous operator such that $\mathcal{C}(D)$ is a relatively compact subset of Y, and*

$$\mathcal{T} : D \times \overline{\mathcal{C}(D)} \to D$$

a continuous operator such that the family $\{\mathcal{T}(\cdot, y) : y \in \overline{\mathcal{C}(D)}\}$ is equicontractive. Then, the operator equation

$$\mathcal{T}(z, \mathcal{C}(z)) = z,$$

admits a solution on D.

Theorem 2.8. ***(Rothe's Fixed Point Theorem)*** *Let E be a Hausdorff topological vector space. Let $B \subset E$ a convex and closed subset such that the zero of E belongs to the interior of B. Let $\mathcal{S} : B \to E$ be a continuous mapping such that $\mathcal{S}(B)$ is relatively compact on E, and $\mathcal{S}(\partial B) \subset B$. Then, there exist a point $x^* \in B$ such that $\mathcal{S}(x^*) = x^*$.*

2.2 Characterization of Dense Range Operators

The following results can be found in Curtain & Pritchard's lectures notes and Curtain & Zwart's book. To apply this result is not necessary to understand the whole proof of it. Nevertheless, for better understanding of the reader we presents here a complete proof of it. These results are linked to concepts of functional analysis such as Hilbert spaces, open mapping theorem, Cauchy-Schwartz inequality, Riesz representation theorem, and perpendicular theorem.

Theorem 2.9. *(Curtain & Pritchard [15] Curtain & Zwart [16]) Let W and Z be Hilbert spaces, $\mathcal{G} \in L(W, Z)$ and $\mathcal{G}^* \in L(Z, W)$. Then, the following statements hold:*

a) $\mathrm{Ran}(\mathcal{G}) = Z$ *if, and only if, there exist α such that $\|\mathcal{G}^* z\|_W \geq \alpha \|z\|_Z$, $z \in Z$,*

b) $\overline{\mathrm{Ran}(\mathcal{G})} = Z$ *if, and only if,* $\ker(\mathcal{G}^*) = \{0\}$.

Proof. Let's prove a).

$\Rightarrow$) Suppose that $\mathrm{Ran}(\mathcal{G}) = Z$.

The proof is going to be done by steps:

Step 1 Assume first that $\mathcal{G}$ is a one-to-one map. Therefore, since we assume that $\mathrm{Ran}(\mathcal{G}) = Z$ we have that $\mathcal{G}$ is a bijection and from a consequence of the open mapping theorem, we get that $\mathcal{G}^{-1} \in L(Z, W)$. Also, since

$$\left(\mathcal{G}^{-1}\right)^* = \left(\mathcal{G}^*\right)^{-1} \in L(W, Z)$$

there exists $\beta > 0$ such that

$$\| \left(\mathcal{G}^*\right)^{-1} w\|_Z \leq \beta \|w\|_W, \quad \forall w \in W.$$

Putting

$$z = \left(\mathcal{G}^*\right)^{-1} w \Longleftrightarrow w = \mathcal{G}^* z,$$

we obtain

$$\|\mathcal{G}^* z\|_W \geq \alpha \|z\|_Z, \quad \forall z \in Z,$$

where

$$\alpha = \frac{1}{\beta}.$$

Step 2 For the general case, we consider the Hilbert space

$$W = X \oplus \ker(\mathcal{G})$$

where

$$X = [\ker(\mathcal{G})]^{\perp}.$$

Since X is a closed sublinear space, then X is also a Hilbert space with the same norm

$$\|w\|_W = \|w\|_X, \qquad w \in X.$$

Now, we define the linear operator

$$\begin{aligned} \hat{\mathcal{G}} \colon X &\to Z \\ w &\mapsto \hat{\mathcal{G}} w = \mathcal{G} w. \end{aligned}$$

Trivially, $\hat{\mathcal{G}}$ is one-to-one. Therefore, $\hat{\mathcal{G}}$ is bijective. Then, analogous to step 1, we obtain

$$\|\hat{\mathcal{G}}^* z\|_X \geq \alpha \|z\|_Z \quad \forall z \in Z,$$

where

$$\alpha = \frac{1}{\beta}.$$

From Riesz representation theorem, we know that

$$\begin{aligned} \|\hat{\mathcal{G}}^* z\|_X &= \sup_{\substack{w \in X \\ \|w\|_X \leq 1}} \left| \langle w, \hat{\mathcal{G}}^* z \rangle_W \right| \\ &= \sup_{\substack{w \in X \\ \|w\|_X \leq 1}} \left| \langle w, \hat{\mathcal{G}}^* z \rangle_W \right| \\ &= \sup_{\substack{w \in X \\ \|w\|_X \leq 1}} \left| \langle \hat{\mathcal{G}} w, z \rangle_Z \right| \\ &= \sup_{\substack{w \in X \\ \|w\|_X \leq 1}} \left| \langle \mathcal{G} w, z \rangle_Z \right| \\ &= \sup_{\substack{w \in X \\ \|w\|_X \leq 1}} \left| \langle w, \mathcal{G}^* z \rangle_W \right| \\ &= \|\mathcal{G}^* z\|_W. \end{aligned}$$

Therefore,

$$\|\mathcal{G}^* z\|_W \geq \frac{1}{\beta} \|z\|_Z, \quad \forall z \in Z.$$

$\Leftarrow$) Now, let's assume that

$$\|\mathcal{G}^* z\|_W \geq \alpha \|z\|_Z, \quad \forall z \in Z,$$

and consider

$$\begin{aligned} \|\mathcal{G}^* z\|_W^2 &\geq \alpha^2 \|z\|_Z^2, \quad \forall z \in Z. \\ \langle \mathcal{G}\mathcal{G}^* z, z \rangle &\geq \alpha^2 \|z\|_Z^2, \quad \forall z \in Z. \end{aligned}$$

Then, by Cauchy-Schwartz inequality, we get that

$$\|\mathcal{G}\mathcal{G}^* z\|_Z \geq \alpha^2 \|z\|_Z, \quad \forall z \in Z. \tag{12}$$

Now, Let's show that $\mathrm{Ran}(\mathcal{G}\mathcal{G}^*) = Z$. First, we shall see that $\mathrm{Ran}(\mathcal{G}\mathcal{G}^*)$ is closed. In fact, let $\tilde{z}$ be an accumulation point of $\mathrm{Ran}(\mathcal{G}\mathcal{G}^*)$. Then, there exist a sequence $\{z_n\}_{n=1}^{\infty} \subset Z$ such that

$$\lim_{n\to\infty} \mathcal{G}\mathcal{G}^* z_n = \tilde{z}. \tag{13}$$

From (12), we get that

$$\|z\|_Z \leq \frac{1}{\alpha^2} \|\mathcal{G}\mathcal{G}^* z\|_Z, \quad \forall z \in Z.$$

So,

$$\|z_n - z_m\|_Z \leq \frac{1}{\alpha^2} \|\mathcal{G}\mathcal{G}^* z_n - \mathcal{G}\mathcal{G}^* z_m\|_Z.$$

Since $\{\mathcal{G}\mathcal{G}^* z_n\}_{n=1}^{\infty}$ is a Cauchy sequence, $\|z_n - z_m\| \to 0$ as $n, m \to \infty$. Hence, $\{z_n\}_{n=1}^{\infty}$ is a Cauchy sequence as well. Thus,

$$\lim_{n\to\infty} z_n = z.$$

Passing to the limit in (13), we get that

$$\mathcal{G}\mathcal{G}^* z = \tilde{z},$$

i.e., $\tilde{z} \in \mathrm{Ran}(\mathcal{G}\mathcal{G}^*)$. Now, suppose that $\mathrm{Ran}(\mathcal{G}\mathcal{G}^*) \subsetneq Z$. Then, by the perpendicular theorem, there exist z_0 with $\|z_0\|_Z = 1$ such that

$$z_0 \perp \mathrm{Ran}(\mathcal{G}\mathcal{G}^*),$$

i.e.,

$$\langle \mathcal{G}\mathcal{G}^* z, z_0 \rangle_Z = 0 \qquad \forall z \in Z.$$

In particular,

$$\langle \mathcal{G}\mathcal{G}^* z_0, z_0 \rangle_Z = 0.$$

From (12), we get that $\|z_0\|^2 = 0$, i.e., $z_0 = 0$, which is a contradiction. So,

$$\mathrm{Ran}(\mathcal{G}\mathcal{G}^*) = Z.$$

Clearly, we have that

$$\mathrm{Ran}(\mathcal{G}\mathcal{G}^*) \subset \mathrm{Ran}(\mathcal{G}).$$

Hence,

$$\mathrm{Ran}(\mathcal{G}) = Z.$$

Let's prove b)

$\Rightarrow$) Suppose that $\overline{\mathrm{Ran}(\mathcal{G})} = Z$ and $\mathcal{G}^*$ is not one-to-one. Then, there exists $z_0 \neq 0$ such that $\mathcal{G}^* z_0 = 0$. Therefore,

$$\begin{aligned} 0 &= \langle w, \mathcal{G}^* z_0 \rangle_W, \quad \forall w \in W \\ &= \langle \mathcal{G} w, z_0 \rangle_Z. \end{aligned}$$

Since $\overline{\mathrm{Ran}(\mathcal{G})} = Z$, there exists a sequence $\mathcal{G} w_n \to z_0$. So that,

$$0 = \langle \mathcal{G} w_n, z_0 \rangle_Z, \quad \forall n \in \mathbb{N}.$$

Now, passing to the limit, as $n \to \infty$, we obtain that

$$\begin{aligned} 0 &= \langle z_0, z_0 \rangle_Z \\ &= \|z_0\|_Z^2, \end{aligned}$$

i.e., $z_0 = 0$, which is a contradiction. Therefore, $\mathcal{G}^*$ is one-to-one, i.e., $\ker(\mathcal{G}^*) = \{0\}$.

$\Leftarrow$) Now, let's assume that $\mathcal{G}^*$ is one-to-one and $\overline{\text{Ran}(\mathcal{G})} \subset Z$. Then, by the perpendicular theorem, there exists $z_0 \neq 0$, $z_0 \in Z$ such that

$$\langle \mathcal{G}w, z_0 \rangle_Z = 0 \implies \langle w, \mathcal{G}^* z \rangle_W = 0 \quad \forall w \in W.$$

So, taking $w = \mathcal{G}^* z_0 \in W$, we get that

$$\begin{aligned} \|\mathcal{G}^* z_0\|_W^2 &= \langle \mathcal{G}^* z_0, \mathcal{G}^* z_0 \rangle_W \\ &= 0, \end{aligned}$$

i.e., $\mathcal{G}^* z_0 = 0$ and $z_0 \neq 0$, which is a contradiction. Hence, $\overline{\text{Ran}(\mathcal{G})} = Z$. □

Corollary 2.10. *If in addition to previous hypotheses,* $\dim(Z) < \infty$*, then the following statements hold:*

a.- $\text{Ran}(\mathcal{G}) = Z$*,*

b.- There exist α *such that* $\|\mathcal{G}z\|_Z \geq \alpha \|z\|_Z$*,* $z \in Z$*, and*

c.- $\ker(\mathcal{G}^*) = \{0\}$*.*

Proof. Since all linear subspace of a finite dimensional linear space is closed, then $\text{Ran}(\mathcal{G}) = \overline{\text{Ran}(\mathcal{G})}$. Therefore, using the theorem 2.9, we obtain the result. □

Corollary 2.11. *Let* W*,* Z *be Hilbert spaces and* $\mathcal{G} \in L(W, Z)$ *such that* $Ran(\mathcal{G}) = Z$*. Then,* $w_z = \mathcal{G}^*(\mathcal{G}\mathcal{G}^*)^{-1} z$ *is the solution of the equation*

$$\mathcal{G}w = z,$$

with minimum norm, i.e.,

$$\|w_z\|_W = \inf\{w \in W : \mathcal{G}w = z\}.$$

Proof. By applying the open mapping theorem and Theorem 2.9 is clear that there exist $(\mathcal{G}\mathcal{G}^*)^{-1}$ and $w_z = \mathcal{G}^*(\mathcal{G}\mathcal{G}^*)^{-1} z$ is the solution of $\mathcal{G}w = z$. □

Remark 2.12. *Under the conditions of Corollary 2.11 the operator* $\Upsilon : Z \to W$ *defined by:*

$$\Upsilon z = \mathcal{G}^*(\mathcal{G}\mathcal{G}^*)^{-1} z,$$

is a right inverse to $\mathcal{G}$*, in the sense that*

$$\mathcal{G}\Upsilon = I.$$

The results developed in this section will be used in the next section where we shall present the controllability of the linear system corresponding to our mathematical model. In addition to presenting the controllability for the case on we are focused, non-autonomous, we shall present in Section 3 an algebraic characterization of the controllability for the autonomous case.

3 Controllability of Linear Systems in Finite Dimensional Spaces

The results that will be developed here can be found in [21, 22, 23].

3.1 Non-Autonomous Systems

Corresponding to system (2), we shall consider the linear initial value problem

$$\begin{cases} z'(t) = \mathcal{A}(t)z(t) + \mathcal{B}(t)u(t), \quad z(t) \in \mathbb{R}^n, \quad t \in (0,\tau], \\ z(0) = z_0, \end{cases} \tag{14}$$

where $z_0 \in \mathbb{R}^n$ and $u \in L^2(0,\tau;\mathbb{R}^m)$. It is well known (see [38]) that (14) admits only one solution given by

$$z(t) = \mathcal{U}(t,0)z_0 + \int_0^t \mathcal{U}(t,s)\mathcal{B}(s)u(s)ds, \ \ t \in [0,\tau], \tag{15}$$

where $\mathcal{U}(t,s) = \Phi(t)\Phi^{-1}(s)$ and $\Phi(t)$ is the fundamental matrix associated to the uncontrollable linear system given by

$$z'(t) = \mathcal{A}(t)z(t). \tag{16}$$

Thus, the matrix $\Phi(t)$ satisfies

$$\begin{cases} \Phi'(t) = \mathcal{A}(t)\Phi(t), \\ \Phi(0) = I_{\mathbb{R}^n}, \end{cases}$$

where $I_{\mathbb{R}^n}$ is the identity matrix of dimension $n \times n$.

Proposition 3.1. *The evolution operator $\mathcal{U}(t,s)$, defined above, satisfies the following trivial properties for all t, r, s in $\mathbb{R}$:*

a) $\mathcal{U}(t,t) = I_{\mathbb{R}^n}$;

b) $\mathcal{U}(t,r)\mathcal{U}(r,s) = \mathcal{U}(t,s)$ *(cocycle property);*

c) $\mathcal{U}'(t,s) = \frac{\partial}{\partial t}\mathcal{U}(t,s) = \mathcal{A}\mathcal{U}(t,s)$;

d) $\mathcal{U}(t,s)$ *is continuous;*

e) There exist constants $M > 0$ and $\omega > 0$ such that:

$$\|\mathcal{U}(t,s)\| \leq Me^{\omega\mathcal{A}(t-s)}, \quad 0 \leq s \leq t \leq \tau; \tag{17}$$

f) $\mathcal{U}^{-1}(s,t) = \mathcal{U}(t,s)$.

