UNIVERSIDAD DE INVESTIGACIÓN DE TECNOLOGÍA EXPERIMENTAL YACHAY

Escuela de Ciencias Matemáticas y Computacionales

Concentration and Multiplicity of Solutions for a Non-Linear Schrödinger Equation with critical Frequency: Exponential Case

Trabajo de integración curricular presentado como requisito para la obtención
del título de Matemático

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Urcuquí, Agosto 2019

Urcuquí, 28 de agosto de 2019

SECRETARÍA GENERAL
(Vicerrectorado Académico/Cancillería)
ESCUELA DE CIENCIAS MATEMÁTICAS Y COMPUTACIONALES
CARRERA DE MATEMÁTICA
ACTA DE DEFENSA No. UITEY-ITE-2019-00008-AD

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To my dearly beloved mom Janeth Cevallos Chávez
and to my beloved Yachay Tech.

Jordy José Cevallos Chávez

Acknowledgments

I would like to express my sincere gratitude to professor Juan Mayorga Zambrano, my advisor, by his guidance in the development of this work. I really learned a lot of things from him not only by addressing my mind, but also by inspiring my heart. To Leonardo Medina, from Escuela Politécnica Nacional, by his support and useful suggestions in the study of the infinite case problem.
Also, to the reviewers of this work; professor Hermann Mena, rector of Yachay Tech and professor Hugo Leiva, dean of my school; by their comments and suggestions that clearly improve the quality of this work.
To all of my professors in my major: Hermann Mena, Juan Mayorga, Zenaida Castillo, Antonio Acosta, Hugo Leiva, Eusebio Ariza, Juan Carlos López, Saba Infante, Cédric M. Campos, Esteban Palomo, Hugo Campos, Raúl Manzanilla, Ricardo Chimentón, Rafael Amaro, who proportionated very important components in my formation as a mathematician; I fell very proud and lucky for being their student.
My special gratitude to professors Nicola Di Teodoro and Alexander López by inspiring me through their love for Mathematics and Physics, respectively; I love both of these fields thanks to them. I always remember how fun, useful and enjoyable were their classes and it makes me remember how interesting and fun the learning process is even in such harder subjects.
To professor Graciela Salum, who has helped, listened, and adviced to me when I needed most. Thanks for everything and everything.
My utmost gratitude to professor Carlos Castillo Chávez, who is an academic visionary, by his very admirable care for the students of Yachay Tech by giving us the opportunity at Arizona State University to show to international students and faculty the quality of students that we are.
To professor Fernando Albericio for encouraging us to keep believing in Yachay Tech.
Infinitely thanks to my family by their unconditional support, specially to my beloved mother.
Last but not least, my gratitude to all my friends and colleagues. My extremely high gratitude to those Yachay mates that care about Yachay Tech and have collaborated to me in the Student Council. Thanks for everything and everygthing. You are the best of Yachay Tech guys!

Resumen

En este trabajo se estudia la existencia y el comportamiento cualitativo de las soluciones de la ecuación:

$$\begin{cases} \varepsilon^2 v''(x) - V_\varepsilon(x)v(x) + \left|v(x)\right|^{p-1} v(x) = 0, \quad x \in \mathbb{R}, \\ \lim\limits_{|x|\to\infty} v(x) = 0. \end{cases} \tag{P$_\varepsilon$}$$

Considerando el caso infinito dado por Byeon y Wang (2002), donde

$$\Omega := \{V = 0\} = \{0\},$$

y V decrece exponencialmente al rededor de este. Además, el potencial verifica las siguientes condiciones:

$(V1)$ V es una función contínua no negativa;

$(V2)$ $V(x) \to \infty$ cuando $|x| \to \infty$;

$(V3)$ Para todo $x \in [-1, 1]\backslash\{0\}$:

$$V(x) = \exp\left(-\frac{1}{a(x)}\right),$$

Donde a es una función asintóticamente-(Ω, b)-cuasi-homogénea.

Para $\rho > 0$ encontramos un $\delta_\rho > 0$ tal que nuestro problema límite, cuando $\varepsilon \to 0$, es dado por

$$\begin{cases} w''(x) + \left|w(x)\right|^{p-1} w(x) = 0 \quad x \in (-\delta_\rho, \delta_\rho), \\ w(-\delta_\rho) = w(\delta_\rho) = 0. \end{cases} \tag{P$_L$}$$

Probamos, por un esquema de Lusternik–Schnirelman y usando el género de Kranoselskii, que el problema original, (P$_\varepsilon$), tiene infinitas soluciones. Además, usando la misma técnica, que el problema límite, (P$_L$), posee infinitas soluciones. De hecho, las soluciones del problema original y del problema límite vienen en pares para cada nivel crítico, por las propiedades del género de Kranoselskii. Adicionalmente, se demostró que los valores críticos de (P$_\varepsilon$) son esencialmente a los valores críticos de (P$_L$), cuando ε tiende a cero. Finalmente, probamos algunos de los resultados de concentración obtenidos por Byeon–Wang (2002), Felmer–Mayorga (2007) y Mayorga–Medina (2019). Particularmente probamos que las soluciones del problema original (P$_\varepsilon$) subconvergen en la norma H^1 a las correspondientes soluciones del problema límite (P$_L$).

Palabras clave: Ecuación No-Lineal de Schrödinger, frecuencia crítica, multiplicidad y comportamiento cualitativo de las soluciones.

Abstract

In this work we study the existence and qualitative behavior of solutions for

$$\begin{cases} \varepsilon^2 v''(x) - V_\varepsilon(x)v(x) + |v(x)|^{p-1} v(x) = 0, & x \in \mathbb{R}, \\ \lim\limits_{|x|\to\infty} v(x) = 0. \end{cases} \tag{P$_\varepsilon$}$$

We consider the infinite case, given by Byeon and Wang (2002), where

$$\Omega = \{V = 0\} = \{0\}$$

and V decreases exponentially around it. Here, the potential also verifies:

($V1$) V is a non-negative continuous function over $\mathbb{R}$;

($V2$) $V(x) \to \infty$ as $|x| \to \infty$;

($V3$) For each $x \in [-1, 1]\backslash\{0\}$:

$$V(x) = \exp\left(-\frac{1}{a(x)}\right),$$

where a is an asymptotically-(Ω, b) quasi-homogeneous function.

For a fixed $\rho > 0$ we find $\delta_\rho > 0$ such that our limit problem, as $\varepsilon \to 0$, is given by

$$\begin{cases} w''(x) + |w(x)|^{p-1} w(x) = 0 & x \in (-\delta_\rho, \delta_\rho), \\ w(-\delta_\rho) = w(\delta_\rho) = 0. \end{cases} \tag{P$_L$}$$

We prove by a Lusternik–Schnirelman scheme (using the Kranoselskii genus) that the original problem, (P$_\varepsilon$), has infinitely many solutions. We also prove multiplicity of solutions for the limit problem, (P$_L$), by using the same technique mentioned before. In fact, by the Kranoselskii's genus properties, the solutions found for (P$_\varepsilon$) and (P$_L$) come in pairs for each critical level. Finally, we prove concentration results obtained by Byeon–Wang (2002), Felmer–Mayorga (2007), and Mayorga–Medina (2019), for several settings with critical frequency. In particular, we proved the H^1-convergence of the solutions of (P$_\varepsilon$) to the corresponding solutions of (P$_L$).

Keywords— Non-linear Schrödinger equation, critical frequency, multiplicity and qualitative behavior of solutions.

Contents

1. Introduction

Quantum Mechanics is a field of Physics which studies the behavior and characteristics of the atomic and subatomic particles such as protons, electrons, photons, etc. These properties differ from the objects and phenomena studied by Classical Mechanics. In Classical Mechanics, a system could be described by its position and velocity, grossly speaking. On the other hand, in Quantum Mechanics is studied the evolution of the probability to find a particle in some physical state. To do it, the Schrödinger equation is used. The typical Schödinguer equation is given by:

$$i\hbar\Psi_t(x,t) + \frac{\hbar^2}{2}\Delta\Psi(x,t) - V_0(x,t)\Psi(x,t) = 0, \quad x \in \mathbb{R}^N, t \geq 0, \tag{SchE}$$

where $\hbar$ denotes the reduced Plank's constant:

$$\hbar = \frac{h}{2\pi} \approx 6.62607004 \times 10^{-34} m^2 kg/s,$$

i is the imaginary unit; V_0 is a potential; Δ denotes the Laplacian operator in cartesian coordinates in $\mathbb{R}^N$:

$$\Delta := \sum_{k=1}^{N} \frac{\partial^2}{\partial x_k^2},$$

and Ψ is the wave function. Here $|\Psi(x,t)|^2$ is the probability at time t to find the particle in the state x, [1].

In this work, we are going to deal with a non-linear version of the Schrödinger equation, which is given by:

$$i\hbar\Psi_t(x,t) + \frac{\hbar^2}{2}\Delta\Psi(x,t) - V_0(x)\Psi(x,t) + \left|\Psi(x,t)\right|^{p-1}\Psi(x,t) = 0, \quad x \in \mathbb{R}^N, t \geq 0, \tag{NSchE}$$

where $p > 1$. The previous equation allows us to analyze and describe some phenomena in nature. For example, when a group of identical particles interact with themselves in ultra cold states as is the case of Bose–Einstein condensates, [2]. It also works for the propagation of light in some nonlinear optical materials, in this case (NSchE) is obtained from Maxwell's equations, [3]. When $p = 3$ in (NSchE), the Gross–Pitaevskii's equation appears.

In order to study (NSchE), we are going to use Semiclassical Mechanics. Semiclassical mechanics is a method to asymptotically approach problems of Quantum Mechanics by passing to the limit when the reduced Planck constant $\hbar$ tends to zero. This method is a very useful tool to analyze molecular collisions because its results are frequently accurate and it deals with transformed problems that are easier from the mathematical point of view, [4].

In particular, when the potential V_0 depends only in the spatial variable it is possible to search for traveling wave solutions:

$$\Psi(x,t) = e^{-\frac{iEt}{\hbar}} v(x), \quad x \in \mathbb{R}^N, t \geq 0, \tag{1}$$

where the function v represents the stationary part of Ψ. When $1 >> \hbar > 0$, the traveling waves are known as semiclassical states. If we substitute (1) in (NSchE), we obtain:

$$\varepsilon^2\Delta v(x) - V(x)v(x) + \left|v(x)\right|^{p-1} v(x) = 0, \qquad x \in \mathbb{R}^N \tag{P}$$

where

$$\varepsilon^2 := \frac{\hbar^2}{2}$$

and

$$V(x) := V_0(x) - E.$$

In [5], Floer and Weinstein prove for the unidimensional case of (P) with V being a bounded function and

$$\inf_{x\in\mathbb{R}} V(x) > 0, \tag{2}$$

which implies that for

$$0 < \varepsilon \ll 1$$

there exists a solution v_ε for (P) such that

$$\liminf_{\varepsilon\to 0} \max_{x\in\mathbb{R}} |v_\varepsilon(x)| > 0. \tag{3}$$

Here the solution is concentred around a non-degenerated critical point of V. In order to get this the authors used the Lyapunov–Schmidt reduction methods. In [6], [7] and [8]; the solutions of (P), for $N > 1$, satisfy (3). Also, those solutions present a concentration phenomena. It is important to mention the diversity of the methods used to find solutions. For example, variational methods, Lyapunov–Schmidt reductions and the combination of both of them were in place. Also, Rabinowitz in [9], prove the existence of a solution v_ε of (P) by using the Mountain Pass Theorem, when

$$\inf_{x\in\mathbb{R}^N} V(x) < \liminf_{x\to\infty} V(x). \tag{4}$$

However, for the condition

$$\inf_{x\in\mathbb{R}^N} V(x) < 0,$$

with $0 < \varepsilon \ll 1$, the Mountain Pass Theorem can not be applied for the problem (P).

On the other hand, Byeon and Wang, [10], found a positive solution of

$$\begin{cases} \varepsilon^2 \Delta v - V(x)v + |v|^{p-1}v = 0, & x \in \mathbb{R}^N, \\ \lim\limits_{|x|\to\infty} v(x) = 0, \end{cases} \tag{P_ε}$$

under the condition:

$$\min_{x\in\mathbb{R}^N} V(x) = 0 \tag{5}$$

Here the problem depends on the local conditions of $V(\cdot)$ around its global minimum, where $V = 0$. This solution does not verify (3). Actually, the maximum value of the solution goes to zero and the velocity of the approximation is given by the nature of the set

$$\Omega := \left\{ x \in \mathbb{R}^N : V(x) = 0 \right\}$$

This is the reason why the authors named (5) as the critical frequency situation for (P). Moreover, they present the following cases:

1. **Flat case:** The interior of Ω is not empty and $\Omega = \operatorname{int}\left(\overline{\Omega}\right)$.
2. **Finite case:** The set $\Omega = \{x_0\}$, $V(x_0) = 0$, and V behaves as a polynomial around it, and
3. **Infinite case:** In which the set $\Omega = \{x_0\}$,$V(x_0) = 0$, and V decreases exponentially around it.

For each of these cases there is a limit problem as $\varepsilon \to 0$, which allows to study the concentration phenomena. In fact, they show that there exists a standing wave which is trapped in a neighborhood of isolated minimum points of V and whose amplitude goes to 0 as $\varepsilon \to 0$.

Felmer and Mayorga, in [4], dealt with the flat case presented by Byeon and Wang in [10] with $N > 2$ for (P). Here V satisfies the following conditions:

$(V1)$ V is a positive continuous function over $\mathbb{R}^N$;

$(V2)$ $V(x) \to \infty$ as $|x| \to \infty$;

$(V3)$ $\Omega = \operatorname{int}\{x \in \mathbb{R}^N | V(x) = 0\} \neq \emptyset$ is connected with smooth boundary.

The limit problem, in this case, is given by

$$\begin{cases} \Delta u(x) + \left|u(x)\right|^{p-1} u(x) = 0 & x \in \Omega, \\ u(x) = 0 \quad x \in \partial\Omega, \end{cases} \tag{P$'$}$$

The existence of infinitely many solutions for the problems (P_ε) and (P$'$) is proved. Those solutions share the topology of their level sets, as seen from the Ljusternik–Schnirelman scheme. Denoting their solutions as $\{v_{k,\varepsilon}\}_{k\in\mathbb{N}}$ and $\{u_k\}_{k\in\mathbb{N}}$, respectively. They show that for fixed $k \in \mathbb{N}$ and, up to rescaling $v_{k,\varepsilon}$, the energy of $v_{k,\varepsilon}$ converges to the energy of u_k. It is also shown that the solutions $v_{k,\varepsilon}$ for (P_ε) concentrate exponentially around Ω and that, up to rescaling and up to a subsequence, they converge to a solution of (P$'$).

Based on the previous analytical work, Mayorga and Carrasco, [11], performed numerical simulations. For a fixed $\varepsilon > 0$ they applied a variation of Ha's shooting scheme to obtain numerically explicit solutions for (P_ε) and (P$'$) for the case when $\Omega = (-2, 2)$, $p = 2$ and

$$V(x) = e^{-(x^2-4)} - 1 \qquad x \notin \Omega.$$

A secant method is applied since the use of two initial slopes provides control on k, the Ljusternik–Schnirelman category. The initial value problem solver was Runge–Kutta 4.
Fixed k, the concentration is numerically shown as

$$\lim_{\varepsilon\to 0} \left\| u_k - \frac{1}{\varepsilon^2} v_{k,\varepsilon} \right\|_{L^2(-3,3)} = 0.$$

Here, additional difficulties come from the existence of at least two solutions for each critical level, fact which in its turn comes from the properties of Krasnoselski's genus.

In a recent work of Medina and Mayorga, motivated by [4], the finite case was addressed up, [12]. They study the qualitative behavior of the one dimensional version of (P$'$) for the finite case. For $\alpha > 0$ we could set a $\delta_\alpha > 0$ such that the limit problem is given by

$$\begin{cases} u''(x) - P(x)u(x) + \left|u(x)\right|^{p-1} u(x) = 0 & x \in (-\delta_\alpha, \delta_\alpha), \\ u(-\delta_\alpha) = u(\delta_\alpha) = 0 \end{cases} \tag{P_Y}$$

where P represents a generalization of an even degree homogeneous polynomial and $(-\delta_\alpha, \delta_\alpha)$ is a neighborhood of $\Omega = \{x_0\}$. They prove the existence of infinitely many solutions of (P_ε) and (P_Y). Moreover, they demonstrate that, via a re-escalling, the solutions of the problem (P_ε) approximate the solutions of (P_Y) as ε goes to zero. Also, the authors demonstrate that the solutions of (P_Y) decribe an exponential decay outside of $(-\delta_\alpha, \delta_\alpha)$.

In this work we are going to study the qualitative behavior of the one dimensional version of (P') for the infinite case, i.e.

$$\Omega = \{x_0\}$$

and the potential V decays exponentially when is close to x_0. For some $\rho > 0$, we could find a $\delta_\rho > 0$ such that the limit problem is given by

$$\begin{cases} u''(x) + |u(x)|^{p-1} u(x) = 0 & x \in (-\delta_\rho, \delta_\rho), \\ u(-\delta_\rho) = u(\delta_\rho) = 0, \end{cases} \tag{P_L}$$

where $(-\delta_\rho, \delta_\rho)$ is a neighborhood of x_0.

In this work we prove:

i) Given $\varepsilon > 0$, the functional J_ε has infinite critical points $\{\hat{w}_{k,\varepsilon}\}_{k\in\mathbb{N}} \subseteq \mathcal{M}_\varepsilon$.

ii) The limit functional J^{δ_ρ} has infinite critical points $\{\hat{w}_k^{\delta_\rho}\}_{k\in\mathbb{N}} \subseteq \mathcal{M}^{\delta_\rho}$.

iii) Given $k \in \mathbb{N}$, there exists a $C_\rho > 0$ such that the critical values satisfy

$$\lim_{\varepsilon\to 0} J_\varepsilon(\hat{w}_{k,\varepsilon}) = (1 + C_\rho)\, J^{\delta_\rho}(\hat{w}_k^{\delta_\rho}).$$

iv) For each fixed $k \in \mathbb{N}$, there exists a subsequence of $w_{k,\varepsilon}$ that sub-converges to, w_k^δ, a solution of P_L.

Our work is organized as follows:

- In Section 2, we present some basic definitions related with normed spaces, Lipschitz continuity and complete spaces. Next, we introduce the concepts and results of semicontinuity. For example, we define the weakly and strong convergence and the implication to have strong convergence and lower semicontinuity. Then, we introduce and state some very important concepts and results related with Hilbert spaces. For instance, the scalar product, coerciveness, continuity of bilinear forms, the Cauchy–Schwarz inequality, triangle inequality, Hölder inequality, the Riesz–Fréchet representation theorem. Also, we present some results of convergence in Hilbert spaces. After that, we define the L^p spaces and state some important properties of them. Also, some important inequalities for integrals are presented such as Minkowski, Hölder and the interpolation inequalities. The last is followed by some important concepts and results in partial differential equations (PDE's) and Sobolev Spaces. For instance, multi-index notation, well-posed problem conditions, a Sobolev Space and some of their properties. Moreover, we are going to state some useful theorems such as Morrey's inequality and Poncaré's inequality and Rellinch–Kondrachov. After that, we present some topics of non-linear analysis such as, manifolds, Palais–Smale condition, the symmetric deformation lemma. It is also introduced Kranolselskii's genus and its properties including some useful results for our work. At the end of this section, we introduce some important concepts and a couple of results related with the problem. For example, we define a *quasi-homogeneous* and a $(\cdot, \Omega)$*quasi-homogeneous* function, which gives information about the behavior of V around x_0.
- In Section 3, we present a short introduction to Quantum Mechanics. In this part, we first present some historical facts, then we shortly describe the principles of Quantum Mechanics; such as wave-particle duality principle, superposition principle, Heisenberg uncertainty principle and correspondence principle. Finally, we state the postulates of Quantum Mechanics.
- In Section 4, we shall study the problems (P_ε) and (P_L). For that, we shall define some spaces and functionals and prove some important properties related with them. Finally, we are going to present the results about the existence of infinitely many solutions for the problems (P_ε) and (P_L). Moreover, thanks to the Semiclassical Mechanics machinery, we shall ilustrate how the problems (P_ε) and (P_L) are related.
- In Section 5, we present some conclusions and recommendations related with this work.

2. Mathematical Framework

In this section we present some mathematical concepts and results that are used to deal with our problem. Along the text, we use standard notation. Our principal sources are [13], [14],[15], [16], [17], [11], [18] and [19]. We include the proof of a number of results that are relevant to our work and, in other cases, we refer the reader to some reference.

2.1. Some basic definitions

Definition 2.1 (Linear or Vector Space)**.** A real linear space, over the field $\mathbb{R}$, is an algebraic structure $(\mathscr{V}, +, \cdot)$ where $(\mathscr{V}, +)$ is an Abelian group, and the operation

$$\cdot : \mathbb{R} \times \mathscr{V} \to \mathscr{V}$$
$$(\rho, x) \mapsto \rho x$$

referred as a scalar-vector multiplication, verifies

1. $\forall \rho \in \mathbb{R}, \forall x, y \in \mathscr{V} : \rho(x + y) = \rho x + \rho y$.
2. $\forall \rho, \beta \in \mathbb{R}, \forall x \in \mathscr{V} : (\rho + \beta)x = \rho x + \beta x$.
3. $\forall \rho, \beta \in \mathbb{R}, \forall x \in \mathscr{V} : \rho(\beta x) = (\rho\beta)x$
4. $\forall x \in \mathscr{V} : 1 \cdot x = x \cdot 1 = x$

The first property is usually referred as distributivity in $\mathscr{V}$. The second property is known as distributivity in $\mathbb{R}$. Given a set X, we denote by $\mathcal{P}(X)$ the set of parts of X, i.e.

$$\mathcal{P}(X) := \{A | A \subseteq X\}$$

Definition 2.2 (Topology and Topological Space)**.** Let X be a nonempty set and $\mathcal{T} \subseteq \mathcal{P}(X)$. Then we say that $\mathcal{T}$ is a topology in X iff the following conditions hold

i) $\emptyset \in \mathcal{T}$ and $X \in \mathcal{T}$;

ii) if $A, B \in \mathcal{T}$, then $A \cap B \in \mathcal{T}$;

iii) if $\{A_i\}_{i \in I}$ is a family of elements of $\mathcal{T}$ then $\bigcup_{i \in I} A_i \in \mathcal{T}$.

In this case the pair $(X, \mathcal{T})$ is called a topological space and the elements of $\mathcal{T}$ are referred as open sets.

Definition 2.3 (Metric Space)**.** Let X be a nonempty set. We say that $d : X \times X \to \mathbb{R}$ is a metric iff the following properties hold:

1. Non-negativity: $\forall x, y \in X : d(x, y) \geq 0$.
2. Symmetry: $\forall x, y \in X : d(x, y) = d(y, x)$
3. Separability: $\forall x, y \in X : d(x, y) = 0 \iff x = y$.
4. Triangle inequality: $\forall x, y, z \in X : d(x, y) \leq d(y, z) + d(z, x)$

In this case, we say that (X, d) is a metric space.

Remark 2.1. A metric defines a topology on X, which is generated by open balls, i.e. by sets of the form

$$\mathscr{B}(x, r) = \{y \in X : d(x, y) < r\},$$

where $r > 0$. This means that any open set can be built as the union of open balls.

Definition 2.4 (Continuous function)**.** Let (E, d_E) and (F, d_F) be metric spaces. We say that a mapping $f : E \to F$ is continuous at x_0 iff

$$\lim_{x \to x_0} f(x) = f(x_0),$$

i.e.

$$\forall \varepsilon > 0, \exists \delta = \delta(x_0, \varepsilon) : d_E(x, x_0) < \delta \implies d_F(f(x) - f(x_0)) < \varepsilon.$$

We say that f is continuous on E, or simply continuous, iff f is continuous at every point of E.

Definition 2.5 (Lipschitz continuity)**.** Given two metric spaces (X, d_X) and (Y, d_Y), a function $f : X \to Y$ is called Lipschitz continuous if there exists a real constant $K > 0$ such that,

$$\forall x_1, x_2 \in X : d_Y(f(x_1) - f(x_0)) \leq K d_X(x_1, x_0).$$

In this case, K is referred to as a Lipschitz constant for the function f. In particular, if $K \in (0, 1)$, then we say that f is a contractive mapping.

Remark 2.2. It is easy to show that Lipschitz continuity implies continuity.

Definition 2.6 (Cauchy sequences and completeness)**.** Let (E,d) be a metric space. We say that $\{x_n\}_{n\in\mathbb{N}} \subseteq X$ is a Cauchy sequence iff

$$\forall\varepsilon > 0, \exists N \in \mathbb{N}, \forall l, k > N \implies d(x_l, x_k) < \varepsilon.$$

equivalently, if

$$\lim_{l,k\to\infty} d(x_l, x_k) = 0$$

Moreover, (E,d) is complete iff every Cauchy sequence converges.

Definition 2.7 (Norm and normed space)**.** A norm over a linear space $\mathscr{V}$ is a mapping $||\cdot|| : \mathscr{V} \to \mathbb{R}$ such that the following properties hold

a. Non-negativity : $\forall x \in \mathscr{V} : ||x|| \geq 0$ with equality iff $x = 0$.

b. Homogeneity: $\forall x \in \mathscr{V}, \forall\rho \in \mathbb{R} : ||\rho x|| = |\rho|||x||$

c. Triangle inequality: $\forall x, y \in \mathscr{V} : ||x + y|| \leq ||x|| + ||y||$

In this case, the pair $(\mathscr{V}, ||\cdot||)$ is referred to as a normed space.

