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Homotopy Theory for Finite Categories

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para la obtención del título de Matemático

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Para Pablo

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Christian Chávez
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Resumen

Reformulamos la teoría de homotopía introducida por Rosero en el contexto de categorías finitas. Esta teoría tiene como objetivo asociar un grupo a una categoría finita por medio de un funtor $\mathbf{Cat}_{\mathbf{Fin}*} \to \mathbf{Grp}$ de tal manera que exhiba la mayoría de las propiedades de $\pi_1 \colon \mathbf{Top}_* \to \mathbf{Grp}$, el funtor inducido por el grupo fundamental de un espacio topológico. A pesar de que esto se puede lograr usando el espacio clasificador de una categoría pequeña, tal método es inherentemente topológico. En contraste, nuestra teoría se desarrolla netamente en un contexto algebraico. Además de esto, hemos replanteado los fundamentos de la versión original de la teoría para mayor claridad y rigurosidad. Específicamente, presentamos notación consistente, resultados revisados, demostraciones originales, y generalizamos algunos de los resultados más importantes de la teoría. También presentamos resultados nuevos que aportan a la teoría una estructura más sólida. Finalmente, demostramos que nuestra teoría produce resultados consistentes en cuanto a la realización geométrica de las categorías $\mathbf{S}^1$ y $\mathbf{T}^2$, cuyos grupos fundamentales categóricos son isomorfos a $\mathbb{Z}$ y $\mathbb{Z}^2$, respectivamente.

Palabras clave: grupo fundamental, teoría de categorías, categorías finitas, realización geométrica, nervio de una categoría, espacio clasificador.

Abstract

We develop a reformulation of the theory of homotopy originally introduced by Rosero within the context of finite categories. This theory aims to associate a group to every finite category by means of a functor $\mathbf{Cat}_{\mathbf{Fin}*} \to \mathbf{Grp}$ in such a way that it has many of the properties of $\pi_1 \colon \mathbf{Top}_* \to \mathbf{Grp}$, the functor induced by the fundamental group of a topological space. Even though this can be achieved using the classifying space of a small category, such approach is inherently topological. In contrast, our theory develops a theoretical framework from scratch based on a purely algebraic setting. Furthermore, we have reformulated the foundations of the original version of the theory for improved clarity and rigor. Specifically, we present coherent notation, revised results, novel proofs, and the generalization of some of the principal results of the theory. In addition, we also prove new results that make the overall organization more robust. Finally, we demonstrate that this theory gives consistent results regarding the geometric realization of the nerve of a small category, for the particular cases of the categories $\mathbf{S}^1$ and $\mathbf{T}^2$, whose categorical fundamental groups are isomorphic to $\mathbb{Z}$ and $\mathbb{Z}^2$, respectively.

Keywords: fundamental group, category theory, finite categories, geometric realization, nerve of a category, classifying space.

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Chapter 1

Preliminaries

The purpose of this chapter is to present elementary concepts and fix the notation and terminology used throughout our work. It consists of two main parts: basic group theory and basic general topology. Most proofs have been omitted because they are standard and can be found in any introductory algebra or topology textbook. Some references consulted include [8, 3, 33, 23, 18].

1.1. Elements of Group Theory

A binary operation on a set X is a function from $X \times X$ to X. A set together with a binary operation is called *magma*. If $(X, \cdot)$ is a magma, and $\cdot$ is associative, that is $x \cdot (y \cdot z) = (x \cdot y) \cdot z$ for every $x, y, z \in X$, then $(X, \cdot)$ is a *semigroup*. If, in addition, there exists an element $e \in X$ such that $x \cdot e = e \cdot x = x$ for all $x \in X$, then $(X, \cdot)$ is a *monoid*. A *group* is a monoid where every element has an *inverse*. To elaborate further, we have the following definition.

Definition 1.1 *Group*

A group is a pair $(G, \cdot)$ where G is a set and $\cdot$ is a binary operation on G such that

(i) $\cdot$ is associative,

(ii) there is $e \in G$ such that for all $g \in G$, $g \cdot e = e \cdot g = g$, and

(iii) for every $g \in G$ there is $g' \in G$ such that $g \cdot g' = g' \cdot g = e$.

Remark 1.2. We usually refer to the group $(G, \cdot)$ with the name of the underlying set only, so we might write *the group* G to mean that G has a group structure when endowed with a binary operation, usually understood from the context.

The symbol for the operation is usually omitted, and juxtaposition is used instead: we write ab for $a \cdot b$. The element e is called the *identity* of G and sometimes it is denoted 1. Note that if there is g' such that

$gg' = g'g = e$, then g' is unique: if there were another $g'' \in G$ such that $gg'' = g''g = e$, then

$$g' = g'e = g'(gg'') = (g'g)g'' = eg'' = g''.$$

Therefore, the element g' is denoted by g^{-1} and is called the *inverse* of g.

If, in addition to the conditions of Definition 1.1, the group operation verifies $gh = hg$ for all $g, h \in G$, we say that G is an *abelian group*. In this case, the symbol $+$ is used instead of $\cdot$ to denote the group operation, and the identity element is denoted by 0.

Let g be an element of a group. The product of g with itself n times is denoted g^n. The product of g^{-1} with itself n times is denoted g^{-n}. We define $g^0 = 1$. If the group is abelian, we write ng instead of g^n and $-ng$ instead of g^{-n}. In this case, we also write $0g = 0$.

A group is *finite* if the underlying set is a finite set; otherwise it is called *infinite group*. The *order* of a group is defined to be the cardinality of the underlying set. We denote the order of a group G by $|G|$. We write $|G| < \infty$ if G is finite and $|G| = \infty$ otherwise.

Subgroups

Definition 1.3 *Subgroup*

Let $(G, \cdot)$ be a group. A subgroup of G is a pair $(H, *)$ where H is a subset of G and $*$ is the restriction of $\cdot$ to H.

Let e be the identity of G. It is clear that G and $\{e\}$ are two subgroups of G. The latter is called the *trivial subgroup* of G.[1] We say H is *nontrivial* if $H \neq \{e\}$. We say H is a *proper subgroup* of G if $H \subseteq G$ and $H \neq G$. Otherwise, we say H is *improper*. We write

$$H \leq G$$

to mean H is a subgroup of G, and $H < G$ to mean that H is a proper subgroup of G.

Let H be a subset of G. Subgroups are characterized by the following property: $H \leq G$ if and only if both $H \neq \varnothing$ and $ab^{-1} \in H$ for all $a, b \in H$.

The intersection of two subgroups is again a subgroup, and in general, if $(H_\lambda)_{\lambda \in \Lambda}$ is a nonempty family of subgroups of G, then $\bigcap_{\lambda \in \Lambda} H_\lambda$ is also a subgroup of G. In contrast, the union of a collection of subgroups may not be a subgroup. However, if $(H_\lambda)_{\lambda \in \Lambda}$ is a nonempty family of subgroups of G such that either $H_\lambda \subseteq H_{\lambda'}$ or $H_{\lambda'} \subseteq H_\lambda$ for all $\lambda, \lambda' \in \Lambda$, then $\bigcup_{\lambda \in \Lambda} H_\lambda$ is in fact a subgroup of G. On the other hand, if H and K are subgroups of G, and if we let

$$HK = \{hk \mid h \in H, k \in K\},$$

then $HK \leq G$ if and only if $HK = KH$.

[1] The trivial subgroup may be denoted with the symbol 0 or 1, depending on whether the ambient group is abelian or not, respectively. However, this convention is not always strictly followed.

Direct Products and Direct Sums of Groups

Let $(G_\lambda)_{\lambda\in\Lambda}$ be a nonempty family of groups. The *direct product* of $(G_\lambda)_{\lambda\in\Lambda}$ is the group whose underlying set is the cartesian product

$$\prod_{\lambda\in\Lambda} G_\lambda$$

endowed with the componentwise multiplication

$$(g_\lambda)_{\lambda\in\Lambda}\cdot(g'_\lambda)_{\lambda\in\Lambda}=(g_\lambda\cdot g'_\lambda)_{\lambda\in\Lambda}.$$

If Λ is finite, say $\Lambda=\{1,\dots,n\}$, then we write $G_1\times\cdots\times G_n$ for $\prod_{\lambda\in\Lambda} G_\lambda$. The direct product is commutative and associative up to isomorphism.

If $(G_\lambda)_{\lambda\in\Lambda}$ is a family of abelian groups, we define its *direct sum* to be the subgroup of $\prod_{\lambda\in\Lambda} G_\lambda$ that consist of those elements $(g_\lambda)_{\lambda\in\Lambda}$ for which the set $\{g_\lambda \mid g_\lambda\neq 0\}$ is finite, or equivalently, such that $g_\lambda=0$ for all but finitely many $\lambda\in\Lambda$. The direct sum of $(G_\lambda)_{\lambda\in\Lambda}$ is denoted $\bigoplus_{\lambda\in\Lambda} G_\lambda$. The direct sum of a finite family of abelian groups is denoted $G_1\oplus\cdots\oplus G_n$. If $(G_\lambda)_{\lambda\in\Lambda}$ is finite, the direct product and the direct sum are exactly the same group.

Homomorphisms of Groups

Definition 1.4 *Group-homomorphism*

Let G and H be groups. A map $\psi\colon G\to H$ is a *homomorphism of groups* from G to H if

$$\psi(ab)=\psi(a)\psi(b)$$

for every $a,b\in G$.

We refer to ψ as a group-homomorphism for brevity, or simply as a *homomorphism*, when there are no other algebraic structures involved, such as rings, modules, or algebras. In those cases, we use the terms ring-homomorphism, module-homomorphism, or algebra-homomorphism, respectively.

Let G and H be groups and $\psi\colon G\to H$ a homomorphism. The kernel of ψ is

$$\operatorname{Ker}\psi=\{a\in G\mid \psi(a)=1\}$$

where 1 denotes the identity element of H. The image of ψ is $\operatorname{Im}\psi=\psi(G)$. Both the kernel and image of a homomorphism are subgroups of its domain and codomain, respectively; that is, $\operatorname{Ker}\psi\leq G$ and $\operatorname{Im}\psi\leq H$. Moreover, ψ is injective if and only if $\operatorname{Ker}\psi=1$, where 1 denotes the trivial subgroup of G. For any subgroup $K\leq G$, its image $\psi(K)$ is a subgroup of H. On the other hand, is it a basic result that $\psi(1)=1$ and $\psi(g^n)=(\psi(g))^n$ for any $g\in G$ and any $n\in\mathbb{Z}$.

An *isomorphism* is a bijective homomorphism. Equivalently, $\psi\colon G \to H$ is an isomorphism if and only if there exists another homomorphism $\psi'\colon H \to G$ such that

$$\psi \circ \psi' = \mathrm{Id}_H \quad \text{and} \quad \psi' \circ \psi = \mathrm{Id}_G.$$

where Id_H and Id_G denote the identity maps on H and G, respectively. If there exists a isomorphism from G to H, we write $G \cong H$ and say that G and H are *isomorphic*. In fact, $\cong$ is an equivalence relation in the class of all groups. An *endomorphism* of G is a homomorphism from G to itself. An *automorphism* is a bijective endomorphism.

Factor Groups

Let $(G, \cdot)$ be a group and $H \leq G$. Fix $g \in G$. The *left coset* of H by g is

$$gH = \{gh \mid h \in H\}.$$

The *right coset* of H by g is

$$Hg = \{hg \mid h \in H\}.$$

The *conjugacy class* of H by g is

$$gHg^{-1} = \left\{ghg^{-1} \mid h \in H\right\}.$$

Proposition 1.5

Let H be a subgroup of a group G and g an element of G. The following statements are equivalent.

(i) For every $g \in G$, $Hg = gH$.

(ii) For every $g \in G$, $gHg^{-1} = H$.

(iii) For every $g \in G$, $gHg^{-1} \subseteq H$.

(iv) There exists a homomorphism with domain G whose kernel is H.

A *normal subgroup* H of G is a subgroup of G such that $gHg^{-1} = H$ for every $g \in G$, i.e., H is invariant under all conjugations. By the proposition just presented, a normal subgroup is a subgroup that satisfies any of the conditions listed in Proposition 1.5. The notation $H \trianglelefteq G$ is used to indicate that H is a normal subgroup of G. A fact of fundamental importance is that the kernel of any homomorphism is normal. Clearly, every subgroup of an abelian group is normal.

Let $N \trianglelefteq G$. The set

$$G/N = \{gN \mid g \in G\}$$

is a group when endowed with the operation defined by

$$(gN)(g'N) = (gg')N$$

where $g, g' \in G$. The fact that N is normal is equivalent to the fact that this operation is well-defined. The group G/N is called the *quotient group* of G by N, sometimes also called *factor group*. The map $\pi\colon G \to G/N : g \mapsto gN$ is a surjective homomorphism called *canonical projection*. If G and H are abelian groups, the cokernel of a homomorphism $\psi\colon G \to H$ is defined as

$$\operatorname{Coker}\psi = H/\operatorname{Im}\psi.$$

Theorem 1.6 *Fundamental Theorem of Group Homomorphisms*

Let $\psi\colon G \to H$ be a group-homomorphism. If N is a normal subgroup of G contained in the kernel of ψ, then there exists a unique homomorphism $\overline{\psi}\colon G/N \to H$ such that $\psi = \overline{\psi} \circ \pi$.

Proof. Define

$$\overline{\psi}\colon G/N \to H \ :\ aN \mapsto \psi(a).$$

Note that $\overline{\psi}$ is well-defined because if $aN = bN$, then $ab^{-1} \in N \leq \ker\psi$, whence $\psi(ab^{-1}) = 1$ and thus $\psi(a) = \psi(b)$. Moreover

$$\overline{\psi}((aN)(bN)) = \overline{\psi}(abN) = \psi(ab) = \psi(a)\psi(b) = \overline{\psi}(aN)\overline{\psi}(bN).$$

By construction, we have $\psi = \overline{\psi} \circ \pi$. $\square$

The conclusion of the previous result can be restated by saying that there exists a unique homomorphism $\overline{\psi}\colon G/N \to H$ that makes the following diagram commute.

$$\begin{tikzcd}
G \arrow[r, "\psi"] \arrow[d, dashed, "\pi"'] & H \\
G/N \arrow[ur, "\overline{\psi}"'] &
\end{tikzcd}$$

Remark 1.7. A diagram consists of a collection of objects and arrows between those objects. A diagram is said to commute if, for any pair of objects, all paths obtained by following the arrows between them are equal.[2] For instance, in the diagram above, the objects are G, G/N, and H and the arrows are ψ, π, and $\overline{\psi}$. That this diagram commutes means $\psi = \overline{\psi} \circ \pi$.

[2] Commutative diagrams are ubiquitous to category theory, as we will see in Chapter 3.

Let K and N be subgroups of a group G with $N \trianglelefteq G$. Then

(i) $N \cap K \trianglelefteq K$,

(ii) $NK = KN$, and

(iii) if K is normal in G and $K \cap N = \{e\}$, then $nk = kn$ for all $k \in K$ and $n \in N$.

With these facts at hand, we can now present four fundamental results that are consequences of Theorem 1.6, called, in order of presentation, the first, second, third, and fourth isomorphism-theorems.

Corollary 1.8 *Isomorphism theorems of groups*

(i) For any homomorphism of groups $\psi\colon G \to H$,

$$G/\operatorname{Ker}\psi \cong \operatorname{Im}\psi.$$

(ii) If $K, N \leq G$ and $N \trianglelefteq G$, then

$$\frac{NK}{N} \cong \frac{K}{N \cap K}.$$

(iii) If $H, K \trianglelefteq G$ and $K \leq H$, then

$$H/K \trianglelefteq G/K \quad \text{and} \quad \frac{G/K}{H/K} \cong \frac{G}{H}$$

(iv) Let $N \trianglelefteq G$. There is a one-to-one correspondence between subgroups of G that contain N and subgroups of G/N. In particular, every subgroup of G/N is of the form A/N with $N \leq A \leq G$. Furthermore, for all $A, B \leq G$ that contain N, it holds

(a) $A \leq B$ if and only if $A/N \leq B/N$, and

(b) $A \trianglelefteq G$ if and only if $A/N \trianglelefteq G/N$.

1.2. Elements of General Topology

We denote the power set of a set X by $\mathcal{P}(X)$.

Definition 1.9 *Topological space*

Let X be a set. A *topology* $\mathcal{T}$ on X is a subset of $\mathcal{P}(X)$ such that

(i) both $\varnothing$ and X belong to $\mathcal{T}$,

(ii) if $(X_\lambda)_{\lambda\in\Lambda}$ is a family of elements of $\mathcal{T}$, then $\bigcup_{\lambda\in\Lambda} X_\lambda \in \mathcal{T}$, and

(iii) if $(X_\lambda)_{\lambda\in\Lambda}$ is a finite family of elements of $\mathcal{T}$, then $\bigcap_{\lambda\in\Lambda} X_\lambda \in \mathcal{T}$.

The pair $(X, \mathcal{T})$ is called *topological space*.

Let $(X, \mathcal{T})$ be a topological space. We say that X is a topological space when the topology $\mathcal{T}$ is understood from the context. We say that U is an open subset of X if and only if $U \in \mathcal{T}$. A subset C of X is *closed* if $X \setminus C$ is an open subset of X.

Let $A \subseteq X$. The *interior* of A, denoted A° or $\operatorname{Int}_X A$, is the union of all open subsets of X contained in A. The *closure* of A, denoted $\overline{A}$ or $\operatorname{Cl}_X A$, is the intersection of all closed subsets of X that contain A. Moreover, (i) A is open if and only if $A = A^\circ$, (ii) A is closed if and only if $A = \overline{A}$, and (iii) A is *everywhere dense* if and only if $\overline{A} = X$. A set $N \subseteq X$ is a *neighborhood* of a point $x \in X$ if there exists an open subset U of X such that $x \in U \subseteq N$.

A *subspace* of $(X, \mathcal{T})$ is a pair $(Y, \mathcal{T}_Y)$ where $Y \subseteq X$ and

$$\mathcal{T}_Y = \{ Y \cap O \mid O \in \mathcal{T} \}.$$

The collection $\mathcal{T}_Y$ is called the *subspace* (or *relative*) *topology* on Y. A subset of a topological space is always assumed to be endowed with the subspace topology unless otherwise stated.

Definition 1.10 *Basis*

Let $\mathcal{T}$ be a topology on X. A subset $\mathcal{B}$ of $\mathcal{T}$ is a *basis* for $\mathcal{T}$ if for every $O \in \mathcal{T}$ there exists $\mathcal{B}' \subseteq \mathcal{B}$ such that

$$O = \bigcup_{B \in \mathcal{B}'} B.$$

Example 1.11. The set of open intervals of $\mathbb{R}$ is a basis for the so called *usual topology* of the real numbers. Thus $\mathbb{R}$ is a topological space, whose open sets are arbitrary unions of open intervals. A remarkable subspace of $\mathbb{R}$ that we will be using a lot is the *unit interval* $I = [0, 1]$.

The elements of a basis are called *basic open sets*. We can define a topology on a set by specifying a collection of distinguished open sets.

Proposition 1.12

Let $\mathcal{B}$ be a collection of subsets of a set X. Then $\mathcal{B}$ is a basis for some topology on X if and only if

(i) $X = \bigcup_{B \in \mathcal{B}} B$, and

(ii) for any $B, B' \in \mathcal{B}$ and $x \in B \cap B'$, there exists $B'' \in \mathcal{B}$ such that $x \in B'' \subseteq B \cap B'$.

In this case, there exists a unique topology on X for which $\mathcal{B}$ is a basis, called the *topology generated by* $\mathcal{B}$.

Let x be a point of a topological space X. A collection $\mathcal{B}_x$ of neighborhoods of x is a *neighborhood basis* for X at x if any neighborhood of x contains an element of $\mathcal{B}_x$. If there exists a countable neighborhood basis at every point of X, we say X is *first countable*. A topological space is said to be *second countable* if it admits a countable basis for its topology, and *separable* if it contains a countable dense subset.

On the other hand, a *cover* of X is a family $(U_\lambda)_{\lambda \in \Lambda}$ of subsets of X such that

$$\bigcup_{\lambda \in \Lambda} U_\lambda = X.$$

The cover is said to be open (closed) if each U_λ is open (closed). If $\Lambda' \subseteq \Lambda$ is finite and $\bigcup_{\lambda \in \Lambda'} U_\lambda = X$, we say $(U_\lambda)_{\lambda \in \Lambda'}$ is a finite subcover of $(U_\lambda)_{\lambda \in \Lambda}$.

Definition 1.13

A space is *compact* if every open cover admits a finite subcover.

Continuity and Homeomorphisms

Definition 1.14 *Continuous function*

A map $f\colon X \to Y$ between topological spaces X and Y is *continuous* if for any open subset O of Y, the set $f^{-1}(O)$ is an open subset of X.

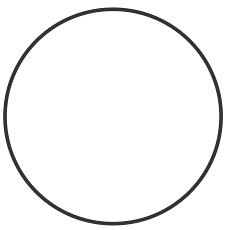
The image of a space under a continuous map is called *continuous image*.

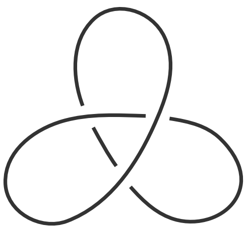
Let X and Y be topological spaces and $f\colon X \to Y$ a continuous map. If there exists a continuous map $f'\colon Y \to X$ such that

$$f \circ f' = 1_Y \quad \text{and} \quad f' \circ f = 1_X$$

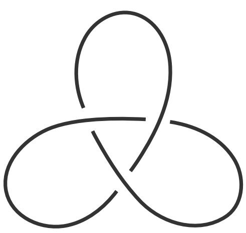
then f is an *homeomorphism*, and X and Y are said to be *homeomorphic*. If X and Y are homeomorphic, we write $X \cong Y$. Being homeomorphic is an equivalence relation in the class of all topological spaces. The composition of homeomorphisms is a homeomorphism. It must be emphasized that a bijective continuous map is not necessarily a homeomorphism. An injective continuous map that is a homeomorphism onto its image is called a *topological embedding*.

Example 1.15. The circle is the set $S^1 = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 = 1\}$ considered as a subspace of $\mathbb{R}^2$. A *knot* is an embedding $S^1 \to \mathbb{R}^3$. Two embeddings $\psi, \psi'\colon X \to Y$ are equivalent if there exist homeomorphisms $\varphi_X\colon X \to X$ and $\varphi_Y\colon Y \to Y$ such that $\psi \circ \varphi_X = \varphi_Y \circ \psi'$. For instance, the knots with images





are equivalent. However, they are not equivalent to the knot



Properties of topological spaces that are preserved under homeomorphisms are called *topological properties* or *invariants*. For instance, cardinality, connectedness, separability, countability, and compactness are some examples of topological properties. Topological properties are useful because, in order to show that two spaces are not homeomorphic, it is enough to find a property that the spaces do not share.

The following result allow us to create a new continuous map given a family of continuous maps defined on smaller open subsets of a topological space: continuous maps can be glued together to create another continuous map.

Theorem 1.16 *Gluing lemma*

Let X and Y be topological spaces, and let $(O_\lambda)_{\lambda\in\Lambda}$ be a finite closed cover of X. Suppose $(f_\lambda\colon O_\lambda \to Y)_{\lambda\in\Lambda}$ is a family of continuous maps such that

$$f_\lambda|_{O_\lambda \cap O_{\lambda'}} = f_{\lambda'}|_{O_\lambda \cap O_{\lambda'}}$$

for all $\lambda, \lambda' \in \Lambda$. Then there exists a unique continuous map $f\colon X \to Y$ such that $f|_{O_\lambda} = f_\lambda$ for every $\lambda \in \Lambda$.

This result, also known as the *pasting lemma*, remains valid if $(O_\lambda)_{\lambda\in\Lambda}$ is assumed to be an arbitrary open cover of X. The gluing lemma will play a crucial role in the next chapter, particularly when we discuss the concatenation of paths in a topological space.

The Product Topology

Let $(X_\lambda, \mathcal{T}_\lambda)_{\lambda\in\Lambda}$ be a family of topological spaces. Let $X = \prod_{\lambda\in\Lambda} X_\lambda$. The collection

$$\left\{ \prod_{\lambda\in\Lambda} O_\lambda \;\middle|\; O_\lambda \in \mathcal{T}_\lambda \text{ for all } \lambda \in \Lambda \text{ and } \{\lambda \in \Lambda \mid O_\lambda \neq X_\lambda\} \text{ is finite} \right\}$$

is a basis for a topology $\mathcal{T}$, called the *product topology* on X. The pair $(X, \mathcal{T})$ is called a *product space*. If Λ is finite, say $\Lambda = \{1, \ldots, n\}$, we write $X_1 \times \cdots \times X_n$ rather than $\prod_{\lambda\in\Lambda} X_\lambda$.

For each $\lambda \in \Lambda$, define the map

$$p_\lambda \colon X \to X_\lambda \quad \text{by} \quad (x_\mu)_{\mu\in\Lambda} \mapsto x_\lambda.$$

We call p_λ the λth *canonical projection* of X onto X_λ. It is clear that p_λ is surjective. Furthermore, it is a standard result that the product topology is the coarsest topology on X with the property that p_λ is continuous, for every $\lambda \in \Lambda$.

The cartesian product $\prod_{\lambda\in\Lambda} X_\lambda$ can be endowed with another topology, called the *box topology*, defined as the topology generated by the basis

$$\left\{ \prod_{\lambda\in\Lambda} O_\lambda \;\middle|\; O_\lambda \in \mathcal{T}_\lambda \text{ for all } \lambda \in \Lambda \right\}.$$

However, the box topology is not suitable for most purposes. The cartesian product of topological spaces is always endowed with the product topology, unless explicitly stated otherwise. Nevertheless, when the index set is finite, both the box and product topology are exactly the same. In any other case, these topologies are different.

Example 1.17. The set $\mathbb{R}^n$ is a product space when endowed with the topology whose basic open sets are products of open intervals of real numbers. Some special subspaces of $\mathbb{R}^n$ include the unit open ball, the unit n-disk, the $(n-1)$-sphere and the n-dimensional cube, defined as follows in order of appearance:

$$\begin{aligned} B &= \{(x_1, \ldots, x_n) \in \mathbb{R}^n \mid x_1^2 + \cdots + x_n^2 < 1\}, \\ D^n &= \{(x_1, \ldots, x_n) \in \mathbb{R}^n \mid x_1^2 + \cdots + x_n^2 \leq 1\}, \\ S^{n-1} &= \{(x_1, \ldots, x_n) \in \mathbb{R}^n \mid x_1^2 + \cdots + x_n^2 = 1\}, \\ I^n &= \{(x_1, \ldots, x_n) \in \mathbb{R}^n \mid 0 \leq x_i \leq 1, \;\; 1 \leq i \leq n\}. \end{aligned}$$

In particular, we call D^2 the *unit disk* and S^1 the *unit circle*, and they will be assumed to be endowed with the subspace topology of $\mathbb{R}^2$

A product space and its projections are characterized, up to unique isomorphism, by the following universal property.

Theorem 1.18 *Universal property of the product topology*

Consider a product space $\prod_{\lambda\in\Lambda} X_\lambda$. For any topological space Y and any family $(\psi_\lambda : Y \to X_\lambda)_{\lambda\in\Lambda}$ of continuous maps, there exists exactly one continuous map $\psi : Y \to \prod_{\lambda\in\Lambda} X_\lambda$ such that the diagram

$$\begin{tikzcd} & \prod_{\lambda\in\Lambda} X_\lambda \arrow[d, "p_\lambda"] \\ Y \arrow[ur, dashed, "\psi"] \arrow[r, "\psi_\lambda"'] & X_\lambda \end{tikzcd}$$

commutes for every $\lambda \in \Lambda$.

Quotient Topology

Definition 1.19 *Quotient topology*

Let $(X, \mathcal{T})$ be a topological space and $\sim$ an equivalence relation on the underlying set X. Let $\pi : X \to X/{\sim}$ be the canonical projection. The collection

$$\mathcal{Q} = \left\{ U \subseteq X/{\sim} \;\middle|\; \pi^{-1}(U) \in \mathcal{T} \right\}$$

is a topology on $X/{\sim}$, called the *quotient topology* on $X/{\sim}$. The *quotient space* of X by $\sim$ is $(X/{\sim}, \mathcal{Q})$.

Alternatively, we could start with a partition of X and define the quotient topology analogously, using the fact that an equivalence relation induces a partition and vice versa.

A quotient space and its canonical projection are characterized, up to unique isomorphism, by the following universal property.

Theorem 1.20 *Universal property of the quotient topology*

Let $X/{\sim}$ be a quotient space. If $\psi : X \to Y$ is a continuous map such that $\psi(x) = \psi(x')$ whenever $x \sim x'$ for all $x, x' \in X$, then there exists a unique continuous map $\overline{\psi} : X/{\sim} \to Y$ such that the diagram

$$\begin{tikzcd} X \arrow[r, "\psi"] \arrow[d, "\pi"'] & Y \\ X/{\sim} \arrow[ur, dashed, "\overline{\psi}"'] & \end{tikzcd}$$

commutes.

The construction we have seen can be generalized in the following sense: if $q\colon X \to Y$ is a surjective map where X is a topological space and Y is any set, then

$$\left\{ U \subseteq Y \;\middle|\; q^{-1}(U) \text{ is open in } X \right\}$$

is a topology on Y, called the quotient topology on Y induced by q.

Definition 1.21 *Quotient map*

Let X and Y be topological spaces. A *quotient map* from X to Y is a surjective function $q\colon X \to Y$ such that Y is endowed with the quotient topology induced by q.

It is straightforward fact that a quotient map is continuous.

Example 1.22 (Quotient spaces).

(i) (Circle) We can define the circle as the set

$$S^1 = \left\{ z \in \mathbb{C} \;\middle|\; |z| = 1 \right\}$$

endowed with the relative topology of $\mathbb{C}$. We have the homeomorphism

$$S^1 \cong I/[1 \sim 0]$$

where $\sim$ is the equivalence relation on $I = [0,1]$ defined by

$$x \sim y \iff x = y \text{ or } \{x,y\} = \{0,1\}.$$

(ii) (Cylinder) The cylinder is the product space $S^1 \times I$, and it is homeomorphic to the quotient

$$I^2/[(0,t) \sim (1,t)]$$

where we identify points of the unit square that are at the same height. Points that are not identified remain as singletons in the quotient.

(iii) (Torus) The 2-torus or just *torus*, denoted T^2, is the product space $S^1 \times S^1$. We have the homeomorphism

$$T^2 \cong I^2/[(0,t) \sim (1,t),\ (t,0) \sim (t,1)].$$

The equivalence relation considered here is the one that identifies the parallel sides of I^2 and leaves the inner points as singletons. In other words, the torus can be obtained from the unit square by identifying the points on the sides that are at the same height or in the same vertical. The n-dimensional torus T^n is just the product space $S^1 \times \cdots \times S^1$ of the circle with itself n times.

(iv) (Möbius strip) The Möbius strip is defined as the quotient space

$$I^2/[(0,t) \sim (1,1-t)].$$

In this case, we identify the vertical sides of I^2 but in the opposite direction.

(v) (Klein bottle) The Klein bottle is the quotient space

$$I^2/[(t,0) \sim (t,1),\ (0,t) \sim (1,1-t)].$$

The Klein bottle can be obtained by pasting together the (single) edges of two Möbius bands.

Connectedness

Definition 1.23 *Connected space*

A topological space is *connected* if it does not admit a cover into two nonempty disjoint open subsets.

Equivalently, a space is connected if has only two subsets that are both open and closed: the empty set and the entire space. A topological space is *disconnected* if it is not connected.

Proposition 1.24 *Properties of connectedness*

(i) The continuous image of a connected space is connected.
(ii) The product of a connected space is connected.
(iii) Every quotient space of a connected space is connected.
(iv) Connectedness is a topological property.

Let X be a topological space and $A \subseteq X$. We say A is connected if A endowed with the subspace topology is connected. More specifically, A is connected if we cannot find two nonempty open subsets U and V of X such that

$$A \subseteq U \cup V \quad \text{and} \quad A \cap U \cap V = \varnothing.$$

If A is connected, then $\overline{A}$ is connected. Moreover, if both $A \subseteq X$ and $B \subseteq X$ are connected and $A \cap B \neq \varnothing$, then $A \cup B$ is connected.

A *connected component* of a space X is a maximal connected subset of X, that is to say, a connected subset that is not properly contained in any other proper connected subset of X. Equivalently, a connected component is one of the equivalence classes generated by the equivalence relation defined by

$$x \sim x' \iff \text{there is } C \subseteq X \text{ connected such that } x, x' \in C$$

for every $x, x' \in X$. Connected components are closed. The number of connected components is a topological invariant. A topological space is *totally disconnected* if all of its connected components are singletons. For instance, any discrete space is totally disconnected.

Path Connectedness

A path in a topological space X is a continuous map from $I = [0,1]$ to X. If γ is a path in X and $\gamma(0) = p$ and $\gamma(1) = q$, we say that γ is a path from p to q.

Definition 1.25 *Path-connected space*

A topological space is *path-connected* if there is a path between any pair of points of the space.