Proof. The proof for a), b), and c), immediately follows from the definition of $\mathcal{U}(t,s)$. The continuity of $\mathcal{U}(t,s)$ can be obtained from the continuity of the fundamental matrix Φ. The inequality (17) can be demonstrated applying the Lemma 2.3, and f) immediately follows from a) and b). □

Now, let us define some concepts in order to characterize the controllability of the linear system (14) in terms of linear operators.

The controllability maps: Let's define the operator $\mathcal{G} : L^2(0,\tau;\mathbb{R}^m) \longrightarrow \mathbb{R}^n$ (for $\tau > 0$) as follows:

$$\mathcal{G}u = \int_0^\tau \mathcal{U}(\tau,s)\mathcal{B}(s)u(s)ds. \tag{18}$$

Then, the adjoint operator $\mathcal{G}^* : \mathbb{R}^n \longrightarrow L^2(0,\tau;\mathbb{R}^m)$ of $\mathcal{G}$ can be computed as follows. By definition of adjoint operator, the following equality holds

$$\langle \mathcal{G}u, z\rangle_{\mathbb{R}^n} = \langle u, \mathcal{G}^*z\rangle_{L^2} \quad z \in \mathbb{R}^n, \quad u \in L^2(0,\tau;\mathbb{R}^m).$$

So that,

$$\begin{aligned} \langle \mathcal{G}u, z\rangle_{\mathbb{R}^n} &= \left\langle \int_0^\tau \mathcal{U}(\tau,s)\mathcal{B}(s)u(s)ds, z \right\rangle_{\mathbb{R}^n} \\ &= \int_0^\tau \left\langle u(s), \mathcal{B}^*(s)\mathcal{U}^*(\tau,s)z\right\rangle_{\mathbb{R}^n} \\ &= \left\langle u, B^*(\cdot)\mathcal{U}^*(\tau,\cdot)z\right\rangle_{\mathbb{L}^2}. \end{aligned}$$

Therefore, the adjoint operator is given by

$$(\mathcal{G}^* z)(s) = \mathcal{B}^*(s)\mathcal{U}^*(\tau, s)z, \quad \forall s \in [0, \tau], \quad \forall z \in \mathbb{R}^n.$$

The Controllability Gramian: Let's define the following operator $\mathcal{W} : \mathbb{R}^n \to \mathbb{R}^n$ as follows:

$$\mathcal{W}z = \mathcal{G}\mathcal{G}^* z = \int_0^\tau \mathcal{U}(\tau, s)\mathcal{B}(s)\mathcal{B}^*(s)\mathcal{U}^*(\tau, s)z ds. \tag{19}$$

Proposition 3.2. *The systems (14) is controllable on* $[0, \tau]$ *if, and only if,* $\mathrm{Ran}(\mathcal{G}) = \mathbb{R}^n$*.*

Proof. $\Rightarrow$) Assume that (14) is controllable on $[0, \tau]$. Let's prove that $\mathcal{G}$ is onto, i.e.,

$$\forall x \in \mathbb{R}^n \quad \exists u \in L^2(0, \tau; \mathbb{R}^m) : \quad \mathcal{G}u = x.$$

Let $x \in \mathbb{R}^n$, consider $z_0, z_1 \in \mathbb{R}^n$ such that $x = z_1 - \mathcal{U}(\tau, 0)z_0$. i.e.,

$$z_1 = x + \mathcal{U}(\tau, 0)z_0.$$

From the controllability, we have that there exist $u \in L^2(0, \tau; \mathbb{R}^m)$ such that the boundary conditions

$$z(0, u) = z_u(0) = z_0 \qquad \textit{and} \qquad z(\tau, u) = z_u(\tau) = z_1,$$

are satisfied and $z_1 = x + \mathcal{U}(\tau, 0)z_0$.

Therefore, from (15), we get

$$\begin{aligned} x + \mathcal{U}(\tau, 0)z_0 &= \mathcal{U}(\tau, 0)z_0 + \int_0^\tau \mathcal{U}(\tau, s)\mathcal{B}(s)u(s)ds \\ x &= \int_0^\tau \mathcal{U}(\tau, s)\mathcal{B}(s)u(s)ds, \end{aligned}$$

i.e.,

$$x = \mathcal{G}u.$$

Thus, we have proved that $\mathcal{G}$ is onto.

$\Leftarrow$) Assume that $\mathrm{Ran}(\mathcal{G}) = \mathbb{R}^n$. Let's prove that (14) is controllable, i.e., there exist $u \in L^2(0, \tau; \mathbb{R}^m)$ such that the boundary conditions

$$z_u(0) = z_0 \qquad \textit{and} \qquad z_u(\tau) = z_1$$

are satisfied. Let $x \in \mathbb{R}^n$. Then, there exist $z_0, z_1 \in \mathbb{R}^n$ such that

$$x = z_1 - \mathcal{U}(\tau, 0)z_0.$$

Since $\mathrm{Ran}(\mathcal{G}) = \mathbb{R}^n$, $\mathcal{G}$ is onto, we know that there exist $u \in L^2(0, \tau; \mathbb{R}^m)$ such that

$$\mathcal{G}u = z_1 - \mathcal{U}(\tau, 0)z_0.$$

Then, from (15),

$$\begin{aligned} z_1 - \mathcal{U}(\tau, 0)z_0 &= \mathcal{G}u \\ z_1 - \mathcal{U}(\tau, 0)z_0 &= \int_0^\tau \mathcal{U}(\tau, s)\mathcal{B}(s)u(s)ds \\ z_1 &= \mathcal{U}(\tau, 0)z_0 + \int_0^\tau \mathcal{U}(\tau, s)\mathcal{B}(s)u(s)ds, \end{aligned}$$

i.e.,

$$z_1 = z_u(\tau)$$

Since $z_u(0) = z_0$ and $z_1 = z_u(\tau)$, we have proved the controllability of (14). This concludes the proof. □

Since the controllability of the linear system is characterized by surjectivity of the operator $\mathcal{G}$, the results obtained in Section 2 can be applied to prove the following lemma.

Lemma 3.3. *(See [39]) The following statements are equivalent.*

a) $\mathrm{Ran}(\mathcal{G}) = \mathbb{R}^n$,

b) $\ker(\mathcal{G}^*) = \{0\}$,

c) $\exists \gamma > 0 / \langle \mathcal{G}\mathcal{G}^* z, z\rangle > \gamma \|z\|^2_{\mathbb{R}^n}$, $z \neq 0$ *in* $\mathbb{R}^n$,

d) $\exists \mathcal{W}^{-1} \in L(\mathbb{R}^n)$,

e) $\forall s \in [0,\tau] \quad \mathcal{B}^*(s)\mathcal{U}^*(\tau,s)z = 0 \Rightarrow z = 0$.

Also, the operator $\Upsilon : \mathbb{R}^n \to L^2(0,\tau;\mathbb{R}^m)$ defined by

$$\Upsilon z = \mathcal{B}^*(\cdot)\mathcal{U}^*(\tau,\cdot)\mathcal{W}^{-1}z = \mathcal{G}^*(\mathcal{G}\mathcal{G}^*)^{-1}z, \tag{20}$$

is called the steering operator and it is a right inverse to $\mathcal{G}$, in the sense that

$$\mathcal{G}\Upsilon = I.$$

Moreover,

$$\|\mathcal{W}^{-1}z\|^2_{\mathbb{R}^n} = \|(\mathcal{G}\mathcal{G}^*)^{-1}z\|^2_{\mathbb{R}^n} \leq \gamma^{-1}\|z\|^2_{\mathbb{R}^n}, \quad z \in \mathbb{R}^n,$$

and a control steering the system (14) from initial state z_0 to a final state z_1 at time $\tau > 0$ is given by

$$u(t) = \mathcal{B}^*(t)\mathcal{U}^*(\tau,t)\mathcal{W}^{-1}(z_1 - \mathcal{U}(\tau,0)z_0) = \Upsilon(z_1 - \mathcal{U}(\tau,0)z_0)(t), \quad t \in [0,\tau]. \tag{21}$$

In order to apply the controllability results obtained in this section to our mathematical model, we will use the following lemma.

Lemma 3.4. *(See [40]) Let S be any dense subspace of $L^2(0,\tau;\mathbb{R}^m)$. Then, system (14) is controllable with control $u \in L^2(0,\tau;\mathbb{R}^m)$ if, and only if, it is controllable with control $u \in S$, i.e.,*

$$\mathrm{Ran}(\mathcal{G}) = \mathbb{R}^n \iff \mathrm{Ran}(\mathcal{G}|_S) = \mathbb{R}^n,$$

where $\mathcal{G}|_S$ is the restriction of $\mathcal{G}$ to S.

Remark 3.5. *According to the previous lemma, if the system is controllable, it is controllable with control functions in $C(0,\tau;\mathbb{R}^m)$ since the set of continuous functions is a dense subspace of $L^2(0,\tau;\mathbb{R}^m)$. Moreover, the operators $\mathcal{G}$, $\mathcal{W}$, and Υ are well defined on the space of continuous functions, and they are given as in* (18)*,* (19)*, and* (20)*, respectively.*

3.2 Autonomous Systems

Here we shall present a known algebraic characterization of controllability for the autonomous case, also known as Kalman rank condition for controllability. This results was one of the first results obtained in the Control Theory by Rudolf E. Kalman in 1963 and this marked the beginning of a new branch of studies, controllability of linear dynamical systems [41].

From the controllability point of view and considering that the matrices $\mathcal{A}(t)$ and $\mathcal{B}(t)$ are constant, the autonomous linear system can be represented as $(\mathcal{A},\mathcal{B})$. Moreover, the fundamental matrix of the system is given by

$$\Phi(t) = e^{\mathcal{A}t}.$$

The solution for the autonomous system is given as (15), where the evolution operator is given by

$$\mathcal{U}(t,s) = e^{\mathcal{A}(t-s)},$$

and the controllability maps by

$$\mathcal{G}u = \int_0^\tau e^{\mathcal{A}(t-s)}\mathcal{B}(s)u(s)ds.$$

3.2.1 Algebraic Characterization of Controllability for Systems $(\mathcal{A}, \mathcal{B})$

Theorem 3.6. *(Kalman condition) The system $(\mathcal{A}, \mathcal{B})$ is controllable on $[0, \tau]$ if, and only if,*

$$\text{rank}\left[\mathcal{B} \quad | \quad \mathcal{A}\mathcal{B} \quad | \quad \cdots \quad | \quad \mathcal{A}^{n-1}\mathcal{B}\right] = n.$$

Proof. $\Rightarrow$) Let's assume that the system $(\mathcal{A}, \mathcal{B})$ is controllable. Then, by proposition (3.2), $\text{Ran}(\mathcal{G}) = \mathbb{R}^n$.

By Cayley-Hamilton theorem, we know that any matrix $\mathcal{A}$ is a root of the characteristic polynomial, i.e., if

$$P(\lambda) = \lambda^n + a_1\lambda^{n-1} + \ldots + a_n\lambda^0$$

is the characteristic polynomial of $\mathcal{A}$, then

$$P(\mathcal{A}) = \mathcal{A}^n + a_1\mathcal{A}^{n-1} + a_2\mathcal{A}^{n-2} + \cdots + a_n\mathcal{A}^0 = 0,$$

and by induction

$$\mathcal{A}^{n+k} = \beta_1^k\mathcal{A}^{n-1} + \beta_2^k\mathcal{A}^{n-2} + \cdots + \beta_n^k\mathcal{A}^0, \quad \forall k \in \mathbb{N},$$

which implies that

$$e^{\mathcal{A}(t-s)} = \sum_{i=0}^{n-1} \alpha_i(t-s)\mathcal{A}^i, \qquad \alpha_i(t-s) \in \mathbb{R}, \qquad i = 0, 1, 2 \ldots, n-1.$$

Then, the operator $\mathcal{G}$ can be written as follows:

$$\begin{aligned} \mathcal{G}u &= \int_0^\tau \left(\sum_{i=0}^{n-1} \alpha_i(\tau - s)\mathcal{A}^i\right) \mathcal{B}u(s)ds \\ &= \sum_{i=0}^{n-1} \mathcal{A}^i\mathcal{B} \int_0^T \alpha_i(\tau - s)u(s)ds. \end{aligned}$$

Taking $y(i) = \int_0^\tau \alpha_i(\tau - s)u(s)ds$, and taking into account that $u(s) \in \mathbb{R}^m$ and $\alpha_i(\tau - s) \in \mathbb{R}$, we get

$$\mathcal{G}u = \sum_{i=0}^{n-1} \mathcal{A}^i\mathcal{B}y(i), \qquad y(i) \in \mathbb{R}^m.$$

Now, let's consider the following operator $\widetilde{\mathcal{G}} : \mathcal{Y} \to \mathbb{R}^n$, $\mathcal{Y} = \mathbb{R}^m \times \mathbb{R}^m \times \cdots \times \mathbb{R}^m$, given by

$$\widetilde{\mathcal{G}}y = \sum_{i=0}^{n-1} \mathcal{A}^i\mathcal{B}y(i), \qquad y = (y(0), y(1), \ldots, y(n-1)).$$

Then, since every element of $\text{Ran}(\mathcal{G})$ can be written as an element of $\text{Ran}(\widetilde{\mathcal{G}})$, we have that

$$\text{Ran}(\mathcal{G}) \subset \text{Ran}(\widetilde{\mathcal{G}}) \subset \mathbb{R}^n,$$

i.e.,

$$\text{Ran}(\widetilde{\mathcal{G}}) = \mathbb{R}^n.$$

So that,

$$\text{span}\left\{\mathcal{B}\mathbb{R}^m, \mathcal{A}\mathcal{B}\mathbb{R}^m, \cdots, \mathcal{A}^{m-1}\mathcal{B}\mathbb{R}^m\right\} = \mathbb{R}^n.$$

Hence,

$$\text{rank}\left[\mathcal{B} \quad | \quad \mathcal{A}\mathcal{B} \quad | \quad \cdots \quad | \quad \mathcal{A}^{n-1}\mathcal{B}\right] = n.$$

$\Leftarrow$) Let's assume that

$$\text{rank}\left[\mathcal{B} \quad | \quad \mathcal{A}\mathcal{B} \quad | \quad \cdots \quad | \quad \mathcal{A}^{n-1}\mathcal{B}\right] = n,$$

and the system (14) is not controllable, then $\ker(\mathcal{G}) \neq \{0\}$ by Lemma 3.3. Therefore, there exist $\eta \in \mathbb{R}^n$ such that $\eta \neq 0$ and

$$(\mathcal{G}^*\eta)(t) = \mathcal{B}^* e^{\mathcal{A}^*(\tau-s)}\eta = 0, \ s \in [0,\tau].$$