The space $(\mathscr{V}, ||\cdot||)$ induces a metric space, by means of

$$d(x, y) = ||x - y||, \qquad x, y \in \mathscr{V}.$$

Moreover, if $(\mathscr{V}, ||\cdot||)$ is complete, then we say that it is a Banach Space.

2.2. Lower Semicontinuity

We are going to introduce briefly some concepts about operators and functionals. After that, we shall present some useful results about convergence and lower semicontinuity. We base our presentation on [20], [21] and [13].

Let E and F be vector spaces. A linear operator $T\colon E \to F$ is a mapping such that satisfies the following condition

$$\forall\alpha \in \mathbb{R}\forall x, y \in E : T(\alpha x + y) = \alpha T(x) + T(y) \tag{6}$$

The space of linear operators is defined by

$$L(E, F) := \{T\colon E \to F/T \text{ is linear}\}.$$

Whenever $F = \mathbb{R}$, we say that T is a *bounded linear functional*. We, also, say that T is a *bounded operator* iff there is a $c > 0$ such that

$$\forall x \in E : ||T(x)||_F \leq c||x||_E \tag{7}$$

The space of the bounded operators is denoted by

$$B(E, F) := \{T \in L(E, F)|T \text{ is bounded }\}.$$

Let's recall that the functional $||\cdot||\colon B(E,F) \to \mathbb{R}$, given by

$$||T|| = \inf(\mathscr{O}_T),$$

where

$$\mathscr{O}_T = \{c > 0\,/\forall u \in E : ||T(u)||_F \leq c||u||_E\}$$

is a norm on $B(E,F)$.
It is easy to show that

$$\forall T \in B(E, F)\forall x \in E : ||T(x)||_Y \leq ||T||_{B(E,F)}||x||_E \tag{8}$$

An operator T is called a *compact linear operator*, or completely continuous linear operator, iff T is linear and for every bounded subset M of X, the image $T(M)$ is relatively compact, that is the closure $\overline{T(M)}$ is compact.

Theorem 2.1 (Compactness criterion)**.** *Let $T\colon E \to F$ be a linear operator. Then T is compact if and only if it maps every bounded sequence $\{x_n\}_{n\in\mathbb{N}} \subseteq E$ onto a sequence $\{T(x_n)\}_{n\in\mathbb{N}} \subseteq F$ which has a convergent subsequence.*

A proof is presented in [21].

Let's denote

$$E^* := B(E, \mathbb{R})$$

the dual space of E, i.e. the space of continuous linear functionals on E. For $\varphi \in E^*$ and $f \in E$ let's write the duality product as follow

$$\langle\varphi, f\rangle = \varphi(f)$$

We say that $\{x_n\}_{n\in\mathbb{N}} \subseteq E$ weakly converges to $x \in E$ iff

$$\forall \varphi \in E^* : \lim_{n\to\infty} \langle \varphi, x_n \rangle = \langle \varphi, x \rangle,$$

which can be denoted by

$$x_n \rightharpoonup x, \text{ as } n \to \infty$$

The sequence $\{x_n\}_{n\in\mathbb{N}} \subseteq E$ converges in norm of E, or strongly, if there exists $x \in E$ such that

$$\lim_{n\to\infty} ||x - x_n|| = 0,$$

which can be denoted by

$$x_n \to x, \text{ as } n \to \infty$$

Proposition 2.1 (Strong convergence implies weak convergence). *Let E be a Banach space and let $\{x_n\}_{n\in\mathbb{N}} \subseteq E$ be a sequence that strongly converges to $x \in E$. Then $\{x_n\}_{n\in\mathbb{N}} \subseteq E$ converges weakly to $x \in E$.*

Proof. We have to prove that

$$\forall \varphi \in E^* : \lim_{n\to\infty} \langle \varphi, x_n \rangle = \langle \varphi, x \rangle, \tag{9}$$

Let $\varphi \in E^*$, generic. Then, by the continuity of φ and (8), we have that

$$\begin{aligned} |\langle \varphi, x - x_n \rangle| &= |\varphi(x - x_n)| \\ &\leq ||\varphi||_{E^*} ||(x - x_n)||_E \end{aligned} \tag{10}$$

We know, by hypothesis, that $\{x_n\}_{n\in\mathbb{N}}$ converges to x in the norm of E, therefore (10) implies that

$$\lim_{n\to\infty} \langle \varphi, x_n - x \rangle = 0$$

By the linearity of φ and the previous expression, we have that

$$\lim_{n\to\infty} \langle \varphi, x_n \rangle = \langle \varphi, x \rangle.$$

By the arbitrariness of φ, we have proved (9). □

Definition 2.8 (Lower Semicontinuity). A functional $\psi : E \to \mathbb{R}$ is lower semicontinuous iff

$$\forall x \in E : \psi(x) < \liminf_{y\to x} \psi(y) \tag{11}$$

where

$$\liminf_{y\to x} \psi(y) = \lim_{\varepsilon\to 0} \left(\inf_{y \in E \cap \mathscr{B}_\varepsilon(x) \setminus \{x\}} \psi(y) \right)$$

Remark 2.3. Note that if ψ is continuous, then it is also lower semicontinuous, i.e.

$$\liminf_{y\to x} \psi(y) = \lim_{y\to x} \psi(y) = \psi(x).$$

Theorem 2.2 (Lower Semicontinuity properties). *Let $\phi\colon E \to \mathbb{R}$ be a lower semicontinuous function. Then, the following properties hold*

(i) For all $x \in E$ and for every $\varepsilon > 0$, there is some neighborhood $\mathscr{V}$ around x such that

$$\phi(y) \geq \phi(x) - \varepsilon, \forall y \in \mathscr{V}.$$

In particular, for every sequence $\{x_k\}_{k\in\mathbb{N}} \subseteq E$ such that $x_k \to x$, we have

$$\liminf_{k\to\infty} \phi(x_k) \geq \phi(x).$$

(ii) If ϕ_1 and ϕ_2 are lower semicontinuous, then $\phi_1 + \phi_2$ is lower semicontinuous.

(iii) If E is compact, then $\inf_{x\in E} \phi(x)$ is achieved.

A proof of this theorem is presented in [13].

2.3. Hilbert spaces

Definition 2.9 (Scalar product)**.** Let $\mathscr{V}$ be a linear space. A scalar product $(\cdot,\cdot)$ is a bilinear form on $\mathscr{V}\times\mathscr{V}$ with values in $\mathbb{R}$ (i.e., a map from $\mathscr{V}\times\mathscr{V}\to\mathbb{R}$ that is linear in both variables) such that the following properties hold

1. Symmetry: $\forall u,v\in\mathscr{V}:(u,v)=(v,u)$.
2. Bilinearity: $\forall u,v,w\in\mathscr{V}:(u+w,v)=(u,v)+(w,u)$.
3. Homogeneity: $\forall u,v\in\mathscr{V},\forall\rho\in\mathbb{R}:(\rho u,v)=\rho(v,u)$.
4. Non-negativity : $\forall u\in\mathscr{V}:||u||\geq 0$ with the equality iff $u=0$.

An inner-product space is also called a pre-Hilbert space or Euclidean space. Moreover, the scalar product induces a norm with

$$(u,u)=||u||^2 \tag{12}$$

In Corollary 2.1 we provide a justification for (12).

Theorem 2.3 (Cauchy–Schwarz inequality)**.** *Let $(\mathscr{H},(\cdot,\cdot))$ be an inner product space. Then,*

$$\forall x,y\in\mathscr{H}:|(x,y)|\leq||x||||y|| \tag{13}$$

Proof. For the case where x and y are zero, then (13) immediately holds. Then, let's take $x,y\in\mathscr{H}\backslash\{0\}$, generic. Let's define

$$u=\frac{x}{||x||}\text{ and }v=\frac{y}{||y||}.$$

It's clear that the norm of u and v is one.

Then, by the definition of inner product we have that

$$\begin{aligned}
0 &\leq (u-(u,v)v,u-(u,v)v)\\
&= (u,u)-(u,(u,v)v)-((u,v)v,u)+((u,v)v,(u,v)v)\\
&= ||u||^2-2(u,(u,v)v)+||(u,v)v||^2\\
&= 1-2(u,v)(u,v)+|(u,v)|^2||v||^2\\
&= 1-2(u,v)^2+|(u,v)|^2\\
&\leq 1-2|(u,v)|^2+|(u,v)|^2\\
&\leq 1-|(u,v)|^2
\end{aligned}$$

which implies that

$$\begin{aligned}
0 &\leq 1-\frac{|(x,y)|}{||x||||y||}\\
\frac{|(x,y)|}{||x||||y||} &\leq 1\\
|(x,y)| &\leq ||x||||y||
\end{aligned}$$

Since x and y were generic, we have proved (13). □

Corollary 2.1 (Triangle inequality)**.** *In every inner product space $\mathscr{H}$, the following conditions holds*

$$\forall x,y\in\mathscr{H}:||x+y||\leq||x||+||y|| \tag{14}$$

Proof. Let $x,y\in\mathscr{H}$, generic. Applying the Cauchy–Schwarz inequality we get

$$\begin{aligned}
||x+y||^2 &= (x+y,x+y)\\
&\leq ||x||^2+||y||^2+2(x,y)\\
&\leq ||x||^2+||y||^2+2||x||||y||\\
&\leq (||x||+||y||)^2
\end{aligned}$$

Since x and y were chosen arbitrarily, (14) is proved. □

Definition 2.10 (Hilbert Space)**.** A Hilbert Space $\mathscr{H}$ is a complete inner-product space.

Theorem 2.4 (Riesz–Fréchet representation theorem)**.** *Let $\mathscr{H}$ be a Hilbert space and given any $\psi \in \mathscr{H}^*$. Then there exists a unique $u \in \mathscr{H}$ such that*

$$\langle \psi, v\rangle = (u, v), \forall v \in \mathscr{H}. \tag{15}$$

Moreover,

$$|u|_{\mathscr{H}} = ||\psi||_{\mathscr{H}^*}$$

Proofs of this important theorem can be found, e.g. in [13] and [14] .

Definition 2.11 (Continuity and coerciveness)**.** Let $\mathscr{H}$ be a Hilbert space, a bilinear form $a(\cdot,\cdot) : \mathscr{H} \times \mathscr{H} \to \mathbb{R}$ is

- **Continuous:** if there exists $c > 0$ such that

$$|a(u, v)| \leq c||u||||v||, \forall u, v \in \mathscr{H}; \tag{16}$$

- **Coercive** if there exists $\rho > 0$ such that

$$a(v, v) \geq \rho||v||^2, \forall v \in \mathscr{H}. \tag{17}$$

Proposition 2.2. *Let $\mathscr{H}$ be a Hilbert space and $\{u_n\}_{n\in\mathbb{N}} \subseteq \mathscr{H}$ weakly convergent to $u \in \mathscr{H}$. Then*

$$||u|| = \liminf_{n\to\infty} ||u_n||.$$

Proof. By the Riesz–Fréchet representation theorem we choose $\psi \in \mathscr{H}^*$ such that

$$\langle \psi, v\rangle = (u, v), \forall v \in \mathscr{H}.$$

Moreover, by the weak convergence of $\{u_n\}_{n\in\mathbb{N}}$ we have

$$\begin{aligned} \liminf_{n\to\infty} \langle \psi, u_n\rangle &= \lim_{n\to\infty} \langle \psi, u_n\rangle \\ &= \langle \psi, u\rangle \\ &= (u, u) \\ &= ||u||^2. \end{aligned} \tag{18}$$

On the other hand, by (8)

$$\begin{aligned} |\langle \psi, u_n\rangle| &\leq ||\psi||_{\mathscr{H}^*}||u_n||_{\mathscr{H}} \\ &\leq ||u||_{\mathscr{H}}||u_n||_{\mathscr{H}} \end{aligned}$$

So that by passing to the lim sup in the last expression,

$$\begin{aligned} \liminf_{n\to\infty} |\langle \psi, u_n\rangle| &= \lim_{n\to\infty} \langle \psi, u_n\rangle \\ &\leq ||u||_{\mathscr{H}} \liminf_{n\to\infty} ||u_n||_{\mathscr{H}} \end{aligned} \tag{19}$$

By (18) and (19) we conclude the proof. □

Proposition 2.3. *Let $\mathscr{H}$ be a Hilbert space and $\{u_n\}_{n\in\mathbb{N}} \subseteq \mathscr{H}$ weakly convergent to $u \in \mathscr{H}$ and such that*

$$||u|| = \lim_{n\to\infty} \|u_n\|, \tag{20}$$

Then $\{u_n\}_{n\in\mathbb{N}}$ strongly converges to u.

Proof. Since $u \in \mathscr{H}$, then by the Riesz–Fréchet representation theorem we choose a $\psi \in \mathscr{H}^*$ such that

$$\langle \psi, v\rangle = (u, v), \forall v \in \mathscr{H}.$$

Moreover, by the weak convergence of $\{u_n\}_{n\in\mathbb{N}} \subseteq \mathscr{H}$; we get

$$\begin{aligned} \lim_{n\to\infty} \langle \psi, u_n\rangle &= \lim_{n\to\infty} (u, u_n) \\ &= (u, u) \\ &= ||u||^2_{\mathscr{H}}. \end{aligned} \tag{21}$$

Let's note that

$$\begin{aligned} ||u_n - u||^2 &= (u_n - u, u_n - u) \\ &= ||u_n||^2 - 2(u, u_n) + ||u||^2 \end{aligned} \tag{22}$$

By (20), (21) and (22), we obtain

$$\begin{aligned}\lim_{n\to\infty} ||u_n - u|| &= \lim_{n\to\infty} ||u_n||^2 - 2(u, u_n) + ||u||^2 \\ &= ||u||^2 - 2(u, u) + ||u||^2 \\ &= 0\end{aligned}$$

which implies that $\{u_n\}_{n\in\mathbb{N}} \subseteq \mathscr{H}$ strongly converges to $u \in \mathscr{H}$. □

Proposition 2.4. *Every bounded sequence in a Hilbert Space $\mathscr{H}$ has a subsequence that weakly converges in $\mathscr{H}$.*

A proof is presented in [17, chap. 3].

Proposition 2.5 (Lax Milgram)**.** *Let $\mathscr{H}$ be a Hilbert space and $a(\cdot,\cdot)$ a continuous, coercive and bilinear form on $\mathscr{H}$. Then, for any $\phi \in \mathscr{H}^*$ there exists a unique $u \in \mathscr{H}$ such that*

$$\forall v \in \mathscr{H} : a(u, v) = (\phi, v).$$

Moreover, if a is symmetric, then u can be characterized by

$$\frac{1}{2}a(u, u) - \langle\psi, u\rangle = \min_{v\in\mathscr{H}} \left\{\frac{1}{2}a(u, u) - \langle\psi, u\rangle\right\}$$

A proof is presented in [13, Chap. 5] .

2.4. $L^p([a, b])$ spaces

In this part we define the L^p spaces, and present some elementary properties and results. We assume the reader is familiarized with the concepts of *measure sets*, *measurable and integrable functions*. We based this section on [13] and [22].

Let $(\Omega, \mathcal{M}, \mu)$ denote a measure space. Where Ω is a non-empty set, and

(i) $\mathcal{M}$ is a *σ-algebra* on Ω, i.e. $\mathcal{M}$ is a collection of subsets of Ω such that:

1. $\emptyset \in \mathcal{M}$,
2. if $A \in \mathcal{M}$, then $A^c \in \mathcal{M}$, and
3. if $\{A_n\}_{n\in I} \subseteq \mathcal{M}$, for every $n \in \mathbb{N}$, then $\bigcup_{n=1}^{\infty} A_n \in \mathcal{M}$.

(ii) μ is a measure, i.e. $\mu\colon \mathcal{M} \to [0, \infty)$ satisfies:

1. $\mu(\emptyset) = 0$, and
2. For a disjoint family of countable sets $\{A_n\}_{n\in I}$, $\mu\left(\bigcup_{n=1}^{\infty} A_n\right) = \sum_{n=1}^{\infty} \mu(A_n)$.

The members of $\mathcal{M}$ are called measurable sets and we frequently denote $|A|$ instead of $\mu(A)$.

(iii) Ω is *σ-finite*, i.e. there exists a countable family $\{\Omega_n\}_{n\in I}$ in $\mathcal{M}$ such that $\Omega = \bigcup_{n=1}^{\infty} \Omega_n$ and for every n $|\Omega| < \infty$.

We say the the set E is a null set iff the property

$$\mu(E) = 0.$$

We also say that some property holds "a.e.", for almost every $x \in \Omega$ if it holds everywhere except perhaps, on a null set. Let $f\colon [a, b] \to \mathbb{R}$ be a measurable function. The essential supremum of f is given by

$$\sup_{x\in[a,b]} \text{ess}\, f(x) = \inf\{c \in \mathbb{R} : f(x) \le c, \text{ a.e. on } [a, b]\}$$

The essential infimum is given analogously by

$$\inf_{x\in[a,b]} \text{ess}\, f(x) = \sup\{c \in \mathbb{R} : f(x) \ge c, \text{ a.e. on } [a, b]\}$$

The space of the integrable functions is

$$\mathcal{L}^1([a, b]) = \left\{f\colon [a, b] \to \mathbb{R}\colon \|f\|_{L^1([a,b])} = \int_a^b |f| dx < \infty\right\}$$

It's clear that an equivalence relation is defined on $\mathcal{L}^1([a,b])$ by:

$$f \sim g \iff f(x) = g(x) \text{ a.e. } [a,b] \tag{23}$$

Then,

$$L^1([a,b]) = \left\{[f]/f \in \mathcal{L}^1([a,b])\right\}. \tag{24}$$

In other words, $L^1([a,b])$ collects the equivalence classes defined by v on $\mathcal{L}^1([a,b])$.
Let $p \in (0,\infty)$. We set

$$\mathcal{L}^p([a,b]) = \left\{f\colon [a,b] \to \mathbb{R}\colon f \text{ is measurable and } |f|^p \in L^1([a,b])\right\} \tag{25}$$

In the same way, as $L^1([a,b])$, is defined $L^p([a,b])$.

Remark 2.4. The abuse of notation

$$f = [f]$$

is very common and does not provide confusion. Therefore, we shall use it.

Let's consider the functional $||\cdot||_{L^p([a,b])}\colon L^p([a,b]) \to \mathbb{R}$ given by

$$||u||_{L^p([a,b])} = \left(\int_a^b |u(t)|^p dt\right)^{1/p} \tag{26}$$

It's clear that (26) satisfies

(a) $\forall u \in L^p([a,b]) : ||u||_{L^p([a,b])} = 0 \iff u = 0,$

(b) $\forall \lambda \in \mathbb{R}, \forall u \in L^p([a,b]) : ||\lambda u||_{L^p([a,b])} = |\lambda| ||u||_{L^p([a,b])}.$

Then is enough to prove the triangle inequality to conclude that $||\cdot||_{L^p([a,b])}$ defines a norm, i.e.

$$\forall u, v \in L^p([a,b]) : ||u+v||_{L^p([a,b])} \leq ||u||_{L^p([a,b])} + ||v||_{L^p([a,b])} \tag{27}$$

The space $L^\infty([a,b])$ is defined by

$$L^\infty([a,b]) = \left\{f\colon [a,b] \to \mathbb{R} \,/\, f \text{ is measurable and } ||f||_{L^\infty([a,b])} = \sup_{x\in[a,b]} \text{ess}\, f(x) < \infty\right\} \tag{28}$$

The space $C_0([a,b])$, the space of continuous functions with compact support, is defined as follows

$$C_0([a,b]) = \left\{f\colon [a,b] \to \mathbb{R} \,/\, f(x) = 0, \forall x \in [a,b]\backslash K, \text{ where } K \text{ is compact}\right\} \tag{29}$$

Let's present some of the most important and known theorems for integrals, because they are going to be useful to prove that (26) defines a norm and other important features in our results.

Theorem 2.5 (Hölder inequality for integrals)**.** *Let $1 \leq p \leq \infty$. For all $u, v \in L^p([a,b])$, the following inequality holds.*

$$||u\cdot v||_{L^1([a,b])} \leq ||u||_{L^p([a,b])} \cdot ||v||_{L^{p'}([a,b])},$$

which for $1 < p < \infty$ can be written as

$$\left(\int_a^b |u(t)\cdot v(t)|^p\right)^{1/p} \leq \left(\int_a^b |u(t)|^p\right)^{1/p} \cdot \left(\int_a^b |v(t)|^p\right)^{1/p'},$$

where

$$1 = \frac{1}{p} + \frac{1}{p'}.$$

A proof is presented in [13, Theorem 4.6].

Theorem 2.6. *[Minkowski inequality for integrals] For all $u, v \in L^p([a,b])$, the following inequality holds.*

$$\left(\int_a^b |u(t)+v(t)|^p\right)^{1/p} \leq \left(\int_a^b |u(t)|^p\right)^{1/p} + \left(\int_a^b |v(t)|^p\right)^{1/p},$$

i.e.

$$||u+v||_{L^p([a,b])} \leq ||u||_{L^p([a,b])} + ||v||_{L^p([a,b])}$$

Proof. Let $u, v \in L^p([a,b])$, generic. Then we have that

$$\begin{aligned} |v(x)+u(x)|^p &\leq (|v(x)|+|u(x)|)^p \\ &\leq 2^p(|v(x)|+|u(x)|)^p \end{aligned}$$

Consequently, $v+u \in L^p$. On the other hand,

$$\begin{aligned} \|v+u\|^p_{L^p([a,b]} &= \int_a^b (|v(x)+u(x)|)^p dx \\ &= \int_a^b (|v(x)+u(x)|)^{p-1}|v(x)+u(x)|dx \\ &\leq \int_a^b (|v(x)+u(x)|)^{p-1}|v(x)| + \int_a^b (|v(x)+u(x)|)^{p-1}|u(x)|dx \end{aligned}$$

Note that $|v+u|^{p-1} \in L^{p'}([a,b])$, by Hölder's inequality, we obtain

$$\|v+u\|^p_{L^p([a,b]} \leq \|v+u\|^{p-1}_{L^p([a,b])}(\|v\|_{L^p([a,b])} + \|u\|_{L^p([a,b])}), \tag{30}$$

which implies $\|v+u\|_{L^p([a,b])} \leq \|v\|_{L^p([a,b])} + \|u\|_{L^p([a,b])}$. By the arbitrariness of u and v we have proved (2.6). □

By the Theorem (2.6), we can conclude that (26) defines a norm.

Theorem 2.7 (Properties of Lebesgue's Spaces). *Let $I \in \mathbb{R}$ be an open set, then*

i) If $1 \leq p \leq \infty$, then $L^p(I)$ is a Banach space.

ii) If $1 < p < \infty$, then $L^p(I)$ is a reflexive space.

iii) If $1 \leq p < \infty$, then $L^p(I)$ is a separable space.

The proof of these results are provided in [13, Chap. 4] .

Theorem 2.8 (Beppo–Levi's monotone convergence theorem). *Let $I \subseteq \mathbb{R}$, open, and $(f_n)_{n\in\mathbb{N}} \subseteq L^1(I)$, such that*

(a) $\forall n \in \mathbb{N}, f_k \leq f_{k+1}$ a.e. in I and

(b) $\sup_{n\in\mathbb{N}} \int_I f(x)dx < \infty$

Then f_k converges a.e. on I to a finite limit, denoted by f, such that $f \in L^1(I)$ and

$$\|f_k - f\|_{L^1(I)} \to 0$$

Proofs of these results are provided in [19, chap. 5].

Theorem 2.9 (Lebesgue's Dominated Convergence Theorem for $L^1(I)$). *Let $\{f_n\}_{n\in\mathbb{N}}$ be a sequence of functions in $L^1(I)$ such that:*

(a) $f_k(x) \to f(x)$ a.e. in I, and

(b) There exists a function $g \in L^1(I)$ such that

$$\forall k \in \mathbb{N} : |f_k(x)| \leq |g(x)|, \text{ a.e. in } I$$

Then, $f \in L^1(I)$ and

$$||f_k - f||_{L^1(I)} \to 0$$

A proof can be found in [19, Chap. 5]

Theorem 2.10 (Density property). *Let $I \in \mathbb{R}$ be an open set and $p \geq 1$, then $C_0(I)$ is dense in $L^p(I)$, i.e.*

$$\forall f \in L^p(I), \forall \varepsilon > 0, \exists \tilde{f} \in C_0(I) : ||f - \tilde{f}||_{L^\infty} < \varepsilon \tag{31}$$

A proof of this results is provided in [13, Chap. 4] .