Proposition 1.26 *Properties of path connectedness*

(i) The continuous image of a path-connected space is path-connected.

(ii) The product of finitely many path-connected spaces is path-connected.

(iii) Every quotient space of a path-connected space is also path-connected.

(iv) Path connectedness is a topological property.

(v) Path connectedness implies connectedness.

A *path-component* is a maximal nonempty path-connected subset, or equivalently, a path-component is one of the equivalence classes generated by the equivalence relation that identifies pairs of points connected by a path. Unlike the case of connectedness, path-components are not necessarily closed. As in the case of connectedness, path-components of a space form a partition. Finally, the number of path-components is a topological invariant, and any path-component of a space X is contained in a single connected component of X.

Chapter 2

Classical Homotopy Theory

Algebraic topology originated from the attempts to construct algebraic invariants of topological spaces. The classical and widely known ones being the *fundamental group* and the *homology groups*. We devote this chapter to the study of the former: the fundamental group of a pointed space. Algebraic invariants are helpful in the classification of topological spaces. For instance, homeomorphic topological spaces have isomorphic fundamental groups; the converse is not necessarily true. Thus, we can determine if two spaces are not homeomorphic by showing that their fundamental groups are not isomorphic.[1]

[1] To be precise, one computes the fundamental group of a topological space respect to a point chosen in some path-connected component.

Throughout this chapter, I denotes the unit interval $[0,1]$ and $\mathcal{C}(X,Y)$ denotes the space of continuous functions from a space X to a space Y. The fundamental results have been consulted in [6, 18, 30, 15, 32].

2.1. Homotopy

The main motivation for introducing homotopy is to define homotopy of paths for then we can define the fundamental group. Homotopy formalizes the idea of continuous deformation, which will allow us to talk about deformation of paths in a topological space.

Definition 2.1 *Homotopy*

Let f and g be two continuous maps from a topological space X into a topological space Y. A *homotopy* from f to g is a continuous map $H\colon X \times I \to Y$ such that

$$\forall x \in X: \quad H(x,0) = f(x) \quad \text{and} \quad H(x,1) = g(x).$$

If there is a homotopy from f to g, then f is said to be *homotopic* to g, which is denoted by $f \simeq g$. If H is such homotopy, we write $H\colon f \simeq g$.

In light of this definition, if we fix $t \in I$, we can define a mapping $h_t\colon X \to Y$ by $h_t(x) = H(x,t)$. This results in a change of viewpoint of H.[2] Indeed, we can think of a homotopy from f to g as a family of continuous maps $(h_t)_{t\in I}$ such that $h_0 = f$ and $h_1 = g$. Continuity of H implies continuity of h_t for all $t \in I$. However, continuity of each map h_t does not necessarily imply that H is continuous since it must be verified, in addition, that the assignment $(x,t) \mapsto h_t(x)$ is continuous.

[2] Think of t as time so that a homotopy between f and g is a "continuous process of deformation" at t goes from 0 to 1.

Remark 2.2. Setting $X \times I$ or $I \times X$ as the domain of a homotopy is a matter of convenience. We stick to the former.

Example 2.3. (Rectilinear homotopy) Let X and Y be topological spaces. Any two continuous maps $f, g\colon X \to \mathbb{R}^n$ are homotopic. This follows from the fact that the map $H\colon X \times I \to \mathbb{R}^n$ defined by the formula

$$H(x,t) = (1-t)f(x) + tg(x)$$

is continuous since f and g are. Therefore $H\colon f \simeq g$. In fact, this result remains true if $\mathbb{R}^n$ is replaced by any convex subset of $\mathbb{R}^n$.

Theorem 2.4 *$\simeq$ is an equivalence relation*

Suppose X and Y are topological spaces. Homotopy of maps is an equivalence relation on $\mathcal{C}(X,Y)$.

Proof. We prove that $\simeq$ is reflexive, symmetric, and transitive.

(i) (Reflexivity) Any map $f \in \mathcal{C}(X,Y)$ is homotopic to itself, namely via the constant homotopy $H\colon X \times I \to Y : (x,t) \mapsto f(x)$. Continuity of H comes from continuity of f. Thus $\simeq$ is reflexive.

(ii) (Symmetry[3]) Suppose $f \simeq g$ where $f, g \in \mathcal{C}(X,Y)$. This means there is a homotopy H from f to g with $H(x,0) = f(x)$ and $H(x,1) = g(x)$ for all $x \in X$. Then $G\colon X \times I \to Y : (x,t) \mapsto H(x,1-t)$ is continuous and satisfies $G(x,0) = g(x)$ and $G(x,1) = f(x)$ for all $x \in X$, i.e. G is a homotopy from g to f. Hence $\simeq$ is symmetric.

[3] Symmetry of $\simeq$ permits us to state that H is a homotopy *between* f and g whenever H is a homotopy from f to g or otherwise.

(iii) (Transitivity) Suppose $f \simeq g$ and $g \simeq h$ where $f, g, h \in \mathcal{C}(X,Y)$. Let $H, G\colon X \times I \to Y$ be homotopies from f to g and from g to h, respectively. Define the map $F\colon X \times I \to Y$ by

$$F(x,t) = \begin{cases} H(x,2t) & \text{if} \quad t \in [0,\frac{1}{2}], \\ G(x,2t-1) & \text{if} \quad t \in [\frac{1}{2},1], \end{cases}$$

for all $x \in X$ and $t \in I$. Then F is a homotopy between f and h. Indeed, notice $F(x,\frac{1}{2}) = H(x,1) = G(x,0) = g(x)$ for all $x \in X$, so F is continuous by the pasting lemma; also $F(x,0) = H(x,0) = f(x)$ and $F(x,1) = G(x,1) = h(x)$ for all $x \in X$. Hence $\simeq$ is transitive.

The proof is complete. $\square$

By Theorem 2.4, we can split the set $\mathcal{C}(X,Y)$ into equivalence classes, which we call *homotopy classes*. We denote by $\pi(X,Y)$ the quotient set of $\mathcal{C}(X,Y)$ by $\simeq$.

Theorem 2.5 *Composition preserves* $\simeq$

Suppose X, Y, and Z are topological spaces. Let $f_1, f_2 \in \mathcal{C}(X,Y)$ and $g_1, g_2 \in \mathcal{C}(Y,Z)$. If $f_1 \simeq f_2$ and $g_1 \simeq g_2$, then $g_1 \circ f_1 \simeq g_2 \circ f_2$

Proof. Suppose $f_1 \simeq f_2$ and $g_1 \simeq g_2$ and let H and G be homotopies from f_1 to f_2 and from g_1 to g_2, respectively. Define $F\colon X \times I \to Y$ by

$$F(x,t) = G(H(x,t),t)$$

for all $x \in X$ and all $t \in I$. Then F is a homotopy from $g_1 \circ f_1$ to $g_2 \circ f_2$. Indeed, F is continuous as it is the composition of continuous functions and for any $x \in X$ we have

$$\begin{aligned} F(x,0) &= G(H(x,0),0) = G(f_1(x),0) = g_1(f_1(x)) \quad \text{and} \\ F(x,1) &= G(H(x,1),1) = G(f_2(x),1) = g_2(f_2(x)). \end{aligned}$$

Therefore $g_1 \circ f_1 \simeq g_2 \circ f_2$. $\square$

Sometimes we may need to work with homotopies that leave some points fixed. Thus we must establish the notion of homotopy respect to a subspace, where deformation does not occur.

Definition 2.6 *Relative homotopy*

Let X and Y be topological spaces and fix a subset $A \subseteq X$. A homotopy H between maps $f, g \in \mathcal{C}(X,Y)$ is called *homotopy relative to* A if

$$\forall x \in A, \forall t \in I: \quad H(x,t) = H(x,0).$$

We say f and g are *homotopic relative to* A.

Notice from the definition above that $H(x,t) = H(x,0)$ is equivalent to $H(x,t) = f(x)$. Moreover, we have $f|_A = g|_A$ since $H(x,1) = g(x)$ for all $x \in X$.

When the homotopy between two maps is not relative to any particular subspace, the maps are said to be *freely homotopic*. We will add the adjective *freely* for emphasis when necessary. As shorthand, we say f and g are A-homotopic if they are homotopic relative to A, which might also be denoted by $f \simeq g$ (rel A). If $A = \{p\}$, we may just write $f \simeq g$ (rel p).

Example 2.7. Suppose X is a topological space, A is a subset of X and $B \subseteq \mathbb{R}^n$ is a convex set. Let's show that any two continuous maps $f, g\colon X \to B$ are homotopic relative to A if $f|_A = g|_A$. Certainly, let $f, g \in \mathcal{C}(X,B)$

agree on A and define $H\colon X \times I \to B$ by

$$H(x,t) = f(x) + t(g(x) - f(x))$$

for all $x \in X$ and $y \in I$. (H is called the *straight-line homotopy* between f and g.) Note that H is continuous and takes values on B as B is convex. Also $H(x,0) = f(x)$ and $H(x,1) = g(x)$ for all $x \in X$. Then H is a homotopy between f and g. Finally, since $f(x) = g(x)$ for all $x \in A$ we have $H(x,t) = f(x) + t0 = f(x)$ for all $x \in X$ and $t \in I$, as desired.

2.2. Paths

We have defined homotopy of continuous functions between topological spaces. Now we consider the particular case when those continuous maps have $I = [0,1]$ as domain, that is, paths. We already know that paths can be used to study topological properties like connectedness (Section 1.2). We will give them another purpose: paths can serve as a tool to detect holes in a space. We will make this precise later on.

Definition 2.8 *Path*

A *path* in a topological space X is a continuous map from I to X. If f is a path in X, we call $x_0 = f(0)$ and $x_1 = f(1)$ the initial and final point of f, respectively. We say f is a path in X from x_0 to x_1.

Note that it is the mapping f that is the path and not its image $f([0,1])$, which is called a *curve* in X.

Remark 2.9. Any continuous map $f\colon [a,b] \to X$ can be reparametrized to get a continuous map with domain $[0,1]$. Indeed, define $g\colon [0,1] \to X$ by $g(t) = f(a + t(b-a))$ for all $t \in [0,1]$. Thus, using $[0,1]$ in Definition 2.8 is a matter of convenience.

Example 2.10. (i) (Constant path) The *constant path* at $x \in X$ is the map $e_x\colon I \to X$ defined by $e_x(t) = x$ for all $t \in I$.

(ii) (Inverse path) Let X be a topological space. For any path $f \in \mathcal{C}(I,X)$ we define the *inverse path* of f by $\overline{f}\colon I \to X : t \mapsto f(1-t)$. Note that $\overline{\overline{f}} = f$. Moreover, if $\psi\colon X \to Y$ is continuous, $\psi \circ f$ is a path in Y, and

$$\overline{\psi \circ f}(t) = \psi \circ f(1-t) = \psi \circ \overline{f}(t)$$

for all $t \in I$, whence $\overline{\psi \circ f} = \psi \circ \overline{f}$.

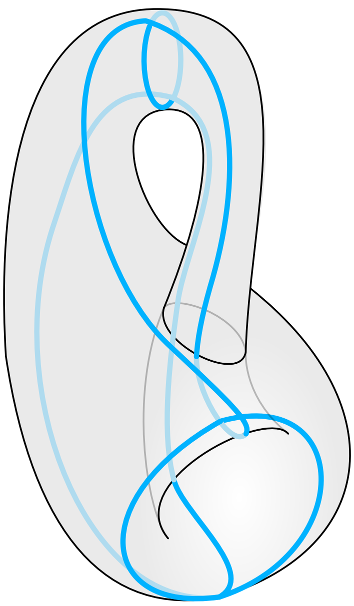


Figure 2.1: Loops on the Klein bottle. The base points are not shown.

Definition 2.11 *Loop*

A *loop* f in a topological space X is a path in X with same initial and terminal point. This common point x_0 is called the *base point* of the loop. We say that f is a loop based at x_0.

The set of loops in a space X based at a point x_0 is denoted by $\Omega_1(X, x_0)$.

We will be interested in those homotopies of continuous maps $[0,1] \to X$ that fix 0 and 1. This notion constitutes a stronger relation than mere homotopy.

Definition 2.12 *Homotopy of paths*

Suppose f and g are two paths in X. A *path-homotopy* between f and g is a homotopy from f to g relative to $\partial I = \{0,1\}$. If there exists such a homotopy, we say that f and g are *path-homotopic* and denote it by $f \sim g$.

In other words, a path-homotopy is one that fixes the endpoints of paths. Thus we may reformulate the definition as follows. Two paths f and g in X are path homotopic if and only if they have the same initial point x_0 and same final point x_1 and, in addition, there is a continuous map $H\colon I^2 \to X$ such that

$$\begin{aligned} H(s,0) &= f(s) \quad \text{and} \quad H(s,1) = g(s), \\ H(0,t) &= x_0 \quad \text{and} \quad H(1,t) = x_1, \end{aligned}$$

for every $(s,t) \in I^2$. Intuitively, we would think of s and t as space and time variables, respectively.

A *null-homotopic* loop is a one that is path-homotopic (and not just homotopic) to a constant loop.

Theorem 2.13 *$\sim$ is an equivalence relation*

Let X be a topological space. Homotopy of paths is an equivalence relation on $\mathcal{C}(I, X)$.

Proof. Notice this does not follow from Theorem 2.4 as path-homotopy is a stronger notion than homotopy alone. Nevertheless, we only need to modify the proof to verify that path-homotopies fix the endpoints of paths.

(i) (Reflexivity) Any path f in X is homotopic to itself via the constant homotopy, which trivially fixes the endpoints of f. Thus $\sim$ is reflexive.

(ii) (Symmetry) Suppose $f \sim g$ where f and g are paths in X such that $x_0 = f(0) = g(0)$ and $x_1 = f(1) = g(1)$. This means there is a homotopy H from f to g relative to $\{0,1\}$. Then $G\colon I^2 \to X : (s,t) \mapsto H(s, 1-t)$ is a homotopy from g to f which satisfies

$$G(0,t) = H(0, 1-t) = x_0 \quad \text{and} \quad G(1,t) = x_1,$$

meaning it is relative to $\{0,1\}$. Thus $g \sim f$. It follows that $\sim$ is symmetric.

(iii) (Transitivity) Let f, g and h are paths in X with same starting point x_0 and same terminal point x_1. Suppose $f \sim g$ and $g \sim h$. Let $H, G\colon I^2 \to X$ be path homotopies from f to g and from g to h, respectively. Define the map $F\colon I^2 \to Y$ by

$$F(s,t) = \begin{cases} H(s,2t) & \text{if} \quad t \in [0, \frac{1}{2}], \\ G(s,2t-1) & \text{if} \quad t \in [\frac{1}{2}, 1], \end{cases}$$

for all $s \in I$ and $t \in I$. Then F is a homotopy between f and h, by Theorem 2.4. Because H and G are relative to $\{0,1\}$, we also have

$$F(0,t) = x_0 \quad \text{and} \quad F(1,t) = x_1,$$

for all $t \in I$. Therefore $f \sim h$, whence we conclude $\simeq$ is transitive.

The proof is complete. $\square$

Remark 2.14. We call the equivalence classes under $\sim$ *path-homotopy* classes. If f is a path we denote its path-homotopy class by $[f]$.

Lemma 2.15

Suppose f is a path in X and $\varphi\colon I \to I$ is a continuous map that fixes 0 and 1. Then $f \circ \varphi \sim f$. We call $f \circ \varphi$ a *reparametrization* of f.

Proof. Define $H\colon I \times I \to I$ by

$$H(s,t) = s + t(\varphi(s) - s).$$

That is, H is the straight line homotopy from the identity map on I to φ. See Example 2.7. Let us verify that the continuous map $f \circ H$ is a path-homotopy from f to $f \circ \varphi$. We have

$$f(H(s,0)) = f(s), \quad f(H(s,1)) = f(\varphi(s)),$$

and

$$f(H(0,t)) = f(t\varphi(0)) = f(0), \quad f(H(1,t)) = f(1).$$

The claim follows. $\square$

Multiplication of Paths

Definition 2.16 *Product of paths*

Let X be a topological space. Suppose f and g are paths in X with $f(1) = g(0)$. In this case we say f and g are *composable*. Define the *product* of f and g as the map $f \cdot g\colon I \to X$ given by

$$f \cdot g(t) = \begin{cases} f(2t) & \text{if} \quad t \in [0, \frac{1}{2}], \\ g(2t-1) & \text{if} \quad t \in [\frac{1}{2}, 1]. \end{cases}$$

It is clear that $f \cdot g$ is a path in X. Note that $f \cdot g$ is continuous since $f(1) = g(0)$, thanks to the gluing lemma, Theorem 1.16. Figure 2.2 illustrates the product of two paths.

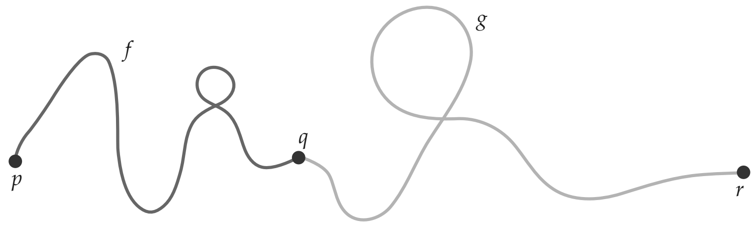


Figure 2.2: A path can be thought of as the motion of a point. The moving point in $f \cdot g$ starts at $p = f(0)$, then traverses f at twice speed up to $q = f(1) = g(0)$, where the point now follows g also at twice speed and finally arrives to $r = g(1)$.

Observe that we cannot formally define $\cdot$ as an operation on the set $\mathcal{C}(I, X)$. This is due to the constrain that paths must agree on one common endpoint so that their product is continuous.[4] Note $\mathcal{C}(I, X)$ is not even closed under $\cdot$ if we not require $f(1) = g(0)$.

[4] This problem can be overcome by working on $\Omega(X, x_0)$ where all paths have the same initial and final point.

The following theorem tell us that $\cdot$ is well defined on the set of homotopy classes of paths. This bring us closer to one of our immediate objectives: to construct a set endowed with a well-defined group operation.

Theorem 2.17 *Product of paths preserves* $\sim$

Let X be a topological space. Let $f, g \in \mathcal{C}(I, X)$ be paths such that $f(1) = g(0)$. Define the product of path-homotopy classes by

$$[f] \cdot [g] = [f \cdot g].$$

Then $\cdot$ is well defined on $\mathcal{C}(I, X)/\sim$. In other words, if $f \sim \hat{f}$ and $g \sim \hat{g}$, then $f \cdot g \sim \hat{f} \cdot \hat{g}$ whenever the products are defined.

Remark 2.18. Unfortunately, the set $\mathcal{C}(I, X)/\sim$ is not a group under product of path-homotopy classes. Notice the additional condition $f(1) = g(0)$. Thus $\cdot$ is not defined for all elements of $\mathcal{C}(I, X)/\sim$ but only for those whose representatives agree on a common endpoint. We can remediate this situation by enforcing all paths to be loops based at a common point. This is the topic of the next section.

Proof. Suppose $f \sim \hat{f}$ and $g \sim \hat{g}$. Denote $x_0 = f(0)$, $x_1 = f(1)$ and $x_2 = g(1)$. Suppose also the product $f \cdot g$ is defined. Let F be a path-homotopy between f and $\hat{f}$. Let G be a path-homotopy between g and $\hat{g}$. Define $H \colon I^2 \to X$ by

$$H(s, t) = \begin{cases} F(2s, t) & \forall s \in \left[0, \frac{1}{2}\right], \forall t \in I, \\ G(2s - 1, t) & \forall s \in \left[\frac{1}{2}, 1\right], \forall t \in I. \end{cases}$$

Note H takes a unique value when $s = 1/2$ as $F(1, t) = x_1 = G(0, t)$. Thus H is well defined and it is continuous by the pasting lemma. Also $\hat{f}$ and

$\hat{g}$ are composable because $\hat{f}(1) = f(1) = g(0) = \hat{g}(0)$. Let $s \in I$. If $s \in \left[0, \frac{1}{2}\right]$, by definition of path multiplication we get

$$\begin{aligned} H(s,0) &= F(2s,0) = f(2s) = f \cdot g(s) \quad \text{and} \\ H(s,1) &= F(2s,1) = \hat{f}(2s) = \hat{f} \cdot \hat{g}(s). \end{aligned}$$

Similarly, if $s \in \left[\frac{1}{2}, 1\right]$,

$$\begin{aligned} H(s,0) &= G(2s-1,0) = g(2s-1) = f \cdot g(s) \quad \text{and} \\ H(s,1) &= G(2s-1,1) = \hat{g}(2s-1) = \hat{f} \cdot \hat{g}(s). \end{aligned}$$

Therefore

$$\forall s \in I : \quad H(s,0) = f \cdot g(s) \quad \text{and} \quad H(s,1) = \hat{f} \cdot \hat{g}(s),$$

so H is a homotopy between $f \cdot g$ and $\hat{f} \cdot \hat{g}$. It is in fact a path-homotopy because it leaves the endpoints fixed: $H(0,t) = F(0,t) = x_0$ and $H(1,t) = G(1,t) = x_2$ for all $t \in I$. □

Not only the consideration of path-homotopy classes guarantees well-definiteness of path multiplication, but also its associativity as the next theorem shows. Since we aim to construct a group endowed with such an operation, we also need an identity element and inverses. Although the next theorem is a step further, we also need closure under the group operation which we still do not have.

Theorem 2.19 *Properties of the product of path-homotopy classes*

Let X be a topological space and $f, g, h \in \mathcal{C}(I, X)$. Suppose $f(0) = p$ and $f(1) = q$.

(i) (Associativity) If either $[f] \cdot ([g] \cdot [h])$ or $([f] \cdot [g]) \cdot [h]$ is defined so is the other, and

$$[f] \cdot ([g] \cdot [h]) = ([f] \cdot [g]) \cdot [h].$$

(ii) (Identities) The constant maps e_p and e_q satisfy

$$[f] \cdot [e_q] = [f] \quad \text{and} \quad [e_p] \cdot [f] = [f].$$

(iii) (Inverses) The inverse path $\overline{f}$ satisfies

$$[f] \cdot [\overline{f}] = [e_p] \quad \text{and} \quad [\overline{f}] \cdot [f] = [e_q].$$

Proof. (i) If $[f] \cdot ([g] \cdot [h])$ is defined, then h starts at $g(1)$ and g starts at $f(1)$. Thus $(f \cdot g) \cdot h$ exists, and so $([f] \cdot [g]) \cdot [h]$ is also defined. The argument also works the other way around. Now, note $[f] \cdot ([g] \cdot [h]) = ([f] \cdot [g]) \cdot [h]$ is equivalent to

$$f \cdot (g \cdot h) \sim (f \cdot g) \cdot h,$$

so it is enough to prove this statement. In regard to the description made at Figure 2.2, the moving point in the path $f \cdot (g \cdot h)$ first follows f at half normal speed, and then follows both g and h at quadruple speed. In contrast, the path $(f \cdot g) \cdot h$ first follows both f and g at quadruple speed, and last h at twice the normal rate. Thus, $f \cdot (g \cdot h)$ is a reparametrization of $(f \cdot g) \cdot h$ and vice versa. Indeed, consider the continuous map

$$\varphi \colon I \to I \ : \ t \mapsto \begin{cases} \frac{t}{2} & \text{if } t \in \left[0, \frac{1}{2}\right], \\ t - \frac{1}{4} & \text{if } t \in \left[\frac{1}{2}, \frac{3}{4}\right], \\ 2t - 1 & \text{if } t \in \left[\frac{3}{4}, 1\right]. \end{cases}$$

Then we have $f \cdot (g \cdot h) \circ \varphi = (f \cdot g) \cdot h$. Since φ fixes 0 and 1, Lemma 2.15 implies that $f \cdot (g \cdot h) \sim (f \cdot g) \cdot h$.

(ii) Let us prove $e_q \cdot f \sim f$. The map $H \colon I^2 \to X$ defined by

$$H(s,t) = \begin{cases} f\left(\frac{2s-t}{2-t}\right) & \text{if } t \in [0, 2s], \\ p & \text{if } t \in [2s, 1], \end{cases}$$

is continuous by the gluing lemma, Theorem 1.16, and satisfies

$$H(s,0) = f(s), \quad H(s,1) = e_p \cdot f(s), \quad H(0,t) = p, \quad H(1,t) = q.$$

Thus, H is a path-homotopy from f to $e_p \cdot f$. Hence $[f] \cdot [e_q] = [f]$. Following a similar reasoning we obtain $[e_p] \cdot [f] = [f]$.

(iii) The map $H \colon I^2 \to X$ defined by

$$H(s,t) = \begin{cases} f(2s) & \text{if} s \in \left[0, \frac{t}{2}\right], \\ f(t) & \text{if} s \in \left[\frac{t}{2}, 1 - \frac{t}{2}\right], \\ f(2-2s) & \text{if } s \in \left[1 - \frac{t}{2}, 1\right] \end{cases}$$

is a path-homotopy from e_p to $f \cdot \overline{f}$. Thus $[f] \cdot [\overline{f}] = [e_p]$. Using the same homotopy but with the roles of f and $\overline{f}$ interchanged it follows analogously that $[\overline{f}] \cdot [f] = [e_q]$.

The proof is complete. □

2.3. The Fundamental Group of a Pointed Space

The fundamental group is an algebraic invariant that we can associate to any topological space. Studying the algebraic situation give us information about the topological one. The following theorem is the one we have been aiming at. It remediates the problem we left unsolved in the last section by only considering those paths that start and end at the same point: loops.

Theorem 2.20

For any topological space X and any $x_0 \in X$, the set $\Omega_1(X, x_0)/\sim$ is a group endowed with the operation of multiplication of path-homotopy classes.

Proof. The fact that $\cdot$ is well defined follows from Theorem 2.17. Associativity, the existence of an identity, and the existence of the inverse of each path class follow as a particular case of Theorem 2.19. $\square$

Definition 2.21 *Fundamental group*

Let X be a topological space and x_0 a point of X. The *fundamental group* of X with base point x_0, denoted $\pi_1(X, x_0)$, is the set $\Omega_1(X, x_0)/\sim$ endowed with the operation of multiplication of path-homotopy classes:

$$[\alpha] \cdot [\beta] = [\alpha \cdot \beta].$$

The fundamental group of a topological space was introduced by Henri Poincaré. The symbol π is also due to Poincaré. Another name is the first homotopy group, which is the reason for the subindex 1 in the notation. In fact, there is an infinite sequence of groups $\pi_n(X, x_0)$ with $n \in \mathbb{Z}^+$. The definition of the higher homotopy groups was due to Hurewicz.

2.3.1. The Role of Base Point

The fundamental group of a topological space greatly depends on the base point chosen in the space. If another base point is chosen in the same path-connected component, the resulting fundamental groups are isomorphic. However, the fundamental groups of a space at base points that belong to different path-connected components may have no relationship to each other. In fact, the group $\pi_1(X, p)$ of a space X at a point $p \in X$, only gives information about the path-connected component to which p belongs. This is the reason why the fundamental group is mainly used to study path-connected topological spaces.

The next result shows that there is a canonical way to change the base point to another point in the same path-connected component. See Figure 2.3.

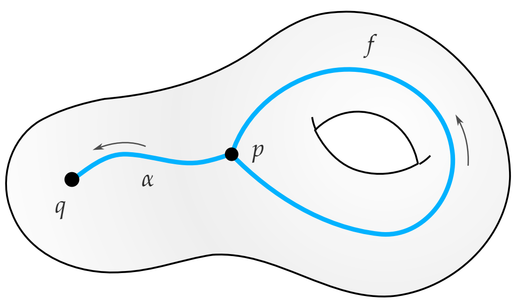


Figure 2.3: Change of base point. In order to change the base point from p to q, we fix a path α from p to q. For any loop f based at p, we traverse as follows: from q to p via $\overline{\alpha}$, then follow f from p back to p, and finally return to q via α. Note the blue loop encloses a hole.

Theorem 2.22 *Change of base point*

Let p and q be two points of a topological space X. Suppose α is a path in X from p to q. Define $\Phi_\alpha\colon \pi_1(X,p) \to \pi_1(X,q)$ by

$$\Phi_\alpha[f] = [\overline{\alpha}] \cdot [f] \cdot [\alpha].$$

Then

(i) Φ_α is a group-isomorphism whose inverse is $\Phi_\alpha^{-1} = \Phi_{\overline{\alpha}}$,

(ii) the groups $\pi_1(X,p)$ and $\pi_1(X,q)$ are isomorphic,

(iii) Φ_{e_p} is the identity map on $\pi_1(X,p)$,

(iv) if α and β are path-homotopic, then $\Phi_\alpha = \Phi_\beta$,

(v) if β is a path from q to r, then $\Phi_{\alpha\cdot\beta} = \Phi_\beta \circ \Phi_\alpha$, i.e., the diagram

$$\begin{tikzcd}
\pi_1(X,p) \arrow[r, "\Phi_\alpha"] \arrow[dr, "\Phi_{\alpha\cdot\beta}"'] & \pi_1(X,q) \arrow[d, "\Phi_\beta"] \\
 & \pi_1(X,r)
\end{tikzcd}$$

commutes.

Remark 2.23. Due to Theorem 2.22, sometimes the imprecise notation $\pi_1(X)$ is used. However, we must be careful to specify the base point even when working in the same path-connected component. The reason is that, even tough $\pi_1(X,p) \cong \pi_1(X,q)$ when p,q lie in the same path-component, the isomorphism may not be canonical. This means that different paths from p to q may give rise to different isomorphisms.

Proof. First note that $\Phi_\alpha[f]$ does indeed define an element of $\pi_1(X,q)$ for any $f \in \Omega(X,p)$.

(i) If α and β are path-homotopic, $[\alpha] = [\beta]$. Thus, for any $f \in \Omega(X,p)$,

$$\Phi_\alpha[f] = [\overline{\alpha}] \cdot [f] \cdot [\alpha] = [\overline{\beta}] \cdot [f] \cdot [\beta] = \Phi_\beta[f].$$

Hence $\Phi_\alpha = \Phi_\beta$.

(ii) Let us first show that Φ_α is a group-homomorphism. If f and f' are any loops based at p, then

$$\begin{aligned}
\Phi_\alpha[f] \cdot \Phi_\alpha[f'] &= ([\overline{\alpha}] \cdot [f] \cdot [\alpha]) \cdot ([\overline{\alpha}] \cdot [f'] \cdot [\alpha]) \\
&= [\overline{\alpha}] \cdot [f] \cdot [e_p] \cdot [f'] \cdot [\alpha] \\
&= [\overline{\alpha}] \cdot [f] \cdot [f'] \cdot [\alpha] \\
&= \Phi_\alpha([f] \cdot [f']).
\end{aligned}$$

This follows from Theorem 2.19. On the other hand,

$$\Phi_{\overline{\alpha}} \circ \Phi_\alpha[f] = [\alpha] \cdot [\overline{\alpha}] \cdot [f] \cdot [\alpha] \cdot [\overline{\alpha}] = [e_p] \cdot [f] \cdot [e_p] = [f]$$

and if g is any loop based at q,

$$\Phi_\alpha \circ \Phi_{\overline{\alpha}}[g] = [\overline{\alpha}] \cdot [\alpha] \cdot [g] \cdot [\overline{\alpha}] \cdot [\alpha] = [e_q] \cdot [g] \cdot [e_q] = [g].$$

This shows $\Phi_{\overline{\alpha}}$ is the inverse of Φ_α, so Φ_α is an isomorphism.

(iii) This is an immediate consequence of (i).

(iv) Let f be a loop based at p. Then

$$\begin{aligned}\Phi_{\alpha\cdot\beta}[f] &= [\overline{\alpha\cdot\beta}][f][\alpha\cdot\beta]\\ &= [\overline{\beta}\cdot\overline{\alpha}]\cdot[f]\cdot[\alpha\cdot\beta]\\ &= [\overline{\beta}]\cdot[\overline{\alpha}]\cdot[f]\cdot[\alpha]\cdot[\beta]\\ &= \Phi_\beta(\Phi_\alpha[f])\end{aligned}$$

and the claim follows.

(v) This is a consequence of Theorem 2.19. Note that

$$\Phi_{e_p}[f] = [\overline{e_p}]\cdot[f]\cdot[e_p] = [e_p]\cdot[f] = [f].$$

The proof is complete. □

Definition 2.24 *Simply connected space*

A topological space X is *simply connected* if X is path-connected and $\pi_1(X,p)$ is trivial for some $p \in X$.

Note that if a space is simply connected, then $\pi_1(X,p)$ is trivial for any $p \in X$. We express the fact that $\pi_1(X,p)$ is the trivial group by writing $\pi_1(X,p) = 0$.

The idea behind simply connectedness is that any loop in X can be continuously deformed to a point, i.e., any loop is null-homotopic to a constant loop. For instance, the torus is not simply connected, since some loops enclose a hole and thus cannot be shrunk to a point. See Figure 2.3.

2.3.2. Circular Loops

Recall that the circle S^1 is the set of points $z \in \mathbb{C}$ such that $|z| = 1$. The map $\omega\colon I \to S^1 : t \mapsto \exp(2\pi i t)$ is a loop in S^1. It traverses once around the circle counterclockwise and maps 0 and 1 in I to $1 \in S^1$ (identified with $(1,0)$ in $\mathbb{R}^2$).