Now, let's do the following change of variables

$$t = \tau - s$$

Thus,

$$\mathcal{B}^* e^{\mathcal{A}^* t}\eta = 0, \ \forall t \in [0,\tau],$$

which implies

$$\langle \eta, e^{\mathcal{A}t}\mathcal{B}z\rangle_{\mathbb{R}^n} = 0, \quad \forall z \in \mathbb{R}^n. \tag{22}$$

If $t = 0$, then

$$\langle \ \eta, \mathcal{B}z \ \rangle_{\mathbb{R}^n} = 0, \qquad \forall z \in \mathbb{R}^n.$$

Now, taking k-th derivative in (22), we get

$$\frac{d^k}{dt^k}\left\langle \eta, e^{\mathcal{A}t}\mathcal{B}z\right\rangle_{\mathbb{R}^n} |_{t=0} = \left\langle \eta, \mathcal{A}^k e^{\mathcal{A}t}\mathcal{B}z\right\rangle_{\mathbb{R}^n} |_{t=0} = \left\langle \eta, \mathcal{A}^k\mathcal{B}z\right\rangle_{\mathbb{R}^n}.$$

So that,

$$\left\langle \eta, \mathcal{A}^k\mathcal{B}z\right\rangle_{\mathbb{R}^n} = 0 \quad \forall k = 0,1,2,\ldots,n-1,$$

i.e.,

$$\left\langle \eta, \operatorname{span}\left\{\mathcal{B}\mathbb{R}^m, \mathcal{A}\mathcal{B}\mathbb{R}^m, \cdots, \mathcal{A}^{m-1}\mathcal{B}\mathbb{R}^m\right\}\right\rangle = 0.$$

If we call π_η the hyperplane by the origin whose normal vector is η, we get

$$\operatorname{span}\left\{\mathcal{B}\mathbb{R}^m, \mathcal{A}\mathcal{B}\mathbb{R}^m, \cdots, \mathcal{A}^{m-1}\mathcal{B}\mathbb{R}^m\right\} \subset \pi_\eta$$

Therefore, $\dim(\operatorname{span}\left\{\mathcal{B}\mathbb{R}^m, \mathcal{A}\mathcal{B}\mathbb{R}^m, \cdots, \mathcal{A}^{m-1}\mathcal{B}\mathbb{R}^m\right\}) < n$, which is a contradiction. □

4 Research Results

This section will be divided into two subsections to present the main results obtained in this thesis and their proofs. First of all, the existence of solutions, the uniqueness, and a prolongation of solutions theorem for semilinear systems with impulses, delays, and nonlocal conditions will be proved. After that, we shall present the results obtained for the control system, the exact controllability and the approximate controllability.

Sitting the Problem

The non-autonomous semilinear system with impulses, delays, and nonlocal conditions is given by

$$\begin{cases} z'(t) = \mathcal{A}(t)z + \mathcal{F}(t, z_t), & t \in (0,\tau], \quad t \neq t_k \\ z(s) + \mathcal{H}(z_{\tau_1}, z_{\tau_2}, \ldots, z_{\tau_q})(s) = \phi(s), & s \in [-r, 0], \\ z(t_k^+) = z(t_k^-) + \mathcal{I}_k(z(t_k)), & k = 1,2,3,\ldots,p. \end{cases}$$

where $0 < t_1 < t_2 < t_3 < \cdots < t_p < \tau$, $0 < \tau_1 < \tau_2 < \cdots < \tau_q < \tau$ and $\mathcal{A}(t)$ is a $n \times n$ continuous matrix; z_t defined as a function from $[-r, 0]$ to $\mathbb{R}^n$ by $z_t(s) = z(t+s)$, $-r \leq s \leq 0$, the functions $\mathcal{F} : [0,\tau]\times C(-r,0;\mathbb{R}^n) \to \mathbb{R}^n$, $\mathcal{H} : C(-r,0;(\mathbb{R}^n)^q) \to C(-r,0;\mathbb{R}^n)$ and $\mathcal{I}_k \in C(\mathbb{R}^n;\mathbb{R}^n)$ which are smooth enough.

The set $C(-r,0;\mathbb{R}^n)$ denotes the space

$$C(-r,0;\mathbb{R}^n) = \{\phi : [-r,0] \to \mathbb{R}^n : \phi \quad \text{is continuous}\}$$

endowed with the norm

$$\|\phi\| = \sup_{-r\le s\le 0} \|\phi(s)\|_{\mathbb{R}^n}.$$

To study this system, we shall consider the following natural Banach space $PC_{t_1..t_p}$ defined by

$$PC_{t_1..t_p}(-r,\tau;\mathbb{R}^n) = \{z : [-r,\tau] \to \mathbb{R}^n : z \in C(J';\mathbb{R}^n) : \forall\ k, z(t_k^+), z(t_k^-) \text{ exist, and } z(t_k) = z(t_k^+)\} \tag{23}$$

with $J' = [-r,\tau]\backslash\{t_1, t_2, \dots, t_p\}$, endowed with the norm

$$\|z\| = \sup_{t\in[-r,\tau]} \|z(t)\|_{\mathbb{R}^n}. \tag{24}$$

Also, we shall denote:

$$(\mathbb{R}^n)^q = \mathbb{R}^n \times \mathbb{R}^n \times \cdots \mathbb{R}^n = \prod_{k=1}^{q} \mathbb{R}^n, \tag{25}$$

endowed with the norm

$$\|y\|_q^n = \sum_{i=1}^{q} \|y_i\|_{\mathbb{R}^n}, \quad y = (y_1, y_2, \dots, y_q)^T \in (\mathbb{R}^n)^q,$$

and the norm in the space $C(-r,0;(\mathbb{R}^n)^q)$ is given by

$$\|y\|_{Cq} = \sup_{t\in[-r,\tau]} \|y(t)\|_q^n = \sup_{t\in[-r,\tau]} \left(\sum_{i=1}^{q} \|y_i(t)\|_{\mathbb{R}^n} \right), \quad \forall y \in C(-r,0;(\mathbb{R}^n)^q). \tag{26}$$

4.1 Existence of Solutions for Impulsive Semilinear Evolution Equations with Delays and Nonlocal Conditions

In order to prove the existence of solutions for the semilinear system (1), first, we shall motivate the formula of the solution as the solution of an integral equation.

4.1.1 Characterization of the Solution for Semilinear Evolution Equations with Impulses, Delays, and Nonlocal Conditions

Proposition 4.1. *The semilinear system* (1) *has a solution* z *if, and only if,*

$$\begin{aligned} z(t) =& \,\mathcal{U}(t,0)\{\phi(0) - \mathcal{H}(z_{\tau_1}, z_{\tau_2}, \dots, z_{\tau_q})(0)\} + \int_0^t \mathcal{U}(t,s)\mathcal{F}(s,z_s)ds \\ &+ \sum_{0<t_k<t} \mathcal{U}(t,t_k)\mathcal{I}_k(z(t_k)), \quad t \in [0,\tau]. \\ z(t) =& \,\phi(t) - \mathcal{H}(z_{\tau_1}, z_{\tau_2}, \dots, z_{\tau_q})(t), \quad t \in [-r,0]. \end{aligned} \tag{27}$$

Proof. $\Rightarrow$) Suppose that z is a solution of the problem (1). If we put

$$z_0 = \phi(0) - \mathcal{H}(z_{\tau_1}, z_{\tau_2}, z_{\tau_3}, ..., z_{\tau_q})(0) \tag{28}$$

and,

$$f(t) = \mathcal{F}(t, z_t), \quad t \neq t_k, \tag{29}$$

z will be a solution of the following initial value problem on $[0,t_1)$

$$\begin{cases} z'(t) = & \mathcal{A}(t)z(t) + f(t), \quad t \in [0,t_1) \\ z(0) = & z_0. \end{cases}$$

Hence, using the method of variation of parameters to find the particular solution for the semilinear ordinary differential equation, we obtain that

$$z(t) = \mathcal{U}(t,0)z_0 + \int_0^t \mathcal{U}(t,s)f(s)ds, \quad t \in [0,t_1).$$

Now, passing to the limit, as $t \to t_1^-$, we get

$$z(t_1^-) = \mathcal{U}(t_1, 0)z_0 + \int_0^{t_1} \mathcal{U}(t_1, s)f(s)ds.$$

On the interval $[t_1, t_2)$ z is the solution of the following initial value problem

$$\begin{cases} z'(t) = & \mathcal{A}(t)z(t) + f(t), \quad t \in [t_1, t_2) \\ z(t_1) = & z(t_1^+). \end{cases}$$

Therefore,

$$z(t) = \mathcal{U}(t, t_1)z(t_1) + \int_{t_1}^{t} \mathcal{U}(t, s)f(s)ds, \quad t \in [t_1, t_2).$$

From here, using the cocycle property of $\mathcal{U}(t, s)$, $t \in [t_1, t_2)$, and $z(t_1^+) = z(t_1^-) + \mathcal{I}_k(z(t_1))$, we get that

$$\begin{aligned} z(t) &= \mathcal{U}(t, t_1)z(t_1^+) + \int_{t_1}^{t} \mathcal{U}(t, s)f(s)ds \\ &= \mathcal{U}(t, t_1)\left\{z(t_1^-) + \mathcal{I}_k(z(t_1))\right\} + \int_{t_1}^{t} \mathcal{U}(t, s)f(s)ds \\ &= \mathcal{U}(t, t_1)\left\{\mathcal{U}(t_1, 0)z_0 + \int_0^{t_1} \mathcal{U}(t_1, s)f(s)ds + \mathcal{I}_k(z(t_1))\right\} + \int_{t_1}^{t} \mathcal{U}(t, s)f(s)ds \\ &= \mathcal{U}(t, 0)z_0 + \int_0^{t_1} \mathcal{U}(t, s)f(s)ds + \mathcal{U}(t, t_1)\mathcal{I}_k(z(t_1)) + \int_{t_1}^{t} \mathcal{U}(t, s)f(s)ds \\ &= \mathcal{U}(t, 0)z_0 + \int_0^{t} \mathcal{U}(t, s)f(s)ds + \mathcal{U}(t, t_1)\mathcal{I}_k(z(t_1)). \end{aligned} \tag{30}$$

In the same way, for $t \in [t_2, t_3)$, z is solution of the following initial value problem

$$\begin{cases} z'(t) = & \mathcal{A}(t)z(t) + f(t), \quad t \in [t_2, t_3) \\ z(t_2) = & z(t_2^+). \end{cases} \tag{31}$$

Passing to the limit, as $t \to t_2^-$ in (30), we obtain that

$$z(t_2^-) = \mathcal{U}(t_2, 0)z_0 + \int_0^{t_2} \mathcal{U}(t_2, s)f(s)ds + \mathcal{U}(t_2, t_1)\mathcal{I}_k(z(t_1)).$$

Since z is solution of (31), z can be written on $[t_2, t_3)$ as

$$z(t) = \mathcal{U}(t, t_2)z(t_2) + \int_{t_2}^{t} \mathcal{U}(t, s)f(s)ds.$$

Then, using that $z(t_2^+) = z(t_2^-) + \mathcal{I}_k(z(t_2))$ and the cocycle property, we get

$$z(t) = \mathcal{U}(t, t_2)z(t_2^+) + \int_{t_2}^{t} \mathcal{U}(t, s)f(s)ds$$

$$= \mathcal{U}(t, t_2)\left\{z(t_2^-) + \mathcal{I}_k(z(t_2))\right\} + \int_{t_2}^{t} \mathcal{U}(t, s)f(s)ds$$

$$= \mathcal{U}(t,t_2)\left\{\mathcal{U}(t_2,0)z_0 + \int_0^{t_2}\mathcal{U}(t_2,s)f(s)ds + \mathcal{U}(t_2,t_1)\mathcal{I}_k(z(t_1)) + \mathcal{I}_k(z(t_2))\right\} + \int_{t_2}^{t}\mathcal{U}(t,s)f(s)ds$$

$$= \mathcal{U}(t,0)z_0 + \int_0^{t_2}\mathcal{U}(t,s)f(s)ds + \mathcal{U}(t,t_1)\mathcal{I}_k(z(t_1)) + \mathcal{U}(t,t_2)\mathcal{I}_k(z(t_2) + \int_{t_2}^{t}\mathcal{U}(t,s)f(s)ds$$

$$= \mathcal{U}(t,0)z_0 + \int_0^{t}\mathcal{U}(t,s)f(s)ds + \sum_{k=1}^{2}\mathcal{U}(t,t_k)\mathcal{I}_k(z(t_k)).$$

In this way, for $t \in [t_3, t_4)$, $t \in [t_4, t_5)$, ..., $t \in [t_p, t_{p+1})$, we get, by induction, that

$$z(t) = \mathcal{U}(t,0)z_0 + \int_0^{t}\mathcal{U}(t,s)f(s)ds + \sum_{k=1}^{p}\mathcal{U}(t,t_k)\mathcal{I}_k(z(t_k)),$$

i.e.,

$$z(t) = \mathcal{U}(t,0)z_0 + \int_0^{t}\mathcal{U}(t,s)f(s)ds + \sum_{0<t_k<t}\mathcal{U}(t,t_k)\mathcal{I}_k(z(t_k)), \quad t \in [0,\tau].$$

So that, changing the variable back, we obtain (27).