Lemma 2.1 (Interpolation Inequality). *Let $u \in L^p([a,b]) \cap L^q([a,b])$ with $1 \leq p \leq \infty$ and $1 \leq q \leq \infty$. Then, $u \in L^r([a,b])$ for any $r \in [\min\{p,q\}, \max\{p,q\}]$ and the following inequality holds*

$$||u||_r \leq ||u||^\rho_p ||u||^{1-\rho}_q \tag{32}$$

where

$$\frac{1}{r} = \frac{\rho}{p} + \frac{1-\rho}{q} \qquad \rho \in [0,1]$$

Proof. Let $u \in L^p(\Omega) \cap L^q(\Omega)$ and $r \in \big[\min\{p,q\}, \max\{p,q\}\big]$, generic. Let $\rho \in [0,1]$ such that

$$\frac{1}{r} = \frac{\rho}{p} + \frac{1-\rho}{q},$$

then,

$$\frac{p}{\rho r} = 1 + \frac{p(1-\rho)}{q\rho} > 1 \quad \text{and} \quad \frac{q}{(1-\rho)r} = 1 + \frac{q\rho}{p(1-\rho)} > 1$$

Then by using Hölder inequality, we have

$$\begin{aligned}
\|u\|_{L^r(\Omega)} &= \left(\int_\Omega |u(x)|^r \, dx\right)^{1/r} \\
&= \left(\int_\Omega |u(x)|^{r\rho} \cdot |u(x)|^{(1-\rho)\cdot r} \, dx\right)^{1/r} \\
&\leq \left\{\left(\int_\Omega (|u(x)|^{r\rho})^{\frac{p}{r\rho}} \, dx\right)^{\frac{r\rho}{p}} \cdot \left(\int_\Omega (|u(x)|^{r(1-\rho)})^{\frac{q}{r(1-\rho)}} \, dx\right)^{\frac{r(1-\rho)}{q}}\right\}^{1/r} \\
&= \|u\|^{\rho}_{L^p(\Omega)} \|u\|^{(1-\rho)}_{L^q(\Omega)}.
\end{aligned}$$

Which proves (32) by the arbitrariness of u, r and ρ. □

Theorem 2.11. *Let $u \in L^p(\Omega)$ and $p > q$, then there is a $C_{p,q,\Omega} > 0$ such that*

$$\|u\|_{L^p(\Omega)} \leq C\|u\|_{L^q(\Omega)} \tag{33}$$

Proof. Let $u \in L^p(\Omega)$ and $p > q$, generic. Then, by Hölder's inequality we obtain

$$\begin{aligned}
\|u\|^p_{L^p(\Omega)} &= \int_\Omega |u|^p dx \\
&= \int_\Omega |u|^p \cdot 1 \\
&\leq \left(\int_\Omega |u|^{p\cdot\frac{q}{p}} dx\right)^{p/q} \left(\int_\Omega 1\right)^{1-p/q} \\
&= \left(\int_\Omega |u|^{p\cdot\frac{q}{p}} dx\right)^{p/q} \left(\int_\Omega 1dx\right)^{1-p/q} \\
&= \left(\int_\Omega |u|^q dx\right)^{p/q} |\Omega|^{1-p/q} \\
&= C_{p,q,\Omega}\|u\|^p_{L^q(\Omega)}
\end{aligned}$$

In fact,

$$\|u\|_{L^p(\Omega)} \leq \|u\|_{L^q(\Omega)} \, |\Omega|^{\frac{1}{p}-\frac{1}{q}}$$

Then, by the arbitrariness of u, p and q. □

2.5. Some topics on Partial Differential Equation and Sobolev spaces

Partial differential equations (PDE) are a kind of mathematical model to represent phenomena coming, e.g. from Physics, Chemistry and Biology. Our main references for this part are [23] and [24]. In the PDE is very common to use the the multi-index notation, i.e. let $\Omega \subseteq \mathbb{R}$ and $u : \Omega \to \mathbb{R}$. We are going to use the following notation:

(a) We say that an element $\rho \in \mathbb{N}^n$ is a multi-index of order

$$|\rho| := \sum_{k=1}^{n} \rho_k$$

(b) Given a multi-index $\rho \in \mathbb{N}^n$, we define

$$D^\rho u(x) := \frac{\partial^{|\rho|} u(x)}{\prod_{k=1}^{n} \partial x_k^{\rho_k}}.$$

(c) If $s \in \mathbb{N}$, we say that

$$D^s u(x) := \{D^\rho u(x) : |\rho| = s\},$$

is the set of all partial derivatives of order s.

Let

$$F : \mathbb{R}^{N_s} \times \cdots \times \mathbb{R}^N \times \mathbb{R} \times \Omega \to \mathbb{R}.$$

The expression

$$F(D^s u(x), \cdots, Du(x), u(x), x) = 0, x \in \Omega$$

is called a s-order PDE, where u is the unknown.
We say that $v : \Omega \to \mathbb{R}$ is a solution of the PDE, if v satisfies:

$$F(D^s v(x), \cdots, Dv(x), v(x), x) = 0, x \in \Omega$$

There are several and well studied types of PDE, such as

(i) A linear PDE has the following form:

$$\sum_{|k| \leq s} \rho_k(x) D^k w(x) = f(x),$$

where each ρ_k and f are given functions. One of the most known examples of linear PDE's is the Helmholtz's equation, given by

$$\Delta w + l^2 w = 0.$$

(ii) If the equation has the form

$$\sum_{|k| \leq s} \rho_k(x) D^k w(x) + \rho_0 \left(D^{s-1} w, \cdots, Du, u, x \right) = f(x),$$

we say that it is a semi-linear PDE. For example, we could mention the Fisher's equation, which models the evolution of a population of density w subject to diffusion and growth of the population. It is given by

$$w_t(x) - G\Delta w = rw(P - w).$$

(iii) The equation is called a quasi-linear, if could be written as

$$\sum_{|k| \leq s} \rho_k \left(D^{s-1} w, \cdots, Du, u, x \right) D^k w(x) + \rho_0 \left(D^{s-1} w, \cdots, Du, u, x \right) = f(x).$$

We could mention, as example, the first order quasilinear Burguer's equation

$$w_t(x) + cw(x) w_x(x) = 0, x \in \mathbb{R}.$$

(iv) An equation is called a non-linear PDE if it does not depends linearly of the higher order derivatives. The Eikonal equation is an example of non-linear PDE, which is given by

$$|\nabla w| = g(x).$$

This equation is well known in the geometric optics.

The main goal of PDE theory is to set up appropriated conditions in the data to have a *well-posed problem.* A problem is said to be well-posed iff the problem satisfies the following conditions:

(i) a solution exists,

(ii) the solution is unique, and

(iii) Stability of solutions, which refers that the solution's behavior changes continuously with the initial conditions.

These conditions are extremely important to face and implement some numerical approaches of the problem. The Sobolev spaces are motivated by the complexity of finding classical, strong, solutions for PDE problems. Then arises the notion of weak solution. This involves Sobolev spaces, which are our basic tools. They are endowed with norms that involve L^p norms and weak derivatives. This part is based on [13], [23] and [24].

Remark 2.5. Let $\Omega \subseteq \mathbb{R}^N$ be open and let $1 \leq p \leq \infty$. We say that a function $f : \Omega \to \mathbb{R}$ belongs to $L^p_{loc}(\Omega)$ if $f\chi_K \in L^p(\Omega)$ for every compact set K contained in Ω. In particular, note that if $f \in L^p_{loc}(\Omega)$, then $f \in L^1_{loc}(\Omega)$.

Let $u, v \in L^1_{loc}(\Omega)$ and k be a multi-index. We say that v is the weak derivative of order k of u, i.e.

$$D^k u = v,$$

if we have the following expression

$$\int_\Omega u D^k \phi dx = (-1)^{|k|} \int_\Omega v \phi dx, \forall \phi \in C_0^\infty(\Omega). \tag{34}$$

Example 2.1. We know that the function

$$\begin{aligned} |\cdot| &: [-1,1] \to \mathbb{R} \\ x \mapsto |x| &= \begin{cases} x & \text{if } x > 0, \\ -x & \text{if } x < 0. \end{cases} \end{aligned}$$

has not strong derivative. So that, let's compute its weak derivative.
Let $\phi \in C_0^\infty([-1,1])$, using integration by parts we have

$$\begin{aligned} \int_{-1}^{1} |x|\phi'(x)dx &= \int_{-1}^{0} -x\phi'(x)dx + \int_{0}^{1} x\phi'(x)dx \\ &= -x\phi(x)|_{-1}^{0} + \int_{-1}^{0} \phi(x)dx + x\phi(x)|_{-1}^{0} - \int_{0}^{1} \phi(x)dx \\ &= \int_{-1}^{0} \phi(x)dx - \int_{0}^{1} \phi(x)dx \\ &= -\int_{-1}^{1} \operatorname{sgn}(x)\phi(x)dx \end{aligned}$$

where

$$\begin{aligned} |\cdot| &: \mathbb{R} \to \mathbb{R} \\ x \mapsto \operatorname{sgn}(x) &= \begin{cases} 1 & \text{if } x > 0, \\ 0 & \text{if } x = 0, \\ -1 & \text{if } x < 0. \end{cases} \end{aligned}$$

With this we conclude that the weak derivative of $|\cdot|$ is the sign function.

Let $s \in \mathbb{N}$ and $1 \leq p \leq \infty$. We define the Sobolev space

$$W^{s,p}(x) := \left\{ u \in L^p(\Omega) \;\middle|\; \begin{array}{l} \forall k \in \mathbb{N}^N : |k| \leq s, \exists D^k u \in L^p(\Omega) \text{ such that} \\ \forall \phi \in C_0^\infty(\Omega) : \int_\Omega f D^k \phi dx = (-1)^{|k|} \int_\Omega \phi D^k f dx. \end{array} \right\} \tag{35}$$

Remark 2.6. When $p = 2$, the space is denoted as

$$H^s(\Omega) := W^{s,2}(\Omega)$$

The functional given by

$$\begin{aligned} \|\cdot\|_{W^{s,p}(\Omega)} &: W^{s,p}(\Omega) \to \mathbb{R} \\ f \mapsto \|f\|_{W^{s,p}(\Omega)} &= \begin{cases} \left(\sum_{|k| \leq s} \|D^k f\|^p_{L^p(\Omega)} \right)^{1/p}, & \text{if } 1 \leq p < \infty. \\ \sum_{|k| \leq s} \|D^k f\|_{L^p(\Omega)}, & \text{if } p = \infty \end{cases}. \end{aligned}$$

defines a norm on the Sobolev spaces $W^{s,p}(\Omega)$. Sobolev spaces satisfy the following conditions as the Lebesge spaces.

Proposition 2.6. *Let $\Omega \subseteq \mathbb{R}^N$, open. Then,*

i) If $1 \leq p \leq \infty$, then $W^{s,p}(\Omega)$ is a Banach space.

ii) If $1 < p < \infty$, then $W^{s,p}(\Omega)$ is a reflexive space.

iii) If $1 \leq p < \infty$, then $W^{s,p}(\Omega)$ is a separable space.

A proof is presented in [13, Chap. 9].

Theorem 2.12 (Morrey)**.** *Let $p > N$, then*

$$W^{1,p}(\mathbb{R}^N) \subseteq L^{\infty}(\mathbb{R}^N) \tag{36}$$

with continuous injection. Furthermore, for all $u \in W^{1,p}(\mathbb{R}^N)$, we have

$$|u(x) - u(y)| \leq C|x-y|^{\rho}\|\nabla u\|_{L^p(\mathbb{R}^N)} \qquad a.e.\ x, y \in \mathbb{R}^N \tag{37}$$

where

$$\rho := 1 - \frac{N}{p}$$

and C is a constant that only depends of p and N.

A proof of these result is presented in [13, Theorem 9.12].

Remark 2.7. The inequality (37) implies the existence of a function

$$\tilde{u} \in C(\mathbb{R}^N)$$

such that

$$u = \tilde{u} \qquad \text{a.e. on } \mathbb{R}^N.$$

In other words, every function $u \in W^{1,p}(\mathbb{R}^N)$ with $p > N$ admits a continuous representative. In our case, $N = 1$ and $p = 2$, every element of the space $W^{1,p}(\mathbb{R})$ possesses a continuous representative.

Theorem 2.13 (Rellich–Kondrachov)**.** *Let $\Omega \subseteq \mathbb{R}^N$ be compact and of class C^1. Then we have the following compact injection*

$$W^{1,p}(\Omega) \subseteq C(\overline{\Omega}),$$

where, $p > N$.

The proof of this result can be found in [13, Theorem 9.16].

The space $H_0^s(\Omega)$ is formed by elements of $H^s(\mathbb{R}^N)$ such that "cancel out" in the boundary of Ω, grossly speaking. The Sobolev space $W_0^{1,p}(\Omega)$ is formed by the elements of $W^{1,p}(\Omega)$ such that they "cancel out" in the boundary of Ω. In particular, $W_0^{1,2}(\Omega)$, denoted by $H_0^1(\Omega)$, is known as the closure of $C_0^{\infty}(\Omega)$ in $W_0^{s,2}(\Omega)$.

Remark 2.8. The subspace $C^{\infty}(\Omega) \cap W^{s,p}(\Omega)$ is dense in $W^{s,p}(\Omega)$.

Theorem 2.14 (Poncaré's inequality)**.** *Suppose that $1 < p < \infty$ and the set Ω is a bounded open set. Then there exists a constant $C_{\Omega,p} > 0$ such that*

$$\forall u \in W_0^{1,p}(\Omega) : \|u\|_{L^p(\Omega)} \leq C_{\Omega,p}\|\nabla u\|_{L^p(\Omega)}. \tag{38}$$

In particular, the expression $\|\nabla u\|_{L^p(\Omega)}$ is a norm of $W_0^{1,p}(\Omega)$ and it is equivalent to the norm $W^{1,p}(\Omega)$.

The proof of this result can be found in [13, Theorem 9.19].

Grossly, we could say that Ω is of class C^1 if the boundary does not present any kings.

Proposition 2.7. *Let Ω be of class C^1 and $u \in L^p(\Omega)$ with $p > 1$. Then the following statements are equivalent.*

(i) $u \in W_0^{1,p}(\Omega)$,

(ii) $\exists C > 0, \forall \phi \in C_c^1(\mathbb{R}^N) : \left|\int_{\Omega} \frac{\partial \phi}{\partial x_i} dx\right| \leq C\|\phi\|_{L^p(\Omega)} \quad \forall i = 1, \cdots, N,$

(iii) The function

$$\overline{u}(x) = \begin{cases} u(x) & \text{if } x \in \Omega, \\ 0 & \text{if } x \in \mathbb{R}^N \backslash \Omega. \end{cases}$$

belongs to $W^{1,p}(\Omega)$ in such case

$$\frac{\partial \overline{u}}{\partial x_i} = \overline{\frac{\partial u}{\partial x_i}}.$$

The proof is provided in [13, Theorem 9.18].

2.6. Topics on non-linear analysis

We base this part on [25],[26] and [23].
Let's state some important concepts that will help to deal with our problem.

2.6.1. Deformable sets

Let (X, τ_X) be a topological space and $A, B \subseteq X$. We would say that A is deformable to B in X if there is a mapping $\eta \in \mathbf{C}([0,1] \times A, X)$ such that

$$\forall x \in A : \eta(0,x) = x \quad \wedge \quad \eta(1,x) \in B \tag{39}$$

in this case, η is called a deformation from A to B in X. In particular, if A is deformable to singleton $\{x_0\} \subseteq X$, then we say that A is contractible in X. For example, the unit sphere

$$\mathbb{S}^{N-1} := \{x \in \mathbb{R}^N : |x| = 1\}$$

is contractible in $\mathbb{R}^N$. In fact, we can use the mapping

$$\begin{aligned} \eta\colon [0,1] \times \mathbb{S}^{N-1} &\to \mathbb{R}^N \\ (t,x) &\mapsto \eta(t,x) = (1-t)x \end{aligned}$$

Observe that η is continuous and

$$\forall x \in \mathbb{S}^{N-1} : \eta(0,x) = x \quad \text{and} \quad \eta(1,x) = 0$$

2.6.2. Fréchet differentiablity

Now, let's define differentiability of a functional defined on a Banach space. Let X and F be normed spaces and $U \subseteq X$ an open set. A mapping $f\colon U \to F$ is Fréchet differentiable at $u \in U$ iff there exists a $\phi \in \mathcal{B}(E,F)$ such that

$$f(u+v) - f(u) = \phi(v) + o(||v||_X), \tag{40}$$

for all $v \in X$ such that $u + v \in U$. Then ϕ is called the differential of f at u. Moreover, if $f'\colon U \to \mathcal{B}(E,F)$ is continuous, then we would say that $f \in \mathcal{C}^1$ in U. In this case, we write $\phi := f'(u)$ as such a mapping is unique. To study more in deep the properties and results about Fréchet differentiability see [27].

Remark 2.9. If f' is of class C^1, we say that f is of class C^2. Following in this way, we can say that f is of class C^3, C^4, etc.

Let $\mathscr{H}$ be a Hilbert Space and $U \subseteq \mathscr{H}$ be an open set and f a differentiable functional. A number $a \in \mathrm{Im}(f)$ will be called a regular value if it is not a critical value.

2.6.3. Manifolds

Definition 2.12 ($\mathcal{C}^k$ Differentiable Manifold)**.** We would say that $\mathscr{M} \subseteq \mathscr{H}$, nonempty, is a $\mathcal{C}^k$ manifold if there exist an open set $U \subseteq \mathscr{H}$, a differentiable functional $f\colon U \to \mathbb{R}$ of class $\mathcal{C}^k$ and a regular value a of f such that $\mathscr{M} = f^{-1}(a)$

Example 2.2. The set

$$\mathbb{S}_{\mathscr{H}} := \{x \in \mathscr{H} : \|x\| = 1\}$$

is a manifold of class C^∞, because the functional

$$\begin{aligned} \zeta\colon \mathscr{H} &\to \mathbb{R} \\ x &\mapsto \zeta(x) = \frac{1}{2}\left(\|x\|^2 - 1\right) \qquad \text{and} \qquad D\zeta(x) = x \end{aligned}$$

is of class C^∞, $\mathbb{S}_{\mathscr{H}} = \zeta^{-1}(0)$ and

$$\forall u \in \mathbb{S}_{\mathscr{H}} : T_u\mathbb{S}_{\mathscr{H}} = \{v \in \mathscr{H} : (u,v) = 0\}.$$

Note that the identification $D\zeta(x) = x$ is possible by the Riesz–Fréchet representation theorem.

Let $\Gamma_u(\mathscr{M})$ be the set of all the C^1 trajectories $\sigma : (-\varepsilon, \varepsilon) \to \mathscr{H}$, with $\varepsilon > 0$, such that $\sigma(0) = u$ and $\sigma(t) \in \mathscr{M}$ for all $|t| \leq \varepsilon$. By the implicit function theorem we can write the tangent space as

$$T_u\mathscr{M} = \{D\sigma(0) : \sigma \in \Gamma_u(\mathscr{M})\},$$

in particular, $T_u\mathscr{M}$ does not depend of f.

Definition 2.13 (Critical point, critical and regular value)**.** Let $\zeta\colon \mathscr{H} \to \mathbb{R}$ be a C^1 mapping and $\mathscr{M}$ be a C^1 manifold of $\mathscr{H}$. A critical point $u \in \mathscr{M}$ of ζ satisfies

$$\forall v \in T_u\mathscr{M} : \langle D\zeta(u), v\rangle = 0. \tag{41}$$

In the case that $\mathscr{M} = \mathscr{H}$, then u still is a critical point, but we also have, by (41), that $D\zeta(u) = 0$. Let $c \in \mathbb{R}$ be a critical value of ζ on $\mathscr{M}$ if $c = \zeta(u)$ for some critical point u of ζ on $\mathscr{M}$. Otherwise, c is called a regular value of ζ on $\mathscr{M}$.

The extremums of a functional in a manifold satisfy the following property.

Proposition 2.8. *Let $J\colon \mathscr{H} \to \mathbb{R}$ be a C^1 functional, $\mathscr{M}$ is a C^1 manifold of $\mathscr{H}$ and u is a extremum of J in $\mathscr{M}$, entonces*

$$\forall v \in T_u\mathscr{M} : \langle DJ(u), v\rangle = 0.$$

Proof. Let $v \in T_u\mathscr{M}$, generic, and $\sigma \in \Gamma(\mathscr{M})$ such that $D\sigma(0) = v$. If u is a minimum of J in $\mathscr{M}$, then 0 is a minimum of $J \circ \sigma\colon (-\varepsilon, \varepsilon) \to \mathbb{R}$. By the chain rule for functionals we have

$$0 = D(J \circ \sigma)(0) = \langle DJ(\sigma(0)), D\sigma(0)\rangle = \langle DJ(u), v\rangle.$$

By the arbitrariness of v we conclude the result for the minimum. The analysis for the maximum is analogous. □

Let's state one of the most important results in Variational Calculus.

Theorem 2.15 (Lagrange's multipliers)**.** *Let $\mathscr{M}$ be a C^1 manifold of $\mathscr{H}$ and $J\colon \mathscr{H} \to \mathbb{R}$ be a C^1 functional, and u is a critical point of J in $\mathscr{M}$, then there exists $\lambda \in \mathbb{R}$ such that*

$$\nabla J(u) = \lambda \nabla f(u)$$

where f is the mapping that defines the manifold.

The proof of this result is presented in [26].

2.6.4. Tangent fields

Let $\mathscr{H}$ be a Hilbert space and $\mathscr{M}$ be a manifold of class C^1 of $\mathscr{H}$. A tangent field to $\mathscr{M}$ is a mapping $\zeta\colon \mathscr{M} \to \mathscr{H}$ such that

$$\forall u \in \mathscr{M} : f(u) \in T_u\mathscr{M}.$$

On the other hand, a field is called locally Lipschitz if

$$\forall u \in \mathscr{M}, \exists r_u, C_u > 0, \forall w, v \in B^{\mathscr{M}}_{r_u}(u) : \|\zeta(v) - \zeta(w)\| \le C_u\|u - v\|,$$

where

$$B^{\mathscr{M}}_{r_u}(u) := \{v \in \mathscr{M} : \|v - u\| < r_u\}.$$

The following theorem is a fundamental result to guarantee the existence and uniqueness of solutions of Cauchy's problems on manifold in a Hilbert space.

Theorem 2.16 (Global uniqueness and existence)**.** *Let $f\colon \mathscr{M} \to \mathscr{H}$ be a locally Lipschitz field tangent to $\mathscr{M}$. Then, for each $u \in \mathscr{M}$ exists an open interval*

$$I(u) := (t^-, t^+)$$

such that contains the origin and a unique $\sigma(\cdot, u) \in C^1(I(u), \mathscr{M})$, which is solution of the Cauchy's problem

$$\begin{cases} \sigma_t(t, u) = f(\sigma(t, u)), \\ \sigma(0, u) = u. \end{cases} \tag{CP}$$

$I(u)$ is the maximal interval, which means that the solutions could not be extended to a larger interval. If $\|f(\sigma(t, u))\| \le C < \infty$, for any $t \in [0, T^+(u))$, then $T^+(u) = \infty$. For $T^-(u)$ we have a similar result.

The domain for σ is then

$$D_\sigma := \{(t, u) \in \mathbb{R} \times \mathscr{M} : t \in I(u)\}$$

open in $\mathbb{R} \times \mathscr{M}$ and the mapping σ, defined by (CP), is continuous.

The mapping $\sigma\colon D_\sigma \to \mathscr{M}$ is called the flow generated by f. If $D_\sigma = \mathbb{R} \times \mathscr{M}$ we say that the flow is global.

The proof is provided in [28].

Now, let's assume that $\mathscr{M} \subseteq \mathscr{H}$ is a manifold of class C^2 and $J \in C^2(\mathscr{H}, \mathbb{R})$. Let $\Psi \in C^2(\mathscr{O}, \mathbb{R})$ defined in a neighborhood $\mathscr{O}$ of $\mathscr{M} \subseteq \mathscr{H}$ such that $\mathscr{M} = \Psi^{-1}(a_0)$ for a regular value a_0 of Ψ.

The gradient field of J on $\mathscr{M}$ is obtained by the orthogonal projection of $\nabla J(u)$ on $T_u\mathscr{M}$ for each $u \in \mathscr{M}$, i.e.

$$\nabla_{\mathscr{M}} J(u) := \nabla J(u) - \frac{\nabla J(u), \nabla\Psi(u)}{||\nabla\Psi(u)||^2}\nabla\Psi(u)$$

Taking $\mathscr{O}$ small enough we could assume without loss of generality that $\Psi(u) \neq 0$ for every $u \in \mathscr{O}$.

The negative gradient flow of J on $\mathscr{M}$ is the solution for the Cauchy's problem

$$\begin{cases} \sigma_t(t,u) = -\nabla_{\mathscr{M}} J(\sigma(t,u)), \\ \sigma(0,u) = u. \end{cases} \tag{NGF}$$

2.6.5. Palais–Smale condition and the symmetric deformation lemma

Let's begin the study of what Palais–Smale condition means and the implications in a non-linear functional that verifies it. This section is mainly based on [29], [27] and [25].

Before presenting the Palais–Smale condition, we introduce the definition of Palais–Smale Sequence.

Definition 2.14 (Palais–Smale sequence)**.** A sequence $\{u_m\}_{m\in\mathbb{N}} \subseteq X$ is a Palais-Smale sequence for ϕ if

$$|\phi(u_m)| \leq c$$

uniformly in m , while

$$\lim_{m\to\infty} ||\phi'(u_m)|| = 0.$$

Also, we say that $\{u_k\}_{k\in\mathbb{N}} \subseteq X$ is a Palais-Smale c sequence for ϕ if

$$|\phi(u_m)| \to c$$

while

$$\lim_{k\to\infty} ||\phi'(u_m)|| = 0.$$

Definition 2.15 (Palais–Smale condition)**.** Let X be a Banach space and $\phi\colon X \to \mathbb{R}$ be a $\mathcal{C}^1$-functional. We say that ϕ satisfies the Palais–Smale condition, denoted (PS), if any Palais–Smale sequence admits a convergent subsequence.

Definition 2.16 ((local) Palais–Smale condition)**.** Let X and ϕ be as in (PS) condition and $c \in \mathbb{R}$. The functional ϕ is said to satisfy the (local) Palais– Smale condition at level c, denoted by $(PS)_c$, if any sequence $\{u_m\}_{m\in\mathbb{N}} \subseteq X$ such that

$$\phi(u_m) \to c \text{ and } \phi'(u_m) \to 0 \tag{42}$$

admits a convergent subsequence.

Remark 2.10. The following conditions are consequences of the definitions

- When (PS) is satisfied, we can check immediately that $(PS)_c$ holds for all $c \in \mathbb{R}$, while the converse is not true in general.
- If a functional ϕ satisfies $(PS)_c$ for all c, then this does not imply that the critical set K of ϕ is bounded.