A continuous map ℓ from S^1 to a topological space X is a *circular loop* at $p \in X$ if $\ell(1) = p$. The composition $\ell \circ \omega\colon I \to X$ is a usual loop at X based at p, in the sense of Definition 2.8. In fact, any loop in X can be factored in this way through ω.

Proposition 2.25 *Loops factor through circular loops*

Let f be a loop in a space X. There exists a unique map $\tilde{f}\colon S^1 \to X$ such that $f = \tilde{f} \circ \omega$, i.e., the diagram

$$\begin{tikzcd} I \arrow[r, "f"] \arrow[d, "\omega"'] & X \\ S^1 \arrow[ur, "\tilde{f}"'] & \end{tikzcd}$$

commutes. We call $\tilde{f}$ the *circle representative* of f.

Proof. We know $S^1 \cong I/_{\{0,1\}}$ through the homeomorphism

$$\varphi\colon S^1 \to I/_{\{0,1\}} : \exp(2\pi i t) \mapsto [t].$$

Let ψ be the map $I/_{\{0,1\}} \to X : [t] \mapsto f(t)$, which is well-defined because the classes in $I/_{\{0,1\}}$ consist of the singletons $\{t\}$ for $0 < t < 1$ and the class $\{0,1\}$, whose image under f is the base point. Then we define $\tilde{f} = \psi \circ \varphi$. Therefore,

$$\tilde{f} \circ \omega(t) = \psi \circ \varphi(\exp(2\pi i t)) = \psi([t]) = f(t)$$

for any $t \in I$. ∎

Lemma 2.26

Let f and g be two loops in a space X that are based at $p \in X$. Then $f \sim g$ if and only if $\tilde{f} \simeq \tilde{g}$ (rel $(1,0)$).

Proof. ($\Rightarrow$) Suppose $\mathcal{H}\colon f \sim g$. We can regard $\mathcal{H}$ as a family of continuous functions $(h_t\colon I \to X)_{t\in I}$ such that $h_0 = f$, $h_1 = g$, and $h_t(0) = h_t(1) = p$ for all $t \in I$. It is clear that h_t is a loop for every $t \in I$. By Proposition 2.25, $h_t = \widetilde{h_t} \circ \omega$ for every $t \in I$. Therefore

$$\left(\widetilde{h_t}\colon S^1 \to X\right)_{t\in I}$$

gives a homotopy from $\widetilde{h_0} = \tilde{f}$ to $\widetilde{h_1} = \tilde{g}$. Moreover, $\widetilde{h_t}(1,0) = \widetilde{h_t}(\omega(1)) = h_t(1) = p$. Hence $\tilde{f} \simeq \tilde{g}$ relative to $(0,1)$.

($\Leftarrow$) Suppose $\mathcal{H}\colon \tilde{f} \simeq \tilde{g}$ (rel $(0,1)$). Regard $\mathcal{H}$ as a family $(h_t\colon S^1 \to X)_{t\in I}$ such that $h_0 = \tilde{f}$, $h_1 = \tilde{g}$, and $h_t(1,0) = p$ for all $t \in I$. Consider the family $(h_t \circ \omega\colon I \to X)_{t\in I}$, and note that

$$h_0 \circ \omega = \tilde{f} \circ \omega = f, \quad h_1 \circ \omega = \tilde{g} \circ \omega = g$$

and $h_t \circ \omega(0) = h_t(1,0) = p$. This shows f and g are path homotopic. ∎

2.3.3. Homomorphisms Induced by Continuous Maps

Let X and Y be topological spaces. We write $f\colon (X, x_0) \to (Y, y_0)$ to denote that $f\colon X \to Y$ is a continuous map such that $y_0 = f(x_0)$, where $x_0 \in X$. We say f is a *map of pointed spaces*. This map induces a map

$$f_\#\colon \Omega(X, x_0) \to \Omega(Y, y_0) \ : \ \alpha \mapsto f \circ \alpha$$

from the set of loops in X based at x_0 to the set of loops in Y based at y_0.

Proposition 2.27

Let $f\colon (X, x_0) \to (Y, y_0)$ be a map of pointed spaces. Let $\alpha, \beta \in \Omega(X, x_0)$ be homotopic loops. Then $f_\#(\alpha) \sim f_\#(\beta)$.

Proof. Let $\mathcal{H}\colon \alpha \sim \beta$. Let us see $f \circ \mathcal{H}$ is a homotopy of paths from $f \circ \alpha$ to $f \circ \beta$. We have

$$\begin{aligned} f \circ \mathcal{H}(s, 0) &= f(\alpha(s)) = f \circ \alpha(s), \\ f \circ \mathcal{H}(s, 1) &= f(\beta(s)) = f \circ \beta(s), \quad \text{and} \\ f \circ \mathcal{H}(0, t) &= y_0 = f \circ \mathcal{H}(1, t) \end{aligned}$$

for all $s, t \in I$. This proves the claim. □

The map $f_\#$ induces in turn a well-defined map of fundamental groups.

Theorem 2.28

Let $f\colon (X, x_0) \to (Y, y_0)$ be a map of pointed spaces. The map

$$f_*\colon \pi_1(X, x_0) \to \pi_1(Y, y_0) \ : \ [\alpha] \mapsto [f \circ \alpha]$$

is well-defined.

Proof. Follows from Proposition 2.25. □

Proposition 2.29

Let $f, g\colon (X, x_0) \to (Y, y_0)$ be maps of pointed spaces that are homotopic relative to x_0. Then $f_* = g_*$.

Proof. Let $\mathcal{H}\colon f \simeq g$ (rel x_0). Let's see that $f \circ \alpha \sim g \circ \alpha$ for any $\alpha \in \Omega(X, x_0)$. Define $\mathcal{G}\colon I^2 \to Y$ by $(s, t) \mapsto \mathcal{H}(\alpha(s), t)$, and note that

$$\begin{aligned} \mathcal{G}(s, 0) &= \mathcal{H}(\alpha(s), 0) = f(\alpha(s)) = f \circ \alpha(s), \\ \mathcal{G}(s, 1) &= \mathcal{H}(\alpha(s), 1) = g(\alpha(s)) = g \circ \alpha(s), \quad \text{and} \\ \mathcal{G}(0, t) &= \mathcal{H}(\alpha(0), t) = y_0 = \mathcal{H}(\alpha(1), t) = \mathcal{G}(1, t). \end{aligned}$$

Hence $\mathcal{G}\colon f \circ \alpha \sim g \circ \alpha$. Therefore $f_*[\alpha] = [f \circ \alpha] = [g \circ \alpha] = g_*[\alpha]$ for all $\alpha \in \Omega(X, x_0)$, whence $f_* = g_*$. The proof is complete. □

As a result of Theorem 2.28, any continuous map $f\colon X \to Y$ induces a well-defined map $f_*\colon \pi_1(X,p) \to \pi(Y, f(p))$ for any point $p \in X$, simply by setting $f_*[\alpha] = [f \circ \alpha]$, for $\alpha \in \Omega(X,p)$. Moreover, f_* preserves the group structure.

Theorem 2.30

Let $f\colon X \to Y$ be a continuous map and fix $p \in X$. Then f_* is a group-homomorphism.

Proof. Let $\alpha \sim \beta$ in $\Omega(X, x_0)$. It is easy to see that $f \circ (\alpha \cdot \beta) = (f \circ \alpha) \cdot (f \circ \beta)$, as follows from the definition of product of paths. Then

$$\begin{aligned} f_*([\alpha]\cdot[\beta]) &= f_*[\alpha \cdot \beta] \\ &= [f \circ (\alpha \cdot \beta)] \\ &= [(f\circ\alpha)\cdot(f\circ\beta)] \\ &= f_*[\alpha]\cdot f_*[\beta]. \end{aligned}$$

This proves the result. □

Moreover, the induced morphism has functorial properties, as stated in the next result. We will discuss what functorial properties are in the next chapter when we arrive at the definition of functor.

Theorem 2.31 *Functorial properties of**

Let $f\colon (X,p) \to (Y,q)$ and $g\colon (Y,q) \to (Z,r)$ be maps of pointed spaces. Then

(i) $(g \circ f)_* = g_* \circ f_*$

(ii) $\left(\mathrm{Id}_{(X,p)}\right)_* = \mathrm{Id}_{\pi_1(X,p)}$

Proof. (i) For any $\alpha \in \Omega(X,p)$ we have

$$(g\circ f)_*[\alpha] = [(g\circ f)\circ\alpha] = [g\circ(f\circ\alpha)] = g_*[f\circ\alpha] = g_*\left(f_*[\alpha]\right).$$

Thus $(g\circ f)_* = g_* \circ f_*$.

(ii) This is obvious.

□

As a result, the next corollary highlights one of the important aspects of the fundamental group: homeomorphic spaces have isomorphic fundamental groups. Therefore, spaces with non-isomorphic fundamental groups are not homeomorphic.

Corollary 2.32

If $f\colon (X,p) \to (Y, f(p))$ is a homeomorphism, then

$$f_*\colon \pi_1(X,p) \to \pi_1(Y, f(p))$$

is a group-isomorphism.

Proof. Suppose f is a homeomorphism with inverse g. Then $f \circ g = \mathrm{Id}_Y$ and $g \circ f = \mathrm{Id}_X$. By the functorial properties of $*$, we obtain

$$f_* \circ g_* = (f \circ g)_* = \mathrm{Id} \quad \text{and} \quad g_* \circ f_* = (g \circ f)_* = \mathrm{Id},$$

where Id denotes the identity of the respective groups. Therefore f_* is a group-isomorphism with inverse g_*. □

It should be noted that although the notation f_* is used for both induced maps $\pi_1(X,p) \to \pi_1(Y, f(p))$ and $\pi_1(X,q) \to \pi_1(Y, f(q))$, they are not equal unless $p = q$. Thus, the notation f_* is a bit ambiguous since does not make any reference to the base point. A more accurate notation would be $(f_p)_*$.

Theorem 2.33

Let $f\colon X \to Y$ be a continuous map and fix two points $p, q \in X$. Suppose γ is a path from p to q. Then $f_* \circ \Phi_\gamma = \Phi_{f \circ \gamma} \circ f_*$, that is, the diagram

$$\begin{tikzcd}
\pi_1(X,p) \arrow[r, "f_*"] \arrow[d, "\Phi_\gamma"'] & \pi_1(Y, f(p)) \arrow[d, "\Phi_{f\circ\gamma}"] \\
\pi_1(X,q) \arrow[r, "f_*"'] & \pi_1(Y, f(q))
\end{tikzcd}$$

commutes.

Proof. Let $\alpha \in \Omega(X,p)$. We have

$$\begin{aligned}
[f \circ (\overline{\gamma} \cdot \alpha \cdot \gamma)] &= [(f \circ \overline{\gamma}) \cdot (f \circ (\alpha \cdot \gamma))] \\
&= [f \circ \overline{\gamma}] \cdot [(f \circ \alpha) \cdot (f \circ \gamma)] \\
&= [f \circ \overline{\gamma}] \cdot [f \circ \alpha] \cdot [f \circ \gamma].
\end{aligned}$$

Since $f \circ \overline{\gamma} = \overline{f \circ \gamma}$, as shown in Example 2.10 (ii), we obtain

$$f_*[\overline{\gamma} \cdot \alpha \cdot \gamma] = [\overline{f \circ \gamma}] \cdot f_*[\alpha] \cdot [f \circ \gamma],$$

which is equivalent to $f_*\left(\Phi_\gamma[\alpha]\right) = \Phi_{f\circ\gamma}\left(f_*[\alpha]\right)$. We conclude by the arbitrariness of α. □

The next result states that the fundamental group is well behaved under products, up to isomorphism. In fact, up to canonical isomorphism.

Theorem 2.34 *Fundamental group of a product space*

Let $(X_1, x_1), \ldots, (X_n, x_n)$ be pointed spaces. Then

$$\pi_1(X_1 \times \cdots \times X_n, (x_1, \ldots, x_n)) \cong \pi_1(X_1, x_1) \times \cdots \times \pi_1(X_n, x_n).$$

Proof. Note that $(X_1 \times \cdots \times X_n, (x_1, \ldots, x_n))$ is itself a pointed space. Let $p_i \colon X_1 \times \cdots \times X_n \to X_i$ be the projection onto the ith factor, for each $1 \leq i \leq n$. Each projection induces a well-defined map

$$p_{i*} \colon \pi_1(X_1 \times \cdots \times X_n, (x_1, \ldots, x_n)) \to \pi_1(X_i, x_i)$$

for all $1 \leq i \leq n$. Now define

$$\begin{aligned} \phi \colon \pi_1(X_1 \times \cdots \times X_n, (x_1, \ldots, x_n)) &\to \pi_1(X_1, x_1) \times \cdots \times \pi_1(X_n, x_n) \\ [\alpha] &\mapsto (p_{1*}[\alpha], \ldots, p_{n*}[\alpha]) \end{aligned}$$

Let us see that ϕ is a group-isomorphism. Since each p_{i*} is a group-homomorphism, so is ϕ. Injectivity of ϕ follows from the fact its kernel is trivial. Indeed, suppose $\phi[\alpha]$ equals the identity of

$$\pi_1(X_1, x_1) \times \cdots \times \pi_1(X_n, x_n),$$

namely $([e_{x_1}], \ldots, [e_{x_n}])$. Expressing α in terms of its component functions as $\alpha(t) = (\alpha_1(t), \ldots, \alpha_n(t))$, we have

$$(p_{1*}[\alpha], \ldots, p_{n*}[\alpha]) = ([e_{x_1}], \ldots, [e_{x_n}]),$$

whence

$$p_{i*}[\alpha] = [p_i \circ \alpha] = [\alpha_i] = [e_{x_i}]$$

for all $1 \leq i \leq n$. Now we can take homotopies $H_i \colon \alpha_i \sim e_{x_i}$ for each $1 \leq i \leq n$. Define $H \colon I^2 \to X_1 \times \cdots \times X_n$ by

$$(s, t) \mapsto (H_1(s, t), \ldots, H_n(s, t)).$$

Note that H is continuous, and

$$\begin{aligned} H(s, 0) &= (H_1(s, 0), \ldots, H_n(s, 0)) = (\alpha_1(s), \ldots, \alpha_n(s)) = \alpha(s), \\ H(s, 1) &= (H_1(s, 1), \ldots, H_n(s, 1)) = (e_{x_1}(s), \ldots, e_{x_n}(s)) = e_{(x_1, \ldots, x_n)}, \\ H(0, t) &= (H_1(0, t), \ldots, H_n(0, t)) = (x_1, \ldots, x_n) = H(1, t). \end{aligned}$$

This shows that H is a path-homotopy from α to $e_{(x_1, \ldots, x_n)}$. Hence $[\alpha]$ is the identity of $\pi_1(X_1 \times \cdots \times X_n, (x_1, \ldots, x_n))$. Therefore, $\operatorname{Ker} \alpha = 0$ as indicated earlier. Let us conclude by showing that ϕ is surjective. Take $([\gamma_1], \ldots, [\gamma_n]) \in \pi_1(X_1, x_1) \times \cdots \times \pi_n(X_1, x_n)$. Define

$$\gamma \colon I \to X_1 \times \cdots \times X_n \ : \ t \mapsto (\gamma_1(t), \ldots, \gamma_n(t)).$$

Notice γ is a loop in the product space $X_1 \times \cdots \times X_n$ based at $(x_1, \ldots, x_n)$. Moreover, we have

$$\phi[\gamma] = (p_{1*}[\gamma], \ldots, p_{n*}[\gamma]) = ([p_1 \circ \gamma], \ldots, [p_n \circ \gamma]) = ([\gamma_1], \ldots, [\gamma_n]).$$

This concludes the proof. $\square$

2.3.4. The Fundamental Groups of $\mathbf{S}^1$ and $\mathbf{T}^n$

The purpose of this short subsection is to highlights two basic results:

(i) **The fundamental group of** S^1. Recall that S^1 is path-connected. The fundamental group of the circle is

$$\pi_1(S^1) \cong \mathbb{Z}.$$

The proof of this fact is rather involved and can be found in [18].

(ii) **The fundamental group of** T^2. We know that T^n is path-connected, as follows from Proposition 1.26. By Theorem 2.34, the fundamental group of T^n is

$$\pi_1(T^n) \cong \pi_1(S^1) \times \cdots \times \pi_1(S^1) \cong \mathbb{Z}^n.$$

In particular, $\pi_1(T^2) \cong \mathbb{Z}^2$.

Although the Seifert-Van Kampen theorem is a powerful tool for computing fundamental groups of various spaces, it is beyond the scope of this work. The interested reader may refer to [18] or [32] for further details.

2.4. Homotopy Equivalence

In this last section we present a criterion under which a continuous map induces an isomorphism of fundamental groups.

Definition 2.35 *Homotopy equivalence*

Let X and Y be two topological spaces. We say that X is *homotopy equivalent* to Y, denoted $X \simeq Y$, if there exists a pair of continuous maps $f\colon X \to Y$ and $g\colon Y \to X$ such that

$$f \circ g \simeq \mathrm{Id}_Y \quad \text{and} \quad g \circ f \simeq \mathrm{Id}_X.$$

Equivalently, we say X and Y have the same *homotopy type*. The pair of maps f and g is called a *homotopy equivalence* between X and Y.

A continuous map $\psi\colon X \to Y$ for which there exists a continuous map $\varphi\colon Y \to X$ such that $\psi \circ \varphi \simeq \mathrm{Id}_Y$ and $\psi \circ \varphi \simeq \mathrm{Id}_X$ is called a *homotopy inverse* for φ. It is evident that homeomorphic spaces are homotopy equivalent. A *homotopy invariant* is a topological property that is preserved under homotopy equivalences.

Proposition 2.36

$\simeq$ is an equivalence relation on the class of all topological spaces. The classes of spaces that are homotopy equivalent are called *homotopy types*.

Proof. Reflexivity and symmetry are straightforward. In order to prove transitivity, suppose $X \simeq Y$ and $Y \simeq Z$. Then there exist continuous maps $f\colon X \to Y$, $g\colon Y \to X$, $h\colon Y \to Z$, and $\delta\colon Z \to Y$ such that

$$g \circ f \simeq \mathrm{Id}_X, \quad f \circ g \simeq \mathrm{Id}_Y, \quad \delta \circ h \simeq \mathrm{Id}_Y, \quad \text{and} \quad h \circ \delta \simeq \mathrm{Id}_Z.$$

Therefore

$$(g \circ \delta) \circ (h \circ f) = g \circ (\delta \circ h) \circ f \sim g \circ \mathrm{Id}_Y \circ f = g \circ f \simeq \mathrm{Id}_X$$

and

$$(h \circ f) \circ (g \circ \delta) = h \circ (f \circ g) \circ \delta \simeq h \circ \mathrm{Id}_Y \circ \delta = h \circ \delta \simeq \mathrm{Id}_Z.$$

Thus, $X \simeq Z$ by definition. □

We conclude this chapter with the following result, whose proof can be found in [18].

Theorem 2.37 $X \simeq Y \implies \pi_1(X, x) \cong \pi_1(Y, y)$

Let $f\colon X \to Y$ be a homotopy invertible map between spaces X and Y. Fix any point $x \in X$. Then

$$f_*\colon \pi_1(X, x) \to \pi_1(Y, f(x))$$

is a group-isomorphism.

Chapter 3

Basic Category Theory

Category theory is essentially a theory about functions, or rather an abstraction of the widely known concept of function. Yet, this abstraction is fundamentally different from its set theoretical definition.

The importance of the subject must be emphasized. It was born in the field of algebraic topology in an attempt to define the idea of *natural transformation* [27, 10]. However, it has reached and found applications in areas far beyond algebraic topology. In fact, it can be used as a foundational theory for mathematics and thus it can replace set theory on this purpose. Fundamentally, the importance and popularity of this theory is due to the fact that functions are everywhere.

In what follows we do not expose a treatment of category theory as an alternative foundational framework for mathematics, nor we aim to present mathematical foundations for category theory. Our objective is to present the theory from an axiomatic point of view by leaving some things undefined. For instance we rely on the (undefined) notion of *collection of objects*, which must not be taken as a synonym of set.[1] A formal treatment of the foundations of category theory can be found in [20, 17, 11, 5, 24, 21]. Discussing such matters is beyond the scope of this work. Nevertheless, a final comment must be said. The Zermelo-Frankel axiomatic set theory together with the axiom of choice (ZFC) is the most popular theory to provide foundations to do *most* of mathematics, but it is not powerful enough to develop category theory because we will need to talk about *large* structures like the "category of all sets," which have no place in ZFC. This justifies the approach we have taken here.

[1] A more appropriate approach would be to work directly with classes as in NBG theory, but we do not need it here.

In this chapter we state the essential concepts and terminology needed in future chapters. Mainly, we lay down the basic requirements for a categorical interpretation of the fundamental group, defined in the previous chapter. This will lead us to the foundation of a theory of homotopy for finite categories. The exposition is intentionally brief and not meant as a full introduction to category theory. The primary references consulted in the preparation of this chapter include [1, 4, 27, 19, 20, 28].

3.1. Categories

In order to define a category, some *constituents* must be specified and then show that they satisfy certain conditions. We never define what they are, but we fully specify all the properties that we want those entities to have by stating how they relate to each other.[2] This approach is illustrated in the following definition.

[2] This is of fundamental importance. A precise treatment of this idea is given by the Yoneda Lemma: an object is completely determined by its relationships to every other object.

Definition 3.1 *Category*

A *category* $\mathbf{C}$ consist of the following data.

(i) A collection $\mathrm{Ob}(\mathbf{C})$, whose elements are called $\mathbf{C}$-*objects* and denoted by A, B, C, etc.

(ii) A collection $\mathrm{Mor}(\mathbf{C})$, whose elements are called $\mathbf{C}$-*morphisms* and denoted by f, g, h, etc.

(iii) For each $\mathbf{C}$-morphism f, there exist unique associated $\mathbf{C}$-objects $\mathrm{dom}(f)$ and $\mathrm{cod}(f)$, called the *domain* and *codomain* of f, respectively. We write $f\colon A \to B$ or $A \xrightarrow{f} B$ to indicate that $A = \mathrm{dom}(f)$ and $B = \mathrm{cod}(f)$. We say that f is a $\mathbf{C}$-morphism from A to B.

(iv) For any two $\mathbf{C}$-morphisms $f\colon A \to B$ and $g\colon B \to C$, there is a $\mathbf{C}$-morphism from A to C denoted $g \circ f$, called the *composition* of f and g. In addition, if $h\colon C \to D$ is any other $\mathbf{C}$-morphism,

$$h \circ (g \circ f) = (h \circ g) \circ f.$$

(v) For each $\mathbf{C}$-object A, there is a $\mathbf{C}$-morphism from A to A called the *identity morphism* of A, denoted by 1_A, such that for every $\mathbf{C}$-morphisms $f\colon A \to B$ and $g\colon B \to A$,

$$f \circ 1_A = f \quad \text{and} \quad 1_A \circ g = g.$$

The definition of a category is broad enough so that many stuff can be regarded as a category. In fact, neither the objects of a category have to be sets nor the morphisms have to be functions in the traditional context of mathematics.

We may omit the prefix $\mathbf{C}$ when it is clear what is the category in context and just talk about *objects* and *morphisms*. Synonyms for morphism include *arrow* and *map*. Usually, as a notational convention, some authors write $A \in \mathbf{C}$ and $f \in \mathbf{C}$ to mean that A is an object of $\mathbf{C}$, i.e. $A \in \mathrm{Ob}(\mathbf{C})$, and that f is a morphism of $\mathbf{C}$, i.e. $f \in \mathrm{Mor}(\mathbf{C})$, respectively. We shall avoid this convention and, whenever possible, use the adjectives $\mathbf{C}$-object and $\mathbf{C}$-morphism in each case.

If f and g are composable $\mathbf{C}$-morphisms, their composition $f \circ g$ is also denoted fg. We shall avoid this convention since some authors write fg to mean $g \circ f$.

Given any two **C**-objects X and Y, the collection of **C**-morphisms from X to Y is denoted $\mathrm{Hom}_{\mathbf{C}}(X, Y)$. It is called the *hom-set* of **C**-morphisms from X to Y.[3] Any two elements of $\mathrm{Hom}_{\mathbf{C}}(X, Y)$ are called parallel morphisms since they have the same domain and codomain. A morphism from a **C**-object X to X is called an *endomorphism* of X. The collection $\mathrm{Hom}_{\mathbf{C}}(X, X)$ is denoted $\mathrm{End}_{\mathbf{C}}(X)$. When the category **C** is clear from the context, we just write $\mathrm{Hom}(X, Y)$.

[3] The name hom-set was inherited from algebra by historical reasons, where the term *homomorphism* is used to refer to the structure preserving functions. However, the hom-sets do not have to be sets, and the morphisms may not be functions.

Below we present some examples of categories. Many of these will be used and explored in future chapters. The usual way to *declare* a category is to specify what the objects and morphisms are and then show that the remaining properties of Definition 3.1 hold.

"In the beginning every axiomatic theory is poor in theorems and rich in definitions which must be clarified by examples." [31]

Example 3.2.

(a) The category **Set** has (i) sets as objects and (ii) functions of sets as morphisms, which (iii) have a clearly specified domain and codomain as required by the usual set-theoretical definition.

(iv) (Composition) Given functions $f\colon A \to B$ and $g\colon B \to C$ between sets A, B and C, define $g \circ f\colon A \to C : a \mapsto g(f(a))$ as the composite of f and g. This is the usual composition of functions.

(v) (Associativity) Let $f\colon A \to B$, $g\colon B \to C$ and $h\colon C \to D$ be functions between sets A, B, C and D. By definition, we have

$$\begin{aligned} h \circ (g \circ f)(a) &= h(g \circ f(a)) \\ &= h(g(f(a))) \\ &= h \circ g(f(a)) \\ &= (h \circ g) \circ f(a) \end{aligned}$$

for every $a \in A$. Thus, the set theoretic definition implies $h \circ (g \circ f) = (h \circ g) \circ f$.

(vi) (Identity morphisms) Every set A has a unique identity function defined by $1_A\colon A \to A : a \mapsto a$. Such function is the identity morphism of A. Moreover, for any function f from a set A to a set B it holds

$$\begin{aligned} f \circ 1_A(a) &= f(1_A(a)) = f(a), \quad \text{and} \\ 1_B \circ f(a) &= 1_B(f(a)) = f(a) \end{aligned}$$

for all $a \in A$. Hence $f \circ 1_A = f = 1_B \circ f$.

(b) Categories of structured sets.

Top The category of topological spaces as objects and continuous functions as morphisms.

- **Top**$_*$ The category of topological spaces with a base point as objects and base-point-preserving continuous functions as morphisms.
- **Grp** The category of groups as objects and homomorphisms of groups as morphisms.
- **Ab** The category of Abelian groups as objects and homomorphisms of groups as morphisms.
- **Rng** The category of rings as objects and homomorphisms of rings as morphisms.
- $\mathbb{K}$-**Vect** The category of vector spaces over a field $\mathbb{K}$ as objects and linear maps as morphisms.
- **Pos** The category of partially ordered sets as objects and order preserving functions as morphisms.
- **Met** The category of metric spaces as objects and contractive maps as morphisms.
- R-**Mod** The category of modules over a ring R (with 1, associative and commutative) as objects and homomorphisms of R-modules as morphisms.

These examples have in common that they are all derived from **Set**. Indeed, in each case the objects are sets with additional structure and the morphisms are functions that preserve such structure. There are many more examples of this kind. Note that the composition of continuous functions is a continuous function, the composition of group homomorphisms is a group homomorphism, the composition of monotone maps is monotone, and so on. In fact, the remaining properties that define a category are immediately inherited from **Set**.

(c) Some finite categories.

- **0** The empty category. It contains no object and hence no morphism.
- **1** It consist of exactly one object and exactly one morphism.
- **2** It consist of exactly two objects and exactly one morphism in between, in addition to the identities.
- **3** It consist of exactly three objects
- $\downdownarrows$ The category with exactly two objects and exactly two arrows with same domain and codomain, i.e., with two parallel arrows.

These categories share the property that there is only one way to compose any pair of arrows in each case. We provide the precise definition of a finite category below.

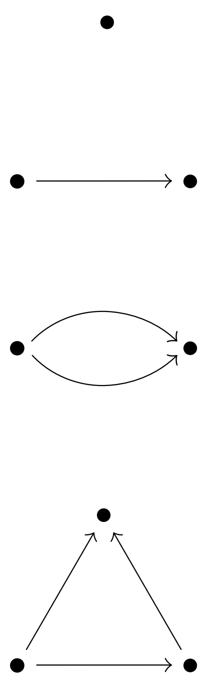


Figure 3.1: Diagrams corresponding to **1**, **2**, $\downdownarrows$ and **3**, shown in sequential order from top to bottom. The empty category has an empty diagram. The objects are depicted as points but this does not mean they are equal. We do not draw the identity arrows and we keep this convention from now and on.

(d) **Every preorder naturally defines a category.** Let $(P, \preceq)$ be a preorder. We can specify a category **P** as follows. The objects of **P** are the elements of P. We declare that there is an arrow from object p to object

q whenever $p \preceq q$. Denote this as $p \to q$. In this way, every morphism has an unequivocally determined domain and codomain. On the other hand, note that we have $p \preceq r$ whenever $p \preceq q$ and $q \preceq r$ by transitivity of $\preceq$. Thus, for any pair of morphisms $p \to q$ and $q \to r$, we define their composite as $p \to r$. Reflexivity of $\preceq$ ensures there is an identity morphism for every object, since $p \preceq p$ for any $p \in P$. Finally, given that there is only one morphism between every pair of objects, associativity holds trivially. Unitality (well-behaved identities) comes from the fact that $p \preceq p \preceq q \preceq q$ is equivalent to $p \preceq q$. Therefore $\mathbf{P}$ is a category. We say $\mathbf{P}$ is a category *induced* by $(P, \preceq)$.

(e) **Every poset induces a category.** Every poset is a preorder, and thus determines a category by the last example. Note this kind of category is not of structured sets. For instance, it is very different from **Pos**, the category of partially ordered sets.

(f) **Every monoid induces a category.** Let $(M, \cdot)$ be a monoid. We can specify a category $\mathbf{M}$ as follows. Take any mathematical object and denote it $*$. We declare that $\mathrm{Ob}(\mathbf{M})$ consist only of $*$. We also state that $\mathrm{Mor}(\mathbf{M}) = M$. In other words, $\mathbf{M}$ is a category with only one object and the morphisms are the elements of M. Since we only have one object, there is only one possible way to associate to each morphism a domain and codomain: every morphism has $*$ both as domain and codomain; in this way, condition (iii) of Definition 3.1 holds. On the other hand, Given any two elements a and b of M, we know $a \cdot b$ is also an element of M. Thus, we define the composite of the morphisms a and b as $a \cdot b$. The identity morphism of $*$ is the identity element of M. Finally, associativity and unitality follow directly from the associativity of the monoid operation and the definition of identity element, respectively.

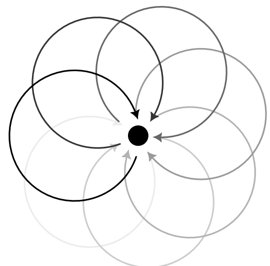


Figure 3.2: A group as a category.

(g) **Every group induces a category.** Every group is a monoid and thus it can be regarded as a category. Be careful not to conflate this kind of category with **Grp**, the category of all groups.

(h) **The homotopy category.** The category **hTop** whose objects are topological spaces and whose morphisms are homotopy classes of continuous functions.

(i) **The opposite category.** If $\mathbf{C}$ is any category, we can build a new category named the opposite of $\mathbf{C}$ and denoted $\mathbf{C}^{\mathrm{op}}$. This category has the same objects as $\mathbf{C}$, but whenever $f \colon A \to B$ is a $\mathbf{C}$-arrow, we have $f^{\mathrm{op}} \colon B \to A$ as a $\mathbf{C}^{\mathrm{op}}$-arrow. Thus $\mathbf{C}^{\mathrm{op}}$ has the same data as $\mathbf{C}$, but the arrows are (formally) turned around.

(j) **Product category.** We define the product of two categories $\mathbf{C}$ and $\mathbf{D}$, denoted $\mathbf{C} \times \mathbf{D}$, as the category whose objects are pairs (C, D), where C is a $\mathbf{C}$-object and D is a $\mathbf{D}$-object, and whose morphisms are pairs (f, g) where $f\colon C \to C'$ is a $\mathbf{C}$-morphism and $g\colon D \to D'$ is a $\mathbf{D}$-morphism. We write $(f, g)\colon (C, D) \to (C', D')$. Composition of $\mathbf{C} \times \mathbf{D}$-morphisms is done component-wise, and the identity morphisms are pairs of the corresponding identities.