$\Leftarrow$) Now, assume that z is the solution of the integral equation (27) and consider the same change of variable (28) and (29), for $t \in [0, t_1)$, we get that

$$z(t) = \mathcal{U}(t,0)z_0 + \int_0^{t}\mathcal{U}(t,s)f(s)ds,$$

$$z(t_1^-) = \mathcal{U}(t_1,0)z_0 + \int_0^{t_1}\mathcal{U}(t_1,s)f(s)ds,$$

and

$$z(t_1^+) = \mathcal{U}(t_1,0)z_0 + \int_0^{t_1}\mathcal{U}(t_1,s)f(s)ds + \mathcal{U}(t_1,t_1)\mathcal{I}_1(z(t_1)).$$

Consequently, $z(t_1^+) = z(t_1^-) + \mathcal{I}_1(z(t_1))$. In the same way, for $t \in [t_1, t_2)$, clearly we have that

$$z(t_2^-) = \mathcal{U}(t_2,0)z_0 + \int_0^{t_2}\mathcal{U}(t_2,s)f(s)ds + \mathcal{U}(t_2,t_1)\mathcal{I}_1(z(t_1))$$

and,

$$z(t_2^+) = \mathcal{U}(t_2,0)z_0 + \int_0^{t_2}\mathcal{U}(t_2,s)f(s)ds + \mathcal{U}(t_2,t_1)\mathcal{I}_1(z(t_1)) + \mathcal{U}(t_2,t_2)\mathcal{I}_2(z(t_2)).$$

So that,

$$z(t_2^+) = z(t_2^-) + \mathcal{I}_2(z(t_2)).$$

Continuing this process, we get for $k = 1, 2, 3, \ldots, p$

$$z(t_k^+) = z(t_k^-) + \mathcal{I}_k(z(t_k)).$$

Now, differentiating with respect to t for $t \in (0, \tau]$, $t \neq t_k$, and using the Proposition (3.1), we get

$$z'(t) = \mathcal{A}(t)\mathcal{U}(t,0)z_0 + \mathcal{A}(t)\int_0^{t}\mathcal{U}(t,s)f(s)ds + f(t) + \mathcal{A}(t)\sum_{0<t_k<t}\mathcal{U}(t,t_k)\mathcal{I}_k(z(t_k))$$

$$= \mathcal{A}(t)\left\{\mathcal{U}(t,0)z_0 + \int_0^{t}\mathcal{U}(t,s)f(s)ds + \sum_{0<t_k<t}\mathcal{U}(t,t_k)\mathcal{I}_k(z(t_k))\right\} + f(t)$$

$$= \mathcal{A}(t)z(t) + f(t).$$

Hence, $z(t)$ is solution of (1). This finishes the proof. □

4.1.2 Existence Theorems.

Now, let us consider the following hypotheses in order to prove the existence of solutions for system (1):

(H1) The function $\mathcal{F} : [0, \tau] \times C(-r, 0; \mathbb{R}^n) \to \mathbb{R}^n$ satisfies the following conditions.

i) $\|\mathcal{F}(t, \phi_1) - \mathcal{F}(t, \phi_2)\| \le \mathcal{K}(\|\phi_1\|, \|\phi_2\|)\|\phi_1 - \phi_2\|, \quad \phi_1, \phi_2 \in C(-r, 0; \mathbb{R}^n)$,

ii) $\|\mathcal{F}(t, \phi)\| \le \Psi(\|\phi\|), \quad \phi \in C(-r, 0; \mathbb{R}^n)$,

where $\mathcal{K} : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ and $\Psi : \mathbb{R}_+ \to \mathbb{R}_+$ are continuous and non decreasing functions of their arguments.

(H2) There exist constants $L_q, d_k > 0, \quad k = 1, 2, \ldots, p$ such that

i) $ML_q q < M\sum_{k=1}^{p} d_k < \frac{1}{2}, \quad \|\mathcal{I}_k(y) - \mathcal{I}_k(z)\|_{\mathbb{R}^n} \le d_k\|y - z\|_{\mathbb{R}^n}, \quad y, z \in \mathbb{R}^n$, where M is a bound for the evolution operator $\mathcal{U}(t, s)$, i.e.,

$$||\mathcal{U}(t, s)|| \le M, \quad 0 \le s \le t \le \tau.$$

ii) We have $\mathcal{H}(0) \equiv 0$ and

$$\|\mathcal{H}(y)(t) - \mathcal{H}(v)(t)\|_{\mathbb{R}^n} \le L_q \sum_{i=1}^{q} \|y_i(t) - v_i(t)\|_{\mathbb{R}^n}, \quad \forall y, v \in C(-r, \tau; (\mathbb{R}^n)^q).$$

(H3) Given $\rho, \tau > 0$, assume that the relation holds,

$$\left(ML_q q + M\sum_{k=1}^{p} d_k\right)(\|\tilde{\phi}\| + \rho) + \tau M\Psi(\|\tilde{\phi}\| + \rho) \le \rho,$$

where the function $\tilde{\phi}$ is defined as follows:

$$\tilde{\phi}(t) = \begin{cases} \mathcal{U}(t, 0)\phi(0), & t \in [0, \tau] \\ \phi(t), & t \in [-r, 0]. \end{cases} \tag{32}$$

(H4) For $\rho, \tau > 0$ the inequality holds,

$$\tau M\mathcal{K}(\|\tilde{\phi}\| + \rho, \|\tilde{\phi}\| + \rho) + M\sum_{k=1}^{p} d_k < \frac{1}{2}.$$

Theorem 4.2. *Suppose that (H1)-(H3) hold. Then, the system (1) has at least one solution on* $[-r, \tau]$.

We shall transform the problem of finding solutions for system (1) into a fixed point problem. Therefore, let's define the following two operators

$$\mathcal{T} : PC_{t_1..t_p}(-r, \tau; \mathbb{R}^n) \times PC_{t_1..t_p}(-r, \tau; \mathbb{R}^n) \to PC_{t_1..t_p}(-r, \tau; \mathbb{R}^n)$$

and,

$$\mathcal{C} : PC_{t_1..t_p}(-r, \tau; \mathbb{R}^n) \to PC_{t_1..t_p}(-r, \tau; \mathbb{R}^n),$$

defined by

$$\mathcal{T}(z, y)(t) = \begin{cases} y(t) + \sum_{0<t_k<t}\mathcal{U}(t, t_k)\mathcal{I}_k(z(t_k)), & t \in [0, \tau] \\ \phi(t) - \mathcal{H}(z_{\tau_1}, z_{\tau_2}, \ldots, z_{\tau_q}))(t), & t \in [-r, 0] \end{cases}$$

and,

$$\mathcal{C}(y)(t) = \begin{cases} \mathcal{U}(t, 0)\{\phi(0) - \mathcal{H}(y_{\tau_1}, y_{\tau_2}, \ldots, y_{\tau_q})(0)\} + \int_0^t \mathcal{U}(t, s)\mathcal{F}(s, y_s)ds, & t \in [0, \tau] \\ \phi(t), & t \in [-r, 0]. \end{cases}$$

Also, let's define the following closed and convex set

$$D = D(\rho,\tau,\phi) = \{y \in PC_{t_1..t_p}([-r,\tau];\mathbb{R}^n) : \|y-\tilde{\phi}\| \le \rho\}, \tag{33}$$

where the function $\tilde{\phi}$ is defined in (32) and $\rho > 0$. Now, finding the solution of problem (1) is reduced to the problem of finding solutions of the operator equation

$$\mathcal{T}(z,\mathcal{C}(z)) = z.$$

First, we shall prove that the operator $\mathcal{C}$ is compact. After that, we shall prove that $\{\mathcal{T}(\cdot,y) : y \in \overline{\mathcal{C}(D)}\}$ is equicontractive, where D is the closed convex set given by (33). So, by applying Theorem 2.7, we get the result.

Proof. The proof of Theorem 4.2 will be given by Claims:

Claim 1: $\mathcal{C}$ is continuous.

In fact, using the hypothesis $i)$ of (**H1**) and $ii)$ of (**H2**), we have the following estimate for $z, y \in PC_{t_1..t_p}(-r,\tau;\mathbb{R}^n)$.

$$\begin{aligned}
\|\mathcal{C}(z)(t)-\mathcal{C}(y)(t)\|_{\mathbb{R}^n} &\le ML_q q\|z-y\| + \int_0^t \|\mathcal{U}(t,s)\|_{\mathbb{R}^n}\|(\mathcal{F}(s,z_s)-\mathcal{F}(s,y_s))\|_{\mathbb{R}^n}ds \\
&\le ML_q q\|z-y\| + M\int_0^t \mathcal{K}(\|z_s\|,\|y_s\|)\|z_s-y_s\|ds \\
&\le ML_q q\|z-y\| + M\tau\mathcal{K}(\|z\|,\|y\|)\|z-y\|.
\end{aligned}$$

Taking supremum, we have:

$$\|\mathcal{C}(z)-\mathcal{C}(y)\| \le \left(ML_q q + M\tau\mathcal{K}(\|z\|,\|y\|)\right)\|z-y\|.$$

Hence, we conclude that $\mathcal{C}$ is continuous. Moreover, $\mathcal{C}$ is locally Lipschitz.

Claim 2: $\mathcal{C}$ maps bounded sets of $PC_{t_1..t_p}(-r,\tau;\mathbb{R}^n)$ into bounded sets of $PC_{t_1..t_p}(-r,\tau;\mathbb{R}^n)$.

It is enough to show that for any $R > 0$ there exists $r > 0$ such that for each $y \in B_R = \{z \in PC_{t_1..t_p} : \|z\| \le R\}$, we have that $\|\mathcal{C}(y)\| \le r$. In fact, taking $y \in B_R$ and using $ii)$ of (**H1**) and $ii)$ of (**H2**), then the following estimates hold

$$\|\mathcal{C}(y)(t)\|_{\mathbb{R}^n} = \|\phi(t)\|_{\mathbb{R}^n} \le \|\phi\|, \quad \forall t \in [-r,0],$$

and,

$$\begin{aligned}
\|\mathcal{C}(y)(t)\|_{\mathbb{R}^n} &\le \|\mathcal{U}(t,0)\{\phi(0)-\mathcal{H}(y_{\tau_1},y_{\tau_2},\dots,y_{\tau_q})(0)\}\| + \int_0^t \|\mathcal{U}(t,s)\mathcal{F}(s,y_s))\|_{\mathbb{R}^n}ds \\
&\le M\{\|\phi(0)\|_{\mathbb{R}^n} + L_q q\|y\|\} + \tau M\Psi(\|y\|) \\
&\le M\{\|\phi(0)\|_{\mathbb{R}^n} + L_q qR\} + \tau M\Psi(R) = l, \quad \forall t \in (0,\tau].
\end{aligned}$$

Taking supremum on t and $r = \max\{l, \|\phi\|\}$, we have that

$$\|\mathcal{C}(y)\| \le r.$$

Hence, Claim 2 holds.

Claim 3: $\mathcal{C}$ maps bounded sets of $PC_{t_1..t_p}(-r,\tau;\mathbb{R}^n)$ into equicontinuous sets of $PC_{t_1..t_p}(-r,\tau;\mathbb{R}^n)$.

In fact, consider B_R as in the foregoing claim. Then, we shall prove that $\mathcal{C}(B_R)$ is equicontinuous on the interval $[-r,\tau]$. Clearly, it is sufficient to prove this on $(0,\tau]$. Considering $ii)$ of (**H1**), $ii)$ of (**H2**), and $y \in B_R$ the following estimate holds:

$$\begin{aligned}
\|\mathcal{C}(y)(t_2)-\mathcal{C}(y)(t_1)\|_{\mathbb{R}^n} \leq\ & \|\mathcal{U}(t_2,0)\{\phi(0)-\mathcal{H}(y_{\tau_1},y_{\tau_2},\ldots,y_{\tau_q})(0)\}+\int_0^{t_2}\mathcal{U}(t_2,s)\mathcal{F}(s,y_s)ds \\
& -\mathcal{U}(t_1,0)\{\phi(0)-\mathcal{H}(y_{\tau_1},y_{\tau_2},\ldots,y_{\tau_q})(0)\}-\int_0^{t_1}\mathcal{U}(t_1,s)\mathcal{F}(s,y_s)ds\|_{\mathbb{R}^n} \\
\leq\ & \|[\mathcal{U}(t_2,0)-\mathcal{U}(t_1,0)]\{\phi(0)-\mathcal{H}(y_{\tau_1},y_{\tau_2},\ldots,y_{\tau_q})(0)\}\|_{\mathbb{R}^n} \\
& +\left\|\int_0^{t_1}\mathcal{U}(t_2,s)\mathcal{F}(s,y_s)ds+\int_{t_1}^{t_2}\mathcal{U}(t_2,s)\mathcal{F}(s,y_s)ds\right. \\
& \left.-\int_0^{t_1}\mathcal{U}(t_1,s)\mathcal{F}(s,y_s)ds\right\|_{\mathbb{R}^n} \\
\leq\ & \|\mathcal{U}(t_2,0)-\mathcal{U}(t_1,0)\|(\|\phi(0)-\mathcal{H}(y_{\tau_1},y_{\tau_2},\ldots,y_{\tau_q})(0)\|_{\mathbb{R}^n}) \\
& +\int_0^{t_1}\|[\mathcal{U}(t_2,s)-\mathcal{U}(t_1,s)]\mathcal{F}(s,y_s)\|ds+\int_{t_1}^{t_2}\|\mathcal{U}(t_2,s)\mathcal{F}(s,y_s)\|ds \\
\leq\ & \|\mathcal{U}(t_2,0)-\mathcal{U}(t_1,0)\|(\|\phi(0)\|_{\mathbb{R}^n}+L_q\sum_{i=1}^{q}\|y_i(t)\|_{\mathbb{R}^n}) \\
& +\Psi(\|y\|)\int_0^{t_1}\|\mathcal{U}(t_2,s)-\mathcal{U}(t_1,s)\|ds+M\Psi(\|y\|)\int_{t_1}^{t_2}ds \\
\leq\ & \|\mathcal{U}(t_2,0)-\mathcal{U}(t_1,0)\|(\|\phi(0)\|_{\mathbb{R}^n}+L_q q\|y\|_{\mathbb{R}^n}) \\
& +\Psi(R)\int_0^{t_1}\|\mathcal{U}(t_2,s)-\mathcal{U}(t_1,s)\|ds+M\Psi(R)(t_2-t_1) \\
\leq\ & \|\mathcal{U}(t_2,0)-\mathcal{U}(t_1,0)\|(\|\phi(0)\|_{\mathbb{R}^n}+L_q qR) \\
& +\Psi(R)\int_0^{t_1}\|(\mathcal{U}(t_2,s)-\mathcal{U}(t_1,s))\|ds+M\Psi(R)(t_2-t_1).
\end{aligned}$$

Since $(\|\phi(0)\|_{\mathbb{R}^n}+L_q qR)$ is bounded and $\mathcal{U}(t,s)$ is continuous, $\|\mathcal{U}(t_2,0)-\mathcal{U}(t_1,0)\|(\|\phi(0)\|_{\mathbb{R}^n}+L_q qR)\to 0$ as $t_2\to t_1$. Also, $\Psi(R)\int_0^{t_1}\|(\mathcal{U}(t_2,s)-\mathcal{U}(t_1,s))\|ds\to 0$, as $t_2\to t_1$ by the continuity of $\mathcal{U}(t,s)$; and clearly $M\Psi(R)(t_2-t_1)$ goes to zero, as t_2 goes to t_1. Therefore, $\|\mathcal{C}(y)(t_2)-\mathcal{C}(y)(t_1)\|_{\mathbb{R}^n}$ goes to zero as t_2 goes to t_1, i.e., $\mathcal{C}(B_R)$ is equicontinuous.

Claim 4: The subset $\mathcal{C}(D)$ is a relatively compact in $PC_{t_1..t_p}(-r,\tau;\mathbb{R}^n)$.