In finite dimensions, a large class of functionals satisfying (PS) can be characterized as follows:

Proposition 2.9 (Characterization of functionals that satisfies (PS))**.** *Suppose $\phi \in C^1(\mathbb{R}^n)$ and assume the function $||D\phi|| + |\phi| : \mathbb{R}^n \to \mathbb{R}$ is coercive. Then (PS) holds for ϕ.*

Proof. If $||D\phi|| + |\phi|$ is coercive, clearly a Palais-Smale sequence will be bounded, hence will contain a convergent subsequence by the Bolzano–Weierstrass theorem. □

Remark 2.11. Let $\mathscr{H}$ be a Hilbert space. We say that $J\colon \mathscr{M} \subseteq \mathscr{H} \to \mathbb{R}$ satisfies $(PS)_{\mathscr{M}}$ if every Palais–Smale sequence for J has a convergent subsequence in $\mathscr{H}$. We, also, say that J satisfies the Palais–Smale condition in the level $c \in \mathbb{R}$, $(PS)_{\mathscr{M},c}$, if every Palais–Smale sequence in the level c has a convergent subsequence in $\mathscr{H}$.

To further study and get more results and some applications of Palais–Smale condition see e.g.[25], [29], [27] and [23]. Let $\mathscr{S}_1$, and $\mathscr{S}_2$ two symmetric sets, i.e. $\mathscr{S}_k = -\mathscr{S}_k$ for $k \in \mathbb{N}$, and $f\colon \mathscr{S}_1 \to \mathscr{S}_2$ we say that f is an odd function if

$$\forall u \in \mathscr{S}_1 \colon f(-u) = -f(u).$$

Also, we say that f is an even function if

$$\forall u \in \mathscr{S}_1 \colon f(-u) = f(u).$$

If $\mathscr{M}$ s symmetric and J is an even mapping, then the gradient over $\mathscr{M}$ of J is odd. In other words,

$$\forall u \in \mathscr{M} : \nabla_{\mathscr{M}} J(-u) = -\nabla_{\mathscr{M}} J(-u).$$

In consequence, the negative gradient σ is odd, i.e.

$$\forall u \in D_\sigma : \sigma(t, -u) = -\sigma(t, u). \tag{43}$$

Theorem 2.17. *Let $f\colon \mathscr{M} \to \mathscr{H}$ be a locally Lipschitz field tangent $\mathscr{M}$ and f an odd mapping. Then the flow generated by f satisfy (43).*

A proof of this theorem is presented in [28].

Lemma 2.2. *If J satisfies $(PS)_{\mathscr{M},c}$ for all $c \in [a, b]$, then for $\varepsilon > 0$ there exist a $\delta > 0$ such that*

$$\forall u \in J^{-1}[a - \delta, b + \delta] \backslash \bigcup_{c \in [a,b]} B_\varepsilon(K_c) : \|D_{\mathscr{M}} J(u)\| \geq \frac{\delta}{\varepsilon}.$$

A proof is presented in [12].

Theorem 2.18 (Symmetric deformation lemma)**.** *If J satisfies the $(PS)_{\mathscr{M},c}$, then given $\varepsilon > 0$ there exists $\delta > 0$ such that $J^{c+\delta} \backslash B_\varepsilon(K_c)$ is deformable to $J^{c-\delta}$ in $\mathscr{M}$. Moreover, the deformation is odd in $u \in \mathscr{M}$. If K_c for every $c \geq a$, then $\mathscr{M}$ is deformable to J^a, then the deformation is odd for $u \in \mathscr{M}$.*

Proofs are presented in [8] and [12].

2.6.6. Kranoselskii's genus

Let E be a Banach space, we define the set

$$\Sigma_E = \{A \in E : A = \bar{A}, A = -A, 0 \notin A\} \tag{44}$$

The Krasnoselski's genus of $A \in \Sigma_E$, denoted by $\gamma(A)$ is the least natural number k such that there exist

$$f \in \mathrm{C}(A, \mathbb{R}^k \backslash \{0\}) \tag{45}$$

If there is not such k, then

$$\gamma(A) = \infty.$$

Also, by definition

$$\gamma(\emptyset) = 0.$$

Remark 2.12. The concept of genus generalizes the notion of dimension.

Proposition 2.10. *For any bounded symmetric neighborhood of Ω of the origin, in $\mathbb{R}^m$, we have $\gamma(\partial\Omega) = m$.*

A proof is presented in [8].

Now, to have a better understanding of the previous results let's present an example presented in [12].

Example 2.3. For $\mathbb{S}^{m-1}$, taking the identity mapping given by

$$\begin{aligned} id\colon \mathbb{S}^{m-1} &\to \mathbb{R}^m \backslash \{0\} \\ x &\mapsto id(x) = x \end{aligned}$$

Then we could observe that

$$\gamma(\mathbb{S}^{m-1}) \leq m.$$

Proposition 2.11. *Let $A, B \in \Sigma_E$ and f be a continuous odd mapping, then the following properties hold:*

(K_1) ***Normalization:*** *If for $x \neq 0$,*

$$\gamma(\{x\} \cup \{-x\}) = 1.$$

(K_2) ***Odd mapping:*** *If $f \in C(A, B)$, then*

$$\gamma(A) \leq \gamma(B).$$

(K_3) ***Monotony:*** *If $A \subseteq B$, then*

$$\gamma(A) \leq \gamma(B).$$

(K_4) ***Subadditivity:***

$$\gamma(A \cup B) \leq \gamma(A) + \gamma(B).$$

(K_5) ***Continuity:*** *Let A be a compact set, then $\gamma(A) < \infty$ and there exists a $\delta > 0$ such that*

$$B_\delta(A) \in \Sigma_E \text{ and } \gamma(B_\delta(A)) = \gamma(A).$$

Proof are presented in [8] and [12].

Theorem 2.19. *Let $\mathscr{H}$ be a Hilbert space, $\mathcal{M} \in \Sigma_H$ a C^1 manifold of $\mathscr{H}$ and $J \in \mathrm{C}^1$ an even functional. Suppose that J satisfies $(PS)_\mathcal{M}$ and $J|_\mathcal{M}$ is bounded from below. Therefore,*

$$\gamma(\mathcal{M}) \leq \sum_{c \in \mathbb{R}} \gamma(K_c), \tag{46}$$

i.e. J has at least $\gamma(\mathcal{M})$ pairs of critical points on $\gamma(\mathcal{M})$. Moreover, the critical values of f from $1 \leq k \leq dim(\mathscr{H})$, are given by

$$C_k(f) = \inf_{A \in \mathcal{A}_k(\mathcal{M})} \max_{u \in A} f(u)$$

where

$$\mathcal{A}_k(\mathcal{M} = \{A \in \Sigma_H \cap \mathcal{M} : \gamma(A) \geq k\}$$

Proofs are showed in [8] and [12].

The following result, which is proved in [12], is crucial to obtain the existence of infinitely many solutions for the original and limiting problem.

Corollary 2.2. *Let $\mathscr{H}$ be a Hilbert space with infinite dimension, i.e. $\dim(\mathscr{H}) = \infty$. Then*

$$\gamma(\mathbb{S}_\mathscr{H}) = \infty.$$

Proof. If $\dim(\mathscr{H}) = \infty$, then for any $m \in \mathbb{N}$ there exists a linear isometry from $\mathbb{R}^m$ to $\mathscr{H}$ that induces an odd continuous mapping from $\mathbb{S}^{m-1}$ to $\mathbb{S}_\mathscr{H}$.
Then by (K_2) and Proposition 2.10, we have

$$m = \gamma(\mathbb{S}^{m-1}) \leq \gamma(\mathbb{S}_\mathscr{H}),$$

for every $m \in \mathbb{N}$. Then we conclude $\gamma(\mathbb{S}_\mathscr{H}) = \infty$.

□

2.7. Quasi-homogeneous functions and subconvergence

Let $a, b \in \mathbb{R}$ such that $a < b$. We choose a continuous mapping $r : \mathbb{R}\backslash\{0\} \to [0, \infty)$, satisfying

$$\begin{aligned} x/t &\in \mathbb{R}\backslash[a, b], & t &\in (0, r(x)) \\ x/t &\in \{a, b\}, & t &\in (0, r(x)) \\ x/t &\in (a, b), & t &\in (r(x), \infty) \end{aligned}$$

Definition 2.17. A continuous function $b : \mathbb{R} \to [0, \infty)$ is called an Ω *quasi-homogeneous function* if

(i) $b(\cdot)$ is strictly increasing on $[0, \infty)$; and

(ii) it holds

$$\lim_{r \to 0} \frac{b(cr)}{b(r)} \begin{cases} < 1 & \text{if } c < 1, \\ > 1 & \text{if } c > 1. \end{cases} \tag{47}$$

Definition 2.18. A continuous function $a : \mathbb{R} \to (0, \infty)$ is called an *asymptotically* (Ω, b)*-quasi-homogeneous function* if there is a *quasi-homogeneous* function b satisfying

$$\lim_{|x| \to 0} \frac{a(x)}{b(x)} = 1. \tag{48}$$

Let's define, for $\varepsilon > 0$,

$$g(\varepsilon) = \frac{1}{b^{-1}\left(\frac{-1}{\ln(\varepsilon^2)}\right)}, \tag{49}$$

As it is mentioned in [10], for some $\rho > 0$ it holds

$$\lim_{r \to 0} \frac{b(r)}{r^\rho} = 0 \tag{50}$$

This shall be used to prove the next result.

Theorem 2.20. *Let b be a Ω quasi-homogeneous function, g defined by (49) and for $\rho > 0$ given in (50). We have*

a) $\lim_{r \to 0} b(r) = 0$;

b) $\lim_{\varepsilon \to 0} g(\varepsilon) = \infty$;

c) $\lim_{\varepsilon \to 0} \frac{g(\varepsilon)}{|\ln(\varepsilon)|^\rho} = 0$.

Proof. (a) Let $r = \frac{1}{x}$. Then, by (50), we have,

$$\lim_{x \to \infty} b\left(\frac{1}{x}\right) x^\rho = 0,$$

which implies that

$$\lim_{x \to \infty} b\left(\frac{1}{x}\right) = 0$$

Then

$$\lim_{r \to 0} b(r) = 0 \tag{51}$$

(b) We have, as a consequence of (49), that

$$\frac{-1}{\ln(\varepsilon^2)} = b\left(\frac{1}{g(\varepsilon)}\right). \tag{52}$$

Thus,

$$\lim_{\varepsilon \to 0} b\left(\frac{1}{g(\varepsilon)}\right) = \lim_{\varepsilon \to 0} \frac{-1}{\ln(\varepsilon^2)} = 0$$

By (51) we can conclude that

$$\lim_{\varepsilon \to 0} \frac{1}{g(\varepsilon)} = 0,$$

and therefore,

$$\lim_{\varepsilon \to 0} g(\varepsilon) = \infty. \tag{53}$$

(c) We also have that

$$\begin{aligned} L &= \lim_{\varepsilon \to 0} \frac{g(\varepsilon)}{|\ln(\varepsilon)|^{\frac{1}{\rho}}} \\ &= \left(\lim_{\varepsilon \to 0} \frac{|-2|(g(\varepsilon))^\rho}{|-2||\ln(\varepsilon)|}\right)^{\frac{1}{\rho}} \\ &= \left(2 \lim_{\varepsilon \to 0} \frac{(g(\varepsilon))^\rho}{|-2\ln(\varepsilon)|}\right)^{\frac{1}{\rho}} \\ &= \left(2 \lim_{\varepsilon \to 0} \left|\frac{(g(\varepsilon))^\rho}{\ln(\varepsilon^2)}\right|\right)^{\frac{1}{\rho}}. \end{aligned}$$

Then, by (52), we have that

$$L = \left(2\left|\lim_{\varepsilon\to 0} b\left(\frac{1}{g(\varepsilon)}\right)(g(\varepsilon))^{\rho}\right|\right)^{\frac{1}{\rho}}.$$

Let's make the following change of variable

$$r = \frac{1}{g(\varepsilon)}.$$

By (53) it's clear that $r \to 0$ as $\varepsilon \to 0$ and by (50) we can conclude that

$$L = \left(2\left|\lim_{\varepsilon\to 0} \frac{b(r)}{r^{\rho}}\right|\right)^{\frac{1}{\rho}} = 0. \tag{54}$$

By (51), (53) and (54) we are done. □

Remark 2.13. Since b is continuous, point (a) in Theorem 2.20 implies that

$$b(0) = 0 \tag{55}$$

and also

$$b^{-1}(0) = 0, \tag{56}$$

as well.

Corollary 2.3. *For any $C > 0$, it holds*

$$\lim_{\varepsilon\to 0} \frac{1}{(g(\varepsilon))^2} \exp\left(\frac{C}{b\left(\frac{1}{g(\varepsilon)}\right)}\right) = \lim_{\varepsilon\to 0} \frac{1}{(\varepsilon^C g(\varepsilon))^2} = \infty$$

Proof. Let's show that

$$\frac{1}{(g(\varepsilon))^2} \exp\left(\frac{C}{b\left(\frac{1}{g(\varepsilon)}\right)}\right) = \frac{1}{(\varepsilon^C g(\varepsilon))^2}$$

By (49) we have

$$\begin{aligned}
\frac{1}{(g(\varepsilon))^2} \exp\left(\frac{C}{b\left(\frac{1}{g(\varepsilon)}\right)}\right) &= \frac{1}{(g(\varepsilon))^2} \exp\left(\frac{C}{\frac{-1}{\ln(\varepsilon^2)}}\right) \\
&= \frac{1}{(g(\varepsilon))^2} \exp\left(-C\ln(\varepsilon^2)\right) \\
&= \frac{1}{(g(\varepsilon))^2} \exp\left(\ln \varepsilon^{-2C}\right) \\
&= \frac{1}{(\varepsilon^C g(\varepsilon))^2}
\end{aligned}$$

Moreover, by (51) and (53) we can conclude that

$$\lim_{\varepsilon\to 0} \frac{1}{(g(\varepsilon))^2} \exp\left(\frac{C}{b\left(\frac{1}{g(\varepsilon)}\right)}\right) = \infty$$

□

Remark 2.14. By taking $C = 1$ in the Corollary 2.3, we have that in particular

$$\lim_{\varepsilon\to 0} (\varepsilon g(\varepsilon))^{\alpha} = 0, \qquad \alpha \in (0,1) \tag{57}$$

Now, let's introduce a crucial definition of convergence.

Definition 2.19 (Subconvergence)**.** A family of functions $\{\zeta_\varepsilon\}_{\varepsilon>0}$ is said to sub-converge in a space E, as $\varepsilon \to 0$, when from any sequence $\{\varepsilon_n\}_{n\in\mathbb{N}}$ converging to zero it is possible to extract a subsequence $\{\varepsilon_{n_k}\}_{k\in\mathbb{N}}$ such that $\left\{\zeta_{\varepsilon_{n_k}}\right\}_{k\in\mathbb{N}}$ converges in E, as $k \to \infty$.

3. A short introduction to Quantum Mechanics

In this chapter we shall present a number of concepts of Quantum Mechanics which are important for our study. Our main reference is N. Zettili, [1]. Quantum Mechanics was originated by the failure of Classical Mechanics to explain some of microscopical phenomena that were observed by scientists such as Albert Einstein, Niels Bohr, Erwin Schrödinger, Werner Heisenberg and Max Plank.

Some historical facts

At the end of the nineteenth century, the scientific community believed that they had managed to completely describe nature through Physics: that all physical phenomena could be explained within the framework of its main theories, [1]. At that time, Physics was divided mainly into three branches: Classical Mechanics, Maxwell's electromagnetism and thermodynamics, [1] and [30]. Classical mechanics was responsible for studying the dynamics of **material bodies**; such explanation was given in terms of **particles**. On the other hand, electromagnetism was responsible for providing a framework to study **radiation** which was explained in terms of **waves**. The interaction of matter and radiation was explained by electromagnetism.

In the early twentieth century, Classical Mechanics was seriously challenged, [1], on two fronts:

- Einstein's 1905 Theory of Relativity claimed that the validity of Newtonian Mechanics ceases at very high speeds **(Relativity Domain)**.
- New experimental techniques showed that the validity of Classical Mechanics ceases at microscopic level **(Microscopic Domain)**.

The failure of Classical Mechanics to explain several microscopic phenomena such as blackbody radiation, the photoelectric effect, atomic stability, and atomic spectroscopy motivated the searching for new ideas outside its purview. New concepts emerged to explain these phenomena; they shall be shortly described, [1].

In 1900 Max Planck introduced the concept of **quantum** of energy. He argued that the energy exchange between an electromagnetic wave of frequency β and matter occurs only in integer multiples of $h\beta$, which he called the energy of a quantum, where h is the fundamental constant called Planck's constant

$$h = 6.62607004 \times 10^{-34}[J \cdot s].$$

The quantization of electromagnetic radiation turned out to be an idea with good far-reaching consequences. Einstein recognized that Planck's idea of the quantization of the electromagnetic waves must be valid for the light as well. He posited that light itself is made of discrete bits of energy called **photons**, each of energy $h\beta$ where β is the frequency of the light, [1]. In 1911 Rutherford experimentally discovered the atomic nucleus. Later, Bohr combined Rutherford's atomic model, Planck's quantum concept, and Einstein's photons, and in 1913 introduced his model of the hydrogen atom in which he argued, atoms can be found only in discrete states of energy and that the interaction of atoms with radiation, [30].

Then, in 1923, Compton made an important discovery that gave the most conclusive confirmation for the corpuscular aspect of light. By scattering X-rays with electrons, he confirmed that the X-ray photons behave like particles with momenta $h\beta/c$; where β is the frequency of the X-rays.

This series of breakthroughs gave not only the theoretical foundations but also the conclusive experimental confirmation for the corpuscular aspect of waves: the concept that waves exhibit a particle behavior at the microscopic scale. At this scale, Classical Mechanics fails not only quantitatively but even qualitatively and conceptually, [1].

The scheme introduced by Planck and the postulates and assumptions adopted by Bohr were considered arbitrary despite the fact that their results coincided with the experimental spectroscopy because they were not based on the principles of a theory. This sparked interest in Heisenberg and Schrödinger to search for a theoretical foundation compatible with these new ideas. As it is mentioned in [1], by 1925 their efforts paid off: they skillfully welded the various experimental findings as well as Bohr's postulates into a refined theory: **Quantum Mechanics**. Heisenberg and Schrödinger independently postulate two formulations of Quantum Mechanics.

- **Matrix mechanics** developed by Heisenberg in 1925 which consisted on the notion that the only allowed values of energy exchange between microphysical systems are those that are discrete: **quanta**.
- **Wave mechanics** developed by Schrödinger in 1926 which describes the dynamics of microscopic matter by means of a wave equation, called the **Schrödinger equation.**

The equation obtained by Schrödinger is a differential equation whose solutions yield the energy spectrum and the wave function of a system under consideration. In 1927 Max Born proposed a probabilistic interpretation of wave mechanics. He took the square modulus of the wave functions that are solutions for the Schrödinger equation and he interpreted them as probability densities, [31].

These two different formulations (Schrödinger's wave formulation and Heisenberg's matrix approach) were shown to be equivalent. Later, Dirac suggested a more general formulation of Quantum Mechanics which deals with abstract objects such as kets, see [1] and [31]. The representation of Dirac's formalism in a continuous basis (the position or momentum representations) gives back Schrödinger's wave mechanics. Dirac combined special relativity with Quantum Mechanics and derived in 1928 an equation which describes the motion of electrons. This equation, known as Dirac's equation, predicted the existence of an antiparticle, the positron, which has similar properties, but opposite charge, with the electron; the positron was discovered in 1932, four years after its prediction by Quantum Mechanics, [1].

In brief, Quantum Mechanics study the dynamics of matter at a microscopic scale. It is the founding basis of all modern physics: solid state, molecular, atomic, nuclear and particle physics, optics, statistical mechanics, etc. Somewhat, it is also considered as the foundation of chemistry and biology. The postulates shall be presented in the Section 3.5.

3.1. Wave-Particle Duality: Complementarity

Some experiments about the nature of light show that the light could be described either as a wave or as particle. In the context of diffraction and the interference phenomena the wave properties of light arise. On the other hand, the particles properties are shown in the photoelectric effect.

The French physicist Louis de Broglie, influenced by the corpuscular phenomena and other Einstein's results about the photoelectric effect, proposed the duality wave-particle not only for light but for other particles such as the electron in 1923, [1]. In other words, Einstein discovered the corpuscular property for light and de Broglie the wave behavior for matter. These experiments show that, at microscopic scale, nature can present particle and wave behavior as well, [30].

Then the question is how could something have both behaviors at the same time? And aren't they mutually exclusive? The answer is no in Classical Mechanics and yes in Quantum Mechanics. This dual behavior can in no way be reconciled within the context of Classical Mechanics, for particles and waves are mutually exclusive entities [30].

To measure particle properties produces a loss of wave properties information in a quantum system. In other words, any measurement gives either one property or the other, but never both at once. Therefore, the microscopic systems are not pure waves nor pure particles but both of them. The manifestations of wave and particle are not contradictory but complementary, according to Bohr. In fact, to describe the true nature of microscopic systems the complementarity of them is crucial, [1] and [30]. Being complementary features of microscopic matter, particles and waves are equally important for a complete description of Quantum systems. This is essence of the complementarity principle. Thanks to this principle Quantum Mechanics can produce accurate results. To get a better understanding of this, see e.g [1], [30] and [31].

3.2. Superposition principle

The state of a system is not necessarily represented by a single wave function but for a *superposition* of waves, i.e. by a linear combination of some waves. If Ψ_1 and Ψ_2 separately satisfy the Schrödinger equation, then the wave function

$$\Psi(\vec{r},t) = \rho_1\Psi_1(\vec{r},t) + \rho_2\Psi_2(\vec{r},t)$$

also satisfies the Schrödinger equation, where ρ_1 and ρ_2 are complex numbers.

Remark 3.1 (Bra-ket: Dirac notation for the scalar product). Dirac denoted the scalar (inner) product, on $L^2 = \mathscr{H}$, by the symbol $\langle | \rangle$, which he called a a bra-ket. For instance, the scalar product (ϕ,ψ) is denoted by the bra-ket $\langle\phi|\psi\rangle$:

$$(\phi,\psi) \to \langle\phi|\psi\rangle$$

Observe that $|\psi\rangle \in L^2$ and $\langle\phi| \in (L^2)^*$. To study the properties and further results of Dirac's notation see [1].

As we mentioned before, the Schrödinger equation is a linear equation. So in general, according to the superposition principle, the linear superposition of many wave functions (which describe the various permissible physical states of a system) gives a new wave function which represents a possible physical state of the system:

$$\Psi(x,t) = \sum_i \rho_i\Psi_i(x,t),$$

where ρ_i are complex numbers. The quantity

$$P = \left|\sum_i \rho_i\Psi_i(x,t)\right|^2,$$

represents the probability density for this superposition. If the states $\Psi_i(x,t)$ are mutually *orthonormal*, the probability will be equal to the sum of the individual probabilities:

$$P = \left|\sum_i \rho_i\Psi_i(x,t)\right|^2 = \sum_i |\rho_i|^2 = P_1 + P_2 + P_3 + \cdots$$

where $P_i = |\rho_i|^2$; P_i is the probability of finding the system in the state $\Psi_i(x,t)$.

3.3. Heisenberg Uncertainty Principle

Grossly speaking, according to Classical Mechanics, given the initial conditions and the forces acting on a system, the future behavior of this physical system can be fully determined, so that *Classical Mechanics* is deterministic.

On the other hand, in Quantum Mechanics the concepts of exact position, exact momentum, and unique path of a particle make no sense. This is the essence of Heisenberg's uncertainty principle proposed in 1927. In its original form, presented in [1], Heisenberg's uncertainty principle states that:

If the x-component of the momentum of a particle is measured with an uncertainty Δp_x, then its $x-$position cannot, at the same time, be measured more accurately than $\Delta x = \hbar/2\Delta p_x$. The three-dimensional form of the uncertainty relations for position and momentum can be written as follows:

$$\Delta p_x \Delta x \geq \hbar/2, \qquad \Delta p_y \Delta y \geq \hbar/2, \qquad \Delta p_z \Delta z \geq \hbar/2$$

This principle indicates that, although it is possible to measure the momentum or position of a particle accurately, it is not possible to measure these two observables simultaneously to an arbitrary accuracy. That is, we cannot localize a microscopic particle without giving to it a rather large momentum. We cannot measure the position without disturbing it; there is no way to carry out such a measurement passively as it is bound to change the momentum, [30].

Heisenberg's uncertainty principle can be generalized to any pair of complementary, or canonically conjugate, dynamical variables: it is impossible to devise an experiment that can measure simultaneously two complementary variables to arbitrary accuracy (if this were ever achieved, the theory of Quantum Mechanics would collapse). To study a bit deeper the importance of this principle and some illustrative examples one can see [30], [32], [33] and [31].

3.4. Correspondence principle

Classical Mechanics is contained in Quantum Mechanics in the form of a certain limiting case, [30]. The "extend of quantization" of the system is required to decide if either a quantum or classical approach is in place.

The formal description of the transition from Quantum Mechanics to Classical Mechanics, corresponding to a large phase, is given by the passage to the limit $\hbar \to 0$. It could be understood in the same way as the transition from wave optics to geometrical optics, i.e. the passage to the limit of zero wavelength, $\lambda \to 0$. This principle is very important for our work because we are going to deal with exactly this asymptotic limit case, when $\hbar \to 0$. In other words, we are going to use the method known as Semiclassical Mechanics, to find a deeper description of this principle, see [30].

3.5. Postulates of Quantum Mechanics

In this section, based on [34] and [30], we are going to state the postulates, which are the foundation of Quantum Mechanics, see also [1], [35], [36] and [37]. These postulates cannot be derived; they came from performed experiments and represent the minimal assumptions needed to develop the theory of Quantum Mechanics. In other words, they are the axioms of the quantum theory.