Definition 3.3 *Subcategories*

Let $\mathbf{C}$ be a category. A *subcategory* $\mathbf{D}$ of $\mathbf{C}$ is a category where every $\mathbf{D}$-object is a $\mathbf{C}$-object, and every $\mathbf{D}$-morphism is a $\mathbf{C}$-morphism In addition,

(i) the identity morphism of every $\mathbf{D}$-object is a $\mathbf{D}$-morphism,

(ii) the domain and codomain of every $\mathbf{D}$-morphism are $\mathbf{D}$-objects, and

(iii) the composition of every pair of (composable) $\mathbf{D}$-morphisms is a $\mathbf{D}$-morphism.

Remark 3.4. For any pair of $\mathbf{D}$-objects X and Y, we always have

$$\mathrm{Hom}_{\mathbf{D}}(X, Y) \subseteq \mathrm{Hom}_{\mathbf{C}}(X, Y).$$

If $\mathrm{Hom}_{\mathbf{D}}(X, Y) = \mathrm{Hom}_{\mathbf{C}}(X, Y)$ for any $\mathbf{D}$-objects X and Y, we say $\mathbf{D}$ is a *full subcategory* of $\mathbf{C}$. Thus, in order to specify a full subcategory, it is enough to state what its objects are.

Example 3.5.

(a) $\mathbf{0}$ is a subcategory of any category.

(b) The category whose objects are sets and whose morphisms are injections (or either surjections or bijections) is subcategory of $\mathbf{Set}$.

(c) The category $\mathbf{Ab}$ of Abelian groups is a subcategory of $\mathbf{Grp}$. In fact, it is a full subcategory of $\mathbf{Grp}$.

(d) We have seen that a group can be regarded as a category. Thus any subgroup of a group determines a category itself, namely a subcategory of the underlying group regarded as a category.

(e) The category of finite sets is a full subcategory of $\mathbf{Set}$.

(f) The category whose objects are sets and whose morphisms are bijections is not a full subcategory of $\mathbf{Set}$.

(g) If K is a field, the category of K-vector-spaces is a full subcategory of K-$\mathbf{Mod}$.

There are foundational concerns regarding the size of categories. Actually, what we have defined as a category corresponds to the concept of a *meta-category*. In the most rigorous sense, a category is is a meta-category whose axioms are interpreted within set theory. This will not be discussed here and we will not use such terminology.

In order to distinguish between categories of different size, we employ several adjectives.

Definition 3.6 *Small and large and categories*

A category is said to be

(i) *small* if both its collection of objects and morphisms are sets,

(ii) *large* if it is not *small*,

(iii) *locally small* if every hom-set is a set,

(iv) *locally finite* if every hom-set is a finite set,

(v) *finite* if both its collection of objects and morphisms are finite sets.

Locally small categories are very common to the extent that that many authors require local smallness to be part of the definition of a category. This avoids the foundational problems mentioned above. Note that finite implies small, which in turn implies locally small. On the other hand, note that in the definition of a finite category, the condition that the collection of objects must be finite can be dropped. This is due to the fact that a finite collection of morphism forces the collection of objects to be finite, as there exists at least one morphism for each object, namely, its identity.

Example 3.7.

(a) **Cat** is the category of all small categories, which is itself a large category. A morphism in **Cat** is a *functor*, a concept to be defined in the next section.

(b) **Set** is a large category. For instance, the collection of all sets is not a set—as otherwise Russell's paradox comes up. Similarly, **Grp**, **Pos**, **Top**, K-**Vect**, R-**Mod** are large.

(c) A category is small if and only if it is locally small and its class of objects is small. Likewise, a category is finite if and only if it is locally small and its class of objects is finite.

(d) **Set** is a locally small category since $\mathrm{Hom}(X, Y) = Y^X$ is a set for any pair of sets X and Y. It follows that any category of structured sets is also locally small.

(e) Any category induced by a preorder is small. The same is true for categories induced by posets, monoids and groups.

(f) The categories **0**, **1**, **2**, **3**, and $\downdownarrows$ are finite.

(g) We define $\mathbf{Cat}_{\mathbf{Fin}}$ to be the category whose objects are finite categories and whose morphisms are functors between finite categories.

3.1.1. Monos and Epis

The concepts covered here serve as reference material only and are not required for future work. They explore abstractions of injectivity and surjectivity.

Definition 3.8 *Monomorphism, Epimorphism*

Let $\mathbf{C}$ be a category.

(i) A morphism $f \in \mathrm{Hom}_{\mathbf{C}}(X, Y)$ is a *monomorphism* if for any $g, h \in \mathrm{Hom}_{\mathbf{C}}(W, X)$, $f \circ g = f \circ h$ implies $g = h$.

(ii) A morphism $f \in \mathrm{Hom}_{\mathbf{C}}(X, Y)$ is an *epimorphism* if for any $g, h \in \mathrm{Hom}_{\mathbf{C}}(Y, Z)$, $g \circ f = h \circ f$ implies $g = h$.

Special arrows are often used for monos and epis. Monos are decorated with $\rightarrowtail$ and epis with $\twoheadrightarrow$.

A monomorphism (or *mono* for short) is the generalized analogue of injection in set theory. Likewise, an epimorphism (or *epi* for short) is the generalized analogue of surjection. The adjectives used to refer to a mono and to an epi are *monic* and *epic*, respectively. The composition of monos is monic and the composition of epis is epic.

A morphisms that is both monic and epic is is called *bimorphism*. It is not necessarily an isomorphism (defined below).

Recall that in set theory a function is injective if and only if it has a left inverse and it is surjective if and only if it has a right inverse. We can settle these characterizations of injectivity and surjectivity in the context of category theory.

Definition 3.9 *Section, Retraction*

Let $\mathbf{C}$ be a category.

(i) A morphism $f \in \mathrm{Hom}_{\mathbf{C}}(X, Y)$ is a *section* if there exists a morphism $g \in \mathrm{Hom}_{\mathbf{C}}(Y, X)$ such that $g \circ f = 1_X$.

(ii) A morphism $f \in \mathrm{Hom}_{\mathbf{C}}(X, Y)$ is a *retraction* if there exists a morphism $g \in \mathrm{Hom}_{\mathbf{C}}(Y, X)$ such that $f \circ g = 1_Y$. The object Y is called a retract of X.

Note that if $g \circ f = 1_X$, then f is a section and g is a retraction. In this case, f is said to be a section of g, and g is called a retraction of f. A more appropriate name for a section is *right inverse*; likewise a better name for retraction is *left inverse*. Thus, a section is a map with a left inverse and a retraction is a map with a right inverse. Every section is a monomorphism and every retraction is an epimorphism.

It is interesting to mention that in **Set**, the statement that every surjection has a section is equivalent to the axiom of choice. The reason is that specifying a section of a surjection $\psi \colon X \to Y$ requires choosing an element of $\psi^{-1}(\{y\})$ for each $y \in Y$.

Definition 3.10 *Isomorphism*

Let $\mathbf{C}$ be a category. A morphism $f \in \mathrm{Hom}_{\mathbf{C}}(X, Y)$ is an *isomorphism* if there exists $f' \in \mathrm{Hom}_{\mathbf{C}}(Y, X)$ such that

$$f \circ f' = 1_Y \quad \text{and} \quad f' \circ f = 1_X.$$

Thus, an isomorphism is a morphism that is both a section and a retraction. In the sense of this definition, the morphism f' is unique: if f' and f'' satisfy $f' \circ f = 1_X$ and $f \circ f'' = 1_Y$, then

$$f' = f' \circ 1_Y = f' \circ (f \circ f'') = (f' \circ f) \circ f'' = 1_X \circ f'' = f''.$$

We thus call f' the inverse morphism of f and denote it f^{-1}. If $f \in \mathrm{Hom}_{\mathbf{C}}(X, Y)$ is an isomorphism, we say X and Y are isomorphic, which is denoted $X \cong Y$. It is easy to verify that the composition of isomorphisms is again an isomorphism and that $\cong$ is an equivalence relation. The collection of all isomorphisms from a $\mathbf{C}$-object X to X is denoted $\mathrm{Aut}_{\mathbf{C}}(X)$. The elements of $\mathrm{Aut}_{\mathbf{C}}(X)$ are called *automorphisms* of X.

When we say a object X is *unique up to isomorphism* regarding certain property, we mean that if X' is any other object that satisfies such property then $X \cong X'$. If there is exactly one isomorphism between X and X', we say X is *unique up to unique isomorphism*.

3.1.2. Initial and Terminal Objects

The following is an abstract characterization of the empty set. Let us recall that for any set A there is exactly one function from $\varnothing$ to A, namely $\varnothing$. If $A = \varnothing$, this function is precisely $\varnothing$ itself.

Definition 3.11 *Initial object*

An object X in a category $\mathbf{C}$ is *initial* if for any $\mathbf{C}$-object Y there is a unique morphism from X to Y.

In other words, X is an initial object if $\mathrm{Hom}_{\mathbf{C}}(X, Y)$ is a singleton for every $\mathbf{C}$-object Y. Necessarily, $\mathrm{Hom}_{\mathbf{C}}(X, X)$ contains only the identity 1_X.

Example 3.12.

(a) As noted at the beginning, the empty set is an initial object in **Set**.
(b) In **Grp**, the trivial group is an initial object.
(c) In **Top**, any empty space is initial.
(d) $\mathbb{Z}$ is an initial object in **Ring**.

A category may not have an initial object, but if it does, then it is unique in a very special way.

Theorem 3.13

Initial objects are unique up to unique isomorphism.

Proof. Suppose a category $\mathbf{C}$ has two initial objects ι and ι'. Since ι is initial, there is a unique morphism $\psi\colon \iota \to \iota'$. Since ι' is initial, there is a unique morphism $\psi'\colon \iota' \to \iota$. Note that $\psi' \circ \psi$ is a morphism from ι to ι. However, as ι is initial there is exactly one morphism from ι to ι, which necessarily has to be 1_ι, because identity morphisms always exist. The uniqueness then implies $\psi' \circ \psi = 1_\iota$. By the same reasoning, $\psi \circ \psi' = 1_{\iota'}$. Thus ψ is an isomorphism and it is unique since is the unique morphism from ι to ι'. □

Now we turn to the analogous concept of an initial object. Let us make the observation that if $*$ is a singleton, then given any set X there is exactly one way to build a map from $*$ to X.

Definition 3.14 *Terminal object*

An object X in a category $\mathbf{C}$ is *terminal* if for any $\mathbf{C}$-object Y there is a unique morphism from Y to X.

In other words, X is a terminal object if $\mathrm{Hom}_{\mathbf{C}}(Y, X)$ is a singleton for every $\mathbf{C}$-object Y. Note that $\mathrm{Hom}_{\mathbf{C}}(X, X)$ contains only the identity 1_X.

Example 3.15.

(a) As noted before, any singleton is a terminal object in $\mathbf{Set}$. In contrast, $\mathbf{Set}$ has only one initial object.

(b) In $\mathbf{Grp}$, the trivial group is a terminal object.

(c) In $\mathbf{Top}$, any empty space is terminal.

(d) In $\mathbf{Ring}$, the zero ring is terminal

A category may not have a terminal object, but if it does, then it is unique in a very special way.

Theorem 3.16

Terminal objects are unique up to unique isomorphism.

Proof. Dual to the proof of Theorem 3.13. □

Definition 3.17 *Zero object*

An object that is both initial and terminal is called a *zero object*.

Again, there are categories that do not have neither an initial nor a terminal object. This is the case, for instance, in the category of fields, in which there are no homomorphisms between fields of different characteristic. A zero object, if it exists, is denoted by 0.

Example 3.18.

(a) In **Grp**, the trivial group is a zero object.

(b) In **Top**, any empty space is a zero object.

(c) In K-**Vect**, the zero dimensional vector space is a zero object.

(d) The zero ring is not a zero object in **Ring** since it is not initial (because it does dot have an identity).

3.2. The Duality Principle

We have already defined the opposite $\mathbf{C}^{\mathrm{op}}$ of a category $\mathbf{C}$. It is the category whose objects are the same as those of $\mathbf{C}$ and whose morphisms are those of $\mathbf{C}$ but formally turned around, that is, $f^{\mathrm{op}}\colon Y \to X$ is a $\mathbf{C}^{\mathrm{op}}$-morphism if and only if $f\colon X \to Y$ is a $\mathbf{C}$-morphism. Identities in $\mathbf{C}^{\mathrm{op}}$ are the same as in $\mathbf{C}$. Composition is defined by $g^{\mathrm{op}} \circ f^{\mathrm{op}} = (f \circ g)^{\mathrm{op}}$ for any pair of composable $\mathbf{C}$-morphisms f and g. Note that $(\mathbf{C}^{\mathrm{op}})^{\mathrm{op}} = \mathbf{C}$.

The principle of duality states that every categorical concept has a dual, obtained by reversing all the arrows around. The importance of this principle relies in the fact that if a statement is true in $\mathbf{C}$, then it also holds in $\mathbf{C}^{\mathrm{op}}$. Thus, since every result has two dual formulations, only one of them needs to be proved, as the other will follow immediately by the principle of duality.

We have already encountered several concepts and their dual notions, such as monomorphism and epimorphism, section and retraction, and initial and terminal objects. We will meet more throughout the remainder of this chapter. Usually, the dual notion of a categorical concept is named by adding the prefix "co" at the beginning.

3.3. Functors

The concept of category was born as an auxiliary step towards the formal definition of functor and natural transformation. This resembles the situation in topology where the definition of a topology was born in order to formalize the idea of continuous function.

The idea of functor is so important that everything in category theory could be stated in terms of functors. We could start all over again and use functors for everything, without need to define what a category is. Roughly speaking, a functor is a morphism between categories that preserves the structure: an assignment of objects to objects and morphisms to morphisms which preserves identities and compositions. Again, note that in the following definition we do not say what a functor *is* but state what it *does*.

Definition 3.19 *Functor*

A *functor* F from a category $\mathbf{C}$ to a category $\mathbf{D}$, denoted $F\colon \mathbf{C} \to \mathbf{D}$, is a mapping that satisfies the following conditions.

(i) To each $\mathbf{C}$-object X, it assigns a unique $\mathbf{D}$-object $F(X)$.

(ii) To each $\mathbf{C}$-morphism $f\colon X \to Y$, it assigns a unique $\mathbf{D}$-morphism $F(f)\colon F(X) \to F(Y)$.

(iii) $F(1_X) = 1_{F(X)}$ for every $\mathbf{C}$-object X.

(iv) $F(g \circ f) = F(g) \circ F(f)$ for all $\mathbf{C}$-morphisms $X \xrightarrow{f} Y$ and $Y \xrightarrow{g} Z$.

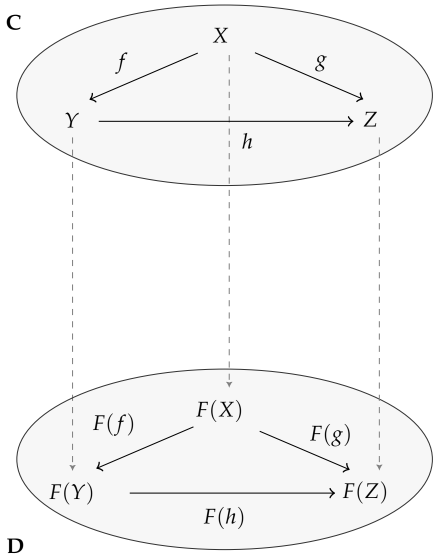


Figure 3.3: Functors preserve commutative triangles. Here $g = h \circ f$. In other words, functors preserve compositions of morphisms.

Remark 3.20. We write

$$F(A \xrightarrow{f} B) \;=\; F(A) \xrightarrow{F(f)} F(B)$$

to indicate the assignment of morphisms to morphisms described above. Some authors write FX instead of $F(X)$ and Ff instead of $F(f)$. If the functor F is understood from the context, it is common to denote $F(f)$ by f_*. This conventions simplify the notation, and we may adopt it when appropriate.

What we have defined as functor is also called *covariant functor*. These terms are synonymous and we may use them interchangeably. Thus, when the noun *functor* is used alone, we mean *covariant functor*. The additional adjective is used in order to distinguish covariant functors from contravariant functors, which we now define.

Definition 3.21 *Contravariant functor*

A *contravariant functor* F from a category $\mathbf{C}$ to a category $\mathbf{D}$ is a functor from $\mathbf{C}^{\mathrm{op}}$ to $\mathbf{D}$.

The above definition was given for the sake of simplicity, but it is worth to mention explicitly that a contravariant functor $F\colon \mathbf{C}^{\mathrm{op}} \to \mathbf{D}$ takes a $\mathbf{C}$-morphism $f\colon A \to B$ to $F(f)\colon F(B) \to F(A)$ and reverses compositions, i.e.,

$$F(g \circ f) = F(f) \circ F(g)$$

for all $\mathbf{C}$-morphisms $f\colon A \to B$.

Example 3.22.

(a) **The identity functor.** Let $\mathbf{C}$ be any category. We can define a functor from $\mathbf{C}$ to $\mathbf{C}$ such that it assigns every object and every morphism to itself. The defining properties of a functor are readily verified. We denote it $\mathbf{1_C}$ and call it the *identity functor* of $\mathbf{C}$.

(b) **The inclusion functor.** The inclusion of a subcategory into its ambient category gives rise to a functor.

(c) A functor $F\colon \mathbf{C} \to \mathbf{D}$ is **constant** if it maps each $\mathbf{C}$-object to a fixed $\mathbf{D}$-object D and each $\mathbf{C}$-morphism to $\mathbb{1}_D$.

(d) **The forgetful functor.** Let $\mathbf{C}$ be a category of structured sets. Define a functor $U\colon \mathbf{C} \to \mathbf{Set}$ as follows:

  (i) to every $\mathbf{C}$-object, U assigns its underlying set, and

  (ii) to very $\mathbf{C}$-morphism, U assigns its underlying function of sets.

  In other words, U removes the structure of the objects and morphisms of $\mathbf{C}$. Sometimes, U is called the *underlying functor*. For a specific example, consider the forgetful functor $U\colon \mathbf{Grp} \to \mathbf{Set}$. In this case,

$$(G, \cdot) \overset{U}{\longmapsto} G$$

  for every group $(G, \cdot)$, and

$$(G, \cdot) \xrightarrow{f} (H, +) \qquad \overset{U}{\longmapsto} \qquad G \xrightarrow{f} H$$

  for any group homomorphism f. Note that, although we use the same symbol, the label f on the left is not the same as the one on the right because they refer to functions that have different domain and codomain. The former is a subset of the cartesian product of the sets $(G, \cdot)$ and $(H, +)$, whereas the latter is a subset of the the cartesian product of the sets G and H.

(e) **Bifunctors.**[4] A *bifunctor* is one whose domain is a product of two categories. Suppose $\mathbf{A}$, $\mathbf{B}$, and $\mathbf{C}$ are categories. Given a bifunctor $F\colon \mathbf{A} \times \mathbf{B} \to \mathbf{C}$ and a fixed $\mathbf{B}$-object B, define the map $F(-, B)\colon \mathbf{A} \to \mathbf{B}$ that sends every $\mathbf{A}$-object A to $F(A, B)$, and every $\mathbf{A}$-morphism ψ to $F(\psi, 1_B)$. Then $F(-, B)$ is a functor from $\mathbf{A}$ to $\mathbf{B}$, which follows immediately from the fact that F itself is a functor. Moreover, $F(-, B)$ is contravariant. Note that, since B is arbitrary, we obtain in this way a family of functors from $\mathbf{A}$ to $\mathbf{B}$. In an entirely similar manner, given any $\mathbf{A}$-object A, we define $F(A, -)\colon \mathbf{B} \to \mathbf{C}$ by mapping any $\mathbf{B}$-object B to $F(A, B)$ and any $\mathbf{B}$-morphism ϕ to $F(1_A, \phi)$. The functor $F(A, -)$ is covariant.

(f) For each topological space with base point (X, x_0), let $\pi_1(X, x_0)$ be its fundamental group. If $\psi\colon (X, x_0) \to (Y, y_0)$ is a $\mathbf{Top}_*$-morphism, define $\pi_1(\psi)\colon \pi_1(X, x_0) \to \pi_1(Y, y_0)$ by $[\gamma] \mapsto [\psi \circ \gamma]$. Then π_1 is a covariant functor from $\mathbf{Top}_*$ to $\mathbf{Grp}$.[5] Indeed, if $f \circ g$ is defined for $\mathbf{Top}_*$-morphisms f and g, then

$$\pi_1(f \circ g)([\gamma]) = [f \circ g \circ \gamma] = \pi_1(f)([g \circ \gamma]) = \pi_1(f) \circ \pi_1(g)([\gamma])$$

  for any loop $\gamma \in \Omega(X, x_0)$.

[4] It is possible to define *multifunctors* for any finite product of categories, in the obvious way. This is the categorical analog of the notion of a function of several variables.

[5] We already discussed the functorial properties of π_1 in chapter 2. The functor π_1 is only one of the tools that algebraic topology deals with. Roughly speaking, algebraic topology studies functors from the category of topological spaces to the category of groups.

Proposition 3.23 *Properties of functors*

(i) Functors preserve isomorphisms.

(ii) Consider two functors $F\colon \mathbf{A} \to \mathbf{B}$ and $G\colon \mathbf{B} \to \mathbf{C}$. Define the *composite* $G \circ F\colon \mathbf{A} \to \mathbf{C}$, of G with F, by

$$(G \circ F)(X \xrightarrow{f} X') = G(F(X)) \xrightarrow{G(F(f))} G(F(X')).$$

Then $G \circ F$ is a functor.

Let $F\colon \mathbf{C} \to \mathbf{D}$ be a functor and X, Y two $\mathbf{C}$-objects. Define

$$F_{X,Y}\colon \operatorname{Hom}_{\mathbf{C}}(X,Y) \to \operatorname{Hom}_{\mathbf{D}}(F(X),F(Y)) : \psi \mapsto F(\psi).$$

Note that $F_{X,Y}$ is a map of sets.

Definition 3.24 *Full and faithful functors*

A functor $F\colon \mathbf{C} \to \mathbf{D}$ is

(i) *faithful* if $F_{X,Y}$ is injective for every pair of $\mathbf{C}$-objects X and Y,

(ii) *full* if $F_{X,Y}$ is surjective for every pair of $\mathbf{C}$-objects X and Y,

(iii) *fully faithful* if F is both full and faithful,

(iv) *surjective on objects* if for any $\mathbf{D}$-object D there is a $\mathbf{C}$-object C such that $F(C) = D$,

(v) *essentially surjective on objects* if for any $\mathbf{D}$-object D there is a $\mathbf{C}$-object C such that $F(C) \cong D$.

Both faithfulness and fullness are local properties. For instance, in the former case, we do not check whether F is injective, but rather whether the induced function $F_{X,Y}$ is injective for each pair of objects X and Y. The same idea applies for the other concepts.

A functor $F\colon \mathbf{C} \to \mathbf{D}$ is *injective on morphisms* provided that for any pair (f, g) of $\mathbf{C}$-morphisms, it holds that

$$F(f) = F(g) \implies f = g.$$

It is clear that a functor that is injective on morphisms is faithful, but not the other way around. A faithful functor that is injective on objects is called an *embedding* If $F\colon \mathbf{C} \to \mathbf{D}$ is such a functor, then $\mathbf{C}$ is identified as a subcategory of $\mathbf{D}$. If $F\colon \mathbf{C} \to \mathbf{D}$ is a fully faithful functor that is injective on objects, then it is a *full embedding* of $\mathbf{C}$ into $\mathbf{D}$. In this case, $\mathbf{C}$ is a *full subcategory* of $\mathbf{D}$.

Remark 3.25. The definition of a faithful functor does not state that if f and g are distinct $\mathbf{C}$-morphisms, then $F(g) \neq F(g)$. This only applies when both f and g are parallel arrows, that is, they have the same domain and codomain.

Proposition 3.26

Let $F\colon \mathbf{C} \to \mathbf{D}$ be a fully faithful functor. For any $\mathbf{C}$-morphism f, if $F(f)$ is an isomorphism, then f is an isomorphism.

Definition 3.27 *Isomorphism of categories*

Let $\mathbf{C}$ and $\mathbf{D}$ be categories.

(i) A functor $F\colon \mathbf{C} \to \mathbf{D}$ is an *isomorphism* if there exists a functor $G\colon \mathbf{D} \to \mathbf{A}$ such that

$$F \circ G = \mathbf{1_D} \quad \text{and} \quad G \circ F = \mathbf{1_C}.$$

(ii) We say $\mathbf{C}$ and $\mathbf{D}$ are *isomorphic* if there is an isomorphism between them and write $\mathbf{C} \cong \mathbf{D}$ in such a case.

Since the functor G in the definition above is uniquely determined by F we write $G = F^{-1}$. On the other hand, $\cong$ is an equivalence relation in the *conglomerate* of all categories.

Remark 3.28. Note that a functor is an isomorphism if and only if it is full, faithful, and bijective on objects.

Definition 3.29 *Concrete category over* **Set**

A category $\mathbf{C}$ is *concrete* if there is a faithful functor $\mathbf{C} \to \mathbf{Set}$.

The functor $\mathbf{C} \to \mathbf{Set}$ should be thought of as a forgetful functor, which assigns to every $\mathbf{C}$-object its “underlying” set and to each $\mathbf{C}$-morphism its “underlying” function.

Example 3.30. Many familiar categories are concrete. For instance,

(i) **Set**,

(ii) **Grp**,

(iii) **Ab**,

(iv) **Top**,

(v) **Ring**,

(vi) K-**Vect**,

(vii) R-**Mod**,

(viii) **Pos**,

are examples of concrete categories. An example of a non-concrete category is **hTop**: despite the fact its objects are sets with additional structure, its morphisms are classes of functions rather than actual functions between them (with extra structure).

3.4. Products and Coproducts

The categorical notions of products and coproducts are abstractions of the notions of cartesian products and direct sums, e.g., of groups.

Definition 3.31 *Product*

Let $(X_\lambda)_{\lambda\in\Lambda}$ be a family of objects in a category $\mathbf{C}$. A *product* of this family is a pair $(X, (\pi_\lambda)_{\lambda\in\Lambda})$ where X is a $\mathbf{C}$-object and $(\pi_\lambda\colon X \to X_\lambda)_{\lambda\in\Lambda}$ is a family of $\mathbf{C}$-morphisms such that if X' is any $\mathbf{C}$-object and $(\pi'_\lambda\colon X' \to X_\lambda)_{\lambda\in\Lambda}$ is a family of $\mathbf{C}$-morphisms, there is a unique $\mathbf{C}$-morphism $\psi\colon X' \to X$ such that $\pi'_\lambda = \pi_\lambda \circ \psi$ for every $\lambda \in \Lambda$, that is, the diagram

$$\begin{tikzcd} X \arrow[r, "\pi_\lambda"] & X_\lambda \\ X' \arrow[u, dashed, "\psi"] \arrow[ur, "\pi'_\lambda"'] & \end{tikzcd}$$

commutes.

In this case X is an attracting object, because for each pair with the described property, there is a unique arrow into X that makes the diagram above commute. The morphisms π_λ are called the *canonical projections*, even though they may not be epimorphisms or projections in the traditional sense.

The uniqueness in this definition means that if ψ and ψ' are two morphisms that satisfy $\pi_\lambda \circ \psi = \pi_\lambda \circ \psi'$ for every $\lambda \in \Lambda$, then necessarily $\psi = \psi'$.

Note that we have defined *a product* and not *the product* of the family $(X_\lambda)_{\lambda\in\Lambda}$. This is due to the fact that, in the strict sense, products are not unique, but they are unique in a very special way.

Proposition 3.32 *Products are unique up to unique isomorphism*

Let $(X_\lambda)_{\lambda\in\Lambda}$ be a family of objects in a category $\mathbf{C}$. A product of $(X_\lambda)_{\lambda\in\Lambda}$, if it exists, is unique up to unique isomorphism.

Proof. Suppose $(X, (\pi_\lambda)_{\lambda\in\Lambda})$ and $(X', (\pi'_\lambda)_{\lambda\in\Lambda})$ are two products of the family $(X_\lambda)_{\lambda\in\Lambda}$. Since $(X', (\pi'_\lambda)_{\lambda\in\Lambda})$ satisfies the universal property of Definition 3.31 respect to $(X, (\pi_\lambda)_{\lambda\in\Lambda})$, there is a unique $\mathbf{C}$-morphism $\psi\colon X' \to X$ such that $\pi'_\lambda = \pi_\lambda \circ \psi$ for every $\lambda \in \Lambda$. Likewise, there is a unique morphism $\psi'\colon X \to X'$ such that $\pi_\lambda = \pi'_\lambda \circ \psi'$ for every $\lambda \in \Lambda$.

We have the following commutative diagram

$$\begin{tikzcd}
X \arrow[r, dashed, "\psi'"] \arrow[d, "\pi_\lambda"] & X' \arrow[r, dashed, "\psi"] \arrow[d, "\pi'_\lambda"] & X \arrow[d, "\pi_\lambda"] \\
X_\lambda \arrow[r, equal] & X_\lambda \arrow[r, equal] & X_\lambda
\end{tikzcd}$$

Thus $\pi_\lambda \circ (\psi \circ \psi') = \pi_\lambda$ for every λ. Since $\pi_\lambda \circ 1_X = \pi_\lambda$ for every λ, the uniqueness implies $\psi \circ \psi' = 1_X$. In an entirely analogous manner we obtain $\psi' \circ \psi = 1_{X'}$. Hence ψ and ψ' are inverses of each other. In particular, ψ is an isomorphism and since is the only morphism from X' to X, we have proven that X and X' are isomorphic up to unique isomorphism. $\square$

This result permit us to speak of *the* product of the family $(X_\lambda)_{\lambda\in\Lambda}$. The product of $(X_\lambda)_{\lambda\in\Lambda}$, if it exists, is denoted by

$$\prod_{\lambda\in\Lambda} X_\lambda.$$

We must emphasize how the uniqueness stated in Definition 3.31 enables us to conclude the equalities $\psi \circ \psi' = 1_X$ and $\psi' \circ \psi = 1_{X'}$, meaning that ψ is an isomorphism. Without such uniqueness, we cannot conclude that an isomorphism exists, let alone a unique one.

Remark 3.33. Uniqueness is a key part of the definition of a product (in fact, of any universal property) because it gives us a *canonical isomorphism* between the objects that satisfy the definition.

Example 3.34.

(a) The category **1** does not admit products.

(b) In **Set**, every family of sets admits a product. The product of a family of sets is its cartesian product.

(c) In **Grp**, every family of groups admits a product: the cartesian product of the underlying sets together with the binary operation of componentwise multiplication. The product of a family of sets is its cartesian product.

(d) In **Top**, the product of a family of topological spaces is the cartesian product of the underlying sets endowed with the product topology, the coarsest topology for which all the canonical projections are continuous.

As expected, there is a notion dual to that of a product.

Definition 3.35 *Coproduct*

Let $(X_\lambda)_{\lambda\in\Lambda}$ be a family of objects in a category $\mathbf{C}$. A *coproduct* of this family is a pair $(X, (\varsigma_\lambda)_{\lambda\in\Lambda})$ where X is a $\mathbf{C}$-object and $(\varsigma_\lambda : X_\lambda \to X)_{\lambda\in\Lambda}$ is a family of $\mathbf{C}$-morphisms such that if X' is any $\mathbf{C}$-object and $(\varsigma'_\lambda : X_\lambda \to X')_{\lambda\in\Lambda}$ is a family of $\mathbf{C}$-morphisms, there is a unique $\mathbf{C}$-morphism $\psi : X \to X'$ such that $\varsigma'_\lambda = \psi \circ \varsigma_\lambda$ for every $\lambda \in \Lambda$, that is, the diagram

$$\begin{tikzcd} X_\lambda \arrow[r, "\varsigma_\lambda"] \arrow[dr, "\varsigma'_\lambda"'] & X \arrow[d, dashed, "\psi"] \\ & X' \end{tikzcd}$$

commutes.

In this case, X is a repelling object. The morphisms π_λ are called the *canonical injections*, even though they may not be monomorphisms or injections in the traditional sense. Analogously to the case of the product, a coproduct, if it exists, is unique up to a unique isomorphism. The proof of this result is completely similar to that of Proposition 3.32, and, in fact, follows directly from the duality principle.

3.5. Fibered Products and Amalgamated Sums

Definition 3.36 *Fibered product*

Let $f : X \to Z$ and $g : Y \to Z$ be two morphisms in a category $\mathbf{C}$. A *fibered product* of f and g is a triple (P, p_1, p_2) that consist of a $\mathbf{C}$-object P and two $\mathbf{C}$-morphisms $p_1 : P \to X$ and $p_2 : P \to Y$ such that

(i) $f \circ p_1 = g \circ p_2$, and

(ii) for any $\mathbf{C}$-object P' and any pair of $\mathbf{C}$-morphisms $p'_1 : P' \to X$ and $p'_2 : P' \to Y$ that satisfy $f \circ p'_1 = g \circ p'_2$, there exists a unique $\mathbf{C}$-morphism $\psi : P' \to P$ such that $p'_1 = p_1 \circ \psi$ and $p'_2 = p_2 \circ \psi$, that is, the diagram

$$\begin{tikzcd} P' \arrow[dr, dashed, "\psi"'] \arrow[drr, bend left, "p'_2"] \arrow[ddr, bend right, "p'_1"'] & & \\ & P \arrow[r, "p_2"] \arrow[d, "p_1"] & Y \arrow[d, "g"] \\ & X \arrow[r, "f"'] & Z \end{tikzcd}$$

commutes.