In order to prove Claim 4, the following lemma will be considered.

Definition 4.3. *Given $y\in PC_{t_1..t_p}(-r,\tau;\mathbb{R}^n)$ and $i=1,2,\ldots,p$, we define the function $\widetilde{y}_i\in C(t_i,t_{i+1};\mathbb{R}^n)$ such that*

$$\widetilde{y}_i(t)=\begin{cases} y(t), & \text{for } \ t\in[t_i,t_{i+1}), \\ y(t_{i+1}^-), & \text{for } \ t=t_{i+1}, \end{cases}$$

and for $W\subset PC_{t_1..t_p}(-r,\tau;\mathbb{R}^n)$ with $i=1,2,\ldots,p$, we define $\widetilde{W}_i=\{\widetilde{y}_i : y\in W\}$.

Lemma 4.4. *A set $W\subset PC_{t_1..t_p}(-r,\tau;\mathbb{R}^n)$ is relatively compact in $PC_{t_1..t_p}(-r,\tau;\mathbb{R}^n)$ if, and only if, each set $\widetilde{W}_i, i=1,2,\ldots,p$, with $t_0=0$ and $t_{p+1}=\tau$, is relatively compact in $C([t_i,t_{i+1}];\mathbb{R}^n)$.*

In fact, let D be a bounded subset of $PC_{t_1..t_p}(-r,\tau;\mathbb{R}^n)$. From Claims 2 and 3, $W=\mathcal{C}(D)$ is bounded and equicontinuous in $PC_{t_1..t_p}(-r,\tau;\mathbb{R}^n)$. Then applying Arzelà-Ascoli Theorem, we get that $\widetilde{W}_i$ is relatively compact in $C(t_i,t_{i+1};\mathbb{R}^n)$. Hence, from Lemma 4.4, we obtain that $W=\mathcal{C}(D)$ is relatively compact, in particular $\mathcal{C}$ is a compact operator by definition of compactness.

Claim 5: The family $\{\mathcal{T}(\cdot,y) : y \in \overline{\mathcal{C}(D)}\}$ is equicontractive and the conditions of Theorem 2.7 are satisfied for the closed and convex set given in (33). In fact, for $z, x \in PC_{t_1..t_p}(-r,\tau;\mathbb{R}^n)$, and using $ii)$ of (**H2**), we obtain

$$\begin{aligned}\|\mathcal{T}(z,\mathcal{C}(y))(t)-\mathcal{T}(x,\mathcal{C}(y))(t)\|_{\mathbb{R}^n} &\leq \|\mathcal{H}(z_{\tau_1},z_{\tau_2},\dots,z_{\tau_q})(t)-\mathcal{H}(x_{\tau_1},x_{\tau_2},\dots,x_{\tau_q})(t)\|_{\mathbb{R}^n}\\ &\leq L_q q\|z-x\|_{\mathbb{R}^n}\\ &\leq ML_q q\|z-x\|_{\mathbb{R}^n} \quad \forall t\in[-r,0],\end{aligned}$$

and using $i)$ of (**H2**), we can get

$$\begin{aligned}\|\mathcal{T}(z,\mathcal{C}(y))(t)-\mathcal{T}(x,\mathcal{C}(y))(t)\|_{\mathbb{R}^n} &\leq \|\sum_{0<t_k<t}\mathcal{U}(t,t_k)\mathcal{I}_k(z(t_k))-\sum_{0<t_k<t}\mathcal{U}(t,t_k)\mathcal{I}_k(x(t_k))\|_{\mathbb{R}^n}\\ &\leq \sum_{0<t_k<t}\|\mathcal{U}(t,t_k)(\mathcal{I}_k(z(t_k))-\mathcal{I}_k(x(t_k)))\|_{\mathbb{R}^n}\\ &\leq M\sum_{k=1}^{p}\|(\mathcal{I}_k(z(t_k))-\mathcal{I}_k(x(t_k)))\|_{\mathbb{R}^n}\\ &\leq M\sum_{k=1}^{p}d_k\|z(t_k)-x(t_k)\|_{\mathbb{R}^n}\\ &\leq (M\sum_{k=1}^{p}d_k)\|z-x\|_{\mathbb{R}^n} \quad \forall t\in(0,\tau].\end{aligned}$$

Taking supremum on t and from $i)$ of (**H2**), we have that

$$\|\mathcal{T}(z,\mathcal{C}(y))-\mathcal{T}(x,\mathcal{C}(y))\| \leq (M\sum_{k=1}^{p}d_k)\|z-x\| \leq \frac{1}{2}\|z-x\|$$

is a contraction independently of $y \in \overline{\mathcal{C}(D)}$. Hence, that the family $\{\mathcal{T}(\cdot,y) : y \in \overline{\mathcal{C}(D)}\}$ is equicontractive.

Finally, we shall prove that

$$\mathcal{T}(\cdot,\mathcal{C})D(\rho,\tau,\phi) \subset D(\rho,\tau,\phi).$$

In fact, let us consider $z \in D(\rho,\tau,\phi)$ and $t \in [-r,0]$. Then,

$$\mathcal{T}(z,\mathcal{C}(z))(t) = \phi(t) - \mathcal{H}(z_{\tau_1},z_{\tau_2},\dots,z_{\tau_q})(t),$$

and for $t \in [0,\tau]$, we get

$$\begin{aligned}\mathcal{T}(z,\mathcal{C}(z))(t) &= \mathcal{U}(t,0)[\phi(0)-\mathcal{H}(z_{\tau_1},z_{\tau_2},\dots,z_{\tau_q})(0)]\\ &\quad + \int_0^t \mathcal{U}(t,s)\mathcal{F}(s,z_s)ds + \sum_{0<t_k<t}\mathcal{U}(t,t_k)\mathcal{I}_k(z(t_k)).\end{aligned}$$

Therefore, for $t \in [-r,0]$, $ii)$ of (**H2**), and (**H3**), we obtain

$$\begin{aligned}\|\mathcal{T}(z,\mathcal{C}(z))(t)-\tilde{\phi}(t)\|_{\mathbb{R}^n} &= \|\mathcal{H}(z_{\tau_1},z_{\tau_2},\dots,z_{\tau_q}))(t)\|_{\mathbb{R}^n}\\ &\leq L_q q\|z\| \leq ML_q q\|z\| \leq ML_q q(\|\phi\|+\rho) < \rho,\end{aligned}$$

and for $t \in (0, \tau]$, moreover, using $ii)$ of (**H1**), $ii)$ of (**H2**), and (**H3**), we have

$$\begin{aligned}
\|\mathcal{T}(z, \mathcal{C}(z))(t) - \tilde{\phi}(t)\|_{\mathbb{R}^n} &\leq M\|\mathcal{H}(z_{\tau_1}, z_{\tau_2}, \dots, z_{\tau_q}))(0)\|_{\mathbb{R}^n} + \int_0^t \|\mathcal{U}(t,s)\mathcal{F}(s, z_s)\|ds \\
&\quad + \sum_{0<t_k<t} \|\mathcal{U}(t, t_k)\mathcal{I}_k(z(t_k))\| \\
&\leq ML_q q\|z\| + M\tau\Psi(\|z\|) + M\sum_{k=1}^{p} d_k\|z\| \\
&\leq ML_q q(\|\tilde{\phi}\| + \rho) + M\tau\Psi(\|\tilde{\phi}\| + \rho) + (M\sum_{k=1}^{p} d_k)(\|\tilde{\phi}\| + \rho) \\
&= \left(ML_q q + M\sum_{k=1}^{p} d_k\right)(\|\tilde{\phi}\| + \rho) + M\tau\Psi(\|\tilde{\phi}\| + \rho) \\
&\leq \rho.
\end{aligned}$$

Hence, $\mathcal{T}(\cdot, \mathcal{B})D(\rho, \tau, \phi) \subset D(\rho, \tau, \phi)$. Since Claim 1, Claim 4 and Claim 5 hold, the proof of Theorem 4.2 immediately follows by applying Theorem 2.7. □

4.1.3 Uniqueness Theorem and Prolongation of Solutions

Theorem 4.5. *In addition to the conditions of Theorem 4.2, suppose that* (**H4**) *holds, then the semilinear system (1) has only one solution on* $[-r, \tau]$.

Proof. Assume that (**H4**) holds and let z_1 and z_2 be two solutions of problem (1). Then, using the hypotheses (**H1**) and (**H2**) we have the following estimate

$$\begin{aligned}
\|z_1(t) - z_2(t)\|_{\mathbb{R}^n} &\leq \|\mathcal{H}(z_{1,\tau_1}, z_{1,\tau_2}, \dots, z_{1,\tau_q})(0) - \mathcal{H}(z_{2,\tau_1}, z_{2,\tau_2}, \dots, z_{2,\tau_q})(0)\|_{\mathbb{R}^n} \\
&\quad + \int_0^t \|\mathcal{U}(t,s)(\mathcal{F}(s, z_1) - \mathcal{F}(s, z_2))\|ds \\
&\quad + \sum_{0<t_k<t} \|\mathcal{U}(t, t_k)(\mathcal{I}_k(z_1(t_k)) - \mathcal{I}_k(z_2(t_k)))\| \\
&\leq \left(M\tau\mathcal{K}(\|\tilde{\phi}\| + \rho, \|\tilde{\phi}\| + \rho) + M\sum_{k=1}^{p} d_k\right)\|z_1 - z_2\|.
\end{aligned}$$

By the assumption, we get

$$\tau M\mathcal{K}(\|\phi\| + \rho, \|\phi\| + \rho) + M\sum_{k=1}^{p} d_k < 1.$$

Taking supremum, we have

$$\|z_1 - z_2\| \leq l\|z_1 - z_2\| \quad \text{with} \quad 0 \leq l < 1,$$

which implies that

$$\|z_1 - z_2\| = 0$$

i.e.,

$$z_1 = z_2.$$

□

Now, we shall consider the following subset $\widetilde{D}$ of $\mathbb{R}^n$

$$\widetilde{D} = \{y \in \mathbb{R}^n : \|y\|_{\mathbb{R}^n} \leq R\}, \quad \text{with} \quad R = \|\tilde{\phi}\| + \rho. \tag{34}$$

Therefore, for all $z \in D$ and for $-r \leq t \leq \tau$, we have $z(t) \in \widetilde{D}$.

Definition 4.6. *We shall say that $[-r, s_1)$ is a maximal interval of existence of the solution $z(\cdot)$ of problem (1) if there is no solution of the (1) on $[-r, s_2)$ with $s_2 > s_1$.*

Theorem 4.7. *Suppose that the conditions of Theorem 4.5 hold. If z is a solution of problem (1) on $[-r, s_1)$ and s_1 is maximal, then either $s_1 = +\infty$ or there exists a sequence $\tau_n \to s_1$ as $n \to \infty$ such that $z(\tau_n) \to \partial\widetilde{D}$.*

Proof. Suppose, for the purpose of contradiction, that there exist a neighborhood N of $\partial\tilde{D}$ such that $z(t)$ does not enter in it, for $0 < s_2 \leq t < s_1$. We can take $N = \tilde{D}\backslash B$, where B is a closed subset of $\tilde{D}$, then $z(t) \in B$ for $0 < t_p < s_2 \leq t < s_1$. We need to prove that $\lim_{t\to s_1} z(t) = z_1 \in B$. In fact, if we consider $0 < t_p < s_2 \leq \ell < t < s_1$, then

$$\begin{aligned}
\|z(t) - z(\ell)\|_{\mathbb{R}^n} &\leq \|\mathcal{U}(t,0) - \mathcal{U}(\ell,0)\|\|\phi(0)\|_{\mathbb{R}^n} + \|\mathcal{U}(t,0) - \mathcal{U}(\ell,0)\|\|\mathcal{H}(z_{\tau_1}, z_{\tau_2}, \ldots, z_{\tau_q})(0)\|_{\mathbb{R}^n} \\
&\quad + \int_0^\ell \|\mathcal{U}(t,s) - \mathcal{U}(\ell,s)\|\|\mathcal{F}(s,z_s)\|ds + \int_\ell^t \|(\mathcal{U}(t,0)\|\|\mathcal{F}(s,z_s)\|ds \\
&\quad + \|\sum_{0<t_k<t} \mathcal{U}(t,t_k)\mathcal{I}_k(z(t_k)) - \sum_{0<t_k<t} \mathcal{U}(\ell,t_k)\mathcal{I}_k(z(t_k))\| \\
&\leq (\|\mathcal{U}(t,0) - \mathcal{U}(\ell,0)\|)\left(\|\phi(0)\|_{\mathbb{R}^n} + L_q q\right) \\
&\quad + \left(\int_0^\ell \|\mathcal{U}(t,s) - \mathcal{U}(\ell,s)\|ds + \int_\ell^t \|(\mathcal{U}(t,0)\|ds\right)\Psi(R) \\
&\quad + \|\mathcal{U}(t,\ell) - I\| \sum_{k=1}^q \|\mathcal{U}(\ell,t_k)\|\|\mathcal{I}_k(z(t_k)\| \\
&\leq (\|\mathcal{U}(t,0) - \mathcal{U}(\ell,0)\|)\left(\|\phi(0)\|_{\mathbb{R}^n} + L_q q\right) \\
&\quad + \left(\int_0^\ell \|\mathcal{U}(t,s) - \mathcal{U}(\ell,s)\|ds + \int_\ell^t \|(\mathcal{U}(t,0)\|ds\right)\Psi(R) \\
&\quad + \|\mathcal{U}(t,\ell) - I\| M R \sum_{k=1}^q d_k.
\end{aligned}$$

Since $\mathcal{U}(t,s)$ is a continuous operator for $t \geq 0$, $\|z(t) - z(\ell)\|_{\mathbb{R}^n}$ goes to zero as $\ell \to s_1$. Therefore, $\lim_{t\to s_1} z(t) = z_1$ exists in $\mathbb{R}^n$, and since B is closed, z_1 belongs to B. This completes the proof.

□

Corollary 4.8. *In the conditions of Theorem 4.5, if the second part of hypothesis* **(H1)** *is changed by*

$$\|\mathcal{F}(t,\phi)\| \leq h(t)(1 + \|\phi(0)\|_{\mathbb{R}^n}), \quad \phi \in C(-r, 0; \mathbb{R}^n),$$

where $h(\cdot)$ is a continuous function on $[-r, \infty)$, then a unique solution of problem (1) exists on $[-r, \infty)$.