According to classical mechanics, the state of a particle is specified, at any time t, by two fundamental dynamical variables: the position $r(t)$ and the moment $p(t)$. Any other physical quantity, relevant to the system, can be calculated in terms of these two dynamical variables. In addition, knowing these variables at a time t, we can predict, using for instance Hamilton's equations

$$\frac{dx}{dt} = \frac{\partial H}{\partial t} \quad \text{and} \quad \frac{dp}{dt} = -\frac{\partial H}{\partial x}$$

the values of these variables at any later time t'.

Now, we introduce the postulates of Quantum Mechanics.

Postulate 1. (State of a System)

Every possible state of a system corresponds to a separable Hilbert Space, on $\mathbb{C}$, denoted by $\mathscr{H}$. The state of any physical system and its information in time is described with a state vector $\Psi(\cdot, t) \in \mathscr{H}$; it is called time dependent state vector.

If $\{\Psi_i(x,t)\}_{i\leq n} \subseteq \mathscr{H}$, then the Superposition Principle holds, i.e.

$$\Psi(x,t) = \sum_i \rho_i \Psi_i(x,t),$$

where ρ_i are complex numbers. Then, $\Psi \in \mathscr{H}$, as well. Moreover, if $\{\Psi_i(x,t)\}_{i\in I}$ constitute a orthonormal basis, then the Parseval's identity holds, i.e.

$$||\Psi(x,t)||^2 = \sum_i |\rho_i|^2,$$

Postulate 2. (Observables operators)

To every physically measurable quantity A, called an observable or dynamical variable, there corresponds a linear Hermitian operator $\hat{A}$ in a Hilbert space, which has a set of orthonormal eigenvectors $\{\Psi_i\}_{i\in\mathbb{N}}$ with their correspondent eigenvalues $\{\lambda_i\}_{i\in\mathbb{N}}$:

$$\hat{A}(\Psi_i) = \lambda_i \Psi_i, i \in \mathbb{N}$$

Postulate 3. (Correspondence Principle)

A quantum observable operator corresponding to a dynamic variable is obtained by substituting the canonical variable in the classical mechanics by the Quantum Mechanics operator. In general, any observable function $f(x, p)$ that depends of the position and momentum can be changed into an operator by substituting x and p by their operators, i.e.

$$F(\hat{x}, \hat{p}) := f(\hat{x}, i\hbar\nabla)$$

Postulate 4. (Probabilistic outcome of measurements)

If an observable operator $\hat{A}$ has eigenvectors $\{\Psi_i\}_{i\in\mathbb{N}}$ with their eigenvalues $\{\lambda_i\}_{i\in\mathbb{N}}$, then the probability to produce the eigenvalue λ_i of measure from the normalized state $\Psi(\cdot)$ is

$$P(\lambda_i) := \left|(\Psi_i, \Psi)_{L^2(\mathbb{R}^N)}\right|^2.$$

Postulate 5. [Time evolution of a system]

The time evolution of the state vector $\psi(\cdot, t)$ of a system is governed by the time-dependent Schrödinger equation

$$i\hbar\psi_t(x, t) = \hat{H}\psi(x, t), x \in \mathbb{R}^N, t \geq 0 \tag{58}$$

where $\hat{H}$ is the Hamiltonian operator corresponding to the total energy of the system.

Remark: The first four postulates could be joint in the stationary category, i.e. time-independet, and the last one in the nonstationary category, i.e. time-dependent.

4. Results

4.1. Preliminaries

The typical one dimensional Schödinguer equation can be writen as

$$i\hbar\Psi_t(x, t) + \frac{\hbar^2}{2}\Psi_{xx}(x, t) - V_0(x, t)\Psi(x, t) = 0, \quad x \in \mathbb{R}, t \geq 0, \tag{SchE}$$

where $\hbar$ denotes the reduced Plank's constant; i is the imaginary unit; V_0 is a potential and Ψ is the wave function. In this grade project, we deal with the following non-linear version of (SchE),

$$i\hbar\Psi_t(x, t) + \frac{\hbar^2}{2}\Psi_{xx}(x, t) - V_0(x)\Psi(x, t) + \left|\Psi(x, t)\right|^{p-1}\Psi(x, t) = 0, \quad x \in \mathbb{R}, t \geq 0, \tag{NSchE}$$

where $p > 1$. In particular, when the potential V_0 depends only in the spatial variable it is possible to search for traveling wave solutions:

$$\Psi(x, t) = \exp\left(-\frac{iEt}{\hbar}\right)v(x), \quad x \in \mathbb{R}, t \geq 0, \tag{59}$$

where the function v represents the stationary part of Ψ. If we substitute (59) in (NSchE), we will have:

$$\begin{aligned}
i\hbar\Psi_t(x, t) + \frac{\hbar^2}{2}\Psi_{xx}(x, t) - V_0(x)\Psi(x, t) + \left|\Psi(x, t)\right|^{p-1}\Psi(x, t) &= 0 \\
-\frac{iE}{\hbar}i\hbar e^{-\frac{iEt}{\hbar}}v(x) + \frac{\hbar^2}{2}e^{-\frac{iEt}{\hbar}}v''(x) - V_0(x)e^{-\frac{iEt}{\hbar}}v(x) + \left|e^{-\frac{iEt}{\hbar}}v(x)\right|^{p-1}e^{-\frac{iEt}{\hbar}}v(x) &= 0 \\
e^{-\frac{iEt}{\hbar}}\left\{Ev(x) + \frac{\hbar^2}{2}v''(x) - V_0(x)v(x)) + \left|e^{-\frac{iEt}{\hbar}}v(x)\right|^{p-1}v(x)\right\} &= 0 \\
e^{-\frac{iEt}{\hbar}}\left\{\frac{\hbar^2}{2}v''(x) - (V_0(x) - E)v(x) + \left|v(x)\right|^{p-1}v(x)\right\} &= 0,
\end{aligned}$$

which implies that for all $x \in \mathbb{R}$, v should verify

$$\varepsilon^2 v''(x) - V(x)v(x) + |v(x)|^{p-1} v(x) = 0, \tag{P}$$

where

$$\varepsilon^2 := \frac{\hbar^2}{2}$$

and

$$V(x) := V_0(x) - E.$$

In our study, we shall use the asymptotic method known as Semiclassical Mechanics, which is based in the the Correspondence Principle, see Postulate 3 in the previous section. Actually, we are interested in the multiplicity and behavior of solutions of (P), as $\varepsilon \to 0$, for the problem

$$\begin{cases} \varepsilon^2 v''(x) - V_\varepsilon(x)v(x) + |v(x)|^{p-1} v(x) = 0, & x \in \mathbb{R}, \\ \lim_{|x| \to \infty} v(x) = 0. \end{cases} \tag{P$_\varepsilon$}$$

We consider the infinite case given by Byeon and Wang, in [10]: $\Omega = \{0\}$; i.e.$V(0) = 0$, and V decreases exponentially around it. This case, as was mentioned in the introduction, has associated a limit problem when ε goes to zero. Then, we are going to study the behavior of localized solutions of (P) around 0. Here, the potential verifies:

$(V1)$ V is a non-negative continuous function over $\mathbb{R}$;

$(V2)$ $V(x) \to \infty$ as $|x| \to \infty$;

$(V3)$ For each $x \in [-1, 1]\backslash\{0\}$:

$$V(x) = \exp\left(-\frac{1}{a(x)}\right), \tag{60}$$

where a is a (Ω, b) quasi-homogeneous function.

Let's recall, from Section 2.7, that a continuous function $b : \mathbb{R} \to [0, \infty)$ is called an Ω *quasi-homogeneous function* if

(i) $b(\cdot)$ is strictly increasing on $[0, \infty)$; and

(ii) the following condition holds

$$\lim_{r \to 0} \frac{b(cr)}{b(r)} \begin{cases} < 1 & \text{if } c < 1 \\ > 1 & \text{if } c > 1 \end{cases} \tag{61}$$

A continuous function $a : \mathbb{R} \to (0, \infty)$ is called an *asymptotically-*(Ω, b)*-quasi-homogeneous* function if there is a *quasi-homogeneous function* b satisfying

$$\lim_{|x| \to 0} \frac{a(x)}{b(x)} = 1. \tag{62}$$

Let's define

$$g(\varepsilon) = \frac{1}{b^{-1}\left(\frac{-1}{\ln(\varepsilon^2)}\right)}, \tag{63}$$

Remark 4.1. In [10] is proved that for a given $\rho > 0$ there is a δ_ρ such that

$$\lim_{\varepsilon \to 0} (\varepsilon g(\varepsilon))^2 \max_{|x| \leq \delta < \delta_\rho} V\left(\frac{x}{g(\varepsilon)}\right) = 0 \tag{64}$$

In other words, for a fixed $\rho > 0$ there is a $\delta_\rho > 0$, small enough, such that for every $0 < \delta < \delta_\rho$ we have

$$|\varepsilon| < \delta_\rho \implies (\varepsilon g(\varepsilon))^2 \max_{|x| \leq \delta} V\left(\frac{x}{g(\varepsilon)}\right) < \rho \tag{65}$$

Remark 4.2. In what remains of this section, ρ is positive and δ_ρ as in Remark 4.1.

Remark 4.3. According [10], our limit problem as $\varepsilon \to 0$, for the infinite case is given by

$$\begin{cases} w''(x) + |w(x)|^{p-1} w(x) = 0 & x \in I_{\delta_\rho}, \\ w(-\delta_\rho) = w(\delta_\rho) = 0. \end{cases} \tag{P$_L$}$$

where $I_{\delta_\rho} := (-\delta_\rho, \delta_\rho)$.

Before presenting the results, let's establish and define some spaces and functionals that provide the environment for work.
Let's first define the functional space where we are going to find the solutions of (P_ε).

$$H_\varepsilon = \{w \in H_0^1(\mathbb{R}) : ||w||_\varepsilon < \infty\}$$

where

$$||w||_\varepsilon = \sqrt{\int_{\mathbb{R}} [|w'(x)|^2 + V_\varepsilon(x)||w(x)|^2]dx}$$

and

$$V_\varepsilon(x) = (\varepsilon g(\varepsilon))^{-2} V\left(\frac{x}{g(\varepsilon)}\right), \tag{66}$$

Proposition 4.1. *The form* $B_\varepsilon(\cdot,\cdot)$ *given by*

$$B_\varepsilon(w,y) = \int_{\mathbb{R}} \Big[w'(x)y'(x) + V_\varepsilon w(x)y(x)\Big]\,dx$$

defines a scalar product on H_ε*, which induces the norm* $||\cdot||_\varepsilon$*.*

Proof. (a) Let's prove that B_ε is a bilinear form, i.e.

$$\forall \rho \in \mathbb{R}, \forall w, y, z \in H_\varepsilon : B_\varepsilon(\rho w + z, y) = \rho B_\varepsilon(w,y) + B_\varepsilon(z,y). \tag{67}$$

Let $\rho \in \mathbb{R}$ and $w, y, z \in H_\varepsilon$, generic. Then,

$$\begin{aligned} B_\varepsilon(\rho w + z, y) &= \int_{\mathbb{R}} \Big[(\rho w'(x) + z'(x))y'(x) + V_\varepsilon \rho w(x) + z(x))y(x)\Big]\,dx \\ &= \int_{\mathbb{R}} \Big[\rho w'(x)y'(x) + V_\varepsilon \rho w(x)y(x)\Big]\,dx + \int_{\mathbb{R}} \Big[z'(x)y'(x) + V_\varepsilon z(x)y(x)\Big]\,dx \\ &= \rho \int_{\mathbb{R}} \Big[w'(x)y'(x) + \rho V_\varepsilon w(x)y(x)\Big]\,dx + \int_{\mathbb{R}} \Big[z'(x)y'(x) + V_\varepsilon z(x)y(x)\Big]\,dx \\ &= \rho B_\varepsilon(w,y) + B_\varepsilon(z,y) \end{aligned}$$

Since ρ, w, y, z were chosen arbitrarily, then we have proved (67).

(b) Let's prove that B_ε is symmetric, i.e.

$$\forall w, y \in H_\varepsilon : B_\varepsilon(w,y) = B_\varepsilon(y,w). \tag{68}$$

Let $w, y \in H_\varepsilon$, chosen arbitrarily. Then,

$$\begin{aligned} B_\varepsilon(w,y) &= \int_{\mathbb{R}} \Big[w'(x)y'(x) + V_\varepsilon w(x)y(x)\Big]\,dx \\ &= \int_{\mathbb{R}} \Big[y'(x)w'(x) + V_\varepsilon y(x)w(x)\Big]\,dx \\ &= B_\varepsilon(y,w). \end{aligned}$$

Since w, y were generic, then we have proved (68).

(c) Now, let's prove that

$$\forall w \in H_\varepsilon : B_\varepsilon(w,w) \geq 0. \tag{69}$$

Let $w \in H_\varepsilon$, generic. Then, by (66) we know that $V_\varepsilon(x) > 0$ for any $x \in \mathbb{R}$. On the other hand, we have by definition of the bilinear form B_ε we will have that

$$\begin{aligned} B_\varepsilon(w,w) &= \int_{\mathbb{R}} \Big[(w'(x))^2 + V_\varepsilon(x)(w(x))^2\Big]\,dx \\ &\geq 0 \end{aligned}$$

Since w was generic, then we have proved (69).
It's immediate that

$$B_\varepsilon(w,w) = 0 \Longleftrightarrow w = 0 \quad a.e.$$

□

Roughly speaking, we shall show that both (P_ε) and (P_L) have an infinite number of solutions which are critical points of functionals working on appopiate Nehari's manifolds.

The following set is a closed C^1 Nehari's manifold, which shall be studied afterwards:

$$\mathcal{M}_\varepsilon := \{w \in H_\varepsilon : ||w||_{L^{p+1}(\mathbb{R})} = 1\} := L_\varepsilon^{-1}(0), \tag{70}$$

where,

$$\begin{aligned} L_\varepsilon \colon H_\varepsilon &\to \mathbb{R} \\ w &\mapsto L_\varepsilon(w) = \frac{||w||^{p+1}_{L^{p+1}(\mathbb{R})} - 1}{p+1}. \end{aligned}$$

The directional derivative of L_ε is given by

$$\left\langle DL_\varepsilon(w), y\right\rangle = \int_{\mathbb{R}} w(x)|w(x)|^{p-1}y(x)dx, \qquad w, y \in H_\varepsilon$$

Associated with (P_ε), is the functional

$$\begin{aligned} J_\varepsilon \colon \mathcal{M}_\varepsilon &\to \mathbb{R} \\ w &\mapsto J_\varepsilon(w) = \frac{1}{2}||w||_\varepsilon^2 = \frac{1}{2}\int_{\mathbb{R}} \left[(w'(x))^2 + V_\varepsilon(x)(w(x))^2\right] dx \end{aligned}$$

As we shall see, the critical points of J_ε are solutions of (P_ε).

Let's define the space where we are going to find weak solutions of (P_L).

Remark 4.4. For a fixed $\rho > 0$, we choose a $\delta_\rho > 0$, as in Remark 4.1 such that

$$\mathrm{H}_0^1(I_{\delta_\rho}) = \{w \in \mathrm{H}_0^1(I_{\delta_\rho}) : \|w\|_{H_0^1(I_{\delta_\rho})} < \infty\}, \tag{71}$$

where

$$\|w\|_{H_0^1(I_{\delta_\rho})} = \left(\int_{I_{\delta_\rho}} |w'(x)|^2 dx\right)^{1/2}$$

and $I_{\delta_\rho} := (-\delta_\rho, \delta_\rho)$.

Proposition 4.2. *The form $B_{e,\delta_\rho}(\cdot,\cdot)$ given by*

$$B_{e,\delta_\rho}(w, y) = \int_{I_{\delta_\rho}} w'(x)y'(x)dx,$$

defines a scalar product on $\mathrm{H}_0^1(I_{\delta_\rho})$, which induces the norm $\|w\|_{H_0^1(I_{\delta_\rho})}$.

The proof is analogous to Proposition 4.1.

The following set is also a Nehari's manifold

$$\mathcal{M}^{\delta_\rho} := \{w \in H_e(I_{\delta_\rho}) : ||w||_{L^{p+1}(I_{\delta_\rho})} = 1\} := (L^{\delta_\rho})^{-1}(0), \tag{72}$$

where,

$$\begin{aligned} L^{\delta_\rho} \colon \mathrm{H}_0^1(I_{\delta_\rho}) &\to \mathbb{R} \\ w &\mapsto L^{\delta_\rho}(w) = \frac{||w||^{p+1}_{L^{p+1}(I_{\delta_\rho})} - 1}{p+1}. \end{aligned}$$

The directional of L_ρ^δ is given by

$$\left\langle DL^{\delta_\rho}(w), y\right\rangle = \int_{I_{\delta_\rho}} w(x)|w(x)|^{p-1}y(x)dx$$

Associated with (P_L), is the functional

$$\begin{aligned} J^{\delta_\rho} \colon \mathcal{M}^{\delta_\rho} &\to \mathbb{R} \\ w &\mapsto J^{\delta_\rho}(w) = \frac{1}{2}\|w\|_{H_0^1(I_{\delta_\rho})} = \frac{1}{2}\int_{I_{\delta_\rho}} (w'(x))^2 dx \end{aligned}$$

4.2. Main result

Theorem 4.1. *Let $p > 1$. Assume that conditions (V1)-(V3) hold. Then, the following statements are true.*

i) Given $\varepsilon > 0$, the functional J_ε has infinitely many critical points $\{\hat{w}_{k,\varepsilon}\}_{k\in\mathbb{N}} \subseteq \mathcal{M}_\varepsilon$.

ii) The limit functional J^{δ_ρ} has infinitely many critical points $\{\hat{w}_k^{\delta_\rho}\}_{k\in\mathbb{N}} \subseteq \mathcal{M}^{\delta_\rho}$.

iii) Given $k \in \mathbb{N}$, there exists a $C_\rho > 0$ such that the critical values satisfy

$$\lim_{\varepsilon\to 0} J_\varepsilon(\hat{w}_{k,\varepsilon}) = (1 + C_\rho)\, J^{\delta_\rho}(\hat{w}_k^{\delta_\rho}).$$

iv) For each fixed $k \in \mathbb{N}$, there exists a subsequence of $w_{k,\varepsilon}$ that sub-converges to, w_k^δ, a solution of P_L.

Remark 4.5. In the context of the Theorem 4.1, the function

$$w_{k,\varepsilon} = (2c_{k,\varepsilon})^{\frac{1}{p-1}}\, \hat{w}_{k,\varepsilon}, \text{ where } c_{k,\varepsilon} = J_\varepsilon(\hat{w}_{k,\varepsilon}),$$

satisfies the following equation

$$\begin{cases} w''(x) - (\varepsilon g(\varepsilon))^{-2} V\left(\dfrac{x}{g(\varepsilon)}\right) + |w(x)|^{p-1}\, w(x) = 0 \quad x \in \mathbb{R} \\ \lim\limits_{|x|\to\infty} w(x) = 0. \end{cases} \tag{P'_ε}$$

4.3. Multiplicity by a Ljusternik–Schnirelman scheme

4.3.1. Kranoselskii's genus

Let E be a Hilbert space, we define the set

$$\Sigma_E = \{A \in E : A = \bar{A}, A = -A, 0 \notin A\} \tag{73}$$

The Krasnoselski's genus of $A \in \Sigma_E$, denoted by $\gamma(A)$ is the least natural number k such that there exist an odd function

$$f \in \mathrm{C}(A, \mathbb{R}^k \backslash \{0\}) \tag{74}$$

If there is not such k that satisfies (74), then we write

$$\gamma(A) = \infty.$$

Also, by convention

$$\gamma(\emptyset) = 0.$$

We denote for $c \in \mathbb{R}$

$$K_c = \{u \in A / f'_{\mathscr{M}}(u) = 0 \wedge f_{\mathscr{M}}(u) = c\}.$$

Theorem 4.2. *Let $\mathscr{H}$ be a Hilbert space, $\mathcal{M} \in \Sigma_H$ a C^1 manifold of $\mathscr{H}$ and $J \in \mathrm{C}^1$ an even functional. Suppose that J satisfies $(PS)_\mathcal{M}$ and that $J|_\mathcal{M}$ is bounded from below. Therefore,*

$$\gamma(\mathcal{M}) \le \sum_{c\in\mathbb{R}} \gamma(K_c). \tag{75}$$

In other words, J has at least $\gamma(\mathcal{M})$ pairs of critical points on $\mathcal{M}$. Moreover, the critical values of f from $1 \le k \le dim(H)$, are given by

$$C_k(f) = \inf_{A\in\mathcal{A}_k(\mathcal{M})} \max_{u\in A} f(u),$$

where

$$\mathcal{A}_k(\mathcal{M}) = \{A \in \Sigma_H \cap \mathcal{M} : \gamma(A) \ge k\}.$$

4.3.2. Some compact injections

Let's state a couple of results that will be useful.

Lemma 4.1. *Suppose $I = (a, b)$. Then we have the following compact injection.*

$$\mathrm{H}_0^1(I) \subseteq C(\bar{I}) \tag{76}$$

This proof is a particular case of [13, Theorem 8.8(6)], where $p = 2$.

The following result is stated in [38] and [4].

Lemma 4.2. *For $1 \leq q \leq \infty$, the inclusions*

$$\mathrm{H}_0^1(\mathbb{R}) \subseteq L^q(\mathbb{R})$$

and

$$H_\varepsilon \subseteq L^q(\mathbb{R})$$

are compact.

4.3.3. Nehari's Manifold

In this section we shall prove the existence of infinitely many solutions for problems (P_ε) and (P_L) applying Theorem 4.2. We denote in the context of (73):

$$\Sigma^{\delta_\rho} := \Sigma_{\mathrm{H}_0^1(I_{\delta_\rho})}$$

and

$$\Sigma_\varepsilon := \Sigma_{H_\varepsilon}$$

We shall prove that $\mathscr{M}^{\delta_\rho}$ and $\mathscr{M}_\varepsilon$ satisfy the properties stated in Theorem 4.2.

Lemma 4.3. *We have that*

$$\forall \delta > 0 : \mathscr{M}^\delta \in \Sigma^\delta. \tag{77}$$

Proof. (i) Let's prove that $\mathscr{M}^\delta = -\mathscr{M}^\delta$. By definition of $\mathscr{M}^\delta$ we have:

$$\begin{aligned} \mathscr{M}^\delta &= \left\{u \in \mathrm{H}_0^1(I_{\delta_\rho}) : ||u||_{L^{p+1}(-\delta_\rho,\delta_\rho)} = 1\right\} \\ &= \left\{-u \in \mathrm{H}_0^1(I_{\delta_\rho}) : ||-u||_{L^{p+1}(-\delta_\rho,\delta_\rho)} = 1\right\} \\ &= -\left\{u \in \mathrm{H}_0^1(I_{\delta_\rho}) : ||u||_{L^{p+1}(-\delta_\rho,\delta_\rho)} = 1\right\} \\ &= -\mathscr{M}^\delta \end{aligned} \tag{78}$$

Thus, $\mathscr{M}^\delta$ is symmetric.

(ii) Reasoning by Reduction to Absurdity, we immediately have that $0 \notin \mathscr{M}$.

(iii) We have to prove that $\mathscr{M}^\delta$ is closed, i.e. $\overline{\mathscr{M}^\delta} = \mathscr{M}^\delta$

$$\forall u \in \overline{\mathscr{M}^\delta} : ||u||_{L^{p+1}(-\delta_\rho,\delta_\rho)} = 1.$$

Let $u \in \overline{\mathscr{M}^\delta}$, generic. Then there exists a sequence $\{u_k\}_{k\in\mathbb{N}} \subseteq \mathscr{M}^\delta$, such that

$$\lim_{k\to\infty} ||u - u_k|| = 0.$$

Since

$$||u_k||_{L^{p+1}(-\delta_\rho,\delta_\rho)} = \left(\int_{-\delta_\rho}^{\delta_\rho} |u_k|^{p+1}\right)^{\frac{1}{p+1}} = 1$$

It follows from the continuity of the norm $||\cdot||_{L^{p+1}(-\delta_\rho,\delta_\rho)}$, that

$$||u||_{L^{p+1}(-\delta_\rho,\delta_\rho)} = 1$$

By the arbitrariness of u we have proved (iii). $\square$

Lemma 4.4. *We have that*

$$\forall \varepsilon > 0 : \mathscr{M}_\varepsilon \in \Sigma_\varepsilon$$

The proof follows the same structure as Lemma 4.3.

Now, we have to prove that $\mathscr{M}^\delta$ is a C^1 manifold over $\mathrm{H}_0^1(-\delta,\delta)$. To start with, let's define the functional

$$\begin{aligned}\tilde{L}^\delta\colon \mathrm{H}_0^1(-\delta,\delta) &\to \mathbb{R}\\ w &\mapsto \tilde{L}^\delta(w) = ||w||_{L^{p+1}(-\delta,\delta)}.\end{aligned}$$

Lemma 4.5. *The functional $\tilde{L}^\delta$ is Lipschitz continuous, i.e*

$$\exists c>0, \forall u,v \in \mathrm{H}_0^1(-\delta,\delta) : \left|\tilde{L}^\delta(v) - \tilde{L}^\delta(u)\right| \leq \|v-u\|_{H_0^1(-\delta,\delta)}\,. \tag{79}$$

Proof. Let $u,v \in \mathrm{H}_0^1(-\delta,\delta)$, generic. Then by the definition of $\tilde{L}^\delta$, by continuity of the norm $||\cdot||_{L^{p+1}(-\delta,\delta)}$, by Lemma 4.1 and by Poincaré's inequality we have

$$\begin{aligned}\left|\tilde{L}^\delta(v) - \tilde{L}^\delta(u)\right| &= \left|||v||_{L^{p+1}(-\delta,\delta)} - ||u||_{L^{p+1}(-\delta,\delta)}\right|\\ &\leq ||v-u||_{L^{p+1}(-\delta,\delta)}\\ &\leq C||v-u||_{C([-\delta,\delta])}\\ &\leq C\,\|v-u\|_{H_0^1(-\delta,\delta)}\,.\end{aligned}$$

Thus, By the arbitrariness of u and v we have proved (79). □

Remark 4.6. From Lemma 4.5 it immediately follows that the functional

$$\begin{aligned}L^\delta\colon \mathrm{H}_0^1(-\delta,\delta) &\to \mathbb{R}\\ w &\mapsto L^\delta(w) = \frac{||w||^{p+1}_{L^{p+1}(-\delta,\delta)} - 1}{p+1}.\end{aligned}$$

is continuous.