Other names for fibered product include *pullback* and *fibre product*. The dual notion of fibered product is that of *amalgamated sum*, also called *fibered coproduct* or *pushout*.

Definition 3.37 *Amalgamated sum*

Let $f\colon X \to Y$ and $g\colon X \to Z$ be two morphisms in a category $\mathbf{C}$. An *amalgamated sum* of f and g is a triple (S, s_1, s_2) where S is a $\mathbf{C}$-object and $s_1\colon Y \to S$ and $s_2\colon Z \to S$ are $\mathbf{C}$-morphisms such that

(i) $s_1 \circ f = s_2 \circ g$, and

(ii) for any $\mathbf{C}$-object S' and any pair of morphisms $s_1'\colon Y \to S'$ and $s_2'\colon Z \to S'$ that satisfy $s_1' \circ f = s_2' \circ g$, there exists a unique $\mathbf{C}$-morphism $\psi\colon S \to S'$ such that $s_1' = \psi \circ s_1$ and $s_2' = \psi \circ s_2$, that is, the diagram

$$\begin{tikzcd}
X \arrow[r, "g"] \arrow[d, "f"'] & Z \arrow[d, "s_2"] \arrow[ddr, bend left, "s_2'"] & \\
Y \arrow[r, "s_1"'] \arrow[drr, bend right, "s_1'"'] & S \arrow[dr, dashed, "\psi"] & \\
& & S'
\end{tikzcd}$$

commutes.

Example 3.38. In **Set**, the amalgamated sum of two functions

$$C \xleftarrow{f} A \xrightarrow{g} B$$

consists of the set $(B \sqcup C)/\sim$, where $\sim$ is the equivalence relation defined by

$$b \sim c \iff \exists a \in A : f(a) = b \text{ and } g(a) = c,$$

and the pair of canonical injections

$$i_1\colon B \to (B \sqcup C)/\sim \quad \text{and} \quad i_2\colon C \to (B \sqcup C)/\sim$$

defined by $x \mapsto [x]_\sim$. In this case we have $f(a) \sim g(a)$ for every $a \in A$, so $i_1 \circ g = i_2 \circ f$. To verify that $((B \sqcup C)/\sim, i_1, i_2)$ is indeed a pushout of f and g, suppose $j_1\colon B \to D$ and $j_2\colon C \to D$ satisfy $j_1 \circ g = j_2 \circ f$. Define $\psi\colon (B \sqcup C)/\sim \to D$ by

$$(x,k) \mapsto \begin{cases} j_1(k) & \text{if } k = 1, \\ j_2(k) & \text{if } k = 2. \end{cases}$$

Here $x \in B \sqcup C$ and k is the indexing variable. We have $\psi \circ i_1 = j_1$ and $\psi \circ i_2 = j_2$. Note ψ is the unique function that verifies this property.

3.6. Natural Transformations

We have given the notion of a category and the notion of map between categories, i.e., the notion of functor. There is a further notion of a map between functors. This concept only makes sense when considering parallel functors.

Definition 3.39 *Natural transformation*

Let $\mathbf{C}$ and $\mathbf{D}$ be two categories and $F, G\colon \mathbf{C} \to \mathbf{D}$ two covariant functors. A *natural transformation* $\eta\colon F \to G$ is a family of $\mathbf{D}$-morphisms $(\eta_X\colon F(X) \to G(X))_{X\in\mathbf{C}}$ such that if $f\colon X \to Y$ is any $\mathbf{C}$-morphism, then the following diagram commutes

$$\begin{tikzcd} X \arrow[d, "f"'] & F(X) \arrow[r, "\eta_X"] \arrow[d, "F(f)"'] & G(X) \arrow[d, "G(f)"] \\ Y & F(Y) \arrow[r, "\eta_Y"] & G(Y) \end{tikzcd}$$

In other words, $\eta_X \circ F(f) = G(f) \circ \eta_X$ for every $f \in \mathrm{Hom}_{\mathbf{C}}(X, Y)$.

The morphisms η_X in the definition above are known as the components of η. Rather than viewing a natural transformation as a family of morphisms, we can consider it as an assignment that, for each $\mathbf{C}$-object X, provides a corresponding $\mathbf{D}$-morphism $\eta_X\colon F(X) \to G(X)$ that makes the diagram above commute. The definition of natural transformation for contravariant functors is entirely similar. A natural transformation $\eta\colon F \to G$ between contravariant functors $F, G\colon \mathbf{C} \to \mathbf{D}$ is a family of $\mathbf{D}$-morphisms

$$(\eta_X\colon F(X) \to G(X))_{X\in\mathbf{C}}$$

such that if $f\colon X \to Y$ is any $\mathbf{C}$-morphism, then $\eta_X \circ F(f) = G(f) \circ \eta_Y$. In other words, the following diagram commutes

$$\begin{tikzcd} X \arrow[d, "f"'] & F(X) \arrow[r, "\eta_X"] & G(X) \\ Y & F(Y) \arrow[r, "\eta_Y"] \arrow[u, "F(f)"] & G(Y) \arrow[u, "G(f)"'] \end{tikzcd}$$

Given two natural transformations $\eta\colon F \to G$ and $\theta\colon G \to H$ between functors either covariant or contravariant (but of the same type), we can define their (vertical) composition $\theta \circ \eta\colon F \to H$ by

$$(\theta \circ \eta)_X = \theta_X \circ \eta_X$$

for all $\mathbf{C}$-objects X. The following diagram illustrates why the definition applies equally to both covariant and contravariant functors.

$$\begin{tikzcd} F(X) \arrow[r, "\eta_X"] \arrow[d, "F(f)"'] & G(X) \arrow[r, "\theta_X"] \arrow[d, "G(f)"'] & H(X) \arrow[d, "H(f)"'] \\ F(Y) \arrow[r, "\eta_Y"'] & G(Y) \arrow[r, "\theta_Y"'] & H(Y) \end{tikzcd}$$

If F, G, H were contravariant, only the vertical arrows would be reversed.

The *functor category* from $\mathbf{C}$ to $\mathbf{D}$, denoted $\mathbf{D}^{\mathbf{C}}$, consist of functors

$$\mathbf{C} \to \mathbf{D}$$

as objects and natural transformations between them as morphisms. Thus, it is possible to speak of natural transformations that are monos, epis, sections, retractions, etc.

Definition 3.40 *Natural isomorphism*

Let $F, G\colon \mathbf{C} \to \mathbf{D}$ be two functors. A natural transformation $\eta\colon F \to G$ is a *natural isomorphism* if $\eta_X\colon F(X) \to G(X)$ is an isomorphism for every $\mathbf{C}$-object X. In this case, F and G are *naturally isomorphic*.

In this case we can define the natural transformation $\eta^{-1}\colon G \to F$ given by $\eta_Y^{-1}\colon G(Y) \to F(Y)$ for each $Y \in \mathbf{D}$. We call η^{-1} the inverse of η. Thus, $\eta\colon F \to G$ is a natural isomorphism if and only if there is $\theta\colon G \to F$ such that

$$\eta \circ \theta = \mathbf{1}_{\mathbf{D}} \quad \text{and} \quad \theta \circ \eta = \mathbf{1}_{\mathbf{C}}.$$

It is immediate that the composition of natural isomorphism is again a natural isomorphism.

Definition 3.41 *Equivalence of categories*

Two categories $\mathbf{C}$ and $\mathbf{D}$ are *equivalent* if if there are two functors

$$F\colon \mathbf{C} \to \mathbf{D} \quad \text{and} \quad \mathbf{D} \to \mathbf{C}$$

and two natural isomorphisms

$$\alpha\colon F \circ G \to \mathbf{1}_{\mathbf{D}} \quad \text{and} \quad \beta\colon G \circ F \to \mathbf{1}_{\mathbf{C}}.$$

In this case, we write $\mathbf{C} \simeq \mathbf{D}$ and we say that F and G are *quasi-inverses* of each other.

Chapter 4

Geometric Realization of a Category

The geometric realization is the process of assigning a simplicial complex to a topological space. A more precise term might be topological realization. In this chapter, we introduce the necessary framework to associate a group to a category. This is achieved by first linking a geometric structure to the category, which being a topological space, allows us to apply the techniques of homotopy theory discussed in Chapter 2 to define its fundamental group. However, this approach relies in topological tools and, as we will explain in the last chapter, we seek to develop a theory that leads to same results but in a purely algebraic manner.

The main references for this section are [25, 14, 12, 26, 22, 13, 35].

4.1. Simplicial Sets

For each nonnegative integer $n \geq 0$, we let $[n] = \{0, 1, \ldots, n\}$. When endowed with the usual ordering of the natural numbers, $[n]$ is a totally ordered set. A function $f\colon [n] \to [m]$ is *order preserving* or *monotone* if

$$f(1) \leq \cdots \leq f(m).$$

We know that the composition of monotone functions is monotone, that this composition is associative, and every finite ordinal has an identity function. Thus, the collection of all finite ordinals $[n]$ together with the order preserving functions between these sets naturally defined a category.

Definition 4.1 *Simplex category*

The *simplex category*, denoted $\mathbf{\Delta}$, is the category that consists of the following data.

(i) Objects: linearly ordered sets $[n]$, for each $n \geq 0$.

(ii) Morphisms: monotone maps between linearly ordered sets.

The category $\mathbf{\Delta}$ is small.

Remark 4.2. The collection of all ordinals can be ordered by declaring $[n] \leq [m]$ if and only if $n \leq m$.

Definition 4.3 *Simplicial set*

A simplicial set is a contravariant functor from $\mathbf{\Delta}$ to **Set**.

Remark 4.4. The definition of simplicial set was first introduced by [9].

There is nothing special about **Set** in this definition. We could go a step further and define *simplicial objects*, which is done by replacing **Set** with an arbitrary category in the above definition. For example, if we consider **Grp**, then we would talk about simplicial groups. If we consider covariant functors instead we get the notion of *cosimplicial objects*: a cosimplicial object in a category **C** is a covariant functor from $\mathbf{\Delta}$ to **C**. On the other hand, there is a category $\mathbf{Set}_{\mathbf{\Delta}}$ of simplicial sets, that consists of the functors $\mathbf{\Delta}^{\mathrm{op}} \to \mathbf{Set}$ and whose morphisms are natural transformations.

Simplicial sets are particularly important for the purposes exposed in this chapter because they help "model" topological spaces, serving as the backbone of such spaces. We will elaborate on this idea in the remaining of this chapter.

Our first step is to describe simplicial objects explicitly, i.e., to identify the essential data that constitutes a simplicial set. Given a simplicial set $X\colon \mathbf{\Delta}^{\mathrm{op}} \to \mathbf{Set}$ and a $\mathbf{\Delta}$-object $[n]$, we denote $X_n := X([n])$. Thus, associated to X there is the sequence of sets $(X_n)_{n\in\mathbb{N}}$. (Here $\mathbb{N} = \mathbb{Z}^+ \cup \{0\}$.) On the other hand, every $\mathbf{\Delta}$-morphism can be described in terms of certain special maps, which serve as building blocks, and that we now define. The image of this maps under X will help us to give the explicit description mentioned above.

Definition 4.5 *Coface maps*

Let $n \in \mathbb{Z}^+$ and fix $i \in [n]$. The $\mathbf{\Delta}$-morphism $d^i\colon [n-1] \to [n]$ defined by

$$d^i(k) = \begin{cases} k & \text{if } 0 \leq k \leq i-1, \\ k+1 & \text{if } i \leq k \leq n-1 \end{cases}$$

is called ith *coface map*.

Note that d^i is injective. When $i = n$, the map $d^i(j)$ is the inclusion. In total, there are $n+1$ injections d^i for each positive integer n. Note that d^i depends on n but we omit n from the notation so that it is not overloaded. The ith coface map deletes the ith element in the image. Analogously, we have surjective maps that duplicate the ith element in the image.

Definition 4.6 *Codegeneracy maps*

Let $n \in \mathbb{Z}_0^+$ and fix $i \in [n]$. The $\mathbf{\Delta}$-morphism $s^i \colon [n+1] \to [n]$ defined by

$$s^i(k) = \begin{cases} k & \text{if } 0 \leq k \leq i, \\ k-1 & \text{if } i+1 \leq k \leq n+1 \end{cases}$$

is called the ith *codegeneracy map*.

Proposition 4.7

The coface and codegeneracy maps satisfy the following relationships.

$$\begin{aligned} d^i \circ d^j &= d^{j+1} \circ d^i && \text{if } i \leq j, \\ s^j \circ s^i &= s^i \circ s^{j+1} && \text{if } i \leq j, \\ s^j \circ d^i &= \begin{cases} d^i \circ s^{j-1} & \text{if } i < j, \\ 1 & \text{if } i \in \{j, j+1\}, \\ d^{i-1} \circ s^j & \text{otherwise.} \end{cases} \end{aligned}$$

Proof. Straight from the definitions. □

Let us present a motivating example that illustrates why introducing coface and codegeneracy maps is useful.

Example 4.8. The monotone map $[5] \to [5]$ given by

$$\begin{tikzcd}
0 \arrow[dr] & 0 & & 0 \arrow[dr] & & 0 \\
1 \arrow[r] & 1 & & 1 \arrow[r] & 0 \arrow[r] & 1 \\
2 \arrow[ur] & 2 & \text{can be factored as} & 2 \arrow[ur] & 1 \arrow[dr] & 2 \\
3 \arrow[r] & 3 & & 3 \arrow[ur] & 2 \arrow[ddr] & 3 \\
4 \arrow[dr] & 4 & & 4 \arrow[ur] & & 4 \\
5 \arrow[r] & 5 & & 5 \arrow[uur] & & 5
\end{tikzcd}$$

Note that we have decomposed the map on the left to the composition of a surjective map followed by an injective map, both order-preserving. It turns out that any monotone map can be decomposed as the composition of surjections and injections.

Lemma 4.9 *Factorization of order-reserving maps*

Any $\mathbf{\Delta}$-morphism can be factored into the composition of coface and codegeneracy maps. More precisely, if $\psi\colon [n] \to [m]$ is monotone, there exists $0 \le i_r < \cdots < i_1 \le m$ and $0 \le j_1 < \cdots < j_s < n$ with $r - s = m - n$ such that

$$\psi = d^{i_1} \circ \cdots \circ d^{i_r} \circ s^{j_1} \circ \cdots \circ s^{j_s}.$$

Moreover, this decomposition is unique.

Proof. See [25]. $\square$

Let X be a simplicial set. We denote $d_i = X(d^i)$ and $s_i = X(s^i)$. The image of any monotone map under X is the composition of d_i's and s_i's. Note that since X is contravariant, we have $d_i\colon X_{n-1} \to X_n$ and $s_i\colon X_n \to X_{n+1}$ for every $0 \le i \le n$. We call d_i the ith *face map* and s_i the ith *degeneracy map*. Face and degeneracy maps satisfy the following relationships dual to those of coface and codegeneracy maps presented in Proposition 4.7:

$$\begin{aligned} d^i \circ d^j &= d^{j+1} \circ d^i && \text{if } i \le j, \\ s^j \circ s^i &= s^i \circ s^{j+1} && \text{if } i \le j, \\ s^j \circ d^i &= \begin{cases} d^i \circ s^{j-1} & \text{if } i < j, \\ 1 & \text{if } i \in \{j, j+1\}, \\ d^{i-1} \circ s^j & \text{otherwise.} \end{cases} \end{aligned}$$

By Lemma 4.9, we can describe a simplicial set X explicitly if we know the collection of sets $(X_n)_{n\in\mathbb{N}}$, and the image under X of the coface and codegeneracy maps. In fact, this information together with the relationships between face and degeneracy maps characterizes completely any simplicial set. Thus, we could have defined a simplicial set as the data consisting of

(i) a sequence of sets $(X_n)_{n\ge 0}$,

(ii) a map $d_i\colon X_{n+1} \to X_n$, for each $n \ge 0$ and $i \in [n]$,

(iii) a map $s_i\colon X_n \to X_{n+1}$, for each $n \ge 0$ and $i \in [n]$,

subject to the following relations:

$$\begin{aligned} d^i \circ d^j &= d^{j+1} \circ d^i && \text{if } i \le j, \\ s^j \circ s^i &= s^i \circ s^{j+1} && \text{if } i \le j, \\ s^j \circ d^i &= \begin{cases} d^i \circ s^{j-1} & \text{if } i < j, \\ 1 & \text{if } i \in \{j, j+1\}, \\ d^{i-1} \circ s^j & \text{otherwise.} \end{cases} \end{aligned} \tag{4.1}$$

Let us now look into a particular example of a simplicial set.

Definition 4.10 *Simplex*

A simplex is a contravariant functor $\mathrm{Hom}_{\mathbf{\Delta}}(-,[n])$ where n is a nonnegative integer. We denote $\mathbf{\Delta}^n = \mathrm{Hom}_{\mathbf{\Delta}}(-,[n])$ and call $\mathbf{\Delta}^n$ n-simplex.

Note that $\mathbf{\Delta}^n$ is a contravariant functor from $\mathbf{\Delta}$ to **Set**, so $\mathbf{\Delta}^n$ is a simplicial set. Following our convention, we denote $\mathbf{\Delta}^n_m = \mathrm{Hom}_{\mathbf{\Delta}}([m],[n])$. Relations (4.1) are readily verified. Therefore, using the characterization of a simplicial set given above, we can specify $\mathbf{\Delta}^n$ by describing

(i) the sequence of sets $(\mathbf{\Delta}^n_m)_{m\geq 0}$,

(ii) the face maps

$$d_i\colon \mathbf{\Delta}^n_m \to \mathbf{\Delta}^n_{m-1} \ : \ \psi \mapsto \psi \circ d^i$$

for every $n \geq 0$ and $i \in [n]$, and

(iii) the codegeneracy maps

$$s_i\colon \mathbf{\Delta}^n_m \to \mathbf{\Delta}^n_{m+1} \ : \ \psi \mapsto \psi \circ s^i$$

for every $n \geq 0$ and $i \in [n]$.

4.2. Geometric Realization of a Simplicial Set

Simplices are important because they are the backbones over which the geometric realization of an arbitrary simplicial set is constructed. First, we have to give the n-simplex $\mathbf{\Delta}^n$ a geometry. In other words, we associate to $\mathbf{\Delta}^n$ a topological space. Recall that $I = [0,1]$.

Definition 4.11 *Standard n-dimensional simplex*

The standard n-dimensional simplex is the topological space that consist of the set

$$|\mathbf{\Delta}^n| = \left\{(x_0,\dots,x_n) \in I^{n+1} : x_0 + \cdots + x_n = 1\right\}$$

endowed with the subspace topology of $\mathbb{R}^n$.

Remark 4.12. An alternative definition of $|\mathbf{\Delta}^n|$ can be given as the convex hull of $n+1$ points in $\mathbb{R}^n$ with zeros every where but a single 1 in some component.

The standard n-dimensional simplex is the *geometric realization* of the n-simplex. As particular examples, $|\mathbf{\Delta}^0|$ is the singleton $\{1\}$, whereas $|\mathbf{\Delta}^1|$ is the unit interval, $|\mathbf{\Delta}^2|$ is a triangle, and $|\mathbf{\Delta}^3|$ is a tetrahedron. See Figure 4.1.

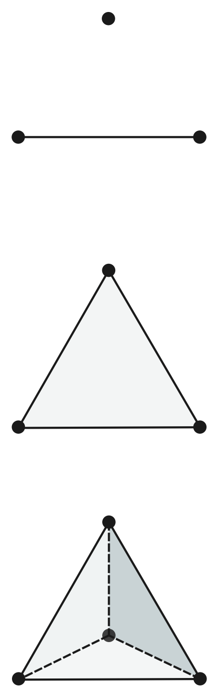


Figure 4.1: Geometric realization of $\mathbf{\Delta}^n$ for $n = 0,1,2,3$ in order from top to bottom.

Definition 4.13 *Geometric realization of a simplicial set*

Let X be a simplicial set and endow every X_n with the discrete topology. The geometric realization of X, denoted $|X|$, is the topological space

$$|X| = \left(\coprod_{n \geq 0} X_n \times |\mathbf{\Delta}^n| \right) \Big/ \sim$$

where $\sim$ is the equivalence relation given by

$$\begin{aligned}(d_i(x), (t_0, \ldots, t_n)) &\sim \left(x, d^i\,(t_0, \ldots, t_n)\right),\\ (s_j(x), (t_0, \ldots, t_n)) &\sim \left(x, s^j\,(t_0, \ldots, t_n)\right).\end{aligned}$$

Here $|X|$ is endowed with the quotient topology.

Remark 4.14. Recall that when working with the disjoint union of topological spaces, we omit the last component corresponding to the label to where the point belongs to. For instance, we should actually have written $(d_i(x), (t_0, \ldots, t_n), n)$ instead of just $(d_i(x), (t_0, \ldots, t_n))$. However, this omission should cause no confusion.

4.3. Geometric Realization as a Functor

The geometric realization of a simplicial set can be regarded as a functor from $\mathbf{Set}_{\mathbf{\Delta}}$, the category of simplicial sets, to **Top**, the category of topological spaces.

We have seen that $|\cdot|$ assigns a topological space to every simplicial set. Moreover, given a morphism

$$\mathcal{F}\colon X \to Y$$

of simplicial sets, there is a naturally induced morphism of topological spaces, that is, a continuous map

$$|\mathcal{F}|\colon |X| \to |Y|$$

defined by

$$[(x, t_0, \ldots, t_n)] \mapsto [(\mathcal{F}(x), t_0, \ldots, t_n)]$$

The properties of this functor are treated in [25].

4.4. The Nerve and the Classifying Space of a Small Category

We now present a method to associate a simplicial set to a small category, called its *nerve*.

Definition 4.15 *Nerve*

The *nerve* of a small category $\mathbf{C}$ is the simplicial set given by

(i) the sequence of sets $(\mathcal{N}\mathbf{C}_n)_{n\geq 0}$ where

$$\mathcal{N}\mathbf{C}_n := \{(\psi_1, \dots, \psi_n) \mid \psi_{i+1} \circ \psi_i \text{ is defined}, 1 \leq i \leq n-1\}.$$

We write an element of $\mathcal{N}\mathbf{C}_n$ as a finite chain

$$A_0 \xrightarrow{\psi_1} A_1 \xrightarrow{\psi_2} A_2 \to \cdots \to A_{n-1} \xrightarrow{\psi_n} A_n.$$

(ii) *Face maps* $d_i \colon \mathcal{N}\mathbf{C}_n \to \mathcal{N}\mathbf{C}_{n-1}$ defined by

$$(\psi_1, \dots, \psi_n) \mapsto (\psi_1, \dots, \psi_{i-1}, \psi_{i+1} \circ \psi_i, \psi_{i+2}, \dots, \psi_n)$$

for all $i \in [n]$.

(iii) *Degeneracy maps* $s_i \colon \mathcal{N}\mathbf{C}_n \to \mathcal{N}\mathbf{C}_{n+1}$ defined, for all $i \in [n]$, by

$$(\psi_1, \dots, \psi_n) \mapsto (\psi_1, \dots, \psi_i, 1_i, \psi_{i+1} \dots, \psi_n)$$

where 1_i is the identity of the ith object of the chain, i.e., A_i.

Remark 4.16. When $i \in \{0, n\}$, the composition $\psi_{i+1} \circ \psi_i$ does not make sense as neither ψ_0 nor ψ_{n+1} exists. However, the convention is that d_i misses ψ_i in this case. Thus

$$d_0 \colon (\psi_1, \dots, \psi_n) \mapsto (\psi_2, \dots, \psi_n) \quad \text{and} \quad d_n \colon (\psi_1, \dots, \psi_n) \mapsto (\psi_1, \dots, \psi_{n-1}).$$

It is readily verified that the relationships between face and degeneracy maps are satisfied. On the other hand, since the nerve of a small category is a simplicial set, we can compute its geometric realization. The geometric realization of the nerve of a small category $\mathbf{C}$ is called the *classifying space* of $\mathbf{C}$, denoted $\mathcal{B}\mathbf{C}$. Thus

$$\mathcal{B}\mathbf{C} = |\mathcal{N}\mathbf{C}|.$$

The classifying space of a small category is a CW complex. Moreover, $\mathcal{B}$ is a functor from **Cat**, the category of small categories, to **Top**. In fact, $\mathcal{B}$ can be regarded as the composition of the functors $\mathcal{N}$ and $|\cdot|$. Furthermore, given a functor $F \colon \mathbf{C} \to \mathbf{D}$ of small categories, it is induced a continuous cellular map of topological spaces $\mathcal{B}F \colon \mathcal{B}\mathbf{C} \to \mathcal{B}\mathbf{D}$ [25]. We will not go into the details here.

Examples

We define special categories and describe the geometric realization of its nerve. Our purpose is not to give a detailed exposition of the computations, but to illustrate the geometric aspect that arises out of a category by following the constructions we have described so far. Specific details can be found in [22] and [25]. The following examples were first introduced by Rosero in [29]. Proofs have been omitted.

The Interval Category $\mathbf{I}_m^n$

Let $m \leq n$ be even integers. The *interval category* $\mathbf{I}_m^n$ consist of

(i) objects: integers k for $k \in \{m, \dots, n\}$;

(ii) morphisms: arrows $k \to k$ for every $k \in \{m, \dots, n\}$, and arrows $i \to j$ for each even integer i and each $j \in \{i-1, i, i+1\}$.

The morphisms of $\mathbf{I}_m^n$ are formal arrows between integers. Alternatively, we could define $\mathbf{I}_m^n$ as the category induced by the following preorder on $\{m, \dots, n\}$:

$$m \preccurlyeq n \iff m = n \text{ or } m \text{ is even and } |m - n| = 1.$$

This category is represented by the diagram

$$\begin{tikzcd}
m \arrow[dr] & & m-2 \arrow[dl] \arrow[dr] & & n-2 \arrow[dl] \arrow[dr] & & n \arrow[dl] \\
& m-1 & & \cdots & & n-1 &
\end{tikzcd}$$

where identity arrows have been omitted.

Remark 4.17. Despite its name, $\mathbf{I}_m^n$ should not be confused with the category that consists of exactly two objects and precisely one morphism between them.

Proposition 4.18

The classifying space of $\mathbf{I}_m^n$ is homeomorphic to the unit interval:

$$\mathcal{B}\mathbf{I}_m^n \cong [0, 1].$$

The Disk Category $\mathbf{D}^2$

The *elementary disk category*, denoted $\mathbf{D}^2$, is the category given by the following commutative diagram.

$$\begin{tikzcd}
& \bullet \arrow[dr] & \\
\bullet \arrow[ur] & & \bullet \arrow[ll]
\end{tikzcd}$$

Formally, $\mathbf{D}^2$ consists of exactly three objects and three non-identity morphisms that make this diagram commute. More explicitly, as shown in Figure 4.2, its objects are $\mathfrak{A}, \mathfrak{B}, \mathfrak{C}$, and its morphisms are the identities of these objects together with the arrows $\mathfrak{f}\colon \mathfrak{A} \to \mathfrak{B}$, $\mathfrak{g}\colon \mathfrak{B} \to \mathfrak{C}$, and $\mathfrak{h}\colon \mathfrak{C} \to \mathfrak{A}$, with the additional conditions that

$$\mathfrak{h} \circ \mathfrak{g} \circ \mathfrak{f} = \mathbb{1}_{\mathfrak{A}} \quad \mathfrak{f} \circ \mathfrak{h} \circ \mathfrak{g} = \mathbb{1}_{\mathfrak{B}} \quad \text{and} \quad \mathfrak{g} \circ \mathfrak{f} \circ \mathfrak{h} = \mathbb{1}_{\mathfrak{C}}.$$

$$\begin{tikzcd} & \mathfrak{B} \arrow[dr, "\mathfrak{g}"] & \\ \mathfrak{A} \arrow[ur, "\mathfrak{f}"] & & \mathfrak{C} \arrow[ll, "\mathfrak{h}"] \end{tikzcd}$$

Figure 4.2: Labeled diagram of $\mathbf{D}^2$. Identities are not shown.

Proposition 4.19

The classifying space of $\mathbf{D}^2$ is homeomorphic to the unit disk:

$$\mathcal{B}\mathbf{D}^2 \cong D^2.$$

The Circle Category $\mathbf{S}^1$

The *circle category*, denoted $\mathbf{S}^1$, is the category given by the diagram

$$\begin{tikzcd} \bullet \arrow[r, bend left] \arrow[r, bend right] & \bullet \end{tikzcd}$$

Formally, $\mathbf{S}^1$ consists of exactly two objects and two parallel non-identity arrows. More explicitly, the objects of $\mathbf{S}^1$ are $\mathfrak{A}, \mathfrak{B}$, and its morphisms are the identities of these objects together with the arrows $\mathfrak{f}\colon \mathfrak{A} \to \mathfrak{B}$ and $\mathfrak{g}\colon \mathfrak{A} \to \mathfrak{B}$. See Figure 4.3.

$$\begin{tikzcd} \mathfrak{A} \arrow[r, bend left, "\mathfrak{f}"] \arrow[r, bend right, "\mathfrak{g}"'] & \mathfrak{B} \end{tikzcd}$$

Figure 4.3: Labeled diagram of $\mathbf{S}^1$.

Proposition 4.20

The classifying space of $\mathbf{S}^1$ is the unit circle:

$$\mathcal{B}\mathbf{S}^1 \cong S^1.$$

The Torus Category $\mathbf{T}^2$

The nth *torus category*, denoted $\mathbf{T}^n$, is the product category $\mathbf{S}^1 \times \cdots \times \mathbf{S}^1$ of $\mathbf{S}^1$ with itself n times. When $n = 2$, we just call $\mathbf{T}^n$ the torus category.

Proposition 4.21

The classifying space of $\mathbf{T}^n$ is the n-dimensional torus, that is

$$\mathcal{B}\mathbf{T}^n \cong T^n.$$

In particular, $\mathcal{B}\mathbf{T}^2 \cong T^2$. This result follows from the fact that the functor induced by the geometric realization preserves products and $\mathcal{B}\mathbf{S}^1 \cong S^1$. As described in [29], there is a similarity between $\mathbf{T}^2$ and T^2 regarding their plane representations, as a commutative diagram and as quotient of the unit square I^2, respectively. See Figure 4.4.

$$\begin{tikzcd} (\mathfrak{A},\mathfrak{A}) \arrow[r, "1"] \arrow[d, "3"'] \arrow[dr] & (\mathfrak{A},\mathfrak{B}) \arrow[d] & (\mathfrak{A},\mathfrak{A}) \arrow[l, "2"'] \arrow[d, "3"] \arrow[dl] \\ (\mathfrak{B},\mathfrak{A}) \arrow[r] & (\mathfrak{B},\mathfrak{B}) & (\mathfrak{B},\mathfrak{A}) \arrow[l] \\ (\mathfrak{A},\mathfrak{A}) \arrow[r, "1"'] \arrow[u, "4"] \arrow[ur] & (\mathfrak{A},\mathfrak{B}) \arrow[u] & (\mathfrak{A},\mathfrak{A}) \arrow[l, "2"] \arrow[u, "4"'] \arrow[ul] \end{tikzcd}$$

Figure 4.4: Flat representation of the torus category $\mathbf{T}^2$. Arrows with the same label are identified.

Remark 4.22. By the results just presented, we have

$$\begin{aligned}
\pi_1(\mathcal{B}\mathbf{D}^2) &\cong \pi_1(D^2) = 0, \\
\pi_1(\mathcal{B}\mathbf{S}^1) &\cong \pi_1(S^1) = \mathbb{Z}, \quad \text{and} \\
\pi_1(\mathcal{B}\mathbf{T}^n) &\cong \pi_1(T^n) = \mathbb{Z}^n.
\end{aligned}$$

We will compare these results with those obtained at the end of Chapter 5.

Chapter 5

Homotopy for Finite Categories

In this chapter we develop a novel theory that aims to associate a group to a finite category in such a way that the framework resembles that of classical homotopy theory presented in Chapter 2. We have seen that this can be achieved by means of the classifying space of a category, as exposed in Chapter 4. However, the key difference is that this will be done in a purely algebraic manner, without relying on any topological method. To begin with, we define the categorical analogs of many concepts of the classical homotopy theory of topological spaces and establish the foundational framework. Subsequently, we demonstrate that the theory gives consistent results with some of the examples presented in Chapter 4. For instance the fundamental group of the topological circle is isomorphic to the fundamental group of the categorical circle as defined in the context of category theory. We show a similar result regarding the fundamental groups of the topological and categorical torus.

The work presented here builds extensively on the original work of Rosero [29], where the fundamental concepts and ideas were introduced. However, we have reformulated most of the definitions and statements due to lack of coherence, well-definiteness and some erroneous proofs. A short attempt in providing a correct formulation of the theory was done by Ajila [2], but not in an exhaustive manner as done here. Apart from that, we present revised results, coherent notation, novel proofs, and generalizations of some of the angular results of the theory. Moreover, we have reorganized the concepts so that the presentation now resembles more that of the classical theory of homotopy, as presented in Chapter 2.