Proof.

$$\begin{aligned}
\|z(t)\|_{\mathbb{R}^n} &\leq M\left(\|\phi(0)\|_{\mathbb{R}^n} + \|\mathcal{H}(z_{\tau_1}, z_{\tau_2}, \ldots, z_{\tau_q})(0)\|_{\mathbb{R}^n}\right) + M\int_0^t \|\mathcal{F}(s,z_s)\|ds \\
&\quad + M\sum_{0<t_k<t} \|\mathcal{I}_k(z(t_k))\|_{\mathbb{R}^n} \\
&\leq M\left(\|\phi(0)\|_{\mathbb{R}^n} + L_q\|\tilde{z}(0)\|_q^n\right) + \int_0^t Mh(s)(1 + \|z(s)\|_{\mathbb{R}^n})ds \\
&\quad + M\sum_{k=1}^p d_k\|z(t_k))\|_{\mathbb{R}^n} \\
&\leq M\left(\|\phi(0)\|_{\mathbb{R}^n} + L_q\|\tilde{z}(0)\|_q^n + \int_0^t h(s)ds\right) + \int_0^t Mh(s)\|z(s)\|_{\mathbb{R}^n}ds + \sum_{k=1}^p Md_k\|z(t_k))\|_{\mathbb{R}^n}.
\end{aligned}$$

Then applying Lemma 2.4, we get the following estimate

$$\|z(t)\|_{\mathbb{R}^n} \leq M \left(\|\phi(0)\|_{\mathbb{R}^n} + L_q \|\tilde{z}(0)\|_q^n + \tau \right) \prod_{t_0 < t_k < t} (1 + M d_k) e^{\int_0^\tau M h(s) ds},$$

$\tilde{z} = (z_{\tau_1}, z_{\tau_2}, \ldots, z_{\tau_q})^T$. This implies that $\|z(t)\|_{\mathbb{R}^n}$ remains bounded as $t \to s_1$ and applying Theorem 4.7, we get the result. □

4.2 Controllability of Semilinear Non-autonomous Systems with Impulses, Delays and Nonlocal Conditions

Sitting the problem

The control system corresponding to the semilinear time varying system of differential equations with impulses, delays, and nonlocal conditions is given by

$$\begin{cases} z'(t) = \mathcal{A}(t)z(t) + \mathcal{B}(t)u(t) + f(t, z_t, u(t)), & t \in (0, \tau], \quad t \neq t_k, \\ z(s) + \mathcal{H}(z_{\tau_1}, z_{\tau_2}, \ldots, z_{\tau_q})(s) = \phi(s), & s \in [-r, 0], \\ z(t_k^+) = z(t_k^-) + \mathcal{I}_k(z(t_k), u(t_k)), & k = 1, 2, 3, \ldots, p, \end{cases}$$

where $0 < t_1 < t_2 < \cdots < t_p < \tau$, $0 < \tau_1 < \tau_2 < \cdots < \tau_q < \tau$, $z(t) \in \mathbb{R}^n$, $u(t) \in \mathbb{R}^m$, z_t defined as a function from $[-r, 0]$ to $\mathbb{R}^n$ by $z_t(s) = z(t+s), -r \leq s \leq 0$, $\mathcal{A}(t)$, $\mathcal{B}(t)$ are continuous matrices of dimension $n \times n$ and $n \times m$ respectively, the control function u belongs to $C(0, \tau; \mathbb{R}^m)$, $\mathcal{H} : C_q = C(-r, 0; (\mathbb{R}^n)^q) \to C(-r, 0; \mathbb{R}^n)$, $\phi \in C(-r, 0; \mathbb{R}^n)$, $f : [0, \tau] \times C(-r, 0; \mathbb{R}^n) \times \mathbb{R}^m \to \mathbb{R}^n, \mathcal{I}_k \in C(\mathbb{R}^n \times \mathbb{R}^m; \mathbb{R}^n)$, $k = 1, 2, 3, \ldots, p$.

In addition to the Banach space given in (23) and its norm (24), the product space $(\mathbb{R}^n)^q$ given in (25), and the norm given in (26); we also consider the following space and the following norms. We shall consider the Banach space

$$PC_{t_1..t_p}(-r, \tau; \mathbb{R}^n) \times C(0, \tau; \mathbb{R}^m),$$

endowed with the norm

$$|||(z, u)||| = \|z\| + \|u\|.$$

In $\mathbb{R}^n \times \mathbb{R}^m$ we consider the norm

$$\|(z, u)\|_1 = \|z\|_{\mathbb{R}^n} + \|u\|_{\mathbb{R}^m}, \quad \forall (z, u) \in \mathbb{R}^n \times \mathbb{R}^m.$$

For all $(z, u) \in PC_{t_1..t_p}(-r, \tau; \mathbb{R}^n) \times C(0, \tau; \mathbb{R}^m)$, we define the following quantity

$$\|f(\cdot, u, z)\| = \sup_{t \in [0, \tau]} \|f(t, u(t), z(t))\|_{\mathbb{R}^n}.$$

4.2.1 Exact Controllability Applying Fixed Point Theorem

We shall transform the controllability problem (2) into a fixed point problem. For this, we define the following operator $\mathcal{S} : PC_{t_1..t_p}(-r, \tau; \mathbb{R}^n) \times C(0, \tau; \mathbb{R}^m) \to PC_{t_1..t_p}(-r, \tau; \mathbb{R}^n) \times C(0, \tau; \mathbb{R}^m)$ by the following formula

$$(y, v) = (\mathcal{S}_1(z, u), \mathcal{S}_2(z, u)) = \mathcal{S}(z, u),$$

where $\mathcal{S}_1$ and $\mathcal{S}_2$ are defined as follow:

$$\mathcal{S}_1 : PC_{t_1..t_p}(-r, \tau; \mathbb{R}^n) \times C(0, \tau; \mathbb{R}^m) \to PC_{t_1..t_p}(-r, \tau; \mathbb{R}^n)$$

and,

$$\mathcal{S}_2 : PC_{t_1..t_p}(-r, \tau; \mathbb{R}^n) \times C(0, \tau; \mathbb{R}^m) \to C(0, \tau; \mathbb{R}^m),$$

such that for $t \in (0, \tau]$, we have that

$$\begin{aligned} y(t) &= \mathcal{S}_1(z,u)(t) \\ &= \mathcal{U}(t,0)\{\phi(0) - \mathcal{H}(z_{\tau_1}, z_{\tau_2}, \dots, z_{\tau_q})(0)\} \\ &\quad + \int_0^t \mathcal{U}(t,s)\mathcal{B}(s)(\Upsilon\mathfrak{L}(z,u))(s)ds \\ &\quad + \int_0^t \mathcal{U}(t,s)f(s,z_s,u(s))ds + \sum_{0<t_k<t} \mathcal{U}(t,t_k)\mathcal{I}_k(z(t_k),u(t_k)), \end{aligned} \tag{35}$$

and

$$v(t) = \mathcal{S}_2(z,u)(t) = (\Upsilon\mathfrak{L}(z,u))(t) = \mathcal{B}^*(t)\mathcal{U}^*(\tau,t)\mathcal{W}^{-1}\mathfrak{L}(z,u), \tag{36}$$

with $\mathfrak{L} : PC_{t_1..t_p}(-r,\tau;\mathbb{R}^n) \times C(0,\tau;\mathbb{R}^m) \to \mathbb{R}^n$ given by

$$\begin{aligned} \mathfrak{L}(z,u) = z_1 &- \mathcal{U}(\tau,0)\{\phi(0) - \mathcal{H}(z_{\tau_1}, z_{\tau_2}, \dots, z_{\tau_q}))(0)\} \\ &- \int_0^\tau \mathcal{U}(\tau,s)f(s,z_s,u(s))ds \\ &- \sum_{0<t_k<\tau} \mathcal{U}(\tau,t_k)\mathcal{I}_k(z(t_k),u(t_k)). \end{aligned} \tag{37}$$

Also, let's define the following operator $\Upsilon : \mathbb{R}^n \to C(0,\tau;\mathbb{R}^m)$ given by

$$\Upsilon z = \mathcal{B}^*(\cdot)\mathcal{U}^*(\tau,\cdot)\mathcal{W}^{-1}z = \mathcal{G}^*(\mathcal{G}\mathcal{G}^*)^{-1}z. \tag{38}$$

The following proposition follows trivially from the definition of the operator $\mathcal{S}$.

Proposition 4.9. *The semilinear system (2) with impulses, delays and nonlocal conditions is controllable on $[0,\tau]$ if, and only if, for all initial state $\phi \in C(-r,0;\mathbb{R}^n)$ and final state $z_1 \in \mathbb{R}^n$ the operator $\mathcal{S}$ given by (35)-(38) has a fixed point, i.e.,*

$$\exists (z,u) \in \mathrm{Dom}(\mathcal{S}) \quad \textit{such that} \quad \mathcal{S}(z,u) = (z,u).$$

Now in order to prove the exact controllability of the system (2), let us consider the following hypothesis.

(H1) The linear system associated to (2) is controllable on $[0,\tau]$,

(H2) The following statements hold:

$$\|f(t,\phi,u)\|_{\mathbb{R}^n} \le a_0\|\phi(-r)\|_{\mathbb{R}^n}^{\alpha_0} + b_0\|u\|_{\mathbb{R}^m}^{\beta_0} + c_0, \quad u \in \mathbb{R}^m, \phi \in C(-r,0;\mathbb{R}^n), \quad t \in [0,\tau]; \tag{39}$$

$$\|\mathcal{I}_k(z,u)\|_{\mathbb{R}^n} \le a_k\|z\|_{\mathbb{R}^n}^{\alpha_k} + b_k\|u\|_{\mathbb{R}^m}^{\beta_k} + c_k, \quad k = 1,2,3,\dots,p, z \in \mathbb{R}^n, u \in \mathbb{R}^m; \tag{40}$$

$$\|\mathcal{H}(z)\| \le e\|z\|^\eta, \quad z \in C_q = C(-r,0;(\mathbb{R}^n)^q)), \quad e \in \mathbb{R}; \tag{41}$$

$$\|\mathcal{H}(z) - \mathcal{H}(w)\| \le K\|z-w\|, \quad z,w \in C_q = C(-r,0;(\mathbb{R}^n)^q)), K \in \mathbb{R}. \tag{42}$$

with $0 < \eta \le 1$, $0 < \alpha_k \le 1$, $0 < \beta_k \le 1$, $a_k \in \mathbb{R}$, $b_k \in \mathbb{R}$, $c_k \in \mathbb{R}$ for $k = 0,1,2,3,\dots,p$, and

$$z(t_k) = z(t_k^+) = \lim_{t \to t_k^+} z(t), \quad z(t_k^-) = \lim_{t \to t_k^-} z(t).$$

By the previous section, we know that, for all $\phi \in C(-r,0;\mathbb{R}^n)$ and $u \in C(0,\tau;\mathbb{R}^m)$, problem (2) associated with u admits only one solution given by

$$\begin{aligned} z(t) &= \mathcal{U}(t,0)\{\phi(0) - \mathcal{H}(z_{\tau_1}, z_{\tau_2}, \dots, z_{\tau_q}))(0)\} + \int_0^t \mathcal{U}(t,s)\mathcal{B}(s)u(s)ds \\ &\quad + \int_0^t \mathcal{U}(t,s)f(s,z_s,u(s))ds \\ &\quad + \sum_{0<t_k<t} \mathcal{U}(t,t_k)\mathcal{I}_k(z(t_k),u(t_k)) \;\; t \in [0,\tau], \\ z(t) &= \phi(t) - \mathcal{H}(z_{\tau_1}, z_{\tau_2}, \dots, z_{\tau_q}))(t), \quad t \in [-r,0]. \end{aligned} \tag{43}$$

Then, the controllability problem will be given by the following theorem

Theorem 4.10. *Suppose that* (**H1**) *and the conditions (39)-(42) of* (**H2**) *hold. If* $0 \leq \alpha_k < 1$, $0 \leq \beta_k < 1$, $k = 0, 1, 2, 3, \ldots, p$, $0 \leq \eta < 1$, *then the nonlinear system (2) is controllable on* $[0, \tau]$. *Moreover, there exists a control* $u \in C(0, \tau; \mathbb{R}^m)$ *such that for a given* $\phi \in C(-r, 0; \mathbb{R}^n)$, $z_1 \in \mathbb{R}^n$ *the corresponding solution* $z(t) = z(t, u)$ *of (2) satisfies*

$$\begin{aligned} z_1 &= \mathcal{U}(\tau, 0)\{\phi(0) - \mathcal{H}(z_{\tau_1}, z_{\tau_2}, \ldots, z_{\tau_q}))(0)\} + \int_0^\tau \mathcal{U}(\tau, s)\mathcal{B}(s)u(s)ds \\ &\quad + \int_0^\tau \mathcal{U}(\tau, s)f(s, z_s, u(s))ds + \sum_{0<t_k<\tau} \mathcal{U}(\tau, t_k)\mathcal{I}_k(t_k, z(t_k), u(t_k)), \end{aligned}$$

with

$$u(t) = \mathcal{B}^*(t)\mathcal{U}^*(\tau, t)\mathcal{W}^{-1}\mathfrak{L}(z, u),$$

Proof. We shall prove this theorem by claims.

Claim 1. The operator $\mathcal{S}$ is continuous.

The continuity of $\mathcal{S}$ is equivalent to the continuity of the operators $\mathcal{S}_1$ and $\mathcal{S}_2$ defined above. The continuity of $\mathcal{S}_1$ follows from the continuity of the nonlinear functions $f(t, z_s, u)$, $\mathcal{I}_k(z, u)$, $\mathcal{H}(z)$, and the following estimate

$$\begin{aligned} \|\mathcal{S}_1(z, u) - \mathcal{S}_1(w, v)\| &\leq K_1\|z - w\| \\ &\quad + K_2 \sup_{s\in J} \|f(s, z_s, u(s)) - f(s, w_s, v(s))\| \\ &\quad + K_3 \sum_{0<t_k<t} \|\mathcal{I}_k(z(t_k), u(t_k)) - \mathcal{I}_k(w(t_k), v(t_k))\|, \end{aligned}$$

where,

$$K_1 = Me^{\omega\tau}K\widehat{K}, \quad K_2 = \frac{M}{w}\widehat{K}, \quad K_3 = M_3\widehat{K}, \quad \textit{with} \quad \widehat{K} = 1 + \frac{M^2}{\omega}\|\mathcal{B}\|^2\|\mathcal{W}^-1\|.$$

The continuity of the operator $\mathcal{S}_2$ follows from the continuity of the operators $\mathfrak{L}$ and Υ defined in (37) and (38), respectively.

Claim 2. $\mathcal{S}$ maps bounded sets of $PC_{t_1..t_p}(-r, \tau; \mathbb{R}^n)$ into equicontinuous sets of $PC_{t_1..t_p}(-r, \tau; \mathbb{R}^n) \times PC_{t_1..t_p}(0, \tau; \mathbb{R}^m)$.