Lemma 4.6. *Let $\delta > 0$. Then, the mapping $DL^\delta\colon \mathrm{H}_0^1(-\delta,\delta) \to (\mathrm{H}_0^1(-\delta,\delta))^*$ defined by*

$$\left\langle DL^\delta(u), v\right\rangle :\, = \int_{-\delta}^{\delta} [u(x)|u(x)|^{p-1}v(x)]dx, \qquad \forall u,v \in \mathrm{H}_0^1(-\delta,\delta), \tag{80}$$

is continuous.

Proof. (a) Let's find an estimate from above for $\left\|DL^\delta(u) - DL^\delta(w)\right\|$ for each $u,w \in \mathrm{H}_0^1(-\delta,\delta)$:

$$\forall u,w \in \mathrm{H}_0^1(-\delta,\delta), \exists c = c(u,w) > 0 : \left\|DL^\delta(u) - DL^\delta(w)\right\|_{(\mathrm{H}_0^1(-\delta,\delta))^*} \leq C$$

Let u, v and $w \in \mathrm{H}_0^1(-\delta,\delta)$, generic. By Lemma 4.1 we have

$$\begin{aligned}\left|\left\langle DL^\delta(u) - DL^\delta(w), v\right\rangle\right| &= \left|\left\langle DL^\delta(u), v\right\rangle - \left\langle DL^\delta(w), v\right\rangle\right|\\ &= \left|\int_{-\delta}^{\delta} u(x)|u(x)|^{p-1}v(x)dx - \int_{-\delta}^{\delta} w(x)|w(x)|^{p-1}v(x)dx\right|\\ &\leq \int_{-\delta}^{\delta}\left|u(x)|u(x)|^{p-1}v(x) - w(x)|w(x)|^{p-1}v(x)\right|dx\\ &\leq c\left[\int_{-\delta}^{\delta}\left|u(x)|u(x)|^{p-1} - w(x)|w(x)|^{p-1}\right|dx\right]\|v\|_{C([-\delta,\delta])}\\ &\leq k\left[\int_{-\delta}^{\delta}\left|u(x)|u(x)|^{p-1} - w(x)|w(x)|^{p-1}\right|dx\right]\|v\|_{H_0^1(-\delta,\delta)}\,.\end{aligned} \tag{81}$$

By the arbitrariness of u, v and w, we are done by choosing

$$C = k\int_{-\delta}^{\delta}\left|u(x)|u(x)|^{p-1} - w(x)|w(x)|^{p-1}\right|dx.$$

(b) We also have to prove that for any $u \in \mathrm{H}_0^1(-\delta,\delta)$ and every sequence $\{u_k\}_{k\in\mathbb{N}} \subseteq \mathrm{H}_0^1(-\delta,\delta)$, such that

$$\lim_{k\to\infty} \|u_k - u\|_{H_0^1(-\delta,\delta)} = 0 \tag{82}$$

we have that

$$\lim_{k\to\infty} \left\| DL^\delta(u) - DL^\delta(u_k) \right\|_{(\mathrm{H}_0^1(-\delta,\delta))^*} = 0 \tag{83}$$

Let $u \in \mathrm{H}_0^1(-\delta,\delta)$, generic, and a sequence $\{u_k\}_{k\in\mathbb{N}} \subseteq \mathrm{H}_0^1(-\delta,\delta)$, such that (82) holds. Then, by the triangular inequality, (81), Lemma 4.1, we have that

$$\begin{aligned}
\left\| DL^\delta(u) - DL^\delta(u_k) \right\|_{(\mathrm{H}_0^1(-\delta,\delta))^*} &= k\left[\int_{-\delta}^{\delta} \left| u(x)|u(x)|^{p-1} - u_k(x)|u_k(x)|^{p-1}\right| dx\right] \\
&\leq C\left[\int_{-\delta}^{\delta} \left| u(x)|u(x)|^{p-1} - u(x)|u_k(x)|^{p-1}\right| dx\right] \\
&+ C\left[\int_{-\delta}^{\delta} \left| u(x)|u_k(x)|^{p-1} - u_k(x)|u_k(x)|^{p-1}\right| dx\right] \\
&= C\left[\int_{-\delta}^{\delta} \left| |u(x)|^{p-1} - |u_k(x)|^{p-1}\right| |u(x)| dx\right] \\
&+ C\left[\int_{-\delta}^{\delta} \left| u(x) - u_k(x)\right| |u_k(x)|^{p-1} dx\right] \\
&\leq C||u||_{C([-\delta,\delta])}\left[\int_{-\delta}^{\delta} \left| |u(x)|^{p-1} - |u_k(x)|^{p-1}\right| dx\right] \\
&+ C||u_k||_{C([-\delta,\delta])}^{p-1}\left[\int_{-\delta}^{\delta} \left| u(x) - u_k(x)\right| dx\right] \\
&\leq C||u||_{C([-\delta,\delta])}\left| \|u\|_{C([-\delta,\delta])}^{p-1} - \|u_k\|_{C([-\delta,\delta])}^{p-1}\right| \\
&+ C\|u_k\|_{C([-\delta,\delta])}^{p-1} \|u-u_k\|_{C([-\delta,\delta])}
\end{aligned} \tag{84}$$

By (82), (84) and by continuity of the norm $\|\cdot\|_{C([-\delta,\delta])}$ we have (83). Since $u \in \mathrm{H}_0^1(-\delta,\delta)$ and $\{u_k\}_{k\in\mathbb{N}}$ were generic, then we have proved (80).

□

Lemma 4.7. *The mapping $DL_\varepsilon \colon H_\varepsilon \to (H_\varepsilon)^*$ defined by*

$$\langle DL_\varepsilon(u), v\rangle : = \int_{\mathbb{R}} [u(x)|u(x)|^{p-1} v(x)] dx, \qquad \forall u, v \in H_\varepsilon, \tag{85}$$

is continuous.

The proof is analogous to that of Lemma 4.6.

4.3.4. Boundedness from bellow of the functionals

Proposition 4.3. *For all $\delta > 0$, there exists a $d > 0$ such that for $u \in$*

$$\forall u \in \mathscr{M}^\delta : 2J^\delta(u) = \|u\|_{H_0^1(-\delta,\delta)} \geq d \tag{86}$$

Proof. Let $\delta > 0$, generic. Then by Lemma 4.1 for all $p \geq 0$ there exists $C > 0$ such that for $u \in \mathrm{H}_0^1(-\delta,\delta)$ we obtain

$$\begin{aligned}
||u||_{L^{p+1}(-\delta,\delta)} &\leq c||u||_{C([-\delta,\delta])} \\
&\leq C\,\|u\|_{H_0^1(-\delta,\delta)}
\end{aligned}$$

Thus,

$$\begin{aligned}
C^{-1} &= C^{-1}||u||_{L^{p+1}(-\delta,\delta)} \\
&\leq \|u\|_{H_0^1(-\delta,\delta)} \\
&= \left(2J^\delta(u)\right)^{1/2} \qquad \forall u \in \mathscr{M}^\delta
\end{aligned}$$

Then, it is enough to take $d = C^{-2}$ and we are done. □

Proposition 4.4. *For all $\varepsilon > 0$, there exists a $l > 0$ such that*

$$\forall u \in \mathscr{M}_\varepsilon : 2J_\varepsilon(u) = ||u||_\varepsilon \geq l \tag{87}$$

This proof is analogous to that of Proposition 4.3.

4.3.5. Palais–Smale Condition

Theorem 4.3. *Let $\delta > 0$. Then, J^δ satisfies (PS) on $\mathscr{M}^\delta$.*

Proof. Let $\delta > 0$ and $\{u_k\}_{k\in\mathbb{N}} \subseteq \mathscr{M}$, such that

(i) $\left\{J^\delta(u_k)\right\}_{k\in\mathbb{N}}$ is bounded, and

(ii) $\lim_{k\to\infty} D_{\mathscr{M}^\delta} J^\delta(u_k) = 0$ in $\mathrm{H}_0^1(-\delta,\delta)$.

We have to prove that $\{u_k\}_{k\in\mathbb{N}}$ has a convergent subsequence in $\mathrm{H}_0^1(-\delta,\delta)$.

(a) We are going to prove that $\{u_k\}_{k\in\mathbb{N}}$ satisfies

$$\lim_{k\to\infty} \left\langle D_{\mathscr{M}^\delta} J^\delta(u_k), u_k \right\rangle = 0. \tag{88}$$

By (ii) and Proposition 2.1, we have the weak convergence of $\left(D_{\mathscr{M}^\delta} J^\delta(u_k)\right)_{k\in\mathbb{N}}$ in $\mathrm{H}_0^1(-\delta,\delta)$.

(b) Let's write DJ^δ in the following form

$$DJ^\delta(u_k) = D_{\mathscr{M}^\delta} J^\delta(u_k) + t_k DL^\delta(u_k), \qquad t_k \in \mathbb{R} \tag{89}$$

where the t_k are the Lagrange's Multipliers. Let's prove that the sequence $\{t_k\}_{k\in\mathbb{N}}$ is bounded. Let's use (88) and do the dual product of (89) and $\{u_k\}_{k\in\mathbb{N}}$, then

$$\begin{aligned} \left\langle D_{\mathscr{M}^\delta} J^\delta(u_k), u_k \right\rangle &= \left\langle DJ^\delta(u_k) - t_k DL^\delta(u_k), u_k \right\rangle \\ &= \|u_k\|^2_{H_0^1(-\delta,\delta)} - t_k \left\langle DL^\delta(u_k), u_k \right\rangle \\ &= \|u_k\|^2_{H_0^1(-\delta,\delta)} - t_k \end{aligned} \tag{90}$$

Using (a) in (90) we will have $\lim_{k\to\infty} \|u_k\|^2_{H_0^1(-\delta,\delta)} - t_k = 0$. By the last expression and the boundedness of $\{u_k\}_{k\in\mathbb{N}}$ we conclude that $\{t_k\}_{k\in\mathbb{N}}$.

(c) Let's prove that $\{u_k\}_{k\in\mathbb{N}}$ has a convergent subsequence.
Since $\{u_k\}_{k\in\mathbb{N}}$ is bounded, by Lemma 4.1, Poincaré's and Sobolev inequalities we have that $\{u_k\}_{k\in\mathbb{N}}$ has a convergent subsequence $\left\{u_{k_j}\right\}_{j\in\mathbb{N}}$ [1],i.e.

$$\begin{aligned} u_{k_j} &\rightharpoonup u \qquad \text{in } \mathrm{H}_0^1(-\delta,\delta) \\ u_{k_j} &\to u \qquad \text{in } C([-\delta,\delta]) \end{aligned}$$

On the other hand,

$$\begin{aligned} \left\langle DL^\delta(u_k), u_k - u \right\rangle &= \left| \int_\delta^\delta u_k |u_k|^{p-1}(u_k - u)dx \right| \\ &\leq \int_\delta^\delta \left| u_k |u_k|^{p-1}(u_k - u) \right| dx \\ &\leq 2\delta ||u_k||^p_{C([-\delta,\delta])} ||u_k - u||_{C([-\delta,\delta])} \end{aligned} \tag{91}$$

Thus, by (91) and the strong convergence in $C([-\delta,\delta])$ we have that

$$\lim_{k\to\infty} \left\langle DL^\delta(u_k), u_k - u \right\rangle = 0 \tag{92}$$

[1] To simplify the notation let's rename $\{u_k\}_{k\in\mathbb{N}}$, again.

Thus by (ii), (92) and (89), we have proved that

$$\lim_{k\to\infty} \left\langle DJ^{\delta}(u_k), u_k - u \right\rangle = 0 \tag{93}$$

Then by weak convergence in $\mathrm{H}^1_0(-\delta,\delta)$

$$\begin{aligned} 0 &= \lim_{k\to\infty} \left\langle DL^{\delta}(u_k), u_k - u \right\rangle \\ &= \lim_{k\to\infty} (u_k, u_k - u)_{\mathrm{H}^1_0(-\delta,\delta)} \\ &= \lim_{k\to\infty} ||u_k||^2_{\mathrm{H}^1_0(-\delta,\delta)} - \lim_{k\to\infty} (u_k - u)_{\mathrm{H}^1_0(-\delta,\delta)} \\ &= \lim_{k\to\infty} ||u_k||^2_{\mathrm{H}^1_0(-\delta,\delta)} - ||u||^2_{\mathrm{H}^1_0(-\delta,\delta)} \end{aligned} \tag{94}$$

Therefore, by (94), the arbitrariness of δ and $\{u_k\}_{k\in\mathbb{N}}$ and by Proposition 2.3 we conclude.

□

Theorem 4.4. *For all $\varepsilon > 0$, J_ε satisfies (PS) on $\mathscr{M}_\varepsilon$.*

This proof is analogous to Proposition 4.3.

Corollary 4.1. *The limit functional J^{δ} possesses infinitely many critical points $\{\hat{w}^{\delta}_k\}_{k\in\mathbb{N}} \subseteq \mathcal{M}^{\delta}$.*

Proof. By Lemma 4.3 we have that $\mathscr{M}^{\delta} \subseteq \Sigma^{\delta}$ is a manifold of class C^1 of $\mathrm{H}^1_0(I_{\delta_\rho})$ and $J^{\delta}(u) = J^{\delta}(-u)$.

The Proposition 4.3 and Theorem 4.3 ensure that J^{δ} are bounded form bellow in $\mathscr{M}^{\delta}$ and J^{δ} satisfies the Palais–Smale condition on $\mathscr{M}^{\delta}$.

Let

$$\mathbb{S}_{\mathrm{H}^1_0(I_{\delta_\rho})} : \{u \in \mathrm{H}^1_0(I_{\delta_\rho}) : ||u||_{\mathrm{H}^1_0(I_{\delta_\rho})} = 1\}$$

the unitary sphere and let the function

$$\begin{aligned} g\colon \mathbb{S}_{\mathrm{H}^1_0(I_{\delta_\rho})} &\to \mathscr{M}^{\delta} \\ u &\mapsto g(u) = \frac{u}{||u||_{L^{p+1}(I_{\delta_\rho})}} \end{aligned}$$

It's clear that g is continuous and an odd function. Therefore, using the properties of the genus we get that

$$\infty = \gamma\left(\mathbb{S}_{\mathrm{H}^1_0(I_{\delta_\rho})}\right) \le \gamma\left(\mathscr{M}^{\delta}\right)$$

We have by the Theorem 4.2 that J^{δ} has a sequence of critical points

$$\{\hat{w}_{k,\varepsilon}\}_{k\in\mathbb{N}} \subseteq \mathcal{M}_\varepsilon$$

□

Remark 4.7. For any $\delta > 0$ and $n \in \mathbb{N}$,we take

$$\mathscr{A}^{\delta}_k := \mathscr{A}_k(\mathscr{M}^{\delta}) \text{ and } c^{\delta}_k := C_k(J^{\delta}) = J^{\delta}(\hat{w}^{\delta}_k) > 0$$

associated to (P_L). Where, c^{δ}_k represents the critical values for the functional J^{δ}.

Corollary 4.2. *Given $\varepsilon > 0$, the functional J_ε possesses infinitely many critical points $\{\hat{w}_{k,\varepsilon}\}_{k\in\mathbb{N}} \subseteq \mathcal{M}_\varepsilon$.*

The proof is analogous to Corollary 4.1.

Remark 4.8. For any $\varepsilon > 0$ and $n \in \mathbb{N}$,we take

$$\mathscr{A}_{k,\varepsilon} := \mathscr{A}_k(\mathscr{M}_\varepsilon) \text{ and } c_{k,\varepsilon} := C_k(J_\varepsilon) = J_\varepsilon(\hat{w}_{k,\varepsilon}) > 0$$

associated to (P_ε). Where, $c_{k,\varepsilon}$ represents the critical values for the functional J_ε.

Remark 4.9. In our study is convenient to have a intermediate problem which is

$$\begin{cases} u''(x) + |u(x)|^{p-1} u(x) = 0 & x \in (-\delta_\rho - \delta, \delta_\rho + \delta), \\ u(-\delta_\rho - \delta) = u(\delta_\rho + \delta) = 0. \end{cases} \tag{$\hat{\mathrm{P}}^{\delta}$}$$

Associated with ($\hat{\mathrm{P}}^{\delta}$) we consider the functional $J^{\delta_\rho+\delta}$.

4.4. Rescaling of solutions

Lemma 4.8. *Let $\delta > 0$ and $\hat{w}_k^\delta$ is a critical value of J^δ. Then*

$$w_k^\delta : = \left(2c_k^\delta\right)^{\frac{1}{p-1}} \hat{w}_k^\delta, \qquad c_k^\delta = J^\delta(\hat{w}_k^\delta) \tag{95}$$

is a weak solution of

$$\begin{cases} u''(x) + \left|u(x)\right|^{p-1} u(x) = 0 & x \in (-\delta, \delta), \\ u(-\delta) = u(\delta) = 0. \end{cases} \tag{P$^\delta$}$$

Proof. We have to prove that

$$\forall v \in \mathrm{H}_0^1(-\delta, \delta) : \int_{-\delta}^{\delta} [-(w_k^\delta)'v' + |w_k^\delta|^{p-1} w_k^\delta v] dx = 0. \tag{96}$$

By using (95), we see that (96) equivalent to

$$\forall v \in \mathrm{H}_0^1(-\delta, \delta) : \int_{-\delta}^{\delta} [-(\hat{w}_k^\delta)'v' + 2c_k^\delta |\hat{w}_k^\delta|^{p-1} \hat{w}_k^\delta v] dx = 0. \tag{97}$$

Since $\hat{w}_k^\delta$ is a critical value of $J^\delta : \mathscr{M}^\delta \to \mathbb{R}$, then by Lagrange's multipliers there exists a parameter $\lambda \in \mathbb{R}$ such that for all $v \in \mathrm{H}_0^1(-\delta, \delta)$

$$\begin{aligned} 0 &= \left\langle DJ^\delta(\hat{w}_k^\delta) - \lambda DL^\delta(\hat{w}_k^\delta), v) \right\rangle \\ &= \int_{-\delta}^{\delta} \left[(\hat{w}_k^\delta)'v' - \lambda |\hat{w}_k^\delta|^{p-1} \hat{w}_k^\delta v\right] dx \end{aligned} \tag{98}$$

In order to find λ, let's take $v = \hat{w}_k^\delta$ in (98). Then we have

$$\begin{aligned} 0 &= \int_{-\delta}^{\delta} \left[\left((\hat{w}_k^\delta)'\right)^2 - \lambda |\hat{w}_k^\delta|^{p-1} (\hat{w}_k^\delta)^2\right] dx \\ \int_{-\delta}^{\delta} \left((\hat{w}_k^\delta)'\right)^2 dx &= \lambda \int_{-\delta}^{\delta} |\hat{w}_k^\delta|^{p-1} (\hat{w}_k^\delta)^2 dx \\ &= \lambda \int_{-\delta}^{\delta} |\hat{w}_k^\delta|^{p+1} dx \end{aligned} \tag{99}$$

By definition we have that

$$\int_{-\delta}^{\delta} |\hat{w}_k^\delta|^{p+1} dx = \|\hat{w}_k^\delta\|_{L^{p+1}(-\delta,\delta)} = 1$$

Therefore, by previous expression and (99) we have that

$$\lambda = \int_{-\delta}^{\delta} \left((\hat{w}_k^\delta)'\right)^2 dx = 2J^\delta(\hat{w}_k^\delta) = 2c_k^\delta \tag{100}$$

Therefore, by substituting (100) in (98) we verify (97). Which implies that (95) is a weak solution of (P$^\delta$). □

Corollary 4.3. *The function $\hat{w}_k^{\delta+\hat{\delta}}$ is a weak solution of ($\hat{\mathrm{P}}^\delta$).*

Proof. It's enough to take $\overline{\delta} = \delta + \hat{\delta}$ and apply the Lemma 4.8. □

Lemma 4.9. *Let $\varepsilon > 0$, then the mapping*

$$w_{k,\varepsilon} : = (2c_{k,\varepsilon})^{\frac{1}{p-1}} \hat{w}_{k,\varepsilon}, \qquad c_{k,\varepsilon} = J_\varepsilon(\hat{w}_{k,\varepsilon}) \tag{101}$$

is a weak solution of

$$\begin{cases} u''(x) - \frac{1}{(\varepsilon g(\varepsilon))^2} V\left(\frac{x}{g(\varepsilon)}\right) u(x) + \left|u(x)\right|^{p-1} u(x) = 0, & x \in \mathbb{R}, \\ \lim\limits_{|x|\to\infty} u(x) = 0. \end{cases} \tag{P$'_\varepsilon$}$$

Where $\hat{w}_{k,\varepsilon}$ is a critical point of J_ε.

The proof is analogous to Lemma 4.8.

Lemma 4.10. *Let $\varepsilon > 0$, then the mapping*

$$v_{k,\varepsilon}(x) : = \left(2(\varepsilon g(\varepsilon))^2 c_{k,\varepsilon}\right)^{\frac{1}{p-1}} \hat{w}_{k,\varepsilon}\left(\frac{x}{g(\varepsilon)}\right), \qquad c_{k,\varepsilon} = J_\varepsilon(\hat{w}_{k,\varepsilon}) \tag{102}$$

is a weak solution of

$$\begin{cases} \varepsilon^2 v''(x) - V(x)v(x) + \left|v(x)\right|^{p-1} v(x) = 0 & \text{in} \quad \mathbb{R}, \\ \lim\limits_{|x|\to\infty} v(x) = 0. \end{cases} \tag{P$_\varepsilon$}$$

Where $\hat{w}_{k,\varepsilon}$ is a critical value of J_ε.

Proof. Let's make the following change of variable

$$z = \frac{x}{g(\varepsilon)} \tag{103}$$

Then, by substituting (103) in (102) we have

$$\begin{aligned} v_{k,\varepsilon}(zg(\varepsilon)) &= \left(2(\varepsilon g(\varepsilon))^2 c_{k,\varepsilon}\right)^{\frac{1}{p-1}} \hat{w}_{k,\varepsilon}(z) \\ \left(\varepsilon g(\varepsilon)\right)^{-\frac{2}{p-1}} v_{k,\varepsilon}(zg(\varepsilon)) &= (2c_{k,\varepsilon})^{\frac{1}{p-1}} \hat{w}_{k,\varepsilon}(z) \end{aligned}$$

By Lemma 4.9 the mapping

$$w_{k,\varepsilon} = (2c_{k,\varepsilon})^{\frac{1}{p-1}} \hat{w}_{k,\varepsilon}$$

is a weak solution for (P$'_\varepsilon$). Substituting the previous expression in (104) we have

$$\left(\varepsilon g(\varepsilon)\right)^{-\frac{2}{p-1}} v_{k,\varepsilon}(zg(\varepsilon)) = w_{k,\varepsilon}(z) \tag{104}$$

and by the change of variable we obtain for any $\phi \in H_0^1(\mathbb{R})$

$$\begin{aligned} 0 &= \int_{\mathbb{R}} \left[(w_{k,\varepsilon})'\phi' + V_\varepsilon w_{k,\varepsilon}\phi - |w_{k,\varepsilon}|^{p-1} w_{k,\varepsilon}\phi\right] dz \\ &= \left(\varepsilon g(\varepsilon)\right)^{-\frac{2}{p-1}} \int_{\mathbb{R}} \left[-(g(\varepsilon))^{-2}(v_{k,\varepsilon})'\phi' + \left(\varepsilon g(\varepsilon)\right)^{-2} V(z) v_{k,\varepsilon}\phi - \left(\varepsilon g(\varepsilon)\right)^{-2} |v_{k,\varepsilon}|^{p-1} v_{k,\varepsilon}\phi\right] dz \\ &= \left(\varepsilon g(\varepsilon)\right)^{-\frac{2}{p-1}-2} \int_{\mathbb{R}} \left[\varepsilon^2 (v_{k,\varepsilon})'\phi' - V(z) v_{k,\varepsilon}\phi + |v_{k,\varepsilon}|^{p-1} v_{k,\varepsilon}\phi\right] dz \end{aligned}$$

With this we conclude that (102) is a weak solution of (P$_\varepsilon$). □

4.5. Limits for critical values

In this part, we prove point (iii) of Theorem 4.1. As we mentioned before, this result is based on the Ljusternik–Schnirelman theory for uniform functionals. The index k of the critical values represents the topological characteristic of the level set, fixed by the kranoselskii's genus. In consequence, we prove that the level sets $J\varepsilon$ and J^{δ_ρ} are by the Lusternik–Schnirelman category, topologically equivalent.

Remark 4.10. In the rest of the document, we shall consider ρ and δ_ρ as in the Remark 4.1. Also, we are going to use $I_{\delta_\rho} := (-\delta_\rho, \delta_\rho)$ and $I_{\delta_\rho+\delta} := (-\delta_\rho - \delta, \delta_\rho + \delta)$ to simplify the notation.

Theorem 4.5. *For all $k \in \mathbb{N}$, there exists $C_\rho > 0$ such that*

$$\lim_{\varepsilon\to 0} c_{k,\varepsilon} = (1 + C_\rho)\, c_k^{\delta_\rho} \tag{105}$$

The proof of this result is divided into several steps given in the following lemmas.