Even though the work of Rosero [29] is based on the work by Larose and Tardif [16], as the author acknowledges, it is nevertheless the original formulation of the theory we explore here. Anything wise and brilliant must be credited to the masterworks that have been revised to write this thesis such as [21, 6, 32, 16, 19, 34, 13, 25, 33, 18, 1, 7]. Anything foolish, assume it is my error.

5.1. The Preordered Category $\mathbf{\Lambda}$

Our first step is to build a category that will play a similar role to that of the unit interval I when we talked about homotopy of paths in Chapter 2. Recall from section 3.1 that any preorder naturally defines a category.

Lemma 5.1

Let $\preccurlyeq$ be the binary relation on $\mathbb{Z}$ defined by

$$m \preccurlyeq n \iff m = n \text{ or } m \text{ is even and } |m - n| = 1.$$

Then $\preccurlyeq$ is a preorder on $\mathbb{Z}$.

Proof. We prove $\preccurlyeq$ is both reflexive and transitive. On one hand, $n \preccurlyeq n$ since $n = n$ for any $n \in \mathbb{Z}$. To see that $\preccurlyeq$ is transitive, suppose $m \preccurlyeq n$ and $n \preccurlyeq p$ where $m, n, p \in \mathbb{Z}$. If either $m = n$ or $n = p$, it is clear that $m \preccurlyeq p$. If this is not the case, then m is even and $|m - n| = 1$, and n is even and $|n - p| = 1$. However, it cannot be true that both m and n are even and $|m - n| = 1$. It follows (vacuously) that $m \preccurlyeq p$. Therefore $\preccurlyeq$ is a preorder on $\mathbb{Z}$. □

Observe that $\preccurlyeq$ is not a linear order. For instance, $\preccurlyeq$ is neither symmetric nor strongly connected. We write $i \prec j$ whenever $i \preccurlyeq j$ and $i \neq j$.

Remark 5.2. Note that if $i \preccurlyeq j$ and $j \leq k$ for some $k \in 2\mathbb{Z}$, then $i \leq k$. Indeed, since this is true when $i = j$, assume $i \neq j$. Then i is even and $j = i - 1$ or $j = i + 1$, so $j < k$ because j is odd. Thus $j - 1 < j + 1 \leq k$, whence $i \leq k$. Similarly, $i \preccurlyeq j$ and $j \geq k$ for some $k \in 2\mathbb{Z}$ imply $i \geq k$. It should also be noted that if $i \preccurlyeq j$, then $-i \preccurlyeq -j$. It does not make sense to ask whether $\preccurlyeq$ is preserved under addition since $\preccurlyeq$ is not strongly connected, meaning not every pair of integers can be compared.

Definition 5.3 *Category* $\mathbf{\Lambda}$

We define $\mathbf{\Lambda}$ to be the category induced by the preordered set $(\mathbb{Z}, \preccurlyeq)$.

Notice that both the collection of objects and morphisms of $\mathbf{\Lambda}$ are countable. Thus, $\mathbf{\Lambda}$ is a small category. We denote any $\mathbf{\Lambda}$-arrow by $i \to j$ where $i, j \in \mathbb{Z}$. Observe that the domain of any nonidentity arrow is an even integer. An odd integer is the domain of exactly one arrow, namely its identity. The domain and codomain of a nonidentity arrow are consecutive integers. Thus any $\mathbf{\Lambda}$-arrow is of the form $n \to n$, $2n \to 2n - 1$, or $2n \to 2n + 1$ with $n \in \mathbb{Z}$. Further, if $i \to j$ is any arrow, there are three possibilities: $i = j$, $i = j - 1$, or $i = j + 1$. This fact implies $i \to j = \mathbb{1}_k$ if and only if $i = j = k$. The following diagram is a visual representation of $\mathbf{\Lambda}$, where the identity arrows have been omitted.

$$\cdots \longleftarrow -2 \longrightarrow -1 \longleftarrow 0 \longrightarrow 1 \longleftarrow 2 \longrightarrow \cdots$$

On the other hand, two $\mathbf{\Lambda}$-arrows $m \to n$ and $p \to q$ are equal if and only if $m = p$ and $n = q$. Finally, note that there are no composable nonidentity arrows; i.e., except by identities, $\mathbf{\Lambda}$ does not have composable arrows. In categorical terminology, this means that $\mathbf{\Lambda}$ is a *thin category* [28]. Hence it is not necessary to verify that a functor from $\mathbf{\Lambda}$ to any other category preserves compositions of arrows. In other words, condition (iv) in Definition 3.19 follows immediately from the fact that a functor maps morphisms to morphisms.

Remark 5.4. Whenever $m \to n$ is used, we implicitly assume $m \preccurlyeq n$, otherwise writing $m \to n$ makes no sense because such an object does not exist.

5.2. Homotopy of Functors

Recall, from Example 3.22 (e) of section 3.3, that a bifunctor is one whose domain is a product of two categories. For clarity, we restate the definition below.

Suppose $\mathbf{A}$, $\mathbf{B}$, and $\mathbf{C}$ are categories. Given a bifunctor $F\colon \mathbf{A} \times \mathbf{B} \to \mathbf{C}$ and a fixed $\mathbf{B}$-object B, define the map

$$F(-, B)\colon \mathbf{A} \to \mathbf{B}$$

which sends every $\mathbf{A}$-object A to $F(A, B)$, and every $\mathbf{A}$-morphism ψ to $F(\psi, \mathbb{1}_B)$. Then $F(-, B)$ is a functor from $\mathbf{A}$ to $\mathbf{B}$. Moreover, since B is arbitrary, we obtain in this way a family of functors from $\mathbf{A}$ to $\mathbf{B}$. In an analogous manner, given any $\mathbf{A}$-object A, we define $F(A, -)\colon \mathbf{B} \to \mathbf{C}$ as the map that sends any $\mathbf{B}$-object B to $F(A, B)$ and any $\mathbf{B}$-morphism ϕ to $F(\mathbb{1}_A, \phi)$.

Let $\mathbf{C}$ and $\mathbf{D}$ be finite categories.

Definition 5.5

Let $F, G\colon \mathbf{C} \to \mathbf{D}$ be covariant functors. A *homotopy* from F to G is a functor $\mathcal{H}\colon \mathbf{C} \times \mathbf{\Lambda} \to \mathbf{D}$ such that, for some even integers $m \leq n$,

(i) $\mathcal{H}(-, k) = F$ for every $k \leq m$, and

(ii) $\mathcal{H}(-, k) = G$ for every $k \geq n$.

If there is a homotopy from F to G, then F is *homotopy equivalent* to G, which is denoted $F \simeq G$.

We write $\mathcal{H}\colon F \simeq G$ to indicate that $\mathcal{H}$ is a homotopy from F to G. Sometimes we say $\mathcal{H}$ is a (m, n)-homotopy to make emphasis on the pair of even integers stated in the definition.

Proposition 5.6

$\simeq$ is an equivalence relation over the objects of $\mathbf{D}^{\mathbf{C}}$.

Proposition 5.6 allows us to say that F and G *are* homotopy equivalent whenever F is homotopy equivalent to G or conversely.

Proof. Let $F, G, H\colon \mathbf{C} \to \mathbf{D}$ be covariant functors.

(i) (Reflexivity) Let $\mathcal{H}\colon \mathbf{C} \times \mathbf{\Lambda} \to \mathbf{D}$ be the functor defined by $\mathcal{H}(C,k) = F(C)$ on objects, and by $\mathcal{H}(\psi,\lambda) = F(\psi)$ on morphisms. Note that $\mathcal{H}(-,k) = F$ for any $k \in \mathbb{Z}$, so $F \simeq F$.

(ii) (Symmetry) Suppose $\mathcal{H}\colon F \simeq G$ is a (m,n)-homotopy. Let $\mathcal{I}\colon \mathbf{C} \times \mathbf{\Lambda} \to \mathbf{D}$ be the functor defined by $\mathcal{I}(C,k) = \mathcal{H}(C,-k)$ on objects and by $\mathcal{I}(\psi,\lambda) = \mathcal{H}(\psi,\lambda)$ on morphisms. It follows that

$$\mathcal{I}(-,k) = \mathcal{H}(-,-k) = G \text{ if } k \leq -n$$

and

$$\hat{\mathcal{H}}(-,k) = \mathcal{H}(-,-k) = F \text{ if } k \geq -m.$$

Thus $\mathcal{I}\colon G \simeq F$.

(iii) (Transitivity) Let $\mathcal{H}\colon F \simeq G$ and $\mathcal{I}\colon G \simeq H$ be (m,n) and (m',n') homotopies, respectively. Define $\mathcal{J}\colon \mathbf{C} \times \mathbf{\Lambda} \to \mathbf{D}$ by

$$\mathcal{J}(C,k) = \begin{cases} \mathcal{H}(C,k+n) & \text{if } k \leq 0, \\ \mathcal{I}(C,k+m') & \text{otherwise,} \end{cases}$$

on objects, and by

Notice that $j \leq 0$ implies $i \leq 0$ and $j > 0$ implies $i \geq 0$. See Remark 5.2.

$$\mathcal{J}(\psi, i \to j) = \begin{cases} \mathcal{H}(\psi, i+n \to j+n) & \text{if } j \leq 0, \\ \mathcal{I}(\psi, i+m' \to j+m') & \text{otherwise,} \end{cases}$$

on morphisms, whenever $i \preccurlyeq j$. Then $\mathcal{J}(-,k) = \mathcal{H}(-,k+n) = F$ if $k \leq m-n$, and $\mathcal{J}(-,k) = \mathcal{I}(-,k+m') = H$ whenever $k \geq n'-m'$. Hence $\mathcal{J}\colon F \simeq H$.

$\square$

Theorem 5.7 $\simeq$ *preserves* $\circ$

Suppose $F \simeq G$ and $F' \simeq G'$ where F, G, F', G' are functors between finite categories. If $F' \circ F$ is defined, so is $G' \circ G$, and

$$F' \circ F \simeq G' \circ G.$$

Proof. Let $\mathcal{H}\colon F \simeq G$ be an (m,n)-homotopy and let $\mathcal{H}'\colon F' \simeq G'$ be a (m',n')-homotopy. Define $\mathcal{I}(-,k) = \mathcal{H}'(-,k+m') \circ \mathcal{H}(-,k+n)$ for each $k \in \mathbb{Z}$. If $k \leq m-n$, then $\mathcal{H}(-,k+n) = F$ since $k+n \leq m$ and $\mathcal{H}'(-,k+m') = F'$ because $k < 0$. Thus $\mathcal{I}(-,k) = F' \circ F$ when $k \leq m-n$. If $k \geq n'-m'$, then $\mathcal{H}'(-,k+m') = G'$, and $\mathcal{H}(-,k+n) = G$ as $k > 0$. Thus $\mathcal{I}(-,k) = G' \circ G$ when $k \geq n'-m'$. It follows $F' \circ F \simeq G' \circ G$. $\square$

5.3. Paths

Suppose throughout that $\mathbf{C}$ is a finite category.

Definition 5.8 *Path*

Let A and B be $\mathbf{C}$-objects. A *path* in $\mathbf{C}$ from A to B is a functor $\alpha\colon \mathbf{\Lambda} \to \mathbf{C}$ such that, for some even integers $m \leq n$,

(i) $\alpha(i \to j) = \mathbb{1}_A$ if $j \leq m$, and

(ii) $\alpha(i \to j) = \mathbb{1}_B$ if $j \geq n$.

Keep in mind Remark 5.2: $i \preccurlyeq j$ and $j \leq m$ imply $i \leq m$. Likewise, if $i \preccurlyeq j$ and $j \geq n$, then $i \geq n$.

By a $\mathbf{C}$-path we mean a path between objects of $\mathbf{C}$. The set of $\mathbf{C}$-paths from A to B is denoted $\Phi_{\mathbf{C}}(A, B)$, or just $\Phi(A, B)$ if the category is known from the context. A *loop based at* A is a path from A to A. We say $\mathbf{C}$ is *path connected* if there is a path between any two $\mathbf{C}$-objects.

The integers m and n in this definition are lower and upper bounds for α, respectively. More precisely, a *lower bound* for α is an even integer l such that

$$\alpha(i \to j) = \mathbb{1}_A \;\text{ for }\; j \leq l.$$

Similarly, an *upper bound* is an even integer u such that

$$\alpha(i \to j) = \mathbb{1}_B \;\text{ for }\; j \geq u.$$

In order to emphasize the lower an upper bounds we sometimes say α is a lu-path. This terminology will ease up the upcoming proofs. Note these integers are not unique: any even integer less than l is also a lower bound, and any even integer greater than u is also an upper bound for α. However, if the sets of lower and upper bounds for α are bounded above and below, respectively, we define $\min \alpha$ to be the greatest lower bound and $\max \alpha$ to be the least upper bound, that is,

$$\begin{aligned} \min \alpha &= \max\{l \in 2\mathbb{Z} : \alpha(i \to j) = \mathbb{1}_A \text{ for all } j \leq l\}, \quad \text{and} \\ \max \alpha &= \min\{u \in 2\mathbb{Z} : \alpha(i \to j) = \mathbb{1}_B \text{ for all } j \geq u\}. \end{aligned}$$

It is clear that $\min \alpha$ and $\max \alpha$ are unique.

The information contained between $\min \alpha$ and $\max \alpha$ is called the *non-trivial part* of α, that is, the objects $\alpha(k)$ for $\min \alpha \leq k \leq \max \alpha$, and the morphisms $\alpha(i \to j)$ for $\min \alpha \leq j \leq \max \alpha$. We define the *length* of a path α to be the cardinality of the set of arrows contained in its nontrivial part, that is, $\max \alpha - \min \alpha$.

We shall illustrate how paths are represented by means of diagrams. Since we are interested only in the nontrivial part, any path α can be given a representation as follows, where $m = \min \alpha$ and $n = \max \alpha$.[1]

$$\underset{(m)}{\alpha(m)} \xrightarrow{\alpha(m \to m+1)} \underset{(m+1)}{\alpha(m+1)} \longleftarrow \cdots \longrightarrow \underset{(n-1)}{\alpha(n-1)} \xleftarrow{\alpha(n \to n-1)} \underset{(n)}{\alpha(n)}$$

[1] We omit the objects and arrows that do not belong to the nontrivial part. Identities are also omitted. The position of the objects is indicated in parentheses below them.

Example 5.9.

(i) (Constant path) A constant functor $\alpha\colon \mathbf{\Lambda} \to \mathbf{C}$ is a path in $\mathbf{C}$ as it can be verified from the definition. We call α a *constant path* at A, where A is the $\mathbf{C}$-object corresponding to the constant value of α. It is denoted $\hat{A}$.

(ii) (Functors preserve paths) If α is a path in $\mathbf{C}$ and $F\colon \mathbf{C} \to \mathbf{D}$ is any functor, then $F \circ \alpha$ is a path in $\mathbf{D}$.

(iii) (Inverse path) If $\alpha\colon \mathbf{\Lambda} \to \mathbf{C}$ is a path, then the functor from $\mathbf{\Lambda}$ to $\mathbf{C}$ defined by $k \mapsto \alpha(-k)$ on objects and by $(i \to j) \mapsto \alpha(-i \to -j)$ on morphisms is also a path in $\mathbf{C}$. We call this functor the *inverse path* of α, and denote it $\overline{\alpha}$. Note that if α is a mn-path from A to B, then $\overline{\alpha}$ is a $(-n, -m)$-path from B to A. On the other hand, note that if $F\colon \mathbf{C} \to \mathbf{D}$ is any functor, then $\overline{F \circ \alpha} = F \circ \overline{\alpha}$.

The first and third examples are the categorical analog notions of constant and inverse paths of classical homotopy theory.

(iv) (A nonexample) Let A be a $\mathbf{C}$-object and $f\colon A \to A$. Let α be the functor that maps every $\mathbf{\Lambda}$-object to A and every $\mathbf{\Lambda}$-arrow to f. Then α is not a path since it does not satisfies the finiteness condition, namely that a path must become eventually constant to the left and to the right.

(v) (Concatenation of paths) If α is a path from A to B and β is a path from B to C, we say α and β are *composable*, or able to be concatenated. In other words, α and β are composable if $\alpha(\max\alpha) = \beta(\min\beta)$. Suppose α and β are composable paths. Let

$$(m, n, p, q) = (\min\alpha, \max\alpha, \min\beta, \max\beta).$$

Define a functor from $\mathbf{\Lambda}$ to $\mathbf{C}$ by[2]

$$k \mapsto \begin{cases} \alpha(k) & \text{if } k \leq n, \\ \beta(k + p - n) & \text{if } k \geq n, \end{cases} \tag{5.1}$$

on objects and by

$$i \to j \quad \mapsto \quad \begin{cases} \alpha(i \to j) & \text{if } j \leq n, \\ \beta(i + p - n \to j + p - n) & \text{if } j \geq n, \end{cases} \tag{5.2}$$

on morphisms.

[2] Notice that there is no problem of definition on the objects when $k = n$ because $\alpha(n) = B = \beta(p)$. A similar argument shows this functor is well-defined on morphisms.

Remark 5.10. By construction, this functor is unique, and it follows from the definition that it is a $(m, q - p + n)$-path from A to C.

Definition 5.11 *Product of paths*

The *product* $\alpha \cdot \beta$ of two composable paths α and β is the path defined by (5.1) and (5.2).

Remark 5.12. With this "operation" we are headed to the categorical notion of the fundamental group of a finite category. Observe that $\cdot$ is not an actual operation on the set of $\mathbf{C}$-paths since it is not defined for every pair of elements of this set. Rather, it should be thought of a way of producing new paths from old ones whenever the concatenation is possible.

Theorem 5.13 *Associativity of* $\cdot$

Let α, β, and γ be paths in $\mathbf{C}$. Then

$$(\alpha \cdot \beta) \cdot \gamma = \alpha \cdot (\beta \cdot \gamma) \tag{5.3}$$

whenever the products are defined.

Proof. Let

$$\begin{aligned} m &= \min \alpha, \quad & p &= \min \beta, \quad & r &= \min \gamma, \\ n &= \max \alpha, \quad & q &= \max \beta, \quad & s &= \max \gamma. \end{aligned}$$

Suppose the products in (5.3) are defined. Note $\max \alpha \cdot \beta = q - p + n$ and $\min \beta \cdot \gamma = p$. By definition,

$$\begin{aligned} (\alpha \cdot \beta) \cdot \gamma(k) &= \begin{cases} \alpha \cdot \beta(k) & \text{if } k \leq q - p + n, \\ \gamma(k + r - q + p - n) & \text{otherwise} \end{cases} \\ &= \begin{cases} \alpha(k) & \text{if } k \leq n, \\ \beta(k + p - n) & \text{if } n \leq k \leq q - p + n, \\ \gamma(k + r - q + p - n) & \text{else} \end{cases} \end{aligned}$$

and

$$\begin{aligned} \alpha \cdot (\beta \cdot \gamma)(k) &= \begin{cases} \alpha(k) & \text{if } k \leq n, \\ \beta \cdot \gamma(k + p - n) & \text{else} \end{cases} \\ &= \begin{cases} \alpha(k) & \text{if } k \leq n, \\ \beta(k + p - n) & \text{if } n \leq k \leq q - p + n, \\ \gamma(k + p - n + r - q) & \text{else.} \end{cases} \end{aligned}$$

A similar computation shows that $(\alpha \cdot \beta) \cdot \gamma$ and $\alpha \cdot (\beta \cdot \gamma)$ agree on morphisms too, completing the proof. □

In view of Theorem 5.13, we will write $\alpha \cdot \beta \cdot \gamma$ to denote either $(\alpha \cdot \beta) \cdot \gamma$ or $\alpha \cdot (\beta \cdot \gamma)$. As a consequence of this result,

$$(\alpha \cdot \beta) \cdot \gamma \sim \alpha \cdot (\beta \cdot \gamma).$$

Now we embark on the task of description of paths in a finite category. This will prove useful later when we define the fundamental group a finite category. The following examples present the type of paths we will deal with in the next section.

Example 5.14.

(i) (Induced paths) Given a $\mathbf{C}$-morphism $f\colon A \to B$, there is a natural way of obtaining new paths out of f. Define a functor from $\mathbf{\Lambda}$ to $\mathbf{C}$ by

$$k \mapsto \begin{cases} A & \text{if } k \leq 2n, \\ B & \text{otherwise,} \end{cases} \qquad i \to j \quad \mapsto \begin{cases} \mathbb{1}_A & \text{if } i,j \leq 2n, \\ f & \text{if } (i,j) = (2n, 2n+1), \\ \mathbb{1}_B & \text{if } i,j \geq 2n+1, \end{cases}$$

on objects and morphisms, respectively, where $n \in \mathbb{Z}$. By construction, this functor, which we denote $f^{(2n)}$, is a path from A to B. Namely, we have "embedded" f into this path at position $2n$, to the right. We call $f^{(2n)}$ the *induced path* in $\mathbf{C}$ by f at position $2n$. This path is represented by the following diagram.[3]

$$f^{(2n)} \; : \qquad \underset{(2n)}{A} \xrightarrow{\;f\;} \underset{(2n+1)}{B} \longleftarrow \underset{(2n+2)}{B}$$

The inverse of $f^{(2n)}$ is given by the diagram

$$\overline{f^{(2n)}} \; : \qquad \underset{(-2n-2)}{B} \longrightarrow \underset{(-2n-1)}{B} \xleftarrow{\;f\;} \underset{(-2n)}{A}$$

which is a path from B to A.

(ii) (Subpaths) Let α be a path in $\mathbf{C}$, and $i \leq j$ two even integers. Define $\alpha_i^j\colon \mathbf{\Lambda} \to \mathbf{C}$ by

$$k \mapsto \begin{cases} \alpha(i) & \text{if } k \leq i, \\ \alpha(k) & \text{if } i \leq k \leq j, \\ \alpha(j) & \text{if } k \geq j, \end{cases} \qquad k \to l \quad \mapsto \begin{cases} \alpha(i \to j) & \text{if } l \leq i, \\ \alpha(k \to l) & \text{if } i \leq k, l \leq j, \\ \alpha(j \to j) & \text{if } l \geq j, \end{cases}$$

on objects and morphisms, respectively. We call any functor of this form a *subpath* of α. Thus, a subpath of α is a path β such that $\beta = \alpha_i^j$ for some even integers $i \leq j$. It is clear from the definition that α_i^j is a ij-path from $\alpha(i)$ to $\alpha(j)$. Note that if α is a mn-path, then $\alpha = \alpha_i^j$ for every pair of even integers $i \leq m$ and $j \geq n$. If $i = \min \alpha_i^j$ and $j = \max \alpha_i^j$, we call α_i^j a proper subpath of α.

(iii) (Translation of paths) Let α be a $\mathbf{C}$-path. Fix an integer n. Let $\beta\colon \mathbf{\Lambda} \to \mathbf{C}$ be the path defined by

$$\beta(k) = \alpha(k+2n) \quad \text{and} \quad \beta(i \to j) = \alpha(i+2n \to j+2n).$$

We say that β has been obtained by a translation of α by $2n$. We denote $\beta = \alpha[2n]$. It follows, by construction, that

$$\alpha[n][m] = \alpha[n+m]$$

for any $m, n \in 2\mathbb{Z}$.

[3] An alternative representation is given by the simplified diagram

$$A_{(2n)} \xrightarrow{\;f\;} B \longleftarrow B$$

Here the subscript $(2n)$ does not represent any property of the object A, but it is a shorthand to denote the position at where f is based. If possible, we will avoid this use of subscripts.

5.4. Homotopy of Paths

Suppose throughout that **C** is a finite category.

In this section we present a relation that allows us to identify paths that are "almost the same." Roughly speaking, two paths are considered to be almost the same if they have the same morphisms contained in their nontrivial parts, regardless of where their nontrivial parts are located, or if one can be obtained from the other by following a commutative diagram.

Definition 5.15 *Homotopy of paths*

Let $\alpha, \beta \in \Phi_{\mathbf{C}}(A, B)$ and $(m, n, p, q) = (\min \alpha, \max \alpha, \min \beta, \max \beta)$. A *path homotopy* $\mathcal{H}$ from α to β is a homotopy (of functors) from α to β that satisfies

(i) $\mathcal{H}(k, -) = \hat{A}$ if $k \leq \min\{m, p\}$, and

(ii) $\mathcal{H}(k, -) = \hat{B}$ if $k \geq \max\{n, q\}$.

If there is a path homotopy from α to β, then α is path homotopic to β, which is denoted $\alpha \sim \beta$.

As in the case of homotopy of functors, we write $\mathcal{H}\colon \alpha \sim \beta$ to indicate that $\mathcal{H}$ is a path homotopy from α to β.

Proposition 5.16

$\sim$ is an equivalence relation in $\Phi_{\mathbf{C}}(A, B)$.

Proof. This proof is similar to that of Proposition 5.6, but here we prove that, in addition, (i) and (ii) of Definition 5.15 hold. Fix $\alpha, \beta, \gamma \in \Phi(A, B)$ and let $(m, n, p, q, r, s) = (\min \alpha, \max \alpha, \min \beta, \max \beta, \min \gamma, \max \gamma)$.

(i) (Reflexivity[4]) Define $\mathcal{H}\colon \mathbf{\Lambda} \times \mathbf{\Lambda} \to \mathbf{D}$ by $\mathcal{H}(i, j) = \alpha(i)$ on objects and by $\mathcal{H}(\phi, \psi) = \alpha(\phi)$ on morphisms. By Proposition 5.6, $\mathcal{H}\colon \alpha \simeq \alpha$. Now notice that $\mathcal{H}(i, j) = A = \hat{A}(j)$ when $k \leq m$, and $\mathcal{H}(i, j) = B = \hat{B}(j)$ for $k \geq n$. Therefore, $\mathcal{H}(k, -) = \hat{A}$ if $k \leq m$ and $\mathcal{H}(k, -) = \hat{B}$ if $k \geq n$. Thus $\mathcal{H}\colon \alpha \sim \alpha$.

(ii) (Symmetry) Suppose $\mathcal{H}\colon \alpha \sim \beta$. Define $\mathcal{H}'\colon \mathbf{\Lambda} \times \mathbf{\Lambda} \to \mathbf{C}$ by

$$\mathcal{H}'(i, j) = \mathcal{H}(i, -j)$$

on objects and by

$$\mathcal{H}'(i \to j, i' \to j') = \mathcal{H}(i \to j, -i' \to -j')$$

on morphisms. As noted in the proof of Proposition 5.6, $\mathcal{H}'\colon \alpha \simeq \beta$. Since $\mathcal{H}'(i, j) = \mathcal{H}(i, -j) = \hat{A}(-j) = A = \hat{A}(j)$ for $i \leq \min\{m, p\}$, we have $\mathcal{H}'(i, -) = \hat{A}$ when $i \leq \min\{m, p\}$. Similarly, $\mathcal{H}'(i, -) = \hat{B}$ if $i \geq \max\{n, q\}$ Therefore $\mathcal{H}'\colon \beta \sim \alpha$.

[4] Note that (i) shows that $\alpha \simeq \alpha$ is equivalent to $\alpha \sim \alpha$ for any path α.

(iii) (Transitivity) Suppose $\alpha \sim \beta$ and $\beta \sim \gamma$. Let $\mathcal{H}$ be a path homotopy from α to β with even integers $p < q$ and $r < s$ such that

$$\mathcal{H}(-,l) = \begin{cases} \alpha & \text{if } l \leq p, \\ \beta & \text{if } l \geq q, \end{cases} \quad \text{and} \quad \mathcal{H}(k,-) = \begin{cases} \hat{A} & \text{if } k \leq r, \\ \hat{B} & \text{if } k \geq s. \end{cases}$$

Likewise, let $\mathcal{I}$ be a path homotopy from β to γ, with even integers $p' < q'$ and $r' < s'$, defined analogously. Define $\mathcal{J} \colon \mathbf{\Lambda} \times \mathbf{\Lambda} \to \mathbf{C}$, as in the proof of Proposition 5.6, by

$$\mathcal{J}(k,l) = \begin{cases} \mathcal{H}(k, l+q) & \text{if } k \leq 0, \\ \mathcal{I}(k, l+p') & \text{if } k > 0, \end{cases}$$

on objects, and with the obvious modifications on morphisms. Then $\mathcal{J}$ is a homotopy from α to γ, meaning $\alpha \simeq \gamma$. Finally, note that if $k \leq \min\{0, r\}$, then $\mathcal{J}(k,l) = \mathcal{H}(k, l+q) = \hat{A}(l+q) = A$ for any $l \in \mathbb{Z}$. Thus $\mathcal{J}(k,-) = \hat{A}$ whenever $k \leq \min\{0, r\}$. Similarly, $\mathcal{J}(k,-) = \hat{B}$ if $k \geq \max\{0, s'\}$. This proves that $\alpha \sim \gamma$.

□

Figure 5.1 shows the structure of $\mathbf{\Lambda} \times \mathbf{\Lambda}$, and that diagram will helps us to better understand the notion of homotopy of paths. We can think of a functor $\mathbf{\Lambda} \times \mathbf{\Lambda} \to \mathbf{C}$ as a diagram that emerges from embedding $\mathbf{\Lambda} \times \mathbf{\Lambda}$ into $\mathbf{C}$.

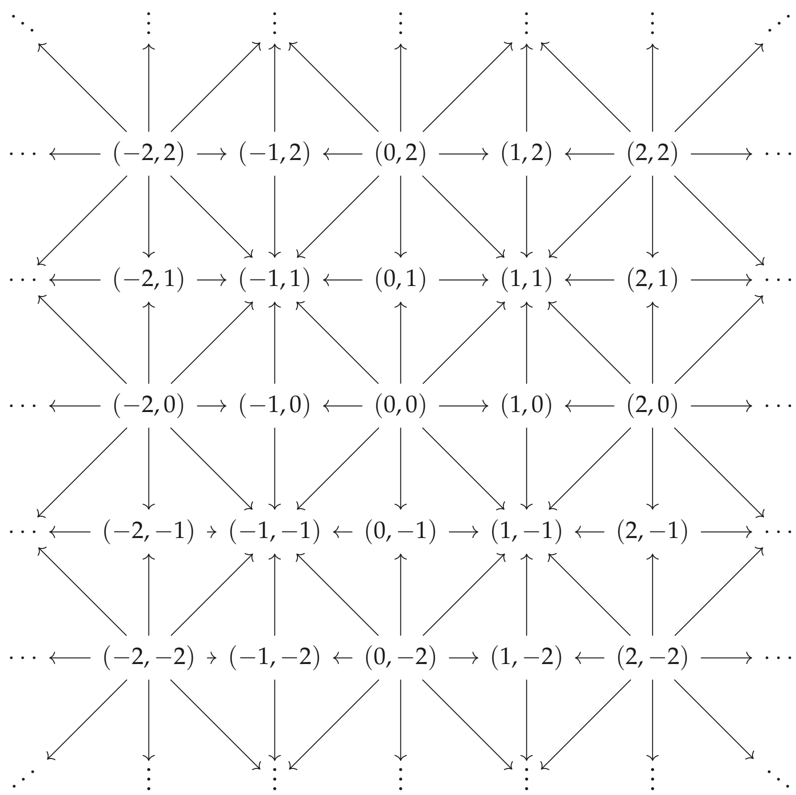


Figure 5.1: Category $\mathbf{\Lambda} \times \mathbf{\Lambda}$.

The following example is very illustrative. It shows why two paths are path-homotopy equivalent if one of them has been obtained by a translation of the other.

Example 5.17. Let α be the path given by the following diagram.

$$\underset{(0)}{A} \xrightarrow{f} B \xleftarrow{g} C \xrightarrow{\phi} D \xleftarrow{\psi} E$$

The commutative diagram of Figure 5.2 gives a homotopy of paths from $\alpha[2]$ to α. The fact that it is commutative follows from the fact that each square commutes.

Remark 5.18. (i) Not only the functor presented in this example gives us an idea of how to reallocate an arbitrary path to any position, but it also suggest a way to eliminate pairs of identity arrows out of a path.

(ii) For instance, the second row is a path which contains a pair of identities of the form $\leftarrow \cdot \rightarrow$, the next row was obtained out of the second by flipping the position of g and the pair $\leftarrow \cdot \rightarrow$, getting a new path with a pair of identities of the form $\rightarrow \cdot \leftarrow$. Continuing this way until the bottom we obtain a new path which is path-homotopic to the original but does not contain any pair of identity arrows.

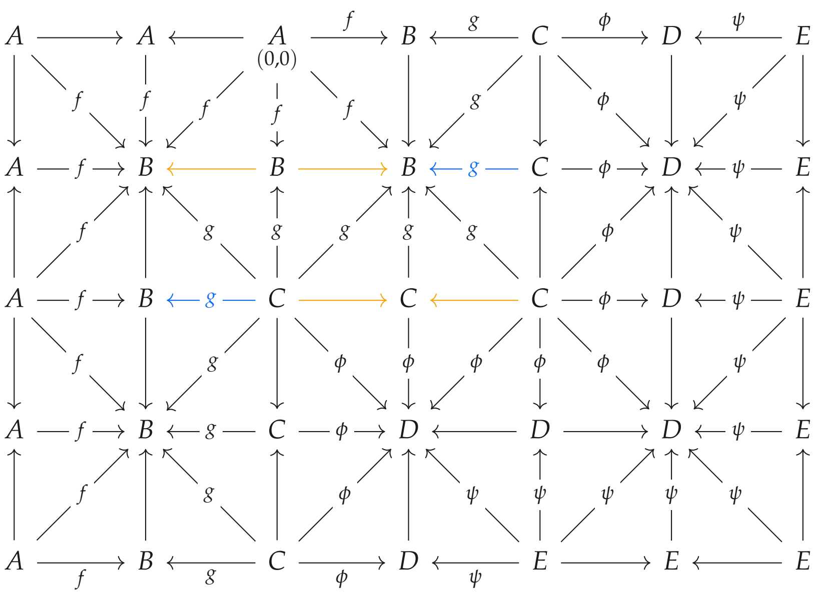


Figure 5.2: Example 5.17.