Consider the following equality

$$\|\mathcal{S}(z, u)(t_2) - \mathcal{S}(z, u)(t_1)\|_1 = \|\mathcal{S}_1(z, u)(t_2) - \mathcal{S}_1(z, u)(t_1)\| + \|\mathcal{S}_2(z, u)(t_2) - \mathcal{S}_2(z, u)(t_1)\|.$$

Let $D \subset PC_{t_1..t_p}(-r, \tau; \mathbb{R}^n)$ be a bounded set. The equicontinuity for $\mathcal{S}(\mathcal{D})$ is given by the equicontinuity of each one of its components $\mathcal{S}_1(D)$, $\mathcal{S}_2(D)$, which are obtained from the continuity of $\mathcal{U}(t, s)$ and the following estimates

$$\begin{aligned} \|\mathcal{S}_1(z, u)(t_2) - \mathcal{S}_1(z, u)(t_1)\| &\leq \|\mathcal{U}(t_2, 0) - \mathcal{U}(t_1, 0)\| \left\{\|\phi(0)\| + \|\mathcal{H}(z_{\tau_1}, z_{\tau_2}, \ldots, z_{\tau_q}))(0)\|\right\} \\ &\quad + \int_0^{t_1} \|\mathcal{U}(t_2, s) - \mathcal{U}(t_1, s)\|\|\mathcal{B}(s)\|\|(\Upsilon\mathfrak{L}(z, u)(s)\|ds \\ &\quad + \int_{t_1}^{t_2} \|\mathcal{U}(t_2, s)\|\|\mathcal{B}(s)\|\|(\Upsilon\mathfrak{L}(z, u)(s)\|ds \\ &\quad + \int_0^{t_1} \|\mathcal{U}(t_2, s) - \mathcal{U}(t_1, s)\|\|f(s, z_s, u(s))\|ds \\ &\quad + \int_{t_1}^{t_2} \|\mathcal{U}(t_2, s)\|\|f(s, z_s, u(s))\|ds \\ &\quad + \sum_{0<t_k<t_1} \|\mathcal{U}(t_2, t_k) - \mathcal{U}(t_1, t_k)\|\|\mathcal{I}_k(t_k, z(t_k), u(t_k))\| \\ &\quad + \sum_{t_1<t_k<t_2} \|\mathcal{U}(t_2, t_k)\|\|\mathcal{I}_k(t_k, z(t_k), u(t_k))\|, \quad \forall(z, u) \in D. \end{aligned}$$

and

$$\|\mathcal{S}_2(z,u)(t_2)-\mathcal{S}_2(z,u)(t_1)\| \le \|\mathcal{B}^*(t_2)\mathcal{U}^*(\tau,t_2)-\mathcal{B}^*(t_1)\mathcal{U}^*(\tau,t_1)\|\|\mathcal{W}^{-1}\mathfrak{L}(z,u)\|, \quad \forall (z,u)\in D,$$

Since $\mathcal{U}(t,s)$ is continuous $\|\mathcal{U}(t_2,s)-\mathcal{U}(t_1,s)\|$ goes to zero as $t_2 \to t_1$ and so does the sum and the integral from t_1 to t_2 which implies that $\mathcal{S}_1(D)$ is equicontinuous. Moreover, the equicontinuity of $\mathcal{S}_2(D)$ follows from the continuity of the evolution operator $\mathcal{U}(t,s)$. Hence, we proved the equicontinuity of $\mathcal{S}(D)$.

Claim 3. The set $\mathcal{S}(D)$ is relatively compact where D is a closed and convex set.
Indeed, let D be a bounded subset of $PC_{t_1..t_p}(-r,\tau;\mathbb{R}^n)\times C(0,\tau;\mathbb{R}^m)$. By the continuity property of f, $\mathfrak{L}$, and $\mathcal{I}_k$, it follows that

$$\|f(\cdot,z,u)\|_0 \le M_5, \quad \|\mathcal{W}^{-1}\mathfrak{L}(z,u)\| \le M_6, \quad \|\mathcal{I}_k(z,u)\|_{\mathbb{R}^n} \le T_k, \quad k=1,2,\dots,p; \quad \forall (z,u)\in D.$$

where M_5, M_6, $T_1,T_2,\dots,T_k\in\mathbb{R}$. Therefore, $\mathcal{S}(D)$ is uniformly bounded. Consequently, if we take a sequence $\{\phi_j : j=1,2,\dots\}$ on $\mathcal{S}(D)$, this sequence is uniformly bounded and from Claim 2 it is equicontinuous on the interval $[-r,t_1]$. Then, applying Arzelà-Ascoli theorem, there is a subsequence $\{\phi_j^1 : j=1,2,\dots\}$ of $\{\phi_j : j=1,2,\dots\}$, which is uniformly convergent on $[-r,t_1]$. Now, consider the sequence $\{\phi_j^1 : j=1,2,\dots\}$ on the interval $[t_1,t_2]$. On this interval the sequence $\{\phi_j^1 : j=1,2,\dots\}$ is uniformly bounded and equicontinuous, and, for the same reason, it has a subsequence $\{\phi_j^2\}$ uniformly convergent on $[-r,t_2]$. In this way, for the intervals $[t_2,t_3]$, $[t_3,t_4]$, $\dots$, $[t_p,\tau]$, we see that the sequence $\{\phi_j^{p+1} : j=1,2,\dots\}$ converges uniformly on the interval $[-r,\tau]$. This means that $\overline{\mathcal{S}(D)}$ is compact, i.e., $\mathcal{S}(D)$ is relatively compact. Note that we can say that the operator $\mathcal{S}$ is compact by definition of compactness.

Claim 4. Let us recall that $0\le\alpha_k<1$, $0\le\beta_k<1$, $k=0,1,2,3,\dots,p$, $0\le\eta<1$. Then, the following limit holds.

$$\lim_{|||(z,u)|||\to\infty}\frac{|||\mathcal{S}(z,u)|||}{|||(z,u)|||}=0,$$

where $|||(z,u)|||=\|z\|+\|u\|$ is the norm on $PC_{t_1..t_p}(-r,\tau;\mathbb{R}^n)\times C(0,\tau;\mathbb{R}^m)$.

Using the conditions (39)-(42) of **(H2)**, we get

$$\|\mathfrak{L}(z,u)\| \le M_1+M_2\{e\|z\|^\eta+a_0\|z\|^{\alpha_0}+b_0\|u\|^{\beta_0}+c_0\}+M_3\sum_{0<t_k<\tau}\{a_k\|z\|^{\alpha_k}+b_k\|u\|^{\beta_k}+c_k\},$$

where

$$M_1=\|z_1\|+M_3\|\phi(0)\|, \quad M_2=M_3+\frac{M}{\omega} \quad \text{and} \quad M_3=Me^{\omega\tau}.$$

So that,

$$\begin{aligned}\|\mathcal{S}_2(z,u)\| &\le \|\mathcal{B}\|M_3M_1\|\mathcal{W}^{-1}\|+\|\mathcal{B}\|M_3M_2\|\mathcal{W}^{-1}\|\{e\|z\|^\eta+a_0\|z\|^{\alpha_0}+b_0\|u\|^{\beta_0}+c_0\}\\ &\quad +\|\mathcal{B}\|M_3^2\|\mathcal{W}^{-1}\|\sum_{0<t_k<\tau}\{a_k\|z\|^{\alpha_k}+b_k\|u\|^{\beta_k}+c_k\},\end{aligned}$$

and,

$$\begin{aligned}\|\mathcal{S}_1(z,u)\| &\le M_3\|\phi(0)\|+\frac{M^2}{\omega}\|\mathcal{B}\|^2\|\mathcal{W}^{-1}\|M_1\\ &\quad +M_2\widehat{K}\{e\|z\|^\eta+a_0\|z\|^{\alpha_0}+b_0\|u\|^{\beta_0}+c_0\}\\ &\quad +M_3\widehat{K}\sum_{0<t_k<\tau}\{a_k\|z\|^{\alpha_k}+b_k\|u\|^{\beta_k}+c_k\}.\end{aligned}$$

Therefore,

$$\begin{aligned}|||\mathcal{S}(z,u)||| &= \|\mathcal{S}_1(z,u)\|+\|\mathcal{S}_2(z,u)\|\\ &\le M_4+\{\|\mathcal{B}\|M_3M_2\|\mathcal{W}^{-1}\|+M_2\widehat{K}\}\{e\|z\|^\eta+a_0\|z\|^{\alpha_0}+b_0\|u\|^{\beta_0}+c_0\}\\ &\quad +\{\|\mathcal{B}\|M_3^2\|\mathcal{W}^{-1}\|+M_3\widehat{K}\}\sum_{0<t_k<\tau}\{a_k\|z\|^{\alpha_k}+b_k\|u\|^{\beta_k}+c_k\},\end{aligned}$$

where M_4 is given by

$$M_4 = M_3\|\phi(0)\| + \|\mathcal{B}\|\|\mathcal{W}^{-1}\|M_1\{M_3 + \frac{M^2}{\omega}\|\mathcal{B}\|\}.$$

Hence,

$$\begin{aligned}\frac{|||\mathcal{S}(z,u)|||}{|||(z,u)|||} \leq & \frac{M_4}{\|z\|+\|u\|} \\ & +\{\|\mathcal{B}\|M_3M_2\|\mathcal{W}^{-1}\| + M_2\widehat{K}\} \\ & \times\{e\|z\|^{\eta-1} + a_0\|z\|^{\alpha_0-1} + b_0\|u\|^{\beta_0-1} + \frac{c_0}{\|z\|+\|u\|}\} \\ & +\{\|\mathcal{B}\|M_3^2\|\mathcal{W}^{-1}\| + M_3\widehat{K}\} \times \\ & \sum_{0<t_k<\tau}\{a_k\|z\|^{\alpha_k-1} + b_k\|u\|^{\beta_k-1} + \frac{c_k}{\|z\|+\|u\|}\},\end{aligned}$$

Consequently,

$$\lim_{|||(z,u)|||\to\infty} \frac{|||\mathcal{S}(z,u)|||}{|||(z,u)|||} = 0.$$

Claim 5. The operator $\mathcal{S}$ has a fixed point.

We know, from Claim 4, that for a fixed $0 < \rho < 1$, there exists $R > 0$ such that

$$|||\mathcal{S}(z,u)||| \leq \rho|||(z,u)|||, \quad |||(z,u)||| = R.$$

It is enough to consider the closed ball of radius $R > 0$, $\mathcal{B}(0,R)$, to see that $\mathcal{S}(\partial\mathcal{B}(0,R)) \subset \mathcal{B}(0,R)$. Then, since $\mathcal{S}$ is continuous, compact and maps the sphere $\partial\mathcal{B}(0,R)$ into the interior of the ball $\mathcal{B}(0,R)$, we can apply Rothe's fixed point Theorem 2.8 to ensure the existence of a fixed point $(z,u) \in \mathcal{B}(0,R) \subset PC_{t_1..t_p}(-r,\tau;\mathbb{R}^n) \times C(0,\tau;\mathbb{R}^m)$ such that

$$\mathcal{S}(z,u) = (z,u).$$

Hence, applying the Proposition 4.9, we get that the nonlinear system (2) is controllable on $[0,\tau]$. Moreover, exists a control $u \in C(0,\tau;\mathbb{R}^m)$ given by $u = \mathcal{B}^*(\cdot)\mathcal{U}^*(\tau,\cdot)\mathcal{W}^{-1}\mathfrak{L}(z,u)$ such that for a given $\phi \in C(-r,0;\mathbb{R}^n)$, $z_1 \in \mathbb{R}^n$ the corresponding solution $z(t) = z(t,u)$ of (2) satisfies:

$$\begin{aligned} z_1 &= \mathcal{U}(\tau,0)\{\phi(0) - \mathcal{H}(z_{\tau_1}, z_{\tau_2}, \ldots, z_{\tau_q}))(0)\} + \int_0^\tau \mathcal{U}(\tau,s)\mathcal{B}(s)u(s)ds \\ &+ \int_0^\tau \mathcal{U}(\tau,s)f(s,z_s,u(s))ds + \sum_{0<t_k<\tau}\mathcal{U}(\tau,t_k)\mathcal{I}_k(t_k,z(t_k),u(t_k)).\end{aligned}$$

It finishes the proof. □

Now, we present another version of the previous theorem, which follows from the estimates considered in Claim 4.

Theorem 4.11. *Suppose that* (**H1**) *and the conditions (39)-(42) of* (**H2**) *hold. The nonlinear system (2) is controllable on* $[0,\tau]$ *if the linear system (14) is controllable on* $[0,\tau]$ *and one of the following statement holds:*

a) $\alpha_0 = 1$, $\max\{\alpha_k : k = 1,2,\ldots,p\} < 1$, $\max\{\beta_k : k = 0,1,2,\ldots,p\} < 1$, $\eta < 1$ *and*

$$\{\|\mathcal{B}\|M_3M_2\|\mathcal{W}^{-1}\| + M_2\widehat{K}\}a_0 < 1.$$

b) $\beta_0 = 1$, $\max\{\beta_k : k = 1,2,\ldots,p\} < 1$, $\max\{\alpha_k : k = 0,1,2,\ldots,p\} < 1$ $\eta < 1$ *and*

$$\{\|\mathcal{B}\|M_3M_2\|\mathcal{W}^{-1}\| + M_2\widehat{K}\}b_0 < 1.$$

c) $\beta_0 = \alpha_0 = 1$, $\max\{\beta_k : k = 1, 2, \ldots, p\} < 1$, $\max\{\alpha_k : k = 1, 2, \ldots, p\} < 1 \quad \eta < 1$ *and*

$$\{\|\mathcal{B}\|M_3M_2\|\mathcal{W}^{-1}\| + M_2\widehat{K}\}(a_0 + b_0) < 1.$$

d) $\beta_0 = \alpha_0 = \eta = 1$, $\max\{\beta_k : k = 1, 2, \ldots, p\} < 1$, $\max\{\alpha_k : k = 1, 2, \ldots, p\} < 1$ *and*

$$\{\|\mathcal{B}\|M_3M_2\|\mathcal{W}^{-1}\| + M_2\widehat{K}\}(e + a_0 + b_0) < 1.$$

e) $\beta_0 < 1, \alpha_0 < 1$, $\max\{\beta_k : k = 1, 2, \ldots, p\} < 1$, $\max\{\alpha_k : k = 1, 2, \ldots, p\} = 1 \quad \eta < 1$ *and*

$$\{\|\mathcal{B}\|M_3^2\|\mathcal{W}^{-1}\| + M_3\widehat{K}\}\sum_{k\in S_\alpha} a_k < 1,$$

where $S_\alpha = \{k : \alpha_k = 1\}$.

f) $\beta_0 < 1, \alpha_0 < 1$, $\max\{\beta_k : k = 1, 2, \ldots, p\} = 1$, $\max\{\alpha_k : k = 1, 2, \ldots, p\} < 1 \quad \eta < 1$ *and*

$$\{\|\mathcal{B}\|M_3^2\|\mathcal{W}^{-1}\| + M_3\widehat{K}\}\sum_{k\in S_\beta} b_k < 1,$$

where $S_\beta = \{k : \beta_k = 1\}$.

g) $\beta_0 < 1, \alpha_0 < 1$, $\max\{\beta_k : k = 1, 2, \ldots, p\} = 1$, $\max\{\alpha_k : k = 1, 2, \ldots, p\} = 1 \quad \eta < 1$ *and*

$$\{\|\mathcal{B}\|M_3^2\|\mathcal{W}^{-1}\| + M_3\widehat{K}\}(\sum_{k\in S_\alpha} a_k + \sum_{k\in S_\beta} b_k) < 1,$$

where

$$M_2 = M_3 + \frac{M}{\omega}, \quad \widehat{K} = 1 + \frac{M^2}{\omega}\|\mathcal{B}\|^2\|\mathcal{W}^-1\|, \textit{and} \quad M_3 = Me^{\omega\tau}.$$

Proof. Let us consider any of the conditions $a) - g)$. Then, from the estimates obtained in Claim 4, we get that

$$\lim_{|||(z,u)|||\to\infty} \frac{|||\mathcal{S}(z, u)|||}{|||(z, u)|||} < \rho < 1.$$

Hence, there exists $R > 0$ such that

$$|||\mathcal{S}(z, u)||| \leq \rho|||(z, u)|||, \quad |||(z, u)||| = R.$$

Then, analogously to the previous theorem the proof of Theorem 4.11 immediately follows by applying Proposition 4.9. □

4.2.2 Aproximate Controllability Using Techniques Avoiding Fixed Point Theorem

In this subsection, the approximate controllability of the problem (2) is proved without using fixed point techniques. Rather than this, we use an alternative technique developed by A.E. Bashirov et al. [42, 43, 44]. In this case the delay help us to prove the approximate controllability of the system by pulling back the control solution to a fixed curve in a short time interval, and from this position, we are able to reach a neighborhood of the final state in time τ by using the exact controllability of the associated linear system (14) on any interval $[\tau - \delta, \tau]$, $0 < \delta < \tau$.