Lemma 4.11. *For all $k \in \mathbb{N}$, there exists $C_\rho > 0$ such that*

$$c_{k,\varepsilon} \leq (1 + C_\rho)\, c_k^{\delta_\rho}, \qquad \forall \varepsilon > 0. \tag{106}$$

Proof. Let $k \in \mathbb{N}$, generic.

(i) We first are going to prove that

$$\forall \varepsilon > 0 : \Sigma^{\delta_\rho} \subseteq \Sigma_\varepsilon \tag{107}$$

Let $\varepsilon > 0$, generic. Let set $u \in \mathrm{H}_0^1(I_{\delta_\rho})$, with its extension by zero outside of $(-\delta_\rho, \delta_\rho)$. Thus, for some $0 < \delta < \delta_\rho$, by using Poincaré's inequality, we have

$$\begin{aligned} \|u\|_\varepsilon &= \int_{\mathbb{R}} |u'|^2 + V_\varepsilon u^2 dx \\ &= \int_{-\delta_\rho}^{\delta_\rho} |u'|^2 + V_\varepsilon u^2 dx \\ &= \int_{-\delta_\rho}^{\delta_\rho} |u'|^2 + (\varepsilon g(\varepsilon))^{-2} V\left(\frac{x}{g(\varepsilon)}\right) u^2 dx \\ &\leq \int_{-\delta_\rho}^{\delta_\rho} |u'|^2 + (\varepsilon g(\varepsilon))^{-2} \max_{|x|\leq\delta} \left\{ V\left(\frac{x}{g(\varepsilon)}\right)\right) \} u^2 dx \end{aligned}$$

$$\begin{aligned} &\leq \int_{-\delta_\rho}^{\delta_\rho} |u'|^2 + \rho u^2 dx \\ &\leq \int_{-\delta_\rho}^{\delta_\rho} |u'|^2 + C_\rho |u'|^2 dx \\ &\leq (1 + C_\rho) \|u\|_{H_0^1(I_{\delta_\rho})}. \end{aligned}$$

By the arbitrariness of u, then we have

$$\mathrm{H}_0^1(I_{\delta_\rho}) \subseteq H_\varepsilon \tag{108}$$

Moreover, By the arbitrariness of ε we have proved (107).

(ii) We shall prove

$$\forall \varepsilon > 0 : \mathscr{M}^{\delta_\rho} \subseteq \mathscr{M}_\varepsilon. \tag{109}$$

Let $\varepsilon > 0$, generic. Let set $u \in \mathrm{H}_0^1(I_{\delta_\rho})$, with its extension by zero outside of I_{δ_ρ}. Thus,

$$\begin{aligned} \|u\|_{L^{p+1}(\mathbb{R})} &= \left(\int_{\mathbb{R}} |u|^{p+1}\right)^{1/p+1} \\ &= \left(\int_{-\delta_\rho}^{\delta_\rho} |u|^{p+1}\right)^{1/p+1} \\ &= \|u\|_{L^{p+1}(-\delta_\rho, \delta_\rho)}. \end{aligned} \tag{110}$$

Then, by (110) we conclude (109).

(iii) Now, let $k \in \mathbb{N}$ and $\varepsilon > 0$. By (107) and (109), we have that

$$\mathscr{A}_k^{\delta_\rho} \subseteq \mathscr{A}_{k\varepsilon}. \tag{111}$$

Then,

$$\begin{aligned} c_{k,\varepsilon} &= \inf_{A \in \mathscr{A}_{k,\varepsilon}} \max_{u \in A} \left\{ J_\varepsilon(u) \right\} \\ &\leq \inf_{A \in \mathscr{A}_k^{\delta_\rho}} \max_{u \in A} \left\{ J_\varepsilon(u) \right\} \\ &\leq \inf_{A \in \mathscr{A}_k^{\delta_\rho}} \max_{u \in A} \left\{ (1 + C_\rho) J^{\delta_\rho}(u) \right\} \\ &= (1 + C_\rho) c_k^{\delta_\rho} \end{aligned}$$

By the arbitrariness of k and ε we have proved (106). □

Proposition 4.5. *Let $k \in \mathbb{N}$ and $\sigma > 0$. Then, for every $0 < \delta \ll 1$, there exists a ε_δ such that*

$$\forall \varepsilon \in (0, \varepsilon_\delta) : c_k^{\delta + \delta_\rho} \leq c_{k,\varepsilon} + \sigma \tag{112}$$

This result is proved in several lemmas.

Lemma 4.12. *Let $k \in \mathbb{N}$ and $\sigma > 0$. Then, for every $\varepsilon > 0$ and $q \geq 1$, there exists*

$$A_\sigma(\varepsilon) \in \mathscr{A}_{k,\varepsilon},$$

$$b_{k,\sigma,\rho} \geq 0,$$

and

$$C_q > 0$$

such that

$$\forall v \in A_\sigma(\varepsilon) : \|v\|_{L^q(\mathbb{R})} \leq C_q \left(b_{k,\sigma,\rho}\right)^{1/2} \tag{113}$$

Proof. Let $\varepsilon > 0$, generic. By the characterization of the infimum of $c_{k,\varepsilon}$, there exists

$$A_\sigma(\varepsilon) \in \mathscr{A}_{k,\varepsilon} : \max_{v\in\ A_\sigma(\varepsilon)} \{J_\varepsilon(v)\} c_{k,\varepsilon} + \frac{\sigma}{3}.$$

Then, by Lemma 4.11 we have that

$$\forall v \in A_{\sigma(\varepsilon)} : J_\varepsilon(v) \leq \left[(1 + C_\rho) c_k^{\delta_\rho} + \frac{\sigma}{3}\right] := b_{k,\sigma,\rho}. \tag{114}$$

Let $v \in A_{\sigma(\varepsilon)}$, generic. By (114) we have

$$b_{k,\sigma,\rho} \geq J_\varepsilon(v) \geq \frac{1}{2}\|v\|^2_{\mathrm{H}_0^1(\mathbb{R})} = \frac{1}{2}\int_{\mathbb{R}} |u'(x)|^2 dx. \tag{115}$$

On the other hand, for $q \geq 1$, by Lemma 4.2, we get a $C_q > 0$ such that

$$\|v\|_{L^q(\mathbb{R})} \leq C_q \|v\|^2_{\mathrm{H}_0^1(\mathbb{R})}. \tag{116}$$

By (116) and (115) we conclude (113). ☐

Lemma 4.13. *Given the conditions in Lemma 4.12. We define for $\beta > 0$*

$$V_\beta := \inf_{x\in\mathbb{R}\setminus(-\delta_\rho-\beta,\delta_\rho+\beta)} \left\{V\left(\frac{x}{g(\varepsilon)}\right)\right\}.$$

Then, for all $\delta > 0$ and $v \in A_{\sigma(\varepsilon)}$ the following property holds

$$\|v\|_{L^2(\mathbb{R}\setminus I_{\delta_\rho+\delta})} \leq \left(\frac{b_{k,\sigma,\rho}}{V_\delta}\right)^{1/2} \cdot \varepsilon g(\varepsilon). \tag{117}$$

Proof. Let $\delta > 0$ and $v \in A_\sigma(\varepsilon)$, generic. By (114) we have that

$$\begin{aligned} b_{k,\sigma,\rho} &\geq \frac{1}{2}\int_{\mathbb{R}\setminus I_{\delta_\rho}} (\varepsilon g(\varepsilon))^2 V\left(\frac{x}{g(\varepsilon)}\right) v^2(x) dx \\ &\geq \frac{1}{2}\int_{\mathbb{R}\setminus I_{\delta_\rho+\delta}} (\varepsilon g(\varepsilon))^{-2} V\left(\frac{x}{g(\varepsilon)}\right) v^2(x) dx \end{aligned}$$

Note that, since $b_{k,\sigma,\rho}$ does not depend on ε, we have

$$\begin{aligned} b_{k,\sigma,\rho} &\geq \frac{1}{2}\int_{\mathbb{R}\setminus(-\delta_\rho-\delta,\delta_\rho+\delta)} (\varepsilon g(\varepsilon))^{-2} V\left(\frac{x}{g(\varepsilon)}\right) v^2(x) dx \\ &\geq \frac{1}{2}\int_{\mathbb{R}\setminus I_{\delta_\rho+\delta}} (\varepsilon g(\varepsilon))^{-2} V_\delta v^2(x) dx \\ &\geq \frac{V_\delta}{2(\varepsilon g(\varepsilon))^2}\int_{\mathbb{R}\setminus I_{\delta_\rho+\delta}} v^2(x) dx \\ &\geq \frac{V_\delta}{2(\varepsilon g(\varepsilon))^2}\|v\|_{L^2(\mathbb{R}\setminus I_{\delta_\rho+\delta})} \end{aligned}$$

which implies that

$$\|v\|_{L^2(\mathbb{R}\setminus I_{\delta_\rho+\delta})} \leq \left(\frac{2b_{k,\sigma,\rho}}{V_\delta}\right)^{1/2} \cdot \varepsilon g(\varepsilon).$$

By the arbitrariness of v and δ, we have proved (117). ☐

Lemma 4.14. *Given the conditions in Lemma 4.12. For all* $\delta > 0$

$$\lim_{\varepsilon\to 0}\max_{v\in A_{\sigma(\varepsilon)}}\{\|v\|_{L^{p+1}(\mathbb{R}\setminus I_{\delta_\rho+\delta})}\} = 0 \tag{118}$$

Proof. Let $\delta > 0$, generic. By the Interpolation Theorem, we choose $\alpha \in (0,1)$ such that

$$\frac{1}{p+1} = \frac{1-\alpha}{2} + \frac{\alpha}{q}$$

Now, by considering (117) and (113), we obtain for all $v \in A_{\sigma(\varepsilon)}$

$$\begin{aligned}
\|v\|_{L^{p+1}(\mathbb{R}\setminus I_{\delta_\rho+\delta})} &\leq \|v\|^{1-\alpha}_{L^2(\mathbb{R}\setminus I_{\delta_\rho+\delta})}\|v\|^{\alpha}_{L^q(\mathbb{R}\setminus I_{\delta_\rho+\delta})} \\
&\leq \left(\frac{2b_{k,\sigma,\rho}}{V_\delta}\right)^{\frac{1-\alpha}{2}}[\varepsilon g(\varepsilon)]^{1-\alpha}(b_{k,\sigma,\rho})^{\alpha} \\
&\leq C\left(\frac{b_{k,\sigma,\rho}}{V_\delta^{1-\alpha}}\right)^{1/2}[\varepsilon g(\varepsilon)]^{1-\alpha}
\end{aligned} \tag{119}$$

In particular,

$$\|v\|_{L^{p+1}(\mathbb{R}\setminus I_{\delta_\rho+\delta})} \leq C\left(\frac{b_{k,\sigma,\rho}}{V_\delta^{1-\alpha}}\right)^{1/2}[\varepsilon g(\varepsilon)]^{1-\alpha} \tag{120}$$

By passing the limit $\varepsilon \to 0$ in (120) and by Corollary 2.14 we conclude (118). ☐

Lemma 4.15. *Under the conditions of Proposition 4.5. For all* $\delta > 0$ *there exists* $\varepsilon_\delta > 0$ *such that*

$$\forall\varepsilon \in (0,\varepsilon_\delta), \forall v \in A_{\sigma(\varepsilon)} : \|v\|_{L^{p+1}(\mathbb{R}\setminus I_{\delta_\rho+\delta})} \geq 1-\delta. \tag{121}$$

Proof. By (118), given $\delta > 0$ and $s > 0$ there exists $\varepsilon_{\delta,s} > 0$ such that there is $\varepsilon \in (0,\varepsilon_{\delta,s})$ such that

$$\max_{v\in A_{\sigma(\varepsilon)}}\{\|v\|_{L^{p+1}(\mathbb{R}\setminus I_{\delta_\rho+\delta})}\} \leq \delta^s \tag{122}$$

Let set $s = 1$. Let $\varepsilon_{\delta,1}$ and $v \in A_{\sigma(\varepsilon)}$, generic. Then, by the triangle inequality, we obtain

$$\begin{aligned}
1 &= \|v\|_{L^{p+1}(\mathbb{R})} \\
&\leq \|v\|_{L^{p+1}(I_{\delta_\rho+\delta})} + \|v\|_{L^{p+1}(\mathbb{R}\setminus I_{\delta_\rho+\delta})} \\
&\leq \|v\|_{L^{p+1}(I_{\delta_\rho+\delta})} + \max_{v\in A_{\sigma(\varepsilon)}}\left\{\|v\|_{L^{p+1}(\mathbb{R}\setminus I_{\delta_\rho+\delta})}\right\} \\
&\leq \|v\|_{L^{p+1}(I_{\delta_\rho+\delta})} + \delta
\end{aligned} \tag{123}$$

Then, by the arbitrariness of ε and v and by (123), we have proved (121). ☐

Now, we define

$$\phi_\delta(x) := \begin{cases} 1 \quad x \in I_* \\ \phi_\delta(x) \in (0,1) \text{ and } |\phi'_\delta(x)| \leq \dfrac{1}{\delta^r} \quad x \in \hat{I} \\ 0 \quad x \in \mathbb{R}\setminus I_{\delta_\rho+\delta}. \end{cases} \tag{124}$$

where, $I_* := (-\delta_\rho - \delta/2, \delta_\rho + \delta/2)$ and $I_{\delta_\rho+\delta} := (-\delta_\rho - \delta, \delta_\rho + \delta)$ and $\hat{I} := (-\delta_\rho - \delta, -\delta_\rho - \delta/2) \cup (\delta_\rho + \delta/2, \delta_\rho + \delta)$. Also,

$$\begin{aligned}
\phi_\delta[\cdot]\colon \mathscr{M}_\varepsilon &\to \mathscr{M}^{\delta_\rho+\delta} \\
w &\mapsto \phi_\delta[w] = \frac{\phi_\delta w}{\|w\|_{L^{p+1}(\mathbb{R})}}.
\end{aligned}$$

Lemma 4.16. *Under the conditions of Proposition (4.5), for all* $\delta \in (0,1)$ *we have*

$$\forall\varepsilon \in (0,\varepsilon_\delta), \forall v \in A_{\sigma(\varepsilon)} : c_k^{\delta_\rho+\delta} \leq \max_{v\in\phi_\delta[A_{\sigma(\varepsilon)}]}\left\{J^{\delta_\rho+\delta}(v)\right\} \tag{125}$$

Proof. Let $\delta \in (0,1)$, $\varepsilon \in (0,\varepsilon_\delta)$ and $v \in A_{\sigma(\varepsilon)}$, generic. By the concentration property given in (121) and the definition (124) we have that

$$\begin{aligned}\int_{I_{\delta_\rho+\delta}} |\phi_\delta v|^{p+1}dx &\geq \int_{I_*} |\phi_\delta v|^{p+1}dx \\ &\geq \int_{I_*} |v|^{p+1}dx \\ &\geq \left(1-\frac{\delta}{2}\right)^{p+1} \end{aligned} \tag{126}$$

So that,

$$\|w\|_{L^{p+1}(I_{\delta_\rho+\delta})} \geq 1-\delta \tag{127}$$

In particular, we see that $\phi_\delta[\cdot]$ is well defined and thus it is continuous. Then, by the definition of $\phi_\delta[\cdot]$ as an odd functional and by the properties of the genus we have that

$$\gamma\left(\phi_\delta[A_{\sigma(\varepsilon)}]\right) \geq k,$$

also,

$$\phi_\delta[A_{\sigma(\varepsilon)}] \in \mathscr{A}_k^{\delta_\rho+\delta}.$$

Then, by the last expression and the definition of $c_k^{\delta_\rho+\delta}$, we have (125). □

Lemma 4.17. *Under the same conditions of Proposition 4.5. For all $\delta \in \left(0,\frac{1}{4}\right)$ and $\varepsilon \in (0,\varepsilon_\delta)$, there exists an element $w \in A_{\sigma(\varepsilon)}$ such that*

$$\max_{v\in\phi_\delta[A_{\sigma(\varepsilon)}]} J^{\delta_\rho+\delta}(v) \leq J_\varepsilon(w) + \frac{\sigma}{3} \tag{128}$$

Proof. Let $\delta \in \left(0,\frac{1}{4}\right)$ and $\varepsilon \in (0,\varepsilon_\delta)$, generic. By the characterization of the supremum we take an element $u \in A_{\sigma(\varepsilon)}$ such that $\bar{v} = \phi_\delta[u]$ and

$$\max_{v\in\phi_\delta[A_{\sigma(\varepsilon)}]} J^{\delta_\rho+\delta}(v) \leq J^{\delta_\rho+\delta}(\bar{v}) + \frac{\sigma}{3} \tag{129}$$

For $\bar{v} = \phi_\delta[u]$, by the definition of ϕ_δ we have

$$\begin{aligned}\|\phi_\delta u\|^2_{L^{p+1}(\mathbb{R})} J^{\delta_\rho+\delta}(\bar{v}) &= \frac{1}{2}\|\phi_\delta u\|^2_{L^{p+1}(\mathbb{R})}\|\bar{v}\|_{H_0^1(I_{\delta_\rho+\delta})} \\ &= \frac{1}{2}\|\phi_\delta u\|_{H_0^1(I_{\delta_\rho+\delta})} \\ &= \frac{1}{2}\int_{I_{\delta_\rho+\delta}} |(\phi_\delta u)'|^2dx \\ &= \frac{1}{2}\int_{I_{\delta_\rho+\delta}} |(\phi_\delta' u+\phi_\delta u')|^2dx \\ &= \frac{1}{2}\int_{I_{\delta_\rho+\delta}} ((\phi_\delta' u)^2+(\phi_\delta u')^2+2|\phi_\delta' u\phi_\delta u'|)dx \\ &\leq \frac{1}{2}\int_{I_{\delta_\rho+\delta}} ((\phi_\delta' u)^2+2|\phi_\delta' u\phi_\delta u'|+(u')^2)dx \\ &\leq \int_{I_{\delta_\rho+\delta}} ((\phi_\delta' u)^2+2|\phi_\delta' u\phi_\delta u'|)dx + \frac{1}{2}\int_{I_{\delta_\rho+\delta}} (u')^2dx \\ &\leq \int_{I_{\delta_\rho+\delta}} ((\phi_\delta' u)^2+2|\phi_\delta' u\phi_\delta u'|)dx + \frac{1}{2}\int_{\mathbb{R}} [(u')^2+V_\varepsilon u^2]dx \\ &\leq \int_{I_{\delta_\rho+\delta}} ((\phi_\delta' u)^2+2|\phi_\delta' u\phi_\delta u'|)dx + \|u\|_\varepsilon \end{aligned} \tag{130}$$

By (130), (126), the definition of ϕ_δ and Cauchy–Schwartz and Poincaré's inequalities, and (115)

$$\begin{aligned}
(1-\delta)^2 J^{\delta_\rho+\delta}(\bar{v}) &\leq \|\phi_\delta u\|^2_{L^{p+1}(\mathbb{R})} J^{\delta_\rho+\delta}(\bar{v}) \\
&\leq \int_{I_{\delta_\rho+\delta}} ((\phi_\delta' u)^2 + 2|\phi_\delta' u\phi_\delta u'|)dx + \|u\|_\varepsilon \\
&\leq J_\varepsilon(u) + \int_{I_{\delta_\rho+\delta}} ((\phi_\delta' u)^2 + 2|\phi_\delta' u\phi_\delta u'|)dx \\
&\leq J_\varepsilon(u) + \int_{\hat{I}} ((\phi_\delta' u)^2 + 2|\phi_\delta' u\phi_\delta u'|)dx + \int_{I_*} ((\phi_\delta' u)^2 + 2|\phi_\delta' u\phi_\delta u'|)dx \\
&\leq J_\varepsilon(u) + \int_{I_*} \left(\frac{1}{\delta^{2r}}|u|^2 + \frac{2}{\delta^r}|uu'|\right)dx \\
&\leq J_\varepsilon(u) + \frac{1}{\delta^{2r}}\|u\|^2_{L^2(I_*)} + \frac{2}{\delta^r}\|u\|^2_{L^2(I_*)}\|u\|^2_{H^1_0(I_*)} \\
&\leq J_\varepsilon(u) + \left(\frac{C_p}{\delta^{2r}} + \frac{2}{\delta^r}\right)\|u\|^2_{L^2(I_*)}\|u\|^2_{H^1_0(I_*)} \\
&\leq J_\varepsilon(u) + \left(\frac{C_p}{\delta^{2r}} + \frac{2}{\delta^r}\right)\sqrt{2}b_{k,\sigma,\rho}\|u\|^2_{L^2(I_*)} \\
&\leq J_\varepsilon(u) + \left(\frac{C}{\delta^{2r}}\right)\|u\|^2_{L^2(I_*)}
\end{aligned}$$

Where,

$$C = \sqrt{2}(C_p + 2)b_{k,\sigma,\rho}.$$

Then, by Hölder's inequality, considering (122) and taking $s > 2r$, we get $\varepsilon^* > 0$ such that

$$\begin{aligned}
(1-\delta)^2 J^{\delta_\rho+\delta}(\bar{v}) &\leq J_\varepsilon(u) + \left(\frac{C}{\delta^{2r}}\right)\|u\|^2_{L^2(I_*)} \\
&\leq J_\varepsilon(u) + \left(\frac{C}{\delta^{2r}}\right)\|u\|^2_{L^2(\mathbb{R}\setminus I_*)} \\
&\leq J_\varepsilon(u) + \left(\frac{C}{\delta^{2r}}\right)\left(\frac{\delta}{2}\right)^s
\end{aligned}$$

Then, we can conclude that

$$\forall\varepsilon \in (0,\varepsilon^*) : (1-\delta)^2 J^{\delta_\rho+\delta}(\bar{v}) \leq J_\varepsilon(u) + C\delta^{s-2r} \tag{131}$$

By (114), we have that

$$\begin{aligned}
\frac{1}{2}J^{\delta_\rho+\delta}(\bar{v}) &\leq \frac{9}{16}J^{\delta_\rho+\delta}(\bar{v}) \\
&= \left(\frac{3}{4}\right)^2 J^{\delta_\rho+\delta}(\bar{v}) \\
&\leq (1-\delta)^2 J^{\delta_\rho+\delta}(\bar{v}) \\
&\leq J_\varepsilon(u) + C\delta^{s-2r} \\
&\leq b_{k,\sigma,\rho} + C\delta^{s-2r}
\end{aligned}$$

The last together with (131) provides

$$\begin{aligned}
J^{\delta_\rho+\delta}(\bar{v}) &\leq J^{\delta_\rho+\delta}(\bar{v}) + \delta^2 J^{\delta_\rho+\delta}(\bar{v}) \\
&\leq J_\varepsilon(u) + C\delta^{s-2r} + 2\delta J^{\delta_\rho+\delta}(\bar{v}) \\
&\leq J_\varepsilon(u) + C\delta^{s-2r} + 4\delta(b_{k,\sigma,\rho} + C\delta^{s-2r})
\end{aligned}$$

Now, by taking $u := w$ we have

$$J^{\delta_\rho+\delta}(\bar{v}) \leq J^{\delta_\rho+\delta}(w) + \frac{\sigma}{3} \tag{132}$$

Thus, we conclude (129) by the arbitrariness of δ and ε.

Proof of Proposition 4.5. By (125), (132) and (130), we have that □

$$
\begin{aligned}
c_k^{\delta_\rho+\delta} &\leq \max_{v\in\phi_\delta[A_\sigma(\varepsilon)]} J^{\delta_\rho+\delta}(v) \\
&\leq J^{\delta_\rho+\delta}(\bar{v}) + \frac{\sigma}{3} \\
&\leq J_\varepsilon(w) + \frac{2}{3}\sigma \\
&\leq \max_{u\in A_{\sigma(\varepsilon)}} J_\varepsilon(u) + \frac{2}{3}\sigma \\
&\leq c_{k,\varepsilon} + \sigma
\end{aligned}
$$

□

Lemma 4.18. *For every $k \in \mathbb{N}$ and $\delta > 0$, the following condition holds*

$$c_k^{\delta_\rho+\delta} \leq c_k^{\delta_\rho} \tag{133}$$

The proof is analogous to that of Lemma 4.11.

Proposition 4.6. *Let $k \in \mathbb{N}$ and $\sigma > 0$. Then, there exists $\delta_\sigma > 0$ such that*

$$\forall \delta \in (0, \delta_\sigma) : c_k^{\delta_\rho} \leq c_k^{\delta_\rho+\delta} + \sigma \tag{134}$$

The proof is divided in several lemmas.

Lemma 4.19. *For every $k \in \mathbb{N}$ and $\delta > 0$, there exists $B_\sigma(\delta) \in \mathscr{A}_k^{\delta_\rho+\delta}$ and $b_{k,\sigma,\rho} \geq 0$ such that*

$$\forall v \in B_\sigma(\delta) : J^{\delta_\rho+\delta}(v) \leq b_{k,\sigma,\rho} \tag{135}$$

Proof. Let $\delta > 0$, generic. Then, by the definition of $c_k^{\delta_\rho+\delta}$ and by the characterization of infimum there exists

$$B_\sigma(\delta) \in \mathscr{A}_k^{\delta_\rho+\delta}$$

such that

$$\max_{v\in B_\sigma(\delta)} J^{\delta_\rho+\delta}(v) \leq c_k^{\delta_\rho+\delta} + \frac{\sigma}{3}. \tag{136}$$

Thus, by Lemma 4.18 we conclude that

$$\forall v \in B_\sigma(\delta) : J^{\delta_\rho+\delta}(v) \leq b_{k,\sigma,\rho}. \tag{137}$$

□

Remark 4.11. Let's proceed as is done by Felmer and Mayorga in [4].