Note that in order to construct the homotopy of paths given by the diagram of Figure 5.2, we have moved each arrow one position at a time, wherein a pair of identity arrows (either of the form $\rightarrow \cdot \leftarrow$ or the form $\leftarrow \cdot \rightarrow$) have been inserted.

We now generalize the idea presented in Example 5.17. To begin with, we shall present a way to obtain new paths from old ones by inserting pairs of identities arrows of the form $\to \cdot \leftarrow$ or the form $\leftarrow \cdot \to$, depending on the position at where the insertion is made. Given a path α in $\mathbf{C}$ and $n \in \mathbb{Z}$, define $\Gamma_n\alpha \colon \mathbf{\Lambda} \to \mathbf{C}$ by

Notice that the insertion is always made to the left of the object $\alpha(n)$, which is the reason why the data is moved two positions to the left.

$$\Gamma_n\alpha(k) = \begin{cases} \alpha(k+2) & \text{if } k \leq n-2, \\ \alpha(n) & \text{if } k = n-1, \\ \alpha(k) & \text{else} \end{cases}$$

on objects. Define $\Gamma_n\alpha$ on morphisms as follows: if n is even,

$$\Gamma_n\alpha(i \to j) = \begin{cases} \alpha(i+2 \to j+2) & \text{if } j \leq n-2, \\ \mathbb{1}_{\alpha(n)} & \text{if } j = n-1, \\ \alpha(i \to j) & \text{if } j \geq n, \end{cases}$$

if n is odd,

$$\Gamma_n\alpha(i \to j) = \begin{cases} \alpha(i+2 \to j+2) & \text{if } i \leq n-3, \\ \mathbb{1}_{\alpha(n)} & \text{if } i = n-1, \\ \alpha(i \to j) & \text{if } i \geq n+1. \end{cases}$$

Remark 5.19. It follows, by construction, that

$$\Gamma_p\alpha = \alpha \quad \text{for all } p \leq \min \alpha.$$

Namely, inserting a pair of identities before $\min \alpha$ leaves the path unchanged because α is constant before $\min \alpha$.

Proposition 5.20

For any path α and any $n \in \mathbb{Z}$,

$$\Gamma_n\alpha \sim \Gamma_{n+1}\alpha.$$

Proof. Let α be a path from A to B. We do the proof for the case when n is odd as the other case follows a similar reasoning. Let $\varphi = \alpha(n+1 \to n)$. Define $\mathcal{H}_n \colon \mathbf{\Lambda} \times \mathbf{\Lambda} \to \mathbf{C}$ by $\mathcal{H}_n(i,j) = \Gamma_n\alpha(i)$ on objects and by

$$\mathcal{H}_n(i \to j, k \to l) = \Gamma_n\alpha(i \to j)$$

on morphisms. Define $\mathcal{H}_{n+1}$ analogously. The reader should notice that $\mathcal{H}_n$ is the homotopy of paths defined in the proof of Proposition 5.16 that proves that $\sim$ is reflexive. Then

$$\mathcal{H}_n \colon \Gamma_n\alpha \sim \Gamma_n\alpha \quad \text{and} \quad \mathcal{H}_{n+1} \colon \Gamma_{n+1}\alpha \sim \Gamma_{n+1}\alpha.$$

Now define $\mathcal{H}\colon \mathbf{\Lambda} \times \mathbf{\Lambda} \to \mathbf{C}$ on objects by

$$\mathcal{H}(i,j) = \begin{cases} \mathcal{H}_n(i,j) & \text{if } j \leq 0, \\ \mathcal{H}_{n+1}(i,j) & \text{otherwise,} \end{cases}$$

and on morphisms by

$$\mathcal{H}(i \to j, i' \to j') = \begin{cases} \mathcal{H}_n(i \to j, i' \to j') & \text{if } i', j' \leq 0 \text{ or } i, j \leq n-2, \\ \mathcal{H}_{n+1}(i \to j, i' \to j') & \text{if } i', j' \geq 1 \text{ or } i, j \geq n+1, \\ \varphi & \text{otherwise.} \end{cases}$$

Note that $\mathcal{H}(i \to j, i' \to j') = \varphi$ precisely when $i, j \in \{n-2, n-1, n, n+1\}$ and $(i', j') = (0, 1)$. Let us prove $\mathcal{H}\colon \Gamma_n\alpha \sim \Gamma_{n+1}\alpha$. This follows immediately from the definition. For any $k \leq 0$, we have

$$\mathcal{H}(i,k) = \mathcal{H}_n(i,k) = \Gamma_n\alpha(i),$$

and also

$$\mathcal{H}(i \to j, \mathbb{1}_k) = \mathcal{H}_n(i \to j, \mathbb{1}_k) = \Gamma_n\alpha(i \to j).$$

Therefore, $\mathcal{H}(-,k) = \Gamma_n\alpha$ for every $k \leq 0$. In an entirely analogous manner it follows that $\mathcal{H}(-,k) = \Gamma_{n+1}\alpha$ for every $k \geq 2$. Up to this point we have shown $\mathcal{H}\colon \Gamma_n\alpha \simeq \Gamma_{n+1}\alpha$. To finish the proof, take $M = \min\{\min \Gamma_n\alpha, \min \Gamma_{n+1}\alpha\}$ and notice that if $i \leq M$, then

$$\mathcal{H}(i,j) = \begin{cases} \Gamma_n\alpha(i) & \text{if } j \leq 0, \\ \Gamma_{n+1}\alpha(i) & \text{otherwise} \end{cases} = \begin{cases} A & \text{if } j \leq 0, \\ A & \text{otherwise.} \end{cases}$$

Thus, $\mathcal{H}(k,-) = \hat{A}$ for every $k \leq M$. Let $N = \max\{\max \Gamma_n\alpha, \max \Gamma_{n+1}\alpha\}$. If $i \geq N$,

$$\mathcal{H}(i,j) = \begin{cases} \Gamma_n\alpha(i) & \text{if } j \leq 0, \\ \Gamma_{n+1}\alpha(i) & \text{otherwise} \end{cases} = \begin{cases} B & \text{if } j \leq 0, \\ B & \text{otherwise.} \end{cases}$$

Thus, $\mathcal{H}(k,-) = \hat{B}$ for every $k \geq N$. The proof is now complete. □

Corollary 5.21

For any path α and any $n \in \mathbb{Z}$,

$$\alpha \sim \Gamma_n\alpha.$$

Proof. Let $m = \min \alpha$. If $n \leq m$, there is nothing to prove because of Remark 5.19. Suppose $n > m$. Since $\alpha = \Gamma_m\alpha$, by Proposition 5.20 we have

$$\alpha \sim \Gamma_{m+1}\alpha \sim \Gamma_{m+2}\alpha \sim \cdots \sim \Gamma_{m+(n-m)}\alpha = \Gamma_n\alpha,$$

as claimed. □

Suppose α is a path in $\mathbf{C}$. Observe that, from a combinatorial point of view, both α and $\alpha[2n]$ encode the same information, for any $n \in \mathbb{Z}$. The only difference being the position of its nontrivial parts. Naturally, we should expect these paths to be path-homotopy equivalent and the following result establishes this claim.

Theorem 5.22 *Invariance of $\sim$ under translation*

For any path α and any $n \in 2\mathbb{Z}$, we have $\alpha \sim \alpha[n]$.

Proof. Suppose α is a path with $m = \min \alpha$ and $p = \max \alpha$. It is enough to prove $\alpha \sim \alpha[2]$ since the result follows inductively from this particular case. By Proposition 5.20,

$$\alpha = \Gamma_m \alpha \sim \Gamma_{m+1} \alpha \sim \cdots \sim \Gamma_p \alpha = \alpha[2],$$

as desired. Now, since α was arbitrary, and using the fact that

$$\alpha[i][j] = \alpha[i+j]$$

for any $i, j \in 2\mathbb{Z}$, we obtain

$$\alpha \sim \alpha[2] \sim \alpha[4] \sim \cdots \sim \alpha[n].$$

The result is established. $\square$

The following definition characterizes paths that do not have any pair of identity arrows of either the form $\rightarrow \cdot \leftarrow$ or the form $\leftarrow \cdot \rightarrow$ in their nontrivial part.

Definition 5.23 *Reduced path*

Let α be a path from A to B. We say α is *reduced* if

$$\alpha \neq \Gamma_n \beta$$

for any $\beta \in \Phi(A, B)$ and every $\min \alpha < n < \max \alpha$.

It follows, vacuously, that any constant path is reduced, because there is no integer n such that $0 < n < 0$.

Example 5.24. Assume the following paths start at the right of 0.

(i) The path

$$X_0 \xrightarrow{f_1} X_1 \xleftarrow{f_2} X_2 \xrightarrow{f_3} X_3 \xleftarrow{f_4} X_4$$

is reduced. Here, we are implicitly assuming that adjacent objects are distinct.

(ii) Neither

$$X_0 \xrightarrow{f_1} X_1 \longleftarrow X_1 \longrightarrow X_1 \xleftarrow{f_2} X_2$$

nor

$$X_0 \xleftarrow{f_1} X_1 \longrightarrow X_1 \longleftarrow X_1 \xrightarrow{f_2} X_2$$

are reduced paths. Both contain a pair of identity arrows. The first contains a pair of the form $\leftarrow \cdot \rightarrow$ and the second contains a pair of the form $\rightarrow \cdot \leftarrow$.

(iii) If α is reduced, then $\alpha[2n]$ is reduced for any $n \in \mathbb{Z}$. This is due to the fact that reallocation of a path does not change its combinatorial data.

Theorem 5.25 *Pruning*

Every path is path-homotopic to a reduced path.

Proof. Let α be a path. If α is either constant or reduced, there is nothing to prove, because, in either case, α is already reduced and path-homotopic to itself by reflexivity of $\sim$. Suppose α is nonconstant and not reduced. We proceed by induction on the number of pairs of identity arrows contained in the nontrivial part of α. For the base case, if α has a pair of identity arrows, then $\alpha = \Gamma_n\beta$ for some integer n and some reduced path β. Then

$$\alpha \sim \beta,$$

by Corollary 5.21. Now, let k be an arbitrary but fixed positive integer, and suppose that if α' has k pairs of identity arrows in its nontrivial part, then α' is homotopic to a reduced path. Let us prove the claim for the case when α has $k+1$ pairs of identity arrows in its nontrivial part. In this case, we have $\alpha = \Gamma_n\gamma$ for some integer n and some path γ, where γ has k pairs of identity arrows in its nontrivial part. By the inductive hypothesis, γ is path-homotopic to a reduced path. By Corollary 5.21, $\alpha \sim \gamma$ and thus α is path-homotopic to a reduced path as well. The principle of mathematical induction proves the result. □

From now and on we refer to Theorem 5.25 as the *pruning theorem*.

Theorem 5.26 $\cdot$ *preserves* $\sim$

If $\alpha \sim \alpha'$ and $\beta \sim \beta'$, and if $\alpha \cdot \beta$ is defined, then $\alpha' \cdot \beta'$ is defined and

$$\alpha \cdot \beta \sim \alpha' \cdot \beta'.$$

Proof. Let α, α' be paths from A to B and β, β' paths from B to C so that $\alpha \cdot \beta$ is defined, where A, B, C are $\mathbf{C}$-objects. If $\alpha \sim \alpha'$, then α' is also a path from A to B, necessarily. Similarly, if $\beta \sim \beta'$, then β is a path from B to C. Thus $\alpha' \cdot \beta'$ is defined. We now proceed to prove the result by considering a special case first. The general case will follow from this case once it is proved.

(i) Suppose $\max \alpha = \max \alpha'$ and $\min \beta = \min \beta'$. Let $\mathcal{H} \colon \alpha \sim \alpha'$ be a aa'-path-homotopy and $\mathcal{G} \colon \beta \sim \beta'$ a bb'-path-homotopy. Denote $n = \max \alpha$ and $p = \min \beta$. Define $\mathcal{F} \colon \mathbf{\Lambda} \times \mathbf{\Lambda} \to \mathbf{C}$ by

$$\mathcal{F}(i,j) = \begin{cases} \mathcal{H}(i,j) & \text{if } i \leq n, \\ \mathcal{G}(i+p-n,j) & \text{otherwise,} \end{cases}$$

on objects, and by

$$\mathcal{F}(i \to j, i' \to j') = \begin{cases} \mathcal{H}(i \to j, i' \to j') & \text{if } j \leq n, \\ \mathcal{G}(i+p-n \to j+p-n, i' \to j') & \text{otherwise,} \end{cases}$$

on morphisms. Let us show $\mathcal{F} \colon \alpha \cdot \alpha' \sim \beta \cdot \beta'$. If $k \leq \min\{a,b\}$, then, for any $i \in \mathbb{Z}$,

$$\mathcal{F}(i,k) = \begin{cases} \alpha(i) & \text{if } i \leq n, \\ \beta(i+p-n) & \text{otherwise,} \end{cases} = \alpha \cdot \beta(i)$$

and, for any $\mathbf{\Lambda}$-arrow $i \to j$,

$$\mathcal{F}(i \to j, \mathbb{1}_k) = \begin{cases} \alpha(i \to j) & \text{if } j \leq n, \\ \beta(i+p-n \to j+p-n) & \text{otherwise,} \end{cases}$$

which equals $\alpha \cdot \beta(i \to j)$, by definition of concatenation of paths (Definition 5.11). Hence,

$$\mathcal{F}(-,k) = \alpha \cdot \beta$$

for every $k \leq \min\{a,b\}$. In an entirely similar manner, we obtain $\mathcal{F}(-,k) = \alpha' \cdot \beta'$ for every $k \geq \max\{a',b'\}$. Finally, we prove

$$\mathcal{F}(k,-) = \begin{cases} \hat{A} & \text{if } k \leq \min\{\min(\alpha \cdot \beta), \min(\alpha' \cdot \beta')\}, \\ \hat{C} & \text{if } k \geq \max\{\max(\alpha \cdot \beta), \max(\alpha' \cdot \beta')\}. \end{cases} \tag{5.4}$$

To this end, notice

$$\min\{\min(\alpha \cdot \beta), \min(\alpha' \cdot \beta')\} = \min\{\min \alpha, \min \alpha'\}$$

by Remark 5.10. Also,

$$\max(\alpha \cdot \beta) = \max \beta - p + n \quad \text{and} \quad \max(\alpha' \cdot \beta') = \max \beta' - p + n,$$

whence

$$\max\{\max(\alpha \cdot \beta), \max(\alpha' \cdot \beta')\} = \max\{\max \beta, \max \beta'\} - p + n.$$

Thus, (5.4) becomes

$$\mathcal{F}(k,-) = \begin{cases} \hat{A} & \text{if } k \le \min\{\min \alpha, \min \alpha'\}, \\ \hat{C} & \text{if } k \ge \max\{\max \beta, \max \beta'\} - p + n. \end{cases}$$

The first equality follows from the fact $\min\{\min \alpha, \min \alpha'\} \le n$. To see the second equality, note that if $k \ge \max\{\max \beta, \max \beta'\} - p + n$, then $k \ge \max \beta - p + n \ge n$, so, for any $j \in \mathbb{Z}$,

$$\mathcal{F}(k,j) = \mathcal{G}(k+p-n,j) = C,$$

where the last equality is due to $k+p-n \ge \max\{\max \beta, \max \beta'\}$. Similarly, we get $\mathcal{F}(\mathbb{1}_k, i \to j) = \mathbb{1}_C$ for any $\mathbf{\Lambda}$-arrow $i \to j$. We have proved this special case.

(ii) Suppose $\alpha \sim \alpha'$ and $\beta \sim \beta'$. By translating α and β to suitable positions, we can apply the result of the case just proved. Indeed, let $m = \max \alpha - \max \alpha'$ and $n = \min \beta - \min \beta'$. Then

$$\max \alpha[m] = \max \alpha' \quad \text{and} \quad \min \beta[n] = \min \beta'.$$

Thus, by (i), $\alpha[m] \cdot \beta[n] \sim \alpha' \cdot \beta'$. However,

$$\alpha[m] \cdot \beta[n] = \alpha[m] \cdot \beta = \alpha \cdot \beta[m].$$

Since $\alpha \cdot \beta \sim (\alpha \cdot \beta)[m]$ by Theorem 5.22, we conclude $\alpha \cdot \beta \sim \alpha' \cdot \beta'$.

End of the proof. □

Theorem 5.27

If α is a path in $\mathbf{C}$ from A to B, then

(i) $\hat{A} \cdot \alpha \sim \alpha$, and

(ii) $\alpha \cdot \hat{B} \sim \alpha$.

Proof. Let $m = \min \alpha$ and $n = \max \alpha$. Recall that the minimum and maximum of a constant path are both equal to zero by convention.[5]

(i) Let us prove $\hat{A} \cdot \alpha = \alpha[m]$. On one hand, by definition of multiplication of paths,

$$\hat{A} \cdot \alpha(k) = \begin{cases} \hat{A}(k) & \text{if } k \le 0, \\ \alpha(k+m) & \text{else.} \end{cases}$$

Note that $\alpha(k+m) = A = \hat{A}(k)$ whenever $k+m \le m$, i.e., whenever $k \le 0$. This means that $\hat{A} \cdot \alpha(k)$ always equals $\alpha(k+m)$ regardless of whether $k \le 0$ or $k > 0$. Thus

$$\hat{A} \cdot \alpha(k) = \alpha(k+m)$$

[5] Note that the effect of multiplying by a constant path on the left produces a translation to the right of the zero.

for every $k \in \mathbb{Z}$. On the other hand, in order to see the equality on morphisms, note that

$$\hat{A} \cdot \alpha(i \to j) = \begin{cases} \hat{A}(i \to j) & \text{if } j \leq 0, \\ \alpha(i+m \to j+m) & \text{else,} \end{cases}$$

and use the fact that

$$\alpha(i+m \to j+m) = \mathbb{1}_A = \hat{A}(i \to j)$$

whenever $j+m \leq m$, i.e., when $j \leq 0$. Then

$$\hat{A} \cdot \alpha(i \to j) = \alpha(i+m \to j+m) = \alpha[m](i \to j)$$

for every $i, j \in \mathbb{Z}$ with $i \preccurlyeq j$. We have proven $\hat{A} \cdot \alpha = \alpha[m]$. Finally, by Theorem 5.22, it follows $\hat{A} \cdot \alpha \sim \alpha$.

(ii) Again, by definition of multiplication of paths, we have

$$\alpha \cdot \hat{B}(k) = \begin{cases} \alpha(k) & \text{if } k \leq n, \\ \hat{B}(k-n) & \text{else.} \end{cases}$$

But $\hat{B}(k-n) = B = \alpha(k)$ for every $k > n$. Thus, $\alpha \cdot \hat{B}(k) = \alpha(k)$ for all $k \in \mathbb{Z}$. Similarly, since

$$\alpha \cdot \hat{B}(i \to j) = \begin{cases} \alpha(i \to j) & \text{if } j \leq n, \\ \hat{B}(i-n \to j-n) & \text{else,} \end{cases}$$

and $\hat{B}(i-n \to j-n) = \mathbb{1}_B = \alpha(i \to j)$ whenever $j > n$, it follows $\alpha \cdot \hat{B}(i \to j) = \alpha(i \to j)$ for any $\mathbf{\Lambda}$-arrow $i \to j$. Therefore, $\alpha \cdot \hat{B} = \alpha$ whence $\alpha \cdot \hat{B} \sim \alpha$.

The proof is complete. □

Lemma 5.28

Let $f\colon A \to B$ be a $\mathbf{C}$-arrow. Then

(i) $f \cdot \overline{f} = \hat{A}$, and

(ii) $\overline{f} \cdot f = \hat{B}$.

Proof. The path-homotopies that show (i) and (ii) are given by the following pair of commutative diagrams, respectively.

$$\begin{tikzcd} \underset{(2n,0)}{A} \arrow[r] \arrow[d] \arrow[dr, "f"] & A \arrow[d, "f"] & A \arrow[l] \arrow[dl, "f"] \arrow[d] \\ A \arrow[r, "f"'] & B & A \arrow[l, "f"] \end{tikzcd} \qquad \begin{tikzcd} B & B \arrow[l] \arrow[r] & B \\ B \arrow[u] & \underset{(2n+2,0)}{A} \arrow[l, "f"] \arrow[r, "f"'] \arrow[u, "f"] \arrow[ul, "f"] \arrow[ur, "f"] & B \arrow[u] \end{tikzcd}$$

□

Theorem 5.29 *Cancellation*

Let α be a path from A to B. Then

(i) $\alpha \cdot \overline{\alpha} = \hat{A}$, and

(ii) $\overline{\alpha} \cdot \alpha = \hat{B}$.

Proof. For (i) note that, since $\alpha \sim \alpha_m^{m+2} \cdot \alpha_{m+2}^{m+4} \cdots \alpha_{n-2}^{n}$,

$$\begin{aligned}
\alpha \cdot \overline{\alpha} &\sim \alpha_m^{m+2} \cdot \alpha_{m+2}^{m+4} \cdots \alpha_{n-4}^{n-2} \cdot \alpha_{n-2}^{n} \cdot \overline{\alpha_{n-2}^{n}} \cdot \overline{\alpha_{n-4}^{n-2}} \cdots \overline{\alpha_m^{m+2}} \cdot \overline{\alpha_m^{m+2}} \\
&\sim \alpha_m^{m+2} \cdot \alpha_{m+2}^{m+4} \cdots \alpha_{n-4}^{n-2} \cdot \overline{\alpha_{n-4}^{n-2}} \cdots \overline{\alpha_m^{m+2}} \cdot \overline{\alpha_m^{m+2}} \\
&\ \vdots \\
&\sim \alpha_m^{m+2} \cdot \overline{\alpha_m^{m+2}} \\
&\sim \hat{A}
\end{aligned}$$

where we have used Lemma 5.28 and Theorem 5.26. The proof for (ii) proceeds analogously. □

5.5. The Fundamental Group of a Finite Category

A category with base object is a pair $(\mathbf{C}, A)$ where $\mathbf{C}$ is a category and A is a $\mathbf{C}$-object. We say that $(\mathbf{C}, A)$ is a pointed category with base object at A.[6] The pointed categories that we consider in what follows are finite.

Let $\mathbf{C}$ be a finite category. For any path α in $\mathbf{C}$, we denote the path-homotopy equivalence class of α by $[\alpha]$ and call it the *path class* of α. In what follows, we are interested only in paths that start and end at the same object. Recall that such a path is called a loop. If α is a loop from A to A, we say α is a loop *based at* A. We call A the *base object* of α. The set of loops in $\mathbf{C}$ based at A is denoted $\Omega(\mathbf{C}, A)$. Recall the constant loop $\hat{A}$ is the path that maps every $\mathbf{\Lambda}$-object to A and every $\mathbf{\Lambda}$-arrow to $\mathbb{1}_A$. Proposition 5.16 says, in particular, that path-homotopy is an equivalence relation on $\Omega(\mathbf{C}, A)$. The set of path classes of loops based at A is denoted

$$\kappa_1(\mathbf{C}, A).$$

Let us see how to give $\kappa_1(\mathbf{C}, A)$ the structure of a group. If α and β are composable paths, it makes sense to define the *product of* their *path classes* to be the path class of their product, that is,

$$[\alpha] \cdot [\beta] = [\alpha \cdot \beta].$$

This operation is only defined when α and β are composable. However, any two loops based at A are composable and their product is again a loop based at A. Thus, such an operation is always defined for any two elements of $\Omega(\mathbf{C}, A)$. We must, nevertheless, verify that it is well defined.

[6] There is a category $\mathbf{Cat}_*$ of pointed small categories, whose objects are pairs $(\mathbf{A}, A)$ where $\mathbf{A}$ is a small category and A is a $\mathbf{A}$-object, referred to as the *base object*. The morphisms of this category are the functors $F\colon (\mathbf{A}, A) \to (\mathbf{B}, B)$ that preserve the base object, i.e., $F(A) = B$.

Proposition 5.30

Let $\cdot$ be the operation on $\kappa_1(\mathbf{C}, A)$ given by

$$[\alpha] \cdot [\beta] = [\alpha \cdot \beta].$$

Then $\cdot$ is well defined.

Proof. Suppose that $[\alpha] = [\alpha']$ and $[\beta] = [\beta']$, or equivalently, that $\alpha \sim \alpha'$ and $\beta \sim \beta'$. Note that α and β are composable since both are loops at A. Then, by Theorem 5.26, it follows $\alpha \cdot \beta \sim \alpha' \cdot \beta'$. Therefore

$$[\alpha \cdot \beta] = [\alpha' \cdot \beta']$$

whence $[\alpha] \cdot [\beta] = [\alpha'] \cdot [\beta']$, by definition. □

Remark 5.31. From now and on we may abbreviate the product of path classes by juxtaposition, that is, we write $[\alpha][\beta]$ instead of $[\alpha] \cdot [\beta]$.

Theorem 5.32

The set $\kappa_1(\mathbf{C}, A)$ is a group under the operation of product of path classes of loops based at A.

Proof. We have already seen that such an operation is defined *on* $\kappa_1(\mathbf{C}, A)$, meaning that $\kappa_1(\mathbf{C}, A)$ is closed under the product of path classes of loops based at A. Let us now prove the associativity of this operation and the existence of an identity and inverses.

(i) (Associativity) This follows from Theorem 5.13. Indeed, if $\alpha, \beta, \gamma \in \Omega(\mathbf{C}, A)$, then

$$\begin{aligned}([\alpha][\beta])[\gamma] = [\alpha \cdot \beta][\gamma] &= [(\alpha \cdot \beta) \cdot \gamma] \\ &= [\alpha \cdot (\beta \cdot \gamma)] \qquad \text{(by Theorem 5.13)} \\ &= [\alpha][\beta \cdot \gamma] \\ &= [\alpha]([\beta][\gamma]).\end{aligned}$$

(ii) (Identity) By Theorem 5.27, for any $\alpha \in \Omega(\mathbf{C}, A)$,

$$[\hat{A}][\alpha] = [\hat{A} \cdot \alpha] = [\alpha] = [\alpha \cdot \hat{A}] = [\alpha][\hat{A}].$$

Thus, the identity is the constant path $\hat{A}$.

(iii) (Inverses) By Theorem 5.29, for any $\alpha \in \Omega(\mathbf{C}, A)$,

$$[\alpha][\overline{\alpha}] = [\alpha \cdot \overline{\alpha}] = [\hat{A}] = [\overline{\alpha} \cdot \alpha] = [\overline{\alpha}][\alpha],$$

whence $[\alpha]^{-1} = [\overline{\alpha}]$.

□

Remark 5.33. By the associativity of $\cdot$, we can write $[\alpha][\beta][\gamma]$ to denote either $([\alpha][\beta])[\gamma]$ or $[\alpha]([\beta][\gamma])$, without any risk of ambiguity.

In light of Theorem 5.32, we call $\kappa_1(\mathbf{C}, A)$ the *fundamental group* of the category $\mathbf{C}$ with base object A.

5.5.1. The Role of Base Object

The following result is the categorical analog of Theorem 2.22.

Theorem 5.34 *Change of base object*

Let A and B be two objects of a finite category $\mathbf{C}$ and let α be a path from A to B. Define

$$\Upsilon_\alpha \colon \kappa_1(\mathbf{C}, A) \to \kappa_1(\mathbf{C}, B) \;:\; [\gamma] \mapsto [\,\overline{\alpha}\,] \cdot [\gamma] \cdot [\alpha].$$

Then

(i) if α and β are path-homotopic, then $\Upsilon_\alpha = \Upsilon_\beta$,

(ii) Υ_α is a group-homomorphism,

(iii) the map $\Upsilon_{\overline{\alpha}} \colon \kappa_1(\mathbf{C}, B) \to \kappa_1(\mathbf{C}, A) \;:\; [\gamma] \mapsto [\alpha] \cdot [\gamma] \cdot [\,\overline{\alpha}\,]$ is a two-sided inverse for Υ_α, so Υ_α is a group-isomorphism, whence $\kappa_1(\mathbf{C}, A) \cong \kappa_1(\mathbf{C}, B)$,

(iv) if α is constant, Υ_α is the identity map on $\kappa_1(\mathbf{C}, A)$,

(v) if β is a path from B to C, then $\Upsilon_{\alpha\cdot\beta} = \Upsilon_\beta \circ \Upsilon_\alpha$, i.e., the diagram

$$\begin{tikzcd}
\kappa_1(\mathbf{C}, A) \arrow[r, "\Upsilon_\alpha"] \arrow[dr, "\Upsilon_{\alpha\cdot\beta}"'] & \kappa_1(\mathbf{C}, B) \arrow[d, "\Upsilon_\beta"] \\
 & \kappa_1(\mathbf{C}, C)
\end{tikzcd}$$

commutes.

Proof. First of all, note that if γ is a loop based at A, then

$$\overline{\alpha} \cdot \gamma \cdot \alpha$$

is a path that goes from B to A (by $\overline{\alpha}$), then from A to B (by γ), and finally from B back to A (by α). Hence, $\Upsilon_\alpha[\gamma]$ does define an element of $\kappa_1(\mathbf{C}, B)$ for any $\gamma \in \Omega(\mathbf{C}, A)$.

(i) If $\alpha \sim \beta$, then $[\alpha] = [\beta]$, so

$$\Upsilon_\alpha[\gamma] = [\,\overline{\alpha}\,] \cdot [\gamma] \cdot [\alpha] = [\,\overline{\beta}\,] \cdot [\gamma] \cdot [\beta] = \Upsilon_\beta[\gamma]$$

for any $\gamma \in \Omega(\mathbf{C}, A)$. Thus $\Upsilon_\alpha = \Upsilon_\beta$.

(ii) For any $\beta, \beta' \in \Omega(\mathbf{C}, A)$, it holds

$$\begin{aligned} \Upsilon_\alpha[\beta] \cdot \Upsilon_\alpha[\beta'] &= ([\,\overline{\alpha}\,] \cdot [\beta] \cdot [\alpha]) \cdot ([\,\overline{\alpha}\,] \cdot [\beta'] \cdot [\alpha]) \\ &= [\,\overline{\alpha}\,] \cdot [\beta] \cdot [\hat{A}] \cdot [\beta'] \cdot [\alpha] \\ &= [\,\overline{\alpha}\,] \cdot [\beta] \cdot [\beta'] \cdot [\alpha] \\ &= \Upsilon_\alpha([\beta] \cdot [\beta']). \end{aligned}$$

Thus Υ_α is a group-homomorphism.

(iii) If $\beta \in \Omega(\mathbf{C}, A)$,

$$\Upsilon_{\overline{\alpha}}(\Upsilon_\alpha[\beta]) = [\alpha] \cdot ([\,\overline{\alpha}\,] \cdot [\beta] \cdot [\alpha]) \cdot [\,\overline{\alpha}\,] = [\,\hat{A}\,] \cdot [\beta] \cdot [\,\hat{A}\,] = [\beta]$$

and if $\gamma \in \Omega(\mathbf{C}, B)$,

$$\Upsilon_\alpha(\Upsilon_{\overline{\alpha}}[\gamma]) = [\,\overline{\alpha}\,] \cdot ([\alpha] \cdot [\gamma] \cdot [\,\overline{\alpha}\,]) \cdot [\alpha] = [\,\hat{B}\,] \cdot [\gamma] \cdot [\,\hat{B}\,] = [\gamma].$$

Thus $\Upsilon_{\overline{\alpha}}$ is a two-sided inverse for Υ_α, whence $(\Upsilon_\alpha)^{-1} = \Upsilon_{\overline{\alpha}}$. In other words, Υ_α is a group-isomorphism that exhibits $\kappa_1(\mathbf{C}, A) \cong \kappa_1(\mathbf{C}, B)$.

(iv) If α is constant, necessarily $A = B$, so $\alpha = \hat{A}$. Moreover, for any $\gamma \in \Omega(\mathbf{C}, A)$,

$$\Upsilon_{\hat{A}}[\gamma] = [\,\overline{\hat{A}}\,] \cdot [\gamma] \cdot [\hat{A}] = [\,\hat{A}\,] \cdot [\gamma] = [\gamma].$$

The claim follows.