From Section 3, on the controllability of linear systems, we know that for all $z_0 \in \mathbb{R}^n$ and $u \in L^2(0, \tau; \mathbb{R}^m)$ the initial value problem

$$\begin{cases} y'(t) = \mathcal{A}(t)y(t) + \mathcal{B}(t)u(t), & t \in [\tau - \delta, \tau], \\ y(\tau - \delta) = z_0, \end{cases} \tag{44}$$

admits only one solution given by

$$y(t) = \mathcal{U}(t, \tau - \delta)z_0 + \int_{\tau-\delta}^{t} \mathcal{U}(t, s)\mathcal{B}(s)u(s)ds, \quad t \in [\tau - \delta, \tau], \tag{45}$$

where $\mathcal{U}(t, s) = \Phi(t)\Phi^{-1}(s)$ and $\Phi(t)$ is the fundamental matrix of the uncontrolled linear system

$$z'(t) = \mathcal{A}(t)z(t).$$

and (44) is controllable. Moreover, a control steering the system (44) from initial state z_0 to a final state z_1 on the interval $[\tau - \delta, \tau]$ is given by

$$v^{\delta}(t) = \mathcal{B}^*(t)\mathcal{U}^*(\tau, t)\mathcal{W}_{\tau\delta}^{-1}(z_1 - \mathcal{U}(\tau, \tau - \delta)z_0) = \Upsilon(z_1 - \mathcal{U}(\tau, \tau - \delta)z_0)(t), \quad t \in [\tau - \delta, \tau].$$

In order to prove the approximate controllability of (2) we shall consider the following hypothesis.

(H1) f satisfies the following estimate

$$|f(t, \phi, u)| \leq \nu(\|\phi(-r)\|), \qquad u \in \mathbb{R}^m, \phi \in C(-r, 0; \mathbb{R}^n),$$

where $\nu : \mathbb{R}_+ \to [0, \infty)$ is a continuous function. In particular, $\nu(z) = a\|z\|^{\beta} + b$, with $\beta \geq 1, a, b \in \mathbb{R}$.

(H2) The linear system (44) is exact controllable on any interval $[\tau - \delta, \tau]$, $0 < \delta < \tau$.

Theorem 4.12. *If the functions $f, \mathcal{I}_k$ are smooth enough,* $(H1)$ *and* $(H2)$ *hold, then the system (2) is approximately controllable on* $(0, \tau]$.

Proof. Given $\phi \in C(-r, 0; \mathbb{R}^n)$, a final state $z_1 \in \mathbb{R}^n$ and $\epsilon > 0$, we want to find a control $u^{\delta} \in L^2(0, \tau; \mathbb{R}^m)$ steering the system to z_1 on $[\tau - \delta, \tau]$. Precisely, for $\delta > 0$ small enough, there exists a control $u^{\delta} \in L^2(0, \tau; \mathbb{R}^m)$ such that corresponding of solutions z^{δ} of (2) satisfies

$$\left\| z^{\delta}(\tau) - z_1 \right\| < \epsilon.$$

In fact, we consider any fixed control $u \in L^2(0, \tau; \mathbb{R}^m)$, the corresponding solution $z(t) = z(t, \phi, u)$ of problem (2) and $K = \max\{\nu(\|z(t)\|) : t \in [0, \tau]\}$. For $\delta > 0$ small enough such that

$$0 < \delta < \min\{\frac{\epsilon}{MK}, \tau - t_p, r\},$$

we define the control $u^{\delta} \in L^2(0, \tau; \mathbb{R}^m)$ as follows:

$$u^{\delta}(t) = \begin{cases} u(t), & \text{if} \quad 0 \leq t \leq \tau - \delta, \\ v^{\delta}(t), & \text{if} \quad \tau - \delta < t \leq \tau, \end{cases}$$

where

$$v^{\delta}(t) = \mathcal{B}^*(t)\mathcal{U}^*(\tau, t)(\mathcal{G}_{\tau\delta}\mathcal{G}_{\tau\delta}^*)^{-1}(z_1 - \mathcal{U}(\tau, \tau - \delta)z(\tau - \delta)), \quad \tau - \delta < t \leq \tau.$$

Since $0 < \delta < \tau - t_p$, then $\tau - \delta > t_p$; and using the cocycle property, the corresponding solution $z^{\delta}(t) = z(t, \phi, u^{\delta})$

of the nonlocal Cauchy problem (2) at time τ can be written as follows:

$$\begin{aligned}
z^\delta(\tau) &= \mathcal{U}(\tau,0)[\phi(0)-\mathcal{H}(z_{\tau_1},\ldots,z_{\tau_q})(0)] + \int_0^\tau \mathcal{U}(\tau,s)\mathcal{B}(s)u^\delta(s)ds \\
&\quad + \int_0^\tau \mathcal{U}(\tau,s)f(s,z_s^\delta,u^\delta(s))ds + \sum_{0<t_k<\tau}\mathcal{U}(\tau,t_k)\mathcal{I}_k((t_k,z(t_k),u^\delta(t_k)) \\
&= \mathcal{U}(\tau,\tau-\delta)\Big\{\mathcal{U}(\tau-\delta,0)[\phi(0)-\mathcal{H}(z_{\tau_1},\ldots,z_{\tau_q})(0)] \\
&\quad + \int_0^{\tau-\delta}\mathcal{U}(\tau-\delta,s)\mathcal{B}(s)(s)u^\delta(s)ds \\
&\quad + \int_0^{\tau-\delta}\mathcal{U}(\tau-\delta,s)f(s,z_s^\delta,u^\delta(s))ds \\
&\quad + \sum_{0<t_k<\tau-\delta}\mathcal{U}(\tau-\delta,t_k)\mathcal{I}_k((t_k,z^\delta(t_k),u^\delta(t_k))\Big\} \\
&\quad + \int_{\tau-\delta}^\tau \mathcal{U}(\tau,s)\mathcal{B}(s)u^\delta(s)ds + \int_{\tau-\delta}^\tau \mathcal{U}(\tau,s)f(s,z_s^\delta,u^\delta(s))ds \\
&= \mathcal{U}(\tau,\tau-\delta)z(\tau-\delta) + \int_{\tau-\delta}^\tau \mathcal{U}(\tau,s)\mathcal{B}(s)v^\delta(s))ds \\
&\quad + \int_{\tau-\delta}^\tau \mathcal{U}(\tau,s)f(s,z_s^\delta,v^\delta(s))ds.
\end{aligned}$$

The corresponding solution $y^\delta(t) = y(t,\tau-\delta,z(\tau-\delta),v^\delta)$ of the initial value problem (44) at time τ, for the control v^δ and the initial condition $z_0 = z(\tau-\delta)$, is given by:

$$y^\delta(\tau) = \mathcal{U}(\tau,\tau-\delta)z(\tau-\delta) + \int_{\tau-\delta}^\tau \mathcal{U}(\tau,s)\mathcal{B}(s)v^\delta(s)ds,$$

and from Lemma 3.3, we get that

$$y^\delta(\tau) = z_1.$$

Therefore,

$$\left\|z^\delta(\tau)-z_1\right\| \leq \int_{\tau-\delta}^\tau \|\mathcal{U}(\tau,s)\|\left\|f(s,z_s^\delta,u^\delta(s))\right\| ds.$$

Now, since $0<\delta<r$ and $\tau-\delta\leq s\leq\tau$, then $s-r\leq\tau-r<\tau-\delta$ and

$$z^\delta(s-r) = z(s-r).$$

Hence, there exists δ small enough such that $0<\delta<\min\{r,\tau-t_p\}$ and

$$\begin{aligned}
\left\|z^\delta(\tau)-z_1\right\| &\leq \int_{\tau-\delta}^\tau \left\|\mathcal{U}(\tau,s)\right\|\nu(\|z^\delta(s-r)\|)ds \\
&= \int_{\tau-\delta}^\tau \left\|\mathcal{U}(\tau,s)\right\|\nu(\|z(s-r)\|)ds \\
&\leq \int_{\tau-\delta}^\tau MKds \\
&< \delta MK \\
&< \epsilon.
\end{aligned}$$

The proof is illustrated graphically in Figure 5.

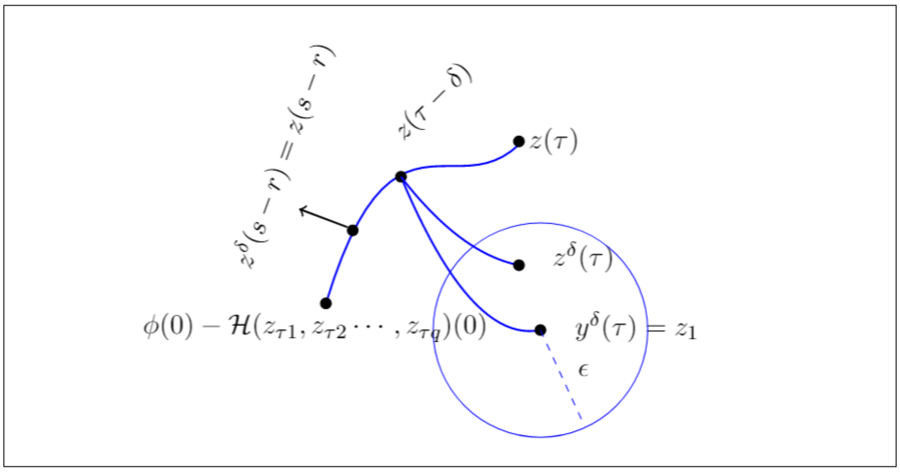

Figure 5: Scheme of the proof

This completes the proof of the theorem. □

5 Conclusions and Future Research

Generally speaking, the study of the existence of solutions is done using fixed point theorems, when finding the appropriated fixed point theorem is not a trivial task. In this work, the existence of solutions for the semilinear system (1) was proved using Karakostas fixed point theorem, and was motivated by the work done in [12, 45]. It should be noted that the fixed point theorem of Karakostas has been used, as far as we know, only in the aforementioned articles. Therefore, to solve the first problem a new technique has been used.

It was proved that the semilinear system with impulses, delays, and nonlocal conditions has at least one solution under some assumptions on $\mathcal{H}, \mathcal{F}$, and $\mathcal{I}_k$. Moreover, we proved that the semilinear system has only one solution on $[-r, \tau]$, which can be extended on $[-r, \infty]$ under some additional conditions. These obtained results about the existence of solutions for this type of systems was applied to study the exact and the approximate controllability of the system.

In order to study the exact controllability of (2), we transformed the controllability problem into a fixed point problem, and moreover, we impose sublinear decay conditions on $\mathcal{H}, f$, and $\mathcal{I}_k$. To solve the fixed point problem was applied the Rothe's fixed point theorem. This technique was strongly motivated by studies done in [40, 46], where the controllability of the semilinear system without delays and nonlocal conditions is proved. Also, it was proved that the semilinear system with impulses, delays, and nonlocal conditions is approximately controllable on $[0, \tau]$ if the functions $f, \mathcal{I}_k$ (non linear term and impulses of the system) are smooth enough and the following inequality holds

$$|f(t, \phi, u)| \leq \nu(\|\phi(-r)\|), \qquad u \in \mathbb{R}^m, \phi \in C(-r, 0; \mathbb{R}^n).$$

To study the approximate controllability, the Bashirov and et al. techniques were employed, avoiding fixed point theorems. Thus, from thesis I learned different techniques to deal with controllability problems. Furthermore, we obtained that the controllability is a robust property under disturbances such as impulses, delays, and nonlocal conditions.

To develop this work, it required knowledge of topics such as functional analysis, differential equations, measure theory and operator theory, which are included in the math curriculum. But in addition to these topics, the study of non curricular topics was required, such as: differential equations with impulses, differential equations with delays, differential equations with nonlocal conditions, characterization of linear dense range operators, fixed point theorems, among others.

The study of each of these non curricular topics, individually, corresponds to a research area. However, the proper guidance of my tutor, which consisted of studying the strictly necessary results for the development of this work, allowed me the completion of this thesis. The study of these evolution equations constitutes a broad area for future research. For which it would be necessary some more advanced studies on functional analysis and partial differential equations.

Lastly, the suggestions for future research will be proposed.

- According to our knowledge, null-controllability, $0 \in \text{int}(\mathcal{C})$ where $\mathcal{C}$ is the domain of null-controllability (the set of points achieving zero), has not yet been studied for this type of systems. Then, it is an open problem worth to study where the open mapping theorem plays an important role.
- Another interesting topic is to study optimal control problems for systems influenced by impulses, delays and nonlocal conditions and also with restrictions on control.
- On the Lyapunov stability theory for dynamical systems with impulses, delays and nonlocal conditions.

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