Now, let's pick for all $\delta > 0$ the diffeomorphism $\Phi_\delta \in C^1(I^{\delta_\rho}, I^{\delta_\rho+\delta})$ such that

$$\forall x \in I^{\delta_\rho} : |\Phi_\delta(x) - x| \leq \mathscr{O}(\delta), |\Phi'_\delta(x) - 1| \leq \mathscr{O}(\delta), \text{ and } \Phi_\delta(|\delta_\rho|) = |\delta_\rho + \delta| \tag{138}$$

And the mapping,

$$
\begin{aligned}
\Gamma_\delta[\cdot]\colon \mathrm{H}_0^1(I_{\delta_\rho+\delta}\backslash\{0\}) &\to \mathrm{H}_0^1(I_{\delta_\rho}) \\
w \mapsto \Gamma_\delta[w] &= \frac{w\circ\Phi_\delta}{\|w\circ\Phi_\delta\|_{L^{p+1}(I_{\delta_\rho})}}.
\end{aligned}
$$

Lemma 4.20. *Given the conditions of Proposition 4.6. There exist*

$$\delta_1 \in (0, \delta_0),$$

$$\delta_0 := \delta_0(\delta_\rho) > 0,$$

such that

$$\forall \delta \in (0, \delta_1) : \Phi_\delta\left[B_\sigma(\delta)\right] \in \mathscr{A}_k^{\delta_\rho} \tag{139}$$

Proof. Note that to prove (139) it's enough to show that Γ_δ is well defined and continuous because it is odd by definition. Let's observe that by (138), for each $\eta > 0$, there is a $\delta_2 := \delta_2(\eta) > 0$ and $\delta_1 \geq \delta_2$ such that

$$\forall x \in I_{\delta_\rho}, \forall \delta \in (0, \delta_2) : 1 - \eta \leq \Phi'_\delta(x) \leq 1 + \eta \tag{140}$$

1. We shall prove that given $\delta \in (0, \delta_2)$ and $v \in \mathrm{H}_0^1(I_{\delta_\rho+\delta}\backslash\{0\})$, then $v \circ \Phi_\delta \neq 0$. Let $\delta \in (0, \delta_2)$ and $v \in \mathrm{H}_0^1(I_{\delta_\rho+\delta}\backslash\{0\})$, generic. From (140) and the change of variable formula for integration we have that:

$$\begin{aligned} \|w \circ \Phi_\delta\|_{L^{p+1}(I_{\delta_\rho})}^{p+1} &= \int_{I_{\delta_\rho}} |w \circ \Phi_\delta|^{p+1} dx \\ &= \int_{I_{\delta_\rho}} |w \circ \Phi_\delta|^{p+1} \frac{\Phi_\delta'}{\Phi_\delta'} dx \\ &\geq \int_{I_{\delta_\rho}} |w \circ \Phi_\delta|^{p+1} \frac{\Phi_\delta'}{|1+\eta|} dx \\ &= \frac{1}{|1+\eta|} \int_{I_{\delta_\rho}} |w \circ \Phi_\delta|^{p+1} \Phi_\delta' dx \\ &= \frac{1}{|1+\eta|} \int_{I_{\delta_\rho+\delta}} |w|^{p+1} dx \end{aligned} \tag{141}$$

On the other hand,

$$\begin{aligned} \|w \circ \Phi_\delta\|_{L^{p+1}(I_{\delta_\rho})}^{p+1} &= \int_{I_{\delta_\rho}} |w \circ \Phi_\delta|^{p+1} dx \\ &= \int_{I_{\delta_\rho}} |w \circ \Phi_\delta|^{p+1} \frac{\Phi_\delta'}{\Phi_\delta'} dx \\ &\leq \int_{I_{\delta_\rho}} |w \circ \Phi_\delta|^{p+1} \frac{\Phi_\delta'}{|1-\eta|} dx \\ &= \frac{1}{|1-\eta|} \int_{I_{\delta_\rho}} |w \circ \Phi_\delta|^{p+1} \Phi_\delta' dx \\ &= \frac{1}{|1-\eta|} \int_{I_{\delta_\rho+\delta}} |w|^{p+1} dx \end{aligned} \tag{142}$$

Therefore, we conclude that $v \circ \Phi_\delta \neq 0$. Moreover, by (141) and (142) we have that

$$\frac{\|v\|_{L^{p+1}(I_{\delta_\rho+\delta})}}{|1+\eta|} \leq \|w \circ \Phi_\delta\|_{L^{p+1}(I_{\delta_\rho+\delta})}^{p+1} \leq \frac{\|v\|_{L^{p+1}(I_{\delta_\rho})}}{|1-\eta|} \tag{143}$$

2. Now, lets prove that

$$\forall v \in \mathrm{H}_0^1(I_{\delta_\rho+\delta}\backslash\{0\}) : \Gamma_\delta[v] \in \mathrm{H}_0^1(I_{\delta_\rho}\backslash\{0\}) \tag{144}$$

Let $w \in C^\infty(I_{\delta_\rho+\delta}\backslash\{0\})$, generic. Then, by Poincaré's inequality and using the change of variable proposed before we obtain

$$\begin{aligned} \left\|\Gamma_\delta[w]\right\|_{H_0^1(I_{\delta_\rho})} &= \int_{I_{\delta_\rho}} (\Gamma_\delta[w]')^2 dx \\ &= \left\| \frac{w \circ \Phi_\delta}{\|w \circ \Phi_\delta\|_{L^{p+1}(I_{\delta_\rho})}} \right\|_{H_0^1(I_{\delta_\rho})} \\ &\leq \frac{(1+\eta)^{\frac{2}{p+1}}}{\|w \circ \Phi_\delta\|_{L^{p+1}(I_{\delta_\rho+\delta})}} \int_{I_{\delta_\rho}} (|w \circ \Phi_\delta|')^2 dx \\ &\leq \frac{(1+\eta)^{\frac{2}{p+1}}}{\|w \circ \Phi_\delta\|_{L^{p+1}(I_{\delta_\rho+\delta})}} \int_{I_{\delta_\rho}} (|w' \circ \Phi_\delta| \Phi_\delta')^2 dx \\ &\leq \frac{(1+\eta)^{\frac{2}{p+1}+1}}{\|w \circ \Phi_\delta\|_{L^{p+1}(I_{\delta_\rho+\delta})}} \int_{I_{\delta_\rho}} (|w' \circ \Phi_\delta|)^2 \Phi_\delta' dx \\ &\leq K \int_{I_{\delta_\rho+\delta}} (|w'|)^2 dx \\ &\leq K \|w\|_{H_0^1(I_{\delta_\rho+\delta})}. \end{aligned}$$

Thus,

$$\Gamma_\delta[w] \in \mathrm{H}_0^1(I_{\delta_\rho}\backslash\{0\}).$$

Moreover, by (138) we have

$$\forall x = |\delta_\rho| : \Gamma_\delta[w](x) = \frac{w(|\delta + \delta_\rho|)}{\|w \circ \Phi_\delta\|_{L^{p+1}(I_{\delta_\rho})}}$$

Therefore, we have also proved that

$$\forall v \in C^\infty(I_{\delta_\rho + \delta}) : \|v\|_{H_0^1(I_{\delta_\rho})} \leq K \, \|v\|_{H_0^1(I_{\delta_\rho+\delta})}, \tag{145}$$

For some $K = K(\eta, \delta)$. By a density criterion we can extend the inequality to all $I_{\delta_\rho+\delta}$. Then, we obtain (143). Finally, by (143) and (145) we obtain (139). □

Lemma 4.21. *Under the conditions of Proposition 4.6. For all $\delta \in (0, \delta_2)$, there exists an element $w \in B_\sigma(\delta)$ such that*

$$\max_{u^* \in \Gamma_\delta[B_\sigma(\delta)]} J^{\delta_\rho}\left(u^*\right) \leq J^{\delta_\rho+\delta}\left(w\right) + \frac{2}{3}\sigma \tag{146}$$

Proof. Let $\delta \in (0, \delta_2)$, for $u = \Gamma_\delta[v]$, then by the change of variable (138) and $\|v\|_{L^{p+1}(I_{\delta_\rho})} = 1$, we obtain

$$\begin{aligned}
J^{\delta_\rho}(u) &= \frac{1}{2}\|u\|_{H_0^1(I_{\delta_\rho})} \\
&= \frac{1}{2\|w \circ \Phi_\delta\|^2_{L^{p+1}(I_{\delta_\rho})}}\|w \circ \Phi_\delta\|^2_{H_0^1(I_{\delta_\rho})} \\
&\leq \frac{(1+\eta)^{\frac{2}{p+1}}}{\|w \circ \Phi_\delta\|^2_{L^{p+1}(I_{\delta_\rho+\delta})}}\int_{I_{\delta_\rho}}(|w \circ \Phi_\delta|')^2 dx \\
&\leq \frac{(1+\eta)^{\frac{2}{p+1}}}{2\|w \circ \Phi_\delta\|^2_{L^{p+1}(I_{\delta_\rho+\delta})}}\int_{I_{\delta_\rho}}(|w' \circ \Phi_\delta|)^2 \Phi_\delta'^2 dx
\end{aligned}$$

$$\begin{aligned}
&\leq \frac{(1+\mathscr{O}(\delta))(1+\eta)^{\frac{2}{p+1}}}{2\|w \circ \Phi_\delta\|^2_{L^{p+1}(I_{\delta_\rho+\delta})}}\int_{I_{\delta_\rho}}(|w' \circ \Phi_\delta|)^2 \Phi_\delta' dx \\
&\leq \frac{(1+\mathscr{O}(\delta))(1+\eta)^{\frac{2}{p+1}}}{2\|w \circ \Phi_\delta\|^2_{L^{p+1}(I_{\delta_\rho+\delta})}}\int_{I_{\delta_\rho+\delta}}(w')^2 dx \\
&\leq \frac{(1+\mathscr{O}(\delta))(1+\eta)^{\frac{2}{p+1}}}{\|w \circ \Phi_\delta\|^2_{L^{p+1}(I_{\delta_\rho+\delta})}} J^{\delta_\rho+\delta}(v)
\end{aligned}$$

By choosing $v = w$ such that

$$J^{\delta_\rho}(u) \leq J^{\delta_\rho+\delta}(w) + \frac{\sigma}{3} \tag{147}$$

By the arbitrariness of $u = \Gamma_\delta[v]$ we conclude (146). □

Proof of Proposition 4.6. Considering (139) and the definition of $c_k^{\delta_\rho}$, we deduce that

$$c_k^{\delta_\rho} \leq \max_{u \in \Gamma_\delta[B_\sigma(\delta)]} J^{\delta_\rho}(u) \tag{148}$$

By (148), (146) and (136) we deduce that

$$\begin{aligned}
c_k^{\delta_\rho} &\leq \max_{u^* \in \Gamma_\delta[B_\sigma(\delta)]} J^{\delta_\rho}\left(u^*\right) \\
&\leq J^{\delta_\rho+\delta}(w) + \frac{2}{3}\sigma \\
&\leq \max_{v \in [B_\sigma(\delta)]} J^{\delta_\rho+\delta}(v) + \frac{2}{3}\sigma \\
&\leq c_k^{\delta_\rho+\delta} + \sigma
\end{aligned}$$

□

Proof of Theorem 4.5. Let $0 < \delta \ll 1$ and by considering the Proposition 4.6, we choose a $\delta \in (0, \delta_{\sigma/2})$. From Proposition 4.5 we have that there exists a $\varepsilon_\delta > 0$ such that

$$c_k^{\delta_\rho} \leq c_k^{\delta_\rho+\delta} + \frac{\sigma}{2} \leq (1 + C_\rho)\, c_{k,\varepsilon} + \sigma, \tag{149}$$

for every $\varepsilon \in (0, \varepsilon_\delta)$. By the Lemma 4.11 and the arbitrariness of $\sigma > 0$, we conclude the proof. □

4.6. Concentration phenomena

Remark 4.12 (Subconvergence)**.** A family of functions $\{\zeta_\varepsilon\}_{\varepsilon>0}$ is said to sub-converge in a space E, as $\varepsilon \to 0$, when from any sequence $\{\varepsilon_n\}_{n\in\mathbb{N}}$ converging to zero it is possible to extract a subsequence $\{\varepsilon_{n_k}\}_{k\in\mathbb{N}}$ such that $\left\{\zeta_{\varepsilon_{n_k}}\right\}_{k\in\mathbb{N}}$ converges in E, as $k\to\infty$.

Let's study the behavior of solutions inside and outside of I_{δ_ρ}.

Lemma 4.22. *Let $k\in\mathbb{N}$.*

(i) As $\varepsilon\to 0$, $w_{k,\varepsilon}$ weakly sub-converges to $u_k\in H^1(\mathbb{R})$,

(ii) $u_k|_{I_{\delta_\rho}}$ is a solution of (P_L), where,

$$J^{\delta_\rho}\left(\hat{u}_k|_{I_{\delta_\rho}}\right)=c_k^{\delta_\rho},$$

and,

$$\hat{u}_k=(2c_k^{\delta_\rho})^{\frac{-1}{p-1}}u_k.$$

Proof. Let $k\in\mathbb{N}$, generic.

(i) Let's prove that for ε_δ

$$\|\hat{w}_{k,\varepsilon}\|_{H^1(\mathbb{R})}\le K', \forall\varepsilon\in(0,\varepsilon_\delta) \tag{150}$$

where, $K'=K'(k)>0$ is a constant.
Let $\varepsilon\in(0,\varepsilon_\delta)$, generic. By Lemma 4.11 we obtain

$$\begin{aligned}\|\hat{w}_{k,\varepsilon}\|^2_{H^1(\mathbb{R})} &= 2J_\varepsilon\left(\hat{w}_{k,\varepsilon}\right)\\ &= 2c_{k,\varepsilon}\\ &\le 2(1+C_\rho)c_k^{\delta_\rho+\delta}.\end{aligned} \tag{151}$$

By Lemma 4.2 and by (151) we obtain

$$\begin{aligned}\|\hat{w}_{k,\varepsilon}\|^2_{H^1(\mathbb{R})} &= C_0\|\hat{w}'_{k,\varepsilon}\|_{L^2(\mathbb{R})}\\ &\le C_1\,\|\hat{w}_{k,\varepsilon}\|_{H^1_0(\mathbb{R})}\\ &\le C_2c_k^{\delta_\rho+\delta}:=K'.\end{aligned}$$

Thus, we conclude (150).
From (150), exists $\hat{u}_k\in\mathrm{H}^1_0(\mathbb{R})$ such that $\hat{w}_{k,\varepsilon}$ weakly and pointwise sub-converges to it, when $\varepsilon\to 0$.

(ii) We are going to prove that $\hat{u}_k|_{I_{\delta_\rho}}$ is a weak solution to (P_L). Since $\hat{w}_{k,\varepsilon}\in\mathscr{M}_\varepsilon$ is a critical value for J_ε, we have

$$\forall\phi\in\mathrm{H}^1_0(\mathbb{R}):\int_{\mathbb{R}}(\hat{w}'_{k,\varepsilon}\phi'-V_\varepsilon\hat{w}_{k,\varepsilon}\phi)dx=\lambda_{k,\varepsilon}\int_{\mathbb{R}}|\hat{w}_{k,\varepsilon}|^{p-1}\hat{w}_{k,\varepsilon}\phi dx, \tag{152}$$

where $\lambda_{k,\varepsilon}=2c_{k,\varepsilon}$ is the Lagrange's multiplier. Passing the limit, $\varepsilon\to 0$, we have

$$\forall\phi\in C_0^\infty(I_{\delta_\rho}):\int_{I_{\delta_\rho}}(\hat{u}'_k\phi')dx=\lambda_k\int_{I_{\delta_\rho}}|\hat{u}_k|^{p-1}\hat{u}_k\phi dx, \tag{153}$$

where $\lambda_k=2c_k$ is the Lagrange's multiplier. In other words, we have that

$$\forall\phi\in C_0^\infty(I_{\delta_\rho}):\lim_{\varepsilon\to 0}\int_{I_{\delta_\rho}}(\varepsilon g(\varepsilon))^{-2}V\left(\frac{x}{g(\varepsilon)}\right)(\hat{w}_{k,\varepsilon}\phi)dx=0,$$

Here, we have used the fact that $\hat{w}_{k,\varepsilon}$ sub-converges to $\hat{u}_k$ in $L^{p+1}(\mathbb{R})$ which comes from Lemma 4.11 and the compact injection $\mathrm{H}^1_0(\mathbb{R})\subseteq L^{p+1}(\mathbb{R})$.

(iii) Let's prove that

$$\hat{u}_k(x)=0,\quad \text{a.e}\quad \mathbb{R}\backslash I_{\delta_\rho}. \tag{154}$$

To prove (154), we proceed by Reduction to Absurdity. We have:

$$\left|\{x\in\mathbb{R}\backslash I_{\delta_\rho}:\hat{u}_k(x)\neq 0,\}\right|>0. \tag{155}$$

We define the sets

$$S_{\delta,\beta}:=\{x\in\mathbb{R}\backslash I_{\delta_\rho+\delta}/\hat{u}_k(x)\ge\beta\},\quad \delta>0,\beta>0,$$

and

$$\{x \in \mathbb{R}\backslash I_{\delta_\rho} : \hat{u}_k(x) \neq 0\} \subseteq \bigcup_{\delta>0,\beta>0} S_{\delta,\beta}$$

Then, by the monotony of the measure, we obtain

$$0 < \left|\{x \in \mathbb{R}\backslash I_{\delta_\rho} : \hat{u}_k(x) \neq 0\}\right| \leq \sum_{\delta>0,\beta>0} |S_{\delta,\beta}| \tag{156}$$

Without loss of generality, let's assume that there exists $\delta^*, \beta^*, \eta > 0$ such that $|S_{\delta^*,\beta^*}| \geq \eta > 0$. Then, by the monotony of the measure, again, we have

$$\forall \delta \in (0, \delta^*) : S_{\delta^*,\beta^*} \subseteq S_{\delta,\beta^*} \implies 0 < \eta < |S_{\delta^*,\beta^*}| \leq |S_{\delta,\beta^*}| \tag{157}$$

By (117) we obtain

$$\|\hat{w}_{k,\varepsilon}\|_{L^2(\mathbb{R}\backslash I_{\delta_\rho+\delta})} \leq \left(\frac{2c_k}{V\delta}\right)^{1/2} \cdot \varepsilon g(\varepsilon), \quad \forall \varepsilon \in (0, \varepsilon_\delta), \forall \delta > 0. \tag{158}$$

For each $\delta > 0$ we pick

$$\varepsilon_\delta^* = \min\left\{\varepsilon_\delta, \left(\frac{V_\delta^2}{2c_k}\right)^{1/2}\right\} \tag{159}$$

Considering (V3), we have $\delta' \in (0, \delta^*)$ such that

$$\forall \delta \in (0, \delta') : V_\delta < \frac{(\beta^*)^2\eta}{2}. \tag{160}$$

Let $\delta_0 \in (0, \delta')$, then we have that

$$\forall x \in S_{\delta_0,\beta^*} : |\hat{u}_k|^2 \geq (\beta^*)^2$$

Integrating the last expression in the domain S_{δ_0,β^*} and by (157), we obtain

$$\begin{aligned} \int_{S_{\delta_0,\beta^*}} |\hat{u}_k(x)|^2 dx &\geq \int_{S_{\delta_0,\beta^*}} (\beta^*)^2 dx \\ &= (\beta^*)^2 |S_{\delta_0,\beta^*}| \\ &\geq (\beta^*)^2\eta \end{aligned} \tag{161}$$

Analogous to the proof of Proposition 4.6, for each $\sigma > 0$ there exist $\varepsilon_\sigma \in (0, \varepsilon_\delta^*)$ such that

$$\|\hat{u}_k\|_{L^2(S_{\delta_0,\beta^*})} \leq \|\hat{w}_{k,\varepsilon}\|_{L^2(S_{\delta_0,\beta^*})} + \sigma$$

Therefore, by $\sigma = \dfrac{(\beta^*)^2\eta}{3}$ and $\varepsilon \in (0, \varepsilon_\sigma)$, by (158), (159) and (160) we have

$$\begin{aligned} \int_{S_{\delta_0,\beta^*}} |\hat{u}_k(x)|^2 dx &\leq \int_{S_{\delta_0,\beta^*}} |\hat{w}_{k,\varepsilon}(x)|^2 dx + \sigma \\ &= \frac{(\beta^*)^2\eta}{3} + \int_{S_{\delta_0,\beta^*}} |\hat{w}_{k,\varepsilon}(x)|^2 dx \\ &\leq \frac{(\beta^*)^2\eta}{3} + \left(\frac{4c_k^{\delta_\rho}}{V_\delta}\right)\varepsilon^2 \\ &\leq \frac{(\beta^*)^2\eta}{3} + \left(\frac{4c_k^{\delta_\rho}}{V_\delta}\right)(\varepsilon_\delta^*)^2 \\ &< \frac{(\beta^*)^2\eta}{3} + V_\delta \\ &< \frac{(\beta^*)^2\eta}{3} + \frac{(\beta^*)^2\eta}{2} = \frac{5}{6}(\beta^*)^2\eta, \end{aligned}$$

which contradicts (161). Then, $|S_{\delta,\beta}| = 0$, for $\delta, \beta > 0$, using (156) we contradicts (155).

(iv) By (ii) we obtain that

$$\hat{u}_k|_{I_{\delta_\rho}} \in \mathrm{H}_0^1(I_{\delta_\rho}).$$

On the other hand, by (153)

$$J_\rho^\delta\left(\hat{u}_k|_{I_{\delta_\rho}}\right) = c_k^{\delta_\rho}.$$

□

Lemma 4.23. *Let $k \in \mathbb{N}$. As $\varepsilon \to 0$ $w_{k,\varepsilon}$ (strongly) sub-converges to u_k in the $H^1(\mathbb{R})$ norm.*

Proof. Let $k \in \mathbb{N}$, generic. By Lemma 4.2, we conclude that $H_\varepsilon \subseteq L^2(\mathbb{R})$, which implies that $\hat{w}_{k,\varepsilon}$ sub-converges in $L^2(\mathbb{R})$ to $\hat{u}_k$ as $\varepsilon \to 0$, i.e.

$$\lim_{\varepsilon\to 0} \int_{\mathbb{R}} (\hat{w}_{k,\varepsilon})^2 dx = \int_{\mathbb{R}} (\hat{u}_k)^2 dx \tag{162}$$

By (151) we have for all $\varepsilon \in (0, \varepsilon_\delta)$, and by using the Poincaré's inequality, that

$$\begin{aligned} \left\| \hat{w}'_{k,\varepsilon} \right\|^2_{H^1_0(\mathbb{R})} &\leq 2c_k^{\delta_\rho} \\ &= \left\| \hat{u}'_k \right\|^2_{H^1_0(I_{\delta_\rho})} \\ &= \int_{I_{\delta_\rho}} |\hat{u}'_k|^2 dx \\ &\leq c \int_{\mathbb{R}} |\hat{u}'_k|^2 dx, \end{aligned}$$

which provides

$$\limsup_{\varepsilon\to 0} \|\hat{w}_{k,\varepsilon}\|_{H^1(\mathbb{R})} = c\, \|\hat{u}_k\|_{H^1(\mathbb{R})} \tag{163}$$

□

5. Conclusions and Recommendations

5.1. Conclusions

In this work, we studied the existence and qualitative behavior of solutions for

$$\begin{cases} \varepsilon^2 v''(x) - V_\varepsilon(x) v(x) + |v(x)|^{p-1} v(x) = 0, & x \in \mathbb{R}, \\ \lim_{|x|\to\infty} v(x) = 0. \end{cases} \tag{P_ε}$$

We considered the infinite case, given by Byeon and Wang in [10], where

$$\Omega = \{V = 0\} = \{0\}$$

and V decreases exponentially around it. Here, the potential verifies:

$(V1)$ V is a non-negative continuous function over $\mathbb{R}$;

$(V2)$ $V(x) \to \infty$ as $|x| \to \infty$;

$(V3)$ For each $x \in [-1, 1]\backslash\{0\}$:

$$V(x) = \exp\left(-\frac{1}{a(x)}\right),$$

where a is a (Ω, b) quasi-homogeneous function.

For a fixed $\rho > 0$ we have a $\delta_\rho > 0$ such that our limit problem, as $\varepsilon \to 0$, is given by

$$\begin{cases} w''(x) + |w(x)|^{p-1} w(x) = 0 & x \in (-\delta_\rho, \delta_\rho), \\ w(-\delta_\rho) = w(\delta_\rho) = 0. \end{cases} \tag{P_L}$$

Then, our main conclusions are the following:

1. We have proved, by a Lusternik–Schnirelman scheme and using the Kranoselskii genus, that the original problem, (P_ε), has infinitely many solutions.
2. We have proved multiplicity of solutions for the limit problem, (P_L) ,using the same technique mentioned before. In fact, by the Kranoselskii's genus the solutions found for (P_ε) and (P_L) come in pairs for each critical level.
3. We have proved concentration results obtained by Byeon Wang, [10], Felmer and Mayorga, [4], and Medina and Mayorga, [12], for several settings with critical frequency. In particular, we proved the H^1-convergence of the solutions of (P_ε) to the corresponding solutions of (P_L).
4. The following courses were determinant for me to be able to deal with this problem: Calculus of Variations, Operators Theory, Functional Analysis, Measure Theory, Partial Differential Equations and Continuous Optimization.
5. All the Non-linear Analysis that was needed for this work goes beyond the topics corresponding to the subjects in YT math career.

5.2. Recommendations

(a) I think that is important to preserve the great academical diversity in YT mathematics, allowing the students to build skills in interdisciplinary fields.

(b) I believe that among the mandatory subjects of the mathematics career there should be at least one more course on mathematical analysis to strengthen the profile of the new mathematicians.

(c) It would be crucial to consolidate the administrative processes to help the students in the graduation activities.

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