(v) Let $\gamma \in \Omega(\mathbf{C}, A)$. We have

$$\begin{aligned} \Upsilon_{\alpha \cdot \beta}[\gamma] &= [\,\overline{\alpha \cdot \beta}\,][\gamma][\alpha \cdot \beta] \\ &= [\,\overline{\beta} \cdot \overline{\alpha}\,] \cdot [\gamma] \cdot [\alpha \cdot \beta] \\ &= [\,\overline{\beta}\,] \cdot [\,\overline{\alpha}\,] \cdot [\gamma] \cdot [\alpha] \cdot [\beta] \\ &= \Upsilon_\beta(\Upsilon_\alpha[\gamma]). \end{aligned}$$

The claim follows.

The proof is complete. □

Corollary 5.35

A path-connected finite category $\mathbf{C}$ has isomorphic fundamental groups for any choice of base object. In other words,

$$\kappa_1(\mathbf{C}, A) \cong \kappa_1(\mathbf{C}, B)$$

for any $\mathbf{C}$-objects A and B.

Proof. Follows immediately from Theorem 5.34 (iii). □

In light of this observation, if $\mathbf{C}$ is path connected, the base object will not be specified, and we just refer to the *the fundamental group* of $\mathbf{C}$, denoted by $\kappa_1(\mathbf{C})$.

Example 5.36.

(i) A consequence of Corollary 5.35 is that the elementary disk category $\mathbf{D}^2$, which is path-connected, has trivial fundamental group for any choice of base object. This category is given by the commutative diagram

$$\begin{tikzcd} & \mathfrak{Y} \arrow[dr, "\mathfrak{b}"] & \\ \mathfrak{X} \arrow[ur, "\mathfrak{a}"] & & \mathfrak{Z} \arrow[ll, "\mathfrak{c}"] \end{tikzcd}$$

By the uniqueness of identities, we have $\mathfrak{c} \circ \mathfrak{b} \circ \mathfrak{a} = \mathbb{1}_{\mathfrak{X}}$ and $\mathfrak{a} \circ \mathfrak{c} \circ \mathfrak{b} = \mathbb{1}_{\mathfrak{Y}}$, so $\mathfrak{a}^{-1} = \mathfrak{c} \circ \mathfrak{b}$. This fact, together with the commutativity of the diagram and Theorem 5.29, implies that any loop at $\mathfrak{Z}$ is homotopic to the constant path at $\mathfrak{Z}$. Therefore,

$$\kappa_1(\mathbf{D}^2, \mathfrak{X}) \cong \kappa_1(\mathbf{D}^2, \mathfrak{Y}) \cong \kappa_1(\mathbf{D}^2, \mathfrak{Z}) \cong 0.$$

(ii) Recall that the circle category $\mathbf{S}^1$ is given by the diagram

$$\begin{tikzcd} \mathfrak{X} \arrow[r, bend left, "\mathfrak{a}"] \arrow[r, bend right, "\mathfrak{b}"'] & \mathfrak{Y} \end{tikzcd}$$

This category is path-connected. Indeed, since $\mathfrak{a}^{(0)}$ is a path from $\mathfrak{X}$ to $\mathfrak{Y}$, then $\overline{\mathfrak{a}^{(0)}}$ is a path from $\mathfrak{Y}$ to $\mathfrak{X}$. Corollary 5.35 implies that

$$\kappa_1(\mathbf{S}^1, \mathfrak{X}) \cong \kappa_1(\mathbf{S}^1, \mathfrak{Y}).$$

Definition 5.37 *Simply connected category*

A finite category $\mathbf{C}$ is *simply connected* if $\mathbf{C}$ is path-connected and

$$\kappa_1(\mathbf{C}, A)$$

is the zero group for some (hence any) $\mathbf{C}$-object A.

5.5.2. Homomorphisms Induced by Functors of Finite Categories

The following constructions resemble those presented in subsection 2.3.3. There, we saw that any continuous map $\psi \colon X \to Y$ between topological spaces induces a well-defined homomorphism

$$\psi_* \colon \pi_1(X, p) \to \pi_1(Y, \psi(p))$$

between the fundamental groups of the pointed spaces (X, p) and $(Y, \psi(p))$.

Let $F\colon \mathbf{C} \to \mathbf{D}$ be a functor between finite categories $\mathbf{C}$ and $\mathbf{D}$. Let A be a $\mathbf{C}$-object. Then F induces a map of between the sets of loops

$$F_{\#}\colon \Omega(\mathbf{C}, A) \to \Omega(\mathbf{C}, F(A)) \;:\; \alpha \mapsto F \circ \alpha.$$

Proposition 5.38

Let $F\colon (\mathbf{C}, A) \to (\mathbf{D}, B)$ be a functor of pointed categories. Let α and β be two homotopic loops in $\mathbf{C}$ based at A. Then

$$F_{\#}(\alpha) \sim F_{\#}(\beta).$$

Proof. Let

$$(m, n, p, q) = (\min \alpha, \max \alpha, \min \beta, \max \beta).$$

Let $\mathcal{H}\colon \alpha \sim \beta$. Then there exist even integers $M \leq N$ such that

$$\begin{aligned} \mathcal{H}(-,k) &= \alpha \text{ if } k \leq M, & \mathcal{H}(k,-) &= \hat{A} \text{ if } k \leq \min\{m,p\}, \\ \mathcal{H}(-,k) &= \beta \text{ if } k \geq N, & \mathcal{H}(k,-) &= \hat{B} \text{ if } k \geq \max\{n,q\}. \end{aligned}$$

Making the composition $F \circ \mathcal{H}$ gives

$$\begin{aligned} F\circ\mathcal{H}(-,k) &= F\circ\alpha \text{ if } k \leq M, & F\circ\mathcal{H}(k,-) &= F\circ\hat{A} \text{ if } k \leq \min\{m,p\}, \\ F\circ\mathcal{H}(-,k) &= F\circ\beta \text{ if } k \geq N, & F\circ\mathcal{H}(k,-) &= F\circ\hat{B} \text{ if } k \geq \max\{n,q\}. \end{aligned}$$

It is clear that both $F \circ \hat{A}$ and $F \circ \hat{B}$ are the constant paths at $F(A)$ and $F(B)$, respectively. Hence $F \circ \mathcal{H}\colon F_{\#}(\alpha) \sim F_{\#}(\beta)$. □

The map $F_{\#}$ induces, in turn, a map of fundamental groups defined by

$$F_*\colon \kappa_1(\mathbf{C}, A) \to \kappa_1(\mathbf{D}, F(A)) : [\gamma] \mapsto [F \circ \gamma].$$

By convention, we write $F_*[\gamma]$ instead of $F_*([\gamma])$.

Theorem 5.39

Let $F\colon (\mathbf{C}, A) \to (\mathbf{C}, B)$ be a functor of pointed categories. Then the map of sets

$$F_*\colon \kappa_1(\mathbf{C}, A) \to \kappa_1(\mathbf{D}, B) : [\gamma] \mapsto [F \circ \gamma]$$

is well-defined.

Proof. Follows from Proposition 5.38. □

On the other hand, as expected, F_* is a group homomorphism.

Proposition 5.40

For any functor between finite categories $F\colon \mathbf{C} \to \mathbf{D}$, the induced map $F_*\colon \kappa_1(\mathbf{C}, A) \to \kappa_1(\mathbf{D}, F(A))$ is a group-homomorphism.

Proof. In order to show $F_*[\alpha \cdot \beta] = F_*[\alpha] \cdot F_*[\beta]$, we must prove

$$F \circ (\alpha \cdot \beta) \sim (F \circ \alpha) \cdot (F \circ \beta),$$

whence the result follows immediately. However, we will actually show the stronger result that $F \circ (\alpha \cdot \beta) = (F \circ \alpha) \cdot (F \circ \beta)$. By definition of concatenation of paths, we have

$$(F \circ \alpha) \cdot (F \circ \beta)(k) = \begin{cases} F \circ \alpha(k) & \text{if } k \leq n, \\ F \circ \beta(k - n + p) & \text{otherwise.} \end{cases}$$

Here $n = \max \alpha$ and $p = \min \beta$. Note that this equals $F((\alpha \cdot \beta)(k))$ for every $k \in \mathbb{Z}$, simply because of the fact that

$$\alpha \cdot \beta(k) = \begin{cases} \alpha(k) & \text{if } k \leq n, \\ \beta(k - n + p) & \text{otherwise.} \end{cases}$$

The same argument on morphisms establishes the result. $\square$

Let us note that $*$ has functorial properties since, for any functors

$$F\colon (\mathbf{C}, A) \to (\mathbf{D}, F(A)) \quad \text{and} \quad G\colon (\mathbf{D}, F(A)) \to (\mathbf{E}, G(F(A))),$$

and for any $\gamma \in \Omega(\mathbf{C}, A)$, we have both

$$(G \circ F)_*[\gamma] = [(G \circ F) \circ \gamma] = [G \circ (F \circ \gamma)] = G_*[F \circ \gamma] = G_*(F_*[\gamma])$$

and

$$\left(\mathbf{1}_{(\mathbf{C},A)}\right)_*[\gamma] = [\mathbf{1}_{(\mathbf{C},A)} \circ \gamma] = [\gamma] = \mathrm{Id}_{\kappa_1(\mathbf{C},A)}[\gamma].$$

Corollary 5.41

Let $F\colon (\mathbf{C}, A) \to (\mathbf{D}, F(A))$ be a functor of pointed categories. Suppose F is an isomorphism. Then

$$F_*\colon \kappa_1(\mathbf{C}, A) \to \kappa_1(\mathbf{D}, F(A))$$

is a group-isomorphism.

Proof. Let $G\colon \mathbf{D} \to \mathbf{C}$ be the inverse of F. Note that both F_* and G_* are group isomorphisms and

$$\begin{aligned} G_* \circ F_* &= (G \circ F)_* = \left(\mathbf{1}_{(\mathbf{C},A)}\right)_* = \mathrm{Id}_{\kappa_1(\mathbf{C},A)}, \\ F_* \circ G_* &= (F \circ G)_* = \left(\mathbf{1}_{(\mathbf{D},F(A))}\right)_* = \mathrm{Id}_{\kappa_1(\mathbf{D},F(A))}. \end{aligned}$$

Therefore $(F_*)^{-1} = G_*$. $\square$

Theorem 5.42

Let $F\colon \mathbf{C} \to \mathbf{D}$ be a functor of finite categories. Let A and B be two $\mathbf{C}$-objects that are connected by a $\mathbf{C}$-path γ. Then the diagram

$$\begin{tikzcd}
\kappa_1(\mathbf{C}, A) \arrow[r, "F_*"] \arrow[d, "\Upsilon_\gamma"'] & \kappa_1(\mathbf{D}, F(A)) \arrow[d, "\Upsilon_{F\circ\gamma}"] \\
\kappa_1(\mathbf{C}, B) \arrow[r, "F_*"'] & \kappa_1(\mathbf{D}, F(B))
\end{tikzcd}$$

commutes.

Proof. Let $\alpha \in \Omega(X, p)$. We have

$$\begin{aligned}
[F \circ (\overline{\gamma} \cdot \alpha \cdot \gamma)] &= [(F \circ \overline{\gamma}) \cdot (F \circ (\alpha \cdot \gamma))] \\
&= [F \circ \overline{\gamma}] \cdot [(F \circ \alpha) \cdot (F \circ \gamma)] \\
&= [F \circ \overline{\gamma}] \cdot [F \circ \alpha] \cdot [F \circ \gamma].
\end{aligned}$$

Since $F \circ \overline{\gamma} = \overline{F \circ \gamma}$, we obtain

$$F_*[\overline{\gamma} \cdot \alpha \cdot \gamma] = [\,\overline{F \circ \gamma}\,] \cdot F_*[\alpha] \cdot [F \circ \gamma],$$

which is equivalent to $F_* \left(\Upsilon_\gamma[\alpha]\right) = \Upsilon_{F\circ\gamma} \left(F_*[\alpha]\right)$. The arbitrariness of α allow us to conclude. □

We saw in Theorem 2.34 that the fundamental group of a product of topological spaces is well behaved under products, up to isomorphism. In our theory, this is also true.

Theorem 5.43 *Fundamental group of a categorical product*

Let $(\mathbf{C}_1, A_1), \ldots, (\mathbf{C}_N, A_N)$ be pointed finite categories. Then

$$\kappa_1(\mathbf{C}_1 \times \cdots \times \mathbf{C}_N, (A_1, \ldots, A_N)) \cong \kappa_1(\mathbf{C}_1, A_1) \times \cdots \times \kappa_1(\mathbf{C}_N, A_N).$$

Proof. Note that $(\mathbf{C}_1 \times \cdots \times \mathbf{C}_N, (A_1, \ldots, A_N))$ is itself a pointed category. Let $p_i\colon \mathbf{C}_1 \times \cdots \times \mathbf{C}_n \to \mathbf{C}_i$ be the projection onto the ith factor, for each $1 \leq i \leq N$. By Theorem 5.39, each projection induces a well-defined map

$$p_{i*}\colon \kappa_1(\mathbf{C}_1 \times \cdots \times \mathbf{C}_N, (A_1, \ldots, A_N)) \to \kappa_1(\mathbf{C}_i, A_i)$$

defined by $[\alpha] \mapsto [p_i \circ \alpha]$, for all $1 \leq i \leq N$. Now define

$$\begin{aligned}
\mathfrak{P}\colon \kappa_1(\mathbf{C}_1 \times \cdots \times \mathbf{C}_N, (A_1, \ldots, A_N)) &\to \kappa_1(\mathbf{C}_1, A_1) \times \cdots \times \kappa_1(\mathbf{C}_N, A_N) \\
[\alpha] &\mapsto (p_{1*}[\alpha], \ldots, p_{N*}[\alpha])
\end{aligned}$$

Let us see that $\mathfrak{P}$ is a group-isomorphism. To begin with, since each p_{i*} is a group-homomorphism, so is $\mathfrak{P}$. Indeed, given $\alpha, \beta \in \Omega(\mathbf{C}_1 \times \cdots \times$

$\mathbf{C}_N, (A_1, \ldots, A_N))$, we have

$$\begin{aligned}
\mathfrak{P}([\alpha] \cdot [\beta]) &= \mathfrak{P}[\alpha \cdot \beta] \\
&= (p_{1*}[\alpha \cdot \beta], \ldots, p_{N*}[\alpha \cdot \beta]) \\
&= (p_{1*}[\alpha] \cdot p_{1*}[\beta], \ldots, p_{N*}[\alpha] \cdot p_{N*}[\beta]) \\
&= (p_{1*}[\alpha], \ldots, p_{N*}[\alpha]) \odot (p_{1*}[\beta], \ldots, p_{N*}[\beta]) \\
&= \mathfrak{P}[\alpha] \odot \mathfrak{P}[\beta].
\end{aligned}$$

Recall that the group operation $\odot$ is defined component-wise.

Injectivity of $\mathfrak{P}$ follows from the fact its kernel is trivial. Indeed, suppose $\mathfrak{P}[\alpha]$ equals the identity of $\kappa_1(\mathbf{C}_1, A_1) \times \cdots \times \kappa_1(\mathbf{C}_N, A_N)$, namely $([\hat{A}_1], \ldots, [\hat{A}_N])$. In other words, $(p_{1*}[\alpha], \ldots, p_{N*}[\alpha]) = ([\hat{A}_1], \ldots, [\hat{A}_N])$. Let α_i be the ith component of α, so that $\alpha = (\alpha_1, \ldots, \alpha_N)$. Then

$$[p_i \circ \alpha] = [\alpha_i] = [\hat{A}_i]$$

whence $\alpha_i \sim \hat{A}_i$, for all $1 \leq i \leq n$. Now we can take homotopies $\mathcal{H}_i \colon \alpha_i \sim \hat{A}_i$ for each $1 \leq i \leq n$. We know that there exist even integers $m_i \leq n_i$ such that

$$\mathcal{H}_i(-, k) = \begin{cases} \alpha_i & \text{if } k \leq m_i \\ \hat{A}_i & \text{if } k \geq n_i \end{cases}$$

and $\mathcal{H}_i(k, -) = \hat{A}_i$ whenever $k \leq \min \alpha_i$ or $k \geq \max \alpha_i$. (Keep in mind that the bounds of a constant path can be taken as convenience.) Our goal is to show that the path-class of α is the identity of $\kappa_1(\mathbf{C}_1 \times \cdots \times \mathbf{C}_N, (A_1, \ldots, A_N))$. Define $\mathcal{H} \colon \mathbf{\Lambda} \times \mathbf{\Lambda} \to \mathbf{C}_1 \times \cdots \times \mathbf{C}_N$ by

$$\begin{aligned}
\mathcal{H}(i, j) &= (\mathcal{H}_1(i, j), \ldots, \mathcal{H}_N(i, j)), \quad \text{and} \\
\mathcal{H}(i \to j, k \to l) &= (\mathcal{H}_1(i \to j, k \to l), \ldots, \mathcal{H}_N(i \to j, k \to l))
\end{aligned}$$

on objects and morphisms, respectively. Let

$$\begin{aligned}
m &= \min\{m_1, \ldots, m_N\}, & m' &= \min\{\min \alpha_1, \ldots, \min \alpha_N\}, \\
n &= \max\{n_1, \ldots, n_N\}, & n' &= \max\{\max \alpha_1, \ldots, \max \alpha_N\}.
\end{aligned}$$

Note that

$$\mathcal{H}(-, k) = (\mathcal{H}_1(-, k), \ldots, \mathcal{H}_N(-, k)) = \begin{cases} (\alpha_1, \ldots, \alpha_N) & \text{if } k \leq m, \\ (\hat{A}_1, \ldots, \hat{A}_N) & \text{if } k \geq n. \end{cases}$$

Moreover,

$$\mathcal{H}(k, -) = (\mathcal{H}_1(k, -), \ldots, \mathcal{H}_N(k, -)) = (\hat{A}_1, \ldots, \hat{A}_N)$$

whenever $k \leq m'$ or $k \geq n'$. It is clear that $(\hat{A}_1, \ldots, \hat{A}_N)$ is the constant path based at $(A_1, \ldots, A_N)$. Therefore $\mathcal{H} \colon \alpha \sim (\hat{A}_1, \ldots, \hat{A}_N)$, whence $[\alpha] = [(A_1, \ldots, A_N)]$. We have proven that $\mathfrak{P}$ is injective.

Let us conclude by showing that $\mathfrak{P}$ is surjective. Take

$$([\gamma_1], \ldots, [\gamma_N]) \in \kappa_1(\mathbf{C}_1, A_1) \times \cdots \times \kappa_1(\mathbf{C}_N, A_N).$$

Define $\gamma\colon \mathbf{\Lambda} \to \mathbf{C}_1 \times \cdots \times \mathbf{C}_N$ by

$$k \mapsto (\gamma_1(k), \ldots, \gamma_N(k)), \qquad (i \to j) \ \mapsto \ (\gamma_1(i \to j), \ldots, \gamma_N(i \to j))$$

on objects and morphism, respectively. Note that γ is a loop in the product $\mathbf{C}_1 \times \cdots \times \mathbf{C}_N$ based at $(A_1, \ldots, A_N)$. Moreover, we have

$$\mathfrak{P}[\gamma] = ({p_1}_*[\gamma], \ldots, {p_N}_*[\gamma]) = ([p_1 \circ \gamma], \ldots, [p_N \circ \gamma]) = ([\gamma_1], \ldots, [\gamma_N]).$$

The proof is complete. □

Corollary 5.44

If $\mathbf{C}$ and $\mathbf{D}$ are path-connected finite categories, then

$$\kappa_1\left(\mathbf{C} \times \mathbf{D}\right) \cong \kappa_1\left(\mathbf{C}\right) \times \kappa_1\left(\mathbf{D}\right).$$

5.6. First Computations

5.6.1. The Fundamental Group of $\mathbf{S}^1$

Recall $\mathbf{S}^1$ is the category with exactly two objects and two (distinct) parallel arrows, whose diagram is given by

$$\begin{tikzcd} \mathfrak{X} \arrow[r, bend left, "\mathfrak{a}"] \arrow[r, bend right, "\mathfrak{b}"'] & \mathfrak{Y} \end{tikzcd}$$

Rosero provided a proof in [29] (p. 101) that $\kappa_1(\mathbf{S}^1, \mathfrak{X}) \cong \mathbb{Z}$, which is based on the following (restated but equivalent) claim.

> Let $\mathbf{C}$ be a finite category and $f, g\colon A \to B$ two parallel $\mathbf{C}$-arrows. Let C_n denote the path obtained from chaining n consecutive paths of the form $A \xrightarrow{f} B \xleftarrow{g} A$, as given by the diagram
>
> $$A \xrightarrow{\ f\ } B \xleftarrow{\ g\ } A \longrightarrow \cdots \longleftarrow A \xrightarrow{\ f\ } B \xleftarrow{\ g\ } A$$
>
> Then $C_n \nsim C_m$ for any $m, n \in \mathbb{Z}_0^+$ distinct.

In other words, paths of the form C_n are not path-homotopy equivalent if they have distinct length. However, this claim is not true, as established by the following counterexample.

Example 5.45. Let $\mathbf{C}$ be a finite category that has a morphism with at least two sections. We know that sided inverses do not have to be unique, so such a category exists. More precisely, suppose $\mathbf{C}$ has at least two parallel morphisms $f, g \colon A \to B$ and a morphism $\varphi \colon B \to A$ such that

$$\varphi \circ f = \mathbb{1}_A \quad \text{and} \quad \varphi \circ g = \mathbb{1}_A.$$

Then, the following diagram commutes.

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, equal] & B \arrow[d, equal] & A \arrow[l, "g"'] \arrow[r, "f"] \arrow[d, equal] & B \arrow[d, equal] & A \arrow[l, "g"'] \arrow[d, equal] \\
A \arrow[r, "f"] \arrow[ur, "f"] \arrow[d, equal] \arrow[dr, "f"] & B \arrow[d, equal] & A \arrow[l, "g"'] \arrow[ul, "g"'] \arrow[r, "f"] \arrow[ur, "f"] \arrow[d, equal] \arrow[dl, "g"'] \arrow[dr, equal] & B \arrow[d, "\varphi"] & A \arrow[l, "g"'] \arrow[ul, "g"'] \arrow[d, equal] \arrow[dl, equal] \\
A \arrow[r, "f"'] & B & A \arrow[l, "g"] \arrow[r, equal] & A \arrow[r, equal] & A
\end{tikzcd}$$

Identities are indicated with a double line. Observe that this diagram gives a path-homotopy from C_1 to C_2. Hence $C_1 \sim C_2$ even tough this paths are of distinct length.

As a result, the consequences of this claim as presented in [29], turn out to be uncertain. Nevertheless, with a slight (but strong) modification of the statement, we establish its validity within the framework presented so far in our version of the theory. The key of the following argument is that the choice of a morphism two right inverses is not possible.

Theorem 5.46

Let C_n denote the path in $\mathbf{S}^1$ that consist of n pairs of arrows of the form $\mathfrak{A} \xrightarrow{\mathfrak{a}} \mathfrak{B} \xleftarrow{\mathfrak{b}} \mathfrak{A}$, as in the diagram

$$\mathfrak{A} \xrightarrow{\ \mathfrak{a}\ } \mathfrak{B} \xleftarrow{\ \mathfrak{b}\ } \mathfrak{A} \longrightarrow \cdots \longleftarrow \mathfrak{A} \xrightarrow{\ \mathfrak{a}\ } \mathfrak{B} \xleftarrow{\ \mathfrak{b}\ } \mathfrak{A}$$

For any pair of positive integers m and n, if $C_n \sim C_m$, then $n = m$. By convention, we define $C_0 = \hat{A}$.

Proof. Notice that C_n is reduced and of minimal length, meaning that it does not contain pairs of identities either of the form $\to \cdot \leftarrow$ or the form $\leftarrow \cdot \to$, and it cannot be simplified further. Hence C_n is itself its minimal representative. Now, by a translation if necessary, we can assume that both C_n and C_m start at the same position (by Theorem 5.22), which without loss of generality we suppose is 0. In order to apply an inductive argument, leave m fixed.

We proceed by induction on n. Suppose $C_1 \sim C_m$. Then, by Theorem 5.26, we have

$$\overline{C_1} \cdot C_1 \sim \overline{C_1} \cdot C_m.$$

Since $\overline{C_1} \cdot C_m \sim C_{m-1}$, by construction, we obtain $\hat{\mathfrak{A}} \sim C_{m-1}$, which is impossible unless C_{m-1} is itself the constant path at $\mathfrak{A}$, that is, unless $m - 1 = 0$. Hence, necessarily we obtain $m = 1$. The base case is established.

For the inductive step, fix $n = k$ for some $k \in \mathbb{Z}^+$, and suppose that $k = k'$ whenever $C_k \sim C_{k'}$. Let us see that $C_{k+1} \sim C_m$ implies $k + 1 = m$. Assume $C_{k+1} \sim C_m$. Then

$$\overline{C}_1 \cdot C_{k+1} \sim \overline{C}_1 \cdot C_m \iff C_k \sim C_{m-1}.$$

By the inductive hypothesis, $k = m - 1$ as desired. Therefore, the principle of mathematical induction proves the claim. Finally, since $\sim$ is symmetric, applying the same argument over m, the validity of the assertion is established for every $m, n \in \mathbb{Z}^+$. □

The proof of the following result relies on the well-known fact from group theory that an infinite cyclic group is isomorphic to $\mathbb{Z}$, the group of integers.

Theorem 5.47

The fundamental group of $(\mathbf{S}^1, \mathfrak{A})$ is isomorphic to the abelian group of integers, that is,

$$\kappa_1(\mathbf{S}^1, \mathfrak{A}) \cong \mathbb{Z}.$$

Proof. Let us prove $\kappa_1(\mathbf{S}^1, \mathfrak{A})$ is an infinite cyclic group, from which the assertion will follow immediately. On one hand, every element of $\kappa_1(\mathbf{S}^1, \mathfrak{A})$ is of the form $[C_n]$ for some $n \in \mathbb{Z}_0^+$, by the previous discussion. By Theorem 5.46, $[C_n] \neq [C_m]$ for $m \neq m$. Hence, we obtain an infinite sequence of path classes: $[C_0], [C_1], [C_2], \ldots$. Thus, $\kappa_1(\mathbf{S}^1, \mathfrak{A})$ is infinite. On the other hand, since

$$C_n \sim C_{n-1} \cdot C_1 \sim C_1 \cdots C_1,$$

where the dots denote the product of C_1 with itself n times, we see that $[C_n] = [C_1]^n$. Furthermore, since

$$\overline{C}_1 \cdot C_1 \sim \hat{\mathfrak{A}} \quad \text{and} \quad C_1 \cdot \overline{C}_1 \sim \hat{\mathfrak{A}},$$

we obtain $[C_1]^{-1} = [\overline{C_1}]$, and inductively we see that $[C_1]^{-n} = [\overline{C_1} \cdots \overline{C_1}]$ for every $n \in \mathbb{Z}^+$. Therefore, $\kappa_1(\mathbf{S}^1, \mathfrak{A})$ is cyclic with generator $[C_1]$, that is,

$$\kappa_1(\mathbf{S}^1, \mathfrak{A}) = \left\{ [C_1]^k \mid k \in \mathbb{Z} \right\}.$$

The proof is complete. □

Remark 5.48. Since $\mathbf{S}^1$ is a connected category, it follows by Corollary 5.35 that

$$\kappa_1(\mathbf{S}^1, \mathfrak{A}) \cong \kappa_1(\mathbf{S}^1, \mathfrak{B}).$$

As a result, we just write $\kappa_1(\mathbf{S}^1) \cong \mathbb{Z}$.

5.6.2. The Fundamental Group of $\mathbf{T}^n$

The 2-torus category, denoted $\mathbf{T}^2$, is defined as the product category $\mathbf{S}^1 \times \mathbf{S}^1$. In general, we define $\mathbf{T}^n$, the n-dimensional categorical torus, as the product category $\mathbf{S}^1 \times \cdots \times \mathbf{S}^1$ of $\mathbf{S}^1$ with itself n times. Note that $\mathbf{T}^n$ is connected because $\mathbf{S}^1$ is.

Theorem 5.49

The fundamental group of the n-dimensional torus is the direct product of $\mathbb{Z}$ with itself n times, that is,

$$\kappa_1(\mathbf{T}^n) \cong \mathbb{Z}^n.$$

Proof. Since $\kappa_1(\mathbf{S}^1) \cong \mathbb{Z}$, Theorem 5.43 implies that

$$\kappa_1(\mathbf{T}^n) \cong \kappa_1(\mathbf{S}^1 \times \cdots \times \mathbf{S}^1) \cong \kappa_1(\mathbf{S}^1) \times \cdots \times \kappa_1(\mathbf{S}^1) \cong \mathbb{Z}^n,$$

as claimed. □

Remark 5.50. Regarding the computations just made, we have obtained consistent results with those presented in Chapter 4, where we discussed the geometric realization of the nerve of a small category. See Remark 4.22.

5.7. Homotopy Equivalence

Recall that the identity functor of a category $\mathbf{C}$, denoted $\mathbb{1}_{\mathbf{C}}$, maps every object and every morphism to itself.

Definition 5.51

Two finite categories $\mathbf{C}$ and $\mathbf{D}$ are *homotopy equivalent* if there exist functors $F\colon \mathbf{C} \to \mathbf{D}$ and $G\colon \mathbf{D} \to \mathbf{C}$ such that

$$G \circ F \simeq \mathbb{1}_{\mathbf{C}} \quad \text{and} \quad F \circ G \simeq \mathbb{1}_{\mathbf{D}}.$$

In this case we write $\mathbf{C} \simeq \mathbf{D}$. Equivalently, we say that $\mathbf{C}$ and $\mathbf{D}$ are of the same *homotopy type*. The pair

$$(F, G)$$

is called *homotopy equivalence* between $\mathbf{C}$ and $\mathbf{D}$.

Proposition 5.52

$\simeq$ is an equivalence relation over the objects of $\mathbf{Cat}_{\mathbf{Fin}}$.

Proof. (i) (Reflexivity) Every category is homotopy equivalent to itself via the identity functor.

(ii) (Symmetry) Suppose $\mathbf{C} \simeq \mathbf{D}$, meaning that there is a pair of functors $F\colon \mathbf{C} \to \mathbf{D}$ and $G\colon \mathbf{D} \to \mathbf{C}$ such that

$$G \circ F \simeq \mathbb{1}_{\mathbf{C}} \quad \text{and} \quad F \circ G \simeq \mathbb{1}_{\mathbf{D}},$$

which is equivalent to

$$F \circ G \simeq \mathbb{1}_{\mathbf{D}} \quad \text{and} \quad G \circ F \simeq \mathbb{1}_{\mathbf{C}}.$$

Thus, by definition, $\mathbf{D} \simeq \mathbf{C}$.

(iii) (Transitivity) Suppose $\mathbf{C} \simeq \mathbf{D}$ and $\mathbf{D} \simeq \mathbf{E}$. Then there are functors $F\colon \mathbf{C} \to \mathbf{D}$, $G\colon \mathbf{D} \to \mathbf{C}$, $H\colon \mathbf{D} \to \mathbf{E}$ and $I\colon \mathbf{E} \to \mathbf{D}$ such that

$$G \circ F \simeq \mathbb{1}_{\mathbf{C}}, \quad F \circ G \simeq \mathbb{1}_{\mathbf{D}}, \quad I \circ H \simeq \mathbb{1}_{\mathbf{D}}, \quad \text{and} \quad H \circ I \simeq \mathbb{1}_{\mathbf{E}}.$$

Therefore

$$(G \circ I) \circ (H \circ F) = G \circ (I \circ H) \circ F \sim G \circ \mathbb{1}_{\mathbf{D}} \circ F = G \circ F \simeq \mathbb{1}_{\mathbf{C}}$$

and

$$(H \circ F) \circ (G \circ I) = H \circ (F \circ G) \circ I \simeq H \circ \mathbb{1}_{\mathbf{D}} \circ I = H \circ I \simeq \mathbb{1}_{\mathbf{E}}.$$

Thus, by definition, $\mathbf{C} \simeq \mathbf{E}$.

$\square$

Conclusion

In this work, we have defined the fundamental group of a finite category in a purely algebraic manner, independently of any topological method. We have stated and proved results analogous to those in the classical theory of homotopy. Furthermore, we discussed how our theory yields consistent results with those obtained by computing the fundamental group of the geometric realization of the nerve of the categorical versions of the disk, the circle, and the torus.

This work builds upon Rosero's [29] original approach, which, despite its originality, lacked the rigor necessary for further development. Our contribution lies in establishing a solid foundation for the theory. While we have preserved its core ideas and spirit, most concepts have been reformulated, along with many results and proofs.

A key aspect of this reformulation is the justification of the computations of the categorical fundamental groups of $\mathbf{S}^1$ and $\mathbf{T}^2$, which we showed to be isomorphic to the classical fundamental groups of the circle and the torus, respectively. These facts were originally established by Rosero using a statement that we have now disproved by providing an explicit counterexample. Nevertheless, we have succeeded in constructing a new proof that confirms the validity of these results within our version of the theory.

The theory can be further expanded in several directions, of which we highlight the statement and proof of a categorical analogue of the Seifert-Van Kampen theorem, as well as a deeper investigation into the potential equivalence of κ_1 and π_1, specifically whether

$$\kappa_1(\mathbf{C}) \cong \pi_1(\mathcal{B}\mathbf{C})$$

for any path-connected finite category $\mathbf{C}$.